Bachelor of Science
(SIXTH SEMESTER)

MT(N)-305
Mechanics

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: MECHANICS

COURSE CODE:MT(N)-305

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Board of Studies-March 2023

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Course Title and Code : MECHANICS, Course Code- MT(N)-305
Copyright : Uttarakhand Open University
Edition : 2025
Published By : Uttarakhand Open University, Haldwani, Nainital- 263139

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MECHANICS

COURSE INFORMATION

The present self-learning material **"Mechanics"** has been designed for B.Sc. (Sixth Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Three Blocks.

First block is Vector Analysis and Introduction of Mechanics. This block is beginning with vector algebra, vector differentiation, and integral theorems, followed by the study of gradient, divergence, and curl. Fundamental concepts of mechanics such as stress, strain, displacement, and equilibrium are introduced. The Second Second block is Equilibrium of Rigid body and Virtual Work. This block is focuses on equilibrium of rigid bodies, virtual work, and conservative forces. Further, the third block is deals with moment of inertia, rotational dynamics, rectilinear motion including SHM, and constrained motion on smooth curves. Finally The last and fourth block of this course is String in two dimension and central orbit, it explores string motion through the common catenary, kinematics in two dimensions, and motion under central forces, including properties of orbits.

COURSE NAME: MECHANICS **COURSE CODE: MT(N)305**
CREDIT: 04

SYLLABUS

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Vector Multiple Product and Differentiation of Vector- Triple product, Geometrical interpretation of scalar triple product, Reciprocal system of vectors, Differentiation of vectors. **Green, Gauss and Stok's Theorem-** Introduction of vector functions, Line integral Surface integral, Volume integr green's theorem, The divergence theorem of Gauss, Stoke's theorem. **Gradiant Divergent and Curl-** Partial derivatives of vectors, Gradient of scalar field, Divergence of vector point function, Curl of vector point function, Laplacian operators. **Introduction of Mechanics-** Basic Concept of Mechanics, what is stress, what is displacement, what is strain, Basic equation of mechanics, Equilibrium Equation, Strain Displacement relation, Compatibility equation, Constitutive relation **Equilibrium of a rigid body**: Moment of a force about a point, Sigh of the moment, General theorem of moments, Couple. **Definition of virtual work,** Principle of virtual work, Force which can be omitted in forming the equation of virtual work. **Conservative forces and inverse square law**: Conservative and non-conservative forces. **Moment of inertia:** Equation of motion, angular momentum vector, Moment of inertia and radius of gyration Physical significance of MI, theorems of parallel and perpendicular axes, Rotational kinetic energy. **Formulation of moment of inertia:** Formulation and derivation of moment of inertia for some simple symmetric systems (rod, rectangular lamina, circular lamina, solid sphere, cylinder). **Rectilinear Motion:** Simple harmonic motion (SHM) and its geometrical representation, SHM under elastic forces, Motion under inverse square law, Motion in resisting media, Concept of terminal velocity, Motion of varying mass.. **Constrained motion in a plane smooth curve**: Motion on a smooth curve in a vertical plane, Discuss the motion of a particle, Use of principle of conservation of energy. Cycloid motion. **Common catenary**: Common catenary Intrinsic and Cartesian equations of the common catenary, Approximations of the catenary. **Kinematics in two Dimension:** Angular Velocity, Rate of change of Unit Vector, Relation Between Angular and Liner Velocity, Component of velocity and acceleration along the coordinate axes in two dimensions. Meaning of central orbit, Differential equation of the orbit, (p, r) equation of the orbit, Apses and apsidal distances, Areal velocity, Characteristics of central orbits

BLOCK I- VECTOR ANALYSIS AND INTRODUCTION OF MECHANICS

UNIT-1: VECTORS MULTIPLE PRODUCTS AND DIFFERENTIATION OF VECTORS

CONTENTS:

1.1 *INTRODUCTION*

There are two forms of vector multiplication. The cross product of two vectors and the dot product of two vectors are the two ways to multiply a vector since a vector contains both magnitude and direction. Given that the resulting value is a scalar quantity, the dot product of two vectors is also known as the scalar product. As the result is a vector that is perpendicular to these two vectors, the cross product is also known as the vector product.

In this unit learners will be learn more about the two-vector multiplication, including its working principle, attributes, applications, and examples.

1.2 OBJECTIVES

After reading this unit learners will be able to

- Memorized about the vector triple product and scalar triple product and also their geometrical representation.
- Analyze about the reciprocal system of vectors.
- Analyze the application of differentiation of vectors.
- Memorized the useful theorems and their application of vector triple product and scalar triple product.

1.3 TRIPLE PRODUCT

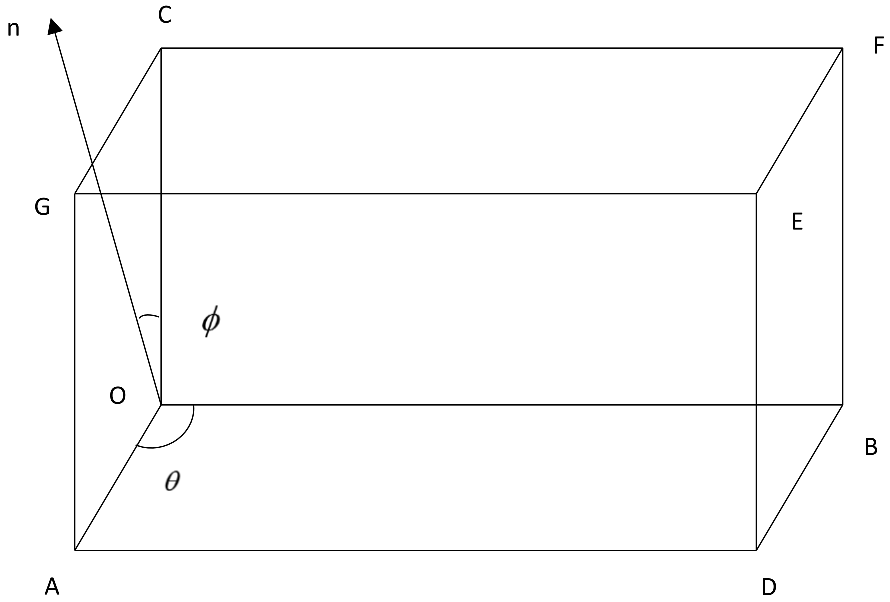
Triple Product: As we know that the vector product $\vec{a}\times\vec{b}$ and scalar product $\vec{a}.\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ are always a vector quantity and scalar quantity respectively. Therefore, if we multiply to these quantities with another vector quantity $\vec{c}$ by both vectorially and scalarly i.e., $(\vec{a}\times\vec{b})\times\vec{c}$ called **vector triple product** similarly $(\vec{a}\times\vec{b}).\vec{c}$ is called **scalar triple product**.

Remark: (***a***) Vector triple product is again a vector quantity.
(***b***) Scalar triple product again is a scalar quantity.
(***c***) $(\vec{a}.\vec{b})\times\vec{c}$ and $(\vec{a}.\vec{b}).\vec{c}$ are meaningless because scalar quantity $(\vec{a}.\vec{b})$ never be product vectorially and scalarly with any vector quantity. Similarly, the product $\vec{a}\times\vec{b}.\vec{c}$ is meaningless so it is meaningful only if it is written in some sense $(\vec{a}\times\vec{b}).\vec{c}$.

Scalar Triple Product: The scalar triple product is the scalar product of two vectors in which one of the vectors is itself vector product of two vectors. Thus if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then, $(\vec{a}\times\vec{b}).\vec{c}$ is called **scalar triple product**.

Some books named scalar triple product as **mixed product** because in this product both 'cross' and 'dot' signs involved.

1.3.1 GEOMETRICAL INTERPRETATION OF SCALAR TRIPLE PRODUCT



To explain geometrically to scalar triple product, we consider a parallelopiped whose edges and length are *OA*, *OB*, *OC* in the direction of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Let the volume of parallelopiped is *V* which is necessarily positive.

Let $\vec{a}\times\vec{b}=\overrightarrow{n}$, then from definition of vector product it is clear that $\overrightarrow{n}$ is perpendicular to the face *OADB* and $\left|\overrightarrow{n}\right|$ is measure as the area of parallelogram OADB. Since, by definition, vectors, $\vec{a}$, $\vec{b}$ and $\overrightarrow{n}$ form right-handed triad.

Let ϕ is the angle between the vectors $\overrightarrow{OC}$ and $\overrightarrow{n}$. Then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form right-handed or a left-handed triad according as ϕ to be acute and obtuse.

Now, $\left(\overrightarrow{a}\times\overrightarrow{b}\right).\overrightarrow{c}=\left|\left(\overrightarrow{a}\times\overrightarrow{b}\right)\right|.\left|\overrightarrow{c}\right|\cos\phi=\left|\overrightarrow{n}\right|\left|\overrightarrow{c}\right|\cos\phi$

= (area of the parallelogram *OADB*).(OC$\cos\phi$) $\left[\because |c|=OC\right]$

So, according to the value of ϕ is acute or obtuse, $OC\cos\phi$ will depend positive or negative. Its absolute value gives the length of the perpendicular from *C* to the plane *OADB*.

So, V = (Area of the Parallelogram *OADB*).(Length of perpendicular from C to the parallelogram *OADB*).

Therefore, if ϕ is acute, $\left(\vec{a}\times\vec{b}\right).\vec{c}=+V$ i.e., if $\vec{a}$, $\vec{b}$ and $\vec{n}$ form right-handed triad.

And, if ϕ is obtuse, $\left(\vec{a}\times\vec{b}\right).\vec{c}=-V$ i.e., if $\vec{a}$, $\vec{b}$ and $\vec{n}$ form left-handed triad.

Since, we know that the vectors $\vec{a}$, $\vec{b}$ $\vec{c}$ are right-handed triad so, vectors $\vec{b}$, $\vec{c}$, $\vec{a}$ and $\vec{c}$, $\vec{a}$, $\vec{b}$ are also right-handed triad. Hence each product $\left(\vec{b}\times\vec{c}\right).\vec{a}$ and $\left(\vec{c}\times\vec{a}\right).\vec{b}$ will have the same value $+V$ or $-V$ according as $\vec{a}$, $\vec{b}$ $\vec{c}$ are left-handed triad.

Thus, $\left(\vec{a}\times\vec{b}\right).\vec{c}=\left(\vec{b}\times\vec{c}\right).\vec{a}=\left(\vec{c}\times\vec{a}\right).\vec{b}$

$\therefore\ \vec{a}.\vec{b}=\vec{b}.\vec{a}$ and $\vec{a}\times\vec{b}=-\vec{b}.\vec{a}$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\vec{c}.\left(\vec{a}\times\vec{b}\right)=\left(\vec{b}\times\vec{c}\right).\vec{a}=\vec{a}.\left(\vec{b}\times\vec{c}\right)=\left(\vec{c}\times\vec{a}\right).\vec{b}=\vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$=-\left(\vec{b}\times\vec{a}\right).\vec{c}=-\vec{c}.\left(\vec{b}\times\vec{a}\right)=-\left(\vec{c}\times\vec{b}\right).\vec{a}$$

$$=-\vec{a}.\left(\vec{c}\times\vec{b}\right)=-\left(\vec{a}\times\vec{c}\right).\vec{b}=-\vec{b}.\left(\vec{a}\times\vec{c}\right)$$

From this we conclude that *value of scalar triple product depends on the cyclic order of the factors and is independent of the position of the dot and cross. These may be interchanged at pleasure. However, an analytic permutation of the three factors changes the value of the product in sign but not in magnitude*. The notation used to write scalar triple product is $\left(\vec{a}\times\vec{b}\right).\vec{c}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left[\vec{a},\vec{b},\vec{c}\right]$. This notation takes into consideration only the cyclic order of three vectors and disregards the unimportant position of dot and cross.

i.e., $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left[\vec{b}\,\vec{c}\,\vec{a}\right]=\left[\vec{c}\,\vec{a}\,\vec{b}\right]=-\left[\vec{c}\,\vec{b}\,\vec{a}\right]$ etc.

Note 1: If $\hat{i}, \hat{j}, \hat{k}$ constitutes an orthogonal right-handed triad of unit vectors, then $\left[\hat{i}\,\hat{j}\,\hat{k}\right] = \left(\hat{i}\times\hat{j}\right).\hat{k} = \hat{k}.\hat{k} - = 1$

2: As nature of $\vec{a}$, $\vec{b}$, $\vec{c}$ are right-handed or left-handed, scalar triple product $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ will be decided to be positive or negative.

Distributive law for vector product:

To prove that $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, where $\vec{a}$, $\vec{b}$ $\vec{c}$ are any three vectors.

Let $\vec{r} \equiv \vec{a}\times(\vec{b}+\vec{c}) - \vec{a}\times\vec{b} - \vec{a}\times\vec{c}$ …
(1)

Now scalar product both side by the vector $\vec{d}$, we get

$\vec{d}.\vec{r} = \vec{d}.\left[\vec{a}\times(\vec{b}+\vec{c}) - \vec{a}\times\vec{b} - \vec{a}\times\vec{c}\right]$ …

(2)

$\vec{d}.\vec{r} = \vec{d}.\left[\vec{a}\times(\vec{b}+\vec{c})\right] - \vec{d}.(\vec{a}\times\vec{b}) - \vec{d}.(\vec{a}\times\vec{c})$ [Scalar product follows the distributive law]

As we know that position of cross and dot can be interchanged without affecting its value.

$\vec{d}.\vec{r} = (\vec{d}\times\vec{a}).(\vec{b}+\vec{c}) - (\vec{d}\times\vec{a}).\vec{b} - (\vec{d}\times\vec{a}).\vec{c}$

$\vec{d}.\vec{r} = (\vec{d}\times\vec{a}).\vec{b} + (\vec{d}\times\vec{a}).\vec{c} - (\vec{d}\times\vec{a}).\vec{b} - (\vec{d}\times\vec{a}).\vec{c}$ [scalar product is distributive]

= 0

$\Rightarrow \vec{d} = 0$ or $\vec{r} = 0$ or $\vec{d}$ is perpendicular to $\vec{r}$. But we had taken $\vec{d}$ as arbitrary. So, we can choose it non zero and not perpendicular to $\vec{r}$.

So, $\vec{r} = 0$ i.e., $\vec{a}\times(\vec{b}+\vec{c}) - \vec{a}\times\vec{b} - \vec{a}\times\vec{c} = 0$

$\Rightarrow \vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, hence proved.

Properties of scalar triple Product:

(1) If two vectors of a scalar product are equal then its value will be zero.

Proof: Let three vectors are $\vec{a}, \vec{a}, \vec{b}$ in which two vectors are equal. So, there scalar product is $\left[\vec{a}\,\vec{a}\,\vec{b}\right] = \left[\vec{a}, \vec{a}, \vec{b}\right] = \vec{a}.(\vec{a}\times\vec{b})$

Since we know that $(\vec{a}\times\vec{b})$ is the perpendicular vector to the plane of $\vec{a}$ and $\vec{b}$. It means dot product of vector $(\vec{a}\times\vec{b})$ with vector $\vec{a}$ and $\vec{b}$ will be zero.

$\Rightarrow \vec{a}.(\vec{a}\times\vec{b}) = 0$ i.e., $\left[\vec{a}\,\vec{a}\,\vec{b}\right] = 0$

(2) If two vectors are parallel then value of scalar triple product will be zero.

Proof: Let three vectors are $\vec{a}, \vec{b}, \vec{c}$ in which vector $\vec{b}$ is parallel to $\vec{a}$ i.e., $\vec{b} = k\,\vec{a}$. So there scalar product is,

$$\left[\vec{a}, \mathrm{k}\,\vec{a}, \vec{b}\right] = \vec{a}.(\mathrm{k}\,\vec{a}\times\vec{b})$$

$$= k\left[\vec{a}.(\vec{a}\times\vec{b})\right]$$

$$= k\left[\vec{a}, \vec{a}, \vec{b}\right] = k.0 = 0$$

(3) The necessary and sufficient condition for three non-parallel and non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be coplanar is that $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$=0

Proof: *Necessary Condition*: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ are three coplanar vectors. As we know that vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{a}$ and $\vec{b}$. Since vector $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar so, vector $\vec{c}$ will also perpendicular to the vector $(\vec{a}\times\vec{b})$.

As we know that if two vectors $\vec{\alpha}$, $\vec{\beta}$ are perpendicular then $\vec{\alpha}.\vec{\beta} = 0$

So, $(\vec{a}\times\vec{b}).\vec{c} = 0$

$\Rightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

Sufficient Condition: Let $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$ i.e., $\left(\vec{a}\times\vec{b}\right).\vec{c} = 0$ it means vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{c}$. Since $(\vec{a}\times\vec{b})$ is also perpendicular to both the vector $\vec{a}$ and $\vec{b}$. Hence vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{a}$, $\vec{b}$, $\vec{c}$. It means vectors are on the same plane i.e., these are coplanar.

(4) As distributive law holds for both vector and scalar product, it holds for the scalar triple product.

Thus $\left[\vec{a},\vec{b}+\vec{d},\vec{c}+\vec{r}\right]=\left[\vec{a},\vec{b},\vec{c}\right]+\left[\vec{a},\vec{b},\vec{r}\right]+\left[\vec{a},\vec{d},\vec{c}\right]+\left[\vec{a},\vec{d},\vec{r}\right]$, the order of cycle of the factor being maintained in each term.

To express the scalar triple product in terms of rectangular component of the vector:

Let $\vec{a}=a_1i+a_2j+a_3k,\quad \vec{b}=b_1i+b_2j+b_3k,\quad \vec{c}=c_1i+c_2j+c_3k$

$$\because \vec{b}\times\vec{c}=\left(b_1i+b_2j+b_3k\right)\times\left(c_1i+c_2j+c_3k\right)=\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=(\mathrm{b}_2c_3-\mathrm{b}_3c_2)i-(\mathrm{b}_1c_3-\mathrm{b}_3c_1)j+(\mathrm{b}_1c_2-\mathrm{b}_2c_1)k$$

$$\vec{a}.(\vec{b}\times\vec{c})=\left(a_1i+a_2j+a_3k\right).(\mathrm{b}_2c_3-\mathrm{b}_3c_2)i-(\mathrm{b}_1c_3-\mathrm{b}_3c_1)j+(\mathrm{b}_1c_2-\mathrm{b}_2c_1)k$$

$$=a_1(\mathrm{b}_2c_3-\mathrm{b}_3c_2)-a_2(\mathrm{b}_1c_3-\mathrm{b}_3c_1)+a_3(\mathrm{b}_1c_2-\mathrm{b}_2c_1)$$

$$\left[\therefore i.i=j.j=k.k=1, i.j=j.k=k.i=0\right]$$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\vec{c}.\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It means position of dot and cross in scalar triple product is independent.

Expression of the scalar triple product in terms of three non-coplanar vectors *l, m, n*:

Let, $\vec{a}=a_1l+a_2m+a_3\,\mathrm{n},\quad \vec{b}=b_1l+b_2m+b_3\,\mathrm{n},\quad \vec{c}=c_1l+c_2m+c_3n$

So,

$$\vec{b}\times\vec{c}=\left(b_1l+b_2m+b_3n\right)\times\left(c_1l+c_2m+c_3n\right)$$

$$=b_1c_1l\times l+b_1c_2l\times m+b_1c_3l\times n+b_2c_1m\times l+b_2c_2m\times m+b_2c_3m\times n+b_3c_1n\times l+b_3c_2n\times m+b_3c_3n\times n$$

$$\left[\because l\times l=0, l\times m=-m\times l\, etc.\right]$$

$$=b_1c_2 l\times m+b_1c_3 l\times n-b_2c_1 l\times m+b_2c_3 m\times n-b_3c_1 l\times n-b_3c_2 m\times n$$

$$=\left(b_1c_2-b_2c_1\right)l\times m+(b_2c_3-b_3c_2)m\times n+(b_1c_3-b_3c_1)l\times n$$

$$=(b_2c_3-b_3c_2)m\times n-(b_1c_3-b_3c_1)n\times l+\left(b_1c_2-b_2c_1\right)l\times m$$

$$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=\left(a_1 l+a_2 m+a_3\,\mathrm{n}\right).\left[(b_2c_3-b_3c_2)m\times n-(b_1c_3-b_3c_1)n\times l+\left(b_1c_2-b_2c_1\right)l\times m\right]$$

$$=a_1(b_2c_3-b_3c_2)[l\,m\,n]-a_2(b_1c_3-b_3c_1)[l\,m\,n]+a_3\left(b_1c_2-b_2c_1\right)[l\,m\,n]$$

[$\because [l\,m\,n]=[m\,n\,l]=[n\,l\,m]$ and scalar triple product in which two vectors are same is equal to zero i.e., $\because [l\,l\,n]=0$]

Hence, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[l\,m\,n]$

Solved Example

Example 1: If vectors $\vec{a}=2i-j+k,\ \ \vec{b}=i+2j-3k,\ \ \vec{c}=3i+pj+5k$ are coplanar then find the value of constant.

Answer: As we know that three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar then

$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

Now, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix}=2(10+3p)+1(5+9)+1(p-6)$ = $7p$ + 28

Since, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

$\Rightarrow 7p+28=0$

$\Rightarrow p=-4$

Example 2: If the vectors $\vec{a}=2i-4j+5k,\ \ \vec{b}=i-j+k,\ \ \vec{c}=3i-5j+2k$ are representing the edges of parallelopiped, then find its volume.

Answer: Since we know that the volume of parallelopiped is equal to the absolute value of scalar triple product of its edges i.e., $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}2 & -4 & 5\\ 1 & -1 & 1\\ 3 & -5 & 2\end{vmatrix}=2(-2+5)+4(2-3)+5(-5+3)$$

$$=6-4-10=-8$$

Since volume of surface never be a negative quantity, so required volume of parallelopiped is 8.

Example 3: Prove that the points $4i+5j+k, -(\mathrm{j}+\mathrm{k}), 3\mathrm{i}+9\mathrm{j}+4\mathrm{k}$ and $4(-i+j+k)$ are coplanar.

Answer: Let *A*, *B*, *C*, *D* are the four points whose position vectors position vectors are given from the origin *O*.

It means, $\vec{OA}=4i+5j+k,\ \vec{OB}=-(\mathrm{j}+\mathrm{k})$, $\vec{OC}=3i+9j+4k$ and $\vec{OD}=4(-i+j+k)$.

If we have to show that four points *A*, *B*, *C*, *D* are coplanar then we have only to prove that the vectors $\vec{AB},\vec{AC},\vec{AD}$ are coplanar.

Now, $\vec{AB}=\vec{OB}-\vec{OA}=-(j+k)-(4i+5j+k)=-4i-6j-2k=a$ (say)

$\vec{AC}=\vec{OC}-\vec{OA}=(3\mathrm{i}+9j+4k)-(4i+5j+k)=-i+4j+3k=b$ (say)

$\vec{AD}=\vec{OD}-\vec{OA}=(-4\mathrm{i}+4j+4k)-(4i+5j+k)=-8i-j+3k=c$ (say)

$$\left[\vec{AB},\vec{AC},\vec{AD}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}=-4(12+3)+6(-3+24)-2(1+32)$$

$$=-60+126-66=0$$

So, we can say that given four vectors are coplanar.

Example 4: Prove that the points $-a+4b-3c,\ 3a+2b-5c,\ -3a+8b-5c$ and $-3a+2b+c$ are coplanar.

Answer: Let *A*, *B*, *C*, *D* are the four points whose position vectors position vectors are given from the origin *O*.

It means, $\vec{OA}=-a+4b-3c,\ \vec{OB}=3a+2b-5c$, $\vec{OC}=-3a+8b-5c$ and $\vec{OD}=-3a+2b+c$.

If we have to show that four points *A*, *B*, *C, D* are coplanar then we have only to prove that the vectors $\vec{AB},\vec{AC},\vec{AD}$ are coplanar.

Now, $\vec{AB}=\vec{OB}-\vec{OA}=3a+2b-5c-(-\mathrm{a}+4\mathrm{b}-3\mathrm{c})=4a-2b-2c$

$$\vec{AC} = \vec{OC} - \vec{OA} = (-3\mathrm{a}+8\mathrm{b}-5\mathrm{c}) - (-\mathrm{a}+4\mathrm{b}-3\mathrm{c}) = -2a+4b-2c$$

$$\vec{AD} = \vec{OD} - \vec{OA} = (-3a+2b+c) - (-\mathrm{a}+4\mathrm{b}-3\mathrm{c}) = -2a-2b+4c$$

$$\left[\vec{AB}, \vec{AC}, \vec{AD}\right] = \begin{vmatrix} -4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} \left[a\ b\ c\right] = \{-4(16-4)+2(-8-4)-2(4+8)\}\left[a\ b\ c\right]$$

$$= \{-48-24-24\}\left[a\ b\ c\right] = 0$$

So, we can say that given four vectors are coplanar.

Example 5: Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$

Answer: $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = \left(\vec{a}+\vec{b}\right).\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]$

$$= \left(\vec{a}+\vec{b}\right).\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right]$$

$$= \vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{c}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)$$

$$+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{c}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right]+\left[\vec{a}, \vec{b}, \vec{a}\right]+\left[\vec{a}, \vec{c}, \vec{c}\right]+\left[\vec{a}, \vec{c}, \vec{a}\right]$$

$$+\left[\vec{b}, \vec{b}, \vec{c}\right]+\left[\vec{b}, \vec{b}, \vec{a}\right]+\left[\vec{b}, \vec{c}, \vec{c}\right]+\left[\vec{b}, \vec{c}, \vec{a}\right]$$

(If two vectors of a scalar product are equal then its value will be zero.)

$$= \left[\vec{a}, \vec{b}, \vec{c}\right]+0+0+0+0+0+0+\left[\vec{b}, \vec{c}, \vec{a}\right]$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right]+\left[\vec{a}, \vec{b}, \vec{c}\right]$$

$$\left[\because \left[\vec{a}, \vec{b}, \vec{c}\right] = \left[\vec{b}, \vec{a}, \vec{c}\right] = \left[\vec{c}, \vec{a}, \vec{b}\right]\right]$$

$$= 2\left[\vec{a}, \vec{b}, \vec{c}\right]$$

Example 6: Prove that $[l\,m\,n][a\,b\,c] = \begin{vmatrix} l.a & l.b & l.c \\ m.a & m.b & m.c \\ n.a & n.b & n.c \end{vmatrix}$

Answer: Let $l = l_1 i + l_2 j + l_3 k,\ m = m_1 i + m_2 j + m_3 k,\ n = n_1 i + n_2 j + n_3 k$

and $a = a_1 i + a_2 j + a_3 k,\ b = b_1 i + b_2 j + b_3 k,\ c = c_1 i + c_2 j + c_3 k$

Now taking LHS = $[l\,m\,n][a\,b\,c] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} l_1 a_1 + l_2 a_2 + l_3 a_3 & l_1 b_1 + l_2 b_2 + l_3 b_3 & l_1 c_1 + l_2 c_2 + l_3 c_3 \\ m_1 a_1 + m_2 a_2 + m_3 a_3 & m_1 b_1 + m_2 b_2 + m_3 b_3 & m_1 c_1 + m_2 c_2 + m_3 c_3 \\ n_1 a_1 + n_2 a_2 + n_3 a_3 & n_1 b_1 + n_2 b_2 + n_3 b_3 & n_1 c_1 + n_2 c_2 + n_3 c_3 \end{vmatrix}$$

As we know that, $l.a = l_1 a_1 + l_2 a_2 + l_3 a_3$

Similarly, we can write

$$L.H.S. = [l\,m\,n][a\,b\,c] = \begin{vmatrix} l.a & l.b & l.c \\ m.a & m.b & m.c \\ n.a & n.b & n.c \end{vmatrix}$$

VECTOR TRIPLE PRODUCT: Vector triple product is the vector product of two vectors in which one is itself the vector product of two vectors. Thus if $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors then the product of the form $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ etc. are called "Vector Triple Products".

Theorem 1: To prove that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \,.\, \vec{c}\right)\vec{b} - \left(\vec{a} \,.\, \vec{b}\right)\vec{c}$

Proof: Let $\vec{r} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and $\vec{b} \times \vec{c} = \vec{d}$

As we know that if $\vec{b} \times \vec{c} = \vec{d}$, it means $\vec{d}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. Also, $\vec{r} = \vec{a} \times \vec{d}$ i.e., $\vec{r}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{d}$. Now the vector $\vec{r}$ is perpendicular to the vector $\vec{d}$, whereas the vector $\vec{d}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. It means $\vec{r}$ must lie in the plane containing $\vec{b}$ and $\vec{c}$.

$\Rightarrow \vec{r} = l\,\vec{b} + m\,\vec{c}$, where, l and m are scalars. ……
(1)

Since $\vec{r}$ is perpendicular to the vector $\vec{a}$, then $\vec{r}\,.\,\vec{a} = \vec{0}$

$$\Rightarrow \left(l\,\vec{b} + m\,\vec{c}\right).\vec{a} = l\left(\vec{b}\,.\,\vec{a}\right) + m\left(\vec{c}\,.\,\vec{a}\right)$$

Let, $\dfrac{l}{\left(\vec{c}\,.\,\vec{a}\right)} = \dfrac{-m}{\left(\vec{b}\,.\,\vec{a}\right)} = \lambda$ (say)

Putting the value of scalars l, m in (1)

$$\vec{r} = \lambda\left(\vec{c}\,.\,\vec{a}\right)\vec{b} - \lambda\left(\vec{b}\,.\,\vec{a}\right)\vec{c} = \lambda\left[\left(\vec{c}\,.\,\vec{a}\right)\vec{b} - \left(\vec{b}\,.\,\vec{a}\right)\vec{c}\right]$$

……(2)

Now we have to find the value of λ.

Consider unit vectors $\hat{j}$ and $\hat{k}$, the first parallel to $\vec{b}$ and second perpendicular to it in the plane containing $\vec{b}$ and $\vec{c}$. Then we may write $\vec{b} = b_2\,\hat{j}$ and $\vec{c} = c_2\,\hat{j} + c_3\,\hat{k}$, then remaining vector may be written as $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$

Now, $\vec{b}\times\vec{c} = b_2\,\hat{j}\times\left(c_2\,\hat{j} + c_3\,\hat{k}\right) = b_2c_2\left(\hat{j}\times\hat{j}\right) + b_2c_3\left(\hat{j}\times\hat{k}\right) = b_2c_3\,\hat{i}$

$$\vec{r} = \vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}\right)\times b_2c_3\,\hat{i}$$

$$= a_1b_2c_3\,\hat{i}\times\hat{i} + a_2b_2c_3\,\hat{j}\times\hat{i} + a_3b_2c_3\,\hat{k}\times\hat{i} = a_3b_2c_3\,\hat{j} - a_2b_2c_3\,\hat{k} \qquad …….(3)$$

$$\left[\because \hat{i}\times\hat{i} = 0,\ \hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{i} = \hat{j}\right]$$

Also,

$$\vec{r} = \lambda\left(\vec{c}\,.\,\vec{a}\right)\vec{b} - \lambda\left(\vec{b}\,.\,\vec{a}\right)\vec{c} = \lambda\left[\left(c_2\,\hat{j} + c_3\,\hat{k}\right).\left(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}\right)\overset{\rightarrow}{b_2}\,\hat{j} - \overset{\rightarrow}{b_2}\,\hat{j}\,.\left(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}\right)\left(c_2\,\hat{j} + c_3\,\hat{k}\right)\right]$$

$$\vec{r} = \lambda\left[c_2a_2b_2\,\hat{j} + c_3a_3b_2\,\hat{j} - b_2a_2c_2\,\hat{j} - b_2a_2c_3\,\hat{k}\right]$$

$$= \lambda\left[c_3a_3b_2\,\hat{j} - b_2a_2c_3\,\hat{k}\right]$$

………. (4)

Now from equation (3) and (4), we conclude that $\lambda = 1$

Hence, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Corollary:

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=-\left[\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right]=-\left[\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}\right]=\left(\vec{c}.\vec{a}\right)\vec{b}-\left(\vec{c}.\vec{b}\right)\vec{a}$$

Solved Example

Example 7: Prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$.

Answer: We know that, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

……..(1)

Similarly, $\vec{b}\times\left(\vec{c}\times\vec{a}\right)=\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}$

…… (2)

and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}$

…… (3)

Now, adding equation (1), (2) and (3)

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}+\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}+\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}$$

As we know that vector scalar product is commutative i.e., $\vec{a}.\vec{b}=\vec{0}$

$$\Rightarrow \vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$$

Example 8: Prove that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right),\vec{b}\times\left(\vec{c}\times\vec{a}\right),\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar.

Answer: Let $\vec{r_1}=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, $\vec{r_2}=\vec{b}\times\left(\vec{c}\times\vec{a}\right)$, $\vec{r_3}=\vec{c}\times\left(\vec{a}\times\vec{b}\right)$.

To prove that these vectors are coplanar, first we have to prove that $\vec{r_1}+\vec{r_2}+\vec{r_3}=0$. In the previous example we have already prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$ i.e., $\vec{r_1}+\vec{r_2}+\vec{r_3}=0$.

It means any one of these vectors can be expressed in terms of other two vectors. Hence these vectors are coplanar.

Example 9: If the vectors $\vec{a} = i - 2j + k,\quad \vec{b} = 2i + j + k,\quad \vec{c} = i + 2j - k$ then find $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$.

Answer: We have $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} . \vec{c}\right)\vec{b} - \left(\vec{a} . \vec{b}\right)\vec{c}$

$$\left(\vec{a} . \vec{c}\right)\vec{b} = \left[(i - 2j + k).(i + 2j - k)\right](2i + j + k) = (1 - 4 - 1)(2i + j + k)$$

$$\left(\vec{a} . \vec{c}\right)\vec{b} = -8i - 4j - 4k$$

Similarly,

$$\left(\vec{a} . \vec{b}\right)\vec{c} = \left[(i - 2j + k).(2i + j + k)\right](i + 2j - k) = (2 - 2 + 1)(i + 2j - k)$$

$$\left(\vec{a} . \vec{b}\right)\vec{c} = i + 2j - k$$

Hence, $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} . \vec{c}\right)\vec{b} - \left(\vec{a} . \vec{b}\right)\vec{c} = (-8i - 4j - 4k) - (i + 2j - k)$

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = -9i - 6j - 3k$$

Example 10: Show that $\left[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\right] = [a\ b\ c]^2$ and also express the result in terms of determinants.

Answer: We know that, $\left[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\right] = \left(\vec{a} \times \vec{b}\right).\left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right]$

Let $\vec{d} = \vec{b} \times \vec{c}$, then $\vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left(\vec{d} . \vec{a}\right)\vec{c} - \left(\vec{d} . \vec{c}\right)\vec{a}$

$$\vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left[\left(\vec{b} \times \vec{c}\right) . \vec{a}\right]\vec{c} - \left[\left(\vec{b} \times \vec{c}\right) . \vec{c}\right]\vec{a} = \left[\vec{b}\,\vec{c}\,\vec{a}\right]\vec{c} - \left[\vec{b}\,\vec{c}\,\vec{c}\right]\vec{a} = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

$$\left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

$$\left(\vec{a} \times \vec{b}\right).\left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \left(\vec{a} \times \vec{b}\right).\left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

Since we know that scalar triple product is the scalar.

$$\Rightarrow \left(\vec{a}\times\vec{b}\right).\left[\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\left\{\left(\vec{a}\times\vec{b}\right).\vec{c}\right\}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]\left[\vec{a}\,\vec{b}\,\vec{c}\right]$$

$$\Rightarrow \left(\vec{a}\times\vec{b}\right).\left[\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$$

Let $\vec{a}=a_1i+a_2j+a_3\text{k}$, $\vec{b}=b_1i+b_2j+b_3\text{k}$, $\vec{c}=c_1i+c_2j+c_3k$, then,

$$\left[a\,b\,c\right]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

again $\vec{a}\times\vec{b}=\begin{vmatrix}i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\left(a_2a_3-b_2a_3\right)i+\left(b_1a_3-a_1b_3\right)j+\left(a_1b_2-a_2b_1\right)k$

similarly, $\vec{b}\times\vec{c}=\begin{vmatrix}i & j & k\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\left(b_2c_3-b_3c_2\right)i+\left(c_1b_3-b_1c_3\right)j+\left(b_1c_2-c_1b_2\right)k$

and $\vec{c}\times\vec{a}=\begin{vmatrix}i & j & k\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}=\left(c_2a_3-a_2c_3\right)i+\left(a_1c_3-a_3c_1\right)j+\left(c_1a_2-a_1c_2\right)k$

Now, $\left[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}\right]=\begin{vmatrix}a_2b_3-b_2a_3 & b_1a_3-a_1b_3 & a_1b_2-a_2b_1\\ b_2c_3-b_3c_2 & c_1b_3-b_1c_3 & b_1c_2-b_2c_1\\ c_2a_3-c_3a_2 & a_1c_3-a_3c_1 & c_1a_2-c_2a_1\end{vmatrix}$

$$\left[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}\right]=\begin{vmatrix}C_1 & C_2 & C_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

where the capital letters A_1, A_2, A_3 etc., denote the cofactor corresponding small letters a_1, a_2, a_3 etc. in the determinant $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

Hence, $\left[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}\right]=\left[a\ b\ c\right]^2=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$

Scalar product of four vectors: If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four vectors then the product $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)$ or $\left(\vec{a}\times\vec{d}\right).\left(\vec{b}\times\vec{c}\right)$ is called scalar product of four vectors.

Theorem 2: Prove that $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix}\vec{a}.\vec{c} & \vec{b}.\vec{c}\\ \vec{a}.\vec{d} & \vec{b}.\vec{d}\end{vmatrix}$

Proof: Let $\vec{a}\times\vec{b}=\vec{r}$. Then $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)=\vec{r}.\left(\vec{c}\times\vec{d}\right)$

As we know that position of dot and cross may be interchanged without altering the value of the product. Therefore,

$$\vec{r}.\left(\vec{c}\times\vec{d}\right)=\left(\vec{r}\times\vec{c}\right).\vec{d}=\left[\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right].\vec{d}=\left[\left(\vec{c}.\vec{a}\right)\vec{b}-\left(\vec{c}.\vec{b}\right)\vec{a}\right].\vec{d}$$

$$=\left(\vec{c}.\vec{a}\right)\left(\vec{b}.\vec{d}\right)-\left(\vec{c}.\vec{b}\right)\left(\vec{a}.\vec{d}\right)$$

$$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix}\vec{a}.\vec{c} & \vec{b}.\vec{c}\\ \vec{a}.\vec{d} & \vec{b}.\vec{d}\end{vmatrix}$$

This relation also known as Lagrange's Identity.

Vector Product of four Vectors: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be four vectors then the vector products of the vectors $\left(\vec{a}\times\vec{b}\right),\left(\vec{c}\times\vec{d}\right)$ i.e., $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ is known as vector product of four products.

Theorem 3: To prove that

(i) $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}$

(ii) $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$

Proof: As we know that $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ is the vector quantity which is either written in terms of $\vec{c}$ and $\vec{d}$ or in terms of $\vec{a}$ and $\vec{b}$. To express the vector $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of, let us put $\vec{a}\times\vec{b}=\vec{l}$. Then

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\vec{l}\times\left(\vec{c}\times\vec{d}\right)=\left(\vec{l}\,.\vec{d}\right)\vec{c}-\left(\vec{l}\,.\vec{c}\right)\vec{d}$$

$$=\left[\left(\vec{a}\times\vec{b}\right).\vec{d}\right]\vec{c}-\left[\left(\vec{a}\times\vec{b}\right).\vec{c}\right]\vec{d}=\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d} \qquad \ldots\ldots..$$

(1)

Similarly, we also express the vector $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of $\vec{a}$ and $\vec{b}$

which is,

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a} \qquad \ldots\ldots\ldots..$$

(2)

equating the equation (1) and (2)

$$\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$$

$$\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}-\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}+\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}=0$$

Which is the required linear relation connecting the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

1.4 *RECIPROCAL SYSTEM OF VECTORS*

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors such that $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\neq 0$, then the three vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ defined as

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\ \text{and}\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

are called reciprocal systems of vectors to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Example 11: To show that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$

Answer: $\vec{a}.\vec{a}' = \vec{a}.\frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\vec{a}.\left(\vec{b}\times\vec{c}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

Similarly, $\vec{b}.\vec{b}' = \vec{b}.\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\vec{b}.\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{b}\ \vec{c}\ \vec{a}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

and $\vec{c}.\vec{c}' = \vec{c}.\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\vec{c}.\left(\vec{a}\times\vec{b}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{c}\ \vec{a}\ \vec{b}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

Example 12: The scalar product of any other pair of vectors, one from each system is zero, i.e., $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$

Answer: $\vec{a}.\vec{b}' = \vec{a}.\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\vec{a}.\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{\left[\vec{a}\ \vec{c}\ \vec{a}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 0$

Similarly, we can prove the other results i.e., $\vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$.

Example 13: The product of the scalar triple product of three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and the scalar triple product of their reciprocal $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ is equal to 1 i.e., $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = 1$.

Answer: First we have to define scalar triple product $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = \vec{a}'.\left(\vec{b}'\times\vec{c}'\right)$

$$\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = \vec{a}'.\left(\vec{b}'\times\vec{c}'\right) = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}.\left[\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\times\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\right] = \frac{\left(\vec{b}\times\vec{c}\right).\left[\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}$$

As we know that

$$\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right) = \left[\vec{c}\ \vec{a}\ \vec{b}\right]\vec{a} - \left[\vec{c}\ \vec{a}\ \vec{a}\right]\vec{b} = \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a} - 0 = \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}$$

So,

$$\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]=\frac{\left(\vec{b}\times\vec{c}\right).\left[\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left(\vec{b}\times\vec{c}\right).\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\left(\vec{b}\times\vec{c}\right).\vec{a}\right\}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}$$

$$=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

So, $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left[\vec{a}\ \vec{b}\ \vec{c}\right]=1$

Remarks 1: Since scalar triple product $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\neq 0$, so we conclude that $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]\neq 0$ i.e., $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are also non coplanar.

2: Since the vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ similarly, we can say that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are also reciprocal to the vector $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$, this is known as symmetry property.

3: The unit vector along three-dimensional co-ordinate axes i.e., the orthonormal vector triads $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a self-reciprocal system.

Solved examples

Example 14: Find the set of reciprocal vectors of the vectors

$\vec{a}=2i+3j-k,\quad \vec{b}=i-j-2k,\quad \vec{c}=-i+2j+2k$.

Solution: Let $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are the system of reciprocal vectors of the given vectors, then

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\text{ and }\vec{c}'=\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix}2 & 3 & -1\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix}=2(-2+4)-3(2-2)-1(2-1)=3\neq 0$$

Now, $\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = (-2+4)i-(2-2)j+(2-1)k = 2i+k$

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{2}{3}i+\frac{1}{3}k$$

$$\vec{c}\times\vec{a} = \begin{vmatrix} i & j & k \\ -1 & 2 & -2 \\ 2 & 3 & -1 \end{vmatrix} = -8i+3j-7k$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{-8i+3j-7k}{3}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = (-6-1)i-(-4+1)j+(-2-3)k = -7i+3j-5k$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{-7i+3j-5k}{3}$$

1.5 *DIFFERENTIATION OF VECTORS*

Vector Function: Let *D* be the subset of real numbers. If we associate unique vector *f*(*t*) to each element *t* of *D*, then this rule defines a **vector function** of the scalar variable *t*. Here *f*(*t*) is the vector quantity and thus *f* is a vector function.

As we know that every vector function can be easily expressed as a linear combination of three fixed non-coplanar vector. Thus, we can write

$$f(t) = f_1(t)i + f_2(t)j + f_3(t)k$$

where i, j, k denote the orthonormal right-handed triad.

Scalar field and vector field

If in the region *R* in the space corresponding to each point *P*(*x*, *y*, *z*) there is unique scalar *f*(*P*), then *f* is called a scalar point function and we say scalar field *f* has been defined in *R*.

e.g., $f(\mathrm{x},\mathrm{y},\mathrm{z}) = x^3 - y^2 - 2xyz$ defines a scalar field.

If corresponding to each point $P(x, y, z)$ in the region R there is unique vector $f(P)$, then f is called a vector point function and we say vector field f has been defined in R.

e.g., $f(\mathrm{x,y,z}) = x^3 i - y^2 j - 2xyz\, k$ defines a scalar field.

Limit and Continuity of a Vector Function:

Definition 1: A real number l is said to be limit of a vector function $f(t)$, when t tends to t_0, if for any small positive number $\in$, there exist a number δ such that

$$\left| f(t) - l \right| < \in \quad \text{whenever} \quad 0 < \left| t - t_0 \right| < \delta$$

If vector function $f(t)$ tends to a limit l as t tends to t_0, we write it mathematically by,

$$\lim_{t \to t_0} f(t) = l$$

Definition 2: A vector function $f(t)$ is said to be continuous at a point t_0 of t if following are satisfied,

(i) $f(t)$ is defined at t_0

(ii) Corresponding to any small positive number $\in$, there exist a number δ such that

$$\left| f(t) - f(t_0) \right| < \in \quad \text{whenever} \quad 0 < \left| t - t_0 \right| < \delta$$

A vector function $f(t)$ is said to be continuous if it is continuous for every value of t for which it has been defined.

Some important theorems:

Theorem 4: The Necessary and Sufficient condition for a vector function $f(t)$ to be continuous at $t = t_0$ is that $\lim_{t \to t_0} f(t) = f(t_0)$.

Theorem 5: The vector function $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$ is continuous if and only if $f_1(t), f_2(t), f_3(t)$ are continuous.

Theorem 6: Let $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$ and $l = l_1 i + l_2 j + l_3 k$, then the necessary and sufficient condition that $\lim_{t \to t_0} f(t) = l$ are

$$\lim_{t \to t_0} f_1(t) = l_1, \quad \lim_{t \to t_0} f_2(t) = l_2, \quad \lim_{t \to t_0} f_3(t) = l_3$$

Theorem 7: If $f(t), \mathrm{g}(t)$ are scalar function of the scalar variable t and let $\psi(t)$ is a scalar function of scalar variable t, then

(i) $\lim_{t \to t_0} \left[f(t) \pm g(t) \right] = \lim_{t \to t_0} f(t) \pm \lim_{t \to t_0} \mathrm{g}(t)$

(ii) $\lim_{t \to t_0} \left[f(t).g(t) \right] = \left[\lim_{t \to t_0} f(t) \right] . \left[\lim_{t \to t_0} \mathrm{g}(t) \right]$

$$\text{(iii)} \lim_{t\to t_0}\left[f(t)\times g(t)\right]=\left[\lim_{t\to t_0} f(t)\right]\times\left[\lim_{t\to t_0} g(t)\right]$$

$$\text{(iv)} \lim_{t\to t_0}\left[\psi(t)g(t)\right]=\left[\lim_{t\to t_0}\psi(t)\right]\left[\lim_{t\to t_0} g(t)\right]$$

$$\text{(v)} \lim_{t\to t_0}\left|f(t)\right|=\left|\lim_{t\to t_0} f(t)\right|$$

Note: Here we use application of these theorems without proof.

Derivative of a vector function with respect to a scalar:

Definition: Let $r = f(t)$ be a vector function of the scalar variable *t*, we define

$$r+\delta r = f(t+\delta t)$$

$$\delta r = f(t+\delta t)-f(t)$$

Consider the vector, $\dfrac{\delta r}{\delta t}=\dfrac{f(t+\delta t)-f(t)}{\delta t}$

If $\lim\limits_{\delta t\to 0}\dfrac{\delta r}{\delta t}=\lim\limits_{\delta t\to 0}\dfrac{f(t+\delta t)-f(t)}{\delta t}$ exist, then the value of this limit is called the derivative of the vector function r with respect to *t*, denoted by $\dfrac{dr}{dt}$ i.e.,

$$\frac{dr}{dt}=\lim_{\delta t\to 0}\frac{\delta r}{\delta t}=\lim_{\delta t\to 0}\frac{(\mathrm{r}+\delta r)-\mathrm{r}}{\delta t}=\lim_{\delta t\to 0}\frac{f(t+\delta t)-f(t)}{\delta t}$$

If $\dfrac{dr}{dt}$ exists, then r is said to be differentiable. Since r is vector quantity so its will also a vector quantity.

If we again differentiate $\dfrac{dr}{dt}$, we get $\dfrac{d^2r}{dt^2}$, the second derivative of r w.r.t. *t*, and so on differentiating successively *n* times we get,

$$\frac{dr}{dt},\ \frac{d^2r}{dt^2},\ \frac{d^3r}{dt^3},\ \frac{d^4r}{dt^4},\ \ldots.,\ \frac{d^nr}{dt^n}.$$

Differentiation Formulae:

Theorem 8: Let $\vec{a},\vec{b}$ and $\vec{c}$ are differentiable vector function of a scalar *t* and $\psi(t)$ is a scalar function of same variable *t*, then

(i) $$\frac{d}{dt}\left[a\pm b\right]=\frac{da}{dt}\pm\frac{db}{dt}$$

(ii) $$\frac{d}{dt}\left[a.b\right]=a.\frac{db}{dt}+\frac{da}{dt}.b$$

(iii) $$\frac{d}{dt}\left[a\times b\right]=a\times\frac{db}{dt}+\frac{da}{dt}\times b$$

(iv) $$\frac{d}{dt}\left[a\,b\,c\right]=\left[\frac{da}{dt}\;b\;c\right]+\left[a\;\frac{db}{dt}\;c\right]+\left[a\;b\;\frac{dc}{dt}\right]$$

(v) $$\frac{d}{dt}\left[\psi a\right]=\psi\frac{da}{dt}+\frac{d\psi}{dt}a$$

(vi) $$\frac{d}{dt}\left\{a\times\left(b\times c\right)\right\}=\frac{da}{dt}\times(b\times c)+a\times\left(\frac{db}{dt}\times c\right)+a\times\left(b\times\frac{dc}{dt}\right)$$

Proof (*i*):

$$\frac{d}{dt}\left[a+b\right]=\lim_{\delta t\to 0}\frac{\left\{\left(a+\delta a\right)+\left(b+\delta b\right)\right\}-\left(a+b\right)}{\delta t}=\lim_{\delta t\to 0}\frac{\delta a+\delta b}{\delta t}=\lim_{\delta t\to 0}\frac{\delta a}{\delta t}+\lim_{\delta t\to 0}\frac{\delta b}{\delta t}=\frac{da}{dt}+\frac{db}{dt}$$

Similarly, we can prove that, $\frac{d}{dt}\left[a-b\right]=\frac{da}{dt}-\frac{db}{dt}$

Proof (*ii*):

$$\frac{d}{dt}\left[a.b\right]=\lim_{\delta t\to 0}\frac{\left\{\left(a+\delta a\right).\left(b+\delta b\right)\right\}-a.b}{\delta t}=\lim_{\delta t\to 0}\frac{a.b+a.\delta b+\delta a.b+\delta a.\delta b-a.b}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{a.\delta b+\delta a.b+\delta a.\delta b}{\delta t}=\lim_{\delta t\to 0}\left\{a.\frac{\delta b}{\delta t}+\frac{\delta a}{\delta t}.b+\frac{\delta a}{\delta t}.\delta b\right\}$$

$$=\lim_{\delta t\to 0}\left\{a.\frac{\delta b}{\delta t}+\frac{\delta a}{\delta t}.b+\frac{\delta a}{\delta t}.\delta b\right\}=\lim_{\delta t\to 0}a.\frac{\delta b}{\delta t}+\lim_{\delta t\to 0}\frac{\delta a}{\delta t}.b+\lim_{\delta t\to 0}\frac{\delta a}{\delta t}.\delta b$$

$$=a.\frac{db}{dt}+\frac{da}{dt}.b+\frac{da}{dt}.0$$

Since $\delta b\to 0$ as $t\to 0$

$$\frac{d}{dt}\left[a.b\right]=a.\frac{db}{dt}+\frac{da}{dt}.b$$

Note: As we know that vector dot product is commutative, then

$$\frac{d}{dt}\left[a.b\right]=a.\frac{db}{dt}+\frac{da}{dt}.b=\frac{d}{dt}\left[b.a\right]$$

Proof (*iii*):

$$\frac{d}{dt}\left[a\times b\right]=\lim_{\delta t\to 0}\frac{\left(a+\delta a\right)\times\left(b+\delta b\right)-a\times b}{\delta t}=\lim_{\delta t\to 0}\frac{a\times b+a\times\delta b+\delta a\times b+\delta a\times\delta b-a\times b}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{a\times\delta b+\delta a\times b+\delta a\times\delta b}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{a\times\frac{\delta b}{\delta t}+\frac{\delta a}{\delta t}\times b+\frac{\delta a}{\delta t}\times\delta b\right\}=\lim_{\delta t\to 0}a\times\frac{\delta b}{\delta t}+\lim_{\delta t\to 0}\frac{\delta a}{\delta t}\times b+\lim_{\delta t\to 0}\frac{\delta a}{\delta t}\times\delta b$$

$$= a\times\frac{db}{dt}+\frac{da}{dt}\times b+\frac{da}{dt}\times 0$$

Since $\delta b\to 0$ as $t\to 0$

$$\frac{d}{dt}\left[a.b\right]=a\times\frac{db}{dt}+\frac{da}{dt}\times b$$

Note: As we know that vector cross product is not commutative, so we must have to maintain the order of the factor *a* and *b*.

Proof (*iv*): $\frac{d}{dt}\left[a\,b\,c\right]=\frac{d}{dt}\left\{a.\left(b\times c\right)\right\}=a.\frac{d}{dt}\left(b\times c\right)+\frac{da}{dt}.\left(b\times c\right)$

$$=a.\left(b\times\frac{dc}{dt}+\frac{db}{dt}\times c\right)+\frac{da}{dt}.\left(b\times c\right)=a.\left(b\times\frac{dc}{dt}\right)+a.\left(\frac{db}{dt}\times c\right)+\frac{da}{dt}.\left(b\times c\right)$$

$$=\left[a\;b\;\frac{dc}{dt}\right]+\left[a\;\frac{db}{dt}\;c\right]+\left[\frac{da}{dt}\;b\;c\right]=\left[\frac{da}{dt}\;b\;c\right]+\left[a\;\frac{db}{dt}\;c\right]+\left[a\;b\;\frac{dc}{dt}\right]$$

Proof (*v*):

$$\frac{d}{dt}\left[\psi a\right]=\lim_{\delta t\to 0}\frac{\left(\psi+\delta\psi\right)\left(a+\delta a\right)-\psi a}{\delta t}=\lim_{\delta t\to 0}\frac{\psi a+\psi\delta a+a\delta\psi+\delta\psi\delta a-\psi a}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\psi\delta a+a\delta\psi+\delta\psi\delta a}{\delta t}=\lim_{\delta t\to 0}\psi\frac{\delta a}{\delta t}+\lim_{\delta t\to 0}a\frac{\delta\psi}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\psi}{\delta t}\delta a$$

$$=\psi\frac{da}{dt}+\frac{d\psi}{dt}a+\frac{d\psi}{dt}0=\psi\frac{da}{dt}+\frac{d\psi}{dt}a$$

Proof (*vi*): $\frac{d}{dt}\left\{a\times\left(b\times c\right)\right\}=a\times\frac{d}{dt}(b\times c)+\frac{da}{dt}\times\left(b\times c\right)$

$$=a\times\left(\frac{db}{dt}\times c+b\times\frac{dc}{dt}\right)+\frac{da}{dt}\times\left(b\times c\right)=a\times\left(\frac{db}{dt}\times c\right)+a\times\left(b\times\frac{dc}{dt}\right)+\frac{da}{dt}\times\left(b\times c\right)$$

$$=\frac{da}{dt}\times\left(b\times c\right)+a\times\left(\frac{db}{dt}\times c\right)+a\times\left(b\times\frac{dc}{dt}\right)$$

Remarks 1: Derivative of a constant vector is always zero.

2: If $\vec{r}$ is a vector quantity and *s* is a scalar quantity, then we write $\frac{d\vec{r}}{dt}=\frac{d\vec{r}}{ds}\frac{ds}{dt}$, which is nothing but the multiplication of the vector $\frac{d\vec{r}}{ds}$ and scalar $\frac{ds}{dt}$.

3: If $\vec{r} = xi + yj + zk$, where the component *x*, *y*, *z* are scalar function of scalar variable *t* and *i*, *j*, *k* are unit vectors along the three co-ordinate axes then,

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

It means when we differentiate a vector, we should differentiate its components.

Sometime, we also use the notation,

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) \text{ and so on.}$$

Some important theorems

Theorem 9: The necessary and sufficient condition for the vector function $a(t)$ to be constant is that $\frac{da}{dt} = 0$.

Proof: The condition is necessary: Let the vector function $a(t)$ is the constant vector corresponding to the scalar variable *t*, it means $a(t + \delta t) = a(t)$

Now, $\frac{da}{dt} = \lim_{\delta t \to 0} \frac{a(t + \delta t) - a(t)}{\delta t} = \lim_{\delta t \to 0} \frac{a(t) - a(t)}{\delta t} = 0$

The condition is sufficient: Let $\frac{da}{dt} = 0$, then, we have to prove that $a(t)$ is the constant vector. Let $a(t) = a_1(t)i + a_2(t)\,\mathrm{j} + a_3(t)\,\mathrm{k}$, then, $\frac{da}{dt} = \frac{da_1}{dt}i + \frac{da_2}{dt}\mathrm{j} + \frac{da_3}{dt}\mathrm{k}$

Since, $\frac{da}{dt} = 0 \Rightarrow \frac{da_1}{dt}i + \frac{da_2}{dt}\mathrm{j} + \frac{da_3}{dt}\mathrm{k} = 0$

Now, equating both side coefficient of *i*, *j*, *k*, we get

$\frac{da_1}{dt} = 0$, $\frac{da_2}{dt} = 0$, $\frac{da_3}{dt} = 0$, Here a_1, a_2, a_3 are constant vector because they are independent from *t*. therefore $a(t)$ is the constant vector function.

Theorem 10: If $\vec{a}$ is differentiable vector function of the scalar variable *t* and if $\left|\vec{a}\right| = a$, then

(i) $$\frac{d\left(\vec{a}\right)^2}{dt} = 2\vec{a}\frac{d\vec{a}}{dt}$$

(ii) $$\vec{a}.\frac{d\vec{a}}{dt} = a\frac{da}{dt}$$

Proof: (*i*) As we know that $\vec{a}^2 = \vec{a}.\vec{a} = \left|\vec{a}\right|\left|\vec{a}\right|\cos 0 = aa = a^2$

Then, $\frac{d\vec{a}^2}{dt} = \frac{da^2}{dt} = 2a\frac{da}{dt}$

(ii) $$\frac{d\vec{a}^2}{dt} = \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 2\vec{a}.\frac{d\vec{a}}{dt}$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 2a\frac{da}{dt}$$

$$\vec{a}.\frac{d\vec{a}}{dt} = a\frac{da}{dt}$$

Theorem 11: If $\vec{a}$ has constant length (fixed magnitude), then $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$.

Proof: We have given that $\left|\vec{a}\right| = a =$ constant. Then $\vec{a}.\vec{a} = a^2 =$ constant.

Since, $\frac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0$$

Since, scalar dot product of two vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ is zero. It means vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular i.e., $\left|\frac{d\vec{a}}{dt}\right| \neq 0$

Theorem 12: The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt} = 0$.

Proof: Let $\vec{a}(t)$ is the vector function of the scalar variable t and it have constant magnitude i.e., $\left|\vec{a}\right| = a =$ constant.

$$\Rightarrow \vec{a}.\vec{a} = a^2 = \text{constant}$$

$$\Rightarrow \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0$$

Which is the necessary condition

Condition is sufficient: Let us assume that $\vec{a}.\frac{d\vec{a}}{dt} = 0$, then we have to prove that $\vec{a}(t)$ is a constant vector.

Since, $\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$

$$\Rightarrow \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0 \quad \Rightarrow \vec{a}.\vec{a} = a^2 = \text{constant}$$

Theorem 13: If $\vec{a}(t)$ is the vector function of the scalar variable t and it is differentiable, then $\frac{d}{dt}\left(a \times \frac{da}{dt}\right) = a \times \frac{d^2a}{dt^2}$

Proof: $\frac{d}{dt}\left(a \times \frac{da}{dt}\right) = \frac{da}{dt} \times \frac{da}{dt} + a \times \frac{d^2a}{dt^2}$

As we know that $a \times a = 0$

$$\Rightarrow \frac{d}{dt}\left(a\times\frac{da}{dt}\right)=0+a\times\frac{d^2a}{dt^2}=a\times\frac{d^2a}{dt^2}$$

Theorem 14: The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant direction $\left(a\times\frac{da}{dt}\right)=0$.

Proof: Let $\vec{a}(t)$ be a vector function corresponding to the scalar variable *t*.

Let **A** be the unit vector in the direction of $\vec{a}(t)$ then, $\vec{a}(t)=\left|\vec{a}(t)\right|A$, If we consider *a* be the magnitude of the vector function $\vec{a}(t)$ i.e., $\vec{a}(t)=aA$

$$\frac{d\vec{a}}{dt}=a\frac{dA}{dt}+\frac{da}{dt}A$$

Hence, $\vec{a}\times\frac{d\vec{a}}{dt}=(aA)\times\left(a\frac{dA}{dt}+\frac{da}{dt}A\right)=a^2A\times\frac{dA}{dt}+a\frac{da}{dt}A\times A$

$$=a^2A\times\frac{dA}{dt}+0=a^2A\times\frac{dA}{dt}$$ [As we know that $a\times a=0$]

……. (1)

The condition is necessary: Suppose $\vec{a}$ has constant direction, then **A** is constant vector because it has constant magnitude as well as constant direction. Therefore $\frac{dA}{dt}=0$

Hence, from (1) we get $\vec{a}\times\frac{d\vec{a}}{dt}=a^2A\times 0=0$

Thus, the condition is necessary.

The condition is sufficient: Let we consider, $\vec{a}\times\frac{d\vec{a}}{dt}=0$

Then from (1) we get, $a^2A\times\frac{dA}{dt}=0$ or $A\times\frac{dA}{dt}=0$ …. (2)

$\therefore$ A is of constant length, then $A.\frac{dA}{dt}=0$ …. (3)

From (2) and (3), we get $\frac{dA}{dt}=0$

Hence A is a constant vector it means direction of A is constant.

Note: (*i*) If $\vec{r}$ represent position vector of a particle at a time *t* with respect to the origin O, then $\delta\vec{r}$ represents small displacement at a particle in time δt. If $\vec{v}$ represents the velocity of the particle at *P*, then

$$\vec{v} = \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

(ii)If $\vec{a}$ represents the acceleration of the particle at time *t*, then

$$\vec{a} = \lim_{\delta t\to 0}\frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

SELF CHEQUE QUESTIONS

Example 15: Find $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$, where

$\vec{r} = (t+1)i + (t^2+t+1)j + (\mathrm{t}^3+t^2+t+1)k$

Solution: We have given that $\vec{r} = (t+1)i + (t^2+t+1)j + (\mathrm{t}^3+t^2+t+1)k$

So, $\frac{d\vec{r}}{dt} = \frac{d}{dt}(t+1)i + \frac{d}{dt}(t^2+t+1)j + \frac{d}{dt}(\mathrm{t}^3+t^2+t+1)k$

$$\frac{d\vec{r}}{dt} = \left(\frac{d}{dt}t + \frac{d}{dt}1\right)i + \left(\frac{d}{dt}t^2 + \frac{d}{dt}t + \frac{d}{dt}1\right)j + \left(\frac{d}{dt}\mathrm{t}^3 + \frac{d}{dt}t^2 + \frac{d}{dt}t + \frac{d}{dt}1\right)k$$

$$\frac{d\vec{r}}{dt} = (1+0)i + (2t+1+0)j + \left(3\mathrm{t}^2+2t+1+0\right)k = i + (2t+1)j + \left(3\mathrm{t}^2+2t+1\right)k$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}i + \left(2\frac{d}{dt}t + \frac{d}{dt}1\right)j + \left(3\frac{d}{dt}\mathrm{t}^2 + 2\frac{d}{dt}t + \frac{d}{dt}1\right)k$$

$$\frac{d^2\vec{r}}{dt^2} = 0 + (2+0)j + (6t+2+0)k = 2j + (6t+2)k$$

Example 16: If $\vec{r} = \sin t\, i + \cos t\, j + t\, k$, then find the following

(i) $\frac{d\vec{r}}{dt}$ **(ii)** $\frac{d^2\vec{r}}{dt^2}$

(iii) $\left|\frac{d\vec{r}}{dt}\right|$ **(iv)** $\left|\frac{d^2\vec{r}}{dt^2}\right|$

Solution: (i) $\frac{d\vec{r}}{dt} = i\frac{d}{dt}\sin t + j\frac{d}{dt}\cos t + k\frac{d}{dt}t = \cos t\, i - \sin t\, j + k$

(ii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\cos t\, i - \sin t\, j + k\right) = i\frac{d}{dt}\cos t - j\frac{d}{dt}\sin t + k\frac{d}{dt}1$

$$\frac{d^2\vec{r}}{dt^2} = -\sin t\, i - \cos t\, j + 0k = -\sin t\, i - \cos t\, j$$

(iii) $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\cos t\right)^2 + \left(\sin t\right)^2 + (1)^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

(iv) $\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{\left(-\sin t\right)^2 + \left(-\cos t\right)^2} = 1$

Example 17: If $\vec{r} = \left(\cos \mathrm{n}t\right)i + \left(\sin nt\right)j$, show that $\vec{r}\times\frac{d\vec{r}}{dt} = nk$, where n is a constant.

Solution: We have given, $\vec{r} = \left(\cos \mathrm{n}t\right)i + \left(\sin nt\right)j$

So, $\frac{d\vec{r}}{dt} = i\frac{d}{dt}\cos \mathrm{nt} + j\frac{d}{dt}\sin nt = -n\sin \mathrm{nt}\, i + n\cos nt\, j$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} i & j & k \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix} = 0i - 0j + \left(n\cos^2 nt + n\sin^2 nt\right)k = nk$$

Example 18: If $\vec{r} = \left(\cos \omega t\right)\vec{a} + \left(\sin \omega t\right)\vec{b}$, where $\vec{a}$, $\vec{b}$ are constant vector and ω is a constant. Then show the following:

(i) $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ and **(ii)**

$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\vec{a}\times\vec{b}$

Solution (*i*): We have given, if $\vec{a}$, $\vec{b}$ are constant vector and ω is a constant it means $\frac{d\vec{a}}{dt} = 0 = \frac{d\vec{b}}{dt}$

Since, $\vec{r} = \left(\cos \omega t\right)\vec{a} + \left(\sin \omega t\right)\vec{b}$

Then, $\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\left(\cos\omega t\right)\vec{a} + \left(\sin\omega t\right)\vec{b}\right) = \frac{d\left(\cos\omega t\right)}{dt}\vec{a} + \frac{d\left(\sin\omega t\right)}{dt}\vec{b}$

$$\frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right) = -\omega^2\cos\omega t\,\vec{a} - \omega^2\sin t\,\omega t\,\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\left(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b}\right) = -\omega^2\,\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} + \omega^2\,\vec{r} = 0$$

(ii) $\vec{r}\times\frac{d\vec{r}}{dt} = \left(\left(\cos\omega t\right)\vec{a} + \left(\sin\omega t\right)\vec{b}\right)\times\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right)$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) - \omega\sin^2\omega t\left(\vec{b}\times\vec{a}\right) = \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) + \omega\sin^2\omega t\left(\vec{a}\times\vec{b}\right)$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\left(\cos^2\omega t + \sin^2\omega t\right)\left(\vec{a}\times\vec{b}\right) = \omega\left(\vec{a}\times\vec{b}\right)$$

Example 19: If $\vec{a} = \left(\cos\theta\right)i + \left(\sin\theta\right)j + \theta k$, $\vec{b} = \left(\cos\theta\right)i - \left(\sin\theta\right)j - 3k$ and $\vec{c} = 2i + 3j - 3k$, find $\frac{d}{d\theta}\left\{a\times\left(b\times c\right)\right\} = \left(\cos\theta\right)i - \left(\sin\theta\right)j - 3k$ at $\theta = \frac{\pi}{2}$.

Answer:

$$b\times c = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix} = \left(3\sin\theta + 9\right)i + \left(3\cos\theta - 6\right)j + \left(3\cos\theta + 2\sin\theta\right)k$$

Now, $a\times\left(b\times c\right) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & 3\cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$

$$= \left(3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta\right)i + \left(3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta\right)j + \left(-6\sin\theta - 9\cos\theta\right)k$$

$$\frac{d\{a\times(b\times c)\}}{d\theta}=\left(-6\cos\theta\sin\theta+2\cos^2\theta-2\sin^2\theta-3\cos\theta+3\theta\sin\theta+6\right)i$$
$$+\left(3\sin\theta+3\theta\cos\theta+9-3\cos^2\theta+3\sin^2\theta-4\sin\theta\cos\theta\right)j+\left(-6\cos\theta+9\sin\theta\right)k$$

Now, putting $\theta=\frac{\pi}{2}$, we get

$$=\left(4+\frac{3}{2}\pi\right)i+15j+9k$$

Example 20: Let a particle moves along the curve $x=t^3+1, y=t^2, z=2t+5$, where t represents the time. Then find the velocity and acceleration of the particle at t = 1, in the direction $i+j+3k$.

Answer: Since, we have given particle moves along the curve $x=t^3+1, y=t^2, z=2t+5$, then

$$\vec{r}=\left(t^3+1\right)i+t^2j+\left(2t+5\right)k$$

So, the velocity of the particle is

$$\vec{v}=\frac{d\vec{r}}{dt}=\left\{\frac{d}{dt}\left(t^3+1\right)\right\}i+\left\{\frac{d}{dt}t^2\right\}j+\left\{\frac{d}{dt}\left(2t+5\right)\right\}k$$

$$\vec{v}=3t^2i+2tj+2k$$

The velocity of particle at t = 1 is $\left(\vec{v}\right)_{t=1}=3i+2j+2k$

Similarly, the acceleration of the particle is

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\left\{\frac{d}{dt}3t^2\right\}i+\left\{\frac{d}{dt}2t\right\}j+\left\{\frac{d}{dt}2\right\}k=6ti+2j$$

The acceleration of particle at t = 1 is $\left(\vec{a}\right)_{t=1}=6i+2j$

Since, the unit vector in the direction of $i+j+3k$ is

$$=\frac{i+j+3k}{\sqrt{1^2+1^2+3^2}}=\frac{i+j+3k}{\sqrt{11}}$$

So, the component of velocity in the direction of given vector

$$=\vec{v}.\vec{b}=\frac{(3i+2j+2k).(i+j+3k)}{\sqrt{11}}=\frac{11}{\sqrt{11}}=\sqrt{11}$$

Similarly, the component of acceleration in the direction of given vector

$$=\vec{a}.\vec{b}=\frac{(6i+2j).(i+j+3k)}{\sqrt{11}}=\frac{8}{\sqrt{11}}$$

Example 21: If $\vec{r}$ is a vector function corresponding to a scalar variable *t*, $\vec{a}$ is a constant vector and *m* is a constant, then differentiate the following with respect to *t*:

(i) $\vec{r}.\vec{a}$ **(ii)** $\vec{r}\times\vec{a}$ **(iii)** $\vec{r}\times\frac{d\vec{r}}{dt}$

(iv) $\vec{r}.\frac{d\vec{r}}{dt}$ **(v)** $\vec{r}^2+\frac{1}{\vec{r}^2}$ **(vi)** $m\left(\frac{d\vec{r}}{dt}\right)^2$

(vii) $\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ **(viii)** $\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$

Answer (i): Let $\vec{R}=\vec{r}.\vec{a}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}.\vec{a}\right)=\frac{d\vec{r}}{dt}.\vec{a}+\vec{r}.\frac{d\vec{a}}{dt}=\frac{d\vec{r}}{dt}.\vec{a}$ $\because\left[\frac{d\vec{a}}{dt}=0\right]$

(ii) Let, $\vec{R}=\vec{r}\times\vec{a}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}\times\vec{a}\right)=\frac{d\vec{r}}{dt}\times\vec{a}+\vec{r}\times\frac{d\vec{a}}{dt}=\frac{d\vec{r}}{dt}\times\vec{a}$

(iii) Let, $\vec{R}=\vec{r}\times\frac{d\vec{r}}{dt}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}=\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$\because\left[\vec{a}\times\vec{a}=0\right]$

(iv) Let, $\vec{R}=\vec{r}.\frac{d\vec{r}}{dt}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}.\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}+\vec{r}.\frac{d^2\vec{r}}{dt^2}=\left(\frac{d\vec{r}}{dt}\right)^2+\vec{r}.\frac{d^2\vec{r}}{dt^2}$

$\because\left[\vec{a}.\vec{a}=\vec{a}^2\right]$

(v) Let, $\vec{R} = \vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}}$

then, $\frac{d\vec{R}}{dt} = \frac{d}{dt}\left(\vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}}\right) = \frac{d}{dt}\vec{r}^{\,2} + \frac{d}{dt}\left(\frac{1}{\vec{r}^{\,2}}\right) = 2\vec{r}\,\frac{d\vec{r}}{dt} - \frac{2}{\vec{r}^{\,3}}\frac{d\vec{r}}{dt}$

(vi) Let, $\vec{R} = m\left(\frac{d\vec{r}}{dt}\right)^2$

Then, $\frac{d\vec{R}}{dt} = \frac{d}{dt}\left\{m\left(\frac{d\vec{r}}{dt}\right)^2\right\} = m\left(2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}\right) = 2m\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}$

$\left[\frac{d\vec{r}^{\,2}}{dt} = 2\vec{r}.\frac{d\vec{r}}{dt}\right]$

(vii) Let, $\vec{R} = \frac{\vec{r}+\vec{a}}{\vec{r}^{\,2}+\vec{a}^{\,2}}$

Then,

$$\frac{d\vec{R}}{dt} = \frac{d}{dt}\left\{\frac{\vec{r}+\vec{a}}{\vec{r}^{\,2}+\vec{a}^{\,2}}\right\} = \frac{1}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)}\frac{d}{dt}\left(\vec{r}+\vec{a}\right) + \left\{\frac{d}{dt}\left(\frac{1}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)}\right)\right\}\left(\vec{r}+\vec{a}\right)$$

$$= \frac{1}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)}\left(\frac{d}{dt}\vec{r} + \frac{d}{dt}\vec{a}\right) - \left\{\frac{1}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)^2}\frac{d}{dt}\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)\right\}\left(\vec{r}+\vec{a}\right)$$

$$= \frac{1}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)}\frac{d\vec{r}}{dt} - \frac{2\vec{r}.\frac{d\vec{r}}{dt}}{\left(\vec{r}^{\,2}+\vec{a}^{\,2}\right)^2}\left(\vec{r}+\vec{a}\right)$$

$$\left[\because \frac{d\vec{a}}{dt} = 0, \frac{d}{dt}\vec{r}^{\,2} = 2\vec{r}.\frac{d\vec{r}}{dt}, \frac{d}{dt}\vec{a}^{\,2} = 0\right]$$

(viii) Let, $\vec{R}=\dfrac{\vec{r}\times\vec{a}}{\vec{r}\,.\,\vec{a}}$

Then, $\dfrac{d\vec{R}}{dt}=\dfrac{d}{dt}\left\{\dfrac{\vec{r}\times\vec{a}}{\vec{r}\,.\,\vec{a}}\right\}=\dfrac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\dfrac{d}{dt}\left(\vec{r}\times\vec{a}\right)+\left\{\dfrac{d}{dt}\left(\dfrac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\right)\right\}\left(\vec{r}\times\vec{a}\right)$

$$=\frac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\left(\frac{d}{dt}\vec{r}\times\vec{a}+\vec{r}\times\frac{d}{dt}\vec{a}\right)-\left\{\frac{1}{\left(\vec{r}\,.\,\vec{a}\right)^2}\frac{d}{dt}\left(\vec{r}\,.\,\vec{a}\right)\right\}\left(\vec{r}\times\vec{a}\right)$$

$$=\frac{\dfrac{d\vec{r}}{dt}\times\vec{a}}{\left(\vec{r}\,.\,\vec{a}\right)}-\left\{\frac{\dfrac{d\vec{r}}{dt}.\vec{a}}{\left(\vec{r}\,.\,\vec{a}\right)^2}\right\}\left(\vec{r}\times\vec{a}\right)$$

1.6 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of vector triple product and scalar triple product.
- The application of differentiation of vectors.

1.7 *GLOSSARY*

- **Vector triple product:** $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ is the vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.
- **Scalar triple product:** $\left(\vec{a}\times\vec{b}\right).\vec{c}$ is the vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

1.8 *REFERENCES*

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1.9 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

1.10 *TERMINAL QUESTION*

Objective Question

1. The value of $i.(j\times k)+j.(k\times i)+k.(i\times j)$

a)	0	**b)**	1
c)	2	**d)**	3

2. The volume of parallelopiped whose edges are given by

$\vec{OA}=2i-3j, \vec{OB}=i+j-k, \vec{OC}=3i-k$ is

a)	1	**b)**	4
c)	2/7	**d)**	None

3. If [*a*, *b*, *c*] is the scalar triple product of three vectors *a*, *b*, *c* then [*a*, *b*, *c*] is equal to

a)	[*b*, *a*, *c*]	**b)**	[*c*, *b*, *a*]
c)	[*b*, *c*, *a*]	**d)**	[*a*, *c*, *b*]

4. If *i*, *j*, *k* are the orthogonal right handed triad of unit vector and $\vec{a}$ is a vector then

$i\times\left(\vec{a}\times i\right)+j\times\left(\vec{a}\times j\right)+k\times\left(\vec{a}\times k\right)$ is equal to

a)	$\vec{a}$	**b)**	$2\vec{a}$
c)	$3\vec{a}$	**d)**	0

5. If $\vec{r} = \vec{a}\, e^{\omega t} + \vec{b}\, e^{-\omega t}$, $\vec{a}, \vec{b}$ are constant vectors then $\dfrac{d^2 \vec{r}}{dt^2} - \omega^2 \vec{r}$ is equal to

a) 1

b) 0

c) 2

d) none of these

6. If a particle moves along the curve $\vec{r} = e^{-t} \cos t\, i + e^{-t} \sin t\, j + e^{-t} k$, then find the magnitude of velocity at $t = 0$ is

a) $2\sqrt{3}$

b) $\dfrac{\sqrt{3}}{2}$

c) $\sqrt{3}$

d) none of these

Fill in the blanks

1. If $\{i, j, k\}$ be a set of orthonormal unit vectors, then $[i, j, k]$ = ……….

2. If $\vec{A}, \vec{B}, \vec{C}$ be three non-coplanar vectors, then $\dfrac{\vec{A}.\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}.\vec{B}} + \dfrac{\vec{B}.\vec{A}\times\vec{C}}{\vec{C}.\vec{A}\times\vec{B}} =$ …………

3. For any three vectors $\vec{a}, \vec{b}, \vec{c}, \vec{a}\times\left(\vec{b}\times\vec{c}\right) + \vec{b}\times\left(\vec{c}\times\vec{a}\right) + \vec{c}\times\left(\vec{a}\times\vec{b}\right) =$ …………

4. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ are reciprocal system of vectors, then find $\vec{a}.\vec{a'} + \vec{b}.\vec{b'} + \vec{c}.\vec{c'} = \ldots$

5. If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\left(\vec{a}\times\vec{b}\right).\vec{c}$ = ………………

6. If $\vec{r} = 3i - 6t^2 j + 4tk,$ then $\dfrac{d\vec{r}}{dt}$ = ……….; $\dfrac{d^2\vec{r}}{dt^2}$ = …………

7. If $\vec{u} = t^2 i - tj + (2t+1)k,\ \vec{v} = (2t-3)i + j - tk,$ then $\dfrac{d}{dt}\left(\vec{u}.\vec{v}\right) =$ ……….

8. If $\vec{r} = (\cos\omega t)i + (\sin\omega t)j,$ then $\vec{r}\times\dfrac{d^2\vec{r}}{dt^2} =$ …………

9. The necessary and sufficient condition for the vector $\vec{a}(t)$ to be constant direction is ...

True or False

1. If $x.\vec{a} = x.\vec{b} = x.\vec{c} = 0$, for some non-zero vectors x, then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

2. If $\{i, j, k\}$ be orthonormal set of unit vectors, then $i \times (j \times k) \neq 0$

3. The orthonormal unit vector triads, $\{i, j, k\}$ form a reciprocal system.

4. A vector is said to be constant only if its direction changes and magnitude is fixed.

5. The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant direction is $\vec{a}.\frac{d\vec{a}}{dt} = 0$

Short answer type question:

1. Prove that the identity $a \times [a \times (a \times b)] = (a\,.\,a)(b \times a)$

2. If a, b, c and a', b', c' are reciprocal system of vectors, prove that

(i) $a \times a' + b \times b' + c \times c' = 0$

(ii) $a' \times b' + b' \times c' + c' \times a' = \frac{a + b + c}{[a\,b\,c]}$

(iii) $a.a' + b.b' + c.c' = 3$

3. Show that $i.j \times k = 1$.

4. Show that $[\lambda a + \mu b, c, d] = \lambda[a, c, d] + \mu[b, c, d]$.

5. Prove that $[i - j, j - k, k - i] = 0$

Long answer type question:

1. Prove that the four points $6a - 4b + 10c, -5a + 3b + 10c, 4a - 6b - 10c$ and $2b + 10c$ are coplanar.

2. Prove that $a \times (b \times a) = (a \times b) \times a$

3. Prove that any three vectors A, B, C ,

$$(A\times B).((B\times C)\times(C\times A))=(A.B\times C)^2 .$$

4. If $r=t^3 i+(2t^3-\frac{1}{5t^2})\,j$, show that $r\times\frac{dr}{dt}=k$

5. If $r=e^{nt}a+e^{-nt}b)$ where a, b are constant vector, show that

$$\frac{d^2r}{dt^2}-n^2r=0$$

1.11 *ANSWERS*

Answer of objective type questions

1. (d) **2.** (b) **3.** (c) **4.** (b) **5.** (b) **6.** (c)

Answer of fill in the blanks

1. 1 **2.** 0 **3.** 0 **4.** 3 **5.** 0

6. $-12tj+4k;-12\,j$ **7.** $6t^2-10t-2$ **8. 0**

9. $\vec{a}\times\frac{d\vec{a}}{dt}=0$

Answer of true and false question.

1. *T* **2.** *F* **3.** *T* **4.** *F* **5.** *T*

UNIT-2: GRADIENT, DIVERGENCE AND CURL

CONTENTS:

2.1 *INTRODUCTION*

A gradient in calculus is the differential operator that is used to create a vector from a three-dimensional vector-valued function. The gradient is denoted by the symbol ∇ (nabla). For instance, if "f" is a function, then "∇f" is used to represent a function's gradient. Let's go into depth about the definition of a function's gradient, directional derivative, characteristics, and solved instances in this unit. Divergence and curl are the two essential operations performed on the vector field in mathematics. Both are important in calculus because they aid in the development of the higher-dimensional version of the calculus fundamental theorem. Divergence often explains the field's behaviour in relation to a point or away from it. The rotational extent of the field around a certain point is also measured using curl.

In rectilinear coordinates, the gradient of a scalar and the divergence and curl of a vector have a straightforward form. If the origin is moved or the coordinates are rotated while using rectilinear coordinates, the form is preserved. However, if you pick arbitrary coordinates, their shapes alter, and they appear fairly different even in polar coordinates in the plane, as well as in cylindrical and spherical coordinates in three dimensions.

2.2 *OBJECTIVES*

After reading this unit learners will be able to

- Implementation of application of vector triple product and scalar triple product in vector calculus
- Memorized about the basic differences and relations between the gradient, divergence and curl operators.
- Application of gradient, divergence and curl operators and there use in vector calculus.
- Memorized the useful theorems and their application of vector triple product and scalar triple product.

2.3 *PARTIAL DERIVATIVES OF VECTORS*

If a vector $\vec{r}$ is depending on two or more variable *i.e.*, $\vec{r} = f(x, y, z)$. Then partial derivative of $\vec{r}$ with respect to *x* is defined as

$$\frac{\partial \vec{r}}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y, z) - f(x, y, z)}{\delta x}$$

if this limit exists. Thus $\frac{\partial \vec{r}}{\partial x}$ is just ordinary differentiation of $\vec{r}$ only with respect to the variable *x* and other variable *y* and *z* are regarded as constant.

Similarly, we can find another partial derivative like $\frac{\partial \vec{r}}{\partial y}$ and $\frac{\partial \vec{r}}{\partial z}$.

Other higher order partial derivative can also define as,

$$\frac{\partial^2 \vec{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^2 \vec{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{r}}{\partial z}\right)$$

$$\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^2 \vec{r}}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{r}}{\partial y}\right)$$

If $\vec{r}$ has continuous partial derivatives of the second order, then,

$$\frac{\partial^2 \vec{r}}{\partial y \partial x} = \frac{\partial^2 \vec{r}}{\partial x \partial y}$$

If $\vec{r} = f(x, y, z)$, then total differential $d\vec{r}$ of $\vec{r}$ is given by,

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy + \frac{\partial \vec{r}}{\partial z} dz$$

The vector differential operator: The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Here, the symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its component along i, j, k.

2.4 *GRADIENT OF SCALAR FIELD*

Let $f(x, y, z)$ be a differentiable function at each point (x, y, z) in the space (i.e., defines a differential scalar field). Then gradient of f, written as *grad* f or ∇f and defined as

$$\nabla f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Note: (*i*) Gradient of f is a vector quantity whose three successive components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

(*ii*) Gradient of a scalar field defines a vector field it means if f is a scalar point functionthen its gradient will be a vector point function.

Theorems involving gradient.

Theorem 1: Gradient of addition of two scalar point function: If f and g are two scalar point function then,

$$grad(f + g) = grad\ f + grad\ g \text{ or } \nabla(f + g) = \nabla f + \nabla g$$

Proof:

$$grad(f+g)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)(f+g)=i\frac{\partial}{\partial x}(f+g)+j\frac{\partial}{\partial y}(f+g)+k\frac{\partial}{\partial z}(f+g)$$

$$=i\frac{\partial}{\partial x}f+i\frac{\partial}{\partial x}g+j\frac{\partial}{\partial y}f+j\frac{\partial}{\partial y}g+k\frac{\partial}{\partial z}f+k\frac{\partial}{\partial z}g$$

$$=\left(i\frac{\partial}{\partial x}f+j\frac{\partial}{\partial y}f+k\frac{\partial}{\partial z}f\right)+\left(i\frac{\partial}{\partial x}g+j\frac{\partial}{\partial y}g+k\frac{\partial}{\partial z}g\right)$$

$$=grad\ f+grad\ g$$

i.e., $\nabla(f+g)=\nabla f+\nabla g$

Similarly, we can prove that $grad(f-g)=grad\ f-grad\ g$ or $\nabla(f-g)=\nabla f-\nabla g$

Theorem 2: Gradient of a constant function: The necessary and sufficient condition for scalar point function to be constant is that $\nabla f_c=0$

Proof: Let us suppose that f_c is a constant function it means its three successive components are zero i.e., $\frac{\partial f_c}{\partial x}=0, \frac{\partial f_c}{\partial y}=0, \frac{\partial f_c}{\partial z}=0$.

$$\because grad\ f=i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$$

Then gradient of constant function is, $grad\ f_c=i.0+j.0+k.0=0$ *i.e.*, $\nabla f_c=0$

Conversely, let $\nabla f_c=0$.

It means, $i\frac{\partial f_c}{\partial x}+j\frac{\partial f_c}{\partial y}+k\frac{\partial f_c}{\partial z}=0i+0j+0k$

equating both side component of *i*, *j*, *k* we get, $\frac{\partial f_c}{\partial x}=0, \frac{\partial f_c}{\partial y}=0, \frac{\partial f_c}{\partial z}=0$

$\Rightarrow$ f_c must be independent from *x*, *y*, *z*.

$\Rightarrow$ f_c must be constant function.

Hence the condition is sufficient.

Theorem 3: Gradient of the product of two scalar point function: If f and g are two scalar point function, then $grad(fg)=f\ grad\ g+g\ grad\ f$ or $\nabla(fg)=f\nabla g+g\nabla f$.

Proof: Wehave

$$\nabla(fg)=grad(fg)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)(fg)=i\frac{\partial}{\partial x}(fg)+j\frac{\partial}{\partial y}(fg)+k\frac{\partial}{\partial z}(fg)$$

$$= i\left(f\,\frac{\partial g}{\partial x} + g\,\frac{\partial f}{\partial x}\right) + j\left(f\,\frac{\partial g}{\partial y} + g\,\frac{\partial f}{\partial y}\right) + k\left(f\,\frac{\partial g}{\partial z} + g\,\frac{\partial f}{\partial z}\right)$$

$$= f\left(i\,\frac{\partial g}{\partial x} + j\,\frac{\partial g}{\partial y} + k\,\frac{\partial g}{\partial z}\right) + g\left(i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z}\right)$$

$\nabla(fg) = f\nabla g + g\nabla f$ or $grad(fg) = f\ grad\ g + g\ grad\ f$

In particular, if c is a constant function, then,

$\nabla(\text{c}\ f) = f\nabla c + c\nabla f = 0 + c\nabla f = c\nabla f$

Theorem 4: Gradient of the quotient of two scalar functions: If f and g are two scalar point function, then $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

Proof:

$$\nabla\left(\frac{f}{g}\right) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(\frac{f}{g}\right) = i\,\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + j\,\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + k\,\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$\because \frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\,\frac{\partial f}{\partial x} - f\,\frac{\partial g}{\partial x}}{g^2}, \frac{\partial}{\partial y}\left(\frac{f}{g}\right) = \frac{g\,\frac{\partial f}{\partial y} - f\,\frac{\partial g}{\partial y}}{g^2}, \frac{\partial}{\partial z}\left(\frac{f}{g}\right) = \frac{g\,\frac{\partial f}{\partial z} - f\,\frac{\partial g}{\partial z}}{g^2}$$

$$\Rightarrow \nabla\left(\frac{f}{g}\right) = i\left(\frac{g\,\frac{\partial f}{\partial x} - f\,\frac{\partial g}{\partial x}}{g^2}\right) + j\left(\frac{g\,\frac{\partial f}{\partial y} - f\,\frac{\partial g}{\partial y}}{g^2}\right) + k\left(\frac{g\,\frac{\partial f}{\partial z} - f\,\frac{\partial g}{\partial z}}{g^2}\right)$$

$$\Rightarrow \nabla\left(\frac{f}{g}\right) = \frac{\left(g\,\frac{\partial f}{\partial x} - f\,\frac{\partial g}{\partial x}\right)i + \left(g\,\frac{\partial f}{\partial y} - f\,\frac{\partial g}{\partial y}\right)j + \left(g\,\frac{\partial f}{\partial z} - f\,\frac{\partial g}{\partial z}\right)k}{g^2}$$

$$= \frac{g\,\frac{\partial f}{\partial x} i - f\,\frac{\partial g}{\partial x} i + g\,\frac{\partial f}{\partial y} j - f\,\frac{\partial g}{\partial y} j + g\,\frac{\partial f}{\partial z} k - f\,\frac{\partial g}{\partial z} k}{g^2}$$

$$= \frac{g\,\frac{\partial f}{\partial x} i + g\,\frac{\partial f}{\partial y} + g\,\frac{\partial f}{\partial z} k - \left(f\,\frac{\partial g}{\partial x} i + f\,\frac{\partial g}{\partial y} j + f\,\frac{\partial g}{\partial z} k\right)}{g^2}$$

$$= \frac{g\nabla f - f\nabla g}{g^2}$$

Solved Example

Example 1: Evaluate the value of $\frac{\partial^2}{\partial x \partial y}(A \times B)$ at (1, 0, -2) where, $A = x^2yzi - 2xz^3 j + xz^2k, B = 2zi + yj - x^2k$.

Answer: First we have to find the value of $A \times B$

So, $A \times B = \begin{vmatrix} i & j & k \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= \left(2x^3z^3 - xyz^2\right)i + \left(2xz^3 + x^4yz\right) j + \left(x^2y^2z + 4xz^4\right)k$$

Now,

$$\frac{\partial^2}{\partial x \partial y}(A \times B) = \frac{\partial^2}{\partial x \partial y}\left\{\left(2x^3z^3 - xyz^2\right)i + \left(2xz^3 + x^4yz\right) j + \left(x^2y^2z + 4xz^4\right)k\right\}$$

$$\frac{\partial^2}{\partial x \partial y}(A \times B) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{\left(2x^3z^3 - xyz^2\right)i + \left(2xz^3 + x^4yz\right) j + \left(x^2y^2z + 4xz^4\right)k\right\}\right]$$

$$= \frac{\partial}{\partial x}\left\{-xz^2i + x^4zj + 2x^2yzk\right\}$$

$$= -z^2i + 4x^3zj + 4xyzk$$

Thus, value of $\frac{\partial^2}{\partial x \partial y}(A \times B)$ at the point (1, 0, -2) evaluated by putting

$x = 1, y = 0, z = -2$

$$\left[\frac{\partial^2}{\partial x \partial y}(A \times B)\right]_{(1,0,-2)} = -4i - 8j$$

Example 2: Evaluate *grad f* at the point (1, -2, -1) where, $f(x, y, z) = 3x^2y - y^3z^2$.

Answer: We have given $f(x, y, z) = 3x^2y - y^3z^2$

So, $grad\ f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(3x^2y - y^3z^2\right)$

$$= i\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right) + j\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right) + k\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)$$

$$= 6xyi + \left(3x^2 - 3y^2z^2\right) j + 2y^3z\,k$$

Thus, value of *grad f* at the point (1, -2, -1)

$$\left(grad\ f\right)_{(1,-2,-1)} = -12i - 9j - 16k$$

Example 3: If $r = \left|\vec{r}\right|$ where, $\vec{r} = xi + yj + zk$, then prove the following:

(i) $\nabla f(\mathrm{r}) = f'(\mathrm{r})\nabla r$ **(ii)** $\nabla r = \frac{1}{r}\vec{r}$ **(iii)**

$\nabla f(\mathrm{r}) \times \vec{r} = 0$

(iv) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ **(v)** $\nabla \log\left|\vec{r}\right| = \frac{\vec{r}}{r^2}$ **(vi)** $\nabla \vec{r^n} = nr^{n-2}\vec{r}$

Answer: Let $\vec{r} = xi + yj + zk$, then

$r = \left|\vec{r}\right| = \sqrt{x^2+y^2+z^2}$ or $r^2 = x^2+y^2+z^2$

(i) $\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r) = i\frac{\partial}{\partial x}f(r) + j\frac{\partial}{\partial y}f(r) + k\frac{\partial}{\partial z}f(r)$

$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\, f'(r)\frac{\partial r}{\partial z}$

$= f'(r)\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right) = f'(r)\nabla r$

(ii) $\nabla \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\sqrt{x^2+y^2+z^2}\right)$

$= i\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + j\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + k\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}$

$= i\frac{2x}{2\sqrt{x^2+y^2+z^2}} + j\frac{2y}{2\sqrt{x^2+y^2+z^2}} + k\frac{2z}{2\sqrt{x^2+y^2+z^2}}$

$= \frac{2xi+2yj+2zk}{2\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{\left|\vec{r}\right|}$

(iii) As we know from **(i)** proof that $\nabla f(\mathrm{r}) = f'(\mathrm{r})\nabla r$ and from (ii) proof that $\nabla r = \frac{1}{r}\vec{r}$

So, $\nabla f(\mathrm{r}) = f'(\mathrm{r})\frac{1}{r}\vec{r}$

Now, $\nabla f(\mathrm{r}) \times \vec{r} = f'(\mathrm{r})\frac{1}{r}\vec{r}\times\vec{r} = f'(\mathrm{r})\frac{1}{r}\left(\vec{r}\times\vec{r}\right) = f'(\mathrm{r})\frac{1}{r}.0 = 0$ [As we know $\vec{r}\times\vec{r} = 0$]

(iv) $\nabla\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= i\frac{1}{r^2}\frac{\partial r}{\partial x} + j\frac{1}{r^2}\frac{\partial r}{\partial y} + k\frac{1}{r^2}\frac{\partial r}{\partial z}$$

$$= \frac{1}{r^2}\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right) = \frac{\nabla r}{r^2}$$

Since we know that $\nabla r = \frac{1}{r}\vec{r}$

So, $\nabla\left(\frac{1}{r}\right) = \frac{\nabla r}{r^2} = \frac{1}{r^2}\left(\frac{1}{r}\vec{r}\right) = -\frac{\vec{r}}{r^3}$

(v) $$\nabla \log\left|\vec{r}\right| = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\log r = i\frac{\partial}{\partial x}\log r + j\frac{\partial}{\partial y}\log r + k\frac{\partial}{\partial z}\log r$$

$$= i\frac{1}{r}\frac{\partial r}{\partial x} + j\frac{1}{r}\frac{\partial r}{\partial y} + k\frac{1}{r}\frac{\partial r}{\partial z} = \frac{1}{r}\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right)$$

$$= i\frac{1}{r}\frac{\partial r}{\partial x} + j\frac{1}{r}\frac{\partial r}{\partial y} + k\frac{1}{r}\frac{\partial r}{\partial z} = \frac{1}{r}\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right)$$

$$= \frac{1}{r}\nabla r = \frac{\vec{r}}{r^2}$$

(vi) $$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) r^n = i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}i + nr^{n-1}\frac{\partial r}{\partial y}j + nr^{n-1}\frac{\partial r}{\partial z}k$$

$$= \left(\frac{\partial r}{\partial x}i + \frac{\partial r}{\partial y}j + \frac{\partial r}{\partial z}k \right) nr^{n-1}$$

$$= nr^{n-1}\nabla r = nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

Example 4: If $f = \left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k$, then verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Answer: We have given, $f = \left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k$

Then, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}\left\{\left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k\right\}$

$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left\{\left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k\right\}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(2x^2y - x^4\right)i + \frac{\partial}{\partial x}\left(e^{xy} - y\sin x\right)j + \frac{\partial}{\partial x}x^2\cos y\,k\right]$$

$$= \frac{\partial}{\partial y}\left\{\left(4xy - 4x^3\right)i + \left(ye^{xy} - y\cos x\right)j + 2x\cos y\,k\right\}$$

$$= \frac{\partial}{\partial y}\left(4xy - 4x^3\right)i + \frac{\partial}{\partial y}\left(ye^{xy} - y\cos x\right)j + \frac{\partial}{\partial y}2x\cos y\,k$$

$$= 4xi + \left(e^{xy} + xye^{xy} - \cos x\right)j - 2x\sin y\,k$$

Similarly, $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2}{\partial x\partial y}\left\{\left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k\right\}$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{\left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + x^2\cos y\,k\right\}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(2x^2y - x^4\right)i + \frac{\partial}{\partial y}\left(e^{xy} - y\sin x\right)j + \frac{\partial}{\partial y}x^2\cos y\,k\right]$$

$$= \frac{\partial}{\partial x}\left\{2x^2i + \left(xe^{xy} - \sin x\right)j - x^2\sin y\,k\right\}$$

$$= \frac{\partial}{\partial x}2x^2i + \frac{\partial}{\partial x}\left(xe^{xy} - \sin x\right)j - \frac{\partial}{\partial x}x^2\sin y\,k$$

$$= 4xi + \left(e^{xy} + xye^{xy} - \cos x\right)j - 2x\sin y\,k$$

Hence, we can easily see that $\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y}$.

Equipotential surfaces or level surfaces: Let a scalar field $f(x,y,z) = c$ in the region *R*. The points which are satisfying the equation $f(x,y,z) = c$, constitutes family of surfaces in the three-dimensional space. The occurred surface of this family is called ***level surfaces***.

Any surface of this family is such a way that the value of the function *f* at any point of it is the same. Hence these surfaces are also called ***iso-f-surfaces***.

2.5 *DIVERGENCE OF A VECTOR POINT FUNCTION*

Let *V* be the differentiable vector point function. Then the divergence of *V* denoted as *divV* or $\nabla.V$ and defined as follows:

$$\nabla.V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).V = i.\frac{\partial V}{\partial x} + j.\frac{\partial V}{\partial y} + k.\frac{\partial V}{\partial z} = \sum i.\frac{\partial V}{\partial x}$$

It should be noted that *divV* is always a scalar quantity.

Solenoidal vector: Adifferentiable vector point function V is said to be solenoidal if $divV = 0$

Theorem 5: If $V = V_1 i + V_2 j + V_3 k$ is differentiable vector point function, then

$$\nabla.V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).\left(V_1 i + V_2 j + V_3 k \right) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \sum \frac{\partial V_1}{\partial x}$$

Proof: We have given that, $V = V_1 i + V_2 j + V_3 k$

Then,

$$\nabla.V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).\left(V_1 i + V_2 j + V_3 k \right) = i\frac{\partial}{\partial x}.\left(V_1 i \right) + j\frac{\partial}{\partial y}.\left(V_2 j \right) + k\frac{\partial}{\partial z}.\left(V_3 k \right)$$

$$= \frac{\partial V_1}{\partial x}\left(i.i \right) + \frac{\partial V_2}{\partial y}\left(j.j \right) + \frac{\partial V_3}{\partial z}\left(k.k \right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Hence, $\nabla.V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

2.6 *CURL OF A VECTOR POINT FUNCTION*

Let F be any differentiable vector point function. Then the curl or sometime called rotation of F denoted as *curlF* or $\nabla \times F$ and defined as follows:

$$\nabla \times F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times F = i \times \frac{\partial F}{\partial x} + j \times \frac{\partial F}{\partial y} + k \times \frac{\partial F}{\partial z} = \sum i \times \frac{\partial F}{\partial x}$$

It should be noted that *curlF* is always a vector quantity.

Irrotational vector: Adifferentiable vector point function F is said to be irrotational if $curl\,F = 0$.

Theorem 6: If $F = f_1 i + f_2 j + f_3 k$ is differentiable vector point function, then

$$curl\,F = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) k$$

Proof: $\nabla\times F=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(f_1 i+f_2 j+f_3 k\right)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)i+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)j+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)k$$

OR

The *curlF* is also prove as,

$$\nabla\times F=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(f_1 i+f_2 j+f_3 k\right)$$

$$=i\frac{\partial}{\partial x}\times\left(f_1 i+f_2 j+f_3 k\right)+j\frac{\partial}{\partial y}\times\left(f_1 i+f_2 j+f_3 k\right)+k\frac{\partial}{\partial z}\times\left(f_1 i+f_2 j+f_3 k\right)$$

as we know that,

$i\times i=0,\ j\times j=0, k\times k=0,\ i\times j=k,\ j\times i=-k, i\times k=j, k\times i=-j,\ j\times k=i, k\times j=-i$

so, $\nabla\times F=\left(\frac{\partial f_2}{\partial x}k-\frac{\partial f_3}{\partial x}j\right)+\left(-\frac{\partial f_1}{\partial y}k+\frac{\partial f_3}{\partial y}i\right)+\left(\frac{\partial f_1}{\partial z}j-\frac{\partial f_2}{\partial z}i\right)$

$$\nabla\times F=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)i+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)j+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)k$$

2.7 *LAPLACIAN OPERATOR*

The Laplacian Operators ∇^2 : The Laplacian operator mathematically denoted as ∇^2 and defined as, $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

If *f* is a scalar point function, then, $\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$, here $\nabla^2 f$ is also a scalar quantity.

If *f* is a vector point function, then, $\nabla^2\vec{f}=\frac{\partial^2\vec{f}}{\partial x^2}+\frac{\partial^2\vec{f}}{\partial y^2}+\frac{\partial^2\vec{f}}{\partial z^2}$, here $\nabla^2\vec{f}$ is also a vector quantity.

Laplace's Equation: The Laplace equation is defined as $\nabla^2 f=0$. A function which satisfied Laplace equation is called **Harmonic function**.

Solved Example

Example 5: Show that $div\,\vec{r}=3$.

Answer: As we know that $\vec{r} = xi + yj + zk$.

So, $div\,\vec{r} = \nabla.\vec{r} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right).\left(x\,i + y\,j + z\,k \right)$

$$= i\frac{\partial}{\partial x}.\left(x\,i + y\,j + z\,k \right) + j\frac{\partial}{\partial y}.\left(x\,i + y\,j + z\,k \right) + k\frac{\partial}{\partial z}.\left(x\,i + y\,j + z\,k \right)$$

$$= \frac{\partial x}{\partial x}(i.i) + \frac{\partial y}{\partial y}(\mathrm{j.\,j}) + \frac{\partial z}{\partial z}(\mathrm{k.k})$$

$$= 1 + 1 + 1 = 3$$

Example 6: Show that $curl\,\vec{r} = 0$.

Answer: $curl\,\vec{r} = \nabla \times \vec{r} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right) \times \left(xi + yj + zk \right)$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right)$$

$$= 0i + 0j + 0k = 0$$

Hence, we can say that $curl\,\vec{r} = 0$

Example 7: If $f = x^2 y\,i - 2xz\,j + 2yz\,k$, then find the following:

(i) $div\,\vec{f}$ **(ii)** $curl\,\vec{f}$ **(iii)** $curl\,curl\,\vec{f}$

Answer: **(i)** We have given that, $f = x^2 y\,i - 2xz\,j + 2y\,k$

Then, $div\,\vec{f} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right).\left(x^2 y\,i - 2xz\,j + 2yz\,k \right)$

$$= i\frac{\partial}{\partial x}.\left(x^2 y\,i - 2xz\,j + 2yz\,k \right) + j\frac{\partial}{\partial y}.\left(x^2 y\,i - 2xz\,j + 2yz\,k \right) + k\frac{\partial}{\partial z}.\left(x^2 y\,i - 2xz\,j + 2yz\,k \right)$$

$$= 2xy + 0 + 2y = 2y(x+1)$$

(ii) $curl\,\vec{f} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right) \times \left(x^2 y\,i - 2xz\,j + 2yz\,k \right)$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix} = i\left(\frac{\partial}{\partial y}2yz + \frac{\partial y}{\partial z}2xz \right) - j\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial z}x^2 y \right) + k\left(-\frac{\partial}{\partial x}2xz - \frac{\partial}{\partial y}x^2 y \right)$$

$$= i\left(2z+2x\right)+k\left(-2z-x^2\right)=2(x+z)i-(2z+x^2)k$$

(iii) $$curl\,curl\,\vec{f}=\nabla\times\left(\nabla\times\vec{f}\right)=\nabla\times\left[2(x+z)i-(2z+x^2)k\right]$$

$$=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2(x+z) & 0 & -(2z+x^2) \end{vmatrix}$$

$$=i\left(\frac{\partial}{\partial y}(-2z-x^2)\right)-\left[\frac{\partial}{\partial x}(-2z-x^2)-\frac{\partial}{\partial z}(2x+2z)\right]j+\left[0-\frac{\partial}{\partial y}(2x+2z)\right]k$$

$$=0i-(-2x-2)j+(0-0)k=(2x+2)j$$

Example 8: If the vector, $V=(x+3y)i+(y-2z)\,j+(x+az)k$, is solenoidal then find the constant a.

Answer: As we know that a vector will be called solenoidal if $divV=0$.

Now, $divV=\left(i\dfrac{\partial}{\partial x}+j\dfrac{\partial}{\partial y}+k\dfrac{\partial}{\partial z}\right).\left[(x+3y)i+(y-2z)\,j+(x+az)k\right]=0$

$$\Rightarrow\frac{\partial}{\partial x}(x+3y)i.i+\frac{\partial}{\partial y}(y-2z)\,j.j+\frac{\partial}{\partial z}(x+az)k.k=0\ \left[\because i.i=j.j=k.k=0\right]$$

$\Rightarrow 1+1+a=0$

$\Rightarrow a=-2$

Example 9: If the vector, $V=(\sin y+z)i+(x\cos y-z)\,j+(x-y)k$, then show that V is irrotational.

Answer: As we know that a vector will be called irrotational if $curlV=0$

.

Now, $curlV=\left(i\dfrac{\partial}{\partial x}+j\dfrac{\partial}{\partial y}+k\dfrac{\partial}{\partial z}\right)\times\left[(\sin y+z)i+(x\cos y-z)\,j+(x-y)k\right]$

$$=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y+z & x\cos y-z & x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]i-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]j+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]k$$

$$=(-1+1)i-(1-1)\,j+\left(\cos y-\cos y\right)k=0$$

Hence, $curlV=0$, which shows the vector V is irrotational.

Example 10: Show that, $\nabla^2\left(\frac{x}{r^3}\right)=0$.

Answer: As we know that, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

So, $\nabla^2\left(\frac{x}{r^3}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)$

First, we evaluate, $\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right]=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}$$

$$\left[\because r^2=x^2+y^2+z^2, then \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}$$

Again, $\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\}$

$$\left[\because r^2=x^2+y^2+z^2, then \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left\{-\frac{3xy}{r^5}\right\}=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}$$

Similarly, we evaluate, $\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$

Therefore,

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{9x}{r^5}+\frac{15x^3}{r^7}+\left(-\frac{3x}{r^5}+\frac{15xy^2}{r^7}\right)+\left(-\frac{3x}{r^5}+\frac{15xz^2}{r^7}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}--\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}\left(x^2+y^2+z^2\right)$$

$$==-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0$$

Hence, $\nabla^2\left(\frac{x}{r^3}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=0$

SOME IMPORTANT RESULTS ON VECTOR IDENTITIES:

1. Prove that $div(A+B)=div\,A+div\,B$ or $\nabla.(A+B)=\nabla.A+\nabla.B$

Proof: $div(A+B)=\nabla.(A+B)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(A+B)$

$$=i.\frac{\partial}{\partial x}(A+B)+j.\frac{\partial}{\partial y}(A+B)+k.\frac{\partial}{\partial z}(A+B)$$

$$=i.\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial x}\right)+j.\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial y}\right)+k.\left(\frac{\partial A}{\partial z}+\frac{\partial B}{\partial z}\right)$$

$$=i.\frac{\partial A}{\partial x}+i.\frac{\partial B}{\partial x}+j.\frac{\partial A}{\partial y}+j.\frac{\partial B}{\partial y}+k.\frac{\partial A}{\partial z}+k.\frac{\partial B}{\partial z}$$

$$=\left(i.\frac{\partial A}{\partial x}+j.\frac{\partial A}{\partial y}+k.\frac{\partial A}{\partial z}\right)+\left(i.\frac{\partial B}{\partial x}+j.\frac{\partial B}{\partial y}+k.\frac{\partial B}{\partial z}\right)$$

$$=\nabla.\mathrm{A}+\nabla.B=div\,A+div\,B$$

2. Prove that $curl(A+B)=curl\,A+curl\,B$ or

$\nabla\times(A+B)=\nabla\times A+\nabla\times B$

Proof: $curl(A+B)=\nabla\times(A+B)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(A+B)$

$$=i\times\frac{\partial}{\partial x}(A+B)+j\times\frac{\partial}{\partial y}(A+B)+k\times\frac{\partial}{\partial z}(A+B)$$

$$=i\times\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial x}\right)+j\times\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial y}\right)+k\times\left(\frac{\partial A}{\partial z}+\frac{\partial B}{\partial z}\right)$$

$$=i\times\frac{\partial A}{\partial x}+i\times\frac{\partial B}{\partial x}+j\times\frac{\partial A}{\partial y}+j\times\frac{\partial B}{\partial y}+k\times\frac{\partial A}{\partial z}+k\times\frac{\partial B}{\partial z}$$

$$=\left(i\times\frac{\partial A}{\partial x}+j\times\frac{\partial A}{\partial y}+k\times\frac{\partial A}{\partial z}\right)+\left(i\times\frac{\partial B}{\partial x}+j\times\frac{\partial B}{\partial y}+k\times\frac{\partial B}{\partial z}\right)$$

$$=\nabla\times\mathrm{A}+\nabla\times B=curl\,A+curl\,B$$

3. If A and ϕ are differentiable vector and scalar function respectively, then

$$div(\phi A)=(grad\,\phi).A+\phi\,divA\ \ or\ \ \nabla.(\phi A)=(\nabla\phi).A+\phi(\nabla.A)$$

Proof: As we know, $div(\phi A)=\nabla.(\phi A)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(\phi A)$

$$= i.\frac{\partial}{\partial x}(\phi A) + j.\frac{\partial}{\partial y}(\phi A) + k.\frac{\partial}{\partial z}(\phi A) = \sum\left\{ i.\left(\frac{\partial}{\partial x}(\phi A) \right) \right\}$$

$$= \sum\left\{ i.\left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum\left\{ i.\left(\frac{\partial \phi}{\partial x} A \right) \right\} + \sum\left\{ i.\left(\phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum\left\{ \left(\frac{\partial \phi}{\partial x} i \right).\mathrm{A} \right\} + \sum\left\{ \phi \left(i.\frac{\partial A}{\partial x} \right) \right\} \left[\because a.(mb) = (ma).b = m(a.b) \right]$$

$$= \left\{ \sum\left(\frac{\partial \phi}{\partial x} i \right) \right\}.A + \phi \sum\left(i.\frac{\partial A}{\partial x} \right) = (\nabla \phi).A + \phi(\nabla.\mathrm{A})$$

4. If A and ϕ are differentiable vector and scalar function respectively, then

$$curl(\phi A) = (grad\,\phi) \times A + \phi\, curlA \quad or \quad \nabla \times (\phi A) = (\nabla \phi) \times A + \phi(\nabla \times A)$$

Proof: As we know, $curl(\phi A) = \nabla \times (\phi A) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times (\phi A)$

$$= i \times \frac{\partial}{\partial x}(\phi A) + j \times \frac{\partial}{\partial y}(\phi A) + k \times \frac{\partial}{\partial z}(\phi A) = \sum\left\{ i \times \left(\frac{\partial}{\partial x}(\phi A) \right) \right\}$$

$$= \sum\left\{ i \times \left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum\left\{ i \times \left(\frac{\partial \phi}{\partial x} A \right) \right\} + \sum\left\{ i \times \left(\phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum\left\{ \left(\frac{\partial \phi}{\partial x} i \right) \times \mathrm{A} \right\} + \sum\left\{ \phi \left(i \times \frac{\partial A}{\partial x} \right) \right\}$$

$\left[\because a \times (mb) = (ma) \times b = m(a \times b) \right]$

$$= \left\{ \sum\left(\frac{\partial \phi}{\partial x} i \right) \right\} \times A + \phi \sum\left(i \times \frac{\partial A}{\partial x} \right) = (\nabla \phi) \times A + \phi(\nabla \times \mathrm{A})$$

5. Prove that

$$div(A \times B) = B.curl\,A - A.curl\,B \quad or \quad \nabla.(A \times B) = B.(\nabla \times A) - A.(\nabla \times B)$$

Proof: As we know that, $div(A \times B) = \nabla.(A \times B) = \sum\left\{ i.\frac{\partial}{\partial x}(A \times B) \right\}$

$$= \sum\left\{ i.\left(\frac{\partial A}{\partial x} \times B + A \times \frac{\partial B}{\partial x} \right) \right\} = \sum\left\{ i.\left(\frac{\partial A}{\partial x} \times B \right) \right\} + \sum\left\{ i.\left(A \times \frac{\partial B}{\partial x} \right) \right\}$$

$$=\sum\left\{\left(i\times\frac{\partial A}{\partial x}\right).\mathrm{B}\right\}-\sum\left\{i.\left(\frac{\partial B}{\partial x}\times A\right)\right\}$$

$$\left[\because a.(b\times c)=(\mathrm{a}\times\mathrm{b}).\mathrm{c}\text{ and }\mathrm{a}.(\mathrm{b}\times\mathrm{c})=-\mathrm{a}.(\mathrm{c}\times\mathrm{b})\right]$$

$$=\left\{\sum\left(i\times\frac{\partial A}{\partial x}\right)\right\}.B-\sum\left\{\left(i\times\frac{\partial B}{\partial x}\right).A\right\}$$

$$=(curl\ A).B-\left\{\sum\left(i\times\frac{\partial B}{\partial x}\right)\right\}.A=(curl\ A).B-A.curl\ B$$

6. Prove that $curl(A\times B)=(B.\nabla)A-B\,div\,A-(A.\nabla)B+A\,div\,B$

Proof: As we know that, $curl(A\times B)=\nabla\times(A\times B)=\sum\left\{i\times\frac{\partial}{\partial x}(A\times B)\right\}$

$$=\sum\left\{i\times\left(A\times\frac{\partial B}{\partial x}+\frac{\partial A}{\partial x}\times B\right)\right\}=\sum\left\{i\times\left(A\times\frac{\partial B}{\partial x}\right)\right\}+\sum\left\{i\times\left(\frac{\partial A}{\partial x}\times B\right)\right\}$$

$$=\sum\left\{\left(i.\frac{\partial B}{\partial x}\right)A\right\}-\sum\left\{(A.i)\frac{\partial B}{\partial x}\right\}+\sum\left\{(B.i)\frac{\partial A}{\partial x}\right\}-\sum\left\{\left(i.\frac{\partial A}{\partial x}\right)B\right\}$$

$$=\left\{\sum\left(i.\frac{\partial B}{\partial x}\right)\right\}A-\left\{A.\sum i\frac{\partial}{\partial x}\right\}B+\left\{B.\sum i\frac{\partial}{\partial x}\right\}A-\left\{\sum\left(i.\frac{\partial A}{\partial x}\right)\right\}B$$

$$=(div\,B)A-(A.\nabla)B+(B.\nabla)A-(div\,A)B$$

7. Prove that $grad(A.B)=(B.\nabla)\mathrm{A}+(\mathrm{A}.\nabla)+\mathrm{B}\times\mathrm{curl\,A}+\mathrm{A}\times\mathrm{curl\,B}$

Proof: We have,

$$grad(A.B)=\nabla(A.B)=\sum i\frac{\partial}{\partial x}(A.B)=\sum i\left(A.\frac{\partial B}{\partial x}+\frac{\partial A}{\partial x}.B\right)$$

$$=\sum\left\{\left(A.\frac{\partial B}{\partial x}\right)i\right\}+\sum\left\{\left(B.\frac{\partial A}{\partial x}\right)i\right\} \qquad \ldots.\ (1)$$

Since we know that,

$$a\times(b\times c)=(a.c)b-(a.b)c\Rightarrow(a.b)c=(a.c)b-a\times(b\times c)$$

Thus, $\sum\left\{\left(A.\frac{\partial B}{\partial x}\right)i\right\}=\sum\left\{(A.i)\frac{\partial B}{\partial x}\right\}+\sum\left\{A\times\left(i\times\frac{\partial B}{\partial x}\right)\right\}$

$$=\left\{A.\sum i\frac{\partial}{\partial x}\right\}B+A\times\sum\left(i\times\frac{\partial B}{\partial x}\right)=(A.\nabla)B+A\times(\nabla\times B) \qquad \ldots.\ (2)$$

Similarly, $\sum\left\{\left(\mathrm{B}.\frac{\partial A}{\partial x}\right)i\right\}=(B.\nabla)A+B\times(\nabla\times A)$ ….

(3)

Now, putting the value of equation (2) and equation (3) in equation (1)

We get, $grad\,(A.B)=\left(A.\nabla\right)B+A\times\left(\nabla\times B\right)+\left(B.\nabla\right)A+B\times\left(\nabla\times A\right)$

8. Prove that $div\,grad\phi=\nabla^2\phi\ \ i.e.,\nabla.\left(\nabla\phi\right)=\nabla^2\phi$

Proof: $div\,grad\phi=\nabla.\nabla\phi=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)i.i+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)j.j+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)k.k$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi$$

9. Prove that, curl of the gradient of ϕ is zero, i.e.,

$\nabla\times\left(\nabla\phi\right)=0\ \ i.e.,curl\ grad\,\phi=0$

Proof: $curl\ grad\,\phi=\nabla\times\left(\nabla\phi\right)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right)$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}=\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)i+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)j+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)k$$

$$-0i+0j+0k=0$$

10. Prove that $div\,Curl\,A=0\ \ i.e,\ \nabla.\left(\nabla\times A\right)=0$

Proof: Let $A=A_1i+A_2j+A_3k$, Then

$$div\,Curl\,A=\nabla.\left(\nabla\times A\right)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left\{\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(A_1i+A_2j+A_3k\right)\right\}$$

First we find out, $\nabla\times A=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(A_1i+A_2j+A_3k\right)$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3\end{vmatrix}=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k$$

Now,

$$div\,Curl\,A = \nabla.(\nabla \times A) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left\{\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)(i.i) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)(j.j) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)(k.k)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}$$

$$= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_2}{\partial x \partial z}\right) = 0$$

SOLVED EXAMPLE

Example 11: Find $\nabla\phi$ and $|\nabla\phi|$, where $\phi = \left(x^2 + y^2 + z^2\right)e^{\left(x^2+y^2+z^2\right)^{1/2}}$.

Solution: Let $r = xi + yj + zk$, then $r^2 = x^2 + y^2 + z^2$.

So, we can write $\phi = r^2 e^{-r}$, then $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

We consider, $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} = \left(2re^{-r} - r^2e^{-r}\right)\frac{\partial r}{\partial x} = \left(2re^{-r} - r^2e^{-r}\right)\frac{x}{r}$

Similarly, $\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} = \left(2re^{-r} - r^2e^{-r}\right)\frac{y}{r}$ and

$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z} = \left(2re^{-r} - r^2e^{-r}\right)\frac{r}{z}$

Now, $\nabla\phi = \left(2re^{-r} - r^2e^{-r}\right)\frac{x}{r}i + \left(2re^{-r} - r^2e^{-r}\right)\frac{y}{r}j + \left(2re^{-r} - r^2e^{-r}\right)\frac{z}{r}k$

$$\nabla\phi = \left(2re^{-r} - r^2e^{-r}\right)\frac{\vec{r}}{r} = (2-r)e^{-r}\,\vec{r}$$

And $|\nabla\phi| = \left|\left(2re^{-r} - r^2e^{-r}\right)\frac{\vec{r}}{r}\right| = (2-r)e^{-r}\left|\vec{r}\right| = (2-r)re^{-r}$

Example 12: Prove that $div\left(r^n\,\vec{r}\right) = (n+3)r^n$.

Solution: We know that, $div\left(\phi\vec{A}\right) = \phi\left(div\,\vec{A}\right) + \vec{A}.\text{grad}\,\phi$

So, $div\left(r^n\,\vec{r}\right) = r^n\left(div\,\vec{r}\right) + \vec{r}.\text{grad}\,r^n$

Since,

$$div\,\vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).\left(xi + yj + zk \right) = \frac{\partial}{\partial x}x(i.i) + \frac{\partial}{\partial y}\mathrm{y(j.\,j)} + \frac{\partial}{\partial z}\mathrm{z(k.k)} = 3$$

And

$$\mathrm{grad}\,r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) r^n = i\frac{\partial}{\partial x}r^n + j\frac{\partial}{\partial y}r^n + k\frac{\partial}{\partial z}r^n = nr^{n-1}\left(\frac{\partial r}{\partial x}i + \frac{\partial r}{\partial y}j + \frac{\partial r}{\partial z}k \right)$$

$$= nr^{n-1}\left(\frac{x}{r}i + \frac{y}{r}i + \frac{z}{r}i \right) = nr^{n-1}\frac{\vec{r}}{r}$$

So, $\mathrm{grad}\,r^n = nr^{n-2}\,\vec{r}$

Now,

$$div\left(r^n\,\vec{r} \right) = r^n\left(div\,\vec{r} \right) + \vec{r}\,.\mathrm{grad}\,r^n = 3r^n + nr^{n-2}\,\vec{r}\,.\,\vec{r} = 3r^n + nr^n = \left(3+n \right) r^n$$

Example 13: Prove that $div\left(\frac{\vec{r}}{r^3} \right) = 0$.

Proof: As we know that, $div\left(r^n\,\vec{r} \right) = \left(n+3 \right) r^n$ …………… (1)

Now putting n = -3 in equation (1)

So, $div\left(r^{-3}\,\vec{r} \right) = \left(3-3 \right) r^{-3} = 0$

Example 14: Prove that $div\left(\hat{r} \right) = \frac{2}{r}$.

Proof: We have $div\left(\frac{\vec{r}}{r} \right) = div\left(r^{-1}\,\vec{r} \right)$

As we know that, $div\left(r^n\,\vec{r} \right) = \left(n+3 \right) r^n$ …………… (1)

Now putting n = -1 in equation (1)

So, $div\left(\hat{r} \right) = div\left(r^{-1}\,\vec{r} \right) = \left(-1+3 \right) r^{(-1)} = \frac{2}{r}$

Example 15: Prove that vector $f(r)\,\vec{r}$ is irrotational.

Proof: As we know that any vector $\vec{A}$ is irrotational if $curl\,\vec{A} = 0$

So, if we have to show that the vector $f(r)\vec{r}$ is irrotational we have to show

$$curl\left[\, f(r)\vec{r}\,\right]=0$$

Since, $curl\left(\phi\vec{A}\right)=\left(grad\,\phi\right)\times A+\phi\, curl\, A$

Now, $curl\left[\, f(r)\vec{r}\,\right]=\left[grad\ f(r)\right]\times\vec{r}+f(r)curl\,\vec{r}$

$$=\left[\, f^{'}(r)grad\,\vec{r}\,\right]\times\vec{r}+f(r).0 \qquad \left[\because curl\,\vec{r}=0\right]$$

$$=\left[\, f^{'}(r)\frac{\vec{r}}{r}\,\right]\times\vec{r}=f^{'}(r)\frac{1}{r}\left(\vec{r}\times\vec{r}\right)=0$$

Example 16: Prove that $\nabla^2 f(r)=f^{''}(r)+\frac{2}{r}f^{'}(r)$.

Proof: As we know that if ϕ is a scalar function, then $\nabla^2\phi=\nabla.\left(\nabla\phi\right)$

So, $\nabla^2 f(r)=\nabla.\left(\nabla f(r)\right)=div\left\{grad\ f(r)\right\}=div\left\{f^{'}(r)\,\text{grad}\,\text{r}\right\}$

$$=div\left\{\frac{1}{r}f^{'}(r)\vec{r}\right\}=\frac{1}{r}f^{'}(r)div\,\vec{r}+\vec{r}.grad\left\{\frac{1}{r}f^{'}(r)\right\}$$

$$=\frac{3}{r}f^{'}(r)+\vec{r}.\left[\frac{d}{dr}\left\{\frac{1}{r}f^{'}(r)\right\}grad\, r\right]=\frac{3}{r}f^{'}(r)+\vec{r}.\left[\left\{\frac{-1}{r^2}f^{'}(r)+\frac{1}{r}f^{''}(r)\right\}\frac{\vec{r}}{r}\right]$$

$$=\frac{3}{r}f^{'}(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f^{'}(r)+\frac{1}{r}f^{''}(r)\right\}\right]\left(\vec{r}.\vec{r}\right)$$

$$=\frac{3}{r}f^{'}(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f^{'}(r)+\frac{1}{r}f^{''}(r)\right\}\right]r^2$$

$$=\frac{3}{r}f^{'}(r)-\frac{1}{r}f^{'}(r)+f^{''}(r)=f^{''}(r)+\frac{2}{r}f^{'}(r)$$

Example 17: If $\nabla^2 f(r)=0$, show that $f(r)=\frac{c_1}{r}+c_2$, where $r^2=x^2+y^2+z^2$ and c_1,c_2 are arbitrary constant.

Answer: From the previous example we know that,

$\nabla^2 f(r)=f^{''}(r)+\frac{2}{r}f^{'}(r)$, where $r^2=x^2+y^2+z^2$

Since, we have given that $\nabla^2 f(r) = 0$, then $f''(r) + \frac{2}{r} f'(r) = 0$

$$\Rightarrow \frac{f''(r)}{f'(r)} = -\frac{2}{r}$$

Integrating with respect to r we get, $\log f'(r) = -2\log r + \log c = \log \frac{c}{r^2}$

$$\Rightarrow f'(r) = \frac{c}{r^2}$$

Again, integrating we get, $f(r) = -\frac{c}{r} + c_2$, where c_2 is a constant

After replacing constant $-c$ by c_1, we get $f(r) = \frac{c_1}{r} + c_2$

SELF CHECK QUESTION

Fill in the blanks:

1. If $F = (x^2 + y^2)i - 2xy\,j$, then $F.d\vec{r} = \ldots\ldots$

2. If $P = e^{xy} i + (x - 2y)\,j + (x\sin y)k,$ then $\frac{\partial P}{\partial x} = \ldots\ldots\ldots$

3. If $\vec{a}$ is a constant vector then $grad\,(\vec{a}.\vec{r}) = \ldots\ldots\ldots$

4. If $\vec{a}$ is a constant vector then $\nabla.(\vec{a}\times\vec{r}) = \ldots\ldots\ldots$

5. If $\vec{r} = xi + yj + zk$, then the value of $div\,\vec{r} =$

6. If $\vec{A} = x^2 zi - 2y^3 z^2 j + xy^2 zk$, then $div\,\vec{A}$ at (1, -1, 1) = ……….

7. If $\vec{r} = xi + yj + zk$, then the value of $curl\,\vec{r} = \ldots\ldots\ldots$

8. For any vector $\vec{A}$, $div\,curl\,\vec{A} = \ldots\ldots$

9. A vector $\vec{V}$ is said to be solenoidal if ……

10. A vector $\vec{F}$ is said to be irrotational if ……

11. If $\phi = x^2 y + 2xy + z^2$, then $curl\,grad\phi = \ldots\ldots$

2.8 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of gradient, divergence, curl operators and Laplacian operators.
- The relations between gradient, divergence and curl operators.

2.9 *GLOSSARY*

- **Gradient of a vector function *f*:** ∇f
- **Divergence of a vector function *f*:** $\nabla . f$
- **Curl of a vector function *f*:** $\nabla \times f$

13.10 *REFERENCES*

- Spiegel, R. Murray (1959), *Vector Analysis*, Schaum's Outline Series.
- N. Saran and S. N. Nigam, *Introduction to vector analysis*, Pothishala Pvt. Ltd. Allahabad.
- Erwin. Kreyszig, "Advanced engineering mathematics, 10th eddition", 2009.
- A. R. Vasishtha, "Vector Calculus", 20th edition, Krishna publication, 2020.

2.11 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

2.12 *TERMINAL QUESTION*

Objective type question:

1. What will be the value of constant ***a***, if the vector

$\vec{V} = (x+3y)i + (y-2z)j + (x+az)k$ is solenoidal?

a. 0 **b.** 1

c. -2 **d.** 2

2. What will be the value of directional derivative of $\phi(x,y,z)=x^2yz+4xz^2$, at the point (1, -2, -1) in the direction of the vector $2i-j-2k$.

a. 37/3 **b.** 1

c. -2 **d.** 2

3. Choose the correct value of $\nabla^2 r^n$, where $\vec{r}=xi+yj+zk$ and $r=\left|\vec{r}\right|$

a. $n(n+1)r^n$ **b.** $n(n+1)r^{n-1}$

c. $n(n+1)r^{n-2}$ **d.** none of these

4. Choose the correct value of $\nabla^2\left(\frac{1}{r}\right)$

a. $-2/r^3$ **b.** $2/r^3$

c. 0 **d.** none of these

5. Choose the correct value of $curl\,(\vec{r}\times\vec{a})$, where $\vec{r}=xi+yj+zk$ and $\vec{a}$ is constant vector.

a. $-\vec{a}$ **b.** $-2\vec{a}$

b. $-3\vec{a}$ **d.** none of these

6. Choose the correct value of $div\,\hat{r}$ is

a. $\frac{2}{r}$ **b.** $\frac{1}{r}$

c. 0 **d.** none of these

Find True and False Statement.

1. The vector $\vec{r}=xi+yj+zk$ is solenoidal

2. $div\vec{V}=0$, if $\vec{V}$ is a constant vector.

3. $\vec{F}=2xyzi+y^2zj-2yz^2k$, is irrotational vector.

4. $curl\,grad\,\phi=0$, if ϕ is a differential scalar function.

5. $div\, grad\, \phi = \nabla^2\phi$, if ϕ is a differential scalar function.

6. $\nabla.\left(A\times B\right) = A.\left(\nabla\times B\right) - B.\left(\nabla\times A\right)$

7. Function which satisfies the Laplace's equation is called harmonic function.

Short answer type question:

1. If $f = x^2y + 2xy + z^2$, then verify that $curl\, grad\, f = 0$

2. Show that $curl\,(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$ further prove that,

$\nabla\psi\times\nabla\phi = -curl\left(\phi\nabla\psi\right)$

3. If $\vec{a}$ is a constant vector then show that $curl\,(\vec{a}.\vec{r})\vec{a} = 0$

4. For the constant vector $\vec{a}$ prove the followings:

(i) $\nabla\left(\vec{a}.\vec{u}\right) = \left(\vec{a}.\nabla\right)\vec{u} + \vec{a}\times curl\,\vec{u}$

(ii) $\nabla.\left(\vec{a}\times\vec{u}\right) = -\vec{a}.curl\,\vec{u}$

(iii) $\nabla\times\left(\vec{a}\times\vec{u}\right) = \vec{a}\,div\,\vec{u} - \left(\vec{a}.\nabla\right)\vec{u}$

5. If $\vec{a}$ is a constant unit vector then show that

$\vec{a}.\left\{\nabla\left(\vec{v}.\vec{a}\right) - \nabla\times\left(\vec{v}\times\vec{a}\right)\right\} = div\,\vec{v}$

6. If $\vec{a}$ is a constantvector then show that $curl\,\vec{a}\,\phi(r) = \frac{1}{r}\phi'(r)\,\vec{r}\times\vec{a}$

Long answer type question:

1. If $\vec{a}$ is a constantvector then show that,

$curl\, r\left[r^n(\vec{a}\times\vec{r})\right] = (n+2)r^n\,\vec{a} - nr^{n-2}\left(\vec{r}.\vec{a}\right)\vec{r}$

2. If $\nabla^2 f(r) = 0$ show that $f(r) = c_1\log r + c_2$, where $r^2 = x^2 + y^2 + z^2$, where c_1, c_2 are arbitrary constant.

3. Show that $\frac{1}{2}\nabla\vec{a}^2 = \left(\vec{a}.\nabla\right)\vec{a} + \vec{a}\times curl\,\vec{a}$

4. Show that $\nabla^2\left[\nabla.\left(\frac{\vec{r}}{r^2}\right)\right]=2r^{-4}$

5. If $\vec{a}$ is a constant vector then find the value of $div\left\{\vec{a}\times\left(\vec{r}\times\vec{a}\right)\right\}$

6. Find $grad\left(div\,\vec{u}\right)$, where $\vec{u}=\left(1/r\right)\vec{r}$

2.13 *ANSWERS*

Answer of self cheque questions:

1. $\left(x^2+y^2\right)dx-2xydy$ **2.** $ye^{xy}i+j+\sin y\,k$

3. $\vec{a}$

4. 0 **5.** 3

6. -3

7. 0 **8.** 0

9. $div\vec{V}=0$

10. $curl\,\vec{F}=0$ **11.** 0

Answer of objective questions:

1. c **2.** a

3. c

4. c **5.** b

6. a

Answer of true and false questions:

1. F **2.** T

3. F

4. T **5.** T

6. F

7. T

Answer of long answer type questions:

Answers: **5.** $2a^2$ **6.** $-\frac{2}{r^3}\vec{r}$

UNIT-3: GREEN’S, GAUSS’S AND STOKE’S THEOREMS

CONTENTS:

3.1 *INTRODUCTION*

Green’s theorem is mainly used for the integration of the line combined with a curved plane. This theorem shows the relationship between a line integral and a surface integral. It is related to many theorems such as Gauss theorem, Stokes theorem. Green’s theorem is used to integrate the derivatives in a particular plane. If a line integral is given, it is converted into a surface integral or the double integral or vice versa using this theorem. In this unit, we will going to learn what is Green’s theorem, its statement, formula, applications and examples in detail.

This unit finally begins to deliver on why we introduced div grad and curl. Two theorems, both of them over two hundred years old, are explained: Gauss’ Theorem enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa. Why would we want to do that? Computational efficiency and/or numerical accuracy! Stokes’ Law enables an integral taken around a closed curve to be replaced by one taken over any surface bounded by that curve.

3.2 *OBJECTIVES*

After reading this unit learners will be able to

- Memorized about the introduction of vector functions, line integrals, surface integrals and volume integrals.
- Analyze about the Green’s theorem and applications of Green’s theorem.
- Analyze about the Gauss divergence theorem and applications of this theorem.
- Analyze about the Stoke’s theorem and applications of Stokes’s theorem.

3.3 *INTRODUCTION OF VECTOR FUNCTIONS*

We shall usually define integration as the reverse process of differentiation. Let two vector functions $f(t)$ and $\mathrm{F}(t)$ of the scalar function t such that

$$\frac{d}{dt}\mathrm{F}(t) = f(t)$$

Here, $\mathrm{F}(t)$ is called the indefinite integral of $f(t)$ with respect to t and symbolically we denote

$$\int f(t)dt = \mathrm{F}(t) \qquad \ldots\ldots. (1)$$

The function $f(t)$ which to be integrated is called integrand. If c is arbitrary constant vector independent from t, then

$$\frac{d}{dt}\{\mathrm{F}(t)+c\} = f(t)$$

Which will equivalent to, $\int f(t)dt = \mathrm{F}(t)+\mathrm{c}$ ……. (2)

From above equation (2) it is obvious that the integral $\mathrm{F}(t)$ of $f(t)$ is indefinite to the extent of an additive arbitrary constant c. Therefore $\mathrm{F}(t)$ is called the indefinite integral of $f(t)$.

If $\frac{d}{dt}\mathrm{F}(t) = f(t)$ for all values of t in the interval $[a,b]$, then the definite integral between the limits $t=a$ and $t=b$ can be defined as,

$$\int_a^b f(t)dt = \int_a^b \left\{\frac{d}{dt}\mathrm{F}(t)\right\}dt = \left[F(t)+c\right]_a^b = F(b)-F(a)$$

Some important rule of integration (without proof)

1. If $f(t) = f_1(t)i + f_2(t)\,\mathrm{j} + f_3(t)\,\mathrm{k}$, then

$$\int f(t)dt = i\int f_1(t)dt + j\int f_2(t)dt + k\int f_3(t)dt$$

2. We have $\frac{d}{dt}(r.s) = \frac{dr}{dt}.s + r.\frac{ds}{dt}$ therefore,

$$\int\left(\frac{dr}{dt}.s + r.\frac{ds}{dt}\right)dt = r.s + c$$

3. We have $\frac{d}{dt}\left(\frac{dr}{dt}\right)^2 = 2\frac{dr}{dt}.\frac{d^2r}{dt^2}$ therefore,

$$\int\left(2\frac{dr}{dt}.\frac{d^2r}{dt^2}\right)dt = \left(\frac{dr}{dt}\right)^2 + c$$

4. $\left(\frac{dr}{dt}\right)^2 = \frac{dr}{dt}.\frac{dr}{dt}$

5. We have $\frac{d}{dt}\left(r\times\frac{dr}{dt}\right) = \frac{dr}{dt}\times\frac{dr}{dt} + r\times\frac{d^2r}{dt^2} = r\times\frac{d^2r}{dt^2}$ therefore,

$$\int\left(r\times\frac{d^2r}{dt^2}\right)dt = r\times\frac{dr}{dt} + c$$

6. We have $\frac{d}{dt}(a\times r)=\frac{da}{dt}\times r+a\times\frac{dr}{dt}=a\times\frac{dr}{dt}$ therefore,

$$\int\left(a\times\frac{dr}{dt}\right)dt=a\times r+c$$

7. We have $\frac{d}{dt}\left(\hat{r}\right)=\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)=\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}$ therefore,

$$\int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)dt=\hat{r}+c$$

8. If c is constant scalar and r a vector function of the scalar t then,

$$\int cr\,dt=c\int r\,dt$$

9. If r and s are two vector function of the scalar t then,

$$\int(r+s)\,dt=\int r\,dt+\int s\,dt$$

Solved Example

Example 1: If $f(t)=\left(t-t^2\right)i+2t^3j-3k$ then find,

(i) $\int f(t)dt$ **(*ii*)** $\int_1^2 f(t)dt$

Answer (i):

$$\int f(t)dt=\int\left\{\left(t-t^2\right)i+2t^3j-3k\right\}dt=i\int\left(t-t^2\right)dt+j\int 2t^3dt+k\int -3dt$$

$$=\left(\frac{t^2}{2}-\frac{t^3}{3}\right)i+\frac{t^4}{2}j-3tk+c$$

(ii): $$\int_1^2 f(t)dt=\int_1^2\left\{\left(t-t^2\right)i+2t^3j-3k\right\}dt=i\int_1^2\left(t-t^2\right)dt+2j\int_1^2 t^3dt-3k\int_1^2 dt$$

$$=\left[\left(\frac{t^2}{2}-\frac{t^3}{3}\right)i+\frac{t^4}{2}j-3tk\right]_1^2+c$$

$$=\left[\left(\frac{t^2}{2}-\frac{t^3}{3}\right)i+\frac{t^4}{2}j-3tk\right]_1^2+c$$

$$=\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_1^2 i+\left(\frac{t^4}{2}\right)_1^2 j-\left(3t\right)_1^2 k=-\frac{5}{6}i+\frac{15}{2}j-3k$$

Example 2: Evaluate the value of r which satisfying the equation $\frac{d^2r}{dt^2}=a$, where a is a constant vector. It is given that at a time $t=0, r=0$ and $\frac{dr}{dt}=u$.

Answer: Given differential equation on r is $\frac{d^2r}{dt^2}=a$.

Now, integrating both side the equation with respect to t we get,

$\frac{dr}{dt}=ta+b$, here b is arbitrary constant vector.

Since it is given that at a time $t=0, r=0$ and $\frac{dr}{dt}=u$.

Then, $u=0a+b$ *i.e.*, $b=u$

$$\Rightarrow \frac{dr}{dt}=ta+u$$

Again, integrating both side with respect to t we get.

$$r=\frac{1}{2}t^2a+tu+c$$

at time $t=0, \quad r=0$

$$\Rightarrow 0=0+0+c \qquad \text{or} \quad c=0$$

So, $r=\frac{1}{2}t^2a+tu$

Example 3: If $r(t)=5t^2i+tj-t^3k$ then show that

$$\int_1^2\left(r\times\frac{d^2r}{dt^2}\right)dt=-14i+75j-15k$$

Answer: As we know that, $\int\left(r\times\frac{d^2r}{dt^2}\right)dt = r\times\frac{dr}{dt}+c$

So, $\int_1^2\left(r\times\frac{d^2r}{dt^2}\right)dt = \left[r\times\frac{dr}{dt}\right]_1^2$

First, we evaluate $r\times\frac{dr}{dt}$.

Since, $r(t)=5t^2i+tj-t^3k$ then $\frac{dr}{dt}=\frac{d}{dt}\left(5t^2i+tj-t^3k\right)=10ti+j-3t^2k$

So, $r\times\frac{dr}{dt}=\left(5t^2i+tj-t^3k\right)\times\left(10ti+j-3t^2k\right)=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$

$$=\left(-3t^3+t^3\right)i-\left(-15t^4+10t^4\right)j+\left(5t^2-10t^2\right)k=-2t^3i+5t^4j-5t^2k$$

Now,

$$\left[r\times\frac{dr}{dt}\right]_1^2=\left[-2t^3i+5t^4j-5t^2k\right]_1^2=\left(-2t^3i+5t^4j-5t^2k\right)_{at\,t=2}-\left(-2t^3i+5t^4j-5t^2k\right)_{at\,t=1}$$

$$=\left(-16i+80j-20k\right)-\left(-2i+5j-5k\right)=-14i+75j-15k$$

Example 4: Prove that $\int_2^3\left(r.\frac{dr}{dt}\right)dt=10$, where

$$r(t)=\begin{cases}2i-j+2k, \; if\;\; t=2\\ 4i-2j+3k, \; if\;\; t=3\end{cases}$$

Answer: We know that, $\int\left(r.\frac{dr}{dt}\right)dt=\frac{r^2}{2}+c$

So, $\int_2^3\left(r.\frac{dr}{dt}\right)dt=\left[\frac{r^2}{2}\right]_2^3$

When $t=2,\; r(t)=2i-j+2k$ then

$r^2=r.r=\left(2i-j+2k\right).\left(2i-j+2k\right)=4+1+4=9$

Similarly, $t=3,\ r(t)=4i-2j+3k$ then

$$r^2=r.r=(4i-2j+3k).(4i-2j+3k)=16+4+9=29$$

Thus, $\int_2^3\left(r.\frac{dr}{dt}\right)dt=\frac{1}{2}(29-9)=10$

Example 5: At a time $t\geq 0$, the acceleration of a particle is given by,

$$a=\frac{dv}{dt}=12\cos 2t\,i-8\sin 2t\ j+16t\,k$$

If at a time $t=0$, the velocity (v) and displacement (r) are zero, then find v and r at any time.

Answer: We have given that acceleration at a time is $a=\frac{dv}{dt}=12\cos 2t\,i-8\sin 2t\ j+16t\,k$

Integrating both side with respect to t we get,

$$v=\int(12\cos 2ti-8\sin 2tj+16tk)dt=\frac{12\sin 2ti}{2}+\frac{8\cos 2tj}{2}+\frac{16t^2k}{2}+c=6\sin 2ti+4\cos 2tj+8t^2k+c$$

At a time $t=0, v=0$, then $0=0+4j+0+c\Rightarrow c=-4j$

So, $v=6\sin 2ti+4\cos 2tj+8t^2k-4j=6\sin 2ti+(4\cos 2t-4)\,j+8t^2k$

Since, $v=\frac{dr}{dt}=6\sin 2ti+(4\cos 2t-4)\,j+8t^2k$

Integrating both side with respect to t we get,

$$r=\int\left(6\sin 2ti+(4\cos 2t-4)\,j+8t^2k\right)dt+d=-\frac{6\cos 2ti}{2}+\left(\frac{4\sin 2\mathrm{t}}{2}-4t\right)j+\frac{8t^3k}{3}+d$$

$$=-3\cos 2ti+(2\sin 2\mathrm{t}-4t)\,j+\frac{8}{3}t^3k+d$$

At a time $t=0, r=0$, then $0=-3+0+0+d\Rightarrow d=3i$

$$r=-3\cos 2ti+(2\sin 2\mathrm{t}-4t)\,j+\frac{8}{3}t^3k+3i$$

$$r=\left(3-3\cos 2t\right)i+\left(2\sin 2\text{t}-4t\right)j+\frac{8}{3}t^3k$$

3.4 *LINE INTEGRAL*

An integral which is to be evaluated along a curve is called a line integral.

Let $r(t)=x(t)i+y(t)j+z(t)k$, be a position vector of $\left(x,y,z\right)$ *i.e.*, $r=xi+yj+zk$, defines a piecewise smooth curve joining two points *A* and *B*. Let at the time $t=t_1$ point be at *A* and at the time $t=t_2$ point is at *B*. Suppose $F\left(x,y,z\right)=F_1i+F_2j+F_3k$ is a vector point function defined and continuous along *C*. If *s* denotes the arc length of the curve *C*, then $\frac{dr}{ds}=t$ is a unit vector along the tangent to the curve *C* at the point *r*. The component of the vector *F* along the tangent is $F.\frac{dr}{ds}$. The integral of $F.\frac{dr}{ds}$ along *C* from *A* to *B* written as

$$\int_A^B\left[F.\frac{dr}{ds}\right]ds=\int_A^B F.dr=\int_C F.dr$$

is an example of a line integral. It is called the tangent line integral of *F* along *C*.

Remarks:

1. If the equation of the curve *C* given in the parametric form *i.e.*, $x=x(t),y=y(t),z=z(t)$

thus, we may write $\int_C F.dr=\int_{t=t_1}^{t=t_2}\left[F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right]dt$.

2. If *r* is the position vector of a point in *C* and let *F* be the force acting on the particle. Then the work done (*W*) by *F* in this displacement is given by the line integral,

$W=\int_C F.dr$, here *r* be taken in the sense of displacement.

3.5 *SURFACE INTEGRAL*

Any integral which evaluated over a surface is called a surface integral.

Let *S* is a finite surface area. Suppose $f(x, y, z)$ is a single valued function defined over *S*. If we divide the area of *S* into *m* small areas like $\delta S_1, \delta S_2, \delta S_3, ..., \delta S_m$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).

We define $f(P_k) = f(x_k, y_k, z_k)$

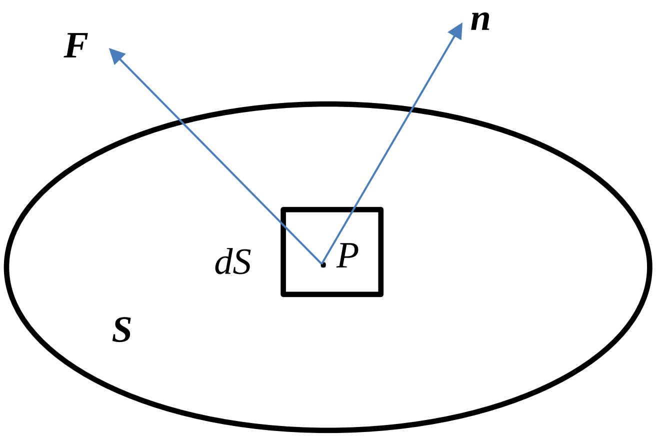
From the sum $\sum_{k=1}^{n} f(P_k)\delta S_k$

Taking limit of this sum as $n \to \infty$ in such a way that largest area δS_k approaches zero. If this limit is exist called surface integral of $f(x, y, z)$ over S and is denoted by

$$\iint_S f(x, y, z)dS$$

If the surface *S* is piecewise smooth and the function is continuous over *S*, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

Flux: Suppose a Piecewise smooth surface *S* and $F(x, y, z)$ is a vector function of defined position and continuous over ***S***. Let *P* be a point on the surface ***S*** and let n be the unit vector at *P* in the direction of outward drawn normal to the surface ***S*** at *P*. Then $F.n$ is the normal component of *F* at *P*. The integral of $F.n$ over ***S*** is,



$\iint_S F.n\ dS$, is called the flux of F over **S.**

Let us associate with the differential of surface area ***dS*** a vector ***dS*** (called vector area) whose magnitude is ***dS*** and direction is that of ***n***. Then ***dS*** **= n** *dS*. Therefore, we can write,

$$\iint_S F.n\ dS = \iint_S F.dS$$

Let we consider at the point P the outward normal to the surface S makes the angle α, β, γ with the positive direction of x, y, z. If the direction cosines of the outward drawn normal are l, m, n, then

$$\text{l} = \cos\alpha, \text{m} = \cos\beta, \text{n} = \cos\gamma$$

Also $n = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$

let $F(x, y, z) = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma = F_1 l + F_2 m + F_3 n$

Therefore, we can write

$$\iint_S F.n\ dS = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma\right) dS$$

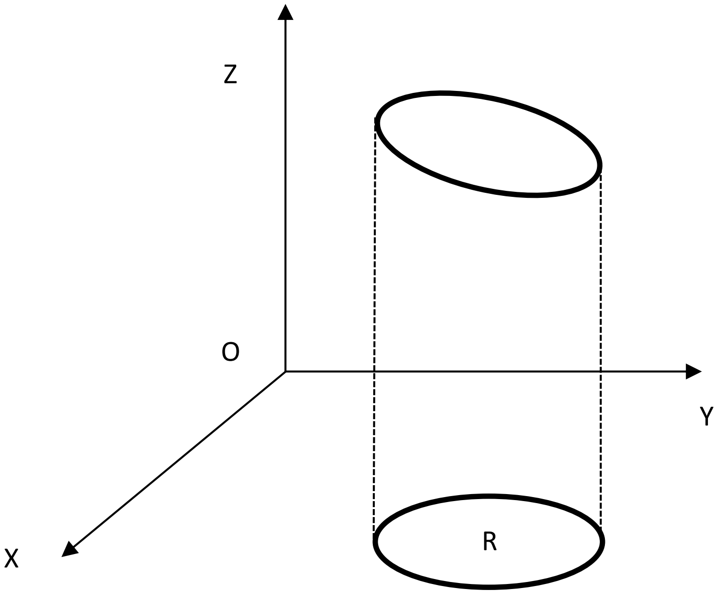
If we define $\iint_S F_1 \cos\alpha dS = \iint_S F_1 dydz$, $\iint_S F_2 \cos\beta dS = \iint_S F_2 dzdx$, $\iint_S F_3 \cos\gamma dS = \iint_S F_3 dxdy$

then,

$$\iint_S F.n\ dS = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy\right) dS$$

Note 1. Other surface integrals are $\iint_S f\ n\ dS, \iint_S F \times dS$

Note 2. Let we consider the surface *S* in such a way that any line perpendicular to the *xy*- plane meets *S* in no more than one point. Then the equation of surface S can be written in the form $z = h(x, y)$



Let *R* be the orthogonal projection of *S* on the *xy*-plane. If γ is the acute angle the undirected normal ***n*** at *P*(*x*, *y*, *z*) to the surface ***S*** makes with *z*-axis, then it can be shown that

$$\cos\gamma\, dS = dx\, dy$$

where *dS* is the small element of area of surface *S* at the point *P*.

Therefore $dS = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|n.k|}$, where k is unit vector along *z*-axis.

Hence $\iint_S F.n\,dS = \iint_R F.\text{n}\frac{dx\,dy}{|n.\text{k}|}$

So, a double integral integrated over R can be used to evaluate the surface integral on S.

3.6 *VOLUME INTEGRAL*

The volume V that is enclosed by surface S. Let $f(x,y,z)$ be a single-valued positional function defined over V. Split the volume V into n volume components $\delta V_1, \delta V_2, ..., \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define

$$f(P_k) = (x_k, y_k, z_k)$$

From the sum $\sum_{k=1}^{n} f(P_k)\delta V_k$

Now taking the limit of this summation as $n \to \infty$ in a manner that the largest of the volumes δV_k approaches zero. This limit exists, is called the volume integral of $f(x,y,z)$ over V and is denoted by

$$\iiint_V f(x,y,z)\,dV$$

If the function $f(x,y,z)$ is continuous and surface is piecewise smooth over *V*, then the above limit exists *i.e.*, is independent of the choice of sub-division and points P_k. If V is the volume of small cuboids, then $dV = dx\,dy\,dz$, so, the will becomes

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

If F(x,y,z) is a vector function, $\iiint_V F\,dV$ is the example of volume integral.

Solved Examples

Example 6: Find $\int_C F.dr$, where $F = x^2 i + y^3 j$ and curve *C* represents the parabola's arc $y = x^2$ in the *x*-*y* plane from (0,0) to (1,1).

Answer: Method 1. Since we have given curve *C* is parabola. First, we have to convert the equation of parabola in parametric form by putting $x = t$ and $y = t^2$.

So, $F = t^2 i + t^6 j$ and we know that $r(t) = xi + yj = ti + t^2 j$

Then $\frac{dr}{dt} = i + 2tj$

Now, $\left(F.\frac{dr}{dt}\right)dt = \left(t^2 i + t^6 j\right).\left(i + 2tj\right)dt = \left(t^2 + 2t^7\right)dt$

At the point (0,0), $t = x = 0$. At the point (1,1), $t = 1$

$$\therefore \int_C \left(F.\frac{dr}{dt}\right)dt = \int_0^1 \left(t^2 + 2t^7\right)dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Method 2. As we know that $r = xi + yj$

$\Rightarrow dr = dxi + dyj$

Thus $F.dr = \left(x^2 i + y^3 j\right).\left(dx\, i + dy\, j\right) = x^2 dx + y^3 dy$

$$\therefore \int_C F.dr = \int_C x^2 dx + y^3 dy$$

Now along the curve *C*, $y = x^2$. Therefore $dy = 2xdx$

$$\therefore \int_C F.dr = \int_0^1 \left[x^2 dx + x^6 (2x) dx\right] = \int_0^1 \left(2x^7 + x^2\right)dx$$

$$= \left[\frac{2x^8}{8} + \frac{x^3}{3}\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Example 7: Find $\int_C F.dr$, where $F = \left(2x + yz\right)i + xz\, j + (xy + 2z)\text{k}$ along the curve $x^2 + y^2 = 1, z = 1$ in the positive direction from (0,1,1) to (1,0,1).

Answer: Let the curve is denoted by *C* and the points *A* and *B* are the points (0,1,1) to (1,0,1) respectively.

As we know the position vector of a point is, $r = xi + yj + zk$

$\Rightarrow dr = dx\, i + dy\, j + dz\, k$

Thus

$$F.dr=\left(\left(2x+yz\right)i+xz\,j+(xy+2z)\mathrm{k}\right).\left(dx\,i+dy\,j+dz\,k\right)=\left(2x+yz\right)dx+xzdy+(xy+2z)dz$$

$$\therefore \int_C F.dr=\int_C\left(2x+yz\right)dx+xzdy+(xy+2z)dz$$

Along the curve from A to B, x varies from 0 to 1, y varies from 1 to 0 and z remains constant i.e., $dz=0$

$$\therefore \int_C F.dr=\int_0^1\left(2x+y\right)dx+\int_1^0 xdy+0$$

$$\left[\int_a^0 f(x)\,dx=-\int_0^a f(x)dx\right]$$

$$=\int_0^1\left(2x+\sqrt{1-x^2}\right)dx-\int_0^1\sqrt{1-y^2}dy$$

$$=\int_0^1 2x+\int_0^1\sqrt{1-x^2}dx-\int_0^1\sqrt{1-y^2}dy$$ (Integration does not depend upon variable)

$$=\int_0^1 2xdx=\left[x^2\right]_0^1=1$$

Example 8: If C is the line segment of the line $y=2x$ in the xy-plane from (-1, -2) to (1, 2), then find

Answer: Since we have given curve C is line. First, we convert the equation of line in parametric form by putting $x=t$ and $y=t$.

So, $r(t)=xi+yj=ti+2tj$

Then $\dfrac{dr}{dt}=i+2\,j$

As we know that, $\dfrac{dr}{dt}=\dfrac{dr}{ds}\dfrac{ds}{dt}$

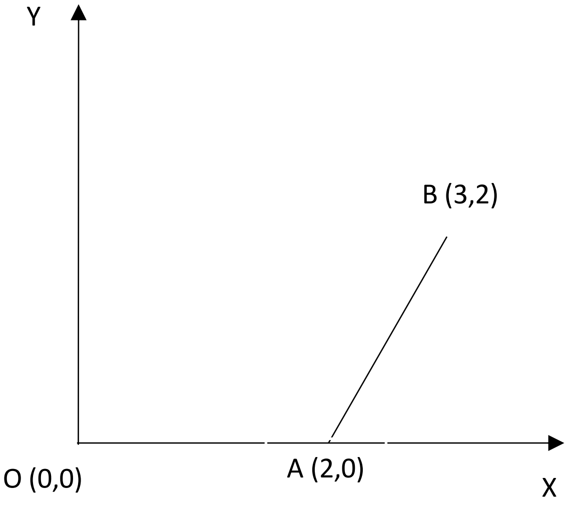
$\therefore \left|\dfrac{dr}{dt}\right|=\left|\dfrac{dr}{ds}\right|\dfrac{ds}{dt}=\dfrac{ds}{dt}$ (Because, $\dfrac{dr}{ds}$ is the unit vector)

$$\therefore \frac{ds}{dt} = \left|\frac{dr}{dt}\right| = |i + 2j| = \sqrt{5}$$

$$\int_C xy^3 ds = \int_C xy^3 \frac{ds}{dt} dt = \int_{-1}^{1} t(2t)^3 \sqrt{5} dt = 8\sqrt{5}\int_{-1}^{1} t^4 dt = \frac{16}{\sqrt{5}}$$

Example 9: Find the value of $\int_C F.dr$, where C is the *xy*-plane curve formed by the straight lines from (0, 0) to (2, 0) and then to (3, 2), where $F = (2x + y)i + (3y - x)j$.

Answer: The figure of the path of curve C in the *xy*-plane is shown below. It consists straight lines *OA* and *AB*.



We have, $\int_C F.dr = \int_C [(2x + y)i + (3y - x)j].(dxi + dyj)$

$$= \int_C (2x + y)dx + (3y - x)dy$$

Along the straight-line *OA*, $y = 0, dy = 0$ and x varies from 0 to 2 and equation of the straight line *AB* is,

$$y - 0 = \frac{2-0}{3-2}(x - 2) \; i.e., \; y = 2x - 4$$

Along *AB*, $y = 2x - 4, dy = 2dx$ and x varies from 2 to 3.

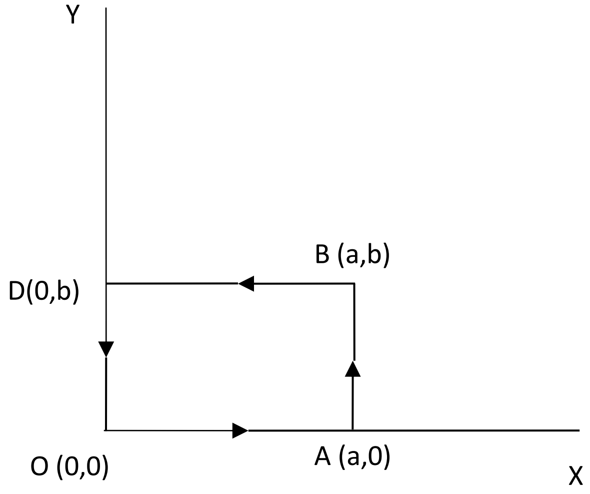
$$\int_C F.dr = \int_0^2 \left[(2x+0)dx+0\right] + \int_2^3 \left[(2x+2x-4)dx + 2\left(6x-12-x\right)dx\right]$$

$$= \left[x^2\right]_0^2 + \int_2^3 \left(14x-28\right)dx = 4 + 14\int_2^3 \left(x-2\right)dx$$

$$= 4 + 14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4+7 = 11$$

Example 10: If C is the rectangle bounded by $y=0, x=a, y=b, \mathrm{x}=0$ in the *xy*-plane then evaluate $\int_C F.dr$, where $F = \left(x^2+y^2\right)i - 2xyj$.

Answer: The path of the integration C has been shown in figure which consists the straight lines *OA*, *AB*, *BD* and *DO*.



Now we have,

$$\int_C F.dr = \int_C \left[\left(x^2+y^2\right)i - 2xyj\right].\left(dxi + dyj\right)$$

$$= \int_C \left(x^2+y^2\right)dx + 2xydy$$

In the line *OA*, $y=0, dy=0$ and x varies from 0 to *a*.

In the line *AB*, $x=a, dx=0$ and y varies from 0 to *b*.

In the line BD, $y = b, dy = 0$ and x varies from a to 0.

In the line DO, $x = 0, dx = 0$ and y varies from b to 0.

$$\int_C F.dr = \int_0^a x^2 dx - \int_0^b 2ay dy + \int_a^0 \left(x^2 + b^2\right) dx + \int_b^0 0 dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2 x\right]_a^0 + 0 = -2ab^2$$

3.7 *GREEN'S THEOREM*

Let R be a closed bounded region in the x-y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. Then $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C \left(Mdx + Ndy\right)$, the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

Greens theorem in the plane (in vector notation):

We know that the position vector of a point is $r = xi + yj$ so that $dr = dxi + dyj$.

Let $F = Mi + Nj$. Then $Mdx + Ndy = \left(Mi + Nj\right).\left(dxi + dyj\right) = F.dr$

Since $curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z} i + \frac{\partial M}{\partial z} j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$

$$\left(\nabla \times F\right).k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Thus, the Green's theorem in plane can be written as

$$\iint_R \left(\nabla \times F\right).k dR = \oint_C F.dr \quad \ldots (1)$$

Where $dR = dxdy$ and k is the perpendicular unit vector to the *xy*-plane.

If s is the arc length of C and t denotes the unit tangent vector to C, then $dr = \frac{dr}{ds}ds = tds$. So, the equation (1) can also be rewritten as,

$$\iint_R (\nabla \times F).kdR = \oint_C F.\mathrm{t}\, ds$$

Solved Examples

Example 11: If *C* is the closed curve of the region bounded by the straight line $y = x$ and the parabola $y = x^2$ then verify the Green's theorem in the plane $\oint_C \left(xy + y^2\right)\mathrm{dx} + \mathrm{x}^2\mathrm{dy}$.

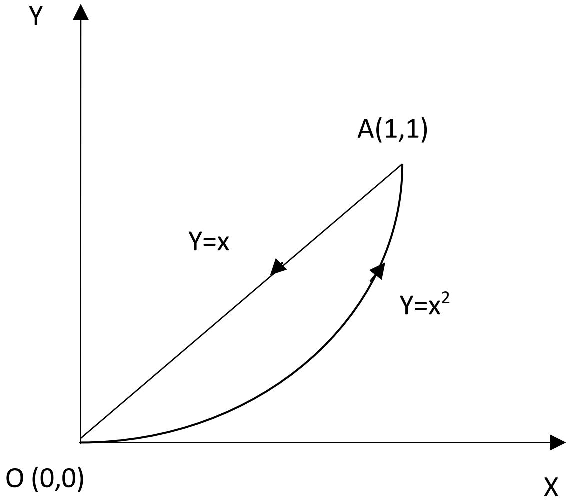
Answer: Since by the Green's function in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \oint_C \left(Mdx + Ndy\right) \quad \ldots (1)$$

So, after comparing equation (1) with $\oint_C \left(xy + y^2\right)\mathrm{dx} + \mathrm{x}^2\mathrm{dy}$ we get,

$$M = xy + y^2,\ N = \mathrm{x}^2$$

The curves $y = x$ and $y = x^2$ intersect at the point (0,0) and (1,1) and positive direction in traversing *C* is as show in figure.



Thus, $\iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint\limits_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}\left(xy + y^2\right)\right]dxdy$

$$= \iint\limits_R \left(2x - x - 2y\right)dxdy = \iint\limits_R \left(x - 2y\right)dxdy$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{x} \left(x - 2y\right)dydx = \int_0^1 \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$\int_0^1 \left[xy - y^2\right]_{y=x^2}^{x} dx = \int_0^1 \left[x^2 - x^2 - x^3 + x^4\right]dx$$

$$= \int_0^1 \left(x^4 - x^3\right)dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Now we evaluate the line integral along *C*. Along the curve $y = x^2, dy = 2xdx$. Thus, along the curve $y = x^2$, the line integral equals

$$\int_0^1 \left[\left\{\left(x\right)\left(x^2\right) + x^4\right\}dx + x^2\left(2x\right)dx\right] = \int_0^1 \left(3x^3 + x^4\right)dx = \frac{19}{20}$$

Along $y = x, dy = dx$. Therefore, along the curve $y = x$, the line integral equals

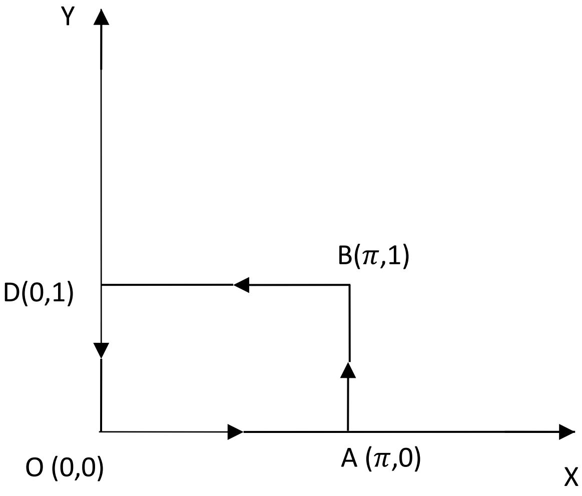
$$\int_1^0 \left[\left\{(x)(x)+x^2\right\}dx + x^2 dx\right] = \int_1^0 3x^4 dx = -1$$

Hence the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

Thus, the theorem is verified.

Example 12: Using the Green's theorem find the value of $\oint_C \left(x^2 - \cosh y\right)dx + \left(y + \sin x\right)dy$, where C is the rectangle having vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Answer: As we know by Green's theorem, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \oint_C \left(Mdx + Ndy\right)$ …. (1)



Comparing the equation (1) with the given equation $\oint_C \left(x^2 - \cosh y\right)dx + \left(y + \sin x\right)dy$ we get,

$$M = x^2 - \cosh y,\ N = y + \sin x$$

$$\therefore \frac{\partial M}{\partial y} = -\sinh y,\ \frac{\partial N}{\partial x} = \cos x$$

Thus, the given line integral is equal to

$$\iint_R (\cos x+\sinh y)dxdy=\int_{x=0}^{x=\pi}\int_{y=0}^{1}(\cos x+\sinh y)dydx$$

$$=\int_{x=0}^{\pi}\left[y\cos x+\cosh y\right]_{y=0}^{y=1}dx=\int_{x=0}^{\pi}\left[\cos x+\cosh 1-1\right]dx$$

$$=\left[\sin x+x\cosh 1-\mathrm{x}\right]_0^{\pi}=\cosh 1-1$$

Example 13: Prove that the area bounded by a simple closed curve *C* is given by $\frac{1}{2}\oint_C(xdy-ydx)$ also find the area generated by the ellipse $x=a\cos\theta, y=b\sin\theta$.

Answer: As we know for the plane region bounded by closed curve *C*, Green's theorem is

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\oint_C(Mdx+Ndy)$$

putting $M=-y,\ N=x$ we get,

$$\oint_C(xdy-ydx)=\iint_R\left[\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right]dxdy$$

$=2\iint_R dxdy=2A$, where $A=\frac{1}{2}\oint_C(xdy-ydx)$ is the area bounded by *C*.

The area of the ellipse $=\frac{1}{2}\oint_C(xdy-ydx)=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\left(a\cos\theta\frac{dy}{d\theta}-b\sin\theta\frac{dx}{d\theta}\right)d\theta$

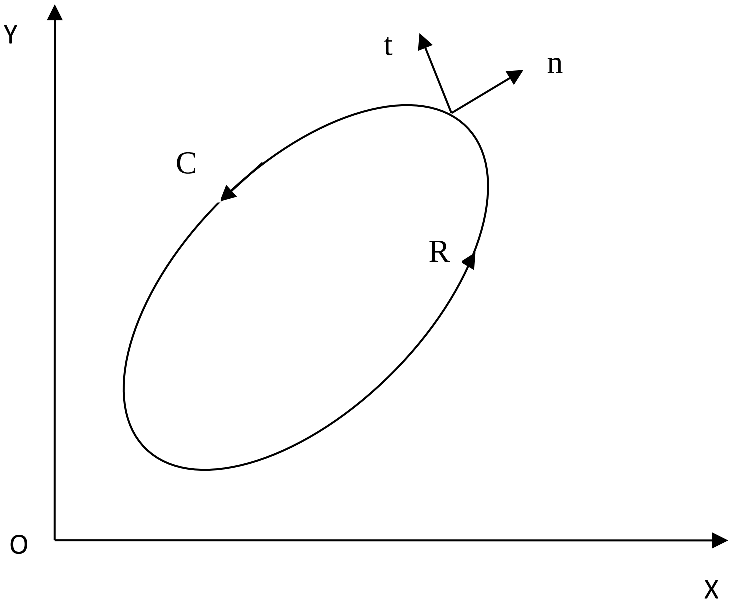
$$=\frac{1}{2}\oint_C(xdy-ydx)=\frac{1}{2}\int_0^{2\pi}\left(ab\cos^2\theta+ab\sin^2\theta\right)d\theta=\frac{ab}{2}\int_0^{2\pi}d\theta=\pi ab.$$

Example 14: Prove that the Green's function may be written as $\iint_R divA\,dxdy=\oint_C A.n\,ds$, where $A=Ni-Mj$, n is the outward unit normal vector to *C* and s denotes the arc length of *C*.

Answer: Since we have, $A = Ni - Mj$

$$\therefore \quad divA = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$\therefore \quad \iint_R divA\, dxdy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C \left(Mdx + Ndy \right)$, by Green's theorem



Since we can write, $Mdx + Ndy = \left(Mi + Nj \right).\left(dxi + dyj \right)$

$$= \left(Mi + Nj \right).\mathrm{dr} = \left\{ \left(Mi + Nj \right).\frac{dr}{ds} \right\} ds$$

Let *t* is unit tangent vector to *C*, then $t = \frac{dr}{ds}$ and we also know that the unit vector *k* is perpendicular to *xy*-plane. Then, $t = k \times n$

So, $Mdx + Ndy = \left[\left(Mi + Nj \right).t \right] ds = \left[\left(Mi + Nj \right).\left(k \times n \right) \right] ds$

$$=\left[\left(Mi+Nj\right)\times k\right].nds=\left(Mi\times k+Nj\times k\right).nds=\left(Ni-Mj\right).nds=A.nds$$

Hence proved.

Note: Putting $A=\nabla\phi$ in previously discussed result, we get

$$\iint_V div\left(\nabla\phi\right)dx\,dy=\oint_C\left(\nabla\phi\right).nds$$

Or $\iint_V \nabla^2\phi dx\,dy=\oint_C \frac{\partial\phi}{\partial n}ds,$ since $\nabla\phi=\frac{\partial\phi}{\partial n}n$

3.8 *THE DIVERGENCE THEOREM OF GAUSS*

Assume that V is the volume enclosed by the closed, piecewise smooth surface S. Assume $F(x,y,z)$ is a continuous vector function of position with a continuous first partial derivative in V. Then

$$\iiint_V \nabla.FdV=\iint_S F.n\,ds$$

where n is the unit normal vector drawn outward from S.

Here, $F.n$ is normal component of vector F, consequently, the divergence theorem may be explained as follows:

The integral of the divergence of a vector F taken over the surface's enclosed volume is the same as the surface integral of the normal component of a vector F taken over a closed surface.

Divergence theorem in cartesian form:

Let $F=F_1i+F_2j+F_3k$ is vector function.

Thus, $\nabla.F=divF=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$.

If $\cos\alpha,\cos\beta,\cos\gamma$ are the direction cosines of the outward drawn unit normal n where α,β,γ are the angles which are taking with the positive direction of $\text{x, y, z}-$ axes.

$n = \cos\alpha i + \cos\beta j + \cos\gamma k$

$\therefore F.n = (F_1 i + F_2 j + F_3 k).(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$

$= F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$

Thus the divergence theorem can be rewritten as,

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dS$$

$$= \iint (F_1 dydz + F_2 dxdx + F_3 dxdy)$$

Remarks: *The divergence theorem is significant because it shows how a surface may be written as a volume integral and vice versa.*

Note: If a region V is surrounded by two closed surfaces S_1 and S_2, one of which falls within the other, the divergence theorem applies to that region.

Some important deduction from divergence theorem:

1. Green's Theorem: Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface, then

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \iint_S \left(\phi\nabla\psi - \psi\nabla\phi\right).nds$$

2. **Harmonic Function:** If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, then ϕ is called harmonic function. If ϕ and ψ are both harmonic functions, then $\nabla^2\phi = 0$, $\nabla^2\psi = 0$.

Since from Green's second identity, we get $\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n} \right) dS = 0$.

Note 1: $\iiint_V \nabla\phi\, dV = \iint_S \phi\, n\, dS$

2: $\iiint_V \nabla \times B\, dV = \iint_S n \times B\, dS$

Solved Examples

Example 15: For any closed surface *S*, prove that $\iint_S curlF\,.\,n\,ds = 0$

Solution: By divergence theorem, we have

$\iint_S curlF\,.\,n\,ds = \iiint_V (div\,curl\,F)\,dV$, where V is the volume enclosed by S

$= 0,$ since $div\,curl\,F = 0$.

Example 16: Evaluate $\iint_S r.\,n\,ds,$ where S is a closed surface.

Solution: $\iint_S r.\,n\,ds = \iiint_V \nabla.\,r\,dV = \iiint_V 3\,dV$, Since $\nabla.\,r = div\,r = 3$

$= 3V,$ where V is the volume enclosed by S .

Example 17: If $F = ax\,i + by\,j + cz\,k, a, b, c$ are constant then prove that

$\iint_S F.\,n\,ds = \frac{4}{3}\pi(a+b+c)$, where S is the surface of unit sphere.

Solution: By the divergence theorem we have,

$\iint_S F.\,n\,ds = \iiint_V (\nabla.\,F)\,dV$, where V is the volume enclosed by $S.$

$$= \iiint_V [\nabla.(ax\,i + by\,j + cz\,k)]dV = \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right]dV$$

$$= \iiint_V (a+b+c)dV = (a+b+c)V = (a+b+c)\frac{4}{3}\pi$$

Since the volume V is enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi(1)^3$ *i.e.*, $\frac{4}{3}\pi.$

Example 18: If n is the unit outward drawn normal to any closed surface *S*, show that

$\iiint_V div\,n\,dV = S$.

Solution: Since we have the divergence theorem,

$$\iiint_V div\, n\, dV = \iint_S n.ndS = \iint_S dS = S$$

Example 19: Prove that, $\iiint_V \frac{dV}{r^2} = \iint_S \frac{r.n}{r^2} dS$.

Solution: $\iint_S \frac{r.n}{r^2} dS = \iint_S \frac{r}{r^2}.ndS = \iiint_V \nabla.\left(\frac{r}{r^2}\right) dV$, (by using divergence theorem)

So, $\nabla.\left(\frac{r}{r^2}\right) = \frac{1}{r^2}(\nabla.r) + r.\nabla\left(\frac{1}{r^2}\right)$

$$= \frac{3}{r^2} + r.\left(-\frac{2}{r^3}\nabla r\right) = \frac{3}{r^2} - \frac{2}{r^3}\left(r.\frac{r}{|r|}\right) = \frac{3}{r^2} - \frac{2}{r^4}r^2 = \frac{1}{r^2}$$

Example 20: Using divergence theorem, prove that the volume V of a region T bounded by a surface S is,

$$\iint_S x\, dydz = \iiint_V \frac{\partial}{\partial x}(x)\, dV = \iiint_V dV = V \qquad \ldots (1)$$

$$\iint_S y\, dzdx = \iiint_V \frac{\partial}{\partial y}(y)\, dV = \iiint_V dV = V \qquad \ldots (2)$$

$$\iint_S z\, dxdy = \iiint_V \frac{\partial}{\partial z}(z)\, dV = \iiint_V dV = V \qquad \ldots (3)$$

Adding equation (1), (2) and (3) we get the result

$$3V = \iint_S (x\, dydz + y\, dzdx + z\, dxdy)$$

$$V = \frac{1}{3}\iint_S (x\, dydz + y\, dzdx + z\, dxdy)$$

Example 21: If $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. Then verify the divergence theorem.

Solution: We have $div\, F = \nabla . F$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z$$

$\therefore$ Volume integral $= \iiint_V \nabla . F\, dV = \iiint_V 2(x+y+z)\, dV$

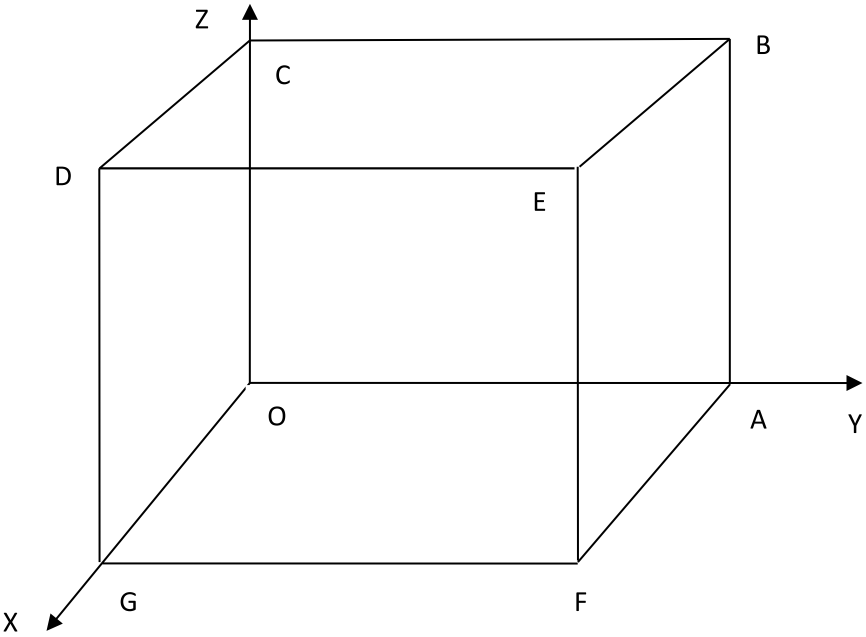
$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(x+y+z)dx\,dy\,dz = 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dydz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2} + ax + az\right]dydz = 2\int_{z=0}^{c}\left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right]dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= [a^2bc + ab^2c + abc^2] = abc(a+b+c)$$

Surface integral: We shall now calculate $\iint_S F.n\, dS$ over the six faces of the rectangular parallelepiped.



Over the face DEG, $\hat{n} = i, x = a$.

Therefore, $\iint_{DEFG} F.n\,dS$

$$= \int_{z=0}^{c}\int_{y=0}^{b}\left[\left(a^2 - yz\right)i + \left(y^2 - za\right)j + \left(z^2 - ay\right)k\right].idydz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}\left(a^2 - yz\right)dydz = \int_{z=0}^{c}\left[a^2y - z\frac{y^2}{2}\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c}\left[a^2b - \frac{zb^2}{2}\right]dz = \left[a^2bz - \frac{z^2}{4}b^2\right]_0^c = a^2bc - \frac{c^2b^2}{4}$$

Over the face ABCO, $\hat{n} = -i, x = 0$. Therefore,

$$\iint_{ABCO} F.n\,dS = \iint[(0 - yz)i + ... + ...].(-i)dydz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yzdydz = \int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_{y=0}^{b} dz = \int_{z=0}^{c}\left[\frac{b^2}{2}z\right]dz = \frac{b^2c^2}{4}$$

Over the face ABEF, $\hat{n} = j, y = b$. Therefore,

$$\iint_{ABEF} F.n\,dS = \int_{z=0}^{c}\int_{x=0}^{a}\left[\left(x^2 - bz\right)i + \left(b^2 - bz\right)j + \left(z^2 - bx\right)k\right].\,jdxdz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a}(b^2 - zx)dxdz = b^2ca - \frac{a^2c^2}{4}$$

Over the face OGDC, $\hat{n} = -j, y = 0$. Therefore,

$$\iint_{OGDC} F.n\,dS = \int_{z=0}^{c}\int_{x=0}^{a} zxdxdz = \frac{c^2a^2}{4}$$

Over the face BCDE, $\hat{n} = k, z = c$. Therefore,

$$\iint_{BCDE} F.n\,dS = \int_{y=0}^{b}\int_{x=0}^{a}(c^2 - xy)dx\,dy = c^2ab - \frac{a^2b^2}{4}$$

Over the face AFGO, $\hat{n} = -k, z = 0$. Therefore,

$$\iint_{AFGO} F.n\,dS = \int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \frac{a^2b^2}{4}$$

Now adding six surface integrals, we get

$$\iint_S F.n\,dS = \left(a^2bc - \frac{c^2b^2}{4} + \frac{c^2b^2}{4}\right) + \left(b^2ca - \frac{a^2c^2}{4} + \frac{a^2c^2}{4}\right) + \left(c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}\right)$$

$$= abc(a+b+c)$$

Hence the theorem is verified.

Example 22: Evaluate $\iint_S x^2 dydz + y^2 dzdx + 2z(xy - x - y)dxdy$

Where S is the surface integral of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

Proof: By using divergence theorem, we convert the given surface integral into volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z(xy - x - y)\}\right]dV$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}[2x + 2y + 2xy - 2x - 2y]dx\,dy\,dz = 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xy\,dx\,dy\,dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}y\right]_{x=0}^{1} dy\,dz = \int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz = \int_{z=0}^{1} 1/2\,dz = 1/2$$

3.9 STOKE’S THEOREM

Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $F(x, y, z)$ be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then

$$\oint_C F.dr = \iint_S (\nabla \times F).nds = \iint_S (curl\ F).ds$$

Where C is traversed in the position direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal n to S, has the surface on the left.

Note: $\oint_C F.dr = \oint_C \left(F.\frac{dr}{ds} \right) ds = \oint_C (F.t) ds,$ where t is unit tangent vector to C. Therefore $F.t$ is the component of $curl\ F$ in the direction of outward drawn normal vector n of S. Therefore in other words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector F taken around a simple closed curve C is equal to the surface integral of the normal component of the $curl$ of F taken over any surface S having C as its boundary.

Cartesian equivalent of stokes theorem:

Let $F = F_1 i + F_2 j + F_3 k$. Let outward drawn normal vector n of S make angles α, β, γ with positive directions of x, y, z axes. Then $n = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.

Also,

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i + \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial x} \right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k$$

$$(\nabla \times F).n = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos\gamma$$

Also, $F.dr = (F_1 i + F_2 j + F_3 k).(dx\, i + dy\, j + dz\, k) = F_1 dx + F_2 dy + F_3 dz$

So, Stoke's theorem can be rewritten as,

$$\oint_C F_1 dx + F_2 dy + F_3 dz = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos\gamma \right] dS$$

Note: *Green's theorem in plane is special case of Stoke's theorem*. If R is a region in the $xy-$plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_S (\nabla \times F).k\,dR = \oint_C F.dr$$

This is nothing but a special case of Stoke's theorem because here $k = n =$ outward drawn unit normal to the surface of region R.

Solved Example

Example 23: Prove that $\oint_C r.dr = 0$

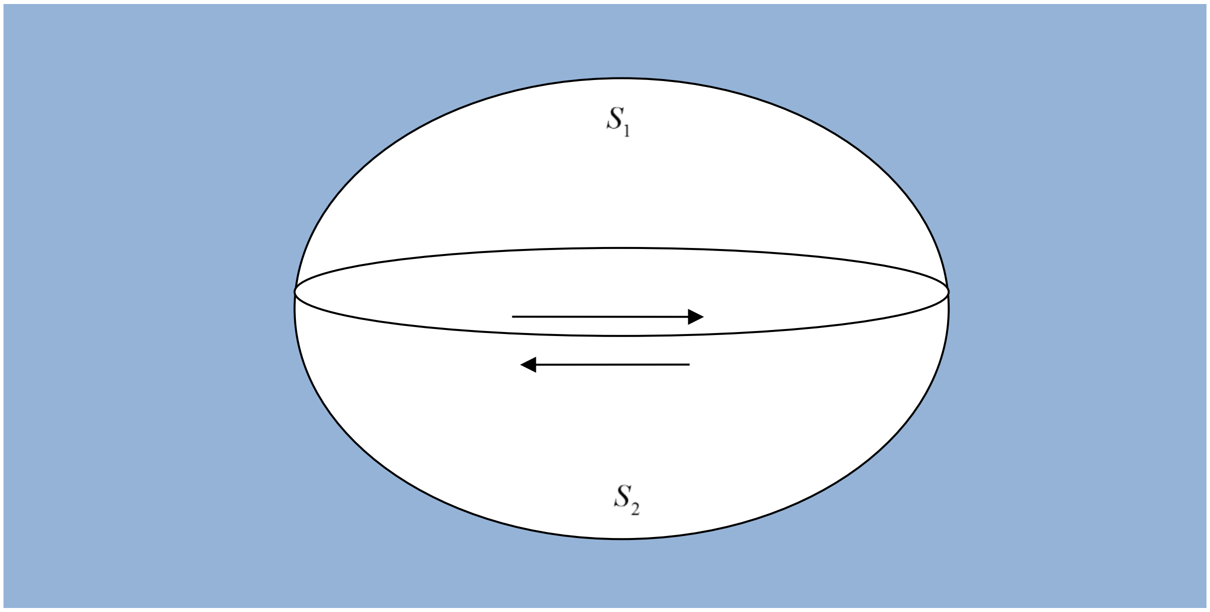
Solution: By Stoke's theorem $\oint_C r.dr = \iint_S (curl\, r).n\, ds = 0$, since $curl\, r = 0$

Example 24: By Stoke's theorem prove that $div\, curl\, F = 0$

Solution: Let V is the volume enclosed by a closed surface. Then using the divergence theorem

$$\iiint_V \nabla.(curl\, F)\, dV = \iint_S (curl\, F).n\, dS$$

Divide the surface S in two section S_1 and S_2 by a closed curve C (as in figure). Then



$$\iint_S (curl\ F).n\,dS = \iint_{S_1} (curl\ F).n\,dS_1 + \iint_{S_2} (curl\ F).n\,dS_2 \qquad \ldots (1)$$

In the right hand side of equation (1) by Stoke's theorem

$$= \oint_C F.dr - \oint_C F.dr = 0$$

Here, the negative sign indicate because the positive directions about the boundaries of the two surfaces are opposite.

$$\iiint_V \nabla.(curl\ F)\,dV = 0$$

Now this equation is true for all volume element V. Therefore we have,

$$\nabla.(curl\ F)\,dV = 0 \quad or \quad div\,curl\ F = 0$$

Example 25: Using Stoke's theorem prove that $curl\ grad\ \phi = 0$.

Solution: Let S be the surface which is enclosed by a simple closed curve C. Then by Stoke's theorem, we have

$$\iint_S (curl\ grad\ \phi).n\,dS = \oint_C grad\phi.dr$$

Now

$$grad\phi.dr = \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right).\left(dx\,i + dy\,j + dz\,k\right) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$\therefore \oint_C grad\phi . dr = \oint_C d\phi = [\phi]_A^A, \text{ where } A \text{ is any point on } C$$

$$= 0$$

Therefore we have $\iint_S (curl\ grad\ \phi) . n\, dS = 0$

Now this equation is true for all surface elements S .

Therefore we have, $(curl\ grad\ \phi) = 0$

Example 26: Verify the Stoke's theorem for $F = yi + zj + xk$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: The boundary C of S is a circle in the $xy-plane$ or radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1, z = 0$.

Let us assume $x = \cos t, y = \sin t, z = 0,\ 0 \le t < 2\pi$ are parametric equation of C . Then

$$\therefore \oint_C F . dr = \oint_C (yi + zj + xk).(dxi + dyj + dzk) = \oint_C (ydx + zdy + xdz) = \oint_C ydx,$$

Since on $C, z = 0$ and $dz = 0$.

$$= \int_0^{2\pi} \sin t \frac{dx}{dt} dt = \int_0^{2\pi} -\sin^2 t dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t) dt = -\frac{1}{2}\left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} = -\pi \qquad \dots (1)$$

Now let us evaluate $\iint_S curl\ F . ndS$. We have

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S curl\ F\,.ndS = \iint_{S_1} curl\ F.k\ dS \qquad \text{[Always remember]}$$

$$= \iint_{S_1} (-i-j-k).k\ dS = \iint_{S_1} (-1)\ dS = -\iint_{S_1}\ dS = -S_1$$

But $S_1 =$ area of a circle of radius $1 = \pi(1)^2 = \pi$

$$\therefore \iint_S curl\ F\,.ndS = -\pi \qquad \dots (2)$$

Now from (1) and (2), the theorem is verified.

Example 27: Verify Stoke's theorem for $F = (2x-y)i - yz^2 j - y^2 zk,$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Solution: The boundary C of S is a circle in the $xy-$plane of radius unity and centre origin. Suppose $x=\cos t, y=\sin t, z=0,\ 0\le t<2\pi$ is parametric equation of C. Then

$$\oint_C F.dr = \oint_C [(2x-y)i - yz^2 j - y^2 zk).(dxi+dyj+dzk) = \oint_C [(2x-y)dx - yz^2 dy - y^2 zdz]$$

$$\oint_C (2x-y)dx\text{, since } z=0 \text{ and } dz=0$$

$$\int_0^{2\pi} (2\cos t-\sin t)\frac{dx}{dt}dt = \int_0^{2\pi} (2\cos t-\sin t)\sin tdt = \int_0^{2\pi} [\sin 2t - \frac{1}{2}(1-\cos 2t)]dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi} = -\left[\left(-\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}(\pi-0) + \frac{1}{4}(0-0)\right] = \pi$$

… (1)

And

$$(\nabla\times F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz+2yz)i - (0-0)j + (0+1)k = k$$

Let S_1 be the plane region bounded by the circle C. If $S^{'}$ is the surface consisting of the surfaces S and S_1, then $S^{'}$ is the closed surface.

Using the application of Gauss divergence theorem, we have

$$\iint_{S^{'}} curl\, F.n\, dS = 0$$

$$\iint_{S} curl\, F.n\, dS = \iint_{S_1} curl\, F.n\, dS = 0 \qquad [\because S^{'} \text{ consists of } S \text{ and } S_1]$$

$$\iint_{S} curl\, F.n\, dS - \iint_{S_1} curl\, F.k\, dS = 0 \qquad [\text{on } S_1, n = -k]$$

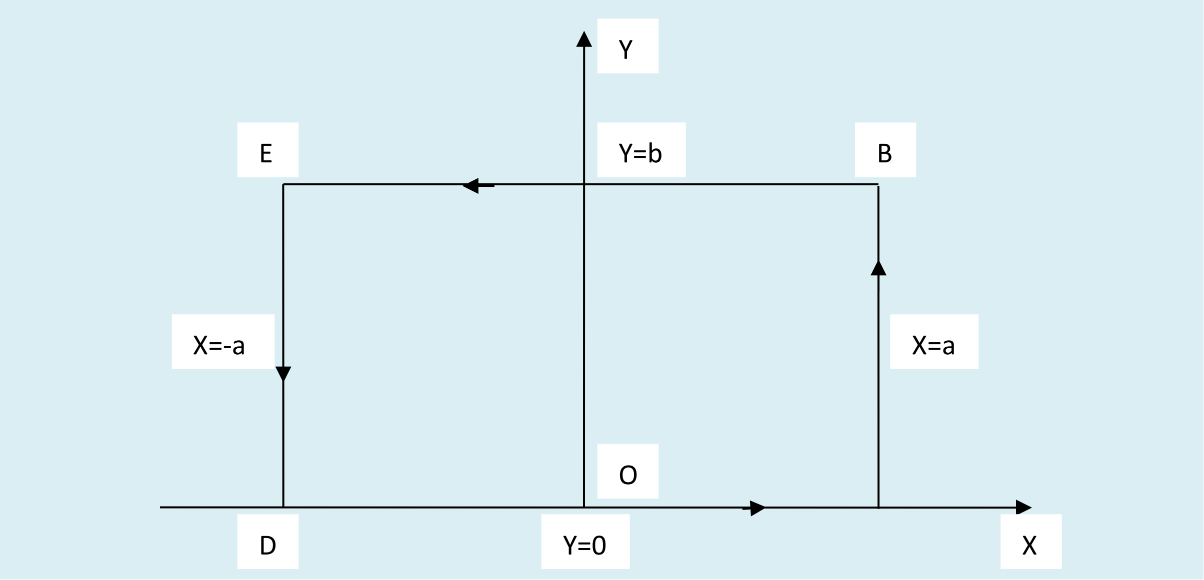
$$\iint_{S} curl\, F.n\, dS = \iint_{S_1} curl\, F.k\, dS$$

$$\therefore \iint_{S} curl\, F.n\, dS = \iint_{S_1} curl\, F.k\, dS = \iint_{S_1} k.k\, dS = \iint_{S_1} dS = S_1 = \pi$$
… (2)

Note that $S_1 =$ area of a circle of radius $1 = \pi(1)^2 = \pi$

Thus by equation (1) and equation (2) Stoke's theorem is verified.

Example 28: Verify Stokes theorem for $F = (x^2 + y^2)i - 2xyj$ taken round the rectangle bounded by $x = \pm a, y = 0, y = b$.



Solution: We have $$curl\ F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)k = -4yk$$

Also $n = k$

$$\therefore \iint_S (curl\ F).n\, dS = \int_{y=0}^{b}\int_{x=-a}^{a}(-4yk).\,k\ dxdy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\, dxdy = -4\int_{y=0}^{b}\left[xy\right]_{x=-a}^{a} dy = -4\int_{y=0}^{b} 2aydy = -4\left[ay^2\right]_0^b = -4ab^2$$

Also $\oint_C F.dr = \oint_C [(x^2+y^2)i - 2xy\ j].(dxi + dy\ j) = \oint_C [(x^2+y^2)dx - 2xydy]$

$$= \oint_{DA}[(x^2+y^2)dx - 2xydy] + \int_{AB} + \int_{BE} + \int_{ED}$$

Along $DA, y=0, dy=0$. Along $AB, x=a, dx=0$. Along $BE, y=b, dy=0$. Along $ED, x=-a, dx=0$.

$$\therefore \oint_C F.dr = \int_{x=-a}^{a} x^2dx + \int_{y=0}^{b} -2aydy + \int_{x=-a}^{a}(x^2+b^2)dx + \int_{y=b}^{0} 2aydy\,.$$

$$= \int_{x=-a}^{a} x^2dx - \int_{x=-a}^{a}(x^2+b^2)dx - 4a\int_0^b ydy$$

$$= -\int_{x=-a}^{a} x^2dx - 4a\int_0^b ydy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2$$

Thus, $\oint_C F.dr = \iint_S (curl\ F).ndS$

Hence the theorem is verified.

SELF CHECK QUESTION

Fill in the blanks:

1. $\int_0^1 [ti + (t^2-2t)\,j]dt = 0$ **………………..**

2. If $F(t)=3t^2i+t\,j+2k$ and $G(t)=6t^2i+(t-1)\,j+3tk$, then

$$\int_0^1\left(\frac{dF}{dt}.G+F.\frac{dG}{dt}\right)dt=0\ \textbf{.........}$$

3. Any integral which is to be evaluated along a curve is called a ……………

4. Any integral which is to be evaluated over a surface is called a ……………

5. $\iint_S F.n\,dS$ is called the ………….. of F over S

6. For any Closed surface S, $\iint_S curlF.n\,dS=$ **……………..**

3.10 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of line, surface and volume integrals.
- The applications of Green's theorem.
- The application of Gauss divergence theorem.
- The application of Stoke's theorem.
- Basic differences between line integral, surface integral, volume integral, Green's, Gauss and Stoke's theorem.

3.11 *GLOSSARY*

- Line integral
- Surface integral
- Volume integral
- Green's theorem
- Gauss divergence theorem
- Stoke's theorem

3.12 *REFERENCES*

- Spiegel, R. Murray (1959), *Vector Analysis*, Schaum's Outline Series.
- N. Saran and S. N. Nigam, *Introduction to vector analysis*, Pothishala Pvt. Ltd. Allahabad.
- Erwin. Kreyszig, "Advanced engineering mathematics, 10th eddition", 2009.

- A. R. Vasishtha, “Vector Calculus”, 20th edition, Krishna publication, 2020.

3.13 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

3.14 *TERMINAL QUESTION*

Objective type question:

1. If C is the curve $x^2 + y^2 = 1, z = y^2$, then by Stoke’s theorem $\oint_C (yzdx + zxdy + xydz)$ is

a. 0
b. 3
c. 5
d. None of these

2. If S denote the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$ then by application of Gauss divergence theorem the value of $\iint_S (xi + yj + zk).n\,dS$ is

a. a^3
b. $2a^3$
c. $3a^3$
d. 0

3. If $F(t) = ti + (t^2 - 2t)j + (3t^2 + 3t^3)k$, then the value of $\int_0^1 F(t)dt$ is

a. $\frac{1}{2}i + \frac{2}{3}j + \frac{7}{4}k$
b. $\frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$
c. $-\frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$
d. none of these

4. If $x^2 i + y^3 j$ and curve C is the arc of the parabola $y = x^2$ in the $xy-$plane from $(0,0)$ to $(1,1)$ then $\oint_C F.dr$ is

a. 7/12
b. 5/12
c. 11/12
d. none of these

5. The work done in moving a particle in a force field $F = 3x^2 i + (2xz - y)j + 3k$ along the line joining $(0,0,0)$ to $(2,1,3)$ is

a. 12 **b.** 16

b. 0 **d.** 20

6. For a closed surface S, the value of $\iint_S r.n\,dS$ is

a. V **b.** $2V$

c. $3V$ **d.** 0

Find True and False Statement.

1. $\int \left(2\frac{dr}{dt}.\frac{d^2r}{dt^2}\right)dt = \left(\frac{dr}{dt}\right)^2 + C$

2. If C is a simple closed curve, then $\oint_C F.dr$ is called the circulation of F about C.

3. If $F = axi + byj + czk, a, b, c$ are constant, then $\iint_S F.n\,dS = \frac{2}{3}\pi(a+b+c)$ where S is the surface of a unit sphere.

4. For any surface S, $\iint_S n\,dS = 0$

5. Green’s theorem in plane is a special case of Stoke’s theorem.

6. Green’s theorem states that “the surface integral of the normal component of a vector F taken over a closed surface is equal to the integral of the divergence of F taken over the volume enclosed by the surface”.

Short answer type question:

1. Evaluate $\int_C F.dr$ where F is $x^2y^2i + yj$ and C is $y^2 = 4x$ in the xy-plane from (0,0) to (4,4)

2. Evaluate $\int (xdy - ydx)$ around the circle $x^2 + y^2 = 1$.

3. Evaluate $\int_C F.dr$, where $F = yzi + zxj + xyk$ and C is the portion of the curve $r = a\cos ti + b\sin tj + ctk$ from $t = 0$ to $t = \pi/2$.

4. Evaluate $\int_C F.dr$, where $F = zi + xj + yk$ and C is the portion of arc of the curve $r = \cos ti + \sin tj + tk$ from $t = 0$ to $t = 2\pi$.

5. Verify Green's theorem in the plane for $\int_C [(2xy - x^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense.

6. Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$.

7. Evaluate by Green's theorem in the plane $\int_C (e^{-x} \sin y dx + e^{-x} \cos y dy]$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi, \pi/2), (0, \pi/2)$ and $y = x^2$.

8. Verify divergence theorem for $F = (2x - z)i + x^2 yj - xz^2 k$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

9. Verify divergence theorem for $F = 4xzi - y^2 yj + yzk$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

10. Prove that $\iint_S r \times n\, dS = 0$, for any closed surface S.

11. By using Gauss divergence theorem evaluate $\iint_S (xi + yj + z^2 k).ndS$, where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$

12. Verify Stoke's theorem for $F = zi + xj + yk$ where curve is the unit circle in the *xy*-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$.

13. Verify Stoke's theorem for $A = 2yi + 3xj - z^2 k$ where S is upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.

Long answer type question.

1. Verify Stoke's theorem for the vector $B = zi + xj + yk$ taken over half of the sphere $x^2 + y^2 + z^2 = a^2$ lying about the $xy-$plane.

2. Evaluate $\oint_C F.dr$ by stoke's theorem where $F = y^2 i + x^2 j - (x + z)k$ and C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0), (1,1,0)$.

3. Verify Stoke's theorem for $F = -y^3 i + x^3 j$, where S is the circular disc $x^2 + y^2 \le 1, z = 0$.

4. Evaluate $\oint_C (xydx + xy^2dy)$ by Stoke's theorem where C is the positively oriented square with vertices $(1,0), (-1,0), (0,1)$ *and* $(0,-1)$

5. Use Gauss divergence theorem to show that $\iint_S \{(x^3 - yz)i - 2x^2yj + 2k\}.ndS = \frac{1}{3}a^5$, where S denotes the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$

6. By Gauss divergence theorem, evaluate $\iint_S (xi + yj + z^2k).ndS$, where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

7. If $F = axi + byj + czk$, where a, b, c are constant, show that $\iint_S n.F\,dS = \frac{4\pi}{3}(a + b + c)$, S being surface of the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$.

8. Verify Green's theorem in the plane to evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the surface enclosed by the *x*-axis and the semi circle $y = (1 - x^2)^{1/2}$.

9. Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$, where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$. .

3.15 *ANSWERS*

Answer of self cheque questions:

1.	$\frac{1}{2}i - \frac{2}{3}j$	**2.**	24	**3.**	Line integral
4.	Surface integral	**5.**	Flux	**6.**	0

Answer of objective questions:

1.	a	**2.**	c	**3.**	b

4. a **5.** b **6.**
c

Answer of true and false questions:

1. T **2.** T **3.**
F
4. F **5.** T **6.**
F

Answer of short answer type questions:

1. 264 **2.** 2π **3.**
0
4. 3π **7.** $2(e^{-\pi}-1)$ **11.**
$7\pi/6$

Answer of long answer type questions:

2. 1/3 **3.** $3\pi/2$ **4.**
1/3
6. $7\pi/6$

UNIT 4- INTRODUCTION OF MECHANICS

CONTENTS:

4.1 INTRODUCTION

This course builds upon the concepts learned in the course “Mechanics of Materials” also known as “Strength of Materials”. In the “Mechanics of Materials” course one would have learnt two new concepts “stress” and “strain” in addition to revisiting the concept of a “force” and “displacement” that one would have mastered in a first course in mechanics, namely “Engineering Mechanics”. Also one might have been exposed to four equations connecting these four concepts, namely strain-displacement equation, constitutive equation, equilibrium equation and compatibility equation. Figure 4.1 pictorially depicts the concepts that these equations relate. Thus, the strain displacement relation allows one to compute the strain given a displacement; constitutive relation gives the value of stress for a known value of the strain or vice versa; equilibrium equation, crudely, relates the stresses developed in the body to the forces and moment applied on it; and finally, compatibility equation places restrictions on how the strains can vary over the body so that a continuous displacement field could be found for the assumed strain field.

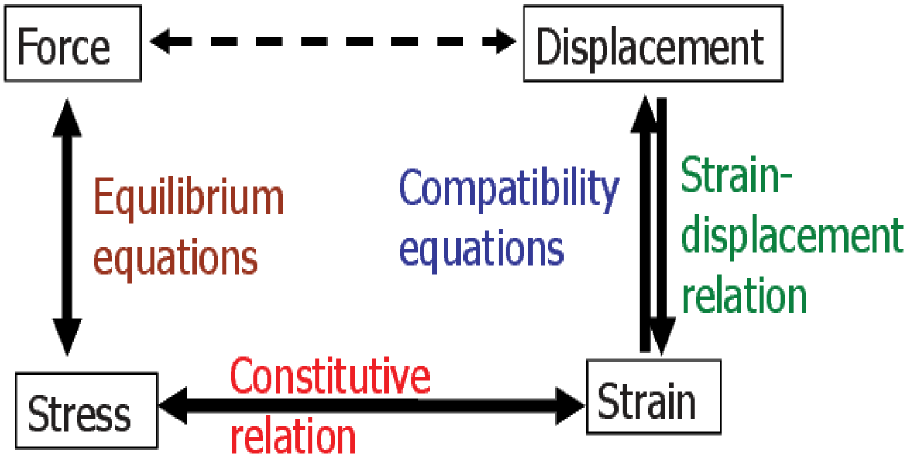


Figure 4.1: Basic concepts and equations in mechanics

In this course too we shall be studying the same four concepts and four equations. While in the "mechanics of materials" course, one was introduced to the various components of the stress and strain, namely the normal and shear, in the problems that was solved not more than one component of the stress or strain occurred simultaneously. Here we shall be studying these problems in which more than one component of the stress or strain occurs simultaneously. Thus, in this course we shall be generalizing these concepts and equations to facilitate three-dimensional analysis of structures.

Before venturing into the generalization of these concepts and equations, a few drawbacks of the definitions and ideas that one might have acquired from the previous course needs to be highlighted and clarified. This we shall do in sections 4.1 and 4.3 Specifically, in section 4.1 we look at the four concepts in mechanics and in section 4.3

4.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of Mechanics as a branch of applied mathematics dealing with motion and forces.
2. To Distinguish between different branches of mechanics, such as: statics, Dynamics, Kinematics.
3. To Explain the physical quantities used in mechanics, including: displacement, velocity, acceleration, Force, mass, momentum
4. To Apply vector algebra to represent forces, velocities, and accelerations.
5. To Understand Newton's laws of motion and their mathematical formulation.

4.3 BASIC CONCEPT OF MECHANICS

4.3.1 What is force?

Force is a mathematical idea to study the motion of bodies. It is not "real" as many think it to be. However, it can be associated with the twitching of the muscle, feeling of the burden of mass, linear translation of the motor, so on and so forth. Despite seeing only displacements we relate it to its cause the force, as the concept of force has now been ingrained.

Let us see why force is an idea that arises from mathematical need. Say, the position[1] ($\mathbf{x}_o$) and velocity ($\mathbf{v}_o$) of the body is known at some time, $t = t_o$, then one is interested in knowing where this body would be at a later time, $t = t_1$. It turns out that mathematically, if the acceleration ($\mathbf{a}$) of the body at any later instant in time is specified then the position of the body can be determined through Taylor's series. That is if

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = \mathbf{f}_a(t),$$

then from Taylor's series

$$\mathbf{x}_1 = \mathbf{x}(t_1) = \mathbf{x}(t_o) + \left.\frac{d\mathbf{x}}{dt}\right|_{t=t_o}(t_1 - t_o) + \left.\frac{d^2\mathbf{x}}{dt^2}\right|_{t=t_o}\frac{(t_1-t_o)^2}{2!} + \left.\frac{d^3\mathbf{x}}{dt^3}\right|_{t=t_o}\frac{(t_1-t_o)^3}{3!} + \left.\frac{d^4\mathbf{x}}{dt^4}\right|_{t=t_o}\frac{(t_1-t_o)^4}{4!} + \dots,$$

which when written in terms of $\mathbf{x}_o$, $\mathbf{v}_o$ and $\mathbf{a}$ reduces to[2]

$$\mathbf{x}_1 = \mathbf{x}_o + \mathbf{v}_o(t_1 - t_o) + \mathbf{f}_a(t_o)\frac{(t_1 - t_o)^2}{2!} + \left.\frac{d\mathbf{f}_a}{dt}\right|_{t=t_o}\frac{(t_1 - t_o)^3}{3!} + \left.\frac{d^2\mathbf{f}_a}{dt^2}\right|_{t=t_o}\frac{(t_1 - t_o)^4}{4!} + \dots$$

Thus, if the function $\mathbf{f}_a$ is known then the position of the body at any other instant in time can be determined. This function is nothing but force per unit mass[3] , as per Newton's second law which gives a definition for the force. This shows that force is a function that one defines mathematically so that the position of the body at any later instance can be obtained from knowing its current position and velocity.

It is pertinent to point out that this function $\mathbf{f}_a$ could also be prescribed using the position, $\mathbf{x}$ and velocity, $\mathbf{v}$ of the body which are themselves function of time, t and hence $\mathbf{f}_a$ would still be a function of time. Thus, $\mathbf{f}_a = \mathbf{g}(\mathbf{x}(t), \mathbf{v}(t), t)$. However, $\mathbf{f}_a$ could not arbitrarily depend on t, $\mathbf{x}$ and $\mathbf{v}$. At this point it suffices to say that the other two laws of Newton and certain objectivity requirements have to be met by this function. We shall see what these objectivity requirements are and how to prescribe functions that meet this requirement subsequently in chapter - 6.

Next, let us understand what kind of quantity is force. In other words is force a scalar or vector and why? Since, position is a vector and acceleration is second time derivative of position, it is also a vector. Then, it follows from equation (4.3) that $\mathbf{f}_a$ also has to be a vector. Therefore, force is a vector quantity. Numerous experiments also show that addition of forces follow vector addition law (or the parallelogram law of addition). In chapter 2 we shall see how the vector addition differs from scalar addition. In fact it is this addition rule that distinguishes a vector from a scalar and hence confirms that force is a vector.

As a summary, we showed that force is a mathematical construct which is used to mathematically describe the motion of bodies.

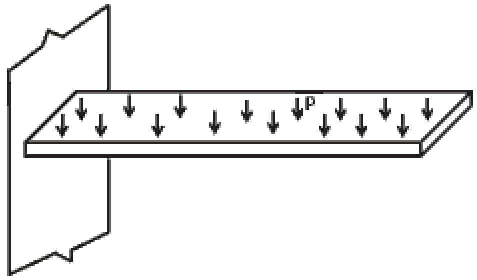
4.3.2 What is stress?

As is evident from figure 4.3, stress is a quantity derived from force. The commonly stated definitions in an introductory course in mechanics for stress are:

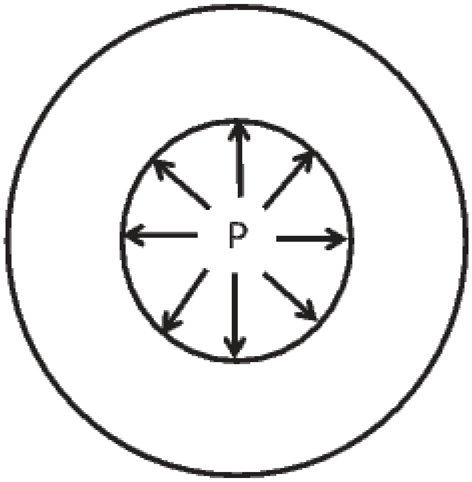
- Stress is the force acting per unit area

- Stress is the resistance offered by the body to a force acting on it

While the first definition tells how to compute the stress from the force, this definition holds only for simple loading case. One can construct a number of examples where definition 1 does not hold. The following two cases are presented just as an example. Case -1: A cantilever beam of rectangular cross section with a uniform pressure, *p*, applied on the top surface, as shown in figure 4.3a. According to the definition 1 the stress in the beam should be *p*, but it is not. Case -2: An annular cylinder subjected to a pressure, *p* at its inner surface, as shown in figure 3b. The net force acting on the cylinder is zero but the stresses are not zero at any location. Also, the stress is not *p*, anywhere in the interior of the cylinder. This being the state of the first definition, the second definition is of little use as it does not tell how to compute the stress. These definitions does not tell that there are various components of the stress nor whether the area over which the force is considered to be distributed is the deformed or the undeformed. They do not distinguish between traction (or stress vector), $\mathbf{t}_{(\mathbf{n})}$ and stress tensor, $\boldsymbol{\sigma}$.



(a) A cantilever beam with uniform pressure applied on its top



surface. (b) An annular cylinder subjected to internal pressure.

Figure 4.3: Structures subjected to pressure loading

Traction is the distributed force acting per unit area of a cut surface or boundary of the body. This traction apart from varying spatially and temporally also depends on the plane of cut characterized by its normal. This quantity integrated over the cut surface gives the net force acting on that surface. Consequently, since force is a vector quantity this traction is also a vector quantity. The component of the traction along the normal direction[4] , **n** is called as the normal stress ($\sigma_{(\mathbf{n})}$). The magnitude of the component of the traction[5] acting parallel to the plane is called as the shear stress ($\tau_{(\mathbf{n})}$).

If the force is distributed over the deformed area then the corresponding traction is called as the Cauchy traction ($\mathbf{t}_{(\mathbf{n})}$) and if the force is distributed over the undeformed or original area that traction is called as the Piola traction ($\mathbf{p}_{(\mathbf{n})}$). If the deformed area does not change significantly from the original area, then both these tractions would have nearly the same magnitude and direction. More details about this traction is presented in chapter 4.

The stress tensor, is a linear function (crudely, a matrix) that relates the normal vector, **n** to the traction acting on that plane whose normal is **n**. The stress tensor could vary spatially and temporally but does not change with the plane of cut. Just like there is Cauchy and Piola traction, depending on over which area the force is distributed, there are two stress tensors. The Cauchy (or true) stress tensor, $\boldsymbol{\sigma}$ and the Piola-Kirchhoff stress tensor (**P**). While these two tensors may nearly be the same when the deformed area is not significantly different from the original area, qualitatively these tensors are different. To satisfy the moment equilibrium in the absence of body couples, Cauchy stress tensor has to be symmetric tensor (crudely, symmetric matrix) and Piola-Kirchhoff stress tensor cannot be symmetric. In fact the transpose of the Piola-Kirchhoff stress tensor is called as the engineering stress or nominal stress. Moreover, there are many other stress measures obtained from the Cauchy stress and the gradient of the displacement which shall be studied in chapter 4.

4.3.3 What is displacement?

The difference between the position vectors of a material particle at two different instances of time is called as displacement. In general, the displacement of the material particle would depend on time; the instances between which the displacement is sought. It is also possible that different particles get displaced differently between the same two instances of time. Thus, displacement in general varies spatially and temporally. Displacement is what can be observed and measured. Forces, traction and stress tensors are introduced to explain (or mathematically capture) this displacement.

The displacement field is at least differentiable twice temporally so that acceleration could be computed. This stems from the observations that the location or velocity of the body does not change abruptly. Similarly, the basic tenant of continuum mechanics is that the displacement field is continuous spatially and is piecewise differentiable spatially at least twice. That is while the displacement field is required to be continuous over the entire body it is required to be twice differentiable not necessarily over the entire body but only on subsets of the body. Thus, in continuum mechanics interpenetration of two surfaces or separation and formation of new surfaces is precluded. The validity of the theory stops just before the body fractures. Notwithstanding this many attempts to use continuum mechanics concepts to understand the process of fracture.

A body is said to undergo rigid body displacement if the distance between any two particles that belongs to the body remains unchanged. That is in a rigid body displacement the particles that belong to a body do not move relative to each other. A body is said to be rigid if it always undergoes only rigid body displacement under action of any force. On the other hand, a body is said to be deformable if it allows relative displacement of its particles under the action of some force. Though, all real bodies are deformable, at times one could idealize a given body as rigid under the action of certain forces.

4.3.4 What is strain?

One observes that rigid body displacements of the body does not give raise to any stresses. Further, stresses are induced only when there is relative displacement of the material particles. Consequently, one requires a measure (or metric) for this relative displacement so that it can be related to the stress. The unique measure of relative displacement is the stretch ratio, $\lambda_{(\mathbf{A})}$, defined as the ratio of the deformed length to the original length of a material fiber along a given direction, **A**. (Note that here **A** is a unit vector.) However, this measure has the drawback that when the body is not deformed the stretch ratio is 1 (by virtue of the deformed length being same as the original length) and hence inconvenient to write the constitutive relation of the form

$$\sigma_{(\mathbf{A})} = f(\lambda_{(\mathbf{A})}),$$

where $\sigma_{(\mathbf{A})}$ denotes the normal stress on a plane who's normal is **A**. Since the stress is zero when the body is not deformed, the function *f* should be such that $f(1) = 0$. Mathematical implementation of this condition that $f(1) = 0$ and that *f* be a one to one function is thought to be difficult when *f* is a nonlinear function of $\lambda_{(\mathbf{A})}$. Consequently, another measure of relative displacement is sought which would be 0 when the body is not deformed and less than zero when compressed and greater than zero when stretched. This measure is called as the strain, $\epsilon_{(\mathbf{A})}$. There is no unique way of obtaining the strain from the stretch ratio. The following functions satisfy the requirement of the strain:

$$\epsilon_{(\mathbf{A})} = \frac{\lambda_{(\mathbf{A})}^{m} - 1}{m}, \qquad \epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})}),$$

where *m* is some real number and ln stands for natural logarithm. Thus, if $m = 1$ then the resulting strain is called as the engineering strain, if $m = -1$, it is called as the true strain, if $m = 2$ it is Cauchy-Green strain. The second function wherein $\epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})})$, is called as the Hencky strain or the logarithmic strain.

Just like the traction and hence the normal stress changes with the orientation of the plane, the stretch ratio also changes with the orientation along which it is measured. We shall see in chapter 3 that a tensor called the Cauchy-Green deformation tensor carries all the information required to compute the stretch ratio along any direction. This is akin to the stress tensor which when known we could compute the traction or the normal stress in any plane.

4.4 BASIC EQUATION IN MECHANICS

Having gained a superficial understanding of the four concepts in mechanics namely the force, stress, displacement and strain, let us look at the four equations that connect these concepts and the reasoning used to obtain them.

4.4.1 Equilibrium equations

Equilibrium equations are Newton's second law which states that the rate of change of linear momentum would be equal in magnitude and direction to the net applied force. Deformable bodies are subjected to two kinds of forces, namely, contact force and body force. As the name suggest the contact force arises by virtue of the body being in contact with its surroundings. Traction arises only due to these contact force and hence so does the stress tensor. The magnitude of the contact force depends on the contact area between the body and its surroundings. On the other hand, the body forces are action at a distance forces. Examples of body force are gravitational force, electromagnetic force. The magnitude of these body forces depends on the mass of the body and hence are generally expressed as per unit mass of the body and denoted by **b**.

On further assuming that the Newton's second law holds for any subpart of the body and that the stress field is continuously differentiable within the body the equilibrium equations can be written as:

$$div(\boldsymbol{\sigma}) + \rho\mathbf{b} = \rho\mathbf{a},$$

where ρ is the density, **a** is the acceleration and the mass is assumed to be conserved. Detail derivation of the above equation is given in chapter 5. The meaning of the operator *div*(·) can be found in chapter 2.

Also, the rate of change of angular momentum must be equal to the net applied moment on the body. Assuming that the moment is generated only by the contact forces and body forces, this condition requires that the Cauchy stress tensor to be symmetric. That is in the absence of body couples, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$, where the superscript $(\cdot)^t$ denotes the transpose. Here again the assumptions made to obtain the force equilibrium equation should hold. See chapter 5 for detailed derivation.

4.4.2 Strain-Displacement relation

The relationship that connects the displacement field with the strain is called as the strain displacement relationship. As pointed out before there is no unique definition of the strain and hence there are various strain tensors. However, all these strains are some function of the gradient of the deformation field, **F**; commonly called as the deformation gradient. The deformation field is a function that gives the position vector of any material particle that belongs to the body at any instance in time with the material particle identified by its location at some time t_o. Then, we show that, the stretch ratio along a given direction **A** is,

$$\lambda_{(\mathbf{A})} = \sqrt{\mathbf{CA} \cdot \mathbf{A}},$$

where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$, is called as the right Cauchy-Green deformation tensor. When the body is undeformed, $\mathbf{F} = \mathbf{1}$ and hence, $\mathbf{C} = \mathbf{1}$ and $\lambda_{(\mathbf{A})} = 1$. Instead of looking at the deformation field, one can develop the expression for the stretch ratio, looking at the displacement field too. Now, the displacement field can be a function of the coordinates of the material particles in the reference or undeformed state or the coordinates in the current or deformed state. If the displacement is a function of the coordinates of the material particles in the reference configuration it is called as Lagrangian representation of the displacement field and the gradient of this Lagrangian displacement field is called as the Lagrangian displacement gradient and is denoted by **H**. On the other hand if the displacement is a function of the coordinates of the material particle in the deformed state, such a representation of the displacement field is said to be Eulerian and the gradient of this Eulerian displacement field is called as the Eulerian displacement gradient and is denoted by **h**. Then it can be shown that (see chapter 3),

$$\mathbf{F} = \mathbf{H} + \mathbf{1}, \qquad \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h},$$

where, **1** stand for identity tensor (see chapter 2 for its definition). Now, the right Cauchy-Green deformation tensor can be written in terms of the Lagrangian displacement gradient as,

$$\mathbf{C} = \mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t\mathbf{H}.$$

Note that the if the body is undeformed then $\mathbf{H} = \mathbf{0}$. Hence, if one cannot see the displacement of the body then it is likely that the components of the Lagrangian displacement gradient are going to be small, say of order 10^{-3}. Then, the components of the tensor $\mathbf{H}^t\mathbf{H}$ are going to be of the order 10^{-6}. Hence, the above equation for this case when the components of the Lagrangian displacement gradient is small can be approximately calculated as,

$$\mathbf{C} = \mathbf{1} + 2\epsilon_L,$$

where

$$\epsilon_L = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^t\right],$$

is called as the linearized Lagrangian strain. We shall see in chapter 3 that when the components of the Lagrangian displacement gradient is small, the stretch ratio (4.7) reduces to

$$\lambda_{\mathbf{A}} = 1 + \epsilon_L \mathbf{A} \cdot \mathbf{A}.$$

Thus we find that ϵ_L contains information about changes in length along any given direction, **A** when the components of the Lagrangian displacement gradient are small. Hence, it is called as the linearized Lagrangian strain. We shall in chapter 3 derive the various strain tensors corresponding to the various definition of strains given in equation.

Further, since $\mathbf{FF}^{-1} = \mathbf{1}$, it follows from that

$$\mathbf{H} = \mathbf{h} + \mathbf{Hh},$$

which when the components of both the Lagrangian and Eulerian displacement gradient are small can be approximated as $\mathbf{H} = \mathbf{h}$. Thus, when the components of the Lagrangian and Eulerian displacement gradients are small these displacement gradients are the same. Hence, the Eulerian linearized strain defined as,

$$\epsilon_E = \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^t\right],$$

and the Lagrangian linearized strain, ϵ_L would be the same when the components of the displacement gradients are small.

Equation (4.14) is the strain displacement relationship that we would use to solve boundary value problems in this course, as we limit ourselves to cases where the components of the Lagrangian and Eulerian displacement gradient is small.

4.4.3 Compatibility equation

It is evident from the definition of the linearized Lagrangian strain, that it is a symmetric tensor. Hence, it has only 6 independent components. Now, one cannot prescribe arbitrarily these six components since a smooth differentiable displacement field should be obtainable from these six prescribed components. The restrictions placed on how these six components of the strain could vary spatially so that a smooth differentiable displacement field is obtainable is called as compatibility equation. Thus, the compatibility condition is

$$curl(curl(\boldsymbol{\epsilon})) = \mathbf{0}.$$

The derivation of this equation as well as the components of the *curl*(·) operator in Cartesian coordinates is presented in chapter 3.

It should also be mentioned that the compatibility condition in case of large deformations is yet to be obtained. That is if the components of the right Cauchy-Green deformation tensor, **C** is prescribed, the restrictions that have to be placed on these prescribed components so that a smooth differentiable deformation field could be obtained is unknown, except for some special cases.

4.4.4 Constitutive relation

Broadly constitutive relation is the equation that relates the stress (and stress rates) with the displacement gradient (and rate of displacement gradient). While the above three equations - Equilibrium equations, strain-displacement relation, compatibility equations - are independent of the material that the body is made up of and/or the process that the body is subjected to, the constitutive relation is dependent on the material and the process. Constitutive relation is required to bring in the dependance of the material in the response of the body and to have as many equations as there are unknowns, as will be shown in chapter 6.

The fidelity of the predictions, namely the likely displacement or stress for a given force depends only on the constitutive relation. This is so because the other three equations are the same irrespective of the material that the body is made up of. Consequently, a lot of research is being undertaken to arrive at better constitutive relations for materials.

It is difficult to have a constitutive relation that could describe the response of a material subjected to any process. Hence, usually constitutive relations are prescribed for a particular process that the material undergoes. The variables in the constitutive relation depends on the process that is being studied. The same material could undergo different processes depending on the stimuli; for example, the same material could respond elastically or plastically depending on say, the magnitude of the load or temperature. Hence it is only apt to qualify the process and not the material. However, it is customary to qualify the material instead of the process too. This we shall desist.

Traditionally, the constitutive relation is said to depend on whether the given material behaves like a solid or fluid and one elaborates on how to classify a given material as a solid or a fluid. A material that is not a solid is defined as a fluid. This means one has to define what a solid is. A couple of definitions of a solid are listed below:

- Solid is one which can resist sustained shear forces without continuously deforming
- Solid is one which does not take the shape of the container

Though these definitions are intuitive they are ambiguous. A class of materials called “viscoelastic solids”, neither take the shape of the container nor resist shear forces without continuously deforming. Also, the same material would behave like a solid, like a mixture of a solid and a fluid or like a fluid depending on say, the temperature and the mechanical stress it is being subjected to. These prompts us to say that a given material behaves in a solid-like or fluid-like manner. However, as we shall see, this classification of a given material as solid or fluid is immaterial. If one appeals to thermodynamics for the classification of the processes, the response of materials could be classified based on (1) Whether there is conversion of energy from one form to another during the process, and (2) Whether the process is thermodynamically equilibrated. Though, in the following section, we classify the response of materials based on thermodynamics, we also give the commonly stated definitions and discuss their shortcomings. In this course, as well as in all these classifications, it is assumed that there are no chemical changes occurring in the body and hence the composition of the body remains a constant.

4.5 SUMMARY

Thus in this unit we introduced the four concepts in mechanics, the four equations connecting these concepts as well as the methodologies used to solve boundary value problems. In the following chapters we elaborate on the same topics. It is not intended that in a first reading of this chapter, one would understand all the details. However, reading the same chapter at the end of this course, one should appreciate the details. This chapter summarizes the concepts that should be assimilated and digested during this course.

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10. A. K. Tayal (2011). **Engineering Mechanics**. Umesh Publications.

4.7 SUGGESTED READING

1. Vector Mechanics for Engineers: Statics – F. P. Beer & E. R. Johnston
A foundational text covering concepts of forces, moments, and equilibrium with clear illustrations.

2. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige
Excellent for understanding theoretical principles and solving analytical problems.

*3.***Engineering Mechanics** – R. K. Bansal
Useful for Indian students; provides numerous solved examples and practice problems.

4. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
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5. **A Textbook of Engineering Mechanics** – S. S. Bhavikatti
Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6

BLOCK-II EQUILIBRIUM OF RIGID BODY AND VIRTUAL WORK

UNIT 5- EQUILIBRIUM OF A RIGID BODY

CONTENTS

5.1 INTRODUCTION

In classical mechanics, the concept of **e**quilibrium plays a fundamental role in understanding the behaviour of physical bodies under the action of forces. **A** rigid body is an idealized object whose shape and size remain unchanged when external forces or moments are applied. The study of equilibrium of a rigid body deal with identifying the conditions under which such a body remains at rest or continues to move with constant velocity.

To maintain equilibrium, a rigid body must satisfy certain necessary and sufficient conditions that ensure the net effect of all applied forces and moments is zero. These principles lead to important tools such as vector addition of forces**,** moments and couples**,** conditions for translational and rotational equilibrium**,** and the use **of** free-body diagrams for problem-solving. Understanding equilibrium is essential for students of mathematics, physics, and engineering because it provides the basis for solving practical problems like stability of buildings, balance of beams, design of machines, and motion of objects. The analytical approach used in this chapter strengthens mathematical reasoning by applying vector algebra, geometry, and calculus to real-world systems.

5.2 OBJECTIVES

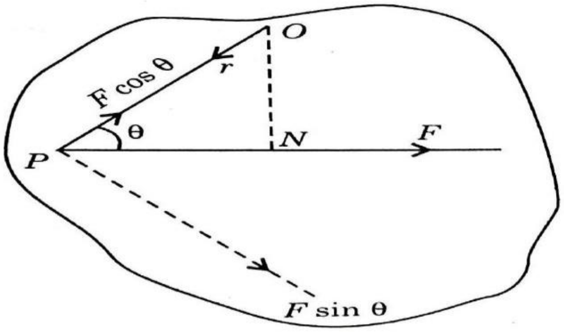
After studying this unit learner will be able

1. To understand the concept of a rigid body and its mathematical representation in classical mechanics.
2. To learn the meaning of equilibrium and distinguish between translational and rotational equilibrium.
3. To study different types of forces acting on a rigid body, including concurrent, coplanar, and non-coplanar forces.
4. To understand the concept of moment and couple and their role in maintaining or disturbing equilibrium.
5. To derive and apply the necessary and sufficient conditions of equilibrium for a system of forces acting on a rigid body.

5.3 MOMENT OF A FORCE ABOUT A POINT

The moment of a force applied on a rigid body about a point is defined as the amount of the tendency of the force to rotate the body about that point.

Let **F** be the force acting at the point P of a rigid body. Let **r** be the position vector of the point P referred to a point O of the body. If θ is the angle between **F** and OP, then the components of **F** along and perpendicular to PO are $F\cos\theta$ and $F\sin\theta$ respectively. The tendency of the component $F\cos\theta$ is to move P along PO. But if the point O is fixed then due to the rigidity of the body the distance PO does not change, and hence the effect of $F\cos\theta$ is nullified. The tendency of the component $F\sin\theta$ is to turn the body perpendicular to OP. Thus when O is fixed, then the net effect of the force F acting at P is to turn P in a direction perpendicular to OP. The amount of the tendency of **F** to turn the body about O is $OP.\ F\sin\theta$.



$\therefore$ Moment of F about $O = OP \cdot F \sin\theta$

$$= F.OP \sin\theta = F.ON.$$

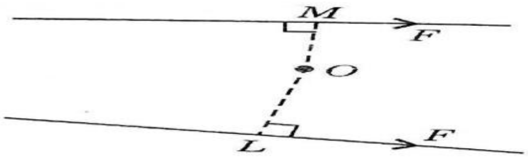
where ON is the perpendicular from O on F. Thus the moment of a force **F** about the point O is the product of the magnitude of the force **F** and the perpendicular distance of O from the line of action of the force **F**. Now the moment of the force F about O

$$= OP \cdot F \sin\theta = |\mathbf{r}||\mathbf{F}| \sin\theta$$

and its direction is perpendicular to the plane of the vectors **r** and **F** in the sense of a right-hand screw rotated from **r** to **F**. Hence the vector moment M of the force F about O is defined by the vector $\mathrm{M} = \mathrm{r} \times \mathrm{F}$. It can be easily seen that the vector moment of a force **F** about any point O is equal to $\mathbf{r} \times \mathbf{F}$ where **r** is the position vector with respect to O of any point P on the line of action of the force.

5.4 SIGN OF THE MOMENT

If O is a point in the plane of the lamina about which it can turn, then the tendency of the forces F_1 and F_2 is to rotate the lamina in opposite directions. The moment of the force F_1 about O has the tendency to rotate



the lamina in the counter-clockwise direction and is taken as positive while the moment of the force F_2 about O having the tendency to rotate the body in clockwise direction is taken as negative.

5.5 GENERAL THEOREMS OF MOMENTS

Theorem 1: The sum of the moments of two like parallel forces acting on a rigid body about any

arbitrary point is equal to the moment of their resultant about the same point.

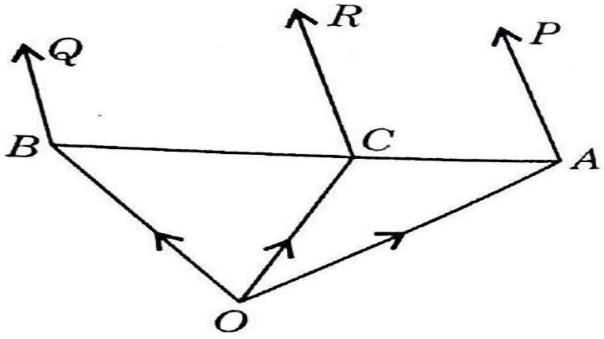
Proof: Let two like and parallel forces **P** and **Q** act at the points A and B of a body. If **R** is the resultant of the force, then

$$R = P + Q$$

acting at C , such that

$$P \cdot AC = Q \cdot BC \tag{1}$$

The sum of the moments of the forces **P** and **Q** about any



arbitrary point O

$$= \overrightarrow{OA} \times \mathbf{P} + \overrightarrow{OB} \times \mathbf{Q}$$

$$= (\overrightarrow{OC} + \overrightarrow{CA}) \times \mathbf{P} + (\overrightarrow{OC} + \overrightarrow{CB}) \times \mathbf{Q}$$
$$= \overrightarrow{OC} \times (\mathbf{P} + \mathbf{Q}) + \overrightarrow{CA} \times \mathbf{P} + \overrightarrow{CB} \times \mathbf{Q}$$
$$= \overrightarrow{OC} \times \mathbf{R} - \overrightarrow{AC} \times \mathbf{P} + \overrightarrow{CB} \times \mathbf{Q} (3)$$

Now $\overrightarrow{AC} \times \mathbf{P}$ and $\overrightarrow{CB} \times \mathbf{Q}$ are both perpendicular to the plane containing $\mathbf{P}, \mathbf{Q}, \overrightarrow{AC}$,
$\overrightarrow{CB}$ and are in the same directions.
Also $|\overrightarrow{AC} \times \mathbf{P}| = AC.P.\sin\theta |\overrightarrow{CB} \times \mathbf{Q}| = CB.Q.\sin\theta$

$$\therefore \quad |\overrightarrow{AC} \times \mathbf{P}| = |\overrightarrow{CB} \times \mathbf{Q}| \quad |\because AC.P = CB.Q. \text{ from (2)}$$
$$\therefore \quad \overrightarrow{AC} \times \mathbf{P} = \overrightarrow{CB} \times \mathbf{Q}$$

Hence from (3), we have
the sum of the moments of the forces **P** and **Q** about O

$$= \overrightarrow{OC} \times \mathbf{R}$$
= moment of the resultant **R** about O. Hence the theorem.

Theorem 2: If a number of coplanar forces acting at a point of a rigid body have a resultant, then the vector sum of the moments of all the forces about any arbitrary point is equal to the moment of the resultant about the point.

Proof: Let the coplanar forces $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n$ acting at a point P of a rigid body have the resultant ***R***

$$\therefore \ \mathbf{R} = \mathrm{F}_1 + \mathrm{F}_2 + \cdots + \mathrm{F}_n \qquad (1)$$

Let O be an arbitrary point and **r** be the position vector of the point p with respect to the point O. Moment of the force **R** about O

$$= \mathbf{r} \times \mathbf{R}$$
$$= \mathbf{r} \times (\mathrm{F}_1 + \mathrm{F}_2 + \cdots + \mathrm{F}_n)$$
$$= \mathbf{r} \times \mathrm{F}_1 \times \mathbf{r} \times \mathrm{F}_2 + \cdots + \mathbf{r} \times \mathrm{F}_n$$

= vector sum of the moments of the forces $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n$ about O. Hence the theorem.

5.6 COUPLE

A system of two equal and unlike parallel forces, whose lines of action are not the same, is called a couple or a torque.

The angle that the resultant R makes with BC

$$= \tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{P\sqrt{\quad}3}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

∴ the resultant is perpendicular to the side BC.
Let the line of action of the resultant meet the side BC at the point D. Since the sum of
moments of the forces about any point is equal to the moment of the resultant about that point, therefore the .um of moments of forces about the point D must be equal $_0$ zero.
i.e.,

$$P \cdot DL - 3P \cdot DM = 0$$

Or $\qquad P.BD\sin 60^\circ - 3P.CD\sin 60^\circ = 0$

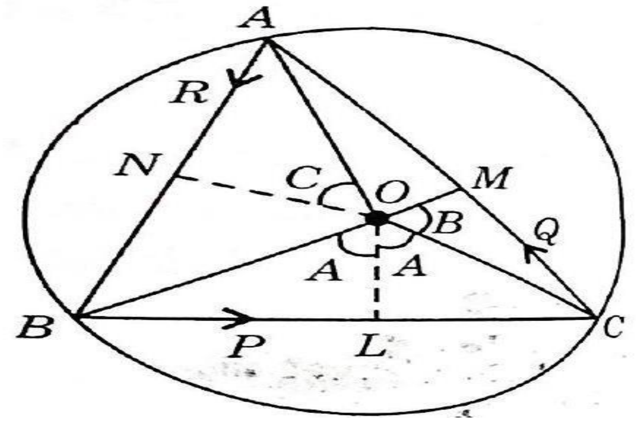
Or $\qquad BD - 3(BD - BC) = 0$

O $\qquad$ r$BD = \frac{3}{2}BC$

Example 1: The resultant of the forces P, Q, R acting along the sides BC, CA, AB respectively of a triangle ABC passes through its circumcenter. Show that

$$\boldsymbol{P\cos A + Q\cos B + R\cos C = 0}$$

Solution: Let the forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC, taken in order. Let OL, OM, ON be the perpendicular bisectors of the sides BC, CA, AB respectively of the $\triangle\, ABC$. Then O is the circum-centre of the $\triangle\, ABC$, and we have $OA = OB = OC$. If the resultant of the forces. P, Q, R passes through the circum-centre O, then the algebraic sum of the moments of these forces about O must be equal to zero.



i.e.,
Now

$$P \cdot OL + Q \cdot OM + R \cdot ON = 0$$

$$\angle BOC = 2\angle BAC = 2A;$$

$\therefore \angle BOL = \angle COL = A$

Similarly, $\angle COM = \angle AOM = B$
and $\angle AON = \angle BON = C$

Substituting in (1), we have

$$OL = OB\cos A, OM = OC\cos B, ON = OA\cos C$$

Or $OB\cos A \cdot P + OC\cos B \cdot Q + OA\cos C \cdot R = 0$

$P\cos A + Q\cos B + R\cos C = 0$

Example 2: Two equal unlike parallel forces acting at fixed points A and B form a couple of moment G. If their lines of action are turned through one right angle they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment $\sqrt{(G^2 + H^2)}$.

Solution: Let two equal unlike parallel forces P, P act at fixed point A and B. Then the moment G of this couple is given by

$$G = P \cdot AM = P \cdot AB\sin\theta \tag{1}$$

where θ is the angle between the force P and the line AB. If the line of action of the forces P, P are turned through one right angle, then the moment H of the new couple is obtained by replacing θ by $90° + \theta$ in (1) and is thus given by

$$H = P \cdot AB\sin(90° + \theta) = P \cdot AB\cos\theta \qquad (2)$$

From (1) and (2), we have

$$\sqrt{(G^2 + H^2)} = P \cdot AB \qquad (3)$$

Also if the forces act at right angles to AB, then the moment of the couple

$$= P \cdot AB = \sqrt{(G^2 + H^2)}, \qquad (3)$$

which is the required result.

CHECK YOUR PROGRESS

MULTIPLE CHOICE QUESTIONS

1. A rigid body is said to be in equilibrium when:

A. Net force is zero
B. Net moment is zero
C. Both net force and net moment are zero
D. Only horizontal forces are zero
2. Translational equilibrium means:

A. Sum of all moments is zero
B. Sum of all forces is zero
C. Body rotates with constant speed
D. Body has no mass
3. The turning effect of a force is called:

A. Torque
B. Pressure
C. Power
D. Work
4. The moment of a force is given by:

A. Force × mass
B. Mass × distance
C. Force × perpendicular distance
D. Force / distance
5. A couple produces:

A. Pure translation
B. Pure rotation
C. No effect on the body
D. Both translation and rotation
6. For rotational equilibrium, which condition must be satisfied?

A. $\Sigma F = 0$
B. $\Sigma M = 0$
C. Both A and B
D. None of the above
7. A system of forces whose lines of action pass through a common point is called:

A. Coplanar forces
B. Concurrent forces
C. Non-coplanar forces
D. Parallel forces
8. A force that balances the resultant of all forces is called:

A. Couple
B. Component force
C. Equilibrant
D. Moment
9. The SI unit of moment is:

A. Newton (N)
B. Joule (J)
C. Newton-meter (N·m)
D. Meter-second (m·s)
10. In equilibrium, the body:

A. Must be at rest
B. Must move in a circle
C. May be at rest or move with constant velocity
D. Must accelerate

5.7 SUMMARY

The equilibrium of a rigid body is a fundamental concept in mechanics that explains the conditions under which a body remains at rest or moves with constant velocity. A rigid body is considered to be in equilibrium when all external forces and moments acting on it balance each other perfectly.

To maintain equilibrium, two essential mathematical conditions must be satisfied: the resultant force acting on the body must be zero, and the resultant moment (torque) about any point must also be zero. These conditions ensure that the body does not undergo translational or rotational motion. By applying vector algebra, force components, and moment principles, one can analyze various physical situations involving beams, structures, supports, and mechanical systems. Understanding these principles strengthens analytical reasoning and forms the foundation for solving practical engineering and mathematical problems related to stability and balance.

5.8 GLOSSARY

1. **Rigid Body**: An ideal body that does not undergo any change in shape or size when forces act on it.
2. **Force:** A push or pull that tends to produce motion, stop motion, or change the direction of motion of a body.
3. **Equilibrium**: A state in which a body remains at rest or moves with constant velocity because the net force and net moment acting on it are zero.
4. **Resultant Force**: The single force that represents the combined effect of all forces acting on a body.
5. **Translational Equilibrium**: Condition in which the sum of all forces along each axis is zero, ensuring no linear motion.
6. **Rotational Equilibrium**: Condition in which the sum of all moments about any point is zero, ensuring no rotation.
7. **Couple**: Two equal and opposite parallel forces whose lines of action do not coincide, creating a pure rotational effect.
8. **Free-Body Diagram (FBD):** A diagram showing all external forces and moments acting on a single isolated body.
9. **Concurrent Forces:** A system of forces whose lines of action pass through a common point.
10. **Coplanar Forces**: Forces that lie in the same plane.

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5.10 SUGGESTED READING

1. Vector Mechanics for Engineers: Statics – F. P. Beer & E. R. Johnston
A foundational text covering concepts of forces, moments, and equilibrium with clear illustrations.

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5. **A Textbook of Engineering Mechanics** – S. S. Bhavikatti *Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6*

6. **Engineering Mechanics**– A. K. Tayal *Straightforward explanations suitable for undergraduate mathematics and engineering courses.*

5.11 TERMINAL QUESTIONS

1. Explain in detail the conditions of equilibrium of a rigid body.

2. Derive the mathematical conditions for translational and rotational equilibrium.

3. Explain the concept of moment and couple with suitable diagrams.

4. Discuss equilibrium of a body acted upon by a system of coplanar concurrent forces.

5. Describe the method of resolving forces into components and explain how it helps in solving equilibrium problems.

6. What is a free-body diagram? Explain its importance and draw an FBD for a block resting on a rough horizontal surface.

7. Discuss the analytical method of determining the resultant of a system of forces and its application in equilibrium.

8. State and prove Lami’s Theorem. How is it used in equilibrium problems?

9. Explain the principle of moments (Varignon’s Theorem) and its use in solving problems of equilibrium.

10. Discuss the equilibrium of a body under three non-parallel forces.

5.12 ANSWERS

1. **Answer: C 2. Answer: B 3. Answer: A 4. Answer: C 5. Answer: B 6. Answer: B 7. Answer: B 8. Answer: C 9. Answer: C 10. Answer: C**

UNIT 6- VERTUAL WORK

CONTENTS:

6.1 INTRODUCTION

In the study of Mathematics, the concept of *virtual work* provides a powerful and elegant method for analysing the equilibrium of mechanical systems. Instead of resolving all forces into components and applying multiple equilibrium equations, the method of virtual work allows us to determine equilibrium by considering the work done by forces during an imagined (virtual) displacement of the system.

This unit introduces the definition of virtual work, explains the principle of virtual work, and identifies the forces that can be omitted while forming the equation of virtual work. These ideas greatly simplify the analysis of systems involving constraints, reactions, and complex force distributions.

6.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of virtual displacement and distinguish it from actual displacement.

2. To Define virtual work and explain its physical significance in mechanical systems.

3. To State and apply the principle of virtual work to problems involving equilibrium.

4. To Identify the conditions of equilibrium using the method of virtual work.

6.3 DEFINITION OF VIRTUAL WORK

When several forces act on a body and cause it to move, they perform actual work. But if the forces are in equilibrium, the body does not move, and therefore no real work is done. However, if we *assume* that the forces in equilibrium undergo a very small imaginary displacement and then calculate the work done during this imagined movement, the displacement is called a virtual displacement, and the work done during it is known as virtual work.

6.4 THE PRINCIPLE OF VIRTUAL WORK

If a system of forces acting on a body is in equilibrium, then the total virtual work done by all the forces during any small imaginary (virtual) displacement of the body is zero.

In other words, when a body is in equilibrium:

Total Virtual Work=0

A virtual displacement is not a real motion; it is only an imagined, infinitesimally small and kinematically possible displacement that is consistent with the constraints of the body.

The principle helps in determining unknown forces or reactions without considering the actual motion of the body.

Theorem : The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation. Proof. Let a system of forces $\mathbf{F}_1, \ldots \ldots, \mathbf{F}_n$ act at the points of a rigid body whose

Proof. Let a system of forces $\mathbf{F}_1, \ldots \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \ldots \ldots, \mathbf{r}_n$. Suppose this system of forces is equivalent to a single force $\mathbf{R} = \Sigma \mathbf{F}_1$ acting at O, together with a couple of moment $\mathbf{G} = \Sigma \mathbf{r}_1 \times \mathbf{F}_1$. Then during any small displacement of the body consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O,the sum of the works done by these forces

$$= \Sigma \mathbf{F}_1 \cdot d\mathbf{r}_1 = \Sigma \mathbf{F}_1 \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r}_1)$$
$$= \mathbf{u} \cdot \Sigma \mathbf{F}_1 + \mathbf{e} \cdot \Sigma \mathbf{r}_1 \times \mathbf{F}_1$$
$$= \mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G}. (1)$$

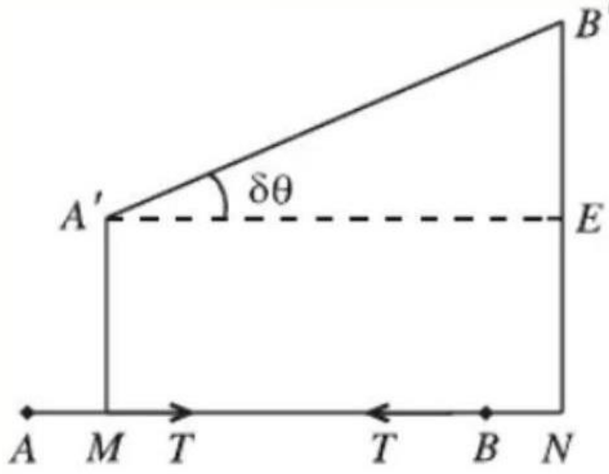
The condition is necessary. Suppose the given system of forces is in equilibrium. Then $\mathbf{R} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$. Therefore, from (1), the sum of the works done by the forces is zero. Hence the condition is necessary. The condition is sufficient. Suppose the sum of the works done by the forces during any small displacement is zero. Then to prove that the forces are in equilibrium. We have, from (1)

$$\mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G} = 0 \qquad (2)$$

6.5 FORCE WHICH CAN BE OMITTED THE EQUATION OF VIRTUAL WORK

The principle of virtual work provides a highly effective technique for solving problems related to the equilibrium of forces. Its major advantage over other methods is that certain forces do not need to be included when writing the virtual work equation, which simplifies the calculations considerably. Below, we discuss and prove the types of forces that can be omitted when formulating the equation of virtual work (i) The work done by the tension of an inextensible string is zero during a small displacement.

Let AB be an inextensible string of length l joining two points A and B of a rigid body. Let T be the tension in the string AB. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is inextensible, therefore $A'B' = AB = l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.



On account of the tension in the string AB, there are two forces each equal to T acting on A and B in opposite directions as shown in the figure. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during this displacement

$$= T \cdot AM - T \cdot BN \text{ [Note that the dis}$$
$$= T \cdot (AB - MB) - T \cdot (MN - MB)$$
$$= T \cdot (AB - MN)$$
$$= T \cdot (AB - A'E) = T \cdot (AB - A'B' \cos \delta\theta)$$

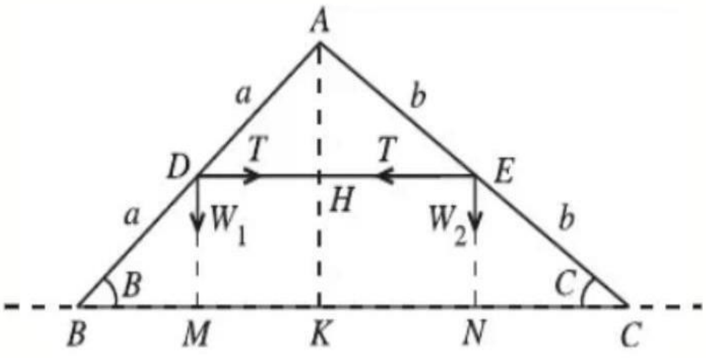
$$= T \cdot l(1 - \cos \delta\theta)$$
$$= T \cdot l\left[1 - \left\{1 - \frac{(\delta\theta)^2}{2!} + \cdots\right\}\right], \text{ expanding } \cos \delta\theta \text{ in powers of } \delta\theta$$
$$= T \cdot l \cdot 0, \text{ to the first order of small quantities}$$
$$= 0.$$

Example 5: Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is

$$\frac{W}{\tan B + \tan C}$$

where W is the total weight of the rods AB and AC.

Solution: AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let



$$AB = 2a \text{ and } AC = 2b$$

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore, the total weight $W = W_1 + W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B + \delta B$ and C changes to $C + \delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE = DH + HE = a\cos B + b\cos C$$

the height of any point of the line DE above BC

$$= DM = a\sin B$$

The equation of virtual work is

$$-T\delta(a\cos B + b\cos C) - W\delta(a\sin B) = 0$$

or $aT\sin B\delta B + bT\sin C\delta C - aW\cos B\delta B = 0$
or $a(W\cos B - T\sin B)\delta B = bT\sin C\delta C.$

From the figure,

$$DM = a\sin B \text{ and } EN = b\sin C$$

Since $DM = EN$, therefore $a\sin B = b\sin C$.

$$\therefore \quad \delta(a\sin B) = \delta(b\sin C)$$
$$\text{or} \quad a\cos B\delta B = b\cos C\delta C. \qquad (2)$$

Dividing (1) by (2), we have

$$\frac{W\cos B - T\sin B}{\cos B} = \frac{T\sin C}{\cos C}$$

or $W - T\tan B = T\tan C$
or $T(\tan B + \tan C) = W$
or $T = \frac{W}{\tan B + \tan C}$.

CHECK YOUR PROGRESS

MULTIPLE CHOICE QUESTIONS

1. Virtual work is defined as the work done by:

A. Actual displacement of the body
B. Imaginary small displacement of the body
C. Gravity only
D. Frictional forces

2. The principle of virtual work states that a body is in equilibrium if:

A. The sum of forces is zero
B. The sum of moments is zero
C. Total virtual work done by all forces is zero
D. The body moves with uniform velocity

3. Which forces can be omitted when applying the virtual work method?

A. Active forces
B. Internal forces and reactions along constraints
C. Gravitational forces
D. All applied forces

4. Virtual displacement is:

A. Always along the direction of motion
B. Infinitesimally small and consistent with constraints
C. A large actual movement
D. The displacement of centre of gravity only

5. In a solid of revolution, the centre of gravity lies:

A. Always at the edge
B. On the axis of symmetry
C. Outside the solid
D. At the centroid of generating curve only

6.6 SUMMARY

1. Definition of Virtual work: Virtual work is the work done by a system of forces during an imagined, infinitesimally small, and kinematically possible displacement of the body, called virtual displacement. This displacement is not real but is assumed only for the purpose of analyzing equilibrium.

2. Principle of Virtual Work: The principle of virtual work states that if a body is in equilibrium under a system of forces, then the total virtual work done by all the forces during any virtual displacement is zero.

Total Virtual Work=0

This principle provides a powerful method to test equilibrium and determine unknown forces.

3. **Forces Which Can Be Omitted in Forming the Equation of Virtual Work**

Certain forces do not contribute to virtual work and can be excluded from the virtual work equation. These include:

1. **Internal forces** in a rigid body, because their virtual work cancels out in pairs.
2. **Reaction forces perpendicular to the virtual displacement**, such as forces at smooth surfaces or hinges.
3. **Constraint forces** that do not cause motion in the direction of the virtual displacement.

Omitting these forces simplifies the analysis and makes the method of virtual work highly efficient for solving equilibrium problems.

6.7 GLOSSARY

Work : The product of a force and the displacement in the direction of that force.
Work = Force × Displacement.

Virtual displacement: A small, imagined, reversible displacement of a system that is consistent with the constraints.

Virtual work: The work done by a force during a virtual displacement.
It is not real work but an imagined work used for analysis.

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Excellent for understanding theoretical principles and solving analytical problems.

*2.***Engineering Mechanics** – R. K. Bansal
Useful for Indian students; provides numerous solved examples and practice problems.

3. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
Provides strong conceptual explanation with real-life engineering applications.

4 . **A Textbook of Engineering Mechanics** – S. S. Bhavikatti
Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6

*5.***Engineering Mechanics**– A. K. Tayal
Straightforward explanations suitable for undergraduate mathematics and engineering courses.

6.10 TERMINAL QUESTIONS

1. Explain the concept of virtual work and derive the condition for equilibrium.

2. Derive the formula for the centre of gravity of a semi-circular lamina**.**

3. Discuss the centre of gravity of a solid of revolution using integration.

4. State and prove the principle of virtual work for a rigid body.

5. Explain **the** forces that can be omitted in the virtual work method, with proof.

6.11 ANSWERS

MCQ1. B MCQ 2. C MCQ3. B MCQ4.B MCQ5 .B

UNIT 7- CONSERVATIVE FORCES AND INVERSE SQUARE LAW

CONTENTS

7.1 INTRODUCTION

As you saw when lifting a book, the work that you do against gravity in lifting is stored somewhere... Physicists say that it is stored in the gravitational field or stored in the Earth/book system and is available for kinetic energy of the book once you let go. Forces that store energy in this way are called conservative forces. Gravity is a conservative force, and there are many others. Elastic (Hooke's Law) forces, electric forces, etc. are conservative forces. As you say when pushing a book, the work that you do against friction is apparently lost - it is certainly not available to the book as kinetic energy! Forces that do not store energy are called nonconservative or dissipative forces. Friction is a nonconservative force, and there are others. Any friction-type force, like air resistance, is a nonconservative force. The energy that it removes from the system is no longer available to the system for kinetic energy. Of course, if energy is a "real thing," the energy taken away by a nonconservative force can't just disappear! I wonder where it goes!

A collision or crash is an event in which two or more bodies exert forces on each other for a relatively short time. Although the most common colloquial use of the word "collision" refers to incidents in which two or more objects collide, the scientific use of the word "collision" implies nothing about the magnitude of the force.

Collision is short-duration interaction between two bodies or more than two bodies simultaneously causing change in motion of bodies involved due to internal forces acted between them, during this event. Collisions involve forces; there is a change in velocity. The magnitude of the velocity difference at impact is called the closing speed. All collisions conserve momentum. What distinguishes different types of collisions is whether they also conserve kinetic energy. The line of impact is the line which is collinear to the common normal of the surfaces that are closest or in contact during impact.

In the early 1600s, Johannes Kepler proposed three laws of planetary motion. Kepler was able to summarize the carefully collected data of his mentor - Tycho Brahe - with three statements that described the motion of planets in a sun-centered solar system. Kepler's efforts to explain the underlying reasons for such motions are no longer accepted; the actual laws themselves are still considered for an accurate description of the motion of any planet and any satellite. Kepler's three laws of planetary motion with applications are discussed in this unit.

In this unit, we shall first understand what we mean by conservative and non-conservative forces. We shall learn how to describe the motion of a body under conservative and non-conservative forces. In this unit we shall also discuss force as gradient of potential energy. We shall discuss the elastic and non-elastic collision and application to variety of situations. We shall define the centre of mass frame and Kepler’s law with their applications to physical world.

7.2 OBJECTIVES

After studying this unit learner will be able
After studying this unit, you should be able to understand-

1. What are conservative forces
2. What are non-conservative forces
3. Applications of conservative and non-conservative forces
4. What are elastic and non-elastic collisions
5. Solution of problems of elastic and non-elastic collisions
6. Application of elastic and non-elastic collisions
7. Kepler’s law and their applications in planetary motion
8. Kepler’s law to solve problems

7.3 WHAT ARE CONSERVATIVE

It is important to know the difference between conservative and nonconservative forces. The work a conservative force does on an object is path-independent; the actual path taken by the object makes no difference. Fifty meters up in the air has the same gravitational potential energy whether you get there by taking the steps or by hopping on a wheel. That's different from the force of friction, which dissipates kinetic energy as heat. When friction is involved, the path you take matters — a longer path will dissipate more kinetic energy than a short one. For that reason, friction is a nonconservative force.

For example, suppose you and some buddies arrive at a majestic peak that rises h meters into the air. You can take two ways up — the quick way or the scenic route. Your friends drive up the quick route, and you drive up the scenic way, taking time out to have a picnic and to solve a few physics problems. You pull out this equation $\Delta P = mg(h_f - h_i)$ .This equation basically states that the actual path you take when going vertically from h_i to h_f doesn't matter. All that matters is your beginning height compared to your ending height. Because the path taken by the object against gravity doesn't matter, gravity is a conservative force.

7.3.1 Conservative Forces

A conservative force is a force that acts on a particle, such that the work done by this force in moving this particle from one point to another is independent of the path taken. To put it another way, the work done depends only on the initial and final position of the particle. Two examples of conservative forces are gravitational forces and elastic spring forces.

Gravitational force is a conservative force: When we throw a ball upward against the gravity, the ball reaches a certain height coming shortly to rest so that its kinetic energy becomes zero. Subsequently starts to come down under gravity. During the down trip the downward pull of the earth provides the kinetic energy to the ball. When, it reaches the starting point, the kinetic energy becomes same, as its initial kinetic energy with which it was thrown.

Elastic Force is a conservative force: when a block is moved from a position on a smooth horizontal plane with a velocity so as to compress a spring. First, it is bought to rest by the elastic force of the string and loses all kinetic energy. Then the compressed spring re-expands and the block moves back under the elastic force gaining kinetic energy. As the block returns to its initial position, it gains the same velocity, thereby attains the same kinetic energy, with which it was compressed. So, we can say that the elastic force is a conservative force.

7.4 NON-CONSERVATIVES FORCES

A non conservative force is a force that acts on a particle or point, such that the work done by this force in moving this particle from one point to another is dependent on the path taken. To put it another way, the work done depends on the path itself. For example, a frictional force is non conservative because the work done by friction always acts in the direction of travel and therefore depends on the length of the path taken. If we slide a stone on a rough floor between two points along different paths, the work done by the friction forces would be different.

Thus, if a system is acted on by a non conservative force such as friction, and the system returns to its original position, then that system will experience a net loss of energy, due to those forces. Energy will thus not be conserved for the system. This makes sense intuitively since we know friction is a source of energy loss. This is why we always try to minimize friction in moving parts and machine components, so as to minimize the energy wasted**.**

7.4.1 Central force is conservative:

Central force is a force which acts upon a particle and always directed towards or away from a point. The magnitude of the central force depends only on the distance of the particle from that point. Gravitational, elastic and electrostatic forces are the example of central forces.

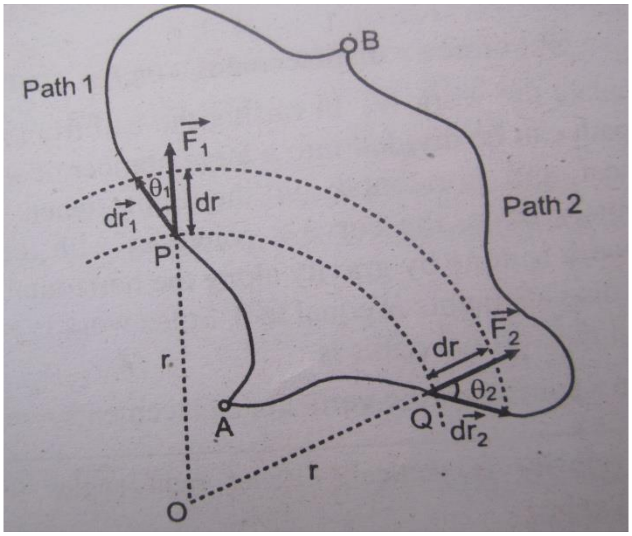

Figure 7.1

Two points A and B are connected by two random path say path1 and path 2. Consider a particle moves from A to B along any path under a central force from point O. let us draw two sectors of radii r and r+dr. consider two forces $\vec{F}_1$ and $\vec{F}_2$ acting on the particles say P and Q. let $d\vec{r}_1$ and $d\vec{r}_2$ be the displacements along the path 1 and path 2. Consider θ_1 and θ_2 be the angles between $\vec{F}_1$ and $d\vec{r}_1$, and $\vec{F}_2$ and $d\vec{r}_2$. Using vector algebra

$$\vec{F}_1.d\vec{r}_1 = F_1 dr_1 \cos\theta_1$$

And $\vec{F}_2.d\vec{r}_2 = F_2 dr_2 \cos\theta_2$

Since, P and Q are at the equal distances from O, so the magnitude of the central forces is equal that is $F_1 = F_2$. The projection of $d\vec{r}_1$ and $d\vec{r}_2$ on $\vec{F}_1$ and $\vec{F}_2$ are equal. Therefore we can write

$$\vec{F}_1.d\vec{r}_1 = \vec{F}_2.d\vec{r}_2$$

If we integrate it throughout the path from A to B , we get

$$\int_A^B \vec{F}_1.d\vec{r}_1 = \int_A^B \vec{F}_2.d\vec{r}_2$$

This shows that the work done

$W=\int_A^B \vec{F}.d\vec{r}$ is independent of the path. So we conclude that the central force is a conservative force.

We can also show that the work done around the closed path is zero.

The work done in moving a particle from A to B is given as

$W_{A\to B}=\int_A^B \vec{F}.d\vec{r}$ along the path 1, similarly the work done B to A

$$W_{B\to A}=\int_B^A \vec{F}.d\vec{r}=-\int_A^B \vec{F}.d\vec{r}$$

For conservative force work done must be same, so

$$W_{A\to B}=W_{B\to A}$$

Hence $\int_A^B \vec{F}.d\vec{r}=\int_A^B \vec{F}.d\vec{r}$

Or, $W_{A\to B}+W_{B\to A}=0$

Thus, conservative force is one which draws or supplies no energy from or to a body in a complete round trip. A conservative force does zero total work on any closed path.

CHECK YOUR PROGRESS

1. True or False statements

(i) Conservative forces depend only on the initial and final positions of the body.

(ii) Conservative force can be expressed as $\vec{F}=-gradU$, where U is the kinetic energy.

(iii) Gravitational force is an example of non-conservative force.

(iv) Force of friction is a conservative force.

(v) Non conservative forces are independent upon path.

7.5 SUMMARY

In this unit, you have studied conservative and non-conservative forces. On which factors these forces depend. You have studied that the central forces are conservative forces. Also, we have derived that force is the negative gradient of potential. You have studied about the collision. We defined there are two types of collision, elastic and non-elastic collision. We also derived that momentum is conserved in elastic collision. You have studied central force, centre of mass coordinates.

7.6 GLOSSARY

Work : The product of a force and the displacement in the direction
Dissipative-wastefully

Intuitively- instinctively

Fundamental- essential

Fright Car- to carry load

Stationary- fixed

Interaction- contact

Ecliptic-apparent path

Constellations- collections

Imaginary- unreal

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7.9 ANSWERS

CYQ1. T CYQ2. F CYQ3. F CYQ4. F CYQ4. F CYQ5. T

BLOCK III- RECTILINEAR MOTION AND MOMENT OF INERTIA

UNIT 8: MOMENT OF INERTIA

Structure

8.1 INTRODUCTION

In the previous units have studied about angular momentum and its conservation, moment of inertia and its physical significance, radius of gyration, equations of angular motion and rotational kinetic energy. In this unit, we shall study some more important concepts of motion and laws of rotational motion. We know that, to find the moment of inertia of a body about a given axis, all that we have to do is to find the sum $\Sigma\, mr^2$ for all particles making up the body by integration or other means. The calculations to find the moment of inertia can be made shorter by the help of some important theorems. In this unit, we shall also study those theorems (theorems of parallel and perpendicular axes).

8.2 OBJECTIVES

After studying this unit, you should be able to-

- Solve problems based on equations of motion
- apply equations of motion
- understand laws of rotational motion
- apply theorems of parallel and perpendicular axes.

8.3 EQUATIONS OF MOTION

If an object is moving in a straight line under a constant acceleration, then relations among its velocity, displacement, time and acceleration can be represented by equations. These equations are called ‘equations of motion’.

Let us consider that a body starts with an initial velocity ‘u’ and has a constant acceleration ‘a’. Suppose it covers a distance‘s’ in time‘t’ and its velocity becomes ‘v’. Then the relations among u, a, t, s and v can be represented by three equations.

First Equation: We know that linear acceleration $a = \frac{dv}{dt}$

or $dv = a\,dt$

Integrating both sides, we get-

$\int_u^v dv = \int_0^t a\,dt$

or $(v - u) = a\,(t - 0)$

or $v = u + at$
.....(1)

This is known as first equation of motion.

Second Equation: We know that linear velocity $v = \frac{ds}{dt}$

But $v = u + at$

Therefore, $u + at = \frac{ds}{dt}$

or $\frac{ds}{dt} = u + at$

or $ds = (u + at)\, dt$

Integrating both sides, we get-

$\int_0^s ds = \int_0^t (u + at)dt$

or $s = u(t - 0) + a(\frac{t^2}{2} - 0)$

or $s = ut + \frac{1}{2} a t^2$
.....(2)

This is called second equation of motion.

Third Equation: By first equation, $v = u + at$

Squaring both sides-

$v^2 = (u + at)^2$

or $v^2 = u^2 + a^2t^2 + 2uat$

$= u^2 + 2a(ut + \frac{1}{2} a t^2)$

or $v^2 = u^2 + 2as$
…..(3)

(using equation 2)

The above equation (3) is known as third equation of motion.

Example 1: A train starting from rest is accelerated by 0.5 m/sec^2 for 10 sec. Calculate its final velocity after 10 sec. Also calculate the distance travelled by train in 10 sec.

Solution: Here, $u = 0$, $a = 0.5$ m/sec^2, $t = 10$ sec

Using first equation of motion $v = u + at$

$v = 0 + 0.5 \times 10 = 5$ m/sec

Using second equation of motion $s = ut + \frac{1}{2} a t^2$

$s = 0\,(10) + \frac{1}{2} \times 0.5\,(10)^2$

$= \frac{1}{2} \times 0.5 \times 100 = 25$ m

Thus the velocity of train after 10 sec is 5 m/sec and the distance travelled is 25 m.

Example 2: A car is accelerated from 8 m/sec to 14 m/sec in 3 sec. What is the acceleration of car ?

Solution: Here, u = 8 m/sec, v = 14 m/sec, t = 3 sec

Using $v = u + at$

$14 = 8 + a \times 3$

$3a = 14 - 8$

$= 6$

$a = 2$ m/sec^2

Self Assessment Question (SAQ) 1: A car is moving with a constant speed of 30 Km/hr. Calculate the distance travelled by car in 1 hr.

Self Assessment Question (SAQ) 2: A particle is shot with constant speed 6×10^6 m/sec in an electric field which produces an acceleration of 1.26×10^{14} m/sec^2 directed opposite to the initial velocity. How far does the particle travel before coming to rest?

Self Assessment Question (SAQ) 3: The initial velocity of a particle is ‘u’ (at t = 0) and the acceleration is given by ‘at^2’. Which of the following relations is valid?

(a) $v = u + at$ (b) $v = u + \frac{at^3}{3}$ (c) $v^2 = u + at^3$ (d) $v = u + \frac{at^3}{2}$

8.4 NEWTON’S LAWS OF ROTATIONAL MOTION

As we know, there are Newton’s three laws of translational motion, similarly we have the following three laws of rotational motion-

First Law: Unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered.

Second Law: The rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque.

Third Law: When a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation.

8.5 GENERAL THEOREMS ON MOMENT OF INERTIA

There are two important theorems on moment of inertia which, in some cases, facilitate the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis be known. These theorems are theorem of parallel axes and theorem of perpendicular axes. Let us discuss these theorems.

8.5.1 Theorem of Parallel Axes

It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two axes.

If ‘I_{cm}’ be the moment of inertia of a body about a parallel axis through its centre of mass, ‘M’ be the mass of the body and ‘r’ be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$

This is the “theorem of parallel axes”.

Proof: Let us consider a plane lamina with ‘C’ as centre of mass. Let ‘I’ be its moment of inertia about an axis PQ in its plane and I_{cm} the moment of inertia about a parallel axis RS passing through C. Let the distance between RS and PQ be ‘r’.

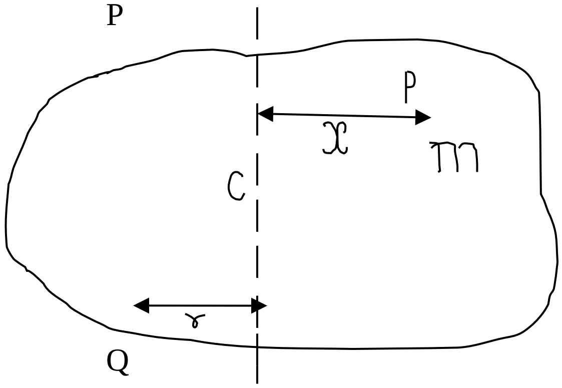


Figure 1

Let us consider a particle P of mass m at a distance x from RS. Its distance from PQ is (r + x) and its moment of inertia about it is $m(r + x)^2$. Therefore, the moment of inertia of the lamina about PQ is given by-

$I = \Sigma\, m(r + x)^2$

$= \Sigma\, m(r^2 + x^2 + 2rx)$

$= \Sigma\, mr^2 + \Sigma\, mx^2 + \Sigma\, 2mrx$

or $I = r^2\, \Sigma\, m + \Sigma\, mx^2 + 2r\, \Sigma\, mx$ …..(4)

(since r is constant)

But $\Sigma\, mx^2 = I_{cm}$, where I_{cm} is the moment of inertia of the lamina about RS, $r^2\, \Sigma\, m = r^2 M$ where M is the total mass of the lamina and $\Sigma\, mx = 0$ because the sum of the moments of all the mass particles of a body about an axis through the centre of mass of the body is zero. Hence, the equation (4) becomes

$I = r^2 M + I_{cm} + 0$

or $I = I_{cm} + M r^2$ …..(5)

It may be seen clearly from equation (5) that the moment of inertia of a body about an axis through the centre of mass is the least. The moment of inertia of the body about an axis not passing through the centre of mass is always greater than its moment of inertia about a parallel axis passing through the centre of mass of the body.

8.5.2 Theorem of Perpendicular Axes

According to this theorem, the moment of inertia of a uniform plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its plane intersecting on the first axis.

If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then

$$I_z = I_x + I_y$$

This is the "theorem of perpendicular axes".

Proof: Let OZ be the axis perpendicular to the plane of the lamina about which the moment of inertia is to be taken. Let OX and OY be two mutually perpendicular axes in the plane of the lamina and intersecting on OZ.

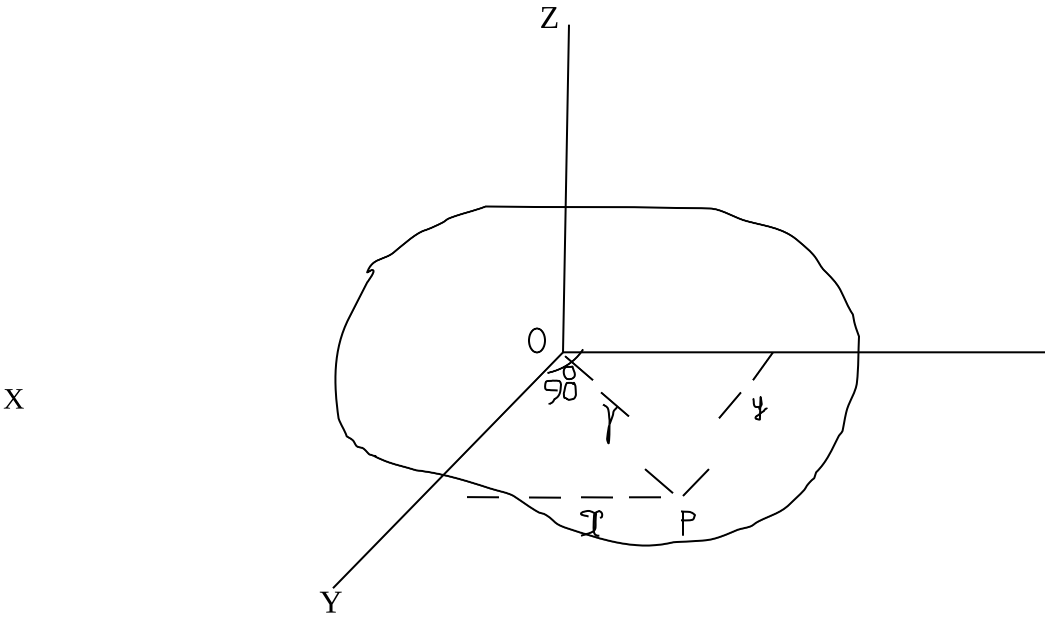


Figure 2

Let us consider a particle P of mass 'm' at a distance of 'r' from OZ. The moment of inertia of this particle about OZ is mr^2. Therefore, the moment of inertia I_z of the whole lamina about OZ is $I_z = \Sigma\, mr^2$

But $r^2 = x^2 + y^2$, where x and y are the distances of P from OY and OX respectively.

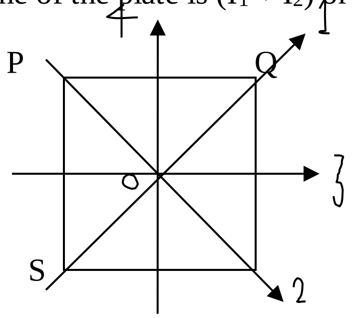
Therefore, $I_z = \Sigma\, m(x^2 + y^2) = \Sigma\, mx^2 + \Sigma\, my^2$

But $\Sigma\, mx^2$ is the moment of inertia I_y of the lamina about OY and $\Sigma\, my^2$ is the moment of inertia I_x of the lamina about OX.

Therefore, $I_z = I_y + I_x$

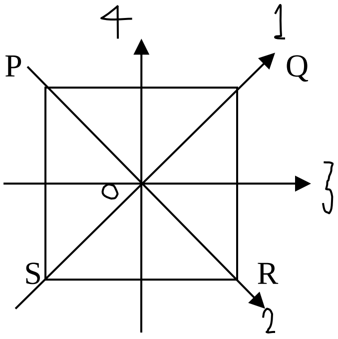
or $I_z = I_x + I_y$

Example 3: Show that the moment of inertia I of a thin square plate PQRS (Figure) of uniform thickness about an axis passing through the centre O and perpendicular to the plane of the plate is $(I_1 + I_2)$ or $(I_3 + I_4)$ or $(I_1 + I_3)$.



R

Solution:



Let I_1, I_2, I_3 and I_4 are the moments of inertia about axes 1, 2, 3 and 4 respectively which are in the plane of the plate.

By the theorem of perpendicular axes, we have-

$I = I_1 + I_2 = I_3 + I_4$

By symmetry of square plate, $I_1 = I_2$ and $I_3 = I_4$

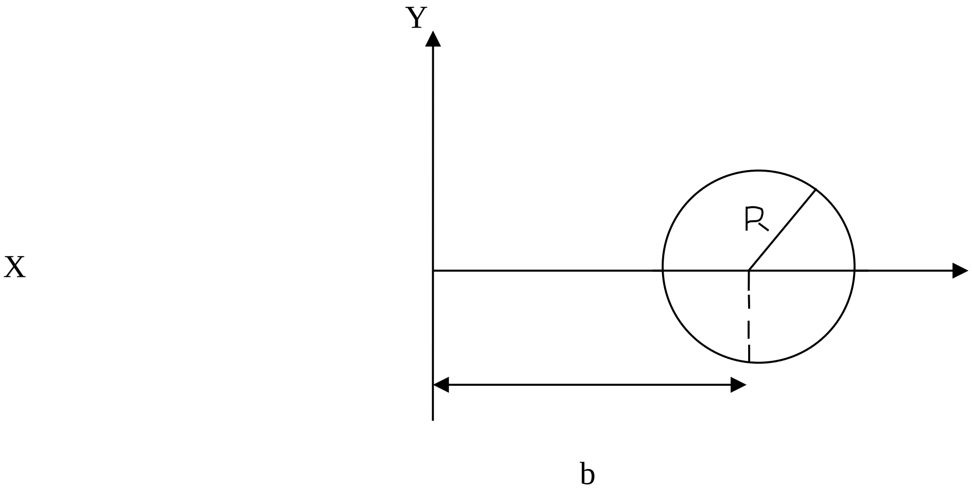
Therefore, $I = 2I_1 = 2\, I_3$

or $I_1 = I_3$

Thus $I_1 = I_2 = I_3 = I_4$

and $I = I_1 + I_2 = I_1 + I_3$

Self Assessment Question (SAQ) 4: The figure represents a disc of mass M and radius R, lying in XY- plane with its centre on X-axis at a distance ‘b’ from the origin. Determine the moment of inertia of the disc about Y-axis if its moment of inertia about a diameter is $\frac{MR^2}{4}$.



8.6 SUMMARY

In the present unit, we have studied about equations of motion and derived all the three equations of motion. In the unit, we have also studied Newton’s laws of rotational motion. According to the first law of rotational motion “unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered” while the second law states “the rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque”. According to Newton’s third law of rotational motion “when a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation”. Sometimes it is difficult to calculate the moments of inertia of some specific bodies. In this unit, we have also studied and derived the general theorems on moment of inertia. These theorems are known as theorem of parallel axes and theorem of perpendicular axes. If ‘I_{cm}’ be the moment of inertia of a body about a parallel axis through its centre of mass, ‘M’ be the mass of the body and ‘r’ be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$. This is the theorem of parallel axes. If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then $I_z = I_x + I_y$. This is the theorem of perpendicular axes. These theorems make easy to find out the moments of inertia of those specific bodies. We have included examples and self assessment questions (SAQs) to check your progress.

8.7 GLOSSARY

Velocity-	a vector physical quantity whose magnitude gives speed
Acceleration-	increase of velocity
Unless-	except, if not
External-	exterior, outer
Rotational-	the action of moving in a circle
Unaltered-	unchanged, unaffected
Facilitate-	make easy, smooth the progress of

8.8 TERMINAL QUESTIONS

1. Explain the equations of motion.

2. What are Newton’s laws of rotational motion? Explain.

3. Discuss and derive general theorems on moment of inertia.

4. Calculate the moment of inertia of mass M and length L about an axis perpendicular to the length of the rod and passing through a point equidistant from its midpoint and one end.

5. Give the statement of theorem of parallel axes. Also derive this theorem.

6. State and establish the theorem of perpendicular axes.

8.9 ANSWERS

Self Assessment Questions (SAQs):

1. Since the car is moving with constant speed, therefore acceleration of car a = 0.

Here, u = 30 Km/hr = 25/3 m/sec, t = 1 hr = 3600 sec

Using second equation of motion, $s = ut + \frac{1}{2} a t^2$

$s = (25/3) \times 3600 + \frac{1}{2} (0) \times (3600)^2 = 3 \times 10^4$ m = 30 Km

2. Given $u = 6 \times 10^6$ m/sec, $a = - 1.26 \times 10^{14}$ m/sec^2, v = 0

Using third equation of motion, $v^2 = u^2 + 2as$

$(0)^2 = (6 \times 10^6)^2 + 2 (- 1.26 \times 10^{14})\, s$

or s = 0.143 m

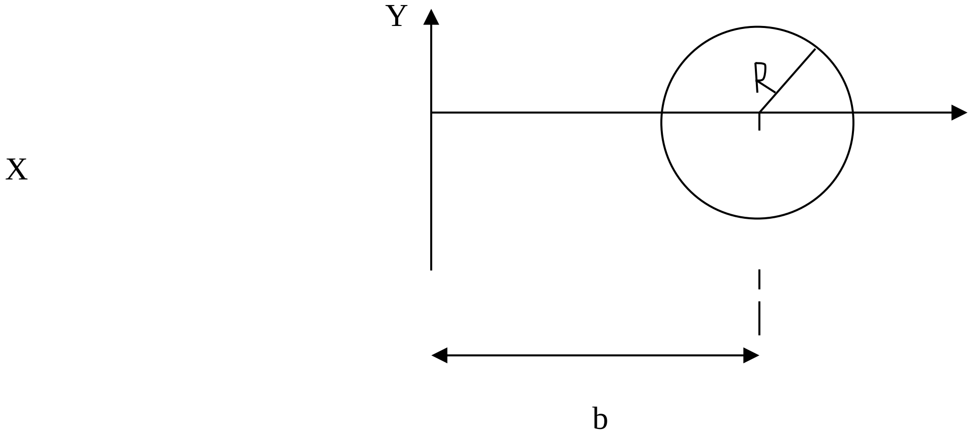
3. Here acceleration = at^2

Using v = u + at

$v = u + (at^2)\, t = u + at^3$

Hence relation (c) is valid.

4.



By the theorem of parallel axes, the moment of inertia of disc about Y-axis is-

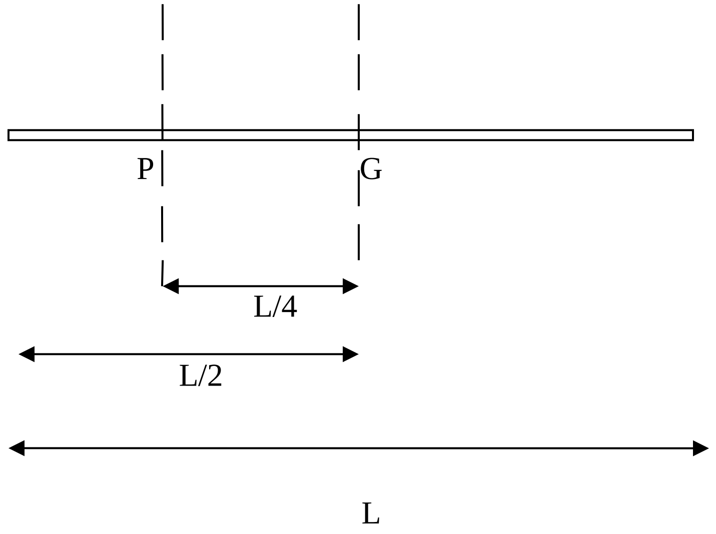
$I = I_{cm} + Mb^2$

$= \frac{MR^2}{4} + Mb^2$

$= M\left(\frac{R^2}{4} + b^2\right)$

Terminal Questions:

4. The moment of inertia of rod about an axis passing through centre of mass and perpendicular to length $I_{cm} = \frac{ML^2}{12}$



The moment of inertia of rod about an axis passing through P and perpendicular to length by theorem of parallel axes is-

$I = I_{cm} + Mr^2$

Here $r = PG = L/4$

Therefore, $I = (ML^2/12) + M(L/4)^2 = (7/48)\, ML^2$

8.10 REFERENCES

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8.11 SUGGESTED READINGS

1. Concepts of Physics, Part I, HC Verma, Bharati Bhawan, Patna

2. Mechanics and Wave Motion – DN Tripathi and RB Singh, Kedar Nath Ram Nath, Meerut

3. Modern Physics, Beiser, Tata McGraw Hill

4. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons

UNIT 9: FORMULATION OF MOMENT OF INERTIA

CONTENTS:

9.1 INTRODUCTION

In the previous unit, we have studied theorem of parallel axes and theorem of perpendicular axes. These theorems make easy the calculations of moment of inertia in some typical cases. In general, the moment of inertia of a body is calculated as the sum $\Sigma\ mr^2$ for all particles making up the body by integration or other means. In this unit, we shall formulate and derive the moment of inertia for some simple symmetric systems like rod, rectangular lamina, circular lamina, solid sphere and cylinder.

9.2 OBJECTIVES

After studying this unit, you should be able to-

- Understand the formulation and derivation of moment of inertia
- Solve problems based on moment of inertia
- Apply the formulae of moment of inertia

9.3 FORMULATION AND DERIVATION OF MOMENT OF INERTIA

The moment of inertia of a continuous homogeneous body of a definite geometrical shape can be calculated by (i) first obtaining an expression for the moment of inertia of an infinitesimal element of the same shape of mass dm about the given axis i.e. $dm.r^2$, where r is the distance of the infinitesimal element from the axis and then (ii) integrating this expression over appropriate limits so as to cover the whole body. In fact sometimes the theorem of parallel and perpendicular axes are also used to calculate the moment of inertia about an axis when the moment of inertia of the body about some other axis has first been calculated. Thus,

$I = \int dm.r^2$

where the integral is taken over the whole body.

9.3.1 Moment of Inertia of a Thin Uniform Rod

(i) About an axis passing through its centre of mass and perpendicular to its length

Let PQ be a thin uniform rod of mass per unit length m. Let RS be the axis passing through the centre of mass C of the rod and perpendicular to its length PQ.

Let us consider an element of length dr at a distance r from centre of mass C.

The mass of the element, dm = m.dr

The moment of inertia of element about the axis through C $= dm.r^2$

$= m\, dr\, r^2$

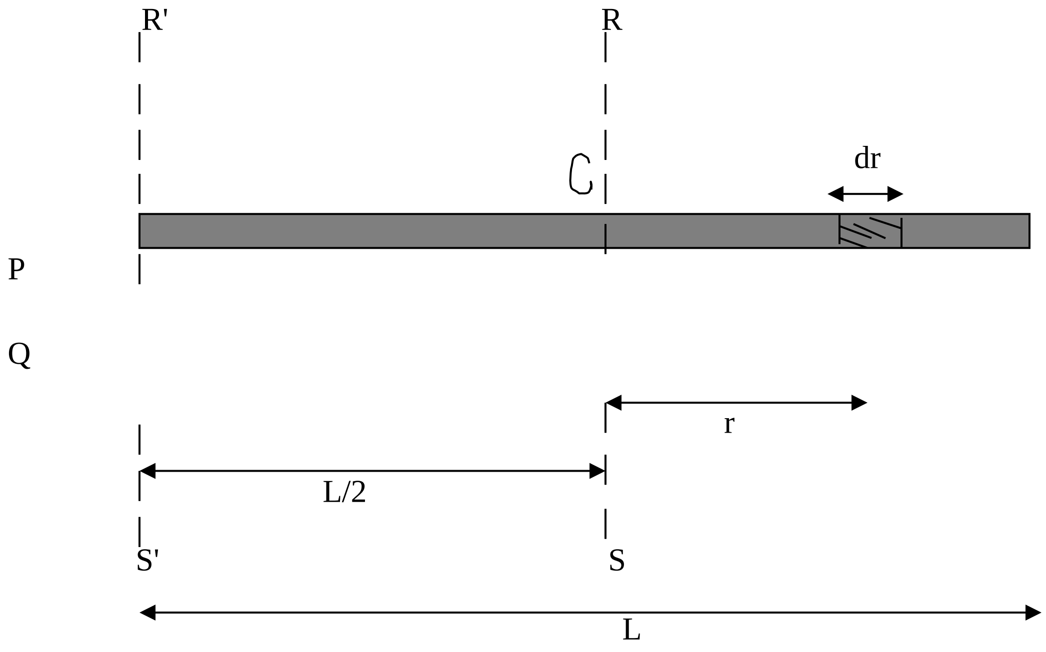


Figure 1

The moment of inertia of the whole rod about axis RS is the sum of the moments of inertia of all such elements lying between r = - L/2 at P and r = L/2 at Q. Hence the moment of inertia

$$I_{cm} = \int_{-L/2}^{L/2} mr^2 dr$$

$$= 2m \int_0^{L/2} r^2 dr = \frac{mL^3}{12} = \frac{(mL)L^2}{12}$$

$$= \frac{ML^2}{12}$$

where mL = M, the total mass of the rod

Thus, $I_{cm} = \frac{ML^2}{12}$
…..(1)

(ii) About an axis passing through its one end and perpendicular to its length

Let $R'\ S'$ be the axis passing through the end P of the rod (Figure 1). The moment of inertia I about a parallel $R'\ S'$ axis passing through one end (using theorem of parallel axes)-

$$I = I_{cm} + M\,(CP)^2$$

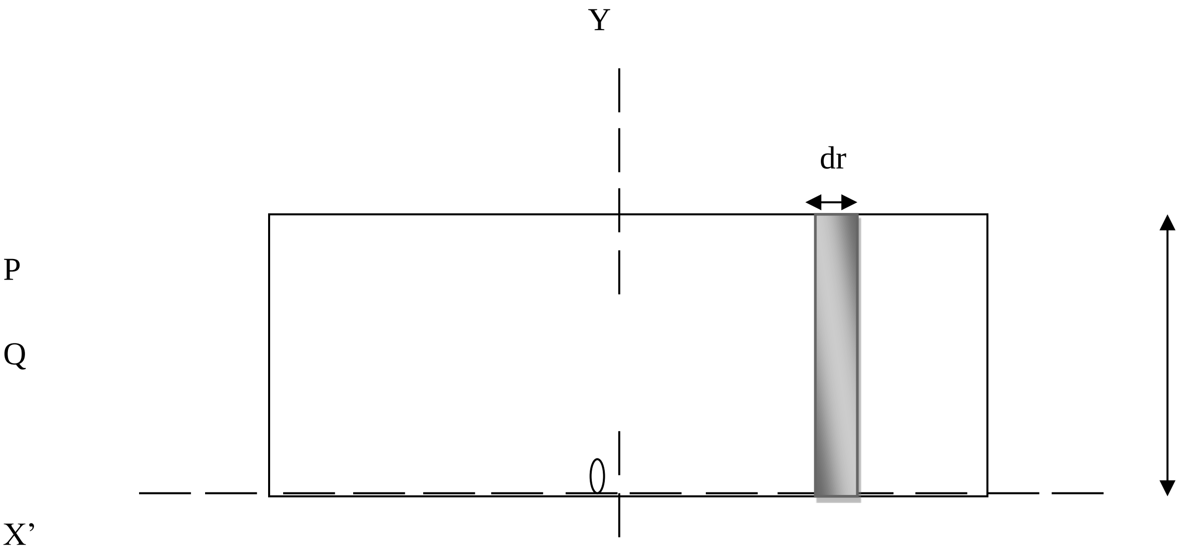
$$= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{3}$$

9.3.2 Moment of Inertia of a Rectangular Lamina

(i) About an axis in its own plane, parallel to one of the sides and passing through the centre of mass

Let PQRS be a rectangular lamina of mass M, length l and breadth b with O as its centre of mass. Let the mass per unit area of the lamina is σ. Let us consider a strip of width dr parallel to the given axis Y' at a distance r from it.



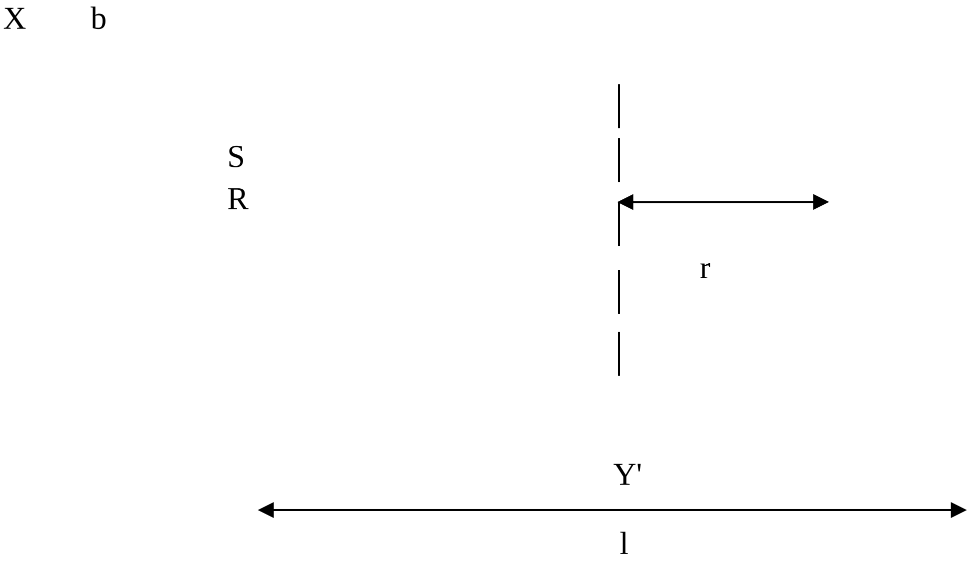


Figure 2

Area of the strip = b dr

Mass of the strip m = (bdr) σ

Moment of inertia of strip about the axis YY' = mr^2

$= (bdr)\,\sigma\, r^2$

$= \sigma\, b\, r^2\, dr$

The moment of inertia of the whole lamina about axis YY′

$I_y = \int_{-l/2}^{l/2} \sigma b r^2 dr = 2\int_0^{l/2} \sigma b r^2 dr$

$= 2\sigma b\int_0^{l/2} r^2 dr = \frac{\sigma b l^3}{12} = \frac{(\sigma bl)l^2}{12}$

i.e. $I_y = \frac{Ml^2}{12}$ …..(1)

where σbl = M, the total mass of the lamina

Similarly, the moment of inertia of the lamina about an axis XX' parallel to the side of length l and passing through the centre will be-

$I_x = \frac{Mb^2}{12}$ …..(2)

(ii) About an axis perpendicular to its plane and passing through the centre of mass

The moment of inertia of the lamina about an axis passing through the centre C and perpendicular to the plane of the lamina

$I = I_x + I_y$ (using theorem of perpendicular axes)

$= \frac{Mb^2}{12} + \frac{Ml^2}{12}$

$= M\left(\frac{l^2+b^2}{12}\right)$ …..(3)

9.3.3 Moment of Inertia of a Circular Lamina

(i) About an axis passing through its centre and perpendicular to its plane

Let O be the centre of the circular lamina and σ the mass per unit area. Let the lamina be supposed to be composed of a number of thin circular strips. Let us consider one such strip of radius r and thickness dr.

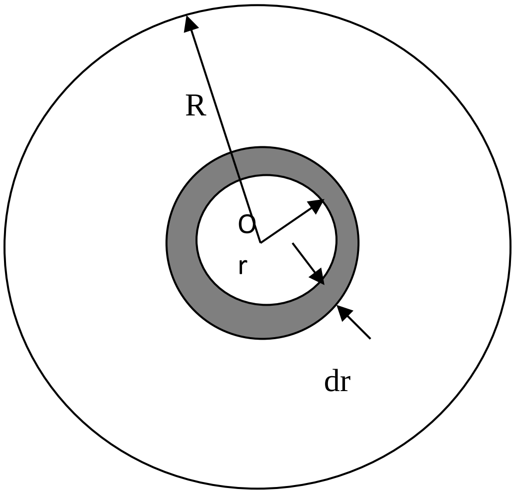


Figure 3

The circumference of the strip = 2π r

Area of the strip = 2π r dr

Mass of the strip m = σ (2π r dr)

The moment of inertia of the strip about an axis passing through O and perpendicular to the plane of the lamina dI = mass × (distance)^2 = m × r^2

$= \sigma(2\pi r\, dr)\, r^2$

$= \sigma\,(2\pi r^3\, dr)$

The moment of inertia of whole lamina $I = \int dI$

$$= \int \sigma\,(2\pi\, r^3\, dr)$$

or $I = 2\pi\sigma \int r^3\, dr$ …..(1)

If circular lamina is solid, then its moment of inertia $I = 2\pi\sigma\int_0^R r^3 dr$

$$= 2\pi\sigma \frac{R^4}{4} = \frac{1}{2}R^2(\pi R^2\sigma)$$

$$= \frac{1}{2} R^2 M$$

where $\pi R^2\sigma = M$, mass of the disc

Thus, $I = \frac{1}{2} M R^2$ …..(2)

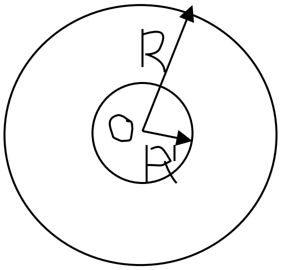
If lamina is having concentric hole and R' and R be the its internal and external radii then the moment of inertia $I = 2\pi\sigma\int_{R'}^{R} r^3 dr$

$$= \frac{2\pi\sigma}{4}[R^4 - R'^4]$$

$$= \frac{2\pi\sigma}{4}(R^2 + R'^2)(R^2 - R'^2)$$

$$= \pi(R^2 - R'^2)\,\sigma \left\{\frac{1}{2}(R^2 + R'^2)\right\}$$

$$= \frac{1}{2} M (R^2 + R'^2)$$



where $\pi(R^2 - R'^2)\,\sigma = M$, mass of the lamina

Figure 4

Thus, $I = \frac{1}{2} M (R^2 + R'^2)$ …..(3)

(ii) About any diameter

Let us consider two mutually perpendicular diameters AB and CD of the lamina. The lamina is symmetrical about both diameters AB and CD.

In accordance with the theorem of perpendicular axis, the moment of inertia about diameter

$I_D = \frac{I}{2}$, where I is the moment of inertia of the disc, about an axis. Through it's center and perpendicular to it's plane.

For solid lamina, $I_D = \frac{1}{2}\left(\frac{1}{2}MR^2\right)$

$$= \frac{1}{4} MR^2$$

For circular lamina with concentric hole,

$I_D = \frac{1}{4} M(R^2 + R'^2)$

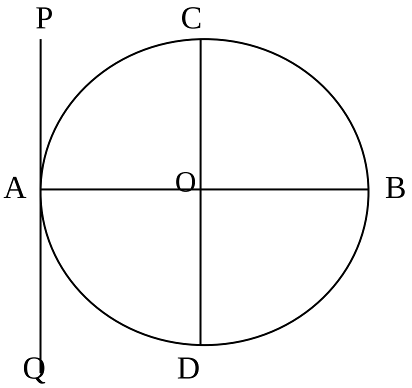


Figure 5

(iii) About a tangent in its plane

Let PQ be a tangent to the lamina in its plane and let it be parallel to the diameter CD.

Using theorem of parallel axes, the moment of inertia of lamina about PQ = Moment of inertia of lamina about CD + $(OA)^2$

or $I_T = I_D + MR^2. M.$

For a solid lamina, $I_T = \frac{1}{4}MR^2 + MR^2$

$= \frac{5}{4}MR^2$

For a lamina having a concentric hole,

$I_T = \frac{1}{4}\ M(R^2 + R'^2) + MR^2$

$= \frac{1}{4}M\ (5R^2 + R'^2)$

(iv) About a tangent perpendicular to its plane

Using theorem of parallel axes, moment of inertia of lamina about a tangent perpendicular to its plane $I'_T = I + MR^2$

For a solid lamina, $I'_T = \frac{1}{2}M\ R^2 + MR^2 = \frac{3}{2}MR^2$

For a circular lamina having concentric hole, $I'_T = \frac{1}{2}M(R^2 + R'^2) + MR^2$

$$= = \frac{1}{2}M(3R^2 + R'^2)$$

9.3.4 Moment of Inertia of a Solid Sphere

(i) About a diameter

Let us consider a sphere of radius R and mass M with centre at O. Let ρ be the density of the material of the sphere.

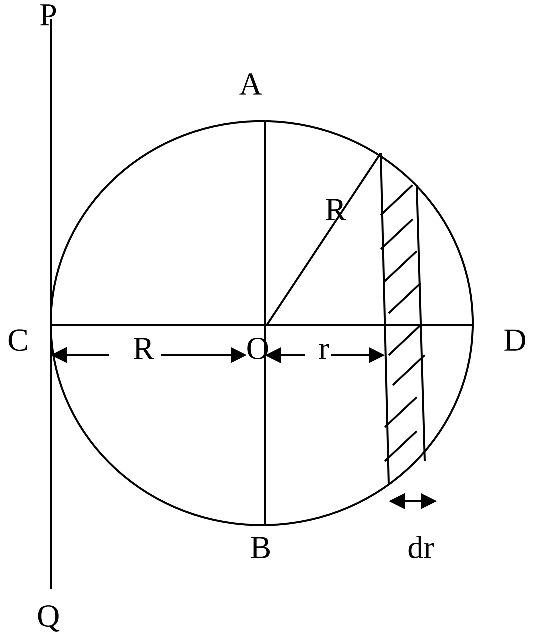


Figure 7

Let us divide the sphere into a number of thin discs by planes perpendicular to the diameter CD and let us consider one such elementary disc of thickness dr at a distance r from O.

Obviously, the radius of the elementary disc = $\sqrt{R^2 - r^2}$

Volume of the elementary disc = its area × its thickness

$= \pi(R^2 - r^2)\, dr$

Mass of the elementary disc m = volume × density

$= \pi(R^2-r^2)\ dr\ \rho$

The moment of inertia of the disc about diameter CD-

$dI = \frac{1}{2}$ [mass of the disc × (radius of the disc)^2]

$= \frac{1}{2}\pi(R^2-r^2)\ dr\ \rho \times (R^2-r^2)$

$= \frac{1}{2}\pi(R^2-r^2)^2\ dr\ \rho$ …..(1)

The moment of inertia of the whole sphere about CD, $I = \int_{-R}^{+R} dI$

$= \int_{-R}^{+R} \frac{1}{2}\pi(R^2-r^2)^2\rho\ dr$

$= \int_{0}^{R} \pi(R^2-r^2)^2\rho\ dr$

$= \pi\rho\int_{0}^{R}(R^4+r^4-\ 2R^2r^2)dr$

$= \frac{8}{15}\pi\rho\ R^5$

$= \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$

$= \frac{2}{5}$ M R^2, where $\frac{4}{3}\pi R^3\rho$ = M, the mass of the sphere

Therefore, $I = \frac{2}{5}$ M R^2

(ii) About a tangent

Let PQ be a tangent to the sphere parallel to the diameter AB and at a distance R (the radius of the sphere) from it.

Using theorem of parallel axes, the moment of inertia of the solid sphere about a tangent PQ-

I_T = moment of inertia of sphere about CD + M×(OC)^2

$= \frac{2}{5} M R^2 + MR^2 = \frac{7}{5} MR^2$

9.3.5 Moment of Inertia of a Solid Cylinder

Let us consider a solid cylinder of mass M, radius R and length L.

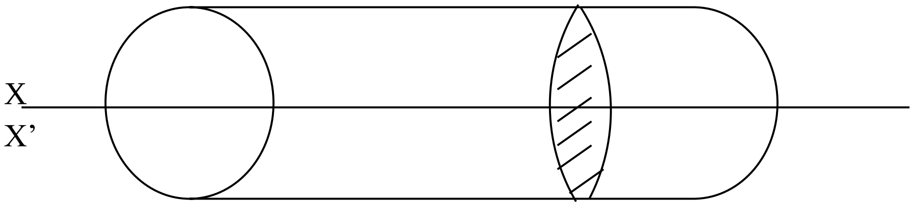


Figure 8

Volume of the cylinder = $\pi R^2 L$

Density of the cylinder $\rho = \frac{M}{\pi R^2 L}$

or $M = (\pi R^2 L)\rho$

(i) Moment of inertia of solid cylinder about geometrical axis

Let us suppose that the cylinder be formed of a large number of co-axial discs of equal radius R, the axis being the geometrical axis (XX') of the cylinder. Let us consider one such disc of mass m and radius R.

Moment of inertia of the disc about geometrical axis XX' = $\frac{1}{2} m R^2$

Therefore, the moment of inertia of the whole cylinder about its geometrical axis XX',

$I_1 = \Sigma\left(\frac{1}{2} mR^2\right)$

$= \frac{1}{2} R^2 \Sigma m$

$= \frac{1}{2} R^2 M$

where Σm = M, the mass of the cylinder

Therefore, $I_1 = \frac{1}{2} MR^2$ …..(1)

(ii) Moment of inertia about an axis passing through centre and perpendicular to geometrical axis

Let us consider an axis YY' perpendicular to geometrical axis XX' and passing through centre O of the cylinder. Let us suppose that the cylinder to be formed of coaxial discs of equal radius R.

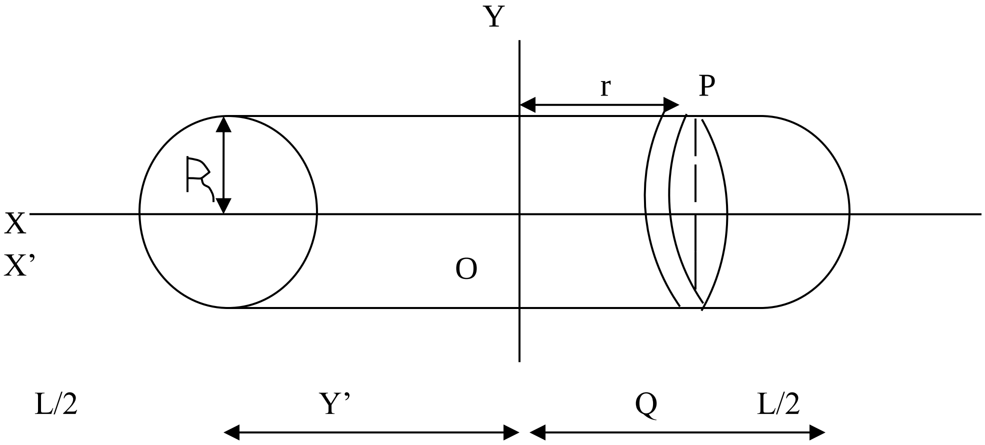


Figure 9

Let us consider one such disc of radius R and thickness dr at a distance r from the axis.

Area of disc = πR^2

Volume of the disc = $(\pi R^2)\,dr$

Mass of the disc $m = (\pi R^2)\,dr\,\rho$

Moment of inertia of disc about its diameter PQ = $\frac{1}{4}$ [mass of the disc × (radius of the disc)^2]

$$= \frac{1}{4}(\pi R^2\,dr\,\rho)\,R^2$$

$$= \frac{1}{4}(\pi R^4\,\rho\,dr)$$

Using theorem of parallel axes, the moment of inertia of the disc about axis YY' –

dI_2 = (Moment of inertia of disc about its diameter PQ) + $m r^2$

$$= \frac{1}{4}\pi R^4\rho\,dr + (\pi R^2\,dr)\,\rho\,r^2$$

$$= \pi R^2\rho\left(\frac{R^4}{4} + r^2\right)dr$$

The moment of inertia of whole cylinder about axis YY'-

$$I_2 = \int_{-L/2}^{+L/2} dI_2$$

$$= \int_{-L/2}^{+L/2} \pi R^2\rho\left(\frac{R^4}{4} + r^2\right)dr$$

$$= 2\,\pi R^2\rho\int_0^{L/2}\left(\frac{R^4}{4} + r^2\right)dr$$

$$= (\pi R^2 L\rho)\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

$$= M\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

where $\pi R^2 L\rho = M$, the mass of cylinder

Therefore, $I_2 = M\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$

Example 1: The mass and radius of a solid circular disc are 500 Kg and 1 metre respectively. Calculate its moment of inertia about its axis.

Solution: The mass of circular disc M = 500 Kg

The radius of circular disc R = 1 m

Moment of inertia of the disc about its axis $I = \frac{1}{2}MR^2$

$$= \frac{1}{2}(500)(1)^2 = 250 \text{ Kg-m}^2$$

Example 2: Determine the moment of inertia of a rectangular lamina of 2 Kg about an axis perpendicular to its plane and passing through the centre of mass. The length and breadth of lamina are 100 cm and 50 cm respectively.

Solution: Mass of lamina M = 2 Kg, l = 100 cm = 1 m, b = 50 cm = 0.5 m

Moment of inertia of lamina about an axis perpendicular to its plane and passing through the centre of mass $I = M\left(\frac{l^2+b^2}{12}\right)$

$$= 2\left[\frac{1^2 + 0.5^2}{12}\right] = 0.21 \text{ Kg-m}^2$$

Self Assessment Question (SAQ) 1: The moment of inertia of a disc about a tangent perpendicular to the plane of disc is-

(a) $\frac{3}{2}$ M R^2 (b) $\frac{5}{4}$ M R^2 (c) M R^2 (d) $\frac{1}{2}$ M R^2

Self Assessment Question (SAQ) 2: The moment of inertia of a thin rod of mass M and length L about an axis passing through one end and perpendicular to length is-

(a) ML^2 (b) $\frac{ML^2}{12}$ (c) $\frac{ML^2}{4}$ (d) $\frac{ML^2}{3}$

9.4 SUMMARY

In the present unit, we have formulated and derived the expressions for moment of inertia of thin uniform rod, rectangular lamina, circular lamina, solid sphere and solid cylinder. Moment of inertia of a thin uniform rod about an axis passing through its centre of mass and perpendicular to its length is expressed as $\frac{ML^2}{12}$, where M is the mass of the rod and L its length while the moment of inertia of the rod about an axis passing through its one end perpendicular to its length is derived as $\frac{ML^2}{3}$. The moment of inertia of a rectangular lamina of mass M, length l and breadth b about an axis perpendicular to its plane and passing through the centre of mass is given as $M\left(\frac{l^2+b^2}{12}\right)$.We have established the expression for moment of inertia of a circular lamina of mass M and radius R about an axis passing through its centre and perpendicular to its plane as $\frac{1}{2}$ M R^2. We have formulated the moment of inertia of a solid sphere about a diameter as $\frac{2}{5}$ M R^2 and about a tangent as $\frac{7}{5}$ M R^2. We have also derived the formulae for moment of inertia of a solid cylinder. After studying the unit, we can solve problems based on moment of inertia. We have also included examples and self assessment questions (SAQs) in the unit to check your progress.

9.5 GLOSSARY

Continuous- unbroken

Homogeneous- uniform

Infinitesimal- microscopic, extremely small

Appropriate- suitable, proper

Uniform- consistent, homogeneous

9.6 TERMINAL QUESTIONS

1. Derive the ratio of moment of inertia of a rectangular bar about an axis passing through one of its ends and through its centre.
2. Establish the expression for moment of inertia of a rectangular lamina about an axis in its own plane parallel to one of the sides and passing through the centre of mass.
3. Calculate the moment of inertia of a solid sphere about (i) a diameter and (ii) a tangent
4. Calculate the moment of inertia of a circular lamina about an axis passing through its centre and perpendicular to its plane. The mass of the lamina is 300 gm and radius 50 cm.
5. Calculate the moment of inertia of a solid cylinder about its own axis.

9.7 ANSWERS

Self Assessment Questions (SAQs):

1. Using theorem of parallel axis, $I = \frac{1}{2} M R^2 + M R^2$

$$= \frac{3}{2} M R^2$$

Hence option (a) is correct

2. Option (d) is correct

Terminal Questions:

4. Mass of lamina M = 300 gm = 0.3 Kg, Radius R = 50 cm = 0.5 m

Required moment of inertia $I = \frac{1}{2} M R^2$

$$= \frac{1}{2} \times 0.3 \times 0.5$$

$$= 0.075 \text{ Kg-m}^2$$

9.8 REFERENCES

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UNIT 10 - SIMPLE HARMONIC MOTION

CONTENTS:

10.1 INTRODUCTION

Any motion which repeats itself after regular interval is called *periodic or harmonic motion* and the time interval after which the motion is repeated (i.e. the position and the velocity of the moving body is the same) is called its time period. Some examples of periodic motion include (see Fig. 1)

- motion of planets around the sun,
- motion of a piston inside a cylinder, used in automobile engines, or

- motion of a ball in a bowl.

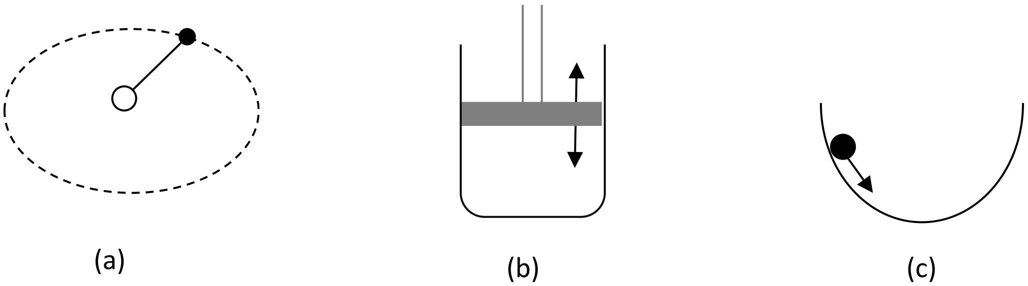

Figure 1: Some examples of periodic motion: (a) motion of the earth around the sun, or moon around the earth; (b) motion of a piston in a cylinder which is used in automobile engines; (c) motion of a ball in a bowl.

If in case of periodic motion, the body moves back and forth repeatedly about a fixed position (called equilibrium or mean position), the motion is said to be *oscillatory or vibratory*. For instance, the motion of the earth around the sun or the motion of the hands of the clock, are examples of periodic motion, but they are not oscillatory in nature. The motion of piston in an automobile engine, motion of a ball in a bowl, motion of needle of sewing machine or the bob of a pendulum clock are all examples of oscillatory motion.

An oscillating body is said to execute *simple harmonic motion* (SHM) if the magnitude of the forces acting on it is directly proportional to the magnitude of its displacement from the mean position and the force (called restoring force) is always directed towards the mean position. Thus, we can see that simple harmonic motion or SHM is actually a special case of oscillatory or vibratory motion. We will study SHM in detail in this unit. Some examples of simple harmonic motion include (see Fig. 2)

- motion of a simple pendulum,
- a vibrating tuning fork, or
- a spring-mass system.

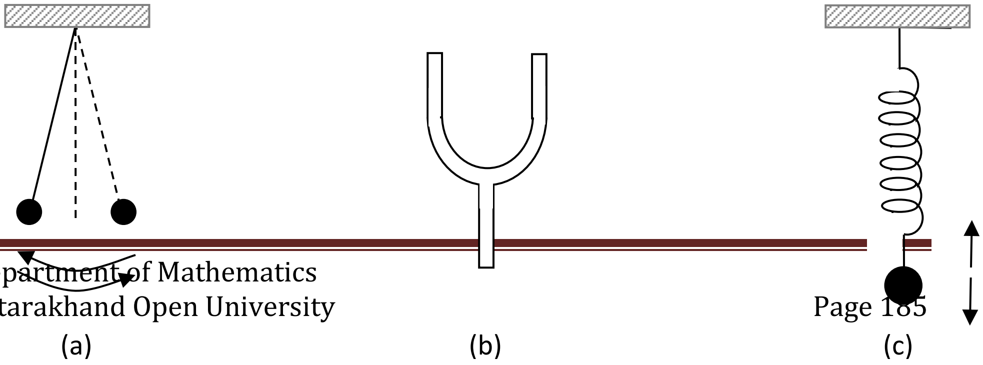

Figure 2: Some examples of SHM: (a) A simple pendulum; (b) a vibrating tuning fork; (c) an oscillating spring-mass system.

10.2 OBJECTIVES

After studying this unit, you should be able to

- describe examples of oscillating systems
- explain what is meant by simple harmonic motion
- explain what is meant by the amplitude and the time period of an oscillating system
- write down the general equation of simple harmonic motion and solve it
- describe how the acceleration, velocity and displacement of an oscillating system change with time
- define angular frequency
- differentiate between vertical and horizontal spring-mass systems
- calculate the time period for composite spring-mass systems

10.3 SIMPLE HARMONIC MOTION

In the previous Section, we discussed two examples of oscillatory motion. Let us now use the spring-mass system to understand simple harmonic motion (SHM). What is SHM? Let us first answer this question.

10.3.1 Definition of SHM

SHM can be defined in a number of ways:

1. If the force acting on the oscillating body is always in the direction opposite to the displacement of the body from the equilibrium or the mean position and its magnitude is proportional to the magnitude of displacement, the body is said to be executing SHM.

2. If the displacement vs. time curve of the oscillating body is sinusoidal in nature, the body is said to be executing SHM. This is another definition of SHM.
3. If the potential energy of the oscillating body is proportional to the square of its displacement with reference to the mean position, the body is said to be executing SHM. This is yet another definition of SHM.

Let us consider the first definition for now. The spring-mass system shown in Fig. 4 (a) is in the position of static equilibrium: the spring is relaxed (neither stretched nor compressed) and there is no force acting on the body. When the body is pulled to the right through a small distance x, the spring starts behaving like an elastic system under stress. You may recall the Young's modulus experiment that you did in your school. We know that if a wire of length L is stretched through a distance x by a force of magnitude F, the Young's modulus Y of the material of the wire is given by

$$Y = \frac{F/\alpha}{x/L} \tag{1.1}$$

Here α is the cross-sectional area of the wire. By rearranging the terms in equation (1.1), we can easily get the following form

$$F = \left(\frac{Y\alpha}{L}\right)x \tag{1.2}$$

We already know that elasticity is the property by virtue of which a body offers resistance to any change in its size or shape or both and makes the body regain its original condition when the deforming force, applied within a certain maximum limit, is removed. In other words, one can say that in the deformed condition, the body develops a restoring force and according to the Newton's third law of motion, this force is equal in magnitude but opposite in direction to the deforming force. Equation (1.2) implies that the restoring force is proportional to the elongation and is directed towards the equilibrium position (relaxed position when there is no restoring force acting).

Similarly, for the spring-mass system, Hooke's Law states that the restoring force is proportional to the displacement of the spring in case of stretched as well as compressed configurations. In our case, the restoring force exerted by the spring on the body is directed to the left [see Fig. 4 (b)] and is given by the following relation:

$$F = -kx \tag{1.3}$$

Since, the restoring force, F is proportional to the displacement[1] and is opposite in sign to the displacement, the resulting motion is simple harmonic. Here k is called the spring constant or stiffness constant. The SI unit of k is Nm^{-1}.

Example 1: If, in a spring-mass system as shown in Fig. 4, the spring constant is 50 Nm^{-1} and the block of mass 1 kg is displaced by 0.01 m to the right before being released, calculate the

(a) restoring force at t = 0,
(b) restoring force when the block travels to the other extreme, and
(c) The restoring force in the static equilibrium position.

Solution:

(a) If x is taken as positive to the right of the mean position, then the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(0.01\text{ m}) = -0.5\ N$$

(b) Similarly, the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(-0.01\text{ m}) = +0.5\ N$$

(c) At the mean position, $x = 0$

$$F = -(50\ Nm^{-1})(0) = 0$$

Self Assessment Question (SAQ) 5: If the displacement x in the above example is halved, how will be the restoring force change in all the three cases.

Self Assessment Question (SAQ) 6: What will happen if instead the initial displacement is doubled?

Self Assessment Question (SAQ) 7: Will the answer change if the mass of the block in the above example is changed to 5 kg?

10.3.2 Basic Characteristics of SHM

Since we now know what SHM is, let us define some of the basic characteristics of SHM. What comes to your mind? The first important

---

[1] The relationship (Eq. (1.3)) is linear only for small values of displacement, x and the elastic force produced in the linear spring is given by $F = -kx$, where x is the change in the length of the spring.

characteristic in SHM is the initial displacement that actually results in oscillations in the first place. The magnitude of the initial displacement, which is also the maximum displacement, is called the ***amplitude (A)*** of oscillations. As we mentioned before, the energy of the system executing SHM alternates between kinetic and potential forms. At the extremities of the oscillations, the kinetic energy is zero as the velocity is zero and the potential energy is the maximum.

Another characteristic of SHM is the ***time period (T)*** which is the time taken for one complete cycle of oscillation. This is the least time taken by an oscillating object to move from a certain position and velocity back to the same position and velocity. Generally, for convenience, we measure the time period from either the mean position or the extreme ends.

Instead of time period, many a times we talk in terms of the ***frequency (ν)*** to characterize SHM. Frequency is the number of complete oscillations executed per second and is the inverse of the time period, i.e.

$$\nu = \frac{1}{T} \tag{1.4}$$

It is expressed in cycles per second or simply s^{-1} or hertz (Hz). We also define a term called ***angular frequency***, denoted by ***ω***, which is given by

$$\omega = 2\pi\nu \tag{1.5}$$

It is expressed in radian per second or simply$rad\ s^{-1}$, since 2π is the angle around a circle in radians and T is in seconds.

Example 2: A mass on a spring oscillates along a vertical line, taking 12 s to complete 10 oscillations. Calculate the

(a) time period, and
(b) the angular frequency.

Solution:

(a) Time period is the time taken for one complete cycle of oscillation; therefore, to complete one oscillation, time needed will be

$$T = \frac{(12\ s)}{(10\ oscillations)} = 1.2\ s$$

(b) The frequency is given by

$$\nu = \frac{1}{T} = \frac{1}{1.2}\ Hz$$

Therefore, the angular frequency is

$$\omega = 2\pi\nu = \frac{2\pi}{1.2} = 5.23\, rad\ s^{-1}$$

Example 3: The motion of a vibrating blade is frozen (when the frequency of the vibrating blade becomes equal to the stroboscope frequency) by illuminating it with a stroboscope (a flashing light). The least stroboscope frequency at which this occurs is 40 Hz. Calculate the

(a) time period, and
(b) the angular frequency of the vibrations.

Solution:

(a) Time period is the inverse of frequency

$$T = \frac{1}{\nu} = \frac{1}{40} = 0.025\ s$$

(b) The angular frequency is given by

$$\omega = 2\pi\nu = 2\pi(40) = 251.2\, rad\ s^{-1}$$

Self Assessment Question (SAQ) 9: An object executes simple harmonic motion with an angular frequency of $1.26 rad\ s^{-1}$. Calculate its time period.

Self Assessment Question (SAQ) 10: If the angular frequency ω is one revolution per minute. Calculate its time period. [Hint: One revolution = (2π) radians]

10.4 DIFFERENTIAL EQUATION OF SHM

Let us now express equation (1.3) in the differential form by using Newton's second law of motion. From Newton's second law of motion, we know that force experienced by a body of mass *m* can be expressed as a function of acceleration,

$$F = ma = m\ddot{x}$$

Therefore, in a spring-mass system, the force can be written as

$$F = m\ddot{x} = -kx$$

Or we can say that

$$m\ddot{x} + kx = 0$$

$$or, \qquad \ddot{x} + \frac{k}{m}x = 0 \qquad (1.6)$$

(**Comment**: Either follow the double dot notation or d^2x/dt^2 notation for double differentiation. In later Units, d^2x/dt^2 notation has been used. For students, it will be better if we follow d^2x/dt^2 notation.)

The above equation is the differential equation of SHM. k is the force constant (for our case of spring-mass system, it is called the spring constant) and has dimensions (MLT^{-2}/L) = MT^{-2}. Therefore, the dimension of k/m is T^{-2}, i.e. square of reciprocal of time. We can replace k/m by ω^2. Thus, the equation (1.6) takes the form

$$\ddot{x} + \omega^2 x = 0 \qquad (1.7)$$

We will find the physical meaning of ω, that it is actually the angular frequency that we already defined earlier, when we solve the differential equation (1.7).

10.4.1 Solution of the Differential Equation of SHM

The second time derivative of displacement ($\ddot{x}$) can be written as

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

Multiplying and dividing by dx in the numerator and the denominator, we get

$$\ddot{x} = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)$$

We already know that $\dot{x}$ or dx/dt actually define the velocity$v$. Therefore, the above expression can take the following form

$$\ddot{x} = v\frac{d}{dx}(v)$$

Since,

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx}$$

We get

$$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) \qquad (1.8)$$

From (1.7) and (1.8), we get

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) + \omega^2 x = 0$$

$$or \;\; \frac{d}{dx}\left(\frac{v^2}{2} + \omega^2\frac{x^2}{2}\right) = 0$$

$$\therefore d(v^2 + \omega^2 x^2) = 0 \qquad (1.9)$$

On integrating both the sides, we get

$$v^2 + \omega^2 x^2 = constant\ (C_1) \qquad (1.10)$$

We already know that on the two extremes, when the magnitude of the displacement is equal to the amplitude ($x = \pm A$), the kinetic energy or the velocity is zero ($v = 0$). Using this boundary condition in equation (1.10), we can calculate the constant (C_1). Thus, C_1 is given by

$$(0)^2 + \omega^2(\pm A)^2 = C_1$$

$$or \;\; C_1 = \omega^2 A^2$$

Using this value in equation (1.10) and rearranging the terms, we get

$$v^2 = \omega^2(A^2 - x^2)$$

$$or \;\; v = \pm\omega\sqrt{(A^2 - x^2)} \qquad (1.11)$$

The above relation is the expression for velocity of a particle executing SHM. We can see how the velocity has a maximum magnitude at $x = 0$ or in other words, the mean position. From (1.11), the maximum velocity is given by

$$|v|_{max} = \omega A \qquad (1.12)$$

Example 4: A 50 g mass vibrates in SHM at the end of a spring. The amplitude of the motion is 12 cm and the period is 0.1 minutes. Find the maximum speed of the mass. What will be the speed at $x = A/2$?

Solution:

$$\omega = 2\pi\nu = 2\pi\left(\frac{1}{0.1 \times 60\ s}\right) = 1.047\ rad\ s^{-1}$$

$$\therefore |v|_{max} = \omega A = (1.047\ rad\ s^{-1})(12 \times 10^{-2}\ m)$$

$$= 0.1256\ m/s$$

From equation (1.11), we get

$$|v| = \omega\sqrt{A^2 - \left(\frac{A}{2}\right)^2} = \frac{3}{4}\omega A$$

$$= \frac{3}{4}(1.047\ rad\ s^{-1})(12 \times 10^{-2}\ m) = 0.0942\ m/s$$

Self Assessment Question (SAQ) 11: In the above question, calculate the speed at $x = 1\ cm$.

Self Assessment Question (SAQ) 12: In the above question, at what location will the speed of the vibrating mass be 5 cm/s?

Now, we will determine the expression for the displacement of a particle executing SHM. From (1.11), we get

$$\frac{dx}{dt} = \pm\omega\sqrt{(A^2 - x^2)}$$

Rearranging the terms, we get

$$\pm\frac{dx}{\sqrt{(A^2 - x^2)}} = \omega dt$$

On integrating both the sides, we get corresponding to the $(+)$ sign

$$\sin^{-1}\frac{x}{A} = \omega t + \delta_1$$

And, corresponding to the $(-)$ sign

$$\cos^{-1}\frac{x}{A} = \omega t + \delta_2$$

where δ_1 and δ_2 are dimensionless constants.

Therefore, we can see that the SHM is defined by a sinusoidal curve

$$x(t) = A\sin(\omega t + \delta) \qquad (1.13)$$

Depending on the value of constant δ and ωt the displacement from the equilibrium position and velocity of the SHM at any instant can be determined.

10.4.2 Angular Frequency of SHM

We know that the displacement $x(t)$ should return to its initial value after one time period T of the motion. Or

$$x(t) = x(t+T)$$

We also know from trigonometry that the sine or cosine function repeats itself when its argument has increased by $2\pi\ rad$. Thus,

$$\omega(t+T) = \omega t + 2\pi$$

Or, we get

$$\omega = \frac{2\pi}{T} = 2\pi\nu \qquad (1.14)$$

The quantity ω is therefore, the angular frequency that we defined earlier. Its SI unit is $rad\ s^{-1}$.

From equation (1.6), we know that

$$\omega^2 = \frac{k}{m}$$

$$\therefore\ \omega = \sqrt{\frac{k}{m}} \qquad (1.15)$$

Example 5: A particle of mass 0.2 kg undergoes SHM according to the equation: $x(t) = 3\sin(\pi t + \pi/4)$. [$t$ is in s and *x* in m]

(a) What is the amplitude of oscillation?
(b) What is the time period of oscillation?
(c) What is the initial value of *x*?
(d) What is the initial velocity when the SHM starts?
(e) At what instants is the particle's energy purely kinetic?

Solution:

(a) Comparing the given equation with $x(t) = A\sin(\omega t + \delta)$, we get the amplitude, $A = 3\ m$.

(b) On comparing, we get $\omega = \pi\ rad\ s^{-1}$. Therefore, from (1.14), we get the time period as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\ s$$

(c) Initial conditions are at $t = 0$

$$x(0) = 3\sin(\pi/4) = 1.5\sqrt{2}\ m$$

(d)

$$\frac{dx}{dt} = v(t) = 3\pi \sin(\pi t + \pi/4)$$

$$v(0) = 3\pi \sin(\pi/4) = \frac{3\pi}{\sqrt{2}}\ m/s$$

(e) The energy is purely kinetic when the particle is at the mean position, i.e. when $x(t) = 0$. Or

$$0 = 3\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$\therefore \pi t + \frac{\pi}{4} = 0, \pi, 2\pi, 3\pi, \ldots$$

$$i.e.\ \ t = -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \ldots$$

Rejecting the negative value of *t*, we get *t* = 3/4, 7/4, 11/4… At these instants, the particle crosses origin and hence its energy is purely kinetic.

Self Assessment Question (SAQ) 13: How are the following characteristics of SHM affected by doubling the amplitude? Explain.

(a) Time period, and (b) maximum velocity.

Self Assessment Question (SAQ) 14: Choose the correct option:

Which of the following functions represent SHM?

(a) $\sin(2\omega t)$ (b) $\sin^{-1}\omega t$ (c) $\sin(\omega t) + 2\cos(\omega t)$ (d) $\sin(\omega t) + \cos(2\omega t)$

10.5 SUMMARY

In this unit, we have studied about what is meant by the periodic motion, the oscillatory motion and what are the conditions and basic characteristics of SHM. We studied about the restoring force that comes in to play due to the displacement from the mean or the equilibrium position and how the restoring force is proportional to the magnitude of the displacement in case of SHM. We studied the two simple systems, simple pendulum and spring-mass system, which are both examples of SHM. We are also aware now that the two properties of inertia and elasticity are responsible for oscillation of a physical system.

10.6 GLOSSARY

Displacement – net change in location of a moving body; in case of SHM, it is measured from the equilibrium position.

Force – anything that can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a vibrating object.

Inertia – the tendency of a physical object to remain still or to continue moving, unless a force is applied to it.

Sound – vibrations in a substance that travel through the substance.

Stiffness – a measure of the force needed to change the shape of an object.

Velocity – speed in a given direction.

10.7 TERMINAL QUESTIONS

1. A horizontal spring-mass system of spring constant k and mass M executes SHM with frequency ν. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Find the new frequency of vibration.

2. A particle executes SHM with amplitude of 0.5 cm and frequency of 100 s^{-1}. What is the maximum speed of the particle?

3. A weight suspended from a spring oscillates up and down. The restoring force in the weight is zero at (a) highest point, (b) lowest point, (c) middle point, (d) none of these.

4. A person goes to bed at sharp 10:00 pm every day. Is it an example of periodic motion? If yes, what is the time period? If no, why?

5. In the above question, is it an example of SHM? If yes, why?

10.8 ANSWERS

Selected Self Assessment Questions (SAQs):

4. (a)

7. No.

10. $\omega = 2\pi/60 \;\; rad/s$ and $T = 1/30 \; s$

13. Period remains unchanged. Maximum velocity is doubled.

14. (a)

15. Hint: The maximum tension in the spring will be when it is stretched to the extreme, which is equal to the sum of the difference of the relaxed length and the equilibrium length of the spring, and the amplitude of the oscillations; i.e. $d + A = (T_0 + \Delta T)/k.$

16. $k = 300 \; Nm^{-1}$

17. The time period for such a system is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{m(\frac{1}{k_1} + \frac{1}{k_2})}$$

Selected Terminal Questions:

1. Original frequency of SHM,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

The new frequency of SHM,

$$\nu_{new} = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}$$

Therefore,

$$\nu_{new} = \nu\sqrt{\frac{M}{m+M}}$$

2. $|v|_{max} = \omega A = (2\pi \times 100)(0.5 \times 10^{-2}) = \pi \; m/s$

3. (c) because at the equilibrium or mean position the restoring force is zero.

4. Yes. Time period = 24 hours.

5. No. SHM is a special case of oscillatory motion, where a body moves back and forth repeatedly about a fixed position. Here nothing like that happens!

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UNIT 11: CONSTRAINED MOTION ON A SMOOTH PLANE CURVES

CONTENTS:

11.1 INTRODUCTION

The study of motion of a particle is a fundamental part of classical mechanics and provides the basis for understanding more complex motions encountered in physical systems. In this chapter, we focus on the motion of a particle under gravity, particularly when it moves along a smooth vertical circular path and along a **c**ycloi**d**, which is an important curve in applied mechanics. The motion of a particle in a smooth vertical circle and along a cycloid provides deep insight into the interplay between force, energy, and geometry. By applying the conservation of energy and understanding constrained motion, students gain a strong conceptual and mathematical foundation essential for higher studies in mechanics.

11.2 OBJECTIVES

After studying this unit learner will able

1. To Understand the concept of motion of a particle and distinguish it from the motion of rigid bodies.

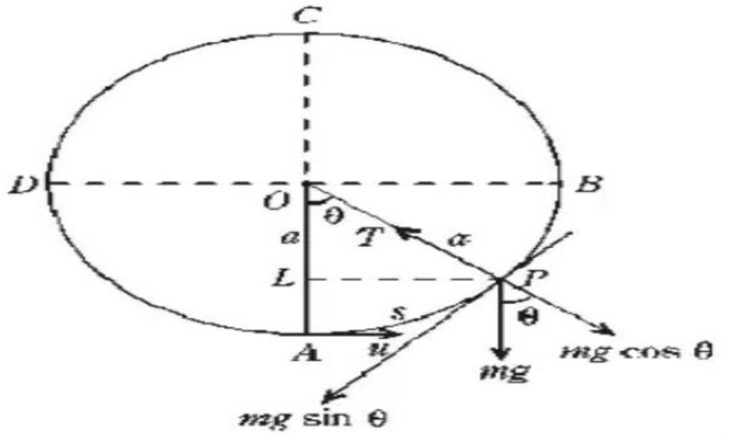
2. To Explain the motion of a particle in a smooth vertical circle**,** identifying the forces acting on the particle at different points of the path.

3. To Analyse the variation of speed of a particle at the highest, lowest, and any intermediate point of a vertical circle.

11.3 MOTION IN A SMOOTH VERTICLE PLANE

A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium; to discuss the subsequent motion.

Let one end of a string of length a be attached to the fixed point O and a particle of mass m be attached at the other end A. Let OA be the vertical position of equilibrium of the string. Let the particle be projected horizontally from A with velocity u. Since the string is inextensible, the particle starts moving in circle whose centre is O and radius a. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc$AP = s$, the forces acting on the particle at P are :



(i) weight mg of the particle acting vertically downwards,
(ii) tension T in the string acting along PO.

If v be the velocity of the particle at P, the tangential and normal accelerations of P are

$$\frac{d^2s}{dt^2} \text{ (in the direction of } s \text{ increasing)}$$

and $\frac{v^2}{\rho}$(along inwards drawn normal at P).
∴ the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \tag{1}$$

and

$$m\frac{v^2}{\rho} = T - mg\cos\theta. \tag{2}$$

Also $s = \text{arc}AP = a\theta$.

$$\therefore\ v = \frac{ds}{dt} = a\frac{d\theta}{dt} \text{ and } \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2} \tag{3}$$

∴ from (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying both sides by $2a\frac{d\theta}{dt}$ and integrating w.r.t. ' t ', we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

where A is constant of integration.
But initially at $A, \theta = 0, v = u$.

$$\begin{aligned} \therefore\quad & A = u^2 - 2ag\cos 0 = u^2 - 2ag \\ \therefore\quad & v^2 = u^2 - 2ag + 2ag\cos\theta \end{aligned} \tag{4}$$

Now for a circle $\rho = a$ (radius).
∴ from (2), we have

$$T = \frac{m}{a}v^2 + mg\cos\theta = \frac{m}{a}(v^2 + ag\cos\theta)$$

Substituting the value of v^2 from (4), we have

If the velocity $v = 0$ at $\theta = \theta_1$, then from (4), we have

$$0 = u^2 - 2ag + 2ag\cos\theta_1$$

or $\cos\theta_1 = \frac{2ag-u^2}{2ag}$.

If h_1 is the height from the lowest point A of the point where the velocity vanishes, then

$$h_1 = OA - a\cos\theta_1 = a - a\cdot\frac{2ag-u^2}{2ag}$$

or $h_1 = \frac{u^2}{2g}$.

Again, if the tension $T = 0$, at $\theta = \theta_2$, then from (5), we have

$$0 = u^2 - 2ag + 3ag\cos\theta_2$$

$$\therefore \quad \cos\theta_2 = \frac{2ag-u^2}{3ag} \tag{8}$$

If h_2 is the height from the lowest point A of the point where the tension vanishes, then

$$h_2 = OA - a\cos\theta_2 = a - a\cdot\frac{2ag-u^2}{3ag}$$

or

$$h_2 = \frac{u^2+ag}{3g} \tag{9}$$

Now the following cases may arise here:
Case I. The velocity $\boldsymbol{v}$ vanishes before the tension T.

This is possible if and only if $h_1 < h_2$
or $\frac{u^2}{2g} < \frac{u^2+ag}{3g}$ or $3u^2 < 2(u^2 + ag)$
or $u^2 < 2ag$ or $u < \sqrt{(2ag)}$.
But when $u < (2ag)$, we have from (6), $\cos\theta_1 = +$ ive i.e., θ_1 is an acute angle.
Thus if the particle is projected with the velocity $u < \sqrt{(2ag)}$., then

it will oscillate about A and will not rise upto the horizontal diameter through O.

Case II. The velocity $\boldsymbol{v}$ and the tension $\boldsymbol{T}$ vanish simultaneously.
This is possible if and only if $h_1 = h_2$
i.e., $\frac{u^2}{2g} = \frac{u^2+ag}{3g}$ i.e., $u^2 = 2ag$ or $u = \sqrt{(2ag)}$.
Also when $u = \sqrt{(2ag)}$, we have from (6) and (8),

$$\theta_1 = \pi/2 = \theta_2.$$

Thus if the particle is projected with the velocity $u = \sqrt{(2ag)}$ .then it will rise upto the level of the horizontal diameter through O and will oscillate about A in the semi-circular arc BAD.

Case III. Condition for describing the complete circle.
At the highest point C, we have $\theta = \pi$. Therefore from (4) and (5), we have at C, At the highest point C, we have $\theta = \pi$. Therefore from (4) and (5), we have at C,

$$v^2 = u^2 - 4ag \text{ and } T = \frac{m}{a}(u^2 - 5ag)$$

If $u^2 > 5ag$ i.e., if $u > \sqrt{(5ag)}$, then neither the velocity v nor the tension T is zero at the highest point C, and so the particle will go on describing the complete circle. And if $u^2 = 5ag$ i.e., if $u = \sqrt{(5ag)}$, then at the highest point C the tension T vanishes whereas the velocity does not vanish.

Hence in this case the string will become momentarily slack at C and the particle will go on describing the complete circle.
Thus the condition for describing the complete circle by the particle is that $u \geq \sqrt{(5ag)}$. In other words, the least velocity of projection for describing the complete circle is $\sqrt{(5ag)}$.

Case IV. The tension T vanishes before the velocity $\boldsymbol{v}$
This is possible if and only if $h_1 > h_2$
i.e., $\frac{u^2}{2g} > \frac{u^2+ag}{3g}$ i.e., $u^2 > 2ag$ or $u > \sqrt{(2ag)}$.
When $u > \sqrt{(2ag)}$, we have from (8), $\cos\theta_2 = -$ ive showing that θ_2 must be $> 90°$.

Now at the point where the tension T is zero, the string becomes slack. Since the velocity v is not zero at that point, therefore the particle will leave the circular path and trace a parabolic path while moving freely under gravity.
Thus if the particle is projected with the velocity usuch that $\sqrt{(2\text{ ag })} < u < \sqrt{(5\text{ ag })}$,then it will leave the circular path at a point somewhere between B and C and trace out a parabolic path.

11.4 DISCUSS THE MOTION OF A PARTICLE

In mechanics, motion of a particle refers to the study of how a body changes its position with time. A particle is an idealized object that has mass but negligible size, so its motion can be completely described by the motion of a single point.

11.5USE OF PRINCIPLE OF CONSERVATION OF ENERGY

The principle of conservation of energy is one of the most powerful tools in classical mechanics and is extensively used to analyze the motion of particles and systems when non-conservative forces like friction are absent**.**

Applications and Use

1. Motion in a Smooth Vertical Circle: As the particle moves upward, kinetic energy decreases **and** potential energy increases. As it moves downward, potential energy decreases **and** kinetic energy increases**.**

The principle is used to: Find the velocity of the particle at any point of the circle. Determine the minimum velocity required at the lowest point to complete the circle. It avoids complicated force resolution along the tangent.

2. Motion Under Gravity: Used in problems of: Free fall, Vertical projection, Motion on smooth inclined planes, Helps in calculating: Maximum height attained, Velocity at any given point

3. Motion Along Curved Paths: Effective in analysing motion along smooth curves like: Circular paths, Cycloids, particularly useful when forces vary with position.

11.6 CYCLOIDAL MOTION

A cycloid is the locus of a point on the circumference of a circle of radius a rolling on a horizontal straight line. Cycloidal motion refers to the motion of a particle along a cycloid, which is the curve traced by a fixed point on the circumference of a circle as it rolls along a straight line without slipping. Cycloidal motion is an important topic in classical mechanics due to its remarkable geometrical and dynamical properties.

Motion of a Particle Along a Smooth Cycloid: When a particle slides down a smooth cycloidal curve under the action of gravity, its motion can be analysed using the principle of conservation of energy**.**

(i) The component of gravitational force along the tangent to the cycloid acts as a **restoring force**.
(ii) This restoring force is proportional to the displacement of the particle from the lowest point.

Cycloidal Motion as Simple Harmonic Motion: It can be shown that the motion of a particle along a cycloid under gravity is simple harmonic motion (SHM)**.**

(i) The acceleration of the particle is directly proportional to its displacement from the lowest point.
(ii) The motion is periodic and oscillatory about the lowest point.

Check your progress

True or False

1. A particle moving in a smooth vertical circle experiences frictional force.
2. In vertical circular motion, the speed of the particle is maximum at the lowest point.
3. The normal reaction on a particle in a vertical circle is the same at all points.
4. The principle of conservation of energy can be applied in vertical circular motion when friction is absent.

5. At the highest point of a vertical circle, the centripetal force is provided only by the weight of the particle.
6. The minimum velocity required to complete a vertical circle is required at the lowest point.

11.7 SUMMARY

This unit deals with the motion of a particle under gravity when it is constrained to move along curved paths such as a smooth vertical circle and a cycloid. The analysis is based on Newton’s laws of motion and the principle of conservation of energy, which together provide a clear understanding of particle dynamics.

In a smooth vertical circular motion, the particle moves under the action of gravity and the normal reaction (or tension) of the path. Since the surface is smooth, friction is absent and the speed of the particle varies at different points of the circle. The velocity is maximum at the lowest point and minimum at the highest point. The principle of conservation of energy is effectively used to relate velocities at different positions, while force analysis helps in determining the reaction and the conditions for maintaining contact. The concept of minimum velocity required to complete the vertical circle is an important result of this study.

The unit also introduces cycloid motion, which is the path traced by a point on the circumference of a rolling circle. When a particle moves along a smooth cycloid under gravity, its motion can be shown to be simple harmonic motion about the lowest point. The cycloid possesses remarkable properties such as the tautochrone property, where the time of descent is independent of the starting point, and the brachistochrone property, where the cycloid represents the curve of quickest descent.

11.8 GLOSSARY

Particle
An idealized body having mass but negligible size, whose motion can be completely described by a single point.

Smooth Surface
A surface on which no friction acts; only normal reaction forces are present.

Vertical Circle
A circular path lying in a vertical plane along which a particle moves under the influence of gravity.

Normal Reaction
The force exerted by a surface on a particle in a direction perpendicular to the surface.

11.9 REFERENCES

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11.10 SUGGESTED READING

1. Vector Mechanics for Engineers: Statics – F. P. Beer & E. R. Johnston
A foundational text covering concepts of forces, moments, and equilibrium with clear illustrations.

2. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige
Excellent for understanding theoretical principles and solving analytical problems.

*3.***Engineering Mechanics** – R. K. Bansal
Useful for Indian students; provides numerous solved examples and practice problems.

4. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
Provides strong conceptual explanation with real-life engineering applications.

11.12 TERMINAL QUESTIONS

1. Discuss the motion of a particle in a smooth vertical circle. Derive expressions for the velocity and normal reaction at any point of the circle.

2. Explain the use of the principle of conservation of energy in the study of motion in a smooth vertical circle.

3. Derive the condition for a particle to just complete a smooth vertical circular motion. Obtain the minimum velocity at the lowest point.

4. Discuss the variation of tension in the string when a particle moves in a vertical circle.

11.13 ANSWERS

CYQ1. False CYQ2. True CYQ3. False CYQ4 . True

CYQ5. True CYQ6. True

BLOCK IV- STRING IN TWO DIMENSION AND CENTRAL ORBIT

UNIT-12: COMMON CATENARY

CONTENTS :

12.1 *INTRODUCTION*

A **common catenary** refers to the natural curve assumed by a perfectly flexible, uniform, inextensible chain or cable hanging freely under its own weight when supported at its ends. The study of this curve forms an important chapter in classical mechanics and calculus, as it beautifully illustrates how physical principles give rise to precise mathematical forms. In this chapter, we explore the derivation of the catenary equation, the conditions under which the curve appears, and its fundamental properties. The common catenary not only provides a real-world example of a hyperbolic cosine function but also demonstrates how variational principles and equilibrium conditions govern physical systems, making it a valuable concept in both mathematics and physics.

12.2 *OBJECTIVE*

The objectives of the chapter **common catenary** in mechanics are:

1. To understand the physical concept of a catenary, including how a uniform, perfectly flexible chain or cable naturally assumes this curve under gravity.
2. To derive the mathematical equation of the catenary using principles of statics and calculus, particularly the relationship between tension, weight, and curve geometry.
3. To study the geometric properties of the catenary, such as its symmetry, shape, and relation to the hyperbolic cosine function.
4. To analyze the forces acting on a hanging chain, including horizontal and vertical components of tension, and understand how they vary along the curve.
5. To apply the catenary equation to practical problems, such as finding the sag, span, or required length of a cable between two points.
6. To compare the catenary with similar curves, especially the parabola, and understand why real chains follow a catenary rather than a parabolic form.
7. To explore engineering applications of catenaries, including their use in suspension bridges, power lines, and architectural structures.

12.3 *FLEXIBLE STRINGS*

In the context of the common catenary, a **flexible string** (or chain, cable, rope) is an idealized physical model with three specific mechanical properties:

1. **Perfect Flexibility:** It cannot resist any bending moment. This means it transmits force only along its tangent (as tension), not sideways or as a stiff beam would. It offers zero resistance to being bent into any shape.
2. **Inextensibility:** Its length is fixed. It does not stretch or contract under the applied tension forces.

3. **Uniform Linear Density:** Its mass is distributed evenly along its length (constant mass per unit length).

These idealizations are crucial because they allow the equilibrium of a small segment of the string to be analyzed using calculus, leading directly to the defining differential equation. The solution to this equation is the **catenary curve**, described by the hyperbolic cosine function $y = c\cosh\left(\frac{x}{c}\right)$.

In short, the "flexible string" is the idealized, uniform, and perfectly limp object whose shape under gravity is the pure mathematical catenary. Real-world chains, cables, and power lines approximate this ideal when their stiffness is negligible compared to their weight and tension.

12.4 *SUSENSION BRIDGES*

A **suspension bridge** is a type of bridge in which the **main load-carrying cables** are hung between tall towers and anchored at both ends. The **deck (roadway)** is supported by many vertical hangers (or suspenders) connected to these main cables. Connection with the Catenary Curve: When the main cable of a suspension bridge hangs freely under its own weight, it naturally takes the shape of a **catenary curve**. However, in actual bridges, because the cable also supports the weight of the deck (which is uniformly distributed horizontally), the shape becomes *close to a parabola* but still fundamentally related to catenary concepts. Figure 1 shows how the suspension bridge shown in practical way.

Figure 1: Suspension bridge
https://en.wikipedia.org/wiki/Suspension_bridge

In other terms: **Suspension bridges are real-world structures whose main cables follow the principles of the catenary curve, making them an important example in this chapter.**

12.4.1 *UNDER CONDITIONS OF VERY HIGH TENSION, THE CATENARY'S SHAPE CONVERGES TO THAT OF A PARABOLA AS A FIRST-ORDER APPROXIMATION*

Using standard notation, in a catenary, *C* denotes the length of the curve whose weight equals the tension at the lowest point. Because the string is stretched tightly, the tension at this lowest point becomes very large, and consequently *C* is also very large. Thus, the equation of the catenary, with its axes drawn through the lowest point *C*, is given by:

$$y+c=\frac{c}{2}[e^{x/c}+e^{x/c}] \quad [\text{since } \cosh\left(\frac{x}{c}\right)=\frac{1}{2}\left(\frac{x}{e^{c}}+\frac{x}{e^{c}}\right)]$$

$$y+c=\frac{c}{2}\left[2+2.\frac{x^2}{2!.c^2}+2.\frac{x^4}{4!.c^4}+...\right]$$

$$y+c=c+\frac{x^2}{2c}+\frac{x^4}{4!.c^3}+... \quad [\text{taking c to be very large quantity, so } \frac{1}{c^3} \text{ is}$$

neglected]

$$\Rightarrow 2cy=x^2$$

$\Rightarrow x^2=2cy$, which represents the equation of parabola.

12.5 *CATENARY OF UNIFORM STRENGTH*

When a heavy, inextensible string is suspended in a vertical plane between two fixed points, and its mass at any point is proportional to the tension there, the curve it forms is called a catenary of uniform strength.

12.6 *COMMON CATENARY*

A uniform string or chain hanging freely under gravity, when supported at two endpoints that are not vertically aligned, takes the shape of a curve known as the common catenary.

Theorem 1: A heavy, uniform, inextensible string hangs freely under gravity from two fixed points that are not vertically aligned. Determine the equation of the curve it assumes.

Proof: Let A and B be the two fixed points from which a heavy, uniform, and inextensible string is suspended. Let *C* denote the lowest point of the curve, and *P* be any point on the string. Let the arc length $CP = s$, and let www represent the weight per unit length of the string. Suppose the tangent at *P* makes an angle ψ with the horizontal.

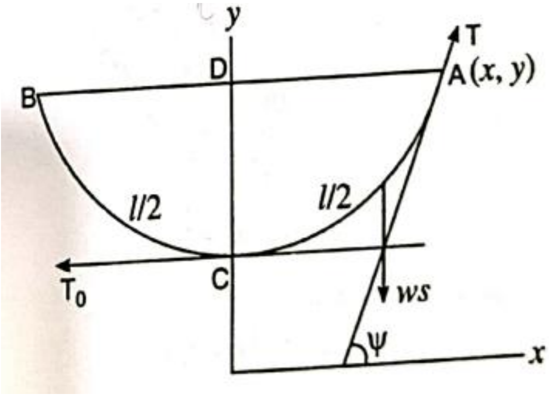

Figure 2

Choose a point *O*, located a vertical distance *c* below the lowest point *C*, as the origin, and let the horizontal and upward vertical lines through *O* be the *x* and *y*-axes, respectively.

The forces acting on the portion of the string are:

1. The tension *T* at *P*, acting along the tangent at *P*.
2. The tension T_0 at *C*, directed horizontally.
3. The weight ws of the arc *CP*, acting vertically downward through its center of gravity *G*.

These three forces meet at a point *D*. By resolving them horizontally and vertically, we obtain the following relations

$T\cos\psi = T_0 = wc$ … (1)

$T\sin\psi = ws$ … (2)

Dividing Eq. (1) to Eq. (2) we get

$$\frac{T\sin\psi}{T\cos\psi} = \frac{ws}{wc} \Rightarrow \tan\psi = \frac{s}{c} \quad \ldots (3)$$

Hence, $s = c\tan\psi$, which is the required intrinsic equation of the catenary.

Now the slope of the tangent at P is, $\frac{dy}{dx} = \tan\psi$. Then by Eq. (3)

$\Rightarrow s = c\frac{dy}{dx}$. Putting $P = \frac{dy}{dx}$ then, $s = cP$. Now differentiating this w.r.to x we get,

$$\frac{ds}{dx} = c\frac{dP}{dx} \qquad \left[\text{Since, } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow \sqrt{1+P^2} = c\frac{dP}{dx}$$

$\Rightarrow \frac{dx}{c} = \frac{dP}{\sqrt{1+P^2}}$. Integrate both side, $\Rightarrow \int \frac{dP}{\sqrt{1+P^2}} = \int \frac{dx}{c}$

$\Rightarrow \sinh^{-1} P = \frac{x}{c} + k$, where k is constant quantity … (4)

At C, $x = 0, P = 0$

$\therefore \sinh^{-1} 0 = 0 + k \Rightarrow k = 0$ $[\because \sinh^{-1} 0 = 0]$

Then Eq. (4) $\Rightarrow \sinh^{-1} P = \frac{x}{c}$

$$\Rightarrow P = \sinh\left(\frac{x}{c}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right) \Rightarrow dy = \sinh\left(\frac{x}{c}\right)dx$$

Now, integrating both side we get, $\Rightarrow \int dy = \int \sinh\left(\frac{x}{c}\right)dx$

$$\Rightarrow y = \frac{\cosh\left(\frac{x}{c}\right)}{\frac{1}{c}} + k_1, \text{ where } k_1 = \text{constant} \qquad \ldots (5)$$

Now, at C, $x = 0, y = c$

$c = c\cosh 0 + k_1$ $\qquad [\because \cosh .0 = 0]$

$\Rightarrow c = c + k_1 \Rightarrow k_1 = 0$

Then by Eq. (5) we get, $y = c\cosh\left(\frac{x}{c}\right)$ $\qquad \ldots (6)$

Which is the required Cartesian equation of the catenary.

Example 1: Prove the following results:

(*i*) $y^2 = c^2 + s^2$

(*ii*) $y = c\sec\psi$

(*iii*) $x = c\log(\sec\psi + \tan\psi)$

(*iv*) $x = c\log\left(\frac{y+s}{c}\right)$

(*v*) $T = wy$

Proof (*i*): As we know that $y = c\cosh\left(\frac{x}{c}\right)$

$$\Rightarrow y^2 = c^2\cosh^2\left(\frac{x}{c}\right) = c^2\left(1+\sinh^2\left(\frac{x}{c}\right)\right) = c^2 + c^2\sinh^2\left(\frac{x}{c}\right) = c^2 + \left(c\sinh\frac{x}{c}\right)^2$$

Since we know that $s = c\tan\psi = c\frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)$ [$\because \frac{dy}{dx} = \sinh\frac{x}{c}$, by previous theorem 1]

Hence, $y^2 = c^2 + s^2$.

(*ii*): Since we have, $y^2 = c^2 + s^2$

Then, $y^2 = c^2 + (c\tan\psi)^2 = c^2(1+\tan^2\psi) = c^2\sec^2\psi$

Hence, $y = c\sec\psi$.

(*iii*): As we have already prove that, $y = c\sec\psi$ and $s = c\tan\psi$

Then using the previously proved statements we can write that,

$$\cosh\frac{x}{c} = \sec\psi \text{ and } \sinh\frac{x}{c} = \tan\psi$$

$$\therefore \cosh\frac{x}{c} + \sinh\frac{x}{c} = \sec\psi + \tan\psi$$

$$\frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2} + \frac{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}{2} = \sec\psi + \tan\psi$$

$$\Rightarrow e^{\frac{x}{c}} = \sec\psi + \tan\psi$$

Taking log both side we get, $\frac{x}{c} = \log(\sec\psi + \tan\psi)$

$$\Rightarrow x = c\log(\sec\psi + \tan\psi)$$

(*iv*): Since we have $x = c\log(\sec\psi + \tan\psi)$ and also we know that, … (7)

$s = c\tan\psi$ and $s = c\sec\psi$.

$\Rightarrow \tan\psi = \dfrac{s}{c}$ and $\sec\psi = \dfrac{y}{c}$

Then by using Eq. (7) we get, $x = c\log\left(\dfrac{y}{c} + \dfrac{s}{c}\right)$

$\Rightarrow x = c\log\left(\dfrac{y+s}{c}\right)$

(*v*): From figure 2 we can easily seen that the component of the tension are,

$T\cos\psi = wc$ and $T\sin\psi = ws$

Now, by adding of squares of these two components,

$T^2\cos^2\psi + T^2\sin^2\psi = (wc)^2 + (ws)^2$

$\Rightarrow T^2(\cos^2\psi + \sin^2\psi) = w^2\left(c^2 + s^2\right)$

$\Rightarrow T^2 = w^2\left(c^2 + s^2\right) = w^2y^2$

Hence $T = wy$

Example 2: A uniform chain of length *l* is hung between two points *A* and *B* that lie on the same horizontal line. If the tension at point *A* is twice the tension at the lowest point of the chain, prove that the horizontal distance between *A* and *B* is $\dfrac{l}{\sqrt{3}}\log(2+\sqrt{3})$.

Solution: Suppose the tangent to the curve at the point *A*(*x*, *y*) forms an angle ψ with the horizontal axis. It is given that the tension at point *A* is double the tension at the lowest point *C* i.e., $T = 2T_0$

$\Rightarrow wy = 2wc$. So we get, $y = 2c$.

As we know that $y = c\sec\psi$ then, $2c = c\sec\psi \Rightarrow \sec\psi = 2$. … (8)

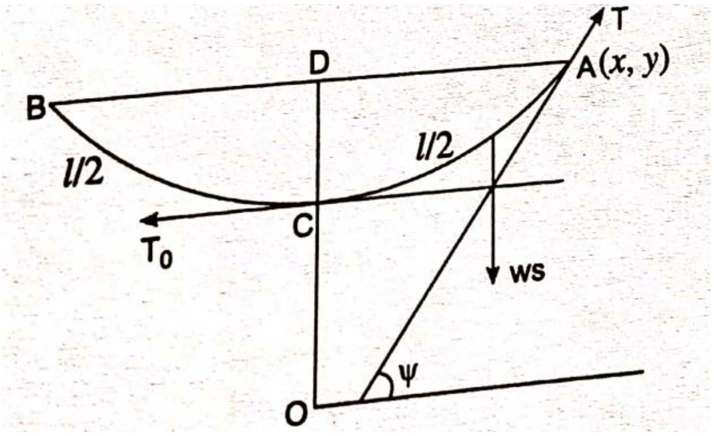

Figure 3

From figure 3, it is clear that the arc $CA = l/2$. By the intrinsic equation of the catenary we know that, $s = c\tan\psi$.

So, at the point A, $s = l/2$

$$\therefore l/2 = c\tan\psi$$

$$\Rightarrow l/2 = c\sqrt{\sec^2\psi - 1} = c\sqrt{4-1} = \sqrt{3}c$$

$$\Rightarrow c = l/2\sqrt{3}$$

Since the span, $AB = 2x = 2c\log(\sec\psi + \tan\psi)$

$$\Rightarrow AB = 2\frac{l}{2\sqrt{3}}\log(2+\sqrt{3})$$

Hence the span $AB = \dfrac{l}{\sqrt{3}}\log(2+\sqrt{3})$

Example 3: A uniform chain of length *l* is suspended between two fixed points A and B that lie on the same horizontal level. If the tension at each end is *n* times the tension at the lowest point of the chain, prove that the horizontal distance between A and B is $AB = \dfrac{l}{\sqrt{n^2-1}}\log(\sqrt{n^2-1}+n)$.

Solution: By previous theorem we know that, $y^2 = c^2 + s^2$.

We have given that, Tension at point $A = n$ times the tension at the point *C*.

i.e., $T = nT_0$.

$\Rightarrow wy = nwc$. So we get, $y = nc$.

As we know that $y = c\sec\psi$ then, $nc = c\sec\psi \Rightarrow \sec\psi = n$.

From figure 3, it is clear that the arc $CA = l/2$. By the intrinsic equation of the catenary we know that, $s = c\tan\psi$.

So, at the point A, $s = l/2$

$\therefore l/2 = c\tan\psi$

$\Rightarrow l/2 = c\sqrt{\sec^2\psi - 1} = c\sqrt{n^2 - 1}$

$\Rightarrow c = l/2\sqrt{n^2 - 1}$

Since the span, $AB = 2x = 2c\log(\sec\psi + \tan\psi)$

$$\Rightarrow AB = 2\frac{l}{2\sqrt{n^2 - 1}}\log(n + \sqrt{n^2 - 1})$$

Hence the span $AB = \dfrac{l}{\sqrt{n^2 - 1}}\log(n + \sqrt{n^2 - 1})$

Example 4: A uniform chain of total length 2*l* is attached at its ends to two fixed points that lie on the same horizontal line. If the vertical sag at the midpoint is *h*, show that the horizontal distance between the two points is $\dfrac{l^2 - h^2}{h}\log\dfrac{l + h}{l - h}$.

Solution: Consider a uniform chain ACB of total length 2*l*, hanging from two points A and B that are positioned at the same horizontal level. Assume that at point A(*x*, *y*), the tangent to the curve forms an angle ψ with the horizontal.

For point A, $y = c + h, s = l$

We have given that the sag in the middle is h.

In a common catenary we know, $y^2 = c^2 + s^2$

$$\Rightarrow (c+h)^2 = c^2 + s^2$$

$$\Rightarrow c^2 + h^2 + 2ch = c^2 + s^2$$

$$\Rightarrow c = \frac{l^2 - h^2}{2h} \qquad \ldots (9)$$

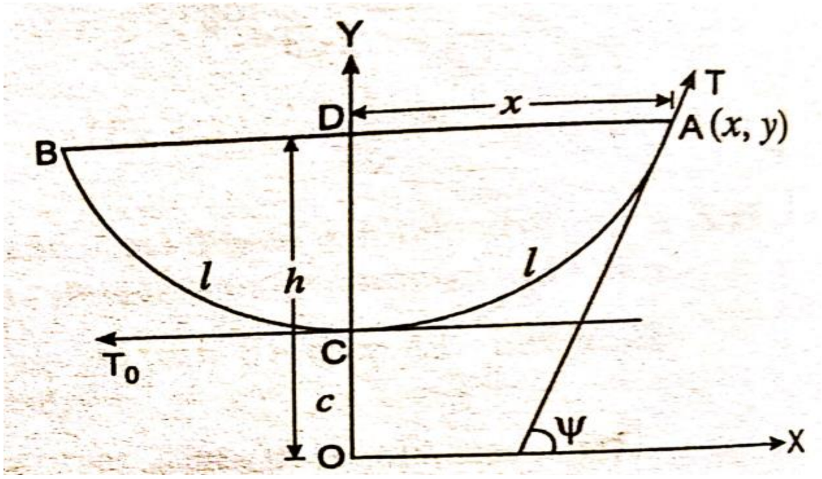

Figure 4

Again, by the intrinsic equation of the common catenary, $s = c\tan\psi$

$$\Rightarrow l = \frac{l^2 - h^2}{2h}\tan\psi \Rightarrow \tan\psi = \frac{2hl}{l^2 - h^2}$$

Similarly we can find, $\sec\psi = \dfrac{l^2 + h^2}{l^2 - h^2}$.

Then the span $AB = 2x = 2c\log(\sec\psi + \tan\psi)$

After putting all the required values in above expression we get,

$$AB = \frac{l^2 - h^2}{h}\log\frac{l+h}{l-h}.$$

Example 5: Let T denote the tension at an arbitrary point P on the catenary, and T_0 represent the tension at the lowest point C. Show that $T^2 - T_0^2 = W^2$, where W being the weight of the arc CP of the catenary.

Solution: Let w denote the weight per unit length of the string and s the length of the arc CP. For a catenary, the tension at the lowest point C is given by, $T_0 = wc$ and the tension at P is, $T = wy$.

Now, $T^2 - T_0^2 = w^2(y^2 - c^2)$ [Since $y^2 = c^2 + s^2$]

$\Rightarrow T^2 - T_0^2 = w^2 s^2 = (ws)^2$.

Since W be the weight of the arc CP so, $W = ws$

Therefore we can write, $\Rightarrow T^2 - T_0^2 = W^2$

Example 6: Show that, for a catenary $y = c\cosh(x/c)$, the perpendicular drawn from the foot of any ordinate to the tangent at the lowest point has a constant length.

Solution: Consider any point $P(x, y)$ on the catenary ACB, where C is the lowest point. Let the tangent drawn at P form an angle ψ with the horizontal. From the point $(x, 0)$, which is the foot of the ordinate, drop a perpendicular MN onto the tangent line PQ.

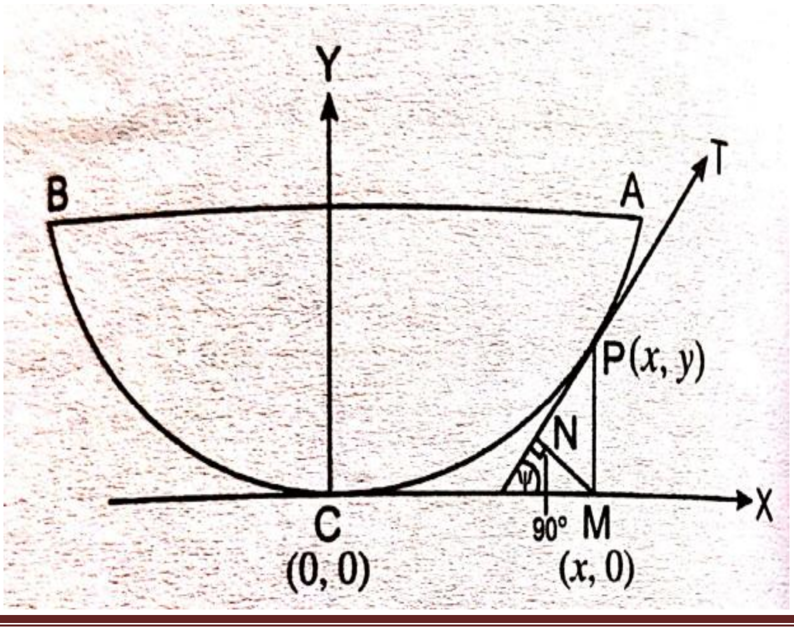

Figure 5

Since the given equation of the common catenary is, $y = c\cosh(x/c)$

Then, $\dfrac{dy}{dx} = \sinh(x/c)$ .Now we can drive the equation of the tangent at point P is,

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow (X - x)\sinh\left(\frac{x}{c}\right) - Y + y = 0\,.$$

Now the length of the perpendicular MN from foot of the ordinate M(x, 0) on the tangent is,

$$\Rightarrow \frac{(x-x)\sinh\left(\frac{x}{c}\right) - 0 + y = 0}{\sqrt{\sinh^2\frac{x}{c} + 1^2}} = \frac{y}{\cosh(x/c)} = \frac{c\cosh(x/c)}{\cosh(x/c)} = c$$

Example 7: The ends of a catenary move freely along a fixed rough horizontal support. Show that the maximum possible span of the catenary, when compared to its total length, has a ratio equal to $\mu\log\left(\dfrac{1+\sqrt{1+\mu^2}}{\mu}\right)$.

Solution: Consider a catenary ACB of length l, whose endpoints slide on a fixed horizontal rod AB. Let (x, y) be the coordinates of point C, and let ψ denote the angle that the tangent at A makes with the horizontal.

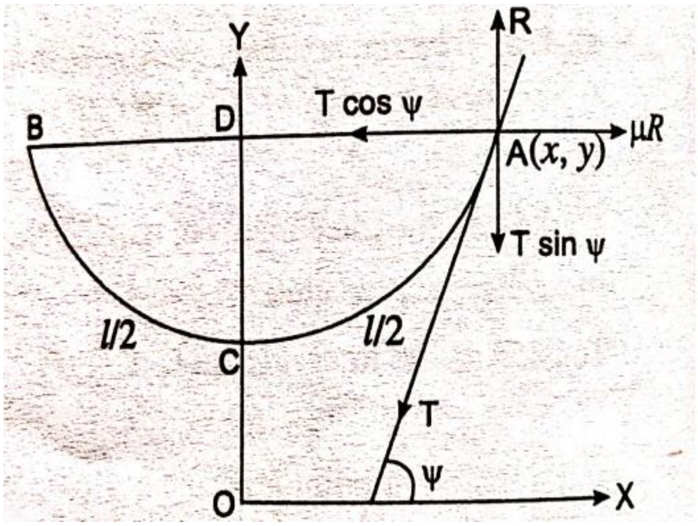

Figure 6

Then, $AC = CB = l/2$ and $AD = DB = x = AB/2$.

Thus, the forces acting at point A are:

(i) The tension T directed along the tangent at A.

(ii) The normal reaction R, perpendicular to the rod AB.

(iii) The frictional force μR, acting along BA.

By resolving these forces into their horizontal and vertical components, we obtain:

$T\cos\psi = \mu R$ and $T\sin\psi = R$. Now, after eliminating T we get,

$$\tan\psi = \frac{1}{\mu} \Rightarrow \sec\psi = \sqrt{1+\frac{1}{\mu^2}} = \frac{\sqrt{\mu^2+1}}{\mu}.$$

By the intrinsic equation of common catenary $s = c\tan\psi$.

At the point $s = l/2$

$$\therefore l/2 = c\frac{1}{\mu} \Rightarrow c = \frac{\mu l}{2}.$$

Here the maximum span

$$AB = 2x = 2c\log[\sec\psi + \tan\psi] = \mu.l\log\left[\frac{\sqrt{\mu^2+1}}{\mu} + \frac{1}{\mu}\right]$$

$$AB = \mu.l \log\left[\frac{\sqrt{\mu^2+1}+1}{\mu}\right]$$

Therefore, the proportion of the maximum span relative to the catenary's

length is $= \dfrac{\mu.l \log\left[\dfrac{\sqrt{\mu^2+1}+1}{\mu}\right]}{l} = \mu.\log\left[\dfrac{\sqrt{\mu^2+1}+1}{\mu}\right]$

Example 8: Consider a uniform chain of length l whose maximum allowable tension is n times its own weight. It is hung between two supports at the same level. Show that the smallest possible sag in the middle is $n-\sqrt{\dfrac{n^2-1}{4}}$.

Solution: Consider a uniform chain of length l suspended in the shape of a catenary curve, denoted as ACB. At any point A with coordinates (x, y) on the curve, let T represent the tension, and let w be the constant weight per unit length of the chain. The **sag** (the vertical drop from the support to the lowest point) will be minimized when the **slope** (or derivative, dy/dx) at the endpoint is at its maximum. The problem states that the maximum permissible tension at the support is equal to n times the total weight of the entire chain, or *nwl*.

So, $T = nwl$ … (1)

At the point A, $T\sin\psi = w\dfrac{l}{2}$ $[T\sin\psi = ws]$

$$\Rightarrow nwl\sin\psi = w\frac{l}{2} \Rightarrow \sin\psi = \frac{1}{2n}$$

$$\therefore \cos\psi = \sqrt{1-\frac{1}{4n^2}} = \frac{\sqrt{4n^2-1}}{2n}.$$

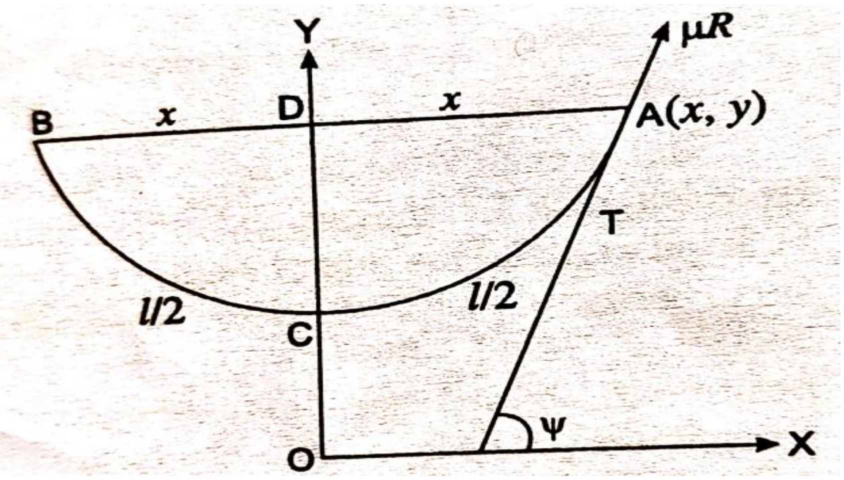

Figure 7

Then, $\therefore \tan\psi = \sqrt{1-\frac{1}{4n^2}} = \frac{1}{\sqrt{4n^2-1}}$.

By the intrinsic equation of common catenary, $s = c\tan\psi$. Then at point A,

$$\frac{l}{2} = \frac{1}{\sqrt{4n^2-1}} \Rightarrow c = \frac{l}{2}\sqrt{4n^2-1}$$

From figure, the sag in the middle is $DC = DO - OC = y - c = c\sec\psi - c$

$$= c\left[\frac{2n}{\sqrt{4n^2-1}} - 1\right] = \frac{l}{2}\sqrt{4n^2-1}\left(\frac{2n}{\sqrt{4n^2-1}} - 1\right) = l.\left(n - \sqrt{n^2 - \frac{1}{4}}\right)$$

Example 9: A uniform chain of length 2*l* has one end fixed at point *A*. The other end is attached to a ring, whose weight is *n* times that of the entire chain. The ring can slide along a rough horizontal rod that also goes through point *A*. Prove that the **maximum possible distance** the ring can be from *A* before slipping is $\frac{2l}{\lambda}\log(\lambda + \sqrt{\lambda^2+1})$, where $\frac{1}{\lambda} = \mu(2n+1)$ and μ denotes the friction coefficient.

Solution: Consider a heavy, uniform chain of total length 2*l*, fixed at one end at point *A*. Its other end is attached to a small but heavy ring, which can slide along a rough horizontal rod that passes through *A*. Let *B* represent the farthest possible stable position of the ring from A before slipping occurs i.e., the **limiting equilibrium** point. The total weight of the chain is denoted by W. If the w is the weight per unit length of the chain then,

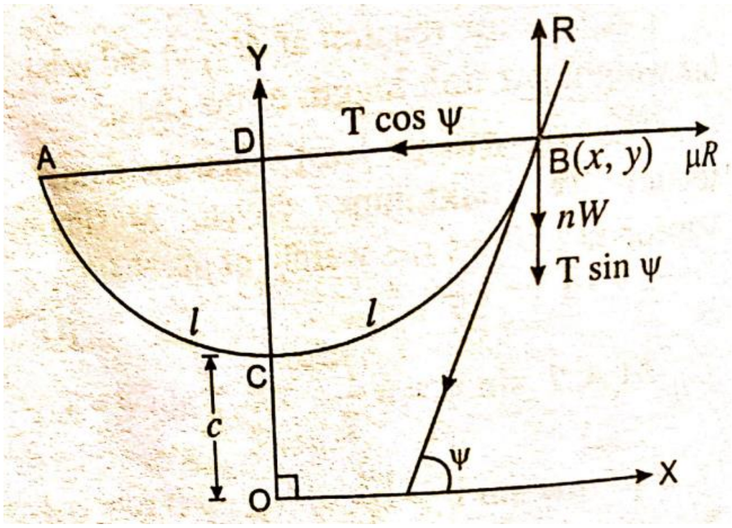

Figure 8

$$W = 2lw \Rightarrow w = \frac{W}{2l}$$

We have given, the weight of the ring is nW.

Under the condition of equilibrium in the ring there are three types of forces are applied.

(i) The weight nW

(ii) Along AB, the frictional force μR

(iii) In the vertically upward direction, the normal reaction R.

So, after resolving the forces horizontal and vertical direction, we get

$$T\cos\psi = \mu R \text{ and } R = nW + T\sin\psi$$

In a catenary, we have $y = c\sec\psi$, for B, we get

$$T = wy = wc\sec\psi .$$

Since $s = c\tan\psi \Rightarrow l = c\tan\psi$

Then, $T\cos\psi = \mu(nW + T\sin\psi) = n\mu W + T\mu\sin\psi$

$$\Rightarrow \frac{W}{2l}c\sec\psi\cos\psi = n\mu W + \frac{W}{2l}c\mu\sin\psi$$

$$\Rightarrow \frac{W}{2l}c = n\mu W + \frac{W}{2l}c\mu\tan\psi$$

$$\Rightarrow c = 2l\mu n + c\mu\tan\psi$$

$$\Rightarrow c = 2\mu nc\tan\psi + c\mu\tan\psi$$

$$\Rightarrow 1 = \mu\tan\psi(2n+1)$$

$$\Rightarrow \frac{1}{\lambda} = \frac{\mu\tan\psi(2n+1)}{\lambda}$$

$\Rightarrow \mu(2n+1) = \dfrac{\mu\tan\psi(2n+1)}{\lambda}$, As we have already given in question $\dfrac{1}{\lambda} = \mu(2n+1)$

$\Rightarrow \tan\psi = \lambda$ then $\sec\psi = \sqrt{1+\lambda^2}$

So, the greatest possible distance of the ring A is
$= AB = 2c\log[\sec\psi + \tan\psi]$

$$= 2c\log[\lambda + \sqrt{1+\lambda^2}] = \frac{2l}{\tan\psi}\log[\lambda + \sqrt{1+\lambda^2}] = \frac{2l}{\lambda}\log[\lambda + \sqrt{1+\lambda^2}]$$

Example 10: The supports of a uniformly strong catenary are separated by a distance *a*, and the chain has length *l*. Demonstrate that the parameter *c* is determined from $\tan\dfrac{l}{4c} = \tan\dfrac{a}{4c}$.

Solution: Let AOB is a catenary of uniform strength of length *l*, where A and B be the points of support, where $AB = a$.

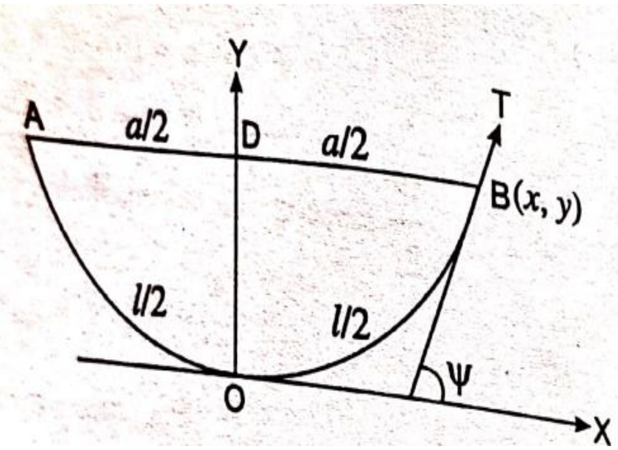

Figure 9

For the point B, $x = \frac{a}{2}, s = \frac{l}{2}$.

Now for catenary of uniform strength, we know that $x = c\psi$ and $s = c\log(\sec\psi + \tan\psi)$

So, $\frac{a}{2} = c\psi \Rightarrow \psi = \frac{a}{2c}$

Then, $l/2 = c\log(\sec\psi + \tan\psi) \Rightarrow \sec\psi + \tan\psi = e^{l/2c}$

$$\Rightarrow \frac{1+\sin\psi}{\cos\psi} = e^{l/2c}$$

$$\Rightarrow \frac{\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\right)^2}{\cos^2\frac{\psi}{2} - \sin^2\frac{\psi}{2}} = e^{l/2c} \Rightarrow \frac{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}} = e^{l/2c}$$

$$\Rightarrow \frac{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}} = \frac{e^{l/4c}}{e^{-l/4c}} \Rightarrow \frac{\cos\frac{\psi}{2}}{\sin\frac{\psi}{2}} = \frac{e^{l/4c} + e^{-l/4c}}{e^{l/4c} - e^{-l/4c}} = \frac{\frac{e^{l/4c} + e^{-l/4c}}{2}}{\frac{e^{l/4c} - e^{-l/4c}}{2}}$$ (By using componendo and dividendo rule)

$$\Rightarrow \cot\frac{\psi}{2} = \coth\frac{l}{4c}$$

$\Rightarrow \tan \frac{a}{4c} = \tanh \frac{l}{4c}$, which gives the parameter c.

Check your progress

Problem 1: A string of total length 2*l* feet hangs between two supports that lie on the same horizontal level, and its lowest point is *d* feet below these supports. If the string has a uniform weight of *wlbs*, determine the horizontal component of the tension in the string.

12.7 *SUMMARY*

The unit on the common catenary in mechanics examines the characteristic curve formed by a uniform, flexible, and inextensible string or chain hanging freely under its own weight when supported at its ends. It introduces the catenary equation, typically expressed as y = c.cosh(x/c), and explains how the parameter c relates to the horizontal tension and the weight per unit length. The chapter discusses key geometric and physical properties of the curve, such as its symmetry, lowest point, and behavior under changes in span or tension. Applications include determining tension components, sag, length between supports, and the conditions for equilibrium. Overall, the chapter provides both the mathematical derivation and physical interpretation of the catenary as a fundamental model in statics and engineering.

12.8 *GLOSSARY*

- Flexible strings
- Suspension bridges
- Catenary of uniform strength
- Common catenary

12.9 *REFERENCES*

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- P. N. Chandiramani & P. R. Godbole (2017). **Engineering Mechanics: Statics and Dynamics**. McGraw-Hill Education India.
- Rajput, R. K. (2013). **Engineering Mechanics**. S. Chand Publishing.
- Narayanaswamy, R. (1997)**. Engineering Mechanics.** Narosa Publishing House.
- A. K. Tayal (2011). **Engineering Mechanics.** Umesh Publications.

12.10 *SUGGESTED READING*

- **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige
  Excellent for understanding theoretical principles and solving analytical problems.
- **Engineering Mechanics** – R. K. Bansal
  Useful for Indian students; provides numerous solved examples and practice problems.
- **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
  Provides strong conceptual explanation with real-life engineering applications.
- **A Textbook of Engineering Mechanics** – S. S. Bhavikatti
  Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6
- **Engineering Mechanics**– A. K. Tayal
  Straightforward explanations suitable for undergraduate mathematics and engineering courses.

12.11 *TERMINAL QUESTION*

Long answer type question

1: A 600-ft kite string makes a 30° angle with the horizontal, and the tension equals the weight of 100 ft of string. Find the kite's vertical height above the flyer's hand.

2: When the tangents drawn at points P and Q on a catenary intersect at a right angle, demonstrate that the tension at the midpoint of the arc PQ is equal to the weight of a segment of the chain whose length is half the length of that arc.

3: A 90-inch uniform string hangs over two smooth pegs at different heights, with vertical drops of 30 and 33 inches. Show that the vertex of the catenary splits the string in the ratio 4:54:54:5, and find the distance between the pegs.

Short answer type question

1: Define common catenary.

2: Derive the intrinsic equation of the common catenary.

3: Prove the following results:

(*i*) $y^2 = c^2 + s^2$

(*ii*) $y = c\sec\psi$

(*iii*) $x = c\log(\sec\psi + \tan\psi)$

(*iv*) $x = c\log\left(\frac{y+s}{c}\right)$

(*v*) $T = wy$

Objective type question:

1. A common catenary is the curve formed by:

(a) A rigid rod under gravity
(b) A flexible string under wind force
(c) A uniform chain freely suspended under gravity
(d) A rotating string

2. The word "catenary" is derived from which language?

(a) Greek
(b) Latin
(c) Sanskrit
(d) French

3. **Which function is used in the equation of a catenary?**

(a) Trigonometric function
(b) Exponential function only

(c) Hyperbolic function
(d) Algebraic function

12.12 *ANSWERS*

Answer of check your progress

Problem 1: The horizontal component of the tension will be $w\frac{l^2-d^2}{2d}$

Answer of long answer type question

1: 555.7 ft nearly

Answer of objective question:

1. c 2. b 3. c

UNIT 13: KINEMATICS IN TWO DIMENSIONS

CONTENTS:

13.1 INTRODUCTION

Motion is a fundamental concept in physics and mathematics, and while the study of motion in a straight line (linear motion) helps us understand basic kinematics, many real-life motions such as the rotation of wheels, motion of planets, ceiling fans, and spinning tops involve rotational motion. To describe such motion accurately, we introduce the important concepts of angular displacement, angular velocity, and angular acceleration**.**

Angular velocit**y** describes how fast an object rotates, while angular acceleration explains how the angular velocity changes with time. These quantities play a vital role in understanding circular motion, rotational dynamics, and engineering applications.

In this unit, we also study the rate of change of unit vectors in a plan**e**, which is essential for describing motion in two dimensions. This concept helps us express velocity and acceleration in vector form when direction continuously changes, as in circular motion.

Further, a very important relationship between angular velocity and linear (tangential) velocity is developed, which allows us to connect rotational motion with linear motion. This relation is extremely useful in practical problems involving wheels, gears, pulleys, and rotating bodies.

Finally, we discuss the components of velocity and acceleration along coordinate axes in two dimensions. Resolving vectors into components simplifies complex motion into manageable parts and helps in solving a wide variety of problems related to projectile motion, circular motion, and plane motion.

Thus, this unit builds a strong mathematical foundation for understanding rotational motion and prepares students for advanced topics in mechanics.

13.2 OBJECTIVES

After studying this unit learners will be able

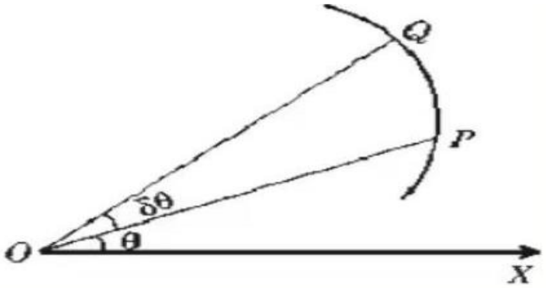
1. To Understand the concept of angular displacement, angular velocity, and angular acceleration and their physical significance.
2. Define and explain angular velocity and angular acceleration in mathematical form.
3. Derive and use expressions for angular velocity and angular acceleration in rotational motion.
4. Understand the rate of change of a unit vector in a plane and apply it in two-dimensional motion.

13.3 ANGULAR VELOCITY

Angular Velocity (Definition): Let P be a point moving in a plane. If O is the fixed point and OX a fixed line through O in the plane of motion, then the angular velocity of the moving point P about O (or the line OP in the plane XOP) is the rate of change of the angle XOP. Let P and Q be the positions of a moving particle at times t and $t + \delta t$ respectively such that $\angle POX = \theta$ and $\angle QOX = \theta + \delta\theta$.

Then the angle turned by the particle in time δt is $\delta\theta$. $\therefore$ Average rate of change of the angle of P about $O = \frac{\delta\theta}{\delta t}$.
$\therefore$ The angular velocity of the point P about O

$$= \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta},$$



where the dot placed over θ denotes differentiation with respect to the time t.
Since the angular velocity has magnitude as well as direction, it is a vector quantity represented by the vector $\vec{\omega}$. The magnitude of the angular velocity is $\frac{d\theta}{dt} (= \dot{\theta} = \omega)$ and its direction is perpendicular to the plane POQ.
Since the angle θ is measured in radians, the unit of angular velocity is radians/sec.

13.4 DEFINITION OF ANGULAR ACCELERATION

The rate of change of the angular velocity is called angular acceleration.
$\therefore$ Angular acceleration $= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}$.
The unit of angular acceleration is radians/ $\sec^2$.

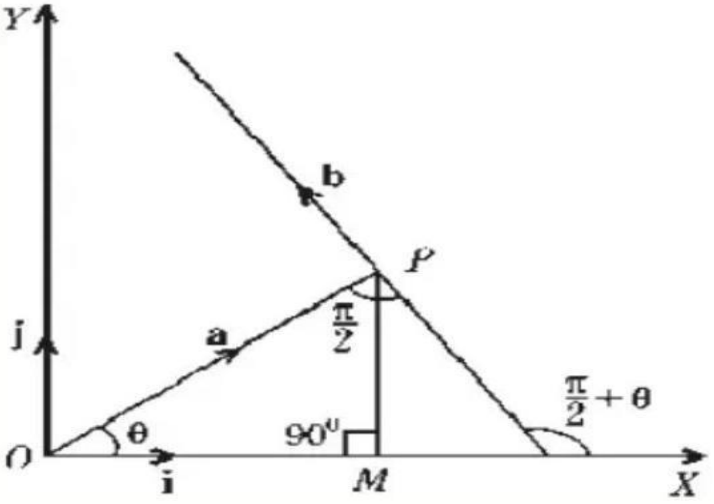
13.5 RATE OF CHANGE OF UNIT VECTOR

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors along two mutually perpendicular fixed lines (say the coordinate axes in the plane).
Let a denote a unit vector $\overrightarrow{OP}$ such that $OP = 1$ and $\angle POX = \theta$.

Then $\mathbf{a} = \overrightarrow{OM} + \overrightarrow{MP} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

The vector $\mathbf{a}$ is a function of θ, where θ is a function of the time t. Differentiating (1) w.r.t. t, we have

$$\frac{d\mathbf{a}}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$



[Note that i and j are constant vectors]

$$= \frac{d\theta}{dt}\left[\cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}\right]$$

where $\mathbf{b} = \cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}$ is a unit vector inclined at an angle $\frac{1}{2}\pi + \theta$ with OX. Therefore, $\mathbf{b}$ is a unit vector perpendicular to OP in the sense in which θ increases. Thus remember that if $\mathbf{a}$ is a unit vector which makes a variable angle θ with OX, then

$$\frac{d\mathbf{a}}{dt} = \frac{d\theta}{dt}\mathbf{b} \tag{2}$$

where **b** is a unit vector perpendicular to **a** in the direction of θ increasing.

Particular case : If **t** and **n** are the unit vectors along the tangent and normal respectively at any point P of a plane curve (as shown in the figure), then

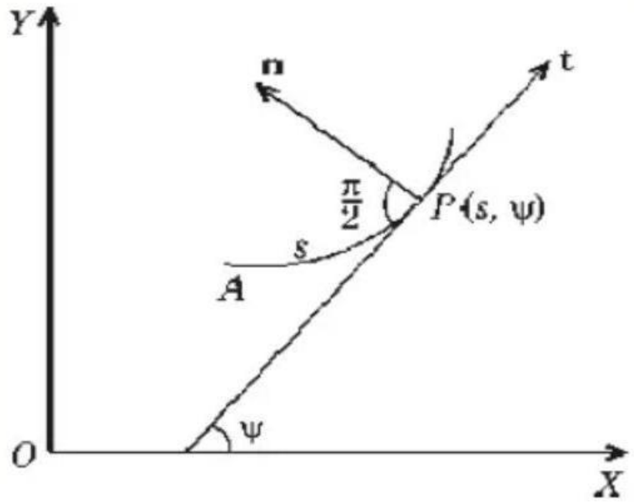
$$\frac{d\mathbf{t}}{dt} = \frac{d\psi}{dt}\mathbf{n} = \dot{\psi}\mathbf{n}$$

where ψ is the angle which the tangent at the point P makes with OX.

Also

$$\frac{d\mathbf{n}}{dt} = -\frac{d\Psi}{dt}\mathbf{t} = -\dot{\psi}\mathbf{t}$$

Here **t** is in the direction of s increasing and **n** is in



13.6 RELATION BETWEEN ANGULAR AND LINER VELOCITY

Now the component of a vector **a** in the direction of a unit vector **b** is given by $\mathbf{a} \cdot \mathbf{b}$. If v_θ is the component of the velocity **v** in the direction perpendicular to OP, then

$$\begin{aligned} v_\theta &= \mathbf{v}\cdot\mathbf{e}_\theta = \left(\frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta\right)\cdot\mathbf{e}_\theta \\ &= r\frac{d\theta}{dt} = r\omega, \qquad [\because \mathbf{e}_r \perp \mathbf{e}_\theta \text{ and } |\mathbf{e}_\theta| = l] \end{aligned}$$

or

$$\omega = \frac{v_\theta}{r} = \frac{\text{component of the velocity } v \text{ at } P \text{ perpendicular to } OP}{OP}.$$

(Remember)
Since the angle between $\mathbf{v}$ and $\mathbf{e}_\theta$ is $\frac{1}{2}\pi - \phi$, therefore

$$v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta = v\cdot 1\cdot\cos\left(\frac{1}{2}\pi - \phi\right) = v\sin\phi.$$

$$\therefore \quad \omega = \frac{v\sin\phi}{r} = \frac{vr\sin\phi}{r^2}$$

$$\text{or} \quad \omega = \frac{d\theta}{dt} = \frac{vp}{r^2}. \qquad [\because p = r\sin\phi]$$

Remark 1: The angular velocity of P about O

$$= \frac{\text{the resolved part of the velocity of } P \perp \text{ to } OP}{OP}.$$

Remark 2: If A and B are both in motion, then the angular velocity of B relative to A

$$= \frac{\text{the resolved part of the velocity of } B \text{ relative to } A \perp \text{ to } AB}{AB}.$$

Alternative method for finding the relation between angular and linear velocities.

Theorem: If v be the velocity of a point P moving in a plane curve and (r, θ) its coordinates referred to the fixed point O in the plane, then the angular velocity of P about O is equal to vp $/r^2$, where p is the perpendicular from O drawn to the tangent at P.
Proof: The angular velocity of P about O

$$= \frac{d\theta}{dt} \quad = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = v\frac{d\theta}{ds} \qquad \left[\because v = \frac{ds}{dt}\right]$$

$$= \frac{v}{r}\cdot\left(r\frac{d\theta}{ds}\right)$$

$$= \frac{v\sin\phi}{r} \qquad \left[\because \sin\phi = r\frac{d\theta}{ds}, \text{from differential calculus}\right]$$

$$= \frac{v}{r}\cdot\frac{p}{r}$$

$$= vp/r^2 \qquad [\because p = r\sin\phi]$$

Example 6: Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.

Solution: We know that the path of a projectile is a parabola whose pedal equation referred to its focus S as the pole is given by

$$p^2 = ar. \tag{1}$$

Let v be the velocity of the projectile at the point $P(r,\theta)$.
Then $MP = PS = r$.
Since the velocity of the projectile at a point of its path is equal to the velocity acquired in falling freely from the directrix to that point, therefore,

$$v = \sqrt{(2g\cdot MP)} = \sqrt{(2gr)} \tag{2}$$

$\therefore$ The angular velocity ω of P about the focus S (i.e., about the pole) is given by

$$\omega = \frac{vp}{r^2} = \frac{\sqrt{(2gr)}\cdot\sqrt{(ar)}}{r^2}$$

$$= \frac{\sqrt{(2ag)}}{r} = \frac{\sqrt{(2ag)}}{SP}.$$

$$\therefore \quad \omega \propto \frac{1}{SP}.$$

13.7 COMPONENT OF VELOCITY AND ACCELERATION ALONG THE COORDINATE AXES IN TWO DIMENSIONS

Let us suppose that P(x, y) be the position of a particle moving in a plane at any time t. If $\overline{OP}$ = r, we have

r = $\overline{OP} = \overline{OM} + \overline{MP}$ = xi + yj.

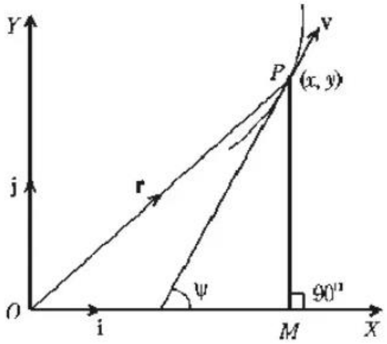
Let v be the vector representing the velocity of the particle at P. Then

v = dr/dt = d/dt (xi + yj)
= (dx/dt)i + (dy/dt)j.

Thus the velocity vector v has been expressed as a linear combination of the vectors i and j.

∴ The x-component of the velocity of P = dx/dt = $\dot{x}$, positive in the direction of vector i, i.e., positive in the direction of x increasing,

and the y-component of the velocity of P = dy/dt = $\dot{y}$, positive in the direction of y increasing.



If v is the resultant velocity of P, we have

$$v = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \frac{ds}{dt}$$

Also, the angle which the direction of v makes with OX

$$= \tan^{-1}\frac{dy/dt}{dx/dt} = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\tan\psi = \psi$$

showing that the resultant velocity at P is along the tangent at P.
If **a** be the acceleration vector of the particle at P, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

$\therefore$ the x-component of the acceleration of $P = \frac{d^2x}{dt^2} = \ddot{x}$, positive in the direction of x increasing, and the y-component of the acceleration of $P = \frac{d^2y}{dt^2} = \ddot{y}$,positive in the direction of y increasing.
The resultant acceleration of

$$P = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

Example 2: The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}$$

If the velocity and displacement are zero at $t = 0$, find the velocity and displacement at any time.
Solution: Here, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}$.
Integrating w.r.t. ' t ', we have

$\mathbf{v} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} + \mathbf{c}_1$, where $\mathbf{c}_1$ is a constant vector.

But at $t = 0, \mathbf{v} = \mathbf{0}$;
$\therefore$ $\mathbf{0} = 4\mathbf{j} + \mathbf{c}_1$ or $\mathbf{c}_1 = -4\mathbf{j}$.
$\therefore$ Velocity $\mathbf{v} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} - 4\mathbf{j}$.
Again $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} - 4\mathbf{j}$.
Integrating, w.r.t. ' t ', we have

$$\mathbf{r} = -3\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j} + \frac{8}{3}t^3\mathbf{k} - 4t\mathbf{j} + \mathbf{c}_2$$

where $\mathbf{c}_2$ is a constant vector.
But at $t = 0, \mathbf{r} = \mathbf{0}$; $\therefore$ $\mathbf{0} = -3\mathbf{i} + \mathbf{c}_2$ or $\mathbf{c}_2 = 3\mathbf{i}$.
$\therefore$ Displacement from the origin is given by

$$\mathbf{r} = -3\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j} + \frac{8}{3}t^3\mathbf{k} - 4t\mathbf{j} + 3\mathbf{i}$$

or

$$\mathbf{r} = 3(1-\cos 2t)\mathbf{i} + 2(\sin 2t - 2t)\mathbf{j} + \frac{8}{3}t^3\mathbf{k}$$

Example 4: A particle moves in the curve $y = a\log\sec\left(\frac{x}{a}\right)$ in such a way that the tangent to the curve rotates uniformly; prove that the resultant acceleration of the particle varies as the square of the radius of curvature.

Solution: Since the tangent to the curve rotates uniformly,

$$\therefore\ \frac{d\psi}{dt} = c \text{ (constant)} \qquad (1)$$

Equation of the path is

$$\therefore \quad y = a\log\sec(x/a).$$

$$\text{OR} \quad \frac{dy}{dx} = \frac{a}{\sec(x/a)}\cdot\sec\frac{x}{a}\tan\frac{x}{a}\cdot\frac{1}{a}$$

$$\therefore \quad \tan\psi = \frac{dy}{dx} = \tan\frac{x}{a}.$$

$$\therefore \quad \Psi = x/a \text{ or } x = a\psi.$$

$$\text{Now} \quad \frac{dx}{dt} = a\frac{d\psi}{dt} = ac \text{ and } \frac{d^2x}{dt^2} = 0.$$

$$\therefore \quad \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \left(\tan\frac{x}{a}\right)\cdot ac. \qquad (2)$$

$$\therefore \quad \frac{d^2y}{dt^2} = ac\sec^2\frac{x}{a}\cdot\frac{1}{a}\cdot\frac{dx}{dt}$$

$$= ac\sec^2\frac{x}{a}\cdot\frac{1}{a}\cdot ac = ac^2\sec^2\frac{x}{a}.$$

$$\therefore \quad \text{Resultant acceleration } = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

$$= \sqrt{\left[(0)^2 + \left(ac^2\sec^2\frac{x}{a}\right)^2\right]} = ac^2\sec^2\frac{x}{a}.$$

Also, the radius of curvature $\rho = \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}$.

But $\frac{dy}{dx} = \tan\frac{x}{a}$ implies that

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\frac{x}{a}.$$

$$\therefore \quad \rho = \frac{[1+\tan^2(x/a)]^{3/2}}{(1/a)\sec^2(x/a)} = \frac{a\sec^3(x/a)}{\sec^2(x/a)} = a\sec(x/a)$$

$$\text{or} \quad \sec(x/a) = \rho/a.$$

∴ from (2), the resultant acceleration $= ac^2(\rho/a)^2 = (c^2/a)\rho^2.$

CHECK YOUR PROGRESS

Multiple Choice Question

1. Angular velocity is defined as:

A. Rate of change of displacement
B. Rate of change of angular displacement
C. Rate of change of linear velocity
D. Rate of change of radial distance

2. The SI unit of angular velocity is:

A. rad
B. rad/s
C. m/s
D. m/s^2

3. If a particle moves in a circle of radius *r* with linear speed *v*, then angular velocity ω is:

A. $\omega = rv$
B. $\omega = v/r$
C. $\omega = r/v$
D. $\omega = v^2r$

4. Angular acceleration is:

A. Rate of change of angular velocity
B. Rate of change of linear velocity
C. Product of ω and r
D. None of these

5. The SI unit of angular acceleration is:

A. rad/s
B. m/s^2
C. rad/s^2
D. rad/m^2

6. When angular velocity is constant, angular acceleration is:

A. Zero
B. Infinite
C. Maximum
D. Constant but non-zero

13.8 SUMMARY

Angular Velocity**:** Angular velocity (ω) measures how fast a body rotates about a fixed point or axis. It is defined as the rate of change of angular displacement.

Angular acceleration: Angular acceleration (α) gives **the** rate at which angular velocity changes**.**

If angular velocity is constant, then angular acceleration is zero**.**

Tangential acceleration in circular motion is given by:at=rα

Rate of Change of a Unit Vector in a Plane: In polar coordinates, unit vectors **change direction** even if the magnitude is fixed.

13.9 GLOSSARY

Angular displacement: The angle through which a body rotates about a fixed point or axis. Measured in radians.

Unit Vector: A vector of magnitude 1, used to represent direction only.

Radial Unit Vector (êr): A unit vector that points outward from the origin toward the particle’s position.

13.10 REFERENCES

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3. P. N. Chandiramani & P. R. Godbole (2017). **Engineering Mechanics: Statics and Dynamics**. McGraw-Hill Education India.
4. Rajput, R. K. (2013). **Engineering Mechanics**. S. Chand Publishing.
5. Narayanaswamy, R. (1997). **Engineering Mechanics**. Narosa Publishing House.
6. A. K. Tayal (2011). **Engineering Mechanics**. Umesh Publications.

13.11 SUGGESTED READING

1. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige *Excellent for understanding theoretical principles and solving analytical problems.*

*2.***Engineering Mechanics** – R. K. Bansal *Useful for Indian students; provides numerous solved examples and practice problems.*

3. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler *Provides strong conceptual explanation with real-life engineering applications.*

4 . **A Textbook of Engineering Mechanics** – S. S. Bhavikatti *Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6*

13.12 TERMINAL QUESTIONS

1. Define angular velocity**.** Derive the relation between linear velocity and angular velocity of a particle moving along a circular path. Illustrate the physical meaning with a diagram.

2. What is angular acceleration? Obtain the expression for angular acceleration and explain how it affects the tangential acceleration of a particle in circular motion.

13.13 ANSWERS

MCQ 1 – B
MCQ 2 – B
MCQ 3 – B
MCQ 4 – A
MCQ 5 – C
MCQ 6 – A

UNIT 14: - Central Orbit

CONTENTS:

14.1 *INTRODUCTION: -*

In classical mechanics, a central orbit is the route taken by a particle traveling under the influence of a central force a force that is always directed toward or away from a fixed point and whose magnitude is determined solely by its distance from that point. This fixed position is referred to as the center of force. Because the force applies along the line connecting the particle and the center, the motion is limited to a plane, and the particle's angular momentum is conserved. Celestial mechanics, atomic physics, and orbital theory all heavily rely on central forces. Common examples include gravitational pull between heavenly entities and electrostatic attraction between charged particles. Depending on the nature of the central force (e.g., inverse-square law), the resulting orbits can take the form of circles, ellipses, parabolas, or hyperbolas. The study of central orbits contributes to a better understanding of planetary motion, scattering issues, and the stability of systems controlled by central interactions.

This unit will discuss about the concepts of central force and central orbit, the differential equation governing a central orbit, the rate of description of sectorial area, the nature of elliptic orbits, and the ideas of apse and apsidal distance.

14.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- To understand the concept of a central Force.
- To study the nature of a central orbit,
- To derive and analyze the differential equation of a central orbit.
- To understand the conservation of angular momentum.
- To examine the rate of description of sectorial area for a particle in a central force field.
- To explore different types of orbits, especially elliptic orbits, and understand the conditions under which they occur.
- To study the concepts of apse and apsidal distance.

14.3 *CENTRE FORCE: -*

A force which is always directed toward a fixed point is called Central force and fix point is known as centre force.

Or

A central force is a force that always acts along the line joining a particle to a fixed point, called the center of force, and whose magnitude depends only on the distance of the particle from that point. This means the force is directed either toward the center (attractive) or away from it (repulsive), and it does not depend on the direction of motion.

Mathematically, a central force F can be written as:

$$\vec{F} = F(r)\hat{r}$$

where F(r) depends only on the radial distance r and $\hat{r}$ is the unit vector toward the center.

Because the force is always central, the torque on the particle is zero, which implies that angular momentum is conserved. As a result, the motion of the particle lies in a plane, and the particle obeys the law of equal areas in equal times, which is fundamental in orbital mechanics.

14.4 *CENTRE ORBIT: -*

A central orbit is the path described by a particle moving under the action of central force. The motion of a planet about the sun is an important example of a central orbit.

Theorem: A central orbit is always a plane curve.

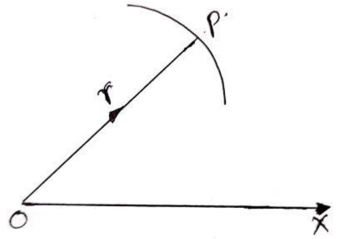


Fig.1

Assume that the center of force O is the vectors' origin. Let P be the particle's position in a central orbit at any time t, and let$\overrightarrow{OP} = r$. The acceleration vector of the particle at point P is $\frac{d^2r}{dt^2}$. This is because the particle moves under the action of the central force, with the center at O. The only force acting on the particle at P is along the line OP or PO, so the acceleration vector of P is parallel to the vector OP.

$$\frac{d^2r}{dt^2} \text{ is parallel to } r \Rightarrow \frac{d^2r}{dt^2} \times r = 0$$

$$\frac{d^2r}{dt^2} \times r + \frac{dr}{dt} \times \frac{dr}{dt} = 0 \qquad \left[\frac{dr}{dt} \times \frac{dr}{dt} = 0\right]$$

$$\frac{d}{dt}\left(\frac{dr}{dt} \times r\right) = 0$$

$$\frac{dr}{dt} \times r = a\ constant\ vector = h,$$

Now taking the product

$$r \cdot \left(\frac{dr}{dt} \times r\right) = r \cdot h$$

But

$$r \cdot h = 0$$

which shows that r is always perpendicular to a constant vector h.

14.5 *DIFFERENTIAL EQUATION OF CENTRE ORBIT: -*

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane two obtained the differential equation of the path.

Let a particle move in a plane with an acceleration P which is always directed to a fix point O in the plane take the centre of force O as the whole lead OX the initial line and R theta the polar coordinate of the position P of the moving particle at any instant t.

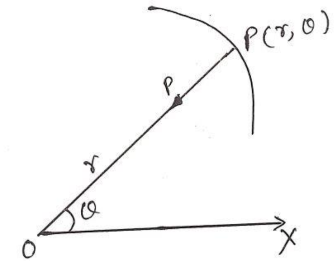


Fig.2

Since the acceleration of the particle is always directed towards the pole O there for the particle has only the radial acceleration and the transfer component of the acceleration of the particle is always zero so the radial acceleration is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \qquad \dots (1)$$

(- Sign has been taken because the radial acceleration P is in the direction of r decreasing) and the transverse acceleration $i.e.,$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \qquad \ldots(2)$$

From (2), we have

Integrating, we obtain

$$r^2\frac{d\theta}{dt} = constant = h$$

Let $r = \frac{1}{u}$

Now

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$$

Also

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}.u^2h = -h\frac{du}{d\theta}$$

and

$$\frac{d^2r}{dt^2} = -h\ \frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h\ \frac{d^2u}{d\theta^2}(u^2h) = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u^2}(u^2h)^2 = -P$$

$$h^2u^2\frac{d^2u}{d\theta^2} + \frac{1}{u^2}h^2u^3 = P$$

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

Hence, which is the differential equation of the central Orbit in polar form.

Pedal Form: If P is the length of the perpendicular drawn from the origin to the tangent at the point P, we obtain

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

But

$$u = \frac{1}{r}, therefore\ \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}\ , \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

So

$$\frac{1}{p^2} = u^2 + \frac{1}{r^4}\left(\frac{du}{d\theta}\right)^2$$

Differentiating w.r.t. x, we get

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$-\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}.\frac{P}{h^2u^2}$$

$$-\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{d\theta} = \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2u^2}\right)$$

$$\frac{1}{p^3}\frac{dp}{dr} = \left(\frac{1}{r^2}\right)\left(\frac{P}{h^2u^2}\right) = u^2.\frac{P}{h^2u^2} = \frac{P}{h^2}$$

$$P = \frac{h^2}{p^3}\frac{dp}{dr}$$

which is the differential equation of a Central Orbit in pedal form.

14.6 *RATE OF DESCRIPTION OF THE SECTORIAL AREA: -*

If every Central Orbit the factorial area traced out by the radius vector to the centre of the force increases uniformly by unit of the time and the linear velocity where is inversely as the perpendicular from the centre upon the tangent to the path.

Let O be the board, and OX be the initial line. Let $P(r,\theta)$ and $Q(r+\delta r, \theta + \delta\theta)$ represent the particle's position in a central orbit at time t and $t+\delta t$ respectively.

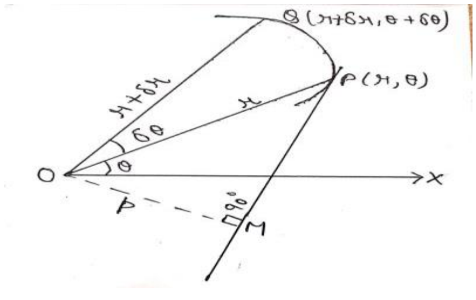


Fig.3

The sectorial area OPQ defined by the particle in time

$$\delta t = \text{area of the } \Delta OPQ$$

$$[\therefore the\ point\ Q\ is\ very\ close\ to\ Pand\ take\ this\ limit\ Q \to P]$$

$$= \frac{1}{2} OP.OQ\ sin < POQ = \frac{1}{2} r(r + \delta r) sin\delta\theta$$

$\therefore$ the rate of description of the sectorial area

$$= \lim_{\delta t \to 0} \frac{sectorial\ area\ OPQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r(r + \delta r) sin\delta\theta}{\delta t}$$

$$= \frac{1}{2} r(r + \delta r). \frac{sind\theta}{\delta\theta}. \frac{\delta\theta}{\delta t} = \frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right) = \frac{1}{2} h$$

$$\left[\because r^2 \left(\frac{d\theta}{dt}\right) = h\right]$$

Thus the rate of sectorial area is constant and is equal to $h/2$.

The rate of the description of the factorial area is also called the aerial velocity of the particle about the fixed point O

Again for a central Orbit we obtain

$$r^2 \left(\frac{d\theta}{dt}\right) = h$$

$$r^2 \left(\frac{d\theta}{ds} \frac{ds}{dt}\right) = h$$

$$r^2 \left(\frac{d\theta}{ds} v\right) = h$$

But

$$r \frac{d\theta}{ds} = sin\phi$$

where ϕ is the angle between the radius vector and tangent.

$$r^2 \left(\frac{d\theta}{ds}\right) = \phi sin\phi = p$$

where p is the length of the perpendicular drawn from the pole O on the tangent at P.

Putting $r^2\left(\frac{d\theta}{ds}\right) = p$

$$\Rightarrow \qquad r^2\left(\frac{d\theta}{ds}v\right) = h \Rightarrow vp = h$$

$$v = \frac{h}{p}$$

$$v \propto 1/p$$

The linear velocity at point P changes inversely with the perpendicular distance from the fixed point to the tangent of the path.

$$v^2 = \frac{h^2}{p^2}$$

But

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

This is the linear velocity at any point of the path of a central orbit.

14.7 *ELLIPTIC ORBIT: -*

A particle moves in ellipse under a for which is always directed towards its focus to find

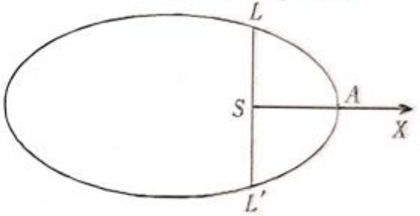
- the law of force
- the velocity at any point of its path
- the periodic time

We know that the polar equation of an ellipse referred to its focus S as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

or

$$u = \frac{1}{l} + \frac{e}{l}\cos\theta, \qquad \ldots(1)$$



where $u = 1/r$. Differentiating, we have

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

(i) **Law of Force:** We know that the differential equation of a central orbit referred to the centre of force as pole is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

where P is the central acceleration assumed to be attractive.

Now here $P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$

$$= h^2u^2\left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right], \text{ substituting for } u \text{ and } d^2u/d\theta^2$$

$$= \frac{h^2u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2},$$

where $\mu = h^2/l$ or $h^2 = \mu l$.

$\therefore$ $P \propto \frac{1}{r^2}$.

Hence the acceleration varies inversely as the square of the distance of the particle from the focus.

(ii) **Velocity:** We know that the velocity in a central orbit is given by

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

$\therefore$ here, $v^2 = h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(\frac{-e}{l}\sin\theta\right)^2\right]$

$$= h^2\left[\frac{1}{l^2} + \frac{2e}{l^2}\cos\theta + \frac{e^2}{l^2}\right] = \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\,\frac{e\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right]$$

$$= \mu\left[2u - \frac{1-e^2}{l}\right] = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right].$$

If $2a$ and $2b$ are the lengths of the major and the minor axes of the ellipse, we have

$$l = \text{the semi-latus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

$$\therefore \qquad \frac{1-e^2}{l} = \frac{1}{a}.$$

$$\therefore \qquad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

which gives the velocity of the particle at any point of its path.

Thus, the aforementioned equation indicates that the velocity's magnitude at any point along the path is only contingent upon the distance from the focus and is unaffected by the direction of motion.

(iii) Periodic time: We know that in a center orbit, the rate of description of the factorial area is constant and equals $h/2.$ Let T be the time period for one complete revolution, which is the time taken by the particle to describe the entire ellipse. The factorial area trace used to describe the entire ellipse is equal to the ellipse's total area.

$$T\left(\frac{h}{2}\right) = the\ whole\ area\ of\ the\ ellipse = \pi ab$$

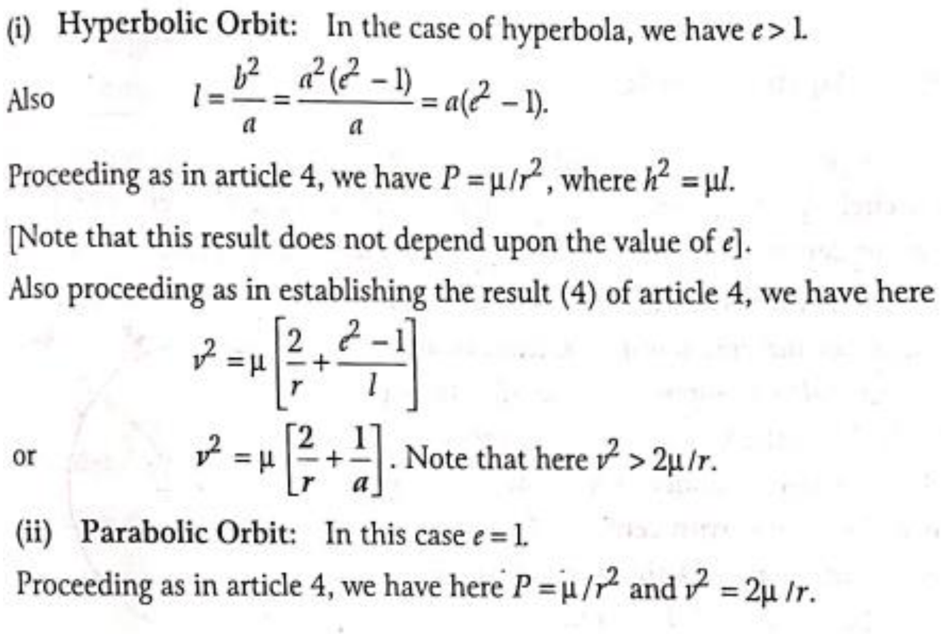
$$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}}$$

$$T = \frac{2\pi ab}{\sqrt{\{\mu(b^2/a)\}}}$$

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}},$$

Hence the time period for one complete revolution to $a^{3/2}$, being a semi-major axis.

14.8 *HYPERBOLIC AND PARABOLIC ORBITS: -*



(i) **Hyperbolic Orbit:** In the case of hyperbola, we have $e > 1$.

Also $$l = \frac{b^2}{a} = \frac{a^2(e^2-1)}{a} = a(e^2-1).$$

Proceeding as in article 4, we have $P = \mu/r^2$, where $h^2 = \mu l$.

[Note that this result does not depend upon the value of e].

Also proceeding as in establishing the result (4) of article 4, we have here

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right]$$

or $$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right].$$ Note that here $v^2 > 2\mu/r$.

(ii) **Parabolic Orbit:** In this case $e = 1$.

Proceeding as in article 4, we have here $P = \mu/r^2$ and $v^2 = 2\mu/r$.

14.9 *VELOCITY FROM INFINITY: -*

In relation with the central orbit, the phrase velocity from infinity at any place refers to the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the influence of the attractive force in accordance with the orbit's law.

Assume a particle moves from rest to infinity in a straight path under the influence of the central attractive acceleration P, which directs it to the centre of force O.

Assume Q represents the particle's position at any time t, with $OQ = r$, and v is its velocity at Q. The expression for acceleration at point Q is $v(dv/dr)$.

$$v\frac{dv}{dr} = -P$$
$$vdv = -Pdr$$

Now the integrating, we get

$$\int_0^V vdv = -\int_\infty^a P\,dr$$
$$\frac{1}{2}V^2 = -\int_\infty^a P\,dr$$
$$V^2 = -2\int_\infty^a P\,dr$$

This gives the velocity from infinity at distance a from the center of the force well moving under the central acceleration P associated with the orbit.

14.10 *VELOCITY IN A CIRCLE: -*

The term velocity in a circle at any point in the central orbit refers to the velocity required to describe a circle passing through that point and falling under the action of the prescribed force connected with the orbit.

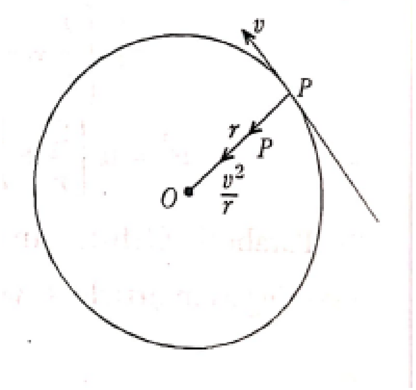


Fig.4

Assume that the center of force O is the pole. let PV the central acceleration directed towards O at any point P of the orbit where op is equal to hour suppose we is the velocity in the circle at P then velocity at the point P of a particle which move under the same Center Acceleration P in a circle with center at o but for a circle with center at the pole the radius vector op is also normal to the circle.

The central radial acceleration P= the inward normal acceleration v^2/ρ.

$$P = v^2/\rho$$

Thus, when moving under a central attractive acceleration P, the velocity V in a circle at a distance a from the center of force is provided by

$$V^2 = a.[P]_{r=a}$$

14.11 *GIVEN CENTRAL ORBIT, TO FIND THE LAW OF FORCE: -*

CaseI: *The equation of the orbit being given in polar form (r, θ).*

We know that the center of the force is referred to as Pole, the differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \qquad \dots (1)$$

where P is the central acceleration as you to be attractive.

From the given orbit equation, we can simply calculate u and $\frac{d^2u}{d\theta^2}$ and substitute the value (1). Thus, if P is positive, the force is attracting; if P is negative, the force is repulsive.

CaseI: *The equation of the orbit being given in padel form (p,r).*

The differential equation of (p, r) form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P \qquad \dots(2)$$

Form the given equation we find the dp/dr and substitute in (2), we find P.

SOLVED EXAMPLES

EXAMPLE1: A particle describe the curve $r^n = a^n cos n\theta$ under a force to the pole. Find the law of the force and obtain the law of force under which a cardioid can described.

SOLUTION: First part: The given equation is

$$r^n = a^n cos n\theta$$

Substituting $r = 1/u$, we get

$$\frac{1}{u^n} = a^n \cos n\theta \quad \text{or} \quad a^n u^n = \sec n\theta.$$

Taking both side logarithm in above equation

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n \sec n\theta \tan n\theta \quad \text{or} \quad \frac{du}{d\theta} = u \tan n\theta.$$

Differentiating again w.r.t. θ, we get

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta} \tan n\theta + u (\sec^2 n\theta) . n$$

$$= u \tan n\theta . \tan n\theta + un \sec^2 n\theta \qquad [\because du/d\theta = u \tan n\theta]$$

$$= u \tan^2 n\theta + un \sec^2 n\theta.$$

The differential equation of Central orbit is given by

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2} .$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\ (u + u\tan^2 n\theta + un\sec^2 n\theta)$$

$$= h^2u^3\ (\sec^2 n\theta + n\sec^2 n\theta) = h^2u^3\ (1+n)\sec^2 n\theta$$

$$= h^2\ (1+n)\,u^3\ .\,(a^n u^n)^2$$

$$= h^2a^{2n}\,(1+n)\,u^{2n+3} = \frac{h^2a^{2n}\,(1+n)}{r^{2n+3}} .$$

Hence $P \propto \frac{1}{r^{2n+3}i}.e.,$ the force varies inversely as the $(2n+3)th$ the power of the distance from the pole.

Second part: Substituting $n = I/2$, we obtain

$$r^{1/2} = a^{1/2}\cos\frac{1}{2}\theta$$

Squaring both side, we obtain

$$r = a\cos^2\frac{1}{2}\theta$$

$$r = \frac{1}{2}a\ .2\cos^2\frac{1}{2}\theta = \frac{1}{2}a\ (1+\cos\theta),$$

which is the equation of Cardioid.

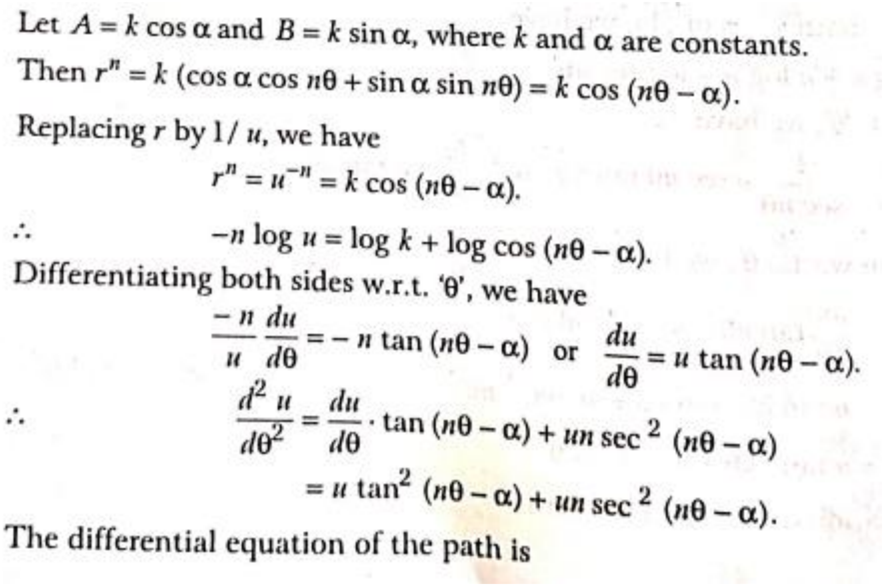
Now putting $n = \frac{1}{2}$ in the value of P, we get

$$P \propto \frac{1}{r^{1+3}}\quad i.e.,\ P \propto \frac{1}{r^4} .$$

EXAMPLE2: A particle describes the curve $r^n = A\,cosn\theta + Bsinn\theta$ under a force to the pole. Find the law of force.

SOLUTION: We know that the given equation is

$$r^n = A\, cosn\theta + Bsinn\theta$$



Let $A = k\cos\alpha$ and $B = k\sin\alpha$, where k and α are constants.

Then $r^n = k\,(\cos\alpha\cos n\theta + \sin\alpha\sin n\theta) = k\cos(n\theta-\alpha)$.

Replacing r by $1/u$, we have

$$r^n = u^{-n} = k\cos(n\theta-\alpha).$$

$$\therefore \quad -n\log u = \log k + \log\cos(n\theta-\alpha).$$

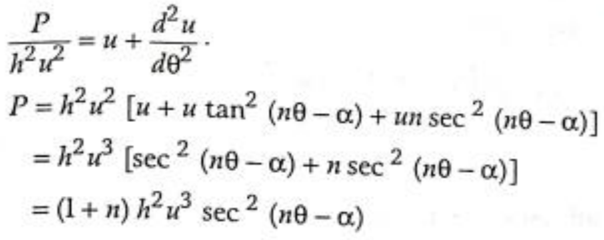
Differentiating both sides w.r.t. 'θ', we have

$$\frac{-n}{u}\frac{du}{d\theta} = -n\tan(n\theta-\alpha) \quad \text{or} \quad \frac{du}{d\theta} = u\tan(n\theta-\alpha).$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\cdot\tan(n\theta-\alpha) + un\sec^2(n\theta-\alpha)$$

$$= u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha).$$

The differential equation of the path is

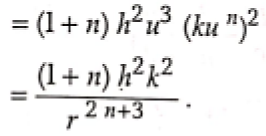


$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$P = h^2u^2\,[u + u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha)]$$

$$= h^2u^3\,[\sec^2(n\theta-\alpha) + n\sec^2(n\theta-\alpha)]$$

$$= (1+n)\,h^2u^3\sec^2(n\theta-\alpha)$$



$$= (1+n)\,h^2u^3\,(ku^n)^2$$

$$= \frac{(1+n)\,h^2k^2}{r^{2n+3}}.$$

Hence $P \propto \frac{1}{r^{2n+3}}.e.$, the force is inversely as the $(2n+3)th$ the power of the distance from the pole.

14.12 *APSE AND APSIDAL DISTANCE: -*

Apse: *An Apse is a point on the central Orbit at which the radius vector from the centre of the force to the point has a maximum and minimum value.*

Apsidal Distance: *The length of the radius vector at an Apps is called an apsidal distance*

Apsidal Angle: *The angle between two consecutive apsidal distances is called an apsidal angle.*

THEOREM: At the radius vector is perpendicular to the tangent i.e., at an apse the article moves at right angles to the radius vector.

SOLUTION: From the definition of apse, r is max. or min. at an apse $i.e.$, $u = 1/r$ is max. or min at an apse.

$$\frac{du}{d\theta} = 0$$

But

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 = \frac{1}{r^2}$$

$$p = r \; or \; r sin\phi = r$$

$$sin\phi = 1 \; or \; \phi = 90°$$

Hence at the radius vector is perpendicular to the tangent i.e., at an apse the article moves at right angles to the radius vector.

14.13 *PROPERTY OF APSE LINE:-*

Theorem: *If the central acceleration P is a single valued function of the distance, every apse-lime divides the orbit into equal and symmetrical portions, and thus there can only be two apsidal distances.*

Proof: Because the central acceleration P is a single-valued function of r, the particle's acceleration remains constant at any given distance r.
The differential equation for a center orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

$$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\frac{P}{u^2}$$

Multiplying both sides by $2du/d\theta$ and integrating w.r.t.$\theta,$ we get

$$v^2=h^2\left[\frac{d^2u}{d\theta^2}+u\right]=2\int\frac{P}{u^2}du+C$$

$$v^2=C-2\int Pdr$$

The equation (1) demonstrates that if P is a single valued function of distance r, the particle's velocity remains constant at the same distance r and is independent of direction of motion.

As a result, we can see that both velocity and acceleration are constant at the same distance from the center. As a result, if the particle's velocity is reversed at an apse, it will form a symmetrical orbit on both sides of the apse line.

Now, when the particle reaches a second apse, the route is symmetrical about this second apsidal distance as well. However, this is only conceivable if the following (third) apsidal distance is equal to the previous one (first) and the angle between the first and second apsidal distances is equal to the angle between the second and third apsidal distances. As a result, if the distance's central acceleration is a single valued function, the apsidal distances are limited to two options. The apsidal angle, which is the angle formed by any two consecutive apsidal distances, is also constant.

14.14 *AREAL VELOCITY: -*

Areal velocity is defined as the rate at which the radius vector of a particle traveling under a central force sweeps away area in relation to the center of force.

$$Areal\,Velocity=\frac{dA}{dt}$$

For motion under a **central force**, the areal velocity is **constant**, i.e.,

$$\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}=constant$$

This result is known as **Kepler's second law** and follows from the **conservation of angular momentum**.

14.15 *CHARACTERISTICS OF CENTRAL ORBITS: -*

A central orbit is the path of a particle moving under a force that is always oriented toward (or away from) a fixed point known as the center of force.

Main Characteristics

- **Force acts along the radius vector:** The force is constantly directed either toward or away from the center.
- **Motion is limited to the plane:** The particle's orbit is in a fixed plane.
- **Conservation of Angular Momentum:** Angular momentum about the center is constant.
- **Constant areal velocity:** The radius vector sweeps out equal areas at regular periods of time.
- **The equation of orbit is second-order:** The second-order differential equation describes the orbit.
- **Possible orbital forms:** Depending on the type of force, the orbit can be circular, elliptical, parabolic, or hyperbolic.

SELF CHECK QUESTIONS

1. Explain why the torque is zero in central force motion.
2. State Kepler's second law and relate it to central force motion.
3. Write the differential equation of a central orbit.
4. What is meant by bounded and unbounded orbits?
5. Distinguish between circular and elliptic orbits.
6. What condition must be satisfied for a circular orbit?
7. Explain the significance of apsidal points.
8. What is the relation between force law and nature of orbit?
9. What is meant by inverse square law of force?
10. Write any two applications of central orbit theory.
11. Show that the areal velocity of a particle moving under a central force is constant.
12. Obtain the expression for angular momentum of a particle in a central orbit.
13. Show that for an inverse square law force, the orbit is a conic section.
14. Find the condition for circular orbit under a central force.
15. Derive the expression for apsidal distance.

14.16 *SUMMARY: -*

In this unit, we have studied the concept of **central force**, which always acts along the radius vector toward or away from a fixed point. We discussed the idea of a **central orbit**, describing the path traced by a particle under such a force, and showed that the motion is confined to a plane with conserved angular momentum. The **differential equation of a central orbit** was derived to determine the shape of the orbit under a given force law. We then examined the **rate of description of the sectorial area**, establishing that the areal velocity remains constant, in accordance with Kepler’s second law. Further, we studied the **elliptic orbit** as an important solution for inverse square law forces, such as gravitation. Finally, the concepts of **apse**, **apsidal distance**, and their properties were explained, which describe the nearest and farthest points of the orbit from the center of force.

14.17 *GLOSSARY: -*

- **Central Force:** A force that always acts along the radius vector toward or away from a fixed point.
- **Center of Force:** The fixed point with respect to which the force on the particle is directed.
- **Central Orbit:** The path traced by a particle moving under the action of a central force.
- **Radius Vector:** The line joining the moving particle to the center of force.
- **Angular Momentum:** The moment of momentum of a particle about the center of force, which remains conserved in central force motion.
- **Areal Velocity:** The rate at which the radius vector sweeps out area with respect to time; it is constant for central orbits.
- **Differential Equation of Orbit:** A second-order differential equation that determines the shape of the orbit under a central force.
- **Sectorial Area:** The area swept by the radius vector in a given interval of time.
- **Elliptic Orbit:** A closed orbit in the shape of an ellipse formed under an inverse square central force.
- **Apse:** A point on the orbit where the particle is at its nearest or farthest distance from the center of force.
- **Apsidal Distance:** The distance of an apse from the center of force.
- **Line of Apsides:** The straight line joining the two apses of an orbit.
- **Periapsis:** The point of minimum distance of the particle from the center of force.
- **Apoapsis:** The point of maximum distance of the particle from the center of force.
- **Inverse Square Law:** A law in which the force varies inversely as the square of the distance from the center.

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14.19 *SUGGESTED READING: -*

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- A.R.Vashishtha(2017-18)- Mechanics.
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14.20 *TERMINAL QUESTIONS: -*

(TQ-1): Find the law of the force towards the pole under which the curve $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is described.

(TQ-2): The velocity at any point of the central orbit is one by $(1/n)$ of words it would be for a circular orbit at the same distance show that the central force varies as $\frac{1}{r^{(2n^2+1)}}$and that the equation of the orbit is

$$r^{n^2-1} = a^{n^2-1}.cos(n^2-1)\theta$$

(TQ-3): A particle describes the curve $r^2 = a^2cos2\theta$ under the force to the pole. Find the law of force.

(TQ-4): A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.

Or

A particle describes the curve $r = 2acos\theta$ under the force to the pole. Find the law of force.

(TQ-5): Find the law of the force towards the pole under which the curve $r^n = a^n cosn\theta$ can be described, if $n = 2$ then show that curve is lemniscate of Bernoulli.

(TQ-6): A particle moves with the central acceleration which varies inversely as the cube of the distance it be projected from an apse at a distance of from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a show that the equation to its path is $r\, cos\left(\frac{\theta}{\sqrt{2}}\right) = a$.

(TQ-7): A particle is projected from an apse at a distance a with the velocity from the Infinity under the action of the central acceleration $\frac{\mu}{r^{2n+3}}$. prove that the equation of the path is $r^n = a^n cosn\theta$.

(TQ-8): Explain the rate of description of sectorial area in a central orbit.

(TQ-9): Obtain the equation of an elliptic orbit under an inverse square law of force.

(TQ-10): Derive the differential equation of a central orbit.

(TQ-11): Write a detailed note on the importance of central force motion in classical mechanics.

(TQ-12): Derive the expression for time period of motion in a central orbit.

14.21 *ANSWERS: -*

(TQ-1): $P \propto 1/r^2$

(TQ-3): $P \propto 1/r^7$

(TQ-4): $P \propto 1/r^5$

(TQ-5): $P \propto 1/r^{2n+3}, P \propto 1/r^7$

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