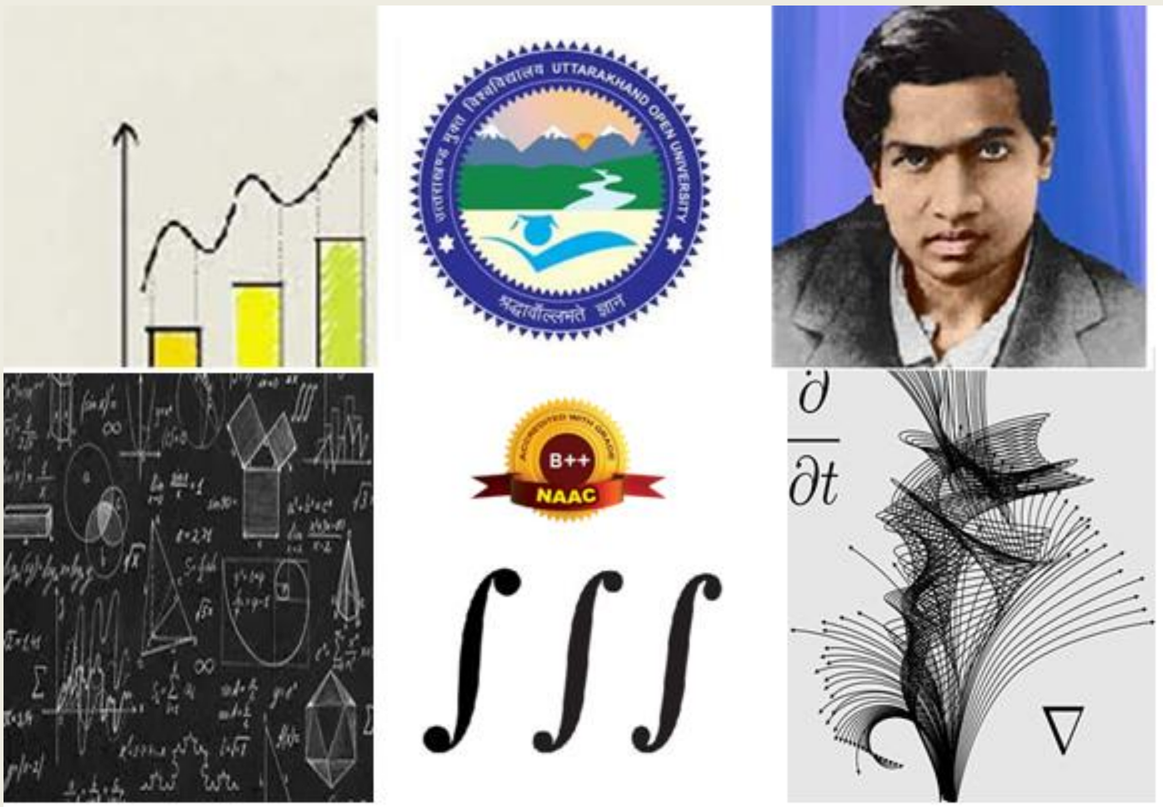
BACHELOR OF SCIENCE
(SIXTH SEMESTER)

MT(N)-302
COMPLEX ANALYSIS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: COMPLEX ANALYSIS

COURSE CODE: MT(N)-302

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School of Science
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<u>COURSE INFORMATION</u>

The self-learning material titled **Complex Analysis** has been carefully designed for B.Sc. (Six Semester) learners at Uttarakhand Open University, Haldwani, to provide convenient access to high-quality academic content. The course is divided into fourteen systematic units that cover the essential areas of complex analysis. **Units 1 and 2** introduce the basic concepts of the complex plane and stereographic projection, forming the foundation for understanding the subject. **Units 3 and 4** explore complex functions, their properties, and the important ideas of limit, continuity, and differentiability. This foundation is further strengthened through **Units 5 and 6**, which focus on analytic functions, followed by Units 7 and 8, which present the theory and applications of complex integration. **Units 9 and 10** extend learning to power series and the expansion of analytic functions, while **Unit 11** explains singularities and the behavior of functions near points of discontinuity. The advanced topics in **Units 12, 13, and 14** cover the residue theorem, its applications, analytic continuation, and the principle of uniqueness.

The material is structured not only to support the academic curriculum but also to help learners prepare for various competitive examinations. It explains fundamental concepts and theorems in a clear and accessible manner, making it suitable for both self-study and revision. Numerous examples, solved problems, and practice exercises have been carefully included to strengthen conceptual understanding and enhance problem-solving skills. Overall, this self-learning material enables students to develop a strong foundation in complex analysis and encourages independent learning through a well-organized and student-friendly approach.

BLOCK I
COMPLEX PLANE AND FUNCTIONS

UNIT 1: - Basics of Complex plane

CONTENTS:

1.1 *INTRODUCTION: -*

Complex numbers extend the real number system to include solutions to equations that have no real solutions, such as $x^2 + 1 = 0$. A complex number is of the form $z = a + ib$ where a the **real part**, b is the **imaginary part**, and i is the **imaginary unit** with $i^2 = -1$. They can be represented on the **complex plane**, with the real part on the horizontal axis and the imaginary part on the vertical axis. The term **"Complex Number"** was coined by C.F. Gauss, and later mathematicians like A.L. Cauchy, B. Riemann, and K. Weierstrass made significant contributions, enriching the subject with their original work. Basic operations with complex numbers, such as addition, subtraction, multiplication, and division, follow specific rules. The modulus and argument provide a polar form, offering an alternative way to express complex numbers, which is particularly useful in advanced mathematics and engineering.

1.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To find the solutions to equations that lack real solutions.
- To represent complex numbers as points or vectors on the complex plane.
- To solved the form of complex numbers.
- To solved the equation of straight line and circle.

1.3 *COMPLEX NUMBERS: -*

Complex numbers were introduced to provide solutions to equations like $x^2 + 1 = 0$, where there are no real solutions. These numbers include both real and imaginary parts and are denoted as $a + ib$, where a, b are the real numbers, is called *Complex Number*. and i represents the imaginary unit, which is defined as the square root of -1, also called i as *imaginary unit.*

If we represent a number in the form $z = x + iy$, then z is called a complex Variable . Here, x and y are called the real and imaginary parts of z respectively. Sometimes we write z as

$$z = (x, y)$$

we also write

$$R(z) = x, I(z) = y$$

If $x = 0, i.e.,\ z = iy,$ then z is known as pure imaginary number.

The complex conjugate of a complex number $z = x + iy$ is denoted as $x + iy$ and is equal to $x - iy$. In other words, it involves changing the sign of the imaginary part while leaving the real part unchanged.

$$z = x + iy \quad or \quad \bar{z} = x + iy$$

Example: the conjugate of $-3 - 5i$ is $3 + 5i$.

It is easy to verify that

$$R(z) = x = \frac{z + \bar{z}}{2}, \quad I(z) = y = \frac{z - \bar{z}}{2i}$$

1.4 *EQUALITY OF COMPLEX NUMBERS: -*

The equality of complex numbers follows the same principles as equality of real numbers. Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are considered equal if and only if both their real parts and imaginary parts are equal, i.e., $x_1 = x_2$ and $y_1 = y_2$.

Formally

$$z_1 = x_1 + iy_1 \;\; or \;\; (x_1, y_1), \;\; z_2 = x_2 + iy_2 \; or \;\; (x_2, y_2)$$

$$z_1 = z_2 \text{ if and only if } x_1 = x_2, \quad y_1 = y_2.$$

Remark: The phrases “greater than” or “less than” have no meaning in the set of complex numbers.

1.5 *ADDITION OF COMPLEX NUMBERS: -*

If $z_1 = x_1 + iy_1$ or (x, y) and $z_2 = x_2 + iy_2$ or (x, y) are any two complex numbers, then the sum of z_1, and $z_2$written as $z_1 + z_2$, is defined by

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Thus $\quad (2 + 4i) + (7 - 9i) = (2 + 7) + i(4 - 9) = 9 - 5i$

Properties of the Addition of complex numbers:

The addition of complex numbers is commutative, associative, admits of identity element and every complex number possesses additive inverse.

Commutativity of Addition in C: To Show that $z_1 + z_2 = z_2 + z_1$, where z_1 and z_2 are any complex numbers.

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, where x_1, y_1, x_2, y_2 are real numbers.

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1)$$

$$= (x_2, y_2) + (x_1, y_1) = (z_2 + z_1)$$

Hence

$z_1 + z_2 = z_2 + z_1$, for all complex numbers z_1 and z_2.

Associativity of Addition in C: To Show that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, where z_1 and z_2 are any complex numbers.

Proof: Let

$$z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3),$$

where $x_1, y_1, x_2, y_2, x_3, y_3$ are real numbers.

$$(z_1 + z_2) + z_3 = \{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$= (\{x_1 + x_2\} + x_3, \{y_1 + y_2\} + y_3)$$

$$= (x_1 + \{x_2 + x_3\}, y_1 + \{y_2 + y_3\})$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= z_1 + (z_2 + z_3)$$

Hence $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $\forall$ complex numbers z_1, z_2 and z_3.

Additive Identity: The complex number $(0,0)$ or $0 + i0$ is additive identity, since for every complex number (x, y), we obtain

$$(x, y) + (0,0) = (x + 0, y + 0) = (0,0) + (x, y)$$

The zero complex number, or complex number $(0,0)$, is represented simply by the symbol 0. A complex number $x + iy$ is considered non-zero if at least one of the variables, x and y, is not zero.

Additive Inverse: The complex number $(-x,-y)$ is the additive inverse of the complex number (x,y) since and also

$$(x,y) + (-x,-y) = (x-x, y-y) = (0,0) = \text{additive identity}$$

and $$(-x,-y) + (x,y) = (0,0)$$

The complex number $(-x,-y)$ is called the negative of the complex number (x,y) and we denote $(-x,-y)$ by $-(x,y)$.

Thus if $z = (x,y)$, then $-z = -(x,y) = (-x,-y)$.

Cancellation law for addition in C. If z_1, z_2 and z_3. are any complex numbers, then

$$z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2$$

1.6 *MULTIPLICATION OF COMPLEX NUMBERS:*

If $z_1 = x_1 + iy_1$ or (x_1, y_1) and $z_2 = x_2 + iy_2$ or (x_2, y_2) are any two complex numbers, then the product of z_1, and z_2 denoted by

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

$$= (x_1, y_1) + (x_2, y_2)$$

$$= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

Ex- $(3+3i)(6+4i) = (3\times 6 - 3\times 4) + i(3\times 4 + 3\times 6)$

$$= 6 + 30i$$

Or using the notation of order pairs, we obtain

$$(3,3)(6,4) = (3\times 6 - 3\times 4, 3\times 4 + 3\times 6) = (6,30)$$

Properties of the Multiplication of complex numbers:

The multiplication of complex numbers is commutative, associative, admits of identity element and every non-zero complex number possesses multiplicatively inverse.

Commutativity of Multiplication in C: To Show that $z_1 z_2 = z_2 z_1$, where z_1 and z_2 are any complex numbers.

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, where x_1, y_1, x_2, y_2 are real numbers, then we get

$$z_1 z_2 = (x_1, y_1)(x_2, y_2)$$

$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

$$= (x_2, y_2)(x_1, y_1) = (z_2 z_1)$$

Hence

$z_1 z_2 = z_2 z_1$, for all complex numbers z_1 and z_2.

Associativity of Multiplication in C: To Show that $(z_1 z_2) z_3 = z_1 (z_2 z_3)$, where z_1 and z_2 are any complex numbers.

Proof: Let

$$z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3),$$

where $x_1, y_1, x_2, y_2, x_3, y_3$ are real numbers.

$$(z_1 z_2) z_3 = \{(x_1, y_1)(x_2, y_2)\}(x_3, y_3)$$

$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)(x_3, y_3)$$

$$= (\{x_1 x_2 - y_1 y_2\} x_3 - \{x_1 y_2 + y_1 x_2\} y_3, \{x_1 x_2 - y_1 y_2\} y_3 + \{x_1 y_2 + y_1 x_2\} x_3)$$

$= (x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3, x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$ By distributive law

Also

$$z_1 (z_2 z_3) = (x_1, y_1)\{(x_2, y_2)(x_3, y_3)\}$$

$$= (x_1, y_1)(x_2 x_3 - y_2 y_3, x_2 y_3 + y_2 x_3)$$

$$= (x_1 \{x_2 x_3 - y_2 y_3\} - y_1 \{x_2 y_3 + y_2 x_3\}, x_1 \{x_2 y_3 + y_2 x_3\} + y_1 \{x_2 x_3 - y_2 y_3\})$$

$= (x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3, x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$ By distributive law

Finally, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$, $\forall$ complex numbers z_1, z_2 and z_3

Multiplicative Identity: The complex number $(1,0)$ or $1+i0$ is multiplicative identity, since for every complex number (x,y), we obtain

$$(x,y)(1,0) = (x.1 - y.0, x.0 + y.1) = (x,y) = (1,0)(x,y)$$

Multiplicative Inverse: The complex number (x,y) is the multiplicative inverse of the complex number (a,b),then we have

$$(x,y)(a,b) = (1,0) simply\ 1$$

$$= (xa - yb, xb + ya) = (1,0)$$

$$xa - yb = 1 \text{ and } xb + ya = 0$$

$$x = \frac{a}{a^2+b^2}, y = \frac{-b}{a^2+b^2}$$

$\Rightarrow$ $a^2 + b^2 \neq 0$, which implies that a and b are not both zero i.e., (a, is a non-zero complex number)

Thus every non-zero complex number possesses multiplicative inverse and the multiplicative inverse of the complex number $(a,b) \neq (0,0)$ is the complex number

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

If z is a non-zero complex number, the multiplicative inverse of 2 is denoted by $1/z$ or z^{-1}.

Cancellation law for multiplication in C. If z_1, z_2 and z_3 . are any complex numbers, then

$$z_1 z_3 = z_2 z_3 \Rightarrow z_1 = z_2$$

1.7 *DIFFERENCE OF COMPLEX NUMBERS: -*

If z_1 and z_2 are two complex number then

$$z_1 - z_2 = z_1 + (-z_2)$$

Thus $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, then we get

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2)$$

$$= (x_1 - x_2,\ y_1 - y_2)$$

1.8 *DIVISION OF COMPLEX NUMBERS: -*

If a complex number (x, y) exists such that a complex number (a, b) is divisible by a complex number (c, d), then $(x, y)(c, d) = (a, b)$.
We get

$$(xc - yd, xd + yc) = (a, b)$$
$$xc - yd = a \ \ and \ xd + yc = b$$

The above equations gives

$$x = \frac{ac + bd}{c^2 + d^2}, \qquad y = \frac{bc - ad}{c^2 + d^2}$$

For all $c^2 + d^2 \neq 0$, which implies that c and d are not both zero.

Or

To divide two complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

If $z_1 = a + ib$, $z_2 = c + id$, the conjugate of z_2 is $\overline{z_2} = c - id$ then

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{(a + ib)(c - id)}{c^2 + d^2}$$
$$= \frac{(ac - bd)(bc - ad)}{c^2 + d^2}$$
$$= \left(\frac{ac-bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right) \text{ if } c^2 + d^2 \neq 0$$

Therefore, in the set of complex numbers, division is always allowed, with the exception of by $(0, 0)$. If z_1 and z_2 are two complex numbers such that $z_1 \neq 0$, then the relation defines the quotient of z_1 and z_2.

$$\frac{z_1}{z_2} = z_1.\frac{1}{z_2} = z_1.(z_2)^{-1}$$

1.9 *MODULUS OF COMPLEX NUMBERS: -*

A complex number's modulus is a measurement of its absolute value or magnitude. Denoted by $\mid z \mid$, the modulus of a complex number $z = x + iy$, where an is the real part and b is the imaginary part, is the square root of the sum of the squares of its real and imaginary components, or $z = \sqrt{}(x^2 + y^2)$.
Clearly, $\mid z \mid = 0$ if and only if $x = 0$ and $y = 0$ That is, if and only if $z = 0$. Additionally, it is easily understood that for any complex number $z, |z| \geq R(z) \ and \ |z| \geq I(z)$.
Recall that we have for every real value of θ, we get

$$|cos\theta + isin\theta| = \sqrt{cos^2\theta + sin^2\theta} = 1$$

Therefore, the complex number $cos\theta + isin\theta$ is referred to as a uni-modular complex number since its modulus is always equal to 1.

If z_1 and z_2 are any two complex numbers, then

$$|z_1 z_2| = |z_1||z_2|$$

If $z_1$is any complex number and z_2 is a complex number that is not zero, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

1.10 *CONJUGATE OF COMPLEX NUMBERS: -*

The complex number $x - iy$ is known as the conjugate of the complex number of z and is represented by the symbol $\bar{z}$. If $z = x + iy$ is any complex number. Therefore, if

$$z = 2 + 3i \text{ and } z = 2 - 3i$$

i.e., $$|z| = |\bar{z}|$$

The following results are given below:

i. $z_1 = z_2$ if and only if $\overline{z_1} = \overline{z_2}$
ii. $\overline{(\bar{z})} = z$.
iii. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}, \overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}\ \forall\ z_2 \neq 0$.
iv. If $z = x + iy$, then
$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2R(z)$$
v. A complex number purely imaginary if and only if $z + \bar{z} = 0$.
vi. If $z = x + iy$, then
$$z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2iI(z)$$
vii. A complex number purely real if and only if $z - \bar{z} = 0$.
viii. If $z = x + iy$, then $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = \left[\sqrt{x^2 + y^2}\right]^2 = |z|^2$

Therefore, the product of two conjugate complex numbers is always ≥ 0, or a totally real number that is never negative.

1.11 *ABSOLUTE VALUE: -*

For a complex number $z = a + ib$, where a the real part is and b is the imaginary part, the absolute value is defined as:

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

$$\therefore \quad |z|^2 = a^2 + b^2 = (a + ib)(a - ib) = z\bar{z}$$

$$|z|^2 = z\bar{z}$$

Also $$\overline{z_1 . z_2} = \bar{z_1}\ \bar{z_2}$$

Properties of the Absolute Value:

- **Non-negativity:** $|\ z\ | \geq 0$
  The absolute value is always non-negative.
- **Zero:** $|\ z\ | = 0$

  if and only if z=0 (i.e., both the real and imaginary parts are zero).

- **Multiplicatively:** $|z_1 . z_2\ | = |z_1||z_2|$
  The absolute value of the product of two complex numbers is the product of their absolute values.

- **Triangle Inequality:** $|z_1 + z_2\ | \leq |z_1| + |z_2|$
  The absolute value of the sum of two complex numbers is less than or equal to the sum of their absolute values.
- **Conjugate:** $|\ z\ | = |\ \bar{z}\ |$
  The absolute value of a complex number is equal to the absolute value of its conjugate.

1.12 *MODULUS AND ARGUMENT POLAR FORM OF COMPLEX NUMBERS*: -

Every non-zero complex numbers $x + iy$ *can always be put in the form* $r(cos\theta + isin\theta)$, *where* r *and* θ *are both real numbers.*

Let $x + iy = r(cos\theta + isin\theta) = rcos\theta + irsin\theta$. Then equating real and imaginary parts on both sides, we obtain

$$x = rcos\theta, y = rsin\theta$$

Then $$r = \sqrt{x^2 + y^2} = |x + iy| = |z|$$

$$\theta = \tan^{-1}\frac{y}{x}$$

It follows

$$z = x + iy = r(cos\theta + isin\theta) = re^{i\theta}$$

where r is known and is equal to the modulus of complex numbers z and that r, θ are called polar coordinates of z.

The **argument** of a complex number $z = x + iy$, denoted by as $\theta = amp(z)$ or $\theta = \arg(z)$, is the angle formed by the line joining the point $P(x, y)$ to the origin with the positive real axis, calculated as $\theta = \tan^{-1}\frac{y}{x}$, and together with the modulus, fully describes the number's position and direction in the complex plane.

- The argument of a complex number z is not unique because it can differ by any integer multiple of 2π.
- The principal value of the argument of a complex number z, denoted as $Arg(z)$, is the value of θ that lies within the interval $-\pi < \theta \leq \pi \; or \; 0 < \theta \leq 2\pi$.
- If $z = 0$, then $arg(z) = arg(0)$ is not defined and $arg(z)$ is defined only if only $z \neq 0$.
- If $Arg(z)$ denoted general value and argument $arg(z)$ denoted principal value, then
$$Arg(z) = arg(z) + 2n\pi \; \forall \; n \in I$$
where $I =$ set of integers.
- If $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}\frac{y}{x}, if \; x > 0, y > 0 \; or \; y \leq 0 \\ \pi + \tan^{-1}\frac{y}{x} if \; x < 0 \; and \; y \geq 0 \\ -\pi + \tan^{-1}\frac{y}{x} if \; x < 0 \; and \; y \geq 0 \\ \frac{\pi}{2} \; if \; x = 0, y > 0 \\ -\frac{\pi}{2} \; if \; x = 0, y < 0 \end{cases}$$

1.13 *GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS*: -

A complex number $z = x + iy$ is defined as an ordered pair of real numbers (x, y), where x the real part is and y is the imaginary part.

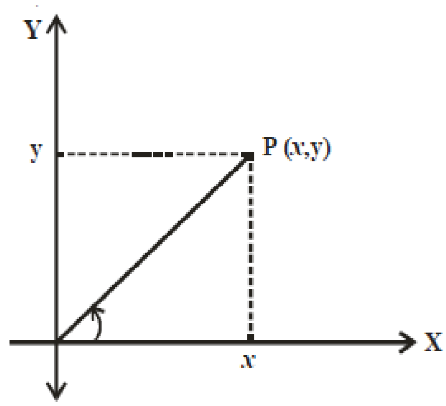

Fig.1

A complex number $z = x + iy$ can be represented by a point P with Cartesian coordinates (x, y) on a rectangular coordinate system, where the X −axis is the real axis and the $Y - a$xis is the imaginary axis.

Each complex number corresponds to a unique point in the plane, and conversely, each point in the plane corresponds to one and only one complex number.

1.14 *COMPLEX PLANE OR ARGAND PLANE*: -

The complex plane, also known as the **Argand plane** or **the z-plane**, is a two-dimensional coordinate system used to represent complex numbers geometrically. **Gauss** was the first to produce in 1799 that complex numbers are represented by points in a plane, then this concept that was developed by **Argand** in 1806. In this plane, each complex number $z = x + iy$ can identify with a point $P = (x, y)$, where x the real part is and y is the imaginary part. The horizontal axis, known as the real axis, contains all points of the form $(x, 0)$, representing real numbers, while the vertical axis, called the imaginary axis, includes points of the form $(0, y)$, representing purely imaginary numbers. Points not on the real axis represent general complex numbers with both real and imaginary parts. The origin $(0,0)$, represents the complex number $0 + i0$. This graphical representation helps in visualizing complex number operations and understanding their properties.

The nonnegative number $|\ z\ |$, called the modulus or absolute value of a complex number $z = x + iy$, represents the distance of the complex number z from the origin in the complex plane. It is calculated using the formula:

$$|z| = \sqrt{x^2 + y^2}$$

This is derived from the Pythagorean theorem, considering $z = (x, y)$ as a point in the xy −plane.(see Fig.2.)

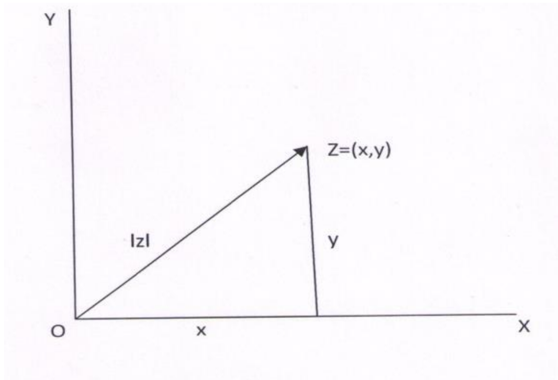

Fig.2.

The distance between two points $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the complex plane is given by the distance formula:

$$|z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula measures the straight-line distance between the points (x_1, y_1) and (x_2, y_2) in the complex plane.

1.15 *PROPERTIES OF PROPERTIES OF MODULUS ARGUMENTS OF COMPLEX NUMBERS*: -

Theorem1: Modulus and argument of the conjugate of two complex numbers, If z is any non-zero complex numbers, then

$$|\bar{z}| = |z| \; and \; arg\bar{z} = -argz$$

Proof: Let $|z| = r$ and $argz = \theta$

Then from modulus argument form a complex number, we get

$$z = r(cos\theta + isin\theta)$$

$$\bar{z} = r(cos\theta - isin\theta) = r[cos(-\theta) + isin(-\theta)]$$

Hence $|\bar{z}| = r = |z|$ and $arg\bar{z} = -argz = -\theta$

Theorem2: Modulus and argument of the product of two complex numbers, If z_1 and z_2 are any two non-zero complex numbers, then

$|z_1.z_2| = |z_1|.|z_2|$ and $arg(z_1z_2) = arg(z_1) + arg(z_2)$

Proof: $|z_1.z_2|^2 = (z_1.z_2)\overline{(z_1.z_2)} = z_1.z_2.\overline{z_1}.\overline{z_2}$

$$= (z_1.\overline{z_1})(z_2.\overline{z_2}) = |z_1|^2.|z_2|^2$$

$$|z_1.z_2|^2 = |z_1|^2.|z_2|^2$$

$\Rightarrow$

$$|z_1.z_2| = |z_1|.|z_2|$$

Let, the complex numbers, we get

$$z_1 = r_1(cos\theta_1 + isin\theta_1)$$
$$z_2 = r_2(cos\theta_2 + isin\theta_2)$$

Now, consider the product z_1z_2:

$$z_1z_2 = [r_1(cos\theta_1 + isin\theta_1)][r_2(cos\theta_2 + isin\theta_2)]$$

$$= r_1r_2[(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2) + i(cos\theta_1sin\theta_2 + sin\theta_1cos\theta_2)]$$

$$= r_1r_2[cos(\theta_1 + \theta_2) + sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = r_1r_2[cos(\theta_1 + \theta_2) + sin(\theta_1 + \theta_2)]$$

This shows that z_1z_2 modulus -argument form, we obtain

$$|z_1.z_2| = r_1r_2 = |z_1|.|z_2|$$

$$\arg(z_1z_2) = arg(z_1) + arg(z_2)$$

Remark: $|z_1| = \left|\frac{z_1}{z_2}.z_2\right| = \left|\frac{z_1}{z_2}\right|.|z_2|,$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$$

Theorem3 : Modulus and argument of the product of two complex numbers, If z_1 and z_2 are any two non-zero complex numbers, then

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Proof: Let z_1 and z_2 be two complex numbers with arguments θ_1 and θ_2 and moduli r_1 and r_2. In polar form, these complex numbers can be expressed as:

$$z_1 = r_1(cos\theta_1 + isin\theta_1)$$

$$z_2 = r_2(cos\theta_2 + isin\theta_2)$$

Now, consider the quotient z_1/z_2:

$$\frac{z_1}{z_2} = \frac{r_1(cos\theta_1 + isin\theta_1)}{r_2(cos\theta_2 + isin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}.\frac{(cos\theta_1 + isin\theta_1)}{(cos\theta_2 + isin\theta_2)}$$

To simplify the fraction, we multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}.\frac{(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2) + i(sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2)}{(cos\theta_2 + isin\theta_2)(cos\theta_2 + isin\theta_2)}$$

$$= \frac{r_1}{r_2}.\frac{(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2) + i(sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2)}{(cos^2\theta_2 + isin^2\theta_2)}$$

$$= \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$$

From this represent $\left|\frac{z_1}{z_2}\right|$ is standard polar form, we get

$$|z_1| = \left|\frac{z_1}{z_2}.z_2\right| = \left|\frac{z_1}{z_2}\right|.|z_2|,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and $$arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = arg(z_1) - arg(z_2)$$

Theorem4: Triangle Inequality

The modulus of the sum of two complex numbers is less than or equal to sum of their moduli.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof: Suppose, $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1 cos\theta_1 + r_2 cos\theta_2) + i(r_1 sin\theta_1 + r_2 sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{(r_1 cos\theta_1 + r_2 cos\theta_2)^2 + (r_1 sin\theta_1 + r_2 sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2} \qquad for\ \ cos(\theta_1 - \theta_2) \leq 1$$

$$= r_1 + r_2 = |z_1| + |z_2|$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Theorem5: The modulus of the difference of two complex numbers is less than or equal to difference of their moduli.

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Proof: Let, $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 - z_2 = r_1(cos\theta_1 + isin\theta_1) - r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1cos\theta_1 - r_2cos\theta_2) + i(r_1sin\theta_1 - r_2sin\theta_2)$$

$$|z_1 - z_2| = \sqrt{(r_1cos\theta_1 - r_2cos\theta_2)^2 + (r_1sin\theta_1 - r_2sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 - 2r_1r_2} \qquad for\ \ cos(\theta_1 - \theta_2) \geq -1$$

$$= r_1 - r_2 = |z_1| - |z_2|$$

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Remark: To prove

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

$$|z_1 - z_2| = |z_1 + (-z_2)|$$

$$\leq |z_1| + |-z_2| \text{ by theorem2}$$

$$= |z_1| + |z_2|$$

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

Hence $\quad |z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

Theorem6: To prove $|z_1 + z_2| \geq |z_1| - |z_2|$.

Proof: : Let, $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1 cos\theta_1 + r_2 cos\theta_2) + i(r_1 sin\theta_1 + r_2 sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{(r_1 cos\theta_1 + r_2 cos\theta_2)^2 + (r_1 sin\theta_1 + r_2 sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 cos(\theta_1 - \theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1 r_2} \qquad for \;\; cos(\theta_1 - \theta_2) \geq -1$$

$$= r_1 - r_2 = |z_1| - |z_2| \;\text{ if }\; r_1 > r_2$$

$$r_1 - r_2 = |z_1| - |z_2|$$

$$|z_1 + z_2| \geq |z_1| - |z_2| \;\text{ if }\; |z_1| > |z_2|$$

Theorem7: Parallelogram Law

The sum of squares of the length of diagonals of a parallelogram is equal to the sum of squares of length of its sides, i.e., prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$$

OR

To prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

Proof: Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1 cos\theta_1 + r_2 cos\theta_2) + i(r_1 sin\theta_1 + r_2 sin\theta_2)$$

$$z_1 - z_2 = (r_1 cos\theta_1 - r_2 cos\theta_2) + i(r_1 sin\theta_1 - r_2 sin\theta_2)$$

Now $|z_1 + z_2|^2 + |z_1 - z_2|^2 = [(r_1 cos\theta_1 + r_2 cos\theta_2)^{\mathbf{2}} + i(r_1 sin\theta_1 + r_2 sin\theta_2)^2] + [(r_1 cos\theta_1 - r_2 cos\theta_2)^2 + (r_1 sin\theta_1 - r_2 sin\theta_2)^2]$

$$= [r_1^2 + r_2^2 + 2r_1r_2cos(\theta_1 - \theta_2)] + [r_1^2 + r_2^2 - 2r_1r_2cos(\theta_1 - \theta_2)]$$

$$= [r_1^2 + r_2^2]$$

$$= 2[|z_1|^2 + |z_2|^2] \qquad \dots(1)$$

Geometrical interpretation: Let P and Q be the points in the Argand diagram representing the complex numbers $z1$and $z2$ respectively. On completing the parallelogram OPRQ, then we get

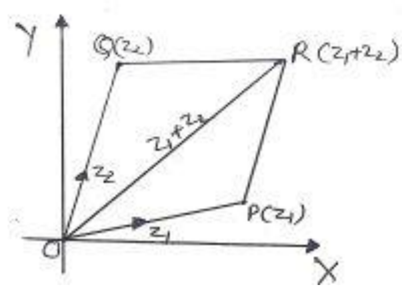

Fig.1

$$z_1 = \overrightarrow{OP}, z_2 = \overrightarrow{OQ}.$$

$$z_1 + z_2 = OP + OQ = OP + PR = OR,$$

$$z_1 - z_2 = OP - OQ = QP,$$

$$|z_1| = OP, \qquad |z_2| = OQ$$

$$|z_1 + z_2| = OR, |z_1 - z_2| = QP$$

From (1), we obtain

$$OR^2 + QP^2 = 2(OP^2 + OQ^2)$$

Theorem8: **(Equation of Straight line)**To find the equation of straight line joining two pointsz_1 and z_2 in the complex plane.

Proof: Let the equation of line AB joining the points A (z_1) and B(z_2), suppose point P(z) on it. So

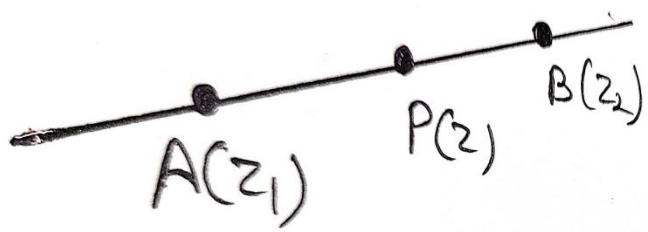

Fig.2.

$$arg\left(\frac{z-z_1}{z_1-z_2}\right) = 0 \text{ or } \pi$$

Consequently $\left(\frac{z-z_1}{z_1-z_2}\right)$ is purely real so that

$$\left(\frac{z-z_1}{z_1-z_2}\right) = \left(\overline{\frac{z-z_1}{z_1-z_2}}\right) = \left(\frac{\bar{z}-\overline{z_1}}{\overline{z_1}-\overline{z_2}}\right)$$

$$(z-z_1)(\overline{z_1}-\overline{z_2}) = (z_1-z_2)(\bar{z}-\overline{z_1})$$

$$z(\overline{z_1}-\overline{z_2}) - \bar{z}(\overline{z_1}-\overline{z_2}) - z_1\overline{z_1} + z_1\overline{z_2} + z_1\overline{z_1} - z_2\overline{z_1} = 0$$

$z(\overline{z_1}-\overline{z_2}) - \bar{z}(\overline{z_1}-\overline{z_2}) + (z_1\overline{z_2} - z_2\overline{z_1}) = 0$ is required equation of line.

Theorem9: (**Equation of a Circle)** To show that the equation of circle in the Argand plane can be put in the form

$$z\bar{z} + \bar{z}b + \bar{b}z + c = 0$$

where c is real and b is complex constant.

Proof: Suppose a be a complex coordinate of the centre C and r be the radius of circle. Consider any point $P(z)$ on the circle.
Then the length of line $CP =$ radius of circle or

$$|z-a| = r$$

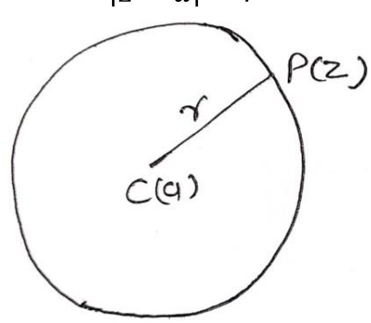

Fig.3.

Squaring both sides, we have

$$|z-a|^2 = r^2$$
$$(z-a)(\bar{z}-\bar{a}) = r^2$$
$$(z\bar{z} - \bar{a}z + a\bar{a} - a\bar{z}) = r^2$$
$$z\bar{z} - \bar{a}z - a\bar{z} + (|a|^2 - r^2) = 0$$

Taking $-a = b$ and $(|a|^2 - r^2) = c = real\ number$

$$z\bar{z} + \bar{z}b + \bar{b}z + c = 0$$

where c is real and b is complex constant.

EXAMPLE1: Find real numbers A and B, if $A + iB = \frac{3-2i}{7+4i}$

SOLUTION: Let we have

$$\frac{3-2i}{7+4i} = \frac{3-2i}{7+4i} \times \frac{7-4i}{7-4i}$$
$$= \frac{21 - 12i - 14i + 8i^2}{49 - 16i^2} = \frac{(21-8) - 26i}{49+16}$$
$$A + iB = \frac{13 - 26i}{65} = \frac{13}{65} - \frac{26}{65}i = \frac{1}{5} - \frac{2}{5}i$$

Equating real and imaginary parts, we obtain

$$A = \frac{1}{5}\ , B = -\frac{2}{5}$$

EXAMPLE2: Prove that $\left|a + \sqrt{a^2 - b^2}\right| + \left|a - \sqrt{a^2 - b^2}\right| = [|a-b| + |a-b|]^2$.

SOLUTION: Suppose $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

$$|z_1 z_2| = |z_1|.|z_2|$$

Now we shall prove the given problem

$$\left[\left|a + \sqrt{a^2 - b^2}\right| + \left|a - \sqrt{a^2 - b^2}\right|\right]^2$$
$$= \left|a + \sqrt{a^2 - b^2}\right|^2 + \left|a - \sqrt{a^2 - b^2}\right|^2 + 2\left|a + \sqrt{a^2 - b^2}\right|\left|a - \sqrt{a^2 - b^2}\right|$$

$$= 2\left[|a|^2 + \left|\sqrt{a^2 - b^2}\right|^2\right] + 2\left|\left[a + \sqrt{a^2 - b^2}\right]\left[a - \sqrt{a^2 - b^2}\right]\right|$$

$$= 2[|a|^2 + |a^2 - b^2|] + 2|a^2 - (a^2 - b^2)|$$

$$= 2[|a|^2 + |a^2 - b^2|] + 2|b^2|$$

$$= 2[|a|^2 + |b^2|] + 2|a^2 - b^2|$$

$$= [|a+b|^2 + |a-b|^2] + 2|a+b|.|a-b|$$

$$= [|a+b|^2 + |a-b|^2]^2$$

Hence

$\left|a + \sqrt{a^2 - b^2}\right| + \left|a - \sqrt{a^2 - b^2}\right| = [|a-b| + |a-b|]^2$ is required the solution.

EXAMPLE3: Determine the regions of Argand diagram given by

$$|z^2 - z| < 1.$$

SOLUTION: Let $z = re^{i\theta}$

Then $z^2 - z = r^2 e^{i2\theta} - re^{i\theta}$

$$= (r^2 cos2\theta - rcos\theta) + i(r^2 sin2\theta - rsin\theta)$$

$$|z^2 - z|^2 = (r^2 cos2\theta - rcos\theta)^2 + (r^2 sin2\theta - rsin\theta)^2$$

$$= r^4 + r^2 - 2r^3 \cos(2\theta - \theta)$$

But $|z^2 - z|^2 < 1$

Hence

$$r^4 + r^2 - 2r^3 \cos\theta < 1$$

or $r^4 + r^2 - 2r^3 \cos\theta - 1 < 0$

Hence

$$r^4 + r^2 - 2r^3 \cos\theta - 1 = 0$$

EXAMPLE4: Determine the region of z −plane for which

$$|z - 1| + |z + 1| \leq 3.$$

SOLUTION: Let $z = x + iy$

$$|z - 1| + |z + 1| = |x + iy - 1| + |x + iy + 1|$$

$$= \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}$$

But $|z - 1| + |z + 1| \leq 3$

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2} \leq 3$$

$$\sqrt{(x-1)^2+y^2} \leq 3-\sqrt{(x+1)^2+y^2}$$

$$(x-1)^2+y^2 \leq 9+(x+1)^2+y^2-6\sqrt{(x+1)^2+y^2}$$

$$0 < 4x+9-6\sqrt{(x+1)^2+y^2}$$

$$6\sqrt{(x+1)^2+y^2} \leq (4x+9)$$

$$36[(x+1)^2+y^2] \leq 16x^2+81+72x$$

$$36x^2+36+36y^2+72x \leq 16x^2+81+72x$$

$$36x^2+36+36y^2 \leq 16x^2+81$$

$$20x^2+36y^2 \leq 45$$

$$\frac{x^2}{(9/4)}+\frac{y^2}{(5/4)} = 1$$

EXAMPLE5: Show that the locus of z such that

$$|z-a|.|z+a| = a^2, a > 0$$

is a lemniscate.

SOLUTION: Let $|z^2-a^2| = a^2$ or $z^2-a^2 = a^2e^{i\lambda}$

Put $z = re^{i\theta}$. Then $r^2e^{i2\theta}-a^2 = a^2e^{i\lambda}$

This $\Rightarrow$ $r^2cos2\theta-a^2 = a^2cos\lambda$

$$r^2sin2\theta = a^2sin\lambda$$

Both above equations are squaring and adding

$$(r^2cos2\theta-a^2)^2+(r^2sin2\theta)^2 = a^4$$

$$r^2(r^2-2a^2cos2\theta) = 0$$

But $r \neq 0$ as $z \neq 0$

$r^2-2a^2cos2\theta = 0$ or $r^2 = 2a^2cos2\theta$

which is lemniscates.

SELF CHECK QUESTIONS

1. What is a complex number? Write its general form.
2. Define the real and imaginary parts of a complex number $z = x + iy$.
3. What is the imaginary unit i? What is the value of i^2?
4. How can every complex number be represented on a complex plane (Argand plane)?
5. What is the origin in the complex plane and what does it represent?
6. How is the modulus geometrically represented in the Argand plane?
7. Define the argument (amplitude) of a complex number.
8. What is the principal value of the argument?

1.16 *SUMMARY*: -

In this unit, we explored the fundamental ideas related to complex numbers and their graphical representation on the complex or Argand plane. A complex number, expressed as $z = x + iy$, combines a real and an imaginary part. We examined the equality of complex numbers and performed the basic operations of addition, subtraction, multiplication, and division. The concepts of modulus or absolute value, conjugate, and argument were introduced to describe the magnitude and direction of a complex number. Further, we expressed complex numbers in polar and exponential forms, which simplify many mathematical operations. The geometrical interpretation on the Argand plane provides a clear visual understanding of these operations. Important properties of modulus and argument were also discussed, followed by the idea of stereographic projection, which maps every point of the complex plane onto a sphere, thus extending the representation of complex numbers to include infinity.

1.17 *GLOSSARY*: -

- **Complex Number:** A number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit with $i^2 = -1$.
- **Real Part:** The component a in a complex number $a + bi$, representing a real number.
- **Imaginary Part:** The component b in a complex number $a + bi$, representing a real number multiplied by the imaginary unit i.
- **Imaginary Unit (i):** A mathematical constant satisfying $i^2 = -1$.

- **Equality of Complex Numbers:** Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.
- **Argand Plane (or Complex Plane):** A plane in which the x-axis represents the real part and the y-axis represents the imaginary part of a complex number.
- **Absolute Value(Modulus):** The distance from the origin to the point (a, b) in the complex plane, calculated as $| z |= \sqrt{a^2 + b^2}$ for a complex number $z = a + bi$.
- **Argument (Amplitude):** The angle θ formed with the positive real axis, calculated using $\theta = \tan^{-1}\frac{b}{a}$, (b/a) for a complex number $z = a + bi$.
- **Polar Form:** A way to expressed as $z = r(cos\theta + isin\theta)$ or $z = re^{i\theta}$.
- **Conjugate of complex number:** The conjugate of a complex number $a + bi$ is $a - bi$.
- **Addition of Complex Numbers:** Combining two complex numbers by adding their real parts and their imaginary parts separately: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- **Multiplication of Complex Numbers:** Multiplying two complex numbers using distributive property and the fact that $i^2 = -1$: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- **Division of Complex Numbers:** Dividing by multiplying the numerator and denominator by the conjugate of the denominator and simplifying: $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$.
- **Properties of Modulus:**

  i. $|z_1 z_2| = |z_1||z_2|$

  ii. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

  iii. $|z| = |\bar{z}|$
- **Properties of Argument:**

  i. $arg(z_1 z_2) = arg(z_1) + arg(z_2)$

  ii. $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

1.18 *REFERENCES*: -

- James Ward Brown and Ruel V. Churchill 2009 (Eighth Edition), Complex Variables and Applications.

- Elias M. Stein and Rami Shakarchi (2003), Complex Analysis..
- Theodore W. Gamelin(2001),Complex Analysis.

1.19 *SUGGESTED READING*: -

- file:///C:/Users/user/Downloads/Paper-III-Complex-Analysis.pdf
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- file:///C:/Users/user/Desktop/1456304480EtextofChapter1Module1%20(1).pdf

1.20 *TERMINAL QUESTIONS: -*

(TQ-1) Define the modulus of a complex number.

(TQ-2) If the complex numbers $sinx + icos2x$ and $cosx - isin2x$ are complex conjugate to each other, then the value of x.

(TQ-3) A relation R on the set of complex numbers is defined by $z_1Rz_2 \Leftrightarrow \frac{z_1-z_2}{z_1+z_2}$ real. Show that R is an equivalence relation.

(TQ-4) Show that the origin and the point representing the roots of the equation $z^2 + pz + q = 0$ form an equilateral if $p^2 = 3q$.

(TQ-5) Represent the complex number $z = 3 + 4i$ on the complex plane.

(TQ-6) Find the modulus and argument of $z = 1 + i$.

(TQ-7) Show that $| z_1z_2 |= |z_1||z_2|$.

(TQ-8) Find the conjugate of $z = 4 - 5i$ and plot it on the Argand plane.

(TQ-9) Determine the locus of points representing complex numbers satisfying $| z |= 2$.

(TQ-10) Show that $|z_1 - z_2| \geq (|z_1| - |z_2|)$

(TQ-11) Find the locus of z such that $Re(z) = 3$.

(TQ-12) Find the locus of z satisfying $| z - 2 |=| z + 2 |$.

(TQ-13) Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and deduce that $\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$ all the numbers concerned being complex.

(TQ-14) Find the principal value of $arg\ 'i'$.

(TQ-15) If $z = -1 + 3i$, find its polar form.

(TQ-16) If $z_1 = 2 + 3i$ and $z_2 = 1 - 4i$, find $z_1 + z_2$ and z_1z_2.

(TQ-17) Find the principal value of $arg\ (1 + i)$.

(TQ-18) Explain how **addition** and **multiplication** of complex numbers are represented geometrically in the complex plane.

(TQ-19) What is the **effect of multiplying a complex number by i** in the complex plane?

(TQ-20) Show that **multiplication by $e^{i\theta}$** corresponds to a **rotation** through an angle θ.

1.21 *ANSWERS: -*

TERMINAL ANSWERS (TQ'S)

(TQ-2) $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \ldots$

(TQ-5)$(x, y) = (3,4)$

(TQ-6) $z = \sqrt{2}e^{\frac{\pi}{4}}$

(TQ-8) $Origional\ point = (4, -5), conjugate\ point = (4,5)$

(TQ-9) $Radius = 2$

(TQ-11) $x = 3$

(TQ-12) $x = 0$

(TQ-15)$\sqrt{10}e^{i(\pi - tan^{-1}3)}$

(TQ-16) $z_1 + z_2 = 3 - i, .z_1z_2 = 14 - 5i$

(TQ-14) $\frac{\pi}{2}$

(TQ-17) $\frac{\pi}{4}$

UNIT2: Stereographic Projection

CONTENTS:

2.1 *INTRODUCTION*: -

Stereographic projection is a method of mapping points from the surface of a sphere onto a plane. It is achieved by projecting points from the North Pole of the sphere onto the equatorial plane**,** creating a one-to-one correspondence between the sphere (excluding the North Pole) and the plane. This projection preserves angles (is conformal) but distorts areas, making it particularly useful in geometry**,** complex analysis**,** cartography**,** and crystallography**.** In complex analysis, it provides a geometric representation of the extended complex plane or the Riemann sphere**,** linking complex numbers with points on the sphere.

2.2 *OBJECTIVES*: -

After studying this unit, the learner's will be able to

- To understand the **concept and definition** of stereographic projection as a method of mapping points from a sphere onto a plane.
- To recognize the **Riemann sphere** as a geometric model for the **extended complex plane** $\mathbb{C} \cup \{\infty\}$.

- To identify the correspondence between the **point at infinity** on the complex plane and the **projection point** on the sphere.

2.3 *EULER'S FORMULA*: -

The Taylor (Maclaurin) series expansions of $e^x, cosx,$ and $sinx$ can be used to obtain Euler's formula:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\cdots\cdots$$

Now putting $x = i\theta$ in e^x

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots\cdots\cdots$$

simplify power of i:

$$e^{i\theta} = \left(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \cdots\cdots\cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} - \cdots\cdots\cdots\right)$$

$$e^{i\theta} = cos\theta + isin\theta$$

when $\theta = \pi$

$$e^{i\pi} = cos\pi + isin\pi = -1 + 0i = -1$$

Hence

$$e^{i\pi} + 1 = 0$$

This is called Euler's Identity, and because it connects five essential constants, it is frequently hailed as the most exquisite mathematical equation:

$$e, i, \pi, 1 \; and \; 0$$

2.4 *nth ROOT OF UNITY*: -

The nth Root of Unity The nth roots of unity are the complex numbers that satisfy the equation

$$z^n = 1$$

where n is a positive integer.

Def: An nth root of unity is any complex number z such that when raised to the power n the result is 1.

In other words, if $z = e^{i\theta}$, then $z^n = 1$ when $\theta = \frac{2k\pi}{n}$, where $k = 0,1,2,3 \dots \dots \dots, n-1$.

Thus, the **nth roots of unity** are given by:

$$z_k = e^{i\frac{2k\pi}{n}} = cos\left(\frac{2k\pi}{n}\right) + isin\left(\frac{2k\pi}{n}\right),$$
$$k = 0,1,2,3 \dots \dots \dots, n-1$$

2.5 *POINT AT INFINITY*: -

The linear transformation $z \to w = f(z)$ is a one-to-one mapping of the finite complex plane onto itself, where $(z) = \lambda z + \mu$, $\lambda \neq 0$. The inversion map $z \to w = 1/z$ does not exhibit this. $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$, where $\rho = 1/r$, are expressed in polar forms. As a result, points in the z-plane near the origin, $r \approx 0$, are mapped onto points in the w-plane distance from the origin. Every point in the z-plane that is inside a disk with a small radius of ε is projected onto every point outside a disk with a big radius of $1/\varepsilon$ in the w-plane. There is no picture of $z = 0$ in the w-plane and the disk in the z-plane shrinks to the origin as $\varepsilon \to 0$. There is no point in the z-plane that can be given $w = 0$ as the image under inversion. Similarly, when the point z moves farther and farther from the origin, its image in the w-plane moves closer and closer to the origin in the w-plane.

2.6 *EXTENDED COMPLEX PLANE: -*

The complex plane C and a symbol ∞ that satisfies the following properties are referred to as the extended complex number system.

(a) If $z \in \mathbb{C}$, then $z + \infty = z - \infty = \infty, z/\infty = 0$.

(a) if$z \in \mathbb{C}$ but $z \neq 0$ $z.\infty = \infty$ and $\frac{z}{0} = \infty$.

(c) $\infty + \infty = \infty.\infty = \infty$

(d)$\infty/z = \infty\ (z \neq 0)$.

The extended complex plane, represented by $\mathbb{C}_\infty$, is the set $\mathbb{C} \cup \{\infty\}$. The use of Riemann's spherical representation of complex numbers, which relies on stereographic projection, greatly clarifies the structure of the Argand plane at the point at infinity.

2.7 *STEREGOGRAPHIC PROJECTION OF COMPLEX NUMBERS*: -

Complex numbers can also be represented by points on a sphere. To do this, one must establish a one-one correspondence between points on the surface of a sphere.

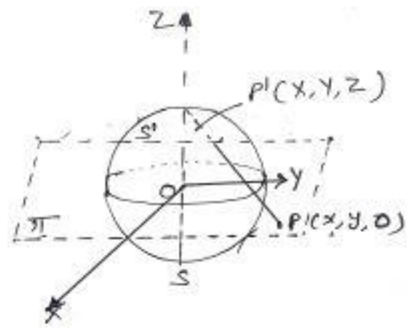

Fig.4

Let S' be a sphere of unit radius with its center at O, and let NS be a diameter of the sphere. Let π be a complex plane passing through O and perpendicular to NS. The points N and S are called the north and south poles, respectively. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ on the plane π. The line joining point N to P intersects the sphere S' at another unique point P', distinct from N. From Figure 1.16, it is also clear that if the point P lies on the plane outside the sphere, then P′ lies on the upper hemisphere of the sphere, and if P lies within the circle (inside the sphere), then P' lies on the lower hemisphere. Thus, for every point on the plane, there exists a unique corresponding point on the sphere S', and conversely, for every point on the sphere S' (except N), there exists a unique corresponding point (complex number) on the complex plane π.

Hence, a one-to-one correspondence is established between all points on the sphere S' (except N) and all points on the plane π. This mapping of the complex plane onto the sphere is called the stereographic projection of complex numbers.

In order to incorporate the sphere's point N into the one-to-one correspondence, we add an extra complex number z', often known as the point at infinity, to the extended complex plane. The renowned mathematician Riemann was the first to present this idea. Consequently, the Riemann Sphere is another name for the sphere S'.

This mapping can be represented analytically as follows:

Let the origin be the center of the sphere. Let two perpendicular lines lie in the plane. The axis is the y-axis. On is the z-axis. Then the equation of the sphere is

$$X^2 + Y^2 + Z^2 = 1 \qquad \dots(1)$$

and equation of the projection plane is

$$Z = 0 \qquad \dots(2)$$

It is clear that the coordinates of N are $(0,0,1)$.

Let $P(x, y, 0)$ be a point on the plane π, and let $P'(X, Y, Z)$ be the corresponding point on the sphere. Since $N(0,0,1), P'(X, Y, Z)$, and $P(x, y, 0)$ lie on the same straight line, therefore,

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = k = \frac{X}{x} = \frac{Y}{y} = \frac{1-Z}{1} \qquad \dots(3)$$

$$X = xk, Y = Yk, Z = 1 - k \qquad \dots(4)$$

where k is a real number. Since the point (X, Y, Z) lies on the unit sphere S', whose equation is

$$x^2k^2 + y^2k^2 + (1-k)^2 = 1$$

$$k = \frac{2}{x^2 + y^2 + 1} \quad (k \neq 0)$$

therefore, from equation (2), the coordinates of P′ are obtained as

$$X = \frac{2x}{x^2 + y^2 + 1}, Y = \frac{2y}{x^2 + y^2 + 1}, Z = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

Thus, the point on the sphere S′ corresponding to the complex number $z = x + iy$ on the plane is

$$P'\left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right)$$

or

$$P'\left(\frac{2x}{|z^2| + 1}, \frac{2y}{|z^2| + 1}, \frac{x^2 + y^2 - 1}{|z^2| + 1}\right)$$

Also, from equation (3), we have

$$x = \frac{X}{1 - Z}, y = \frac{Y}{1 - Z}$$

$$z = x + iy = \frac{X + iy}{1 - Z}$$

Hence, the complex number in the complex plane π corresponding to a point (X, Y, Z) on the sphere S′ is

$$Z = \frac{X + iY}{1 - Z}$$

Only the top point of the sphere, $N(0,0,1)$, has no corresponding point in the complex plane. The point corresponding to $(0,0,1)$ on the complex plane is defined as the point at infinity..

Theorem1: A stereographic projection, every circle on the Riemann sphere is mapped to either a circles or straight line in the complex plane.

Proof: Let a circle on the Riemann sphere S' be formed by the intersection of the Riemann sphere S' with a plane

$$X^2 + Y^2 + Z^2 = 1 \qquad ...(5)$$

and

$$aX + bY + cZ + d = 0 \qquad ...(6)$$

Let the coordinates of a point $P'(X, Y, Z)$ on the Riemann sphere be (X, Y, Z), and let the corresponding point $P(x, y, 0)$ on the plane under stereographic projection have coordinates $(x, y, 0)$. The projection vertex N has coordinates (0,0,1).

Since the points $N(0,0,1), P'(X, Y, Z)$ and $P(x, y, 0)$ lie on the same straight line, therefore,

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = k = \frac{X}{x} = \frac{Y}{y} = \frac{1-Z}{1} \qquad ...(7)$$

where k is a real number.

Since the point (X, Y, Z) lies on both equations (5) and (6), therefore,

$$x^2k^2 + y^2k^2 + (1-k)^2 = 1 \qquad ...(8)$$

and

$$k(ax + by) + c(1-k) + d = 0 \qquad ...(9)$$

On eliminating k from equations (8) and (9), we obtain

$$(c + d)(x^2 + y^2) + 2ax + 2by + d - c = 0 \quad ...(10)$$

which is the relation satisfied by the projection points of the points of the circle lying on the plane (6).

If $c + d \neq 0$, then equation (10) represents a circle, whereas if $c + d = 0$, it represents a straight line.

However, $c + d = 0$ when the plane (6) passes through the north pole $N(0,0,1)$. Hence, if the circle defined by (5) and (6) on the Riemann sphere passes through the north pole N, its projection on the plane π will be a straight line; otherwise, it will be a circle.

In the special case when the plane (6), corresponding to projection (10), is parallel to the projection plane, then $a = 0, b = 0$; therefore, from projection (10), we get

$$(c + d)(x^2 + y^2) = c - d$$

which clearly represents a circle whose center is at the origin.

On the other hand, if the plane (6) passes through both poles N and S, then $c = d = 0$. In that case, the corresponding projection (10) gives

$$ax + by = 0$$

which represents a **straight line** passing through the origin.

SOLVED EXAMPLE

EXAMPLE1: Find all nth roots of unit.
SOLUTION: The number 1 can be written in polar form as:

$$1 = cos0 + isin0$$

Since the complex exponential repeats every 2π, we can also

$$1 = cos2k\pi + isin2k\pi, \quad where\ k = 0,1,2, \dots . n - 1$$

Let

$$z = (cos\theta + isin\theta)$$

Then

$$z^n = (cosn\theta + isinn\theta)$$

From the equation $z^n = 1$, we obtain two conditions:

$$r^n = 1 \text{ and } n\theta = 2k\pi$$

Thus

$r = 1$ and $\theta = \frac{2k\pi}{n}$

Therefore, the nth roots of unity are:

$$z_k = cos\frac{2k\pi}{n} + isin\frac{2k\pi}{n}, \qquad k = 0,1,2, \dots . n - 1$$

or, in exponential (Euler) form:

$$z_k = e^{\frac{2k\pi}{n}}$$

EXAMPLE2: For any two nonzero complex numbers z_1 and z_2 prove that

$$|z_1 + z_2| \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right| \leq 2(|z_1| + |z_2|)$$

SOLUTION: Now we get,

$$|z_1 + z_2| \left| \frac{z_1}{|z_1|} + \frac{z_2}{|z_2|} \right|$$

$$= |z_1 + z_2| \frac{|z_1 z_2|| + z_2|z_1||}{|z_1 z_2|}$$

$$= \frac{|z_1 + z_2|}{|z_1 z_2|}(z_1 z_2|| + z_2|z_1|)$$

$$= 2\frac{|z_1 + z_2|}{|z_1 z_2|}|z_1 z_2|$$

$$= 2|z_1 + z_2|$$

$$\leq 2(|z_1 + z_2|)$$

EXAMPLE3: Prove that the relation

$$\frac{n}{2^{n-1}} = \prod_{k=1}^{n-1} sin\left(\frac{k\pi}{n}\right), n \geq 2$$

SOLUTION: Suppose $1, \rho_1, \rho_2, \dots \dots \rho_{n-1}$be the n roots of unity, where $\rho_k = e^{\frac{2k\pi i}{n}}, k = 1,2, \dots \dots \dots \dots \dots (n-1)$, then

$$z^n - 1 = (z-1)(z-\rho_1)(z-\rho_2) \dots \dots (z-\rho_n)$$

Dividing both sides by $z - 1$ and letting $z \rightarrow 1$, we obtain

$$n = (1-\rho_1)(1-\rho_2) \dots \dots (1-\rho_{n-1})$$

Taking conjugate of both sides, we have

$$n = (1-\overline{\rho_1})(1-\overline{\rho_2}) \dots \dots (1-\overline{\rho_{n-1}})$$

Therefore

$$n^2 = \prod_{k=1}^{n-1}\left(1 - e^{\frac{2k\pi i}{n}}\right)\left(1 - e^{-\frac{2k\pi i}{n}}\right)$$

$$= \prod_{k=1}^{n-1} 2\left(1 - cos\frac{2k\pi}{n}\right)$$

$$= 4sin^2\left(\frac{k\pi}{n}\right)$$

We get the desired outcome by taking the nonnegative square root of each side.

EXAMPLE4: Evaluate $e^{i\pi/4}$ and write it in the form $a + ib$.
SOLUTION: The Euler's Formula is

$$e^{i\theta} = cos\theta + isin\theta \qquad \dots (1)$$

For $\theta = \frac{\pi}{4}$, we have

$$e^{i\frac{\pi}{4}} = cos\frac{\pi}{4} + isin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

EXAMPLE5: Let $f(z) = z^2$. Find the real and imaginary parts of $f(z)$.

SOLUTION: Suppose $f(z) = x + iy,$ then

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2xyi$$

So, the real and imaginary parts of $f(z)$ are

$$u(x,y) = x^2 - y^2, \; v(x,y) = 2xy$$

EXAMPLE6: Let $P = \left(\frac{3}{4}, \frac{4}{5}, 0\right)$ be a point on the unit sphere $X^2 + Y^2 + Z^2 = 1$ stereographic projection onto the complex plane using projection from the north pole $N = (0,0,1)$.

SOLUTION: Let the given points are:

$$X = \frac{3}{4}, Y = \frac{4}{5}, Z = 0$$

Using stereographic projection formula,

$$z = \frac{X + iY}{1 - Z}$$

Putting the values are

$$z = \frac{\frac{3}{5} + i.\frac{4}{5}}{1 - 0} = \frac{3}{5} + i.\frac{4}{5}$$

So, the image of the point P on the complex is $z = \frac{3}{5} + i.\frac{4}{5}$.

SELF CHECK QUESTIONS

1. What is stereographic projection? Define it in your own words.
2. From which point on the sphere is the stereographic projection usually taken?
3. What is the projection plane in a stereographic projection of a sphere?
4. How is a point on the sphere (except the projection point) represented on the plane under stereographic projection?

5. Derive the formula for stereographic projection from a sphere of radius 1 onto the plane.

2.8 *SUMMARY*: -

Stereographic projection is a method for projecting points from the surface of a sphere to a plane. In this projection, each point P on the sphere (except the North Pole) is connected to the North Pole by a straight line. The intersection of this line with the equatorial plane represents the image of P on the plane. It establishes a one-to-one connection between the points on the sphere (excluding the projection point) and the entire complex plane, with the North Pole representing the point at infinity.This projection preserves angles (is conformal) and transforms circles on the sphere into circles or straight lines on the plane. It is widely used in complex analysis to relate the extended complex plane to the geometry of the sphere, known as the Riemann sphere.

2.9 *GLOSSARY*: -

- **Stereographic Projection:** A mapping in which points from a sphere's surface are projected onto a plane using lines drawn from a fixed point (typically the North Pole).
- **Projection Point (North Pole):** A fixed point on the sphere from which lines can be formed to project other places onto the plane.
- **Projection Plane (Equatorial Plane):** The plane onto which the sphere's points are projected, commonly denoted as $z = 0$.
- **Unit Sphere:** A sphere with radius 1 and center at the origin, commonly used to define stereographic projection.
- **Complex Plane:** The plane that represents complex numbers, with each point corresponding to the complex number $z = x + iy$.
- **Riemann Sphere:** A sphere used in stereographic projection to represent the extended complex plane, including the point of infinity.
- **Point at Infinity:** The picture of the projection point (North Pole) after stereographic projection; it depicts the "infinite" point on the complex plane.
- **Inverse Stereographic Projection:** The process of projecting a point from the complex plane back onto the surface of the sphere.
- **Conformal Mapping:** A sort of mapping that keeps the angles between curves. Stereographic projections are conformal.
- **Circle Preservation Property:** In stereographic projection, circles on the sphere transfer to circles or straight lines on the plane.

- **Equator:** The circle on the sphere that is in the projection plane; it maps onto the unit circle in the complex plane when the sphere has radius one.
- **Latitude & Longitude Circles:** Circles on the sphere that are parallel or perpendicular to the equator and project into circles or plane lines.
- **Extended Complex Plane:** The complex plane plus the point at infinity, represented by $C \cup \{\infty\}$.

2.10 *REFERENCES*: -

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- Ruel V. Churchill and James Ward Brown, Complex Variables and Applications (2013), McGraw-Hill Education, 9th Edition.
- Erwin Kreyszig, Advanced Engineering Mathematics (2011), Wiley, 10th Edition.

2.11 *SUGGESTED READING*: -

- file:///C:/Users/user/Desktop/1456304516EtextofChapter1Module2%20(1).pdf
- Murray R. Spiegel (2009) – Schaum's Outline of Complex Variables, 2nd Edition.
- R. Narayanaswamy (2005) – Theory of Functions of a Complex Variable, S. Chand & Company Ltd.
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.

2.12 *TERMINAL QUESTIONS: -*

(TQ-1) Define stereographic projection. Describe the process of projecting a point from the sphere onto the plane.

(TQ-2) Derive the mathematical formula for stereographic projection from the unit sphere onto the complex plane.

(TQ-3) Discuss how the point at infinity in the extended complex plane corresponds to the North Pole of the sphere.

(TQ-4) Explain the relationship between the modulus of the complex number $z = x + iy$ and the height Z of the corresponding point on the sphere.

(TQ-5) Discuss how stereographic projection helps visualize functions of complex variables geometrically.

(TQ-6) Show that under stereographic projection, the unit circle in the complex plane corresponds to the equatorial circle of the sphere.

(TQ-7) Prove that for any point z on the complex plane, the coordinates (X, Y, Z) on the unit sphere are given

$$X = \frac{2x}{|z|^2 + 1}, Y = \frac{2y}{|z|^2 + 1}, Z = \frac{|z|^2 - 1}{|z|^2 + 1},$$

UNIT3: Complex Function and their properties

CONTENTS:

3.1 *INTRODUCTION*: -

A complex function is a mathematical rule that associates each complex number $z = x + iy$ with another complex value $f(z) = u(x, y) + iv(x, y)$. Unlike real functions, complex functions operate on a two-dimensional plane, making their behavior richer and more geometric. The study of these functions forms the core of complex analysis, an important branch of mathematics with deep theoretical results and wide applications in physics, engineering, fluid dynamics, electromagnetism, and number theory. Understanding complex functions requires examining how they behave with respect to limits, continuity, and differentiability. A unique feature of complex functions is that differentiability is governed by the Cauchy–Riemann equations, which impose strict conditions and lead to the powerful concept of analytic (holomorphic) functions. Such functions possess remarkable properties: they are infinitely differentiable, equal to their Taylor series, and preserve angles through conformal mappings. The study of zeros, singularities, poles, and mapping behavior further enhances the understanding of how complex functions transform regions of the complex plane. Thus, complex functions and their properties provide a deep framework to explore both algebraic and geometric behavior within the complex plane.

3.2 *OBJECTIVES*: -

After studying this unit, the learner's will be able to

- To understand the basic concept of a complex function and express it in terms of real and imaginary parts.
- To analyze standard complex functions such as exponential, trigonometric, logarithmic, and Möbius transformations and their properties.

3.3 *FUNCTION OF COMPLEX VARIABLE*: -

A complex variable, represented by z, is any element of a set S contained on the complex plane C. A function $f: S \to C$ is a rule that assigns a unique complex value $f(z)$ to each $z \in S$, indicated as $w = f(z)$, where z is the independent variable and w the dependent variable. This function f translates elements from the domain S to the complex plane C, which is commonly represented by another complex plane known as the w plane. If S is a subset of the real line, f is considered a complicated function of a real variable. The set S is designated as the domain of f, and the collection of all $f(z)$ for z in S is recognized as the range of f.

Or

A function $f: A \to B$ assigns a unique complex number $w_0 = u_0 + iv_0 \in B$ to each non-empty subset of the complex numbers $z_0 = x_0 + iy_0 \in A$. The integer w_0 represents the value of f at z_0, indicated by $f(z_0) = w_0$. As z varies, so does $f(z) = w$ in B. This function is a complex-valued function of a complex variable, with the dependent variable w and the independent variable z. If S is a subset of A, the image of S under f is $f(S) = \{f(z) \mid , z \in S\}$, while the range of f is $R = \{f(z) \mid z \in A\}$.

For each non-zero complex number $z \in \mathbb{C} - \{0\}$, the polar form is given by $z = re^{i\theta}$, where $r = \mid z \mid$ is the modulus and $\theta \in [-\pi, \pi]$ is the argument of z. This can be written as $z = z(r, \theta) = re^{i\theta}$. If we increase the argument θ by 2π, we get:

$$z(r,\theta+2\pi) = re^{i(\theta+2\pi)} = re^{i\theta}.e^{2\pi i} = re^{i\theta} = z(r,\theta)$$

Thus, $z(r,\theta+2\pi)$ returns to its original value, demonstrating the periodicity of the complex exponential function with period 2π.

Definition. A function f is said to be single-valued if it satisfies $f(z) = f\big(z(r,\theta)\big) = z(r,\theta+2\pi)$, meaning the function's value remains unchanged when the argument θ is increased by 2π.

Otherwise, f is said to be a multiple valued function.

Example: $f(z) = z^n, n \in Z$ is said to a single valued function.

Solution: $f(z) = f\big(z(r,\theta)\big) = \left(re^{i\theta}\right)^n$

$$f\big(z(r,\theta+2\pi)\big) = \left[re^{i(\theta+2\pi)}\right]^n = r^n e^{in\theta} e^{2\pi ni}$$

$$= r^n e^{in\theta} e^{2\pi ni}$$

$$\{\because \quad e^{2\pi ni} = 1,\ n \in \mathbb{Z}\}$$

$$= \left(re^{i\theta}\right)^n = f\big(z(r,\theta)\big)$$

Note: If $n \notin \mathbb{Z}$ then $f(z) = z^n$ is multiplied valued function.

$\because\ e^{2\pi ni} \notin 1$, when $n \notin \mathbb{Z}$.

3.4 *SINGLE VALUED AND MULTIVALUED FUNCTION*: -

If each value of z corresponds to exactly one value of w, then w is referred to as a uniform or one-valued function of z. A multi-valued function of z is one in which two or more values of w correspond to some or all values of z. A multi-valued function can be viewed as a collection of numerous one-valued functions. Each one-valued function in the collection is referred to as a branch function of the multi-valued function, with a specific member of the collection serving as the major branch function of the multi-valued function. The value of the function at this branch function is known as the primary value.

Example 1. If $w = z^2$, then each value of z corresponds to a specific value of w. Therefore, w is a one-valued complex function of z.

Example 2. If $w = z^{1/2}$, then each value of z corresponds to two values of w. Therefore, w is a multi-valued function of z.

3.5 *ELEMENTARY COMPLEX FUNCTION*: -

1. **Polynomial** : A function of the form $w = a_0 z^n + a_1 z^{n-1} + \cdots \cdots + a_{n-1} z + a_n$ where n∈N and a′ ≠ 0, a_1, a_2, a are complex constants is called a polynomial in the complex plane. It is called n exponential polynomial.

   The particular case $w = az + b$ is known as linear transformation.
2. **Rational algebraic function**: A function of the form $w = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ is a multiple term is called a rational algebraic function.
   The special case where $ad - bc \neq 0$ is called bilinear (one-one) transformation or $w = \frac{az+b}{cz+d}$ fractional linear transformation.
3. **Exponential function:** The functions defined as follows are called complex exponential functions.
   $$w = e^z = e^{x+iy} = e^x(cosx + isiny)$$
   $a \epsilon R$ if $a^x = e^{zloga}$ where $e = 2.71828$
4. **Trigonometrical functions:** Trigonometrical functions are defined as follows.
   $$sinx = \frac{e^{ix} - e^{-ix}}{2i}, cosx = \frac{e^{ix} + e^{-ix}}{2i}$$
   $$cosecx = \frac{1}{sinx} = \frac{2i}{e^{ix} - e^{-ix}}$$
   $$secx = \frac{1}{cosx} = \frac{2i}{e^{ix} - e^{-ix}}$$
   $$tanx = \frac{sinx}{cosx} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$
   $$cotz = \frac{cosx}{sinx} = \frac{i\left(e^{ix} + e^{-ix}\right)}{e^{ix} - e^{-ix}}$$

Some Standard Trigonometrical Results:

a. $cos^2x + sin^2x = 1$
b. $cos(-x) = cosx$
c. $cos2x = cos^2x - sin^2x$
d. $sin3x = 3sinx - 4sin^3x$
e. $cos3x = 4cos^3x - 3cosx$
f. $sinx + siny = 2\,sin\frac{1}{2}(x+y)cos\frac{1}{2}(x-y)$
g. $cosx - cosy = 2\,sin\frac{1}{2}(x+y)\,sin\frac{1}{2}(y-x)$
h. $sin(x \pm y) = sinx\,cosy \pm cosx\,siny$
i. $cos(x \pm y) = cosx\,cosy \mp cosx\,siny$

j. $sec^2x = 1 + tan^2x$

SOLVED EXAMPLES

EXAMPLE1: Prove that e^z is a periodic function, where z is a complex quantity.

SOLUTION: If $z = x + iy$, then

$$e^z = e^{x+iy} = e^x.e^{iy}$$
$$= e^x(cosy + isiny)$$
$$= e^x[cos(2n\pi + y) + isin(2n\pi + y)]$$

where n is any integer.

Hence e^z is a parodic function of period $2\pi i$.

EXAMPLE2: Prove that $sinz, cosz, tanz\ etc$. Are periodic functions, where z is a complex quantity.

SOLUTION: Let we get

$\cos(z + 2n\pi) = cosz\ cos2n\pi - sinz\ sin2n\pi = cosz, n$ being integer

$sin(z + 2n\pi) = sinz\ cos2n\pi - cosz\ sin2n\pi = sinz, n$ being integer

$\tan(z + n\pi) = \frac{sin(z+n\pi)}{cos(z+n\pi)} = \frac{sinz\ cosn\pi + cosz\ sinn\pi}{cosz\ cosn\pi - sinz\ sinn\pi} = tanz, n$ being integer

From this, we conclude that $cosz$ and $sinz$ are periodic functions with period 2π, and $tan\ z$ is a periodic function with period π.

EXAMPLE 3: Show that $exp(\pm i\ \pi/2) = \pm i$.

SOLUTION: Let $exp(\pm i\theta) = cos\theta \pm isin\theta$, we get

$$exp(\pm i\ \pi/2) = \frac{cos\pi}{2} \pm \frac{isin\pi}{2} = 0 \pm i.1 = \pm i$$

EXAMPLE4: Prove that $\{sin(\alpha - \theta) + e^{-\alpha i}sin\theta\}^n = sin^{n-1}\alpha\{sin(\alpha - n\theta) + e^{-\alpha i}sinn\theta\}$.

SOLUTION: $L.H.S = \{sin(\alpha - \theta) + e^{-\alpha i}sin\theta\}^n$

$$= \{sin\alpha\ cos\theta - cos\alpha\ sin\theta + (cos\alpha - isin\alpha)sin\theta\}^n$$
$$= \{sin\alpha\ cos\theta - cos\alpha\ sin\theta + cos\alpha sin\theta - isin\alpha sin\theta\}^n$$
$$= \{sin\alpha\ cos\theta - isin\alpha sin\theta\}^n = sin^n\alpha\{cos\theta - isin\theta\}^n$$

$= sin^n\alpha\{cosn\theta - isinn\theta\}$ by De-Moiver's theorem

and $R.H.S. = sin^{n-1}\alpha\{sin(\alpha - n\theta) + e^{-\alpha i}sinn\theta\}$

$$= sin^{n-1}\alpha\{sin\alpha\ cosn\theta - cos\alpha\ sinn\theta + (cos\alpha - isin\alpha)sinn\theta\}$$
$$= sin^{n-1}\alpha\{sin\alpha\ cosn\theta - cos\alpha\ sinn\theta + cos\alpha\ sinn\theta - isin\alpha\ sinn\theta\}$$
$$= sin^{n-1}\alpha.sin\alpha.\{\ cosn\theta - isinn\theta\}$$
$$= sin^n\alpha\{cosn\theta - isinn\theta\}$$

$$\boldsymbol{L.H.S. = R.H.S.}$$

5. **Hyperbolic function:** Hyperbolic function is defined as follows.

$$sinhx = \frac{e^x - e^{-x}}{2}, coshx = \frac{e^x + e^{-x}}{2}$$
$$cosechx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}, sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$$

$$tanz = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Properties of Hyperbolic Functions:

a. $cosh^2x - sinh^2x = 1$
b. $sinh2x = 2sinhxcoshx$
c. $cosh2x = cosh^2x + sinh^2x = 1 + 2sinh^2x = 2cosh^2x - 1$
d. $sech^2x = 1 - tanh^2x$
e. $tan2hx = \frac{2tanhx}{(1+tanh^2x)}$
f. $sinh(x+y) = sinhx\ coshy + coshx\ sinhy$
g. $cosh(x+y) = coshx\ coshy + sinhx\ sinhy$
h. $e^x = coshx + sinhx$ and $e^{-x} = coshx - sinhx$
i. $sinh3x = 3sinhx + 4sinh^3x$
j. $cosh3x = 4cosh^3x - 3coshx$
k. $tanh3x = \frac{3tanhx+tanh^3x}{1+3tanh^2x}$

SOLVED EXAMPLES

EXAMPLE5: Show that

$sin\,h(x+y)\cosh(x-y) = \frac{1}{2}(sinh2x + sinh2y)$.

SOLUTION: Let $L.H.S. = sin\,h(x+y)\cosh(x-y)$

$$.= \frac{1}{2}\left[e^{(x+y)} - e^{-(x+y)}\right]\frac{1}{2}\left[e^{(x-y)} - e^{-(x-y)}\right]$$

$$= \frac{1}{4}\left[e^{(2x)} - e^{(2y)} - e^{(-2y)} - e^{-(-2x)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}(e^{2x} - e^{-2x}) - \frac{1}{2}(e^{2y} - e^{-2y})\right]$$

$$= \frac{1}{2}[sinh2x + sinh2y] = R.H.S.$$

EXAMPLE6: Show that

$$co\,s(\alpha + i\beta) + isin(\alpha + i\beta) = e^{-\beta}(cos\alpha + isin\alpha)$$

SOLUTION: Let $L.H.S. = co\,s(\alpha + i\beta) + isin(\alpha + i\beta)$

$$= cos\alpha\ cosi\beta - sin\alpha\ sini\beta + isin\alpha\ cosi\beta + icos\alpha\ sini\beta$$

$$= cos\alpha(cosi\beta + isini\beta) + isin\alpha(cosi\beta + isini\beta)$$

$$= (cosi\beta + isini\beta)(cos\alpha + isin\alpha)$$

$= (cosh\beta - sinh\beta)(cos\alpha + isin\alpha) \quad \because cosi\beta = cosh\beta, sini\beta = isinh\beta$

$$= e^{-\beta}(cos\alpha + isin\alpha) = R.H.S.$$

EXAMPLE7: If $cosh\alpha = sec\theta$, that $tanh^2\frac{1}{2}\alpha = tan^2\frac{1}{2}\theta$.

SOLUTION: Let we get $cosh\alpha = sec\theta$

$$\frac{cosh\alpha}{1} = \frac{1}{cos\theta}$$

Applying componendo and dividendo, we have

$$\frac{cosh\alpha - 1}{cosh\alpha + 1} = \frac{1 - cos\theta}{1 + cos\theta}$$

$$\frac{2sinh^2\frac{1}{2}\alpha}{2cosh^2\frac{1}{2}\alpha}=\frac{2sin^2\frac{1}{2}\theta}{2cos^2\frac{1}{2}\theta}$$

$$tanh^2\frac{1}{2}\alpha=tan^2\frac{1}{2}\theta$$

Note: The componendo and dividendo, is that if

$\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

EXAMPLE8: If $tan\theta = tanhx\, coty$ and $tan\phi = tanhx\, tany$, show that

$$\frac{sin2\theta}{sin2\phi}=\frac{cosh2x+cos2y}{cosh2x-cos2y}$$

SOLUTION: $L.H.S.=\frac{sin2\theta}{sin2\phi}=\frac{2tan\theta/(1+tan^2\theta)}{2tan\phi/(1+tan^2\phi)}$

$$=\frac{tan\theta}{1+tan^2\theta}\times\frac{1+tan^2\phi}{tan\phi}=\frac{tan\theta}{tan\phi}\cdot\frac{1+tan^2\phi}{1+tan^2\theta}$$

$$=\frac{tanhx\,coty}{tanhx\,tany}\times\frac{1+tanh^2x tan^2y}{1+tanh^2x cot^2y}$$

Substituting the given values of $tan\theta$ and $tan\phi$

$$=\frac{cos^2y}{sin^2y}\cdot\frac{cosh^2xcos^2y+sinh^2xsin^2y}{cosh^2xcos^2y}\cdot\frac{cosh^2xsin^2y}{cosh^2xsin^2y+sinh^2xcos^2y}$$

$$=\frac{cosh^2xcos^2y+sinh^2xsin^2y}{cosh^2xsin^2y+sinh^2xcos^2y}$$

$$=\frac{(2cosh^2x)(2cos^2y)+(2sinh^2x)(2sin^2y)}{(2cosh^2x)(2sin^2y)+(2sinh^2x)(2cos^2y)}$$

$$=\frac{(1+cosh2x)(1+cos2y)+(cosh2x-1)(1-cos2y)}{(1+cosh2x)(1-cos2y)+(cosh2x-1)(1+cos2y)}$$

$$[\because 2cosh^2x=1+cosh2x; 2sinh^2x=cosh2x-1]$$

$$=\frac{2(cosh2x+cos2y)}{2(cosh2x-cos2y)}=\frac{cosh2x+cos2y}{cosh2x-cos2y}=R.H.S.$$

6. **Logarithmic function:** If $z = e^w$, $w = Inz$ or $\log_e z$is written. Which is known as the natural factor of z. Hence,
$$w=logz=logr+i(2n\pi+\theta);n = 0,1,2\cdots\cdots\cdots[r=|z|,\theta=amp(z)]$$
represents a multivalued function. Its primary branch function .The value for $n = 0$ is obtained.
Similarly, the logarithmic function is defined based on any real number a.
If $z=a^w$, then $w=\log_a z\,(a>0, a\neq 0,1)$
Clearly, $w=\log_a z=\frac{Inz}{Ina}$

SOLVED EXAMPLES

EXAMPLE9: Find the principal of general value of $log(-1+i)$

SOLUTION: Let $-1+i = r(cos\theta + isin\theta)$

$$rcs\theta = -1, rsin\theta = 1$$

Squaring and adding, we get

$$r^2 = 2, i.e., r = \sqrt{2}$$

Now $cos\theta = -1/\sqrt{2}$ and $sin\theta = 1/\sqrt{2}$

$$\theta = \frac{3}{4}\pi$$

$$-1+i = \sqrt{2}\left(cos\frac{3}{4}\pi + isin\frac{3}{4}\pi\right) = \sqrt{2}e^{\left(\frac{3\pi}{4}\right)i}$$

The general value is

$$log(-1+i) = log\left\{\sqrt{2}e^{\left(\frac{3\pi}{4}\right)i}e^{2n\pi i}\right\}$$

$$log\sqrt{2} + \frac{3}{4}\pi i + 2n\pi = \frac{1}{2}log2 + \left(2n + \frac{3}{4}\right)\pi i$$

Substituting $n = 0$, the principal value is given by

$$log(-1+i) = \frac{1}{2}log2 + \frac{3}{4}\pi i$$

EXAMPLE10: Find the principal of general value of $log(-3)$.

SOLUTION: Let $-3 = -3 + i0 = r(cos\theta + isin\theta)$

So $$rcos\theta = -3, rsin\theta = 0$$

These give $r^2 = 9\ i.e., r = 3$.

Substituting $r = 3$, we obtain

$cos\theta = -1$ and $sin\theta = 0$, giving $\theta = \pi$

$$-3 = 3(cos\pi + isin\pi) = 3.e^{i\pi}$$

$$Log(-3) = log\{3.e^{i\pi}.e^{2n\pi i}\}$$

$$= log3 + log\ e^{(2n\pi+\pi)i}$$

$$= log3 + (2n+1)\pi i$$

Hence the principal value of $log(-3)\ i.e.$, $log(-3)$ is obtained by putting $n = 0$, then

$$Log(-3) = log3 + i\pi$$

EXAMPLE11: Find the principal of general value of $log(-i)$.

SOLUTION: Let $-i = \left(cos\frac{\pi}{2} - isin\frac{\pi}{2}\right) = e^{-i\pi/2}$

So that

$\log(-i) = loge^{-i\pi/2} = -i\pi/2$, giving the principal value.

$Log(-i) = \log(-i) + 2n\pi i = -\left(\frac{i\pi}{2}\right) + 2n\pi i = \frac{1}{2}(4n-1)\pi i.$

EXAMPLE12: Express $log(1+i)^{(1-i)}$in the form $A + iB$.

SOLUTION: Let

$$log(1-i)^{(1-i)} = (1-i)log(1+i)$$

$$= (1-i)\left[\frac{1}{2}\log(1^2+1^2) + itan^{-1}1\right]$$

$$= (1-i)\left[\frac{1}{2}log2 + i\frac{\pi}{4}\right]$$

$$= \left[\frac{1}{2} log2 + \frac{\pi}{4}\right] + -1 \left[\frac{\pi}{4} - \frac{1}{2} log2\right]$$

which is the form $A + iB$.

7. **Inverse Function:** If $z = sin\, w$, then $w = sin^1 z$ is called the inverse of z. Similarly, other trigonometric inverse functions $cos\, z$, $tan\, z, cosec\, z$ and $cot^1 z$ are defined. All these functions are multi-valued functions and can be expressed in terms of logarithms as follows.

$$sin^{-1} z = \frac{1}{i} log \left\{iz + \sqrt{1 - z^2}\right\}$$
$$cos^{-1} z = \frac{1}{i} log \left\{z + \sqrt{z^2 - 1}\right\}$$
$$tan^{-1} z = \frac{1}{2i} log \left(\frac{1 + iz}{1 - iz}\right)$$
$$cosec^{-1} z = \frac{1}{i} \log \left(\frac{1 + \sqrt{z^2 - 1}}{z}\right)$$
$$sec^{-1} z = \frac{1}{i} \log \left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$
$$cot^{-1} z = \frac{1}{2i} log \left(\frac{z + i}{z - i}\right)$$

8. **Inverse Hyperbolic function:** If $z = sinhw$ then The $sinh$ inverse of $w = sinh$ is called the inverse. Similarly, the other hyperbolic inverse functions $coshz, tanhz, sechz, cosechz,$ and $cot\, hz$ are defined. All of these functions are also multivalued functions. They can be expressed in terms of logarithms as follows.

$$sinh^{-1} z = log \left\{z + \sqrt{1 - z^2}\right\}$$
$$cosh^{-1} z = log \left\{z + \sqrt{z^2 - 1}\right\}$$
$$tanh^{-1} z = \frac{1}{2} log \left(\frac{1 + z}{1 - z}\right)$$
$$cosec^{-1} z = log \left(\frac{1 + \sqrt{z^2 - 1}}{z}\right)$$
$$sec^{-1} z = log \left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$
$$coth^{-1} z = \frac{1}{2i} log \left(\frac{z + 1}{z - 1}\right)$$

SOLVED EXAMPLE

EXMPLE13: Express $\cos^{-1}(x + iy)$ in the form of $A + iB$.

SOLUTION: Let us consider $\cos^{-1}(x+iy) = A + iB$

$$\cos(A+iB) = x+iy$$
$$cosAcos(iB) - sinAsin(iB) = x + iy$$
$$cosAcoshB - sinAsinhB = x + iy$$

Equating real and imaginary parts on both sides, we obtain

$$cosAcoshB = x \qquad \ldots(1)$$
$$sinAsinhB = -y \qquad \ldots(2)$$

From (1) and (2), we get

$$coshB = \frac{x}{cosA}$$
$$sinhB = -\frac{y}{sinA}$$
$$\frac{x^2}{cos^2A} - \frac{y^2}{sin^2A} = cosh^2B - sinh^2B = 1$$
$$x^2sin^2A - y^2cos^2A = cos^2A\, sin^2A$$
$$x^2sin^2A - y^2(1 - sin^2A) = (1 - sin^2A)\, sin^2A$$
$$x^2sin^2A - y^2 + y^2sin^2A = (sin^2A - sin^4A)$$
$$sin^4A + (x^2 + y^2 - 1)sin^2A - y^2 = 0$$

So that

$$sin^2A = \frac{-(x^2+y^2-1) \pm \sqrt{(x^2+y^2-1)^2 + 4y^2}}{2}$$

Since sin^2A must be positive, therefore neglecting the $-ive\ sign,$ we obtain

$$sin^2A = \frac{\sqrt{(x^2+y^2-1)^2 + 4y^2} - (x^2+y^2-1)}{2}$$
$$sinA = \pm\left[\frac{\sqrt{(x^2+y^2-1)^2 + 4y^2} - (x^2+y^2-1)}{2}\right]^{1/2}$$
$$A = \pm sin^{-1}\left[\frac{\sqrt{(x^2+y^2-1)^2 + 4y^2} - (x^2+y^2-1)}{2}\right]^{1/2} \qquad \ldots(3)$$

From (1) and (2), we have

$$cosA = \frac{x}{coshB}$$
$$sinA = -\frac{y}{sinhB}$$
$$\frac{x^2}{cosh^2B} + \frac{y^2}{sinh^2B} = cos^2A + sin^2A = 1$$
$$x^2sinh^2B + y^2cosh^2B = cosh^2Bsinh^2B$$
$$x^2sinh^2B + y^2(1 + sinh^2B) = (1 + sinh^2B)sinh^2B$$
$$x^2sinh^2B + y^2 + y^2sinh^2B = (sinh^2B + sinh^4B)$$
$$sinh^4B + (1 - x^2 - y^2)sinh^2B - y^2 = 0$$

So that

$$sinh^2B = \frac{-(1-x^2-y^2) \pm \sqrt{(1-x^2-y^2)^2 + 4y^2}}{2}$$

But $sin^2 A sinh^2 B$ must be positive, therefore neglecting the $-ive\ sign,$ we obtain

$$sinh^2 B = \frac{\sqrt{(1-x^2-y^2)^2+4y^2}-(1-x^2-y^2)}{2}$$

$$sinhB = \pm\left[\frac{\sqrt{(x^2+y^2-1)^2+4y^2}-(1-x^2-y^2)}{2}\right]^{1/2}$$

$$B = \pm sinh^{-1}\left[\frac{\sqrt{(x^2+y^2-1)^2+4y^2}-(1-x^2-y^2)}{2}\right]^{1/2} \qquad \dots(4)$$

Note : The general value of $\cos^{-1}(x+iy)$ $i.e., \cos^{-1}(x+iy)$ is given By

$$\cos^{-1}(x+iy) = 2n\pi \pm \cos^{-1}(x+iy) = 2n\pi \pm (A+iB)$$

where A and B are found in equation (3) and (4).

EXMPLE14: Express $\sin^{-1}(x+iy)$ in the form of $A+iB$.

SOLUTION: Proceed exactly as in Ex.1.

If we already found $\cos^{-1}(x+iy)$, then $sin^{-1}(x+iy)$ can also be deduced from it shown that

Now, we get

$$sin^{-1}(x+iy) = \frac{\pi}{2} - \cos^{-1}(x+iy)$$

$$= \frac{\pi}{2} \pm sin^{-1}\left[\frac{\sqrt{(x^2+y^2-1)^2+4y^2}-(x^2+y^2-1)}{2}\right]^{1/2}$$

$$\pm sinh^{-1}\left[\frac{\sqrt{(1-x^2-y^2)^2+4y^2}-(1-x^2-y^2)}{2}\right]^{1/2},$$

EXMPLE15: Express $cosh^{-1}(x+iy)$ in the form of $\alpha+i\beta$.

SOLUTION: Now we have

$$cosh^{-1}(x+iy) = \alpha + i\beta$$

$$x+iy = cos\,h(\alpha+i\beta) = \cos\{i(\alpha+i\beta)\} = \cos(i\alpha-\beta)$$

$$cosi\alpha\ cos(\beta) + sini\alpha\ sin\beta = cosh\alpha\ cos\beta + isinh\alpha\ sin\beta$$

Equating real and imaginary parts on both sides, we have

$$cosh\alpha\ cos\beta = x \qquad \dots(1)$$

$$sinh\alpha\ sin\beta = y \qquad \dots(2)$$

From (1) and (2), we get

$cos\beta = x/cosh\alpha$ and $sin\beta = y/sinh\alpha$

$$\frac{x^2}{cosh^2\alpha} + \frac{y^2}{sinh^2\alpha} = cos^2\beta + sin^2\beta = 1$$

$$x^2 sinh^2\alpha + y^2 cosh^2\alpha = cosh^2\alpha\ sinh^2\alpha$$

Proceed exactly as in Ex.1, we obtain

$$\alpha = \pm sinh^{-1}\left[\frac{\sqrt{(1-x^2-y^2)^2+4y^2}-(1-x^2-y^2)}{2}\right]^{\frac{1}{2}}$$

Again find β, we get

$$cosh\alpha \ = x/cos\beta$$
$$sinh\alpha \ = y/sin\beta$$
$$\frac{x^2}{cos^2\beta} - \frac{y^2}{sin^2\beta} = cosh^2\alpha - sinh^2\alpha = 1$$
$$x^2 sinh^2\alpha + y^2 cosh^2\alpha = cosh^2\alpha \, sinh^2\alpha$$

Proceed exactly as in Ex.1, we obtain

$$\alpha = \pm sin^{-1}\left[\frac{\sqrt{(x^2+y^2-1)^2+4y^2}-(x^2+y^2-1)}{2}\right]^{\frac{1}{2}}$$

EXMPLE 16: Express $tan^{-1}(x+iy)$ in the form of $A+iB$.

SOLUTION: Let $\tan^{-1}(x+iy) = A+iB$, then

$$tan(A+iB) = x+iy$$

and

$$tan(A-iB) = x-iy$$

Now equating complex conjugates, we get

$$tan2A = tan[(A+iB)+(A-iB)]$$
$$= \frac{tan[(A+iB)+(A-iB)]}{1-tan(A+iB)\,tan(A-iB)}$$
$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$
$$= \frac{2x}{1-(x^2+y^2)} = \frac{2x}{1-x^2-y^2}$$
$$2A = tan^{-1}\frac{2x}{1-x^2-y^2}$$
$$A = \frac{1}{2}tan^{-1}\frac{2x}{1-x^2-y^2}$$

Again

$$tan(2iB) = tan[(A+iB)-(A-iB)]$$
$$= \frac{tan[(A+iB)-(A-iB)]}{1+tan(A+iB)\,tan(A-iB)} = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$
$$= \frac{2iy}{1+x^2+y^2} = \frac{2iy}{1+x^2+y^2}$$
$$itanh2B = \frac{2iy}{1+x^2+y^2}$$

$[\because tan(i\theta) = itanh\theta]$

$$tanh2B = \frac{2y}{1+x^2+y^2}$$
$$2B = tanh^{-1}\frac{2y}{1+x^2+y^2}$$

$$B = \frac{1}{2} tanh^{-1} \frac{2y}{1 + x^2 + y^2}$$

Hence

$$tan^{-1}(x + iy) = A + iB$$

$$= \frac{1}{2} tan^{-1} \frac{2x}{1 - x^2 - y^2} + \frac{i}{2} tanh^{-1} \frac{2y}{1 + x^2 + y^2}$$

Note : The general value of $tan^{-1}(x + iy)$ $i.e., tan^{-1}(x + iy)$ is given By

$$tan^{-1}(x + iy) = n\pi \pm tan^{-1}(x + iy)$$

$$= 2n\pi + \frac{1}{2} \frac{2x}{1 - x^2 - y^2} + \frac{i}{2} tanh^{-1} \frac{2y}{1 + x^2 + y^2}$$

EXMPLE 17: Prove that $sinh^{-1}(cotx) = log(cotx + cosecx)$

SOLUTION: Let $\sinh^{-1}(cotx) = y$, then

$$sinhy = cotx$$

$$\therefore \quad coshy = \sqrt{1 + sinh^2 y} = \sqrt{1 + cot^2 x} = cosecx$$

Adding above equations, we get

$$sinhy + coshy = cotx + cosecx$$

Or

$$e^y = cotx + cosecx$$

Or

$$y = log(cotx + cosecx)$$

$$\therefore \quad sinh^{-1}(cotx) = log(cotx + cosecx)$$

3.6 *COMPLEX FUNCTION AS REAL-VALUED PAIRS*: -

A complex function $f(z)$ takes a complex number as input and gives another complex number as output. Since every complex number can be expressed in terms of its real part and imaginary part, it becomes easier to analyze and understand complex functions by separating them into these two components

Consider the complex function denoted by $f(z) = z^2$ and replace z by $x + iy$ and can be expressed as

$$f(z) = (x^2 - y^2) + i(2xy)$$

Now if we substitute $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, then

$$f(z) = u(x, y) + iv(x, y)$$

Where $u: \mathbb{R}^2 \to \mathbb{R}$ and $v: \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions between real variables x and y. Simply put, $Re\, f(z)$ and $Im\, f(z)$ are real-valued functions of two variables. The advantage of working with a complex function represented in the form (13) is that we are familiar with real valued functions with several variables.

Additionally, a complex number (z) can have a polar representation $(r\ \theta)$. A complicated function, $f(z)$, can be represented as

$$f(r\ \theta) = u(r\ \theta) + iv(r\ \theta)$$

For Example, let $f(z) = z^3$, in polar coordinates can be expressed as

$$f(r\ \theta) = [r(cos\theta + isin\theta)]^3$$

$$= r^3(cos\theta + isin\theta)^3$$

$$= r^3(cos3\theta + isin3\theta)$$ [using De Moivre's theorem]

Where $u(r\ \theta) = r^3cos3\theta$ and $v(r\ \theta) = r^3sin3\theta$.

SOLVED EXAMPLES

EXAMPLE18: Separate e^zinto real and imaginary parts.

SOLUTION: Now we have $e^z = e^{x+iy}$

$$= e^x . e^{iy} = e^x(cosy + isiny)$$

$$= e^x cosy + ie^x siny$$

Which is the form of $a + ib$, where a and b are real.

EXAMPLE19: Separate $tanz$ into real and imaginary parts.

SOLUTION: Now, $tanz = \frac{sinz}{cosz} = \frac{sin(x+iy)}{cos(x+iy)}$

$$= \frac{sin(x+iy)}{cos(x+iy)}$$

$$= \frac{sin(x+iy)}{cos(x+iy)} \times \frac{2\cos(x-iy)}{2\cos(x-iy)}$$

$$= \frac{sin[(x+iy)+(x-iy)] + sin[(x+iy)+(x-iy)]}{cos[(x+iy)+(x-iy)] + cos[(x+iy)+(x-iy)]}$$

$$= \frac{sin2x + sin(2iy)}{cos2x + cos(2iy)}$$

$$= \frac{sin2x + isinh(2y)}{cos2x + cosh(2y)}$$

$$= \left(\frac{sin2x}{cos2x + cosh(2y)}\right) + \left(\frac{sinh2y}{cos2x + cosh(2y)}\right)$$

EXAMPLE20: Separate $tanhz$ into real and imaginary parts.

SOLUTION: Now, $tanhz = \frac{sinhz}{coshz} = \frac{sin(x+iy)}{cos(x+iy)}$

$$= \frac{1}{i}\left[\frac{sini(x+iy)}{cosi(x+iy)}\right]$$

$$= \frac{1}{i}\frac{sin(ix-y)}{cos(ix-y)}$$
$$= -\frac{1}{i}\frac{sin(y-ix)}{cos(y-ix)}$$
$$= -\frac{i}{i^2}\frac{2sin(y-ix)\cos(y+ix)}{2cos(y-ix)\cos(y+ix)}$$
$$= i\,\frac{sin[(y-ix)+(y+ix)+sin[(y-ix)-(y+ix)]}{cos[(y-ix)+(y+ix)+cos[(y-ix)-(y+ix)]}$$
$$[\because 2sinA\,cosA = sin(A+B)+sin(A-B)\,,$$
$$2cosA\,cosB = cos(A+B)+cos(A-B)]$$
$$= i\,\frac{sin\,2y+sin(-2ix)}{co\,s\,2y+cos(-2ix)}$$
$$= i\,\frac{sin\,2y-isinh2x}{cos\,2y+cosh2x}$$
$$= \frac{isin\,2y-i^2sinh2x}{cos\,2y+cosh2x}$$
$$= \frac{isinh\,2x+i\,sin2y}{cos\,2y+cosh2x}$$
$$= \frac{sinh\,2x}{cos\,2y+cosh2x}+i\,\frac{sin2y}{cos\,2y+cosh2x}$$

EXAMPLE21: Separate $sechz$ into real and imaginary parts.
SOLUTION: Now,

$$sechz = \frac{1}{coshz}$$
$$= \frac{1}{\cosh(x+iy)} = \frac{1}{\text{cosi}(x+iy)}$$
$$= \frac{1}{cosi(ix-y)} = \frac{1}{cos(y-ix)}$$
$$= \frac{2cos(y+ix)}{2cos(y+ix)cos(y-ix)}$$
$$= 2\,\frac{cosycos(ix)-sinysin(ix)}{cos[(y+ix)+(y-ix)]+cos[(y+ix)-(y-ix)]}$$
$$= 2\,\frac{cosy\,coshx-isiny\,sinhx}{cos[(y+ix)+(y-ix)]+cos[(y+ix)-(y-ix)]}$$
$$= 2\,\frac{cosy\,coshx-isiny\,sinhx}{cos[(y+ix)+(y-ix)]+cos[(y+ix)-(y-ix)]}$$
$$= 2\,\frac{cosy\,coshx}{cos2y+\cos(2ix)}$$
$$= 2\,\frac{cosy\,coshx}{cosh2x+\cos 2\text{y}} - i\,\frac{2cosy\,coshx}{cosh2x+\cos 2\text{y}}$$

3.7 *PROPERTIES OF COMPLEX NUMBERS*: -

1. **Domain and Range:** The **domain of $\boldsymbol{f(z)}$** is the set of all complex numbers z for which $f(z)$ is defined. The **range of $\boldsymbol{f(z)}$** is the set of all values that $f(z)$ can take.

   Example: If $f(z) = z_1$, then the domain is all $z \neq 0$, and the range is all complex numbers except 0.

2. **Limit of a Complex Function:** A limit exists when the value approaches the same number from all directions in the complex plane. Complex numbers have infinitely many approach paths, so limits are more restrictive.
3. **Continuity:** A function f(z) is continuous at z_0 if
$$\lim_{z \to z_0} f(z) = f(z_0)$$
   This is similar to real analysis, but applies to all plane directions.
4. **Differentiability:** A complex function is differentiable at z_0 if the limit
$$\boldsymbol{f'(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
   exists (and is the same from all directions). Complex differentiability is very strong.
5. **Cauchy's Riemannian Equations:** For a differentiability of $f(z) = u + iv$:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
6. **Algebra of Complex Functions:** If f and g are complex function, then
   a. $f + g$
   b. $f - g$
   c. fg
   d. $\frac{f}{g}$

   are complex Functions.

7. **Analyticity (Holomorphic Functions):** A function is analytic at a point if it is differentiable in a neighborhood of that point.
   Analytic functions have powerful properties:
   - infinitely differentiable
   - represented by power series harmonic
   - real and imaginary parts
8. **Harmonic Property:** If $f(z) = u + iv$ is analytic, then both u and v satisfy:

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

Therefore, both parts are harmonic functions.

9. **Zeros, Poles, and Singularities:**
   - Zeros: points where $f(z) = 0$.
   - Poles: points where the function blows up to infinity.
   - Isolated singularities classify into removable, pole, or essential.
10. **Mapping Property:** A complex function transforms one region of the plane into another.
    - Lines can become circles.
    - Circles can become other curves.
    - Analytic functions preserve angles (conformal mapping).

The above properties will be explained in the next unit, where you will study the behavior of complex functions including limits, continuity, differentiability, analyticity, Cauchy-Riemann equations, singularities, and mappings to gain a deeper understanding of their structure and behavior.

3.8 *FUNCTIONS OF A COMPLEX VARIABLE AS MAPPINGS*: -

Consider the function $f : \mathbb{C} \to \mathbb{C}$, defined as $f(z) = z^2$, where z is a complex variable or a point in the complex plane. This is a mapping between a complex valued function and a complex variable.

Sometimes it is also represented as $w = z^2$, where z is member of z-plane (domain of definition) and w its value, a complex number, member of w -plane (see Fig.1).

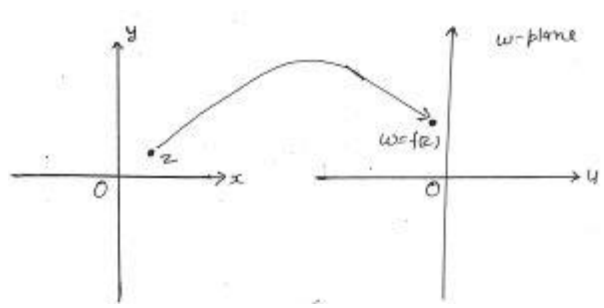

Fig.1

Thus, point z in the z-plane is mapped to point w in the w-plane. The image of a point z in the domain of definition is the point $w = f(z)$. The image of the complete domain of definition is known as the range of f. To see a graphical representation of a mapping) $w = f(z)$, let us set $z = x + iy, w = u + iv$.

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi$$

The equation

$$u = x^2 - y^2, \qquad v = 2xyi$$

is a transformation from the y -plane to the uv –plane.

SELF CHECK QUESTIONS

1. What is a complex function? Write an example.
2. If $f(z) = u(x,y) + iv(x,y)$, identify the real and imaginary parts.
3. Define the domain and range of a complex function.
4. Write the real and imaginary parts of e^z.
5. State the principal value of $logz$.
6. Find $sin(z)$ in terms of x and y.

3.9 *SUMMARY*: -

A complex function is a rule that assigns each complex number $z = x + iy$ a unique complex value $f(z) = u(x,y) + iv(x,y)$ where u and are real-valued functions of two variables. The study of complex functions focuses on their limits, continuity, differentiability, and analytic behavior. A function is continuous if its value approaches $f(z_0)$ from every direction in the complex plane, and it is differentiable only when it satisfies the Cauchy–Riemann equations, which makes complex differentiability much stronger than real differentiability. Functions that are differentiable on a region are called analytic and possess powerful properties such as infinite differentiability, representation by power series, and conformal (angle-preserving) mappings. Complex functions also include important families like exponential, trigonometric, logarithmic, and Möbius transformations, each having special geometric and algebraic properties that make complex analysis rich and widely applicable.

3.10 *GLOSSARY*: -

- **Complex Function:** A rule that assigns each complex number z a complex value $f(z)$.
- **Real Part & Imaginary Part:** For $f(z) = u + iv, u$ is the real part and v is the imaginary part.

- **Domain:** Set of complex numbers where the function is defined.
- **Range:** Set of output complex values of the function.
- **Limit:** The value a function approaches as z approaches a point from any direction.
- **Continuity:** $f(z)$ is continuous at z_0 if $f(z) = f(z_0)$.
- **Differentiability:** A function has a derivative at z_0 if the limit
$$\frac{f(z) - f(z_0)}{z - z_0}$$
exists.
- **Cauchy–Riemann Equations:** Conditions $u_x = v_y$ and $u_y = -v_x$.
- **Analytic Function:** A function differentiable on a region; behaves smoothly and equals its power series.
- **Modulus:** Magnitude of a complex number, $|f(z)| = \sqrt{u^2 + v^2}$.
- **Argument:** Angle a complex number makes with the positive real axis.
- **Singularity:** A point where the function is not analytic.
- **Zero:** A point where $f(z) = 0$.
- **Pole:** A point where $f(z)$ becomes infinite.
- **Möbius Transformation:** A mapping $\frac{cz+d}{az+b}$ that sends lines/circles to lines/circles.
- **Conformal Mapping:** Angle-preserving mapping; analytic functions with non-zero derivatives.

3.11 *REFERENCES*: -

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- S. Ponnusamy & H. Silverman (2021), Complex Variables with Applications. Birkhäuser/Springer.

3.12 *SUGGESTED READING*: -

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- Murray R. Spiegel (2009) – Schaum's Outline of Complex Variables, 2nd Edition.
- R. Narayanaswamy (2005) – Theory of Functions of a Complex Variable, S. Chand & Company Ltd

3.13 *TERMINAL QUESTIONS: -*

(TQ-1) Prove that for all values of x, y, real or complex, the following are true

k. $cos^2x + sin^2x = 1$
l. $cos(-x) = cosx$
m. $cos2x = cos^2x - sin^2x$
n. $sin3x = 3sinx - 4sin^3x$
o. $cos3x = 4cos^3x - 3cosx$
p. $sinx + siny = 2\, sin\frac{1}{2}(x+y)cos\frac{1}{2}(x-y)$
q. $cosx - cosy = 2\, sin\frac{1}{2}(x+y)\, sin\frac{1}{2}(y-x)$
r. $sin(x \pm y) = sinx\; cosy \pm cosx\; siny$
s. $cos(x \pm y) = cosx\; cosy \mp cosx\; siny$
t. $sec^2x = 1 + tan^2x$
u. $cosec^2x = 1 + cot^2x$

(TQ-2) Prove that for all values of x, y, real or complex.

a. $cosh^2x - sinh^2x = 1$
b. $sinh2x = 2sinhxcoshx$
c. $cosh2x = cosh^2x + sinh^2x = 1 + 2sinh^2x = 2cosh^2x - 1$
d. $sech^2x = 1 - tanh^2x$
e. $tan2hx = \frac{2tanhx}{(1+tanh^2x)}$
f. $sinh(x+y) = sinhx\; coshy + coshx\; sinhy$
g. $cosh(x+y) = coshx\; coshy + sinhx\; sinhy$
h. $e^x = coshx + sinhx$ and $e^{-x} = coshx - sinhx$
i. $sinh3x = 3sinhx + 4sinh^3x$
j. $cosh3x = 4cosh^3x - 3coshx$
k. $tanh3x = \frac{3tanhx + tanh^3x}{1+3tanh^2x}$

(TQ-3) Show that $\frac{1+tanhx}{1-tanhx} = cosh2x + sinh2x$

(TQ-4) Split into real and imaginary parts $\frac{e^{i\theta}}{(\mathbf{1} - \boldsymbol{ke^{i\phi}})}$.

(TQ-5) Resolve $e^{\sin(x+iy)}$ into real and imaginary parts.

(TQ-6) If $E\left(\frac{x-a+iy}{x+a+iy}\right) = P + iQ$, find P and Q.

(TQ-7) Show that $\tanh(x+y) = \frac{tanhx + tanhy}{1+tanhx\; tanhy}$

(TQ-8) If $\sin(\theta + i\phi) = cos\alpha + isin\alpha$, then prove that

$$cos^2\theta = sinh^2\emptyset = \pm sin\alpha$$

(TQ-9) Separate $\frac{cos(x+iy)}{(x+iy)+1}$ into real and imaginary parts.

(TQ-10) If $sin(\theta + i\phi) = \rho(cos\alpha + isin\alpha)$, prove that

$$\rho^2 = \frac{1}{2}[cosh2\phi - cos2\theta] \text{ and } tan\alpha = tanh\phi\; cot\theta$$

(TQ-11) If $tan(\theta + i\phi) = (tan\alpha + isec\alpha)$, prove that

$$e^{2\phi} = \pm cot\frac{1}{2}\alpha \text{ and } 2\theta = n\pi + \frac{\pi}{2} + \alpha$$

(TQ-12) Prove that

$$log\left(\frac{1}{1-e^{i\alpha}}\right) = log\left(\frac{1}{2}cosec\frac{\alpha}{2}\right) + i\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

(TQ-13) Prove that

$$log\,tan\left(\frac{\pi}{4}+i\frac{\alpha}{2}\right) = itan^{-1}(sinh\alpha)$$

(TQ-14) Prove that

$$sinh^{-1}x = tanh^{-1}\frac{x}{\sqrt{1+x^2}}$$

(TQ-15) Prove that

$$tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$$

(TQ-16) Prove that

$$coth^{-1}(2/x) = sinh^{-1}\frac{x}{\sqrt{4-x^2}}$$

(TQ-17) If $coshx = sec\theta$, then prove that $x = log(sec\theta \pm tan\theta)$

(TQ-18) If $cosh^{-1}(x+iy) + cosh^{-1}(x-iy) = cosh^{-1}a$, show that

$$2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1.$$

(TQ-19) Show that $sin^{-1}(ix) = n\pi + i(-1)^n\,log\{x+\sqrt{1+x^2}\}$.

(TQ-20) Prove that $tan^{-1}\left[i\frac{x-a}{x+a}\right] = -\frac{1}{2}ilog\left(\frac{a}{x}\right)$.

(TQ-21) Prove that $i^i = e^{-(4n+1)/2}$

(TQ-22) If $i^{\alpha+i\beta} = e^x(cosy + isiny)$,then prove that

$x = -\frac{1}{2}(4n+1)\pi\beta$ and $y = \frac{1}{2}(4n+1)\pi\alpha$.

(TQ-23) $= m^{x+iy}$, then prove that

$$\frac{y}{x} = \frac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$$

where only principal values considered.

(TQ-24) Prove that $(1+itan\alpha)^{1+itan\beta}$can have real values, one of them is $(sec\alpha)^{sec^2\beta}$.

Prove that the following

a. $sinh^{-1}z = log\{z+\sqrt{1-z^2}\}$
b. $cosh^{-1}z = log\{z+\sqrt{z^2-1}\}$
c. $tanh^{-1}z = \frac{1}{2}log\left(\frac{1+z}{1-z}\right)$
d. $cosec^{-1}z = log\left(\frac{1+\sqrt{z^2-1}}{z}\right)$
e. $sec^{-1}z = log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$
f. $coth^{-1}z = \frac{1}{2i}log\left(\frac{z+1}{z-1}\right)$

3.14 *ANSWERS: -*

<u>TERMINAL ANSWERS (TQ'S)</u>

(TQ-4) $\left(\frac{\cos\theta - k\cos(\theta-\phi)}{1-2k\cos\phi+k^2}\right) + i\left(\frac{\sin\theta - k\sin(\theta-\phi)}{1-2k\cos\phi+k^2}\right)$

(TQ-5) $e^{\sin x\cosh y} . e^{i\cos x\sinh y}$

(TQ-6) $P = e^p\cos q$ and $Q = e^p\sin q$, where $p = \frac{x^2+y^2-a^2}{(x+a)^2+y^2}$ and $q = \frac{2ay}{(x+a)^2+y^2}$

UNIT4: Limit, Continuity and Differentiability

CONTENTS:

4.1 *INTRODUCTION*: -

Limit, continuity, and differentiability are key notions in complex analysis that help us understand the behavior of complicated functions. The limit of a complex function is the value that the function approaches as the input variable approaches a specific point in the complex plane. A function is considered to be continuous at a point if its limit exists and equals the function's value, ensuring that there are no abrupt leaps or breaks. Differentiability of a complex function at a point necessitates not just continuity but also the presence of a unique complex derivative, which is a much greater requirement than in real analysis. This leads to analyticity and the solution of the Cauchy-Riemann equations. These principles, when combined, serve as the foundation for more advanced conclusions in complex analysis, such as analytic continuation, contour integration, and conformal mapping.

4.2 *OBJECTIVES*: -

After studying this unit, the learner's will be able to

- Understand the concept of limits.
- Understand the definition of continuity.
- Classify different types of discontinuities.
- Understand differentiability.
- Apply the concepts of limit, continuity, and differentiability.
- Develop problem-solving skills related to evaluating limits, checking continuity, and finding derivatives in various contexts.

4.3 *POINT SET: -*

A point set in the complex plane refers to a gathering of points, each representing a distinct element within the set. These points, commonly referred to as numbers or elements of the set, collectively constitute the spatial arrangement of the set within the two-dimensional complex plane.

4.4 *NEIGHBORHOOD: -*

In the Argand plane (also known as the complex plane), the neighborhood of a point z_0 is defined as the set of points z such that the distance between z_0 and z (denoted as $|z - z_0| < \varepsilon$ is less than some positive real number ε. Mathematically, it can be expressed as $\{z \in \mathbb{C}: |z - z_0| < \varepsilon \}$, where $\mathbb{C}$ represents the complex numbers. This neighborhood represents an open set around the point z, where all points within a certain distance ε from z are included.

The **neighborhood of the point at infinity** in the complex plane is the set of points z s.t. $|z| < k$ where k is any positive real number.

4.5 *LIMIT POINT*: -

The limit of a function $f(z)$, defined in a deleted (punctured) neighborhood of a point z_0, is said to tend to l if for every arbitrary number $\varepsilon > 0$ (however small), there exists a $\delta > 0$ such that

$$0 <| z - z_0 |< \delta \text{ implies } | f(z) - L |< \varepsilon.$$

Symbolically, the limit is written as

$$\lim_{z \to z_0} f(z) = L$$

In the limit of a real function, when $x \geq a$, then $x \geq a$ moves along the x axis, whether to the left or to the right of a. However, in the limit of a complex function, when $x \geq a$, then a can move along any curved path in the planar domain. The limit exists only if the limit of the function is the same in each case.

SOLVED EXAMPLES

EXAMPLE1: If $f(z) = z^2$, then prove that $\lim\limits_{z \to z_0} z_0^2$.

SOLUTION: Here we have to show that for any given $\delta\, \epsilon > 0$, it can be obtained (depending on ϵ) such that

$$|z - z_0| < \epsilon$$

While

$$0 <| z - z_0 |< \delta$$

If $\delta \leq 1$,then

$$0 <| z - z_0 |< \delta \Rightarrow | z - z_0 |=| z - z_0 || z + z_0 |$$

$$< \delta \mid z - z_0 + 2z_0 \mid$$

$$< \delta\{| z - z_0 | +|2z_0|\}$$

$$< \delta\{1 + 2|z_0|\}$$

Let $$\delta = min\left(1, \frac{\epsilon}{1+2|z_0|}\right)$$

$$|z^2 - z_0^2| < \epsilon \; while \mid z - z_0 \mid< \delta$$

$\therefore$ $$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} z_0^2$$

EXAMPLE2: Prove that

$$\lim_{z \to z_0} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i$$

SOLUTION: Here we have to show that $\epsilon > 0$ can be obtained for any arbitrary $\delta > 0$, so that

$$\left|\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} - (4 - 4i)\right| < \epsilon, when\; 0 < |z - i| < \delta$$

$\therefore$ $z \neq 1$, so that

$$\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = \frac{[3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i](z - i)}{z - i}$$

$$= 3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i$$

If $\delta \leq 1$, then

$$= |3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i|$$

$$= |z - i||3z^2 - (6i - 2)z - 1 - 4i|$$

$$= |z - i|3(z - i + i)^2 - (6i - 2)(z - i + i) - 1 - 4i$$

$$= |z - i||3(z - i)^2 - (12i - 2)(z - i) - 10 - 6i|$$

$$=< \delta\{3|z - i|^2 + |12i - 2||z - i| + |-10 - 6i|\}$$

$$=< \delta(3 + 13 + 12) = 28\delta$$

So if $\delta = min\left[1, \frac{\epsilon}{28}\right]$, then we get desired Result.

EXAMPLE3: (Theorems of limits): If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then prove that

i. $\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = A + B$
ii. $\lim_{z \to z_0} [f(z) - g(z)] = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} g(z) = A - B$
iii. $\lim_{z \to z_0} [f(z)g(z)] = \left[\lim_{z \to z_0} f(z)\right]\left[\lim_{z \to z_0} g(z)\right] = AB$
iv. $Lim\ \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{A}{B}$

SOLUTION:

i. Suppose $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, we get, for given $\epsilon > 0, \exists\ \delta_1(> 0)$ and $\delta_2(> 0)$ such that
$|f(z) - A| < \frac{\epsilon}{2}$ whenever $0 < |z - z_0| < \delta_1$
$|f(z) - B| < \frac{\epsilon}{2}$ whenever $0 < |z - z_0| < \delta_2$

then $\forall\ z$ with $0 < |z - z_0| < \delta = min\{\delta_1, \delta_2\}$, we obtain

$$|f(z)+g(z)-(A+B)| \le |f(z)-A|+|g(z)-B|$$

$$< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

$$\Longrightarrow \qquad \lim_{z\to z_0}[f(z)+g(z)] = A+B$$

ii. Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$, we get, for given $\epsilon > 0, \exists\ \delta_1(>0)$ and $\delta_2(>0)$ such that
$|f(z)-A| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_1$
$|f(z)-B| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_2$

then $\forall\ z$ with $0 < |z-z_0| < \delta = min\{\delta_1, \delta_2\}$, we obtain

$$|f(z)+g(z)-(A-B)| \le |f(z)-A|+|g(z)-B|$$

$$< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Since for every $\varepsilon > 0$ such that δ exists, the definition of limit gives:

$$\Longrightarrow \qquad \lim_{z\to z_0}[f(z)-g(z)] = A-B$$

iii. Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$, we get, for given $\epsilon > 0, \exists\ \delta_1(>0)$ and $\delta_2(>0)$ such that
$|f(z)-A| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_1$
$|f(z)-B| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_2$

then $\forall\ z$ with $0 < |z-z_0| < \delta = min\{\delta_1, \delta_2\}$, we obtain

$$|f(z)g(z)-(AB)| \le |B(f(z)-A)+A(g(z)-B)+ (f(z)-A)(g(z)-B)|$$

$$\le |B||f(z)-A|+|A||f(z)-B|+|f(z)-A||g(z)-B|$$

$$< \epsilon(|A|+|B|)\sqrt{\epsilon}+\epsilon$$

Since $\epsilon(>0)$ is arbitrary, we get

$$f(z)g(z) = (AB)$$

iv. Suppose $\underset{z\to z_0}{Lim}\, g(z) = B$ but $B \neq 0$ we get, for given $\epsilon > 0$, $\exists\, \delta(> 0)$ such that

$$|g(z) - B| < \epsilon \text{ whenever}$$
$$0 < |z - z_0| < \delta$$

Let us consider $\epsilon = \frac{|B|}{2}$, the above inequality reduces to

$$\frac{|B|}{2} < |g(z)| < \frac{3|B|}{2} \text{ whenever}$$
$$0 < |z - z_0| < \delta$$

Now for $0 < |z - z_0| < \delta$, we get

$$\left|\frac{1}{g(z)} - \frac{1}{B}\right| = \left|\frac{g(z) - B}{Bg(z)}\right|$$
$$= \frac{|g(z) - B|}{|B||g(z)|} \leq \frac{2|g(z) - B|}{|B|^2} < \frac{2\epsilon}{|B|^2}$$

Since $\epsilon(> 0)$ is arbitrary, we obtain

$$\underset{z\to z_0}{Lim}\frac{1}{g(z)} = \frac{1}{B}$$

Now using(iii) we get

$$\underset{z\to z_0}{Lim}\frac{f(z)}{g(z)} = \underset{z\to z_0}{Lim} f(z) \underset{z\to z_0}{Lim}\ \frac{1}{g(z)}\ = \frac{A}{B}.$$

EXAMPLE4: Find the following limits using the above limit rules:

a. $\lim_{x\to c}(x^3 + 4x^2 - 3)$
b. $\lim_{x\to c}\frac{(x^4+x^2-1)}{x^2+5}$
c. $\lim_{x\to c}(2x + 5)$
d. $\lim_{t\to 6} 8(t - 5)(t - 7)$
e. $\lim_{x\to 2}\frac{(x+2)}{x^2+5x+6}$

SOLUTION:

a. $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 = c^3 + 4c^2 - 3.$
b. $\lim_{x\to c}\frac{(x^4+x^2-1)}{x^2+5} = \frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)} = \frac{\left(\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1\right)}{\left(\lim_{x\to c}x^2+\lim_{x\to c}5\right)} = \frac{c^4+c^2-1}{c^2+5}.$
c. $\lim_{x\to c}(2x + 5) = \lim_{x\to c} 2x + \lim_{x\to c} 5 = 2c + 5.$
d. $\lim_{t\to 6} 8(t - 5)(t - 7) = 8\left(\lim_{x\to c} t - \lim_{x\to c} 5\right)\left(\lim_{x\to c} t - \lim_{x\to c} 7\right) = 8 \times (6 - 5)(6 - 7) = 8 \times (1) \times (-1) = -8.$

e. $\lim\limits_{x\to 2}\frac{(x+2)}{x^2+5x+6} = \frac{\lim\limits_{x\to 2}(x+2)}{\lim\limits_{x\to 2}(x^2+5x+6)} = \frac{2+2}{2^2+5\times 2+6} = \frac{4}{4+10+6} = \frac{4}{20} = \frac{1}{5} = 0.2.$

EXAMPLE5: If $\lim\limits_{z-z_0} f(z)$ exists, prove that it is unique.

SOLUTION: Let $\lim\limits_{z-z_0} f(z) = l_1$ and $\lim\limits_{z-z_0} f(z) = l_2$, where $l_1 \neq l_2$.

Now

$$\lim_{z-z_0} f(z) = l_1 \Rightarrow for\ every\ \delta_1 > 0, \epsilon > 0$$

So that

$$|f(z) - l_1| < \frac{\epsilon}{2}\ while\ 0 < |z - z_0| < \delta_1$$

Similarly

$$\lim_{z-z_0} f(z) = l_2 \Rightarrow for\ every\ \delta_2 > 0, \epsilon > 0$$

So that

$$|f(z) - l_2| < \frac{\epsilon}{2}\ while\ 0 < |z - z_0| < \delta_2$$

Let $\delta = min(\delta_1, \delta_2)$, then $for\ every\ \delta > 0, \epsilon > 0$
so that

$|f(z) - l_1| < \frac{\epsilon}{2}\ and\ |f(z) - l_2| < \frac{\epsilon}{2}$ while $0 < |z - z_0| < \delta$

Now

$$|l_1 - l_2| = |l_1 - f(z) - l_2|$$
$$\leq |l_1 - f(z)||f(z) - l_2|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ while } 0 < |z - z_0| < \delta$$

$\therefore$ $$|l_1 - l_2| < \epsilon$$

Since this is true for every $\in >\ 0$ and is arbitrary. Hence

$$|l_1 - l_2| = 0 \Rightarrow l_1 = l_2$$

Therefore, $\lim\limits_{z-z_0} f(z)$ if it exists, will be unique.

EXAMPLE6: Show that the $\lim\limits_{z\to 0}\left(\frac{\bar{z}}{z}\right)$ does not exist.

SOLUTION: If the limit exists, it must depend on the path z of approaching 0.

But when along the x −axis $z \to 0$ then $y = 0$.

$\therefore$ $z = x + iy = x$ and $\bar{z} = x - iy = x$

Now

$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{y\to 0}\frac{x}{x} = 1 \qquad \dots(1)$$

But when along the y −axis $z \to 0$ then $x = 0$.

$z = x + iy = iy$ and $\bar{z} = x + iy = -iy$

Now

$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{y\to 0}\frac{-iy}{iy} = -1 \qquad \ldots(2)$$

Since (1) and (2) are unequal. Hence $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

EXAMPLE7: If $f(z) = z^2$, prove that $\lim_{z\to 0} f(z) = z^2$.

SOLUTION: Let $\epsilon > 0$ given, to find $\delta > 0$ s.t. $|z^2 - {z_0}^2| < \in$ whenever $0 < |z - z_0| < \delta$

Consider $\quad |z^2 - {z_0}^2| = |(z - z_0)(z + z_0)|$

$$|z + z_0||z - z_0| < \delta|z + z_0|$$

$$= |z - z_0 + 2z_0| \le \delta|z - z_0| + 2\delta|z_0| < \delta\delta + 2\delta|z_0| = \in$$

$\because$ now $\delta > 0$s.t. $\min\left\{\frac{\in}{1+2|z_0|}, 1\right\}$

$\Rightarrow \quad |z^2 - {z_0}^2| < \in$

$\Rightarrow \quad \lim_{z\to 0} f(z) = {z_0}^2$

4.6 *CONTINUITY*: -

A function $f: \mathbb{C} \to \mathbb{C}$ is said to be **continuous** at a point $z_0 \in \mathbb{C}$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $| z - z_0 | < \delta$, it follows that $| f(z) - f(z_0) | < \epsilon$. In other words, small changes in the input z near z_0 result in small changes in the output $f(z)$. The function f is continuous on a set $S \subseteq \mathbb{C}$ if it is continuous at every point in S.

OR

A function $f: D \to \mathbb{C}$ is said to be **continuous** at a point $z_0 \in D$ iff $\forall \in > 0, \exists\ \delta > 0$ s.t $| f(z) - f(z_0) | < \epsilon$ whenever $z \in D$ and $| z - z_0 | < \in$.

Continuity from left and continuity from right: Let f be a function defined on an open interval I and let $a \epsilon I$. We say that f is continuous from the left at a if $\lim_{x\to a-0} f(x)$ exists and is equal to $f\ (a)$. Similarly according to be continuous from the right at a if lim if $\lim_{x\to a+0} f(x)$ exists and is equal to $f\ (a)$.

Continuous function: A function f is said to be continuous if, for every point in its domain, the limit of the function equals its value.

Continuity in an Open interval: A function f is said to be a continuous in the open interval $]a, b[$ if it is continuous at each point of the interval.

Or

A function f is continuous on an open interval (a, b) if f is continuous at every point $c \in (a, b)$.

Since (a, b) is open, every point c in it has points on both sides, so that

$$\lim_{x \to c} f(x) = f(c) \ \forall \ c \in (a, b)$$

Continuity in a Closed interval:

Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous a^+ a if it is continuous from the right at a and also the f is continuous at b if it is continuous from the left at b. Further, f is said to be continuous on the closed interval $[a, b]$, if (i) it is continuous from the right at a, (ii) continuous from the left at b and (iii) continuous on the open interval $]a, b[$.

Or

A function f is continuous on a closed interval $[a, b]$ then,

- f is continuous at every interior point c of $[a, b]$ $i.e.$, at $c \epsilon]a, b[$ if $f(c - 0) = f(c) = f(c + 0)$ $i.e.$, if
$$\lim_{x \to c-0} f(x) = f(c) = \lim_{x \to c+0} f(x).$$
- f is continuous at right-hand limit at the left end point b, if
$$f(b) = f(b - 0) \ i.e., \ \lim_{x \to b-0} f(x)$$
- f is continuous at left-hand limit at the left end point b, if
$$f(a) = f(a + 0) \ i.e., \ \lim_{x \to a+0} f(x).$$

4.7 *DISCONTINUITY*: -

If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of the function.

Type of Discontinuity:

1. **Removable Discontinuity:** A function is known as Removable discontinuity at a point a if $\lim_{x \to a} f(x) = f(a)$., but not equal to $f(a)$.

2. **Finite Discontinuity:** The left-hand and right-hand limits exist but are not equal:

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$$

3. **Infinite Discontinuity:**
- The function approaches $\pm\infty$ as x approaches the point.
- There is a vertical asymptote at the point.

4. **Oscillatory Discontinuity:**
- The function oscillates infinitely as $x \to a$.
- So the limit does **not** exist.

SOLVED EXAMPLES

EXAMPLE1: A function $f(x)$ defined as follow:

$$f(x) = \begin{cases} \left(\frac{x^2}{a}\right) - a, & when\ x < a \\ 0, & when\ x = a \\ a - \left(\frac{x^2}{a}\right) & when\ x > a \end{cases}$$

Prove that the function $f(x)$ is continuous at $x = a$.

SOLUTION: Let $f(a+0) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} f\left[a - \frac{a^2}{(a+h)}\right]$

$$\left[\because f(x) = a - \left(\frac{x^2}{a}\right)\ for\ x > a\right]$$

$$= \left[a - \frac{a^2}{(a)}\right] = a - a = 0$$

$$f(a-0) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} f\left[\frac{(a-h)^2}{a} - a\right]$$

$$\left[\because f(x) = \left(\frac{x^2}{a}\right) - a\ for\ x < a\right]$$

$$= \left(\frac{a^2}{a}\right) - a = a - a = 0$$

Then also have $f(a) = 0$.

Hence $f(a+0) = f(a-0) = f(a)$, therefore $f(x)$ is continuous at $x = 0$

EXAMPLE2: Examine the function defined below for continuity at $x = a$:

$$f(x) = \frac{1}{x-a} cosec\left(\frac{1}{x-a}\right), x \neq a$$

$$f(x) = 0, x = a$$

SOLUTION: Let

$$f(a+0) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} \frac{1}{a+h-a} cosec \frac{1}{a+h-a} = \lim_{h\to 0} \frac{1}{hsin\left(\frac{1}{h}\right)}$$

$$= +\infty, \qquad \text{since } hsin\left(\frac{1}{h}\right) \to 0 \; as \; h \to 0.$$

$$f(a-0) = \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0} \frac{1}{a-h-a} cosec \frac{1}{a-h-a} = \lim_{h\to 0^-} -\left[\frac{1}{hsin\left(-\frac{1}{h}\right)}\right]$$

$$= \lim_{h\to 0} \frac{1}{hsin\left(\frac{1}{h}\right)}$$

$$= +\infty, \qquad \text{since } hsin\left(\frac{1}{h}\right) \to 0 \; as \; h \to 0.$$

Also $\qquad f(a) = 0$

Since $f(a+0) = f(a-0) \neq f(a)$ the function $f(x)$ is discontinuous at $x = a$.

EXAMPLE3: Examine the function defined below for continuity at $x = a$:

$$f(x) = \frac{sin^2 ax}{x^2} for \; x \neq 0, f(x) = 1 \; for \; x = 0.$$

SOLUTION: Let $f(0) = 1$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{sin^2 ah}{h^2}$$

$$= \lim_{h\to 0} \frac{sin^2 ah}{(ah)^2} a^2 = 1. a^2 = a^2$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{sin^2(-ah)}{(-h^2)}$$

$$= \lim_{h\to 0} \frac{sin^2 ah}{(h)^2} = a^2$$

Now $f(x)$ is continuous at $x = 0$ iff

$$f(0+0) = f(0-0) = f(0)$$

Hence f(x) is discontinuous at $x = 0$ unless $a = 1$.

EXAMPLE4: A function $f(x)$ is defined as follows:

$$f(x) = 1 + x \; if \; x \le 2 f and \; f(x) = 5 - x \; if \; x \ge 2.$$

Is the function continuous at $x = 2$?

SOLUTION: Here $f(2) = 1 + 2 \; or \; 5 - 2 = 3$

$$f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [5 - (2+h)]$$

$$= \lim_{h\to 0} [3 - h] = 3$$

and

$f(2-0) = \lim_{h\to 0} f(2-h)$, where h is $+ive$ and very small

$$= \lim_{h\to 0} [1 - (2-h)], \quad [\because 2 - h < 2 \text{ and } f(x) = 1 + x \text{ if } x < 2]$$

$$= \lim_{h\to 0} [3 - h] = 3$$

Thus

$$f(2+0) = f(2-0) = f(2).$$

Hence the $f(x)$ is continuous at $x = 2$.

4.8 *DIFFERENTIBILITY*: -

Let I denote the open interval $]a, b[$ in R and let $x_0 \epsilon I$. Then a function $f: I \to R$ is said to be differential (or derivable) at x_0 iff

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Or

$$\lim_{h\to 0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists finitely and this limit, if it exists finitely, is called the **differential coefficient** or derivative of f with respect to x at $x = x_0$.

Progressive and regressive derivatives:

The progressive derivative of f at $x = x_0$ is obtained by

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}, h > 0$$

It is also called the right hand differential coefficient of f at $x = x_0$ and denoted by $\boldsymbol{R\, f'(x_0)}$ **or by** $\boldsymbol{f'(x_0+0)}$**.**

The progressive derivative of f at $x = x_0$ is obtained by

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{-h}, h > 0$$

It is also known as the left hand differential coefficient of f at $x = x_0$ and denoted by $\boldsymbol{L\, f'(x_0)}$ **or by** $\boldsymbol{f'(x_0+0)}$.

Open interval$]\boldsymbol{a}, \boldsymbol{b}[$**:** A function $f:]a, b[\to R$ is called differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.

Closed interval$[\boldsymbol{a}, \boldsymbol{b}]$: A function $f: [a, b] \to R$ is called differentiable in $[a, b]$ iff $Rf'(a)$ exist $Lf'(b)$ exists and f is differentiable at every point of $]a, b[$.

Alternative definition of differentiability: Let f be a function defined on an interval I and let a be an interior point of I. then by the definition of $f'(a)$, assuming it to exist, we get

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

i.e., $f'(a)$ *exists if for a given* $\epsilon > 0, \exists \delta > 0$, *such that*

$$\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon, whenever\ 0 < |x-a| < \delta$$

Or

$$x\epsilon]a-\delta, a+\delta[\Rightarrow f'(a)-\epsilon < \frac{f(x)-f(a)}{x-a} < f'(a)+\epsilon$$

THEOREM1: If a function f is differentiable at a point $x_0$and c is any real number, then the function cf is also differentiable at x_0 and $(cf)'(x_0) = cf'^{(x_0)}$.

PROOF: By the definition of $f'(x)$, we get

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

Now

$$= \lim_{x\to x_0} \frac{f(cf)(x)-(cf)(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{cf(x)-cf(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0} \left[c\frac{f(x)-cf(x_0)}{x-x_0}\right]$$

$$= \boldsymbol{c} \lim_{x\to x_0} \frac{f(x)-cf(x_0)}{x-x_0} = cf'(x_0)$$

Hence cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

THEOREM2: Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \epsilon I$, then so also is $f+g$ and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

PROOF: Let f and g are differentiable at (x_0), then

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

$$\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0)$$

Now

$$\lim_{x \to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}\right]$$

$$= \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0}$$

As the limit of a sum is equal to the sum of the limits.

$$= f'(x_0) + g'(x_0)$$

Hence $f+g$ is differentiable at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

THEOREM3: : Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \epsilon I$, then so also is $f+g$ and

$$(fg)'(x_0) = f'\,(x_0)g(x_0) + f(x_0)g'(x_0)$$

PROOF: Let f and g are differentiable at (x_0), then

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

$$\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0)$$

$$= \lim_{x \to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0}\; g(x) + f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]$$

$$= \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \lim_{x \to x_0} g(x) + f(x_0) \lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0}$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Now that fact

$$\lim_{x \to x_0} g(x) = g(x_0)$$

Hence fg is differentiable at x_0

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

SOLVED EXAMPLES

EXAMPLE1: Prove that $f(x) = |x|$ is continuous at x=0, but not differentiable at $x = 0$ where $|x|$means the numerical value or the absolute value of x.

SOLUTION: we get $f(0) = |0|$

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$\therefore \quad f(0) = f(0+0) = f(0-0).$

Hence $f(x)$ is continuous at $x = 0$.

Also, we have $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{h}{h}, \text{ (} h \text{ being positive)}$$

$$= \lim_{h \to 0} 1 = 1,$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h}, \text{ (} h \text{ being positive)}$$

$$= \lim_{h \to 0} -1 = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

To draw the graph of the function $f(x) = |x|$.

$$\text{We have } f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0. \end{cases}$$

Let $y = f(x)$. Then the graph of the function consists of the following straight lines:

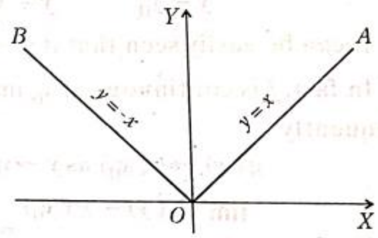
$y = x, \quad x \geq 0$

$y = -x, \quad x \leq 0.$

Let $y = f(x)$. Then the graph of the function consists of the following straight lines :

$$y = x, \quad x \geq 0$$
$$y = -x, \quad x \leq 0.$$

The graph is as shown in the figure. From the graph we observe that the function is continuous at the point O *i.e.*, at the point $x = 0$ but it is not differentiable at this point. The tangent to the curve at the point O from the right is the straight line OA and from the left is the straight line OB. Thus the tangent to the curve at O does not exist and so the function is not differentiable at O.



EXAMPLE2: Show that the function $f(x) = |x| + |x - 1|$ is not differentiable at $x = 0$ and $x = 1$.

SOLUTION: If $x < 0$, then

$$|x| = -x \text{ and } |x - 1| = |1 - x| = 1 - x;$$

if $0 \leq x \leq 1$, then $|x| = x$ and $|x - 1| = |1 - x| = 1 - x$;

and if $x > 1$, then $|x| = x$ and $|x - 1| = x - 1$.

$\therefore$ the function $f(x)$ is given by

$$f(x) = \begin{cases} 1 - 2x, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x \leq 1 \\ 2x - 1, & \text{if } x > 1. \end{cases}$$

At $x = 0$. We have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1 - 1}{h}, \text{ as } f(x) = 1 \text{ if } 0 \leq x \leq 1$$

$$= \lim_{h \to 0} 0 = 0,$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{[1 - 2(-h)] - 1}{-h}$$

$[\because f(x) = 1 - 2x, \text{ if } x < 0]$

$$= \lim_{h \to 0} \frac{2h}{-h} = \lim_{h \to 0} -2 = -2.$$

$\because$ $Rf'(0) \neq Lf'(0)$, so the given function is not differentiable at $x = 0$.

At $x = 1$. We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1 + h) - 1] - 1}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 1 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2,$$

and $$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{1-1}{-h} = \lim_{h\to 0} 0 = 0.$$

$\because$ $Rf'(1) \neq Lf'(1)$, so the given function $f(x)$ is not differentiable at $x = 1$.

EXAMPLE3: Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.

SOLUTION: Let $f(x)$ be an even function, so $f(-x) = f(x)\ \forall x$.

$f'(0)$ exists $\Rightarrow Rf'(0) = Lf'(0) = f'(0)$.

$$\text{Now } f'(0) = Rf'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h}, h > 0$$

$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{h} \qquad [\because f(-x) = f(x)]$$

$$= -\lim_{h\to 0}\frac{f(-h)-f(0)}{-h} = -Lf'(0) = -f'(0).$$

$\therefore$ $2f'(0) = 0 \Rightarrow f'(0) = 0.$

EXAMPLE4: Let $f(x) = x\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, x \neq 0; f(0) = 0$. Show that $f(x)$ is continuous but derivable at $x = 0$.

SOLUTION: We have $f(0) = 0$;

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} h\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}$$

$$= \lim_{h\to 0} h\frac{1-e^{-2/h}}{1+e^{-2/h}}, \text{ dividing the Nr. and Dr. by } e^{1/h}$$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} -h\frac{e^{1/-h}-e^{-1/-h}}{e^{1/-h}+e^{-1/-h}} = \lim_{h\to 0} -h\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}$$

$$= \lim_{h\to 0} -h\frac{e^{-2/h}-1}{e^{-2/h}+1} = 0 \times \frac{0-1}{0+1} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

$$\text{Now } Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\left[h\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}} - 0\right]\Big/ h = \lim_{h\to 0}\frac{1-e^{-2/h}}{1+e^{-2/h}} = \frac{1-0}{1+0} = 1,$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left[(-h) \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0 \right] \Big/ (-h)$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0-1}{0+1} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function is not derivable at $x = 0$.

4.9 *ROLLE'S THEOREM*: -

If a function $f(x)$ is such that

i. $f(x)$ is continuous in the closed interval $[a, b]$,
ii. $f(x)$ exists for every point in the open interval $]a, b[$,
iii. $f(a)\ f(b)$, then there exists at least one value of x, say c, where $a < c < h$ such that $f'(c) = 0$.

Proof: Since f is continuous on $[a, b]$, it is bounded on $[a, b]$. Let M and m be the supremum and infimum off respectively in the closed interval $[a, b]$. Now either $M = m$ or $M \neq m$.

If $M = m$, then f is a constant function over $[a, b]$ and consequently $f(x) = 0$ for all x in $[a, b]$. Hence the theorem is proved in this case.

If $M \neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$. For the sake of definiteness, let $M = f(a)$.

Since every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a, b]$ such that $f(c) = M$. Also, since $f(a) = M = f(b)$, therefore, c is different from both a and b. This implies that $c \epsilon]a, b[$.

Now $f(c)$ is the supremum off on $[a, b]$, therefore,

$$f(x) \leq f(c) \forall x \ in \ [a, b] \qquad \ldots (1)$$

In particular, for all positive real numbers h such that $c - h$ lies in $[a, b]$.

$$f(c - h) \leq f(c)$$

$$\frac{f(c-h)-f(c)}{-h} \geq 0. \qquad \dots (2)$$

Since $f'(x)$ exists at each point of $]a,b[$ and hence, in particular $f'(c)$ exists,so taking limit as $h \to 0$,abovw equation gives $Lf'(c) \geq 0$.

Similarly, from (1)

$$f(c+h) \leq f(c)$$

By the same argument as above, we get

$$Rf'(c) \leq 0$$

Since $f'(c)$ exists, hence

$$Lf'(c) = f'(c) = Rf'(c)$$

Now the above equations we conclude that $f'(c) = 0$.

Similarly

$$M = f(a) \neq m$$

Note 1: There may be more than one point like cat which $f\ (x)$ vanishes.
Note 2: Rolle's theorem will not hold good

i. if $f(x)$ is discontinuous at some point in the interval $a \leq x \leq b$.
ii. if $f(x)$ does not exist at some point in the interval $a < x < b$.
iii. $f(a) \neq f(b)$.

Note3: It can be seen that the conditions of Rolle's theorem are not necessary for $f'(x)$ to vanish at some point in $]a,b[$. For example, $f(x) = cos\,(1/x)$ is discontinuous at $x = O$ in the interval $[-1,2]$ but $f'(x)$ vanishes at an infinite number of points in the interval.

<u>SOLVED EXAMPLES</u>

EXAMPLE1: Discuss the applicability of Rolle’s theorem in the interval $[-1,1]$ to the function $f(x) = |x|$.

SOLUTION: Given $f(x) = |x|, here$

$$f(-1) = |-1| = 1, f(1) = |1| = 1$$

So that

$$f(-1) = f(1)$$

EXAMPLE2: Discuss the applicability of Rolle's theorem to $f(x) = log\left[\frac{x^2+ab}{(a+b)x}\right]$, in the interval $[a, b], 0 < a, b$.

SOLUTION: Here

$$f(a) = \log\left[\frac{a^2 + ab}{(a+b)\,a}\right] = \log 1 = 0,$$

$$f(b) = \log\left[\frac{b^2 + ab}{(a+b)\,b}\right] = \log 1 = 0.$$

Thus $f(a) = f(b) = 0.$

Also $Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)\,x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\frac{(x^2 + 2xh + h^2 + ab)(a+b)\,x}{(a+b)(x+h)(x^2+ab)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2 + 2xh + h^2 + ab)}{x^2 + ab} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{1 + \frac{2xh + h^2}{x^2 + ab}\right\} - \log\left\{1 + \frac{h}{x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2xh + h^2}{x^2 + ab} - \frac{h}{x} + \ldots\right],$$

$$\left[\because \log(1+y) = y - \tfrac{1}{2}y^2 + \ldots\right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $Lf'(x) = \lim_{h \to 0} \left[\frac{f(x-h)-f(x)}{-h}\right]$

$$= \lim_{h \to 0} \frac{1}{(-h)} \left[\frac{-2hx+h^2}{x^2+ab} - \frac{(-h)}{x} + \ldots\right], \text{ replacing } h \text{ by } -h \text{ in (1)}$$

$$= \frac{2x}{x^2+ab} - \frac{1}{x}.$$

Since $Rf'(x) = Lf'(x)$, $f(x)$ is differentiable for all values of x in $[a, b]$. This implies that $f(x)$ is also continuous for all values of x in $[a, b]$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]a, b[$.

Now $f'(x) = 0 \Rightarrow \frac{2x}{x^2+ab} - \frac{1}{x} = 0$ or $2x^2 - (x^2 + ab) = 0$

or $x^2 = ab$ or $x = \sqrt{(ab)}$,

which being the geometric mean of a and b lies in the open interval $]a, b[$. Hence the Rolle's theorem is verified.

4.10 *LAGRANGE'S MEAN VALUE THEOREM*: -

If a function $f(x)$ is

i. continuous in a closed interval $[a, b]$, and
ii. differentiable in the open interval $]a, b[$ $i.e., a < x < b$, then there exists at least one value $'c'$ of x lying in the open interval $]a, b[$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof: Let the function $\emptyset(x)$explained by

$$\emptyset(x) = f(x) + Ax \qquad \ldots(1)$$

Where A is a constant to be chosen such that $\phi(a) = \phi(b)$

$$f(a) + Aa = f(b) + Ab$$

$$A = -\frac{f(b)-f(a)}{b-a}$$

(i) Now the function f is given to be continuous on $[a, b]$ and the mapping $x \to Ax$ is continuous on $[a, b]$, therefore ϕ is continuous on $[a, b]$.

(ii) Also, since f is given to be differentiable on $]a, b[$ and the mapping $x \to Ax$ is differentiable on $]a, b[$, therefore, ϕ is differentiable on $]a, b[$.

(iii) By our choice of A, we have $\phi(a) = \phi(b)$.

From (i), (ii) and (iii), we find that ϕ satisfies all the conditions of Rolle's theorem on $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $]a, b[$, such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$, from (1).

$\therefore \quad \phi'(c) = 0 \Rightarrow f'(c) + A = 0$

or $\quad f'(c) = -A = \dfrac{f(b) - f(a)}{b - a}$, from (2).

This proves the theorem. It is usually known as the '**First Mean Value Theorem of Differential Calculus**'.

SOLVED EXAMPLES

EXAMPLE1: If $f(x) = (x-1)(x-2)(x-3)$ and $a = 0, b = 4$, find c using Lagrange's mean value theorem.

SOLUTION: we get

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

$\therefore \quad f(a) = f(0) = -6$ and $f(b) = f(4) = 6.$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{6 - (-6)}{4 - 0} = \frac{12}{4} = 3.$$

Also $f'(x) = 3x^2 - 12x + 11$ gives $f'(c) = 3c^2 - 12c + 11.$

Putting these values in Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c), (a < c < b), \text{ we get}$$

$$3 = 3c^2 - 12c + 11 \quad \text{or} \quad 3c^2 - 12c + 8 = 0$$

$$c = \frac{12 \pm \sqrt{(144 - 96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

EXAMPLE2: Let $f: [0,1] \to \boldsymbol{R}$ be defined by

$$f(x) = (x-1)^2 + 2 \ \forall x \epsilon [0,1]$$

Find the equation of the tangent to graph of this curve which is parallel to the chord joining the points $(0,3)$ and $(1,2)$ of the curve.

SOLUTION: Because $f(x)$ is a polynomial function, it is continuous on [0,1] and differentiable in$]0,1[$. According to Lagrange's mean value theorem, there exists some $c\epsilon]0,1[$ such that

$$\frac{f(1)-f(0)}{1-0} = f'(c) \ or \ \frac{2-3}{1} = f'(c) \ or -1 = f'(c)$$

Now

$$f'(x) = 2(x-1) obtains \ f'(c) = 2(c-1)$$

Thus $2(c-1) = -1 \ i.e., c = \frac{1}{2}$

$\therefore f(c) = \frac{9}{4}$, so that the point of contact of the tangent is $\left(\frac{1}{2}, \frac{9}{4}\right)$ and slope is $f'(c) = -1$. Hence

$$y - \frac{9}{4} = -1\left(x - \frac{1}{2}\right) \ or \ 4x + 4y = 11$$

SELF CHECK QUESTIONS

1. State the formal $(\varepsilon - \delta)$ definition of limit.
2. What is the condition for a function to be continuous at a point?
3. Write the relationship between continuity and differentiability.
4. Give one example of a function that is continuous but not differentiable.
5. Explain why a function with a “corner” cannot be differentiable.

4.11 *SUMMARY*: -

In this Unit, we learned that limits, continuity, and differentiability are basic calculus notions that characterize the behavior and smoothness of functions. The limit of a function describes how it behaves when the input approaches a specific value, laying the groundwork for defining continuity and derivatives. A function is said to be continuous at a point when its limit exists and equals the function's value at that point, showing that the graph has no breaks or jumps. Differentiability, on the other hand, refers to the presence of a single derivative at a given location, indicating that the function changes smoothly with no corners, cusps, or discontinuities. While every differentiable function is continuous, the opposite is not always true: a function might be continuous but not differentiable. Overall, these principles aid in understanding how functions vary, behave locally, and promote the study of instantaneous rates of change in mathematics and applied sciences.

4.12 *GLOSSARY*: -

- **Limit:** The value a function approaches as the input gets closer to a particular point.
- **Left-Hand Limit (LHL):** The limit of a function as the input approaches a point from the left side.
- **Right-Hand Limit (RHL):** The limit of a function as the input approaches a point from the right side.
- **Existence of Limit:** A limit exists if LHL and RHL are equal.
- **Continuity:** A function is continuous at a point if the limit exists and equals the function's value at that point, meaning no break or jump in the graph.
- **Discontinuity:** A point where a function is not continuous; can be a jump, removable, or infinite discontinuity.

- **Differentiability:** A function is differentiable at a point if its derivative exists there, meaning the graph is smooth without corners or cusps.
- **Derivative:** The instantaneous rate of change of a function or the slope of the tangent line at a point.
- **Non-Differentiable Point:** A point where the derivative does not exist, often due to a corner, cusp, discontinuity, or vertical tangent.
- **Smooth Function:** A function that is continuous and has a derivative at every point in its domain.
- **First Principle of Derivative:** Definition of derivative using the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Relationship Between Concepts:** Every differentiable function is continuous, but a continuous function may not be differentiable.

4.13 *REFERENCES*: -

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- S. Ponnusamy & H. Silverman (2021), Complex Variables with Applications. Birkhäuser/Springer.

4.14 *SUGGESTED READING*: -

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- Murray R. Spiegel (2009), Schaum's Outline of Complex Variables, 2nd Edition.
- R. Narayanaswamy (2005), Theory of Functions of a Complex Variable, S. Chand & Company Ltd
- A.R. & A.K. Vishishtha (2016-2017), Differential calculus.

4.15 *TERMINAL QUESTIONS: -*

(TQ-1) Discuss the continuity and Discontinuity of the following functions:

i. $f(x) = x^3 - 3x$
ii. $f(x) = x + x^{-1}$
iii. $f(x) = e^{-1/x}$
iv. $f(x) = sinx$
v. $f(x) = cos\left(\frac{1}{x}\right) when\ x \neq 0, f(0) = 0.$
vi. $f(x) = sin\left(\frac{1}{x}\right) when\ x \neq 0, f(0) = 0.$
vii. $f(x) = \frac{sinx}{x} when\ x \neq 0, f(0) = 1.$
viii. $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}\ when\ x \neq 0, f(0) = 1.$
ix. $f(x) = \frac{e^{1/x}}{e^{1/x}+1}\ when\ x \neq 0, f(0) = 0.$
x. $f(x) = \frac{xe^{1/x}}{e^{1/x}+1} + \sin(1/x)\ when\ x \neq 0, f(0) = 0.$
xi. $f(x) = sinx\, cos(1/x)\ when\ x \neq 0, f(0) = 0.$

(TQ-2) A function f defined on $[0,1]$ is given by

$$f(x) = \begin{cases} x & if\ x\ is\ rational \\ -1 & if\ x\ is\ irrational \end{cases}$$

Show that f takes every value between 0 and 1, but it is continuous only at the point $x = \frac{1}{2}$.

(TQ-3) Prove that the function f defined by

$$f(x) = \begin{cases} \dfrac{1}{2} & if\ x\ is\ rational \\ \dfrac{1}{3} & if\ x\ is\ irrational \end{cases}$$

is discontinuous everywhere.

(TQ-4) Show that the function f defined by $f(x) = \frac{xe^{1/x}}{1+e^{1/x}}, x \neq 0, f(0) = 1$ is not continuous at $x = 0$ and also show how the discontinuity can be removed.

(TQ-5) Show that the function $f(x) = 3x^2 + 2x - 1$ is continuous for $x = 2$.

(TQ-6) Show that the function $f(x) = (1 + 2x)^{1/x}, x \neq 0, f(x) = e^2, x = 0$ is continuous at $x = 0$.

(TQ-7) Show that the function f defined by $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}, x \neq 0$ and $f(0) = 0$ is discontinuous at $x = 0$.

(TQ-8) Show that the following function is continuous at $x = 0$.

$$f(x) = \frac{sin^{-1}x}{x}, x \neq 0, f(0) = 1.$$

(TQ-9) Prove that the function $f(x) = \frac{|x|}{x}$ for $x \neq 0, f(0) = 0$ is continuous at all points except $x = 0$.

(TQ-10) Discuss the continuity of $f(x) = (1/x)cos(1/x)$

(TQ-11) Discuss the continuity of $f(x) = \frac{1}{1-e^{1/x}}$, when $x \neq 0$ and $f(0) = 0$ for all values of x.

(TQ-12) If function f is continuous on $[a, b]$, differentiable on $]a, b[$ and if $f'(x) = 0$ for all x in $]a, b[$, then prove that $f(x)$ has a constant value throughout $[a, b]$.

(TQ-13) If $f(x)$ and $\phi(x)$ are functions continuous on $[a, b]$ and differentiable on $]a, b[$ and if $f'(x) = \phi'$ throughout the interval $]a, b[$, then prove that $f(x)$ and $\phi(x)$ differ only by a constant.

(TQ-14) If $f'(x) = k$ for each point x of $[a, b]$, k being a constant, the prove that

$$f(x) = kx + C \ \forall \ x\epsilon[a, b]$$

where C is a constant.

(TQ-15) If f is continuous on $[a, b]$ and $f'(x) \geq 0$ in $]a, b[$, then prove that f is increasing in $[a, b]$

(TQ-16) State and prove Rolle's theorem.

(TQ-17) State and prove Lagrange's mean value theorem.

(TQ-18) Show that $\frac{x}{1+x} < \log(1 + x) < x \ for \ x > 0$.

(TQ-19) Show that between any two roots of $e^x cosx = 1$ $\exists$ at least one root of $e^x sinx - 1 = 0$.

(TQ-20) If $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $]a, b[$.

(TQ-21) Let $f(x) = x\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, x \neq 0; f(0) = 0$. Show that $f(x)$ is continuous but not derivable at $x = 0$.

(TQ-22) Let $f(x) = e^{-1/x^2}\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$. Show that at every point f has a differential coefficient and this is continuous at $x = 0$.

4.16 *ANSWERS: -*

TERMINAL ANSWERS (TQ'S)

(TQ-1)

i. Continuous at $x = 0$.
ii. Discontinuous at $x = 0$.
iii. Discontinuous at $x = 0$.
iv. Continuous for all x.
v. Discontinuous at $x = 0$.
vi. Discontinuous at 0.
vii. Continuous for all x.
viii. Discontinuous at 0.
ix. Discontinuous at 0.
x. Discontinuous at 0.
xi. Continuous for all x.

BLOCK II
ANALYTIC FUNCTIONS AND COMPLEX INTEGRATION

UNIT5: Analytic Function-I

CONTENTS:

5.1 *INTRODUCTION*: -

Dear learners, in the previous units, we studied the basics of the complex plane, stereographic projection, complex functions and their properties, as well as limits, continuity, and differentiability. In this unit, we will discuss the basic concept and basic properties of analytic functions. In the study of complex analysis, the idea of an analytic function is very important.

An analytic function is a function of a complex variable that is differentiable at every point within a region of the complex plane. This requirement is far stronger than differentiability for real-valued functions, because differentiability in the complex sense implies smooth and consistent behaviour in all directions in the plane.

Analytic functions - also known as holomorphic or regular functions - possess remarkable mathematical properties. Analytic functions have wide-ranging applications beyond pure mathematics. In physics and engineering**,** they arise naturally in problems involving fluid flow**,** electromagnetic fields, and heat conduction**,** where the potential functions satisfy the same conditions as those of analytic functions. Their ability to model smooth, continuous, and interdependent quantities makes them essential tools for describing real-world phenomena.

5.2 *OBJECTIVES*: -

- Describe the meaning and characteristics of an *analytic (holomorphic)* function in complex analysis.

- Explain and derive, the *Cauchy–Riemann equations* for given complex functions to determine analyticity
- Apply analytic function properties to solve simple problems.

5.3 *ANALYTIC FUNCTIOS*: -

A function f(z) of a complex variable $z = x + iy$ is said to be *analytic* at a point z_0 if:

i. It is differentiable at z_0.
ii. It remains differentiable in some neighborhood of z_0 (that is, within some open region around that point).

In symbols:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must exist and the limit must be independent of the direction from which $\Delta z \to 0$ in the complex plane.

Cauchy–Riemann Equations:

If $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions of two real variables, then f is analytic if and only if the partial derivatives of u and v exist, are continuous, and satisfy:

i. $\partial u/\partial x = \partial v/\partial y$
ii. $\partial u/\partial y = -\partial v/\partial x$

Examples:

- $f(z) = z^n$ (for integer n)
- $f(z) = e^z$
- f(z) = sin z, cos z
- f(z) = 1/z (analytic except at z = 0)
- f(z) = log z (analytic except along its branch cut)

5.3.1 *THEOREM*: -

Theorem 1. If $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Provided the four partial derivatives u_x, u_y, v_x, v_y exist.

Proof: Let $f(z) = u + iv$ is analytic in a domain D, then $\frac{dw}{dz}$ exists so that $\frac{dw}{dz}$ has the same value along every path,

i. **Along $\boldsymbol{x-axis}$**: $\delta z = \delta x$.

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta x \to 0} \frac{\delta w}{\delta x} = \frac{\partial w}{\partial x} \dots\dots (1)$$

ii. **Along $\boldsymbol{y-axis}$**: $\delta z = i\delta y$.

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta y \to 0} \frac{\delta w}{i\delta y} = -i\frac{\partial w}{\partial y} \dots\dots (2)$$

Equating (1) to (2),

$$\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} == -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations are known as Cauchy Riemann equations.
This is necessary condition for $f(z)$ to be analytic.

Note: It is mandatory, that if $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the Cauchy Riemann equations. But converse is not true.

Theorem 2. If $f(z) = u + iv$ is analytic in a domain D, if

- **i.** u, v are differentiable in D and $u_x = v_{y,}\, u_y = v_{x,}$
- **ii.** The partial derivatives $u_x, v_{x,}, u_y, v_y$ all are continuous in domain D.

Proof: Let $f(z) = u + iv$ is analytic in a domain D. Where $u = u(x, y), v = v(x, y)$.
Therefore $f(z) = u + iv = u(x, y) + iv(x, y) = f(x, y)$.
Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

It means

$$u_x = v_{y,}\, u_y = -v_{x,} \qquad \dots (1)$$

Also let these derivatives be continuous.

Let the increments $\delta z, \delta u, \delta z, \delta w$ of z, u, v, w correspond to the increments $\delta x, \delta y$ of x
and y. Continuity of

$$u_x \Rightarrow \ \delta u = u_x \delta x + u_y \delta y + \alpha \delta x + \beta \delta y.$$

Similarly

$$v_x \Rightarrow \ \delta v = v_x \delta x + v_y \delta y + \alpha_1 \delta x + \beta_1 \delta y.$$

Where $\alpha, \beta, \alpha_1, \beta_1$ all tend to zero as $\delta x \to 0, \delta y \to 0$,

$$\frac{\delta w}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y} \qquad \dots (2)$$

$$\delta u + i\delta v = \delta x(u_x + iv_x) + \delta y\left(u_y + iv_y\right) + (\alpha + i\alpha_1)\delta x + (\beta + i\beta_1)\delta y$$
$$= \delta x(u_x + iv_x) + i\ \delta y\left(-iu_y + v_y\right) + \alpha'\ \delta x + \beta' \delta y,$$

where $\alpha' = \alpha + i\alpha_1, \beta' = \ \beta + i\beta_1$.
Using (1), $\delta u + i\delta v = (u_x + iv_x)(\delta x + \ i\delta y) + \alpha'\ \delta x + \beta' \delta y$.
Dividing by $\delta x + \ i\delta y$ and then using (2),

$$\frac{\delta w}{\delta z} = u_x + iv_x + \frac{\alpha' \delta x}{\delta x + i\delta y} + \frac{\beta' \delta x}{\delta x + i\delta y}$$

or

$$\left|\frac{\delta w}{\delta z} - (u_x + iv_x)\right| = \left|\frac{\alpha' \delta x}{\delta z} + \frac{\beta' \delta y}{\delta z}\right| \leq |\alpha'|.\left|\frac{\delta x}{\delta z}\right| + |\beta'|.\left|\frac{\delta y}{\delta z}\right|$$

$\leq |\alpha'| + |\beta'|$ as $|\delta x| \leq |\delta x + i\delta y|$

or

$$\left|\frac{\delta w}{\delta z} - \frac{\partial w}{\partial x}\right| \leq |\alpha| + \ |\alpha_1| + |\beta| + \ |\beta_1|$$

as $\alpha' = \ \alpha + i\alpha_1$.
But when $\delta z \to 0$, the R.H.S. $\to 0$.

Hence,

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} - \frac{\partial w}{\partial x} = 0 \text{ or } \frac{dw}{dz} = \frac{\partial w}{\partial x} = u_x + iv_x.$$

But u_x, v_x exist. Hence $\frac{dw}{dz}$ exists so that w is analytic in D.
Remark 1: If $f(x)$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$, then

$$f(x + h) = f(x) + hf'(x + \theta h), 0 < \theta < 1.$$

Remark 2: $\delta u = \delta x. u_x(x + \theta\delta x, y + \delta y) + \delta y. u_y(x, y + \theta' \delta y)$ where $0 < \theta < 1, 0 < \theta' < 1$.

Theorem 3. If $f(z) = u + iv$ is an analytic function in a domain D and $z = re^{i\theta}$ where u, v, r, θ are all real, show that the Cauchy Riemann equations are:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Or

To derive the necessary and sufficient condition for $f(z)$ to be analytic in polar coordinates.

Proof: Let $f(z) = u + iv$ is an analytic functions in a domain D. So that Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \dots(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \dots(2)$$

are satisfied.

To prove that polar form of (1) and (2) are:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

We have $x = r sin\theta, y = r cos\theta.$

Then $r^2 = x^2 + y^2, tan\,\theta = \frac{y}{x},\ \theta = tan^{-1}(y/x).$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = cos\theta, \frac{\partial r}{\partial y} = \frac{y}{r} = sin\theta.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} . \left(\frac{-y}{x^2}\right) = \frac{-sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} . \left(\frac{1}{x}\right) = \frac{cos\theta}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} . \frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial x} = cos\theta . \frac{\partial u}{\partial r} - \frac{sin\theta}{r}\frac{\partial u}{\partial \theta} \dots\dots\dots\dots\dots(3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} . \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} . \frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial y} = sin\theta \frac{\partial v}{\partial r} + \frac{cos\theta}{r}\frac{\partial v}{\partial \theta} \dots\dots\dots\dots\dots\dots(4)$$

By virtue of (1), (3) and (4) give,

$$\cos\theta.\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$
……………………………………………………………(5)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial y}$$
$$= \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}.\frac{\partial \theta}{\partial x}$$
$$= \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}$$

By virtue of (2), the last two equations give,

$$\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \ldots\ldots (6)$$

$(5) \times \cos\theta + (6) \times \sin\theta$ gives

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

……………………………….(7)

$$\sin\theta\frac{\partial v}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

……………………………………….(8)

From (7) and (8), the require result follows.

Theorem 4. Derivative of w in polar form.

To prove that,

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$$

Proof. Cauchy Riemann equations in polar form are:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

We have $x = rcos\theta, y = rsin\theta,$

$$r^2 = x^2 + y^2,$$

$$\theta = tan^{-1}(y/x)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y^2}{x^2}\right)}.\left(\frac{-y}{x^2}\right) = \frac{-sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y^2}{x^2}\right)}.\left(\frac{1}{x}\right) = \frac{cos\theta}{r}$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}.\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial w}{\partial r}.cos\theta - \frac{sin\theta}{r}\frac{\partial w}{\partial \theta}$$

or

$$\frac{dw}{dz} = cos\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) - \frac{sin\theta}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

$$= (cos\theta\text{-}isin\theta)\left(\frac{\partial w}{\partial r}\right) = e^{-i\theta}\frac{\partial w}{\partial r}$$

or

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}.$$

Again from (1),

$$\frac{dw}{dz} = cos\theta\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right) - \frac{sin\theta}{r}\left(\frac{\partial w}{\partial \theta}\right)$$

$$= -\frac{i}{r}(cos\theta\text{-}isin\theta),$$

$$\frac{\partial w}{\partial \theta} = \frac{-i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$$

or

$$\frac{dw}{dz} = \frac{-i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$$

Theorem 5. Continuity is necessary but not sufficient condition for the existence of a finite derivative.

Proof. I. A function which is differentiable is necessary continuous**.**

Suppose $f(z)$ is differentiable at $z = z_0$.

To prove that $f(z)$ is continuous at $z = z_0$.

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h\to 0}\frac{f(z_0-h) - f(z_0)}{h}$$

………………..(1)

By assumption, this limit exists and is unique.
From (1), we have,

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} + \varepsilon$$

and

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0-h)-f(z_0)}{-h} + \varepsilon'$$

where $\varepsilon, \varepsilon' \to 0$ as $h \to 0$.
Consequently $hf'(z_0) = f(z_0+h) - f(z_0) + h\,\varepsilon$
and

$$-hf'(z_0) = f(z_0-h) - f(z_0) - h\varepsilon'$$

Making $h \to 0$, we obtain,

$$0 = \lim_{h\to 0} f(z_0+h) - f(z_0) \text{ and } 0 = \lim_{h\to 0} f(z_0-h) - f(z_0).$$

This implies $\lim_{h\to 0} f(z_0+h) = f(z_0) = \lim_{h\to 0} f(z_0-h)$.
This implies $f(z)$ is continuous at $z = z_0$.

II. If a function is continuous, then it is not necessarily differentiable**.**
We shall prove this by solved examples.

Theorem 6. an analytic function in a region R, with constant modulus is constant.
Proof. Let $f(z) = u + iv$ be an analytic function with constant modulus.
Then,

$$|f(z)| = |u+iv| = constant$$

This implies $\sqrt{u^2+v^2}$=constant= c .

Squaring both sides, we get $u^2 + v^2 = c^2$ ... (1)

Differentiating equation (1) partially with respect to x, we get,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

This implies,

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad \text{………(2)}$$

Again, differentiating equation (1) partially with respect to y, we get,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

This implies,

$$u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y} = 0$$

Since,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x} = 0$$

………(3)

Squaring and adding (2) and (3), we get,

$$(u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=0 \qquad \text{………(4)}$$

Since $u^2+v^2=c^2\neq 0$ and $f'(z)=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial x}$.

Therefore $|f'(z)|^2=0$.

This implies $|f'(z)|=0$.

This implies $f(z)$ is constant.

SOLVED EXAMPLES

EXAMPLE 1: Explain that $w=f(z)=|z|^2$ is analytic function or not.

SOLUTION: Consider the function $f(z)=|z|^2$.

To prove that $f(z)$ is continuous everywhere but not differentiable everywhere except at $z=0$,

For a function $f(z)$ is continuous at a point z_0.

i. It must be defined at z_0.

ii. It's limit must exist at z_0.

i. $f(z_0)=\lim_{z\to z_0} f(z)$.

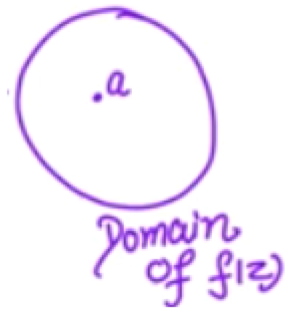
Given $f(z)=|z|^2=|x+iy|^2=(\sqrt{x^2+y^2})^2=x^2+y^2+i0$.

It implies that $u=x^2+y^2, v=0$.

Let $z=a$ be any point in the domain of $f(z)$ where $a\in\mathbb{C}$.

Then

$$\lim_{z\to a} f(z)=\lim_{z\to a}|z|^2=f(a)=|a|^2.$$



$$\lim_{z \to a} f(z) = f(a).$$

Hence function $f(z)$ is continuous at this domain $\mathbb{C}$ since a is arbitrary

Differentiability of $f(z) = |z|^2$.

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{h}$$

$$= \lim_{\delta z \to 0} \frac{|z_0 + \delta z|^2 - |z_0|^2}{h}$$

$$= \lim_{\delta z \to 0} \frac{(z_0 + \delta z)\left(\overline{z_0} + \overline{\delta z}\right) - z_0 \bar{z}}{h}$$

$$= \lim_{\delta z \to 0} \frac{z_0 \overline{\delta z} + \delta z\left(\overline{z_0} + \overline{\delta z}\right)}{\delta z}$$

$$f'(z_0) = \lim_{\delta z \to 0} \left(z_0 \frac{\delta \bar{z}}{\delta z} + \overline{z_0} + \delta \bar{z}\right) \quad \ldots (1)$$

If this limit will exist, then it will be independent of the path along which $\delta z \to 0$.

Case I: Let $\delta z \to 0$ along real axis so that $\delta \bar{z} = \delta z = \delta x, \delta y = 0$ and $\delta x \to 0$ as $\delta z \to 0$

Now (1) becomes

$$f'(z_0) = \lim_{\delta x \to 0} \left(z_0 + \overline{z_0} + \delta x\right)$$

$$f'(z_0) = z_0 + \overline{z_0} \quad \ldots (2)$$

Case II: Let $\delta z \to 0$ along imaginary axis so that $\delta \bar{z} = \delta z = i\, \delta y$, and $\delta y \to 0$ as $\delta z \to 0$.

Now (1) becomes:

$$f'(z_0) = \lim_{\delta y \to 0} \left(z_0 \left(\frac{-i\delta y}{i\delta y}\right) + \overline{z_0} - i\, \delta y\right)$$

$$f'(z_0) = \overline{z_0} - z_0 \qquad \text{... (3)}$$

From (2) and (3), we see that $f'(z_0)$ along the two paths are different except at $z = 0$.
Hence $f'(z_0)$ does not exist everywhere except at $z = 0$. Consequently $f(z)$ is not differentiable except at $z = 0$.
Since the definition of analytic function: A function f(z) of a complex variable $z = x + iy$ is said to be *analytic* at a point z_0 if:

i. It is differentiable at z_0.
ii. It remains differentiable in some neighborhood of z_0 (that is, within some open region around that point).

In symbols:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must exist and the limit must be independent of the direction from which $\Delta z \to 0$ in the complex plane.
The function $f(z) = |z|^2$ is differentiable at $z = 0$. But $f'(z_0)$ does not exist everywhere except at $z = 0$. So the function is no where analytic.
The function $f(z) = |z|^2$ is not differentiable at neighbourhood of $z = 0$. Whether the function $f(z) = |z|^2$ is satisfied the C-R equations at $z = 0$.
The C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are
Given $f(z) = |z|^2 = |x + iy|^2 = (\sqrt{x^2 + y^2})^2 = x^2 + y^2 + i0$.
It implies that $u(x, y) = x^2 + y^2, v(x, y) = 0$.

At $z = 0$,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x^2 - 0}{x} = 0.$$
$$\frac{\partial u}{\partial y} = \lim_{x \to 0} \frac{u(0, y) - u(0,0)}{y} = \lim_{y \to 0} \frac{y^2 - 0}{y} = 0.$$
$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0,0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0.$$
$$\frac{\partial v}{\partial y} = \lim_{x \to 0} \frac{v(0, y) - v(0,0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0.$$

Hence Cauchy -Riemann equations are satisfied at $z = 0$.

EXAMPLE 2: Prove that the function $f(z) = u + iv$, where

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z\neq 0, f(0)=0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin yet $f'(z)$ does not exist.

SOLUTION: $u+iv=f(z)$

$$=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$=\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}, z\neq 0.$$

This implies that $u=\frac{(x^3-y^3)}{x^2+y^2}$, $v=\frac{(x^3+y^3)}{x^2+y^2}$, where $x\neq 0, y\neq 0$.

I. To prove that $f(z)$ is continuous everywhere.
When $z\neq 0, u$ and v both are rational functions of x and y with non - zero denominators.
It follows that u, v and therefore and $f(z)$ are continuous functions everywhere except $z=0$.
To test the continuity of u,, v at $z=0$, we change u,, v to polar co-ordinates:
$u=r(cos^3\theta-sin^3\theta), v=r(cos^3\theta+sin^3\theta)$.
As $z\to 0, r\to 0$.
Evidently, $\lim_{r\to 0} u=0=\lim_{r\to 0} v$.
This implies $\lim_{z\to 0} f(z)=0=f(0)$.
This implies $f(z)$ is continuous everywhere.

II. To show that Cauchy-Riemann equations are satisfied at $z=0$.

$$f(0)=0\Rightarrow u(0,0)+iv(0,0)$$
$$=0$$
$$\Rightarrow u(0,0)=0=v(0,0).$$

Recall that, At $z=0$

$$\frac{\partial u}{\partial x}=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{x}$$

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x-0}{x}=1.$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1.$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y - 0}{y} = 1.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy -Riemann equations are satisfied at $z = 0$.

III. To prove that $f'(0)$ does not exist:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{f(z) - 0}{z}$$

$$f'(0) = \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the path $y = x$,

then,

$$f'(0) = \lim_{x \to 0} \frac{(x^3 - x^3) + i(x^3 + x^3)}{(x^2 + x^2)(x + ix)} = \frac{i}{1 + i}$$

Let $z \to 0$ along the path $x -$ axis, then

$$f'(x) = \lim_{x \to 0} \frac{(x^3 - 0) + i(x^3 + 0)}{(x^2 + 0)(x + i0)}$$

Since $y = 0$.

$$f'(x) = 1 + i$$

$$f'(0) = \begin{cases} i/(1 + i) \text{ along the path } y = x \\ 1 + i \text{ along the path } y = 0 \end{cases}$$

Since the values of $f'(0)$ are not unique along different paths. Hence $f'(0)$ does not exist. As a result of which $f(z)$ is not analytic at $z = 0$.

EXAMPLE 3: Show that the function:

$f(z) = e^{-z^{-4}}, z \neq 0$ and $f(0) = 0$, is not analytic at $z = 0$, although Cauchy Riemann equations are satisfied at the point. How would you explain this?

SOLUTION: To show that Cauchy Riemann equation are satisfied at $z = 0$.

$$w = f_1(z) = u(x, y) + iv(x, y)$$

It is given that

$$0 = f(0) = u(0,0) + iv(0,0)$$

This implies

$$u(0,0) = 0 = v(0,0)$$

If $z \neq 0$, $f(z) = exp(-z^{-4}) = u + iv$

or

$$\exp[-(x + iy)^{-4}] = u(x, y) + iv(x, y)$$

this implies

$$u(x, 0) + iv(x, 0) = exp(-(x)^{-4}),$$

and

$$u(0, y) + iv(0, y) = exp(-(y)^{-4}),$$

$$u(x, 0) = \ exp(-(x)^{-4}), , v(x, 0) = 0.$$

and

$$u(x, 0) = exp(-(x)^{-4})) , v(x, 0) = 0.$$

and $u(0, y) = exp(-(y)^{-4})) , v(0, y) = 0.$

and

$$u(0, y) + iv(0, y) = exp(-(y)^{-4}),$$

Recall that,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x + h, 0) - u(x, y)}{x}$$

At $z = 0$,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h, 0) - u(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{exp(-(x)^{-4})) \ - 0}{x}$$

$$=\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{exp\left(\frac{1}{x^4}\right)}\right]$$

$$=\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{1+\frac{1}{x^4}+\frac{1}{x^4 2!}+\cdots\ldots\ldots}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{1+\frac{1}{x^4}+\frac{1}{x^4 2!}+\cdots\ldots\ldots}\right]=\frac{1}{\infty}=0.$$

At $z=0$,

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}$$

$$=\lim_{x\to 0}\frac{exp(-(y)^{-4}))\ -0}{y}$$

$$=\lim_{y\to 0}\left[\frac{exp\left(-\frac{1}{y^4}\right)-0}{y}\right]=0$$

$$=\lim_{y\to 0}\frac{1}{x}\left[\frac{1}{1+\frac{1}{y^4}+\frac{1}{y^8 2!}+\cdots\ldots\ldots}\right]$$

$$=\lim_{y\to 0}\left[\frac{1}{1+\frac{1}{y^4}+\frac{1}{y^4 2!}+\cdots\ldots\ldots}\right]=\frac{1}{\infty}=0.$$

At $z=0$,

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0.$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Hence Cauchy -Riemann equations are satisfied at $z=0$.

II. To show that $f(z)$ is not analytic at $z=0$,

$$\lim_{z\to 0} f(z) = 0 = \lim_{z\to 0}[exp(-z^{-4})].$$

Let $z \to 0$, along the path $z = re^{i\pi/4} \to 0$.

So that,

$r \to 0$ as $z \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}[exp(-z^{-4})] = \lim_{r\to 0} exp\left[-r^{-4}e^{-i\pi}\right]$$

$$= \lim_{r\to 0} exp[-r^{-4}]$$

$$= \lim_{r\to 0}\left[exp\left(\frac{1}{r^4}\right)\right] = 0^{\infty} = \infty.$$

It shows that $\lim_{z\to 0} f(z)$ does exist. Consequently $f(z)$ is not continuous at $z = 0$. Hence $f(z)$ is not necessarily differentiable at $z = 0$. Therefore $f(z)$ is not analytic at $z = 0$. So the function:$f(z) = e^{-z^{-4}}, z \neq 0$ and $f(0) = 0$, is not analytic at $z = 0$, although Cauchy Riemann equations are satisfied at the point.

EXAMPLE 4: Show that the function:
$f(z) = |xy|^{1/2}$ is not analytic (regular) at $z = 0$, although Cauchy Riemann Equations are satisfied at the point. How would you explain this?

SOLUTION: $u + iv = f(z) = |xy|^{1/2}$,

Hence $u(x, y) = |xy|^{1/2}, v(x, y) = 0$.

At $z = 0$, $\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0.$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0, y) - u(0,0)}{y} = \lim_{y\to 0}\frac{0 - 0}{y} = 0.$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x, 0) - v(0,0)}{x} = \lim_{x\to 0}\frac{0 - 0}{x} =.$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0, y) - v(0,0)}{y} = \lim_{y\to 0}\frac{0 - 0}{y} = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy -Riemann equations are satisfied at $z = 0$.

To prove that $f'(0)$ does not exist:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{|xy|^{1/2} - 0}{x + iy}$$

$$= \lim_{x \to 0} \frac{x\sqrt{m} - 0}{x + imx} \text{ along } y = mx.$$

$$= \frac{\sqrt{m}}{1 + im}$$

Since the values of $f'(0)$ are not unique along different paths. Limit depends on m and so on it is not unique. Hence $f'(0)$ does not exist. As a result of which $f(z)$ is not analytic (regular) at $z = 0$.

5.4 *SUMMARY*: -

The concept of an analytic function is central to the study of Complex Analysis. A complex function $f(z)$, where $z = x + iy$, is said to be analytic (or holomorphic) at a point if it is differentiable in a neighborhood of that point in the complex plane. Analytic functions play a role in complex analysis similar to that of differentiable functions in real analysis, but they possess much stronger and more elegant properties due to the nature of complex differentiation.

For a function $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions representing the real and imaginary parts respectively, the Cauchy–Riemann (C–R) equations provide the necessary and sufficient conditions for analyticity. These equations are: $u_x = v_y$ and $u_y = -v_x$. If these partial derivatives exist and are continuous in a region, then $f(z)$ is analytic in that region. These equations link the behavior of the real and imaginary parts of the function, ensuring that the derivative of $f(z)$ is independent of the direction of approach in the complex plane.

In summary, the Analytic Function Unit introduces learners to the core ideas of complex differentiability, the Cauchy–Riemann equations. These concepts form the foundation for deeper topics such as complex integration, conformal mapping, and the residue theorem. Analytic functions are not only central to pure mathematics but also find significant applications in physics and engineering, particularly in potential theory, fluid flow, and electromagnetic field analysis.

5.5 *GLOSSARY*: -

- **Complex Numbers:** In mathematics, a complex number is an element of a number system that extends the real numbers with a specific element denoted *i*, called the imaginary unit and satisfying the equation $i^2 = -1$; every complex number can be expressed in the form $a \pm ib$ where *a* and *b* are real numbers. Because no real number satisfies the above equation, *i* was called an imaginary number by René Descartes. For the complex number $a \pm ib$, *a* is called the real part, and *b* is called the imaginary part. The set of complex numbers is denoted by either of the symbols ℂ or C.
- **Limit:** A function $f(z)$ tends to the limit l as z tends to z_0 along any path, if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$ and we write $\lim_{z \to z_0} f(z) = l$, where l is finite.
- **Continuity:** For a function $f(z)$ is continuous at a point z_0. It must be defined at z_0. It's limit must exist at z_0. $f(z_0) = \lim_{z \to z_0} f(z)$.

iii. Differentiability: In symbols:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must exist and the limit must be independent of the direction from which $\Delta z \to 0$ in the complex plane.

- **A function that is differentiable at every point** in some region of the complex plane.
- **Has a complex derivative** that exists and is continuous.
- **Can be expressed as a power series** (Taylor series) around any point in its domain.
- **Also called holomorphic function.**

CHECK YOUR PROGRESS

CYP 1. At $z = 0$. the function $f(z) = \bar{z}$ is not differentiable. True\False
CYP 2. $f(z) = 1/(z-1)^3$, $z \neq 1$ is analytic. True\False.
CYP3. Continuity is the necessary nut not the sufficient condition for the existence of a finite derivative. True\False.
CYP4. If $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the Cauchy Riemann equations. True\False.
CYP5. Cauchy Riemann equations are sufficient for a function to be analytic True\False.
CYP6. The function $f(z) = xy + iy$:

i. Everywhere continuous

ii. Analytic
iii. Everywhere Continuous but not analytic
iv. None of these

CYP7. The analytic function whose real part is $e^x cosy$ is:
i. $e^z + ci$
ii. e^{2z}
iii. xe^z
iv. None of these

CYP8. If $f(z)$ is an analytic function whose real part is constant, the $f(z)$is a ……

CYP9. If $f(z)$and $\overline{f(z)}$ are both analytic, that $f(z)$is a ………………

CYP10. An analytic function in a region R, with constant modulus is …………..

5.6 *REFERENCES*: -

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- Goyal, J. N., & Gupta, K. P. (2017). *Theory of functions of a complex variable.* Krishna Prakashan Media Pvt. Ltd.

5.7 *SUGGESTED READING*: -

- Ahlfors, L. V. (1979). *Complex analysis* (3rd ed.). New York, NY: McGraw-Hill Education.
- Conway, J. B. (1978). *Functions of one complex variable I* (2nd ed.). New York, NY: Springer.
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5.8 *TERMINAL QUESTIONS*: -

TQ1: State the basic difference between the limit of a function of a real variable and that of a complex variable.

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……………

TQ 2: If $f(z)$and $\overline{f(z)}$ are both analytic, show that $f(z)$is a constant.

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TQ3: If $f(z)$ is an analytic function whose real part is constant, prove that f $f(z)$is a constant function.

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…………………..

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…………………..

TQ4: Show that for the function

$$f(z) = \begin{cases} (\bar{z})^2/z, & z \neq 0. \\ 0, z = 0 \end{cases}$$

the C-R equations are satisfied at origin. Does $f'(0)$ exist?

..……………………………………………………………………
………..

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………….

5.9 *ANSWERS*: -

CHECK YOUR PROGRESS:

CYP1: True.

CYP2: False.

CYP3: True.

CYP4: True**.**

CYP5: False.

CYP6: (iii)

CYP7: (i)

CYP8: constant function
CYP9: constant function.
CYP10: constant function

TERMINAL QUESTIONS:
TQ4: No.

UNIT6: Analytic Function-II

CONTENTS:

6.1 *INTRODUCTION*: -

Complex analysis is a fundamental branch of mathematics that studies functions of complex variables. It provides powerful tools and concepts that are widely applied in physics, engineering, and applied sciences.

Dear learner's in previous unit we have studied the basics of analytic function and it's properties. The topics covered in this unit ranging from definitions to advanced concepts like branch points and multi-valued functions—form the backbone of understanding analytic functions and their properties.

The concept traces back to studies of harmonic functions by Euler and d'Alembert, arising from physics problems in heat flow, fluid flow, and mechanics. However, the idea of a harmonic conjugate was not yet formalized. Augustin-Louis Cauchy established the foundation of complex analysis and introduced the Cauchy–Riemann equations. He proved that if $f(z) = u(x, y)+iv(x, y)$ is analytic, then u and v must satisfy the Cauchy–Riemann equations. This formalized the concept of a harmonic conjugate: a function v paired with u to form a holomorphic function.

Bernhard Riemann expanded the geometric interpretation of complex functions. He studied how harmonic functions and their conjugates correspond to orthogonal families of curves and showed that analytic functions preserve angles (conformal maps). His work solidified the importance of harmonic conjugates in complex function theory. Harmonic conjugates became widely used in potential theory, electrostatics, hydrodynamics, and conformal mapping methods such as the Schwarz–Christoffel transformation. These applications linked mathematical theory to physical problems. Today, the harmonic conjugate is a key concept in: analytic function theory, PDEs and potential theory, conformal geometry, Fourier analysis.

Ref: *https://darwinthenandnow.com/wpcontent/uploads/2016/05/Riemann-Bernhard.jpg*

Georg Friedrich Bernhard Riemann Born in Germany, 1826-1866 Mathematician Established the mathematics for general relativity and pioneered contributions to differential geometry; formulated the Riemann integral, Fourier series, prime-counting function, contributed to complex analysis including Riemann surfaces, Riemann hypothesis, and the analytic number theory.

Fig 1.1.

6.2 *OBJECTIVES*: -

- Describe the meaning and characteristics of an Conjugate function, Harmonic function and Orthogonal system.
- Explain the Construction of an analytic function.
- Discuss Branch point, Branch cut and branch of multi-valued functions.

6.3 *DEFINITION*: -

The given definitions are useful for the study of analytic functions.

6.3.1 *CONJUGATE FUNCTION*: -

If $f(z) = u + iv$ is analytic in a domain D, and u and v satisfy the Laplace's equations

$$\nabla^2 V = 0,$$

then u and v are called conjugate harmonic functions or conjugate functions simply.

$$\nabla^2 V = 0,$$

$$\text{means } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V = 0.$$

$$\text{Where, } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Del Operator: The gradient operator, Del (∇), in Cartesian Coordinates

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}.$$

$$\nabla^2 = \nabla \cdot \nabla.$$

6.3.2 *HARMONIC FUNCTION*: -

A function $u(x, y)$ is called harmonic function if first and second order partial derivatives of u are continuous and u satisfy Laplace's equation,

$$\nabla^2 V = 0$$

A complex function is harmonic iff its real and imaginary parts are harmonic. Thus, it suffices to treat only the real valued functions, in the study of harmonic functions. From the linearity of the differential operator, ∇^2 it follows that the set of all harmonic maps on a domain forms a vector space. In particular all linear functions $(ax + by)$ are harmonic. However, it is not true that product of two harmonic functions is harmonic. For example, xy is harmonic but x^2y^2 is not harmonic.

EXAMPLE:

Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

$$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \qquad (1)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y)$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \qquad (2)$$

Adding (1) and (2) yields $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and u is harmonic.

6.3.3 *ORTHOGONAL SYSTEM*: -

Two families of curves $u(x, y) = c_1, v(x, y) = c_2$, are said to form an Orthogonal system if they intersect at right angles at each of their points of intersection.

6.4 *CONSTRUCTION OF ANALYTIC FUNCTION*: -

Method I. Milne's Thomson's Method:

We have, $z = x + iy$ so that $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$.

$$w = f(z) = u + iv = u(x, y) + iv(x, y)$$

or

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

In fact, this relation is formal identity in two independent variables z and $\bar{z}$.

By setting $x = z, y = 0$ so that $z = \bar{z}$, we obtain:

$$f(z) = u(z, 0) + iv(z, 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We know that $f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

(by Cauchy Riemann equations)

Taking

$$\frac{\partial u}{\partial x} = \emptyset_1(x, y) = \emptyset_1(z, 0)$$

$$\frac{\partial u}{\partial y} = \emptyset_2(x, y) = \emptyset_2(z, 0)$$

We get $$f'(z) = \emptyset_1(z, 0) - i\emptyset_2(z, 0)$$

Integration yields the result,

$$f(z) = \int [\emptyset_1(z, 0) - i\emptyset_2(z, 0)]dz + c,$$

where c is a constant.

We can calculate $f(z)$ directly if u is known.

Similarly if $v(x, y)$ is given, then it can be proved that:

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]dz + c',$$

Where

$$\frac{\partial v}{\partial y} = \psi_1(x, y)$$

$$\frac{\partial v}{\partial x} = \psi_2(x, y).$$

EXAMPLE:

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

Step I:

. **Derivatives:**

$$u_x = 3x^2 - 3y^2 + 6x$$

$$u_y = -6xy - 6y$$

Step II:

Substitute:

Set $x = z, y = 0$.

$$u_x(z,0) = 3z^2 - 0 + 6z = 3z^2 + 6z$$

$$u_y(z,0) = 0 - 0 = 0$$

Step III:

Formula: $f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$

since $v_y = u_x$ and $v_x = -u_y$

Step IV:

Integrate:

$$f(z) = \int (3z^2 + 6z)dz - i\int 0\,dz + C$$

$$f(z) = (z^3 + 3z^2) - 0 + C$$

$$f(z) = z^3 + 3z^2 + C.$$

Method II. Suppose $f(z) = u + iv$ is analytic and u is known. To determine $f(z)$.

Firstly we shall determine v.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy,$$

By Cauchy-Riemann equations.

Taking $M = -\frac{\partial u}{\partial y}, N = \frac{\partial u}{\partial x}$, we get,

$dv = Mdx + Ndy$ ……………………..(1)

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -\nabla^2 u = 0$ (For u satisfies Laplace's equation)

or

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Consequently (1) is exact differential equation.

So equation (1) can be integrated and v can be determined from the equation $f(z) = u + iv$.

EXAMPLE:

Given $u = x^3 - 3xy^2$.

$u_x = 3x^2 - 3y^2, u_y = -6xy.$

$v_x = u_y = -6xy, v_y = -u_x = -(3x^2 - 3y^2) = 3y^2 - 3x^2.$

$dv = (-6xy)dx + (3y^2 - 3x^2)dy.$

(using integration with respect to x as y constant)

$f(z) = (x^3 - 3xy^2) + i(-3x^2y + y^3 + C).$

6.5 *THEOREMS*: -

THEOREM 1: Suppose $f(z) = u + iv$ is analytic on domain D in $\mathbb{C}$, then $h = Re\ f(z)$ is harmonic on D

PROOF: A function $u(x, y)$ is called harmonic function if first and second order partial derivatives of u are continuous and u satisfy Laplace's equation,

$$\nabla^2 V = 0,$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V = 0.$$

Since $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \dots (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \dots . (2)$$

Since differentiating equation (1) with respect to y and differentiating equation (2) with respect to x and adding (1) and (2),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

Thus, the real part of any analytic function is a harmonic function. The imaginary part is also harmonic for the same reason.

THEOREM 2: If $f(z) = u + iv$ is analytic on domain D in $\mathbb{C}$, prove that the curves $u = constant,\ v = constant,$ form two orthogonal families.

PROOF: Since $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \dots (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \dots . (2)$$

To prove that the curves $u(x, y) = constant = c_1,\ v(x, y) = constant = c_2,$ form two orthogonal families..
Let $m_1 =$ slop of the tangent to the curve $u = c_1$.
$m_2 =$ slop of the tangent to the curve $v = c_2$.
If we show that $m_1 m_2 = -1$.
Taking differential of the curve $u = c_1$ and the curve $v = c_2$.

$$du = 0, dv = 0.$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$m_1 = \frac{dy}{dx} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}, = -\frac{u_x}{u_y}$$

$$m_2 = \frac{dy}{dx} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}, = -\frac{v_x}{v_y}$$

$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = \frac{u_x v_x}{u_y v_y} = \frac{u_x v_x}{(-v_x)(u_x)} = -1$$

So the curves $u(x,y) = constant = c_1,\ v(x,y) = constant = c_2,$ form two orthogonal families.

THEOREM 3: To prove that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}}$$

PROOF: Let $z = x + iy$.
Then $\bar{z} = x - iy$.

$$x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i}$$

$$y = -\tfrac{i}{2}(z - \bar{z}).$$

This implies

$$\frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}}$$

$$\frac{\partial y}{\partial \bar{z}} = \frac{i}{2} = -\frac{\partial y}{\partial z}$$

Let $f = f(x,y)$.

Then $f = f(x,y) = f(z,\bar{z})$, also,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

Or

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$$

Or

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

THEOREM 4: If $f(z) = u + iv$ is analytic on domain D in $\mathbb{C}$, prove that,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\mathrm{Rf(z)}|^2 = 2|\mathrm{f}'(\mathrm{z})|^2.$$

PROOF: $\boldsymbol{f}(z) = u + iv, Rf(z) = u.$

$$\frac{\partial u^2}{\partial x^2} = 2u\frac{\partial u}{\partial x}$$

……………………………..(1)

After differentiation equation (1).

$$\frac{\partial^2 u^2}{\partial x^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$

…………………………………(2)

$$\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$

………………………………(3)

Adding equation (2) and (3),

$$\frac{\partial u^2}{\partial x^2}+\frac{\partial u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]$$

But u satisfies Laplace's equation,

So, by using $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \text{……………(4)}$$

But $\mathrm{f}'(\mathrm{z}) = \frac{\mathrm{dw}}{\mathrm{dz}} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

From equation (4),

$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}=2|\mathrm{f}'(\mathrm{z})|^2.$

Or

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\mathrm{Rf(z)}|^{\mathbf{2}}=2|\mathrm{f}'(\mathrm{z})|^2$$

THEOREM 5: If $w=f(z)=u+iv$ is a analytic on domain D in $\mathbb{C}$, $\mathrm{f}'(\mathrm{z})\neq 0$, prove that,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)log|\mathrm{f(z)}|=0.$$

If $|\mathrm{f(z)}|$ is the product of a function of x and a function of y, show that

$$\mathrm{f(z)}=\mathrm{e}^{\alpha z^2+\beta z+\gamma},$$

where α is real and β,γ are complex constants.

PROOF: Recall that,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ log|\mathrm{f(z)}|=4\frac{\partial^2}{\partial z\partial \bar{z}}\ log|\mathrm{f(z)}|$$

$$=4\frac{\partial^2}{\partial z\partial \bar{z}}\ log|\mathrm{f}'(\mathrm{z})|^2$$

$$=2\frac{\partial^2}{\partial z\partial \bar{z}}\ log\mathrm{f}'(\mathrm{z})\mathrm{f}'(\bar{z})\text{ as }|\mathrm{z}|^2=\mathrm{z}\bar{\mathrm{z}}$$

$$=2\left[\frac{\partial^2}{\partial z\partial \bar{z}}\ log\mathrm{f}'(\mathrm{z})+\frac{\partial^2}{\partial z\partial \bar{z}}log\mathrm{f}'(\bar{z})\right]$$

$$=2\left[\frac{\partial}{\partial z}\ \frac{\mathrm{f}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}+\frac{\partial}{\partial z}\frac{\mathrm{f}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})}\right]=2[0+0\}=0.$$

It follows from the fact $f(z)$ is treated as constant in differentiating with respect to z and $f(\bar{z})$ is treated as constant in differentiating with respect to z.

Second Part:

Let $|\mathrm{f(z)}|=\emptyset(x)\psi(y)$.

By the first part,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ log|\mathrm{f(z)}|=0$$

Or

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\ log\emptyset(x)\psi(y).$$

$$= \frac{\partial^2}{\partial x^2}(log\emptyset + log\psi) + \frac{\partial^2}{\partial y^2}(log\emptyset + log\psi)$$
$$= 0$$

or

$$\frac{\partial^2}{\partial x^2} log\emptyset(x) + \frac{\partial^2}{\partial y^2}\ log\psi(y)$$
$$= \frac{d^2}{\partial x^2} log\emptyset(x) + \frac{d^2}{\partial y^2}\ log\psi(y)$$
$$= 0.$$

or

$$\frac{d^2}{\partial x^2} log\emptyset(x) = -\frac{d^2}{\partial y^2}\ log\psi(y) = 2p, say$$

[L.H.S. and R.H.S. both are independent of each other].

$$\frac{d^2}{\partial x^2} log\emptyset = 2p,$$

gives on integration

$$\frac{d(log\emptyset)}{dx} = 2px + q.$$

Again integrating, $log\emptyset(x) = px^2 + qx + r$.
Similarly $-log\psi(y) = py^2 + q_1 y + r_1$.

$$log\emptyset\psi = log\emptyset + log\psi = p(x^2 - y^2) + (qx - q_1 y) + (r - r_1)$$

Or

$|f'(z)| = \emptyset(x)\psi(y) = \exp[p(x^2 - y^2) + (qx - q_1 y) + (r - r_1)] \ldots\ldots (1)$

Now since, $|\exp[\alpha z^2 + \beta z + y]]| = |\exp[\alpha(x + iy)^2 + \beta(x + iy) + y]]|$

$$= |\exp[\alpha(x^2 - y^2) + 2i\alpha xy + (a + ib)(x + iy) + (c + id)]|$$

as α is real.

$$= |\exp[\alpha(x^2 - y^2) + ax - by + c].\exp[i\,(2\alpha xy\,) + bx + ay + d)]]|$$
$$= \exp[\alpha(x^2 - y^2) + ax - by + c].$$

(as $|e^{ip} = 1|$ for any real p)

6.1 which is of the same from as (1),

Hence we can write,

$$f'(z) = exp(\alpha z^2 + \beta z + y).$$

6.6 *ANALYTICITY AND ZEROS DIFFERENT FUNCTIONS*: -

- **Analytic Function:** A function complex-differentiable at every point in an open set (domain).
- **Zero of an Analytic Function:** A point z_0 where $f(z_0) = 0$.
- **Multiplicity (Order) of a Zero:** If $f(z_0) = 0$, but $f'(z_0) \neq 0$, it's a **simple zero** (order 1). If $f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$, it's a zero of **order** $\boldsymbol{m}$, meaning $f(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$.
- **Isolated Zeros:** For a non-trivial analytic function, zeros cluster at a finite distance; they don't accumulate (have no limit points) within the domain, ensuring they are distinct points.

Examples Across Functions

- **Polynomials ($\boldsymbol{f(z) = z^2 - 1}$):** Zeros at $z = 1, -1$, each simple. Zeros are finite.
- **Trigonometric ($\boldsymbol{f(z) = \sin(z)}$):** Zeros at $z = n\pi$ (integers), each simple (like $z = \pi$ in), infinite but isolated (can't get closer than π to another zero).
- **Exponential ($\boldsymbol{f(z) = e^z - 1}$):** Zeros at $z = 2\pi i n$ (integers times $2\pi i$), infinite and isolated.
- **Analytic vs. Non-Analytic:** $f(x) = |x|$ (real analysis) has a zero at $x = 0$, but it's not differentiable (not analytic in C), so concepts like order and isolation behave differently.

6.7 *BRANCH POINTS*: -

The idea of branch points arose from attempts to understand multi-valued analytic functions such as $\sqrt{z}$, $log z$ and $z^{1/n}$ mathematicians noticed that these functions could not be made single-valued on the whole complex plane, leading to the identification of special points—later called branch points—where values cycle when circling the point. Key contributors include Argand, Wessel, and Cauchy, who laid foundational work in analytic continuation and geometric interpretations. Bernhard Riemann revolutionized the concept by introducing Riemann surfaces, showing that multi-valued functions become single-valued on appropriately constructed multi-layered surfaces. Branch points were defined as locations where sheets of these surfaces meet. Riemann distinguished algebraic and logarithmic branch points and deeply connected the idea to analytic continuation. Following Riemann, Weierstrass, Puiseux, Schwarz, and Poincaré made the notion more rigorous through work on analytic continuation, series expansions, monodromy, and differential equations. Branch points became standardized objects in algebraic geometry and complex analysis. In the modern framework, branch points are studied in complex manifolds, algebraic curves, monodromy theory, and singularity theory. They also play key roles in physics, including quantum mechanics, statistical mechanics, and quantum field theory, where functions often possess branch cuts linked to physical observables.

For a complex number $z = re^{i\theta}$, we define $\log z = \log r + i\theta$. There are thus infinitely many values, or "branches", of $\log z$, for θ may take an infinity of values. For example,

$$\log i = \frac{\pi i}{2} \text{ or } \frac{5\pi i}{2} \text{ or } -\frac{3\pi i}{2} \text{ or} \ldots,$$

depending on which choice of θ we make.

Consider the three curves shown in the diagram. On C_1, we could choose θ to be always in the range $(0, \frac{\pi}{2})$, and then $\log z$ would be continuous and single-valued going round C_1. On C_2, we could choose $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\log z$ would again be continuous and single-valued. But for C_3, which encircles the origin, there is no such choice; whatever we do, $\log z$ cannot be made continuous around C_3 (it must either "jump" somewhere or be multi-valued). A *branch point* of a function – here, the origin – is a point which it is impossible to encircle with a curve upon which the function is continuous and single-valued. The function is said to have a *branch point singularity* at that point.

EXAMPLES:

(i) $\log(z - a)$ has a branch point at $z = a$.

(ii) $\log(z^2 - 1) = \log(z + 1) + \log(z - 1)$ has two branch points, at ± 1.

(iii) $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$ has a branch point at the origin.

6.8 *BRANCH CUT AND BRANCH OF MULTIVALUED FUNCTION*: -

The concept of a branch cut in complex analysis emerged during the 19th century as mathematicians struggled to understand multi-valued functions such as $\sqrt{z}$, $logz$ and $z^{1/n}$ Early work by Argand, Wessel, and Cauchy clarified analytic continuation and the behavior of functions on the complex plane, but these functions still exhibited discontinuities when encircling certain points. To handle this, mathematicians began deliberately "cutting" the plane so a multi-valued function could be treated as single-valued on the remaining domain. The decisive advance came from Bernhard Riemann in the 1850s. Rather than viewing branch cuts as artificial slits, Riemann introduced Riemann surfaces**,** where branch points join different sheets of the surface. In this geometric picture, a branch cut represents the projection onto the complex plane of where sheets are connected. Riemann's viewpoint clarified that branch cuts are not intrinsic to the function itself but are chosen by the analyst to define a single-valued branch. By the late 19th and early 20th centuries, the concept became fully formalized in the works of Weierstrass, Schwarz, and Poincaré. Today, branch cuts are standard tools in complex analysis, differential equations, and theoretical physics, especially in quantum field theory and analytic continuation.A branch cut is a curve or line removed from the complex plane to make a multi-valued complex function single-valued on the remaining domain. A branch cut is a curve (or connected set) in the complex plane chosen so that a multi-valued function can be restricted to a domain where it becomes single-valued and analytic**.** The cut prevents closed loops around branch points, which would otherwise cause the function to change value. A branch cut connects one or more branch points (or a branch point to infinity). The cut itself is not unique; different choices produce different *branches* of the same function. Across a branch cut, the function has a jump discontinuity, corresponding to switching between different sheets of its Riemann surface.

Complex Logarithm: $\log z$

Branch points: 0 and ∞

Typical branch cut: the negative real axis

$$\mathbb{C} \setminus (-\infty, 0] \setminus$$

Reason: Circling the origin changes the argument by 2π, creating infinitly many values.

Square Root: $\sqrt{z}$

Branch points: 0 and ∞

Typical branch cut: the negative real axis

$$\sqrt{z} = r^{1/2}e^{i\theta/2}, \quad \theta \in (-\pi, \pi]$$

Behavior: Crossing the cut flips the sign of the square root.

Inverse Trigonometric Functions

$\arcsin z$

- **Branch points:** -1 and 1
- **Typical branch cuts:**

$$(-\infty, -1] \quad \text{and} \quad [1, \infty)$$

$\arccos z$

- Same branch points and cuts as $\arcsin z$.

$\arctan z$

- **Branch points:** i and $-i$
- **Typical branch cuts:** vertical lines from i and $-i$ to infinity:

$$i[1, \infty), \quad -i[1, \infty)$$

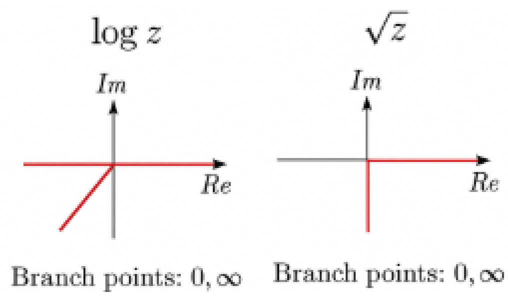

Fig 1.2.

SOLVED EXAMPLES

EXAMPLE1: Find the analytic function $f(z) = u + iv$ which the real part is

$$u = e^x(xcosy - ysiny)$$

Solve by Milne's Thomson's method and differential equation method?

SOLUTION:
Method I. Milne's Thomson's Method:

$$\frac{\partial u}{\partial x} = \emptyset_1(x, y) = \emptyset_1(z, 0) = e^z(z, 0) = e^z(z + 1).$$

$$\frac{\partial u}{\partial y} = \emptyset_2(x, y) = \emptyset_2(z, 0) = 0$$

Integration yields the result,

$$f(z) = \int [\emptyset_1(z, 0) - i\emptyset_2(z, 0)]dz + c,$$

$$= \int [e^z(z+1) - i.0]dz + c,$$

$$= \int (ze^z + e^z)dz + c,$$

$$= (z-1)e^z + e^z + c = ze^z + c$$

Method II. Suppose $f(z) = u + iv$ is analytic and u is known. To determine $f(z)$.

Firstly we shall determine v.

$$\frac{\partial u}{\partial y} = -e^x(-xsiny - ycosy - siny)$$

$$\frac{\partial u}{\partial x} = e^x(xcosy - ysiny + cosy)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy,$$

$$v = \int \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy =$$

$$v = \int e^x(xsiny + ycosy + siny)dx\ (treating\ y\ as\ constant) +$$

$$\int (those\ terms\ which\ do\ not\ contain\ x)dy + c$$

$$v = siny.\int xe^x dx + (ycosy + siny)\int e^x\, dx + \int 0dy + c$$

$$= [(x-1)siny + ycosy + siny]e^x + c$$

$$v = [xsiny + ycosy]e^x + c$$

$$f(z) = u + iv = e^x(xcosy - ysiny) + i[xsiny + ycosy]e^x + c$$

$$= xe^x(cosy + ysiny) + iy[cosy + isiny]e^x + ic$$

$$f(z) = (x+iy)e^x e^{iy} + c' = ze^z + c'$$

EXAMPL2: If $f(z) = u + iv$ is analytic function and $u - v = e^x(cosy - siny)$,

find $f(z)$ in terms of z.

SOLUTION: Given $f(z) = u + iv$ …..(1)
$u - v = e^x(cosy - siny)$……(2)
By (1), $if(z) = iu - v$……………(3)
Adding (1) and (3),

$$(1 + i)f(z) = (u - v) + i(u + v).$$

Taking $u - v = U, u + v = V, (1 + i)f = F(z)$,
We obtain
$F(z) = U + iV$
$f(z) = u + iv$ is analytic it implies $F(z) = U + iV$ is analytic.
By (2), $U = e^x(cosy - siny)$

$$\frac{\partial u}{\partial x} = \emptyset_1(x, y) = e^x(cosy - siny), \emptyset_1(z, 0) = e^z(z, 0) = e^z,$$

$$\frac{\partial u}{\partial y} = \emptyset_2(x, y) = \emptyset_2(z, 0) = -e^z.$$

By Milne's method,

$$F(z) = \int [\emptyset_1(z, 0) - i\emptyset_2(z, 0)]dz + c,$$

$$= \int [e^z + ie^z]dz + c,$$

$$= \int [1 + i]e^z dz + c,$$

$$(1 + i)f = (1 + i)e^z$$

Or

$$f(z) = c_1 + e^z.$$

6.9 *SUMMARY: -*

This unit deals with analytic functions in complex analysis. An analytic function is one which is differentiable at every point in a region of the complex plane. Such functions have real and imaginary parts that are closely related through certain conditions known as the Cauchy–Riemann equations. The real and imaginary parts of an analytic function are called conjugate functions. Each of these parts satisfies Laplace's equation and hence they are known as harmonic functions. The curves represented by constant values of these functions intersect each other at right angles and therefore form an orthogonal system. The unit also explains how to construct an analytic function when either its real part or imaginary part is given. This is done by checking whether the given part is harmonic and then determining its conjugate using the Cauchy–Riemann equations. Important theorems related to analytic functions are discussed, such as conditions for analyticity, properties of harmonic functions, and the behavior of zeros of analytic functions. The unit emphasizes that zeros of analytic functions are isolated unless the function is identically zero everywhere in the region. Finally, the unit introduces multi-valued functions like logarithmic and root functions. Concepts such as branch point, branch cut, and branch are explained to make these functions single-valued. A branch point is a point where the function changes value when encircled, while a branch cut is a curve removed from the plane to avoid this ambiguity. A branch refers to a specific single-valued version of a multi-valued function.

6.10 *GLOSSARY: -*

- **Complex Numbers:** In mathematics, a complex number is an element of a number system that extends the real numbers with a specific element denoted *i*, called the imaginary unit and satisfying the equation $i^2 = -1$; every complex number can be expressed in the form $a \pm ib$ where *a* and *b* are real numbers. Because no real number satisfies the above equation, *i* was called an imaginary number by René Descartes. For the complex number $a \pm ib$, *a* is called the real part, and *b* is called the imaginary part. The set of complex numbers is denoted by either of the symbols ℂ or C.
- **Limit:** A function $f(z)$ tends to the limit l as z tends to z_0 along any path, if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$ and we write $\lim_{z \to z_0} f(z) = l$, where l is finite.
- **Continuity:** For a function $f(z)$ is continuous at a point z_0. It must be defined at z_0. It's limit must exist at z_0. $f(z_0) = \lim_{z \to z_0} f(z)$.
- **Differentiability:** In symbols:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must exist and the limit must be independent of the direction from which $\Delta z \to 0$ in the complex plane.

- **Analytic function:** A function f(z) of a complex variable z = x + iy is said to be *analytic* at a point z_0 if:
  a. It is differentiable at z_0.
  b. It remains differentiable in some neighborhood of z_0 (that is, within some open region around that point).

In symbols:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must exist and the limit must be independent of the direction from which $\Delta z \to 0$ in the complex plane.

CHECK YOUR PROGRESS

Fill in the Blanks

CYP1: If $f(z) = u + iv$ is analytic in a domain D, and u and v satisfy ……………………………………….then u and v are called conjugate harmonic functions or conjugate functions simply.

CYP2: Two families of curves $u(x, y) = c_1, v(x, y) = c_2$, are said to form ……………. ……………if they intersect at right angles at each of their points of intersection.

CYP3: Suppose $f(z) = u + iv$ is ……………..on domain D in $\mathbb{C}$, then $h = Re\, f(z)$ is harmonic on D

CYP4:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \cdots$$

CYP5: $\log(z - 2)$ has a branch point at ………….

CYP6: An analytic function with constant modulus is:

(a) Variable **(b)** May be variable or constant **(c)** Constant **(d)** None of these.

6.11 *REFERENCES: -*

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6.12 *SUGGESTED READING: -*

- Ahlfors, L. V. (1979). *Complex analysis* (3rd ed.). New York, NY: McGraw-Hill Education.
- Conway, J. B. (1978). *Functions of one complex variable I* (2nd ed.). New York, NY: Springer
- Copson, E. T. (1978). *Theory of functions of a complex variable* (2nd ed.). Oxford, England: Oxford University Press.
- https://nptel.ac.in/courses/111106084

6.13 *TERMINAL QUESTIONS: -*

TQ1: For what values of z the function w defined by the equation ceases to be analytic?

$$z = log\rho + i\emptyset, w = \rho(cos\emptyset + isin\emptyset).$$

TQ2: For what values of z the function w defined by the equation

$$z = sinhu.\,cosv + icoshu.\,sinv, w = u + iv.$$

ceases to be analytic?

TQ3: If $u = (x-1)^3 - 3xy^2 + 3y^2$ determine v so that $u + iv$ is a regular function of $x + iy$.

TQ4: If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.

6.14 *ANSWERS: -*

CHECK YOUR PROGRESS:

CYP1: the Laplace's equations $\nabla^2 V = 0$,

CYP2: an Orthogonal system

CYP3: analytic.

CYP4: $\frac{4\partial^2}{\partial z \partial \bar{z}}$

CYP5: $z = 2$
CYP6: c

TERMINAL QUESTIONS:

TQ1. w is analytic function in any finite domain.
TQ2.$z = \pm i$.
TQ3. $v = 3y(x^2 - 2x + 1) - y^3 + c$
TQ4.$f(z) = c_1 - iz^3$.

UNIT 7: Complex Integration-I

CONTENTS:

7.1 *INTRODUCTION:-*

In mathematics, a line integral is an integral where the function to be integrated is evaluated along a curve. The terms path integral, curve integral, and curvilinear integral are also used; contour integral is used as well, although that is typically reserved for line integrals in the complex plane. A complex integral is the process of computing integrals of a complex-valued function over a path in the complex plane, similar to how real integrals are computed over intervals on the real line. These integrals are fundamentally line integrals and are often defined by parameterizing the path, breaking the complex integral into a pair of real line integrals, and using multivariate calculus. The result can depend on the specific path taken, but for some functions, the integral is independent of the path, which is determined by applying the complex version of the Fundamental Theorem of Calculus.

7.2 *OBJECTIVES:-*

After studying this unit, learner will be able to

(i) Domain and Contour

(ii) Complex line integral

(iii) Cauchy's Theorem

7.3 *DOMAIN AND CONTOUR:-*

- **DOMAIN (REGION)**

A set S of points in the Agarnd plane is said to be connected set if any two of its points can be joined by a continuous curve, all of whose points belong to S.

An open connected set is called an open domain. If the boundary points of S are also added to an open domain, then it is called closed domain.

- **CONTOUR**

By contour, we mean a continuous chain of a finite number of regular arcs. If the contour is closed and does not intersect itself, then it is called closed contour.

Examples. Boundaries of triangles and rectangles.

- **SIMPLY AND MULTIPLY CONNECTED DOMAINS**

A domain in which every closed curve can be shrunk to a point without passing out of the region is called a simply connected domain. If a domain is not simply connected. then it is called multiply connected domain.

- **WEIERSTRASS M-TEST**

If $|u_n(z)| \leq M_n$ where M_n is independent of z in a domain R and ΣM_n, the series of positive constants is convergent, then the series $\Sigma u_n(z)$ is uniformly convergent.

Remark: A uniformly convergent series of continuous functions can be integrated by term.

7.4 *COMPLEX LINE INTEGRAL:-*

Suppose $f(z)$ is continuous at every point of a curve C having a finite length, i.e. C is a rectifiable curve.
Divide C into n parts by means of points

$$z_0, z_{\mathrm{i}}, z_2, \dots, z_n,$$

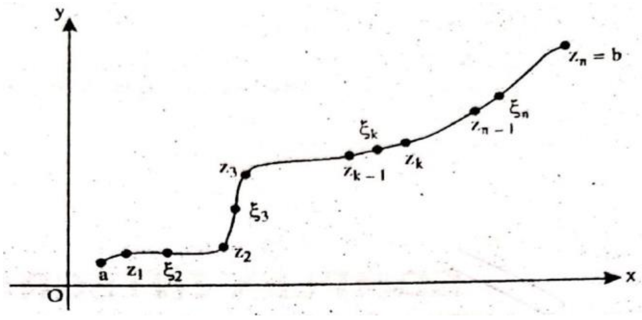

Fig: 7.4

Let

$$a = z_0, b = z_n.$$

We choose a point ξ_k on each arc joining z_{k-1} to z_k. Form the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1}).$$

Suppose maximum value of $(z_r - z_{r-1}) \to 0$ as $n \to \infty$. Then the sum S_n tends to a fixed limit which does not depend upon the mode of subdivision and denote this limit by

$$\int_a^b f(z)dz \text{ or } \int_C f(z)dz$$

which is called the complex line integral or line integral of $f(z)$ along C. An evaluation of integral by such method is also called ab-initio method.

Remark 1:

To define complex line integrals, we will need the following ingredients:

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \le t \le b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

Remark 2: Line integrals are also called path or contour integrals. Given the ingredients we define the complex line integral

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt. \qquad (1a)$$

you should note that this notation looks just like integrals of a real variable. We don't need the vectors and dot products of line integrals in $\mathbb{R}^2$. Also, make sure you understand that the product $f(\gamma(t))\gamma(t)$ is just a product of complex numbers. An alternative notation uses $dz = dx + idy$ to write

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) \qquad (1b)$$

Let's check that Equations 1a and 1b are the same. Equation 1b is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_\gamma f(z)\,dz = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t)]\ \left(x'(t) + iy'(t)\right) dt$$

But,

$$u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$$

and

$$x'(t) + iy'(t) = \gamma'(t)$$

so the right hand side of this equation is

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

That is, it is exactly the same as the expression in Equation 1a.

Example 1.

Compute $\int_\gamma \bar{z}\,dz$ along the straight line from 0 to $1+i$.

Solution: We parametrize the curve as $\gamma(t) = t(1+i)$ with $0 \le t \le 1$. So $\gamma'(t) = 1+i$. The line integral is

$$\int z^2 dz = \int_0^1 t^2(1+i)^2(1+i)\,dt = \frac{2i(1+i)}{3}.$$

Example 2.

Compute $\int_\gamma \bar{z}\,dz$ along the straight line from 0 to $1+i$.

Solution: We can use the same parametrization as in the previous example. So,

$$\int_\gamma \bar{z}\,dz = \int_0^1 t(1-i)(1+i)\,dt = 1.$$

Example 3.

Compute $\int_\gamma z^2\,dz$ along the unit circle.

Solution: We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$, where $0 \le \theta \le 2\pi$. We have $\gamma'(\theta) = ie^{i\theta}$. So, the integral becomes

$$\int_\gamma z^2\,dz = \int_0^{2\pi} e^{2i\theta} ie^{i\theta}\,d\theta = \int_0^{2\pi} ie^{3i\theta}\,d\theta = \left.\frac{e^{i3\theta}}{3}\right|_0^{2\pi} = 0.$$

Notation. By the symbol $\int_C f(z)dz$ we mean the integral of $f(z)$ along a boundary C in the positive sense. In case of closed paths, the positive direction is anti-clockwise. The integral along C is often called a contour integral.

7.5 *CAUCHY'S THEOREM:-*

Cauchy's Theorem in complex analysis, named for Augustin-Louis Cauchy (1789-1857), states that the integral of a holomorphic (analytic) function around a simple closed loop in a simply connected region is zero, a discovery he made in the early 1800s, though his initial proofs involved continuity assumptions {1, 2, 4, 5}. Later, Édouard Goursat (1858-1936) proved the theorem without requiring differentiability of the derivative (f' being continuous), leading to the modern Cauchy - Goursat theorem, forming the bedrock for complex analysis, Cauchy's Integral Formula, and the powerful Residue Theorem.

Statement: If a function $f\ (z\ )$ is analytic and single valued inside and on a simple closed contour C; then $\int_C f(z)dz = 0$

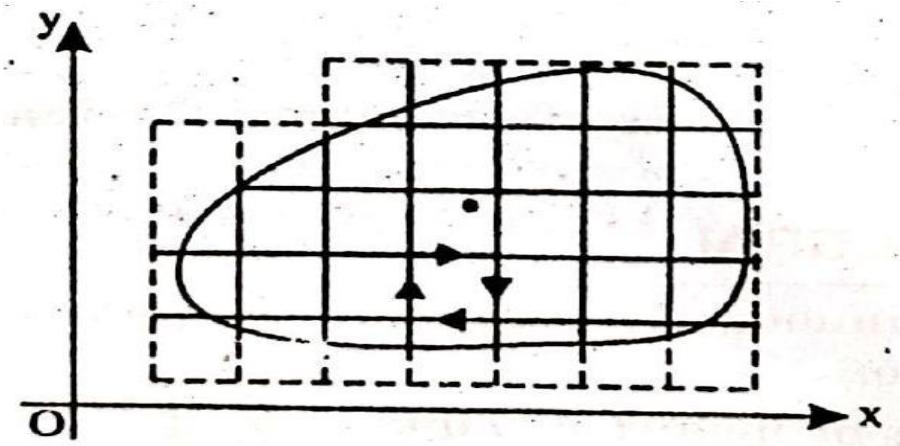

Fig. 7.5

Proof. Firstly, we shall prove a lemma:
Lemma. Given $\varepsilon > 0$ it is possible to divide the region inside C into finite number of meshes either complete square C_n or partial square D_n such that within each mesh there exists a point z_0 for which

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon\ \forall\ z \text{ in the mesh.} \qquad \dots\dots \quad (1)$$

Proof of the lemma.

Gourset's Lemina. Suppose the lemma is not true. It means that the lemma fails at least in one mesh. Subdivide this mesh by means of lines joining the middle points of the opposite sides. If there is still at least one part which does not satisfy the condition (1). Again, subdivide that part in the same way. This process comes to an end after a finite number of steps, when the condition (1) is satisfied for every subdivision, or the process may go on indefinitely. In the second case, we obtain a sequence of squares (each contained in the preceding ones) which has z_0 as its limit point at which the condition (1) is not satisfied. Of course, z_0 is an interior point of C. Since the condition (1) is not satisfied at z_0 and so

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right| \nless \varepsilon \text{ where } |z-z_0|<\delta,$$

δ being a small number of depending upon ε. This declares that $f(z)$ is not differentiable at z_0 so that $f(z)$ is not analytic at z_0, contrary to the initial assumption that $f(z)$ is analytic at every interior point of C. Hence the lemma is true. From the lemma,

$$\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)=\eta(z) \text{ where } |\eta|<\varepsilon \qquad \ldots\ldots(2)$$

and $\eta \to 0$ as $z \to z_0$.
Thus $f(z)=(z-z_0)\eta(z)+f(z_0)+(z-z_0)f'(z_0)$

Proof of the Cauchy's theorem. Divide the interior of C into complete squares $C_1, C_2, \ldots, C_n$ and partial squares $D_1, D_2, \ldots, D_m$, part of whose boundaries are parts of C.
Consider the integral

$$\sum_{r=1}^{n}\int_{C_r} f(z)dz+\sum_{r=1}^{m}\int_{D_r} f(z)dz$$

where the path of every integral being in anti-clockwise direction. In the complete sum, integration along each straight side of each square (complete or partial) happens to be taken twice in opposite directions and so all the integrals along straight sides of squares cancel. The integrals, which remain, are taken along curved boundaries of partial squares because these are described only once. The integrals which are left behind sum equal to

$$\int_C f(z)dz$$

$$\therefore \quad \int_C f(z)dz=\sum_{r=1}^{n}\int_{C_r} f(z)dz+\sum_{r=1}^{m}\int_{D_r} f(z)dz \qquad \ldots\ldots\ldots \quad (3)$$

In view of (2),

$$\int_{C_r} f(z)dz=\int_{C_r}[f(z_0)+(z-z_0)\eta+(z-z_0)f'(z_0)]dz$$

$$=[f(z_0)-z_0f'(z_0)]\int_{C_r} dz+f'(z_0)\int_{C_r} zdz+\int_{C_r}(z-z_0)\eta(z)dz$$

Using the fact

$$\int_{C_r} dz=0=\int_{C_r} zdz$$

we obtain $\int_{C_r} f(z)dz=\int_{C_r}(z-z_0)\eta dz$

In view of this, (3) becomes

$$\int_C f(z)dz=\sum_{r=1}^{n}\int_{C_r}(z-z_0)\eta dz+\sum_{r=1}^{m}\int_{D_r}(z-z_0)\eta dz$$

$$\left|\int_C f(z)dz\right| \le \sum_{r=1}^{n} \left|\int_{C_r} (z-z_j)\eta dz\right| + \sum_{r=1}^{n} \left|\int_{D_r} (z-z_0)\eta dz\right|$$

$$\le \sum_{r=1}^{n} \int_{C_r} |(z-z_0)||\eta||dz| + \sum_{r=1}^{m} \int_{D_r} |z-z_0||\eta||dz|$$

$< \sum_{r=1}^{n} \varepsilon \int_{C_r} |z-z_0||dz| + \sum_{r=1}^{m} \varepsilon \int_{D_r} |z-z_0||dz|$ (4)

as $|\eta| < \varepsilon.$

Let l_n, A_n be respectively the length of the side and area of square C_n. Similarly $l_n{}', A_n{}'$ denote respectively length and area of square D_n. Then (4) takes the form

$$\left|\int_C f(z)dz\right| < \sum_{r=1}^{n} \varepsilon l_r\sqrt{2}\int_{C_r} |dz| + \sum_{r=1}^{n} \varepsilon l_r'\sqrt{2}\int_{D_r} |dz|$$

[Since $|z-z_0| \le l_r\sqrt{2}$ = diagonal of square C_r]

$$= \sum_{r=1}^{n} \varepsilon l_r\sqrt{2}.4l_r + \sum_{r=1}^{m} \varepsilon l_r'\sqrt{2}.(4l_r' + s_r)$$

For $\int_{C_r} |dz|$ = perimeter of square C_r]

$$= 4\varepsilon\sqrt{2}\left[\sum_{r=1}^{n} A_r^2 + \sum_{r=1}^{m} A_r'\right] + \varepsilon\sqrt{2}\sum_{r=1}^{m} l_r's_r$$

$$= 4\varepsilon\sqrt{2}\cdot A + \varepsilon\sqrt{2}\sum_{r=1}^{m} l_r's_r$$

where A = total area of squares of l with which the region was originally covered. Also let L be total length of boundary of C. Then

$$\left|\int_C f(z)dz\right| = 4\varepsilon\sqrt{2}\mathrm{A} + \varepsilon\sqrt{2}\sum_{r=1}^{m} ls_r$$

$$= 4\varepsilon\sqrt{2\mathrm{A}} + \varepsilon\, l\, L\,\sqrt{2}$$

or

$$\left|\int_C f(z)dz\right| < \varepsilon|4\sqrt{2}\mathrm{A} + l\,\mathrm{L}\,\sqrt{2}|$$

Since ε is arbitrary and so making $\varepsilon \to 0$, we get

$$\int_C f(z)dz = 0$$

Alternate Proof of Cauchy's theorem. Here we use Green's theorem to prove the present theorem.

Green's Theorem. If $P(x,y), Q(x,y), \frac{\partial Q}{\partial \dot{x}}, \frac{\partial P}{\partial y}$ all are continuous functions of x and y in a closed contour C, then $\int_C (Pdx + Qdy) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \ldots *$

Cauchy's Theorem. If $f(z)$ is analytic function of z and if $f'(z)$ is continuous at each point within and on a closed contour C, then $\int_C f(z)dz = 0$.

Proof. $f(z) = u + iv$ is analytic and so it is continuous on contour C and also $f'(z)$ exists. It means that u, v, u_x, u_y, v_x, v_y all are continuous in C.

$$f(z) = u + iv = \text{ analytic } \Rightarrow u_x = v_y, u_y = -v_x$$

$$u_x - v_y = 0 \quad \cdots(1), \qquad u_y + v_x = 0 \quad \ldots\ldots(2)$$

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

using (*), we get

$$= \iint_C \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right)\cdots dy + i\iint_C \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \text{ using (1) } d(2)$$

$$= -\iint_C 0dxdy + i\iint_C 0dxdy = 0$$

Note. Goursat showed that for the truth of the theorem the assumption of the continuity of $f'(z)$ is not necessary and so Cauchy's theorem holds iff $f(z)$ is analytic writhin and on C.

Cauchy- Goursat Theorem

In 1900 the French mathematician Edouard Goursat proved that the assumption of continuity of $f'$is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the Cauchy - Goursat Theorem. as we can expect, with fewer hypotheses, the proof of this version of Cauchy's theorem is more complicated than the one just presented.

Statement: If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Example: The function f(z) = e^z is entire and consequently is analytic at all points within and on any simple closed contour C. It follows from the Cauchy-Goursat Theorem that $\int_C e^z dz = 0$

Question 1. If $f(z) = \frac{z^2+5z+6}{z-2}$, does Cauchy's theorem apply

(i) When path of integration is a circle C of radius 3 and centre at origin.

(ii) When path of integration is a circle C of radius 1 and centre at the origin.

Solution. (i). When path is circle C given by $|z - 0| = 3$, then $z = 2$ lies inside c and so $f(z) = \frac{z^2+5z+6}{z-2}$ is not analytic inside c. Hence Cauchy's theorem is not applicable and so $\int_C f(z)dz \neq 0$.

(ii) When C is circle $|z - 0| = 1$, then $z = 2$ lies outside $C. \therefore f(z) = \frac{z^2+5z+6}{z-2}$ is analytic inside C and hence $\int_C f(z)dz = 0$

Question 2. Verify Cauchy's theorem for the function $f(z) = z^3 - iz^2 - 5z + 2i$ if path is circle given by $|z-1| = 2$.

Solution. Evidently $f(z)$ is analytic within and on C. Hence, by Cauchy's theorem,

$$\int_C f(z)dz = 0 \qquad \ldots\ldots..(1)$$

For $C: |z-1| = 2, z - 1 = 2e^{i0}, dz = 2i^i e^{i0} d\theta$

$$\int_C f(z)dz = \int_C^{2\pi} (z^3 - iz^2 - 5z + 2i)dz$$

$$= \int_0^{2\pi} \left[(1+2e^{i\theta})^3 - i(1+2e^{i\theta})^2 - 5(1+2e^{i\theta}) + 2i\right] 2e^{i0} id\theta = 0$$

or $\int_C f(z)dz = 0$... (2).

For $\int_C^{2\pi} e^{ik\theta} d0 = 0$ if k is any non-zero integer.

(1) and (2) ⇒ Cauchy's theorem is verified.

Question 3. Verify Cauchy's Theorem and integrating e^{iz} along the boundary of the triangle with vertices at the points $1+i, -1+i$ and $-1-i$.

Solution. $f(z) = e^{iz}$ is analytic within and upon closed contour $\triangle ABC$ and so by Cauchy's

Theorem $\int_C f(z)dz = 0$

Let $I = \int_C f(z)dz = I_1(\overrightarrow{AB}) + I_2(\overrightarrow{BC}) + I_3(\overrightarrow{CA}) \qquad \ldots\ldots.(1)$

Then $I = \int_C e^{iz}dz = \left(\frac{e^{iz}}{i}\right)$,

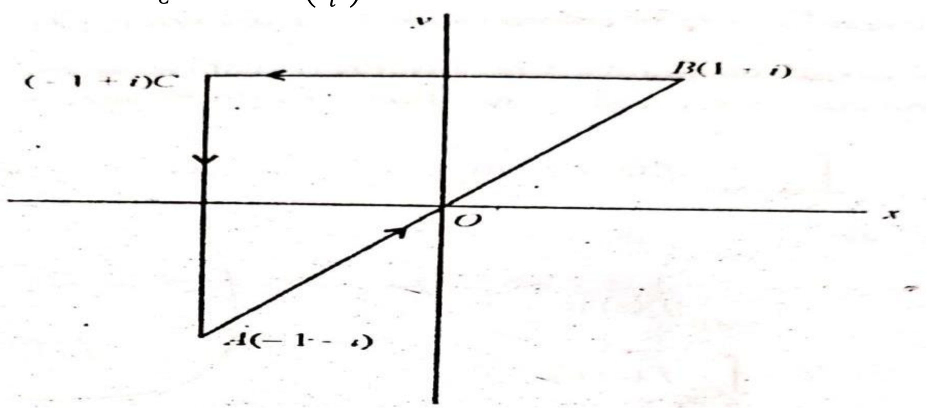

Fig. 7(a)

For I_1, equation of $\overrightarrow{AB}$ is $y = x, z = x + iy = (1+i)x$

$$I_1(\overrightarrow{AB}) = \left[\frac{e^{iz}}{i}\right]_C = \left[\frac{e^{i(1+i)x}}{i}\right]_{x=-1}^{x=1}$$

$$= \frac{1}{i}\left[e^{(i-1)} - e^{-(i-1)}\right]$$

For I_2, equation of BC is $y = 1, z = x + iy = x + i$

$$I_2 = \left[\frac{e^{iz}}{i}\right]_C = \left[\frac{e^{i(x+i)}}{i}\right] = \left[\frac{e^{ix'-1}}{i}\right]_{x=1}^{x=-1}$$

$$= \frac{1}{i}\left(e^{-i-1} - e^{i-1}\right)$$

For I_3: equation of $\overrightarrow{CA}, x = -1, z = x + iy = -1 + iy$

$$I_3 = \left[\frac{e^{iz}}{i}\right]_C = \left[\frac{e^{i(-1+iy)}}{i}\right]_C = \left[\frac{e^{-i-y}}{i}\right]_{y=1}^{y=-1}$$

or, $I_3 = \frac{1}{i}\left(e^{-i+1} - e^{-i-1}\right)$

$$I = I_1 + I_2 + I_3 = \frac{1}{i}\left[\left(e^{i-1} - e^{1-i}\right) + \left(e^{-i-1} - e^{i-1}\right) + \left(e^{-i+1} - e^{-i-1}\right)\right]$$

$$= 0$$

or,

$$\int_C f(z)dz = 0 \qquad \dots\dots\dots (2)$$

(1) & (2) ⇒ Cauchy's theorem is verified.

7.6 *EXTENSION OF CAUCHY'S THEOREM:-*

Suppose $f(z)$ is analytic in a simply connected domain D. Then the integral along any rectifiable curve in D joining any given points of D is the same, i.e., it does not depend upon the curve joining the two points.

Proof. Suppose the two points $A(z_1)$ and $B(z_2)$ of the simply connected domain D are joined by the curves C_1 and C_2 as shown in the diagram. Then, by Cauchy's theorem

or

$$\int_{ALBMA} f(z)dz = 0$$

or $\int_{ALB} f(z)dz + \int_{BMA} f(z)dz = 0$

or

$$\int_{ALlB} f(z)dz \cdots \int_{AMB} f(z)dz = 0$$

or

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

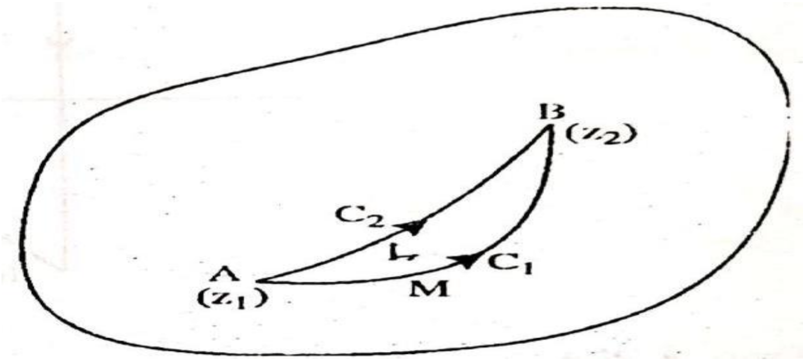

Fig. 7(c)

or

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

This proves the required result.

Question 4. Show that $\int_C e^{-2z}dz$ is independent of the path C joining the points $1-\pi i$ to $3+\pi i$ and determine its value.

Solution. Let $I = \int_C f(z)dz$, where $f(z) = e^{-2z}$ and C is a straight-line joining point $1-\pi i$ to $3+\pi i$. Evidently $f(z)$ is differentiable every where in z-plane. Hence $f(z)$ is analytic in entire z-plane ∴ By corollary 1 of Cauchy's theorem $\int_C f(z)dz$ is independent of path of integration.

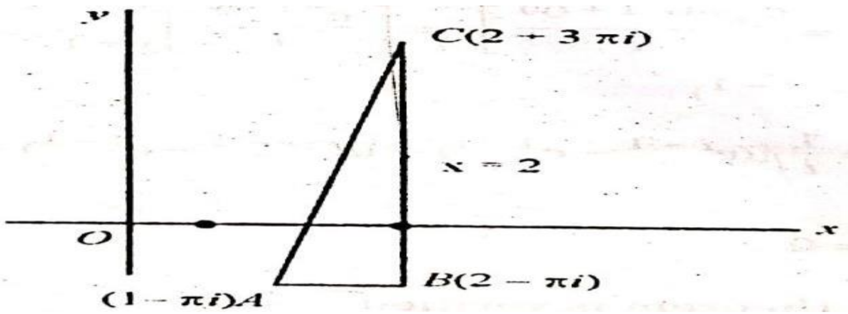

Fig. 7(d)

$$\text{Also} \int_C f(z)dz = \int_{1-\pi i}^{2+3\pi i} e^{-2z}dz = -\frac{1}{2}\left(e^{-2z}\right)_{z=1}^{z=2+3\pi i}$$

$$= -\frac{1}{2}\left[e^{-2(2+3\pi i)} - e^{-2(1-\pi i)}\right] = -\frac{1}{2}\left(e^{-4} - e^{-2}\right)$$

$$\text{as } e^{-6\pi i} = 1 = e^{2\pi i}$$

$$\therefore \int_C f(z)dz = \frac{1}{2}\left(e^{-2} - e^{-4}\right) \quad \ldots\ldots(1)$$

Let$I = \int_C f(z)dz = I_1(\overrightarrow{AB}) + I_2(\overrightarrow{BC})$

For I_1: equation of AB is $y = -\pi, z = x + iy = x - i\pi, dx = dz$

$$I_1 = \int_1^2 e^{-2(x-i\pi)}dx = e^{2i\pi}\int_1^2 e^{-2x}dx = 1.\int_1^2 e^{-2x}dx$$

$$= \left(\frac{e^{-2x}}{-2}\right)_{x=1}^{x=2} = -\frac{1}{2}(e^{-4} - e^{-2})$$

For I_2: equation of BC is $x = 2, z = 2 + iy, dz = idy$

$$I_2 = \int_{-\pi}^{3\pi} e^{-2(2+iy)}idy = \frac{ie^{-4}}{-2}\left(e^{-i2y}\right)_{y=-\pi}^{y=3\pi}$$

$$= -\frac{i}{2e^4}\left(e^{-6i\pi} - e^{-i2\pi}\right) = 0 \text{ as } e^{-6i\pi} = 1 = e^{-2\pi}$$

$$I = \int_C f(z)dz = I_1(\overrightarrow{AB}) + I_2(\overrightarrow{BC})$$

$$= \frac{1}{2}(e^{-2} - e^{-4}) + 0(2)$$

This integral is the same along two different paths:
(i) line $\overrightarrow{AC}$
(ii) line $\overrightarrow{AB}$ + line $\overrightarrow{BC}$
This ⇒ Integral is independent of path.

Corollary 1. Let a closed contour C contain another closed contour C_1. Let $f(z)$ be analytic at every point lying in the ring-shaped domain bounded by C and C_1.
Then $\int_C f(z)dz = \int_{C_1} f(z)dz$

Proof. We make a cross at joining a point Λ of the contour C to a point E of C_1. By Cauchy's theorem,

$$\int_{\text{ABCDAEFGEA}} f(z)dz = 0$$

or

$$\int_{ABCDA} f(z)dz + \int_{AE} f(z)dz + \int_{EFGE} f(z)dz + \int_{EA} f(z)dz = 0$$

But $\int_{AE} f(z)dz = -\int_{EA} f(z)dz$

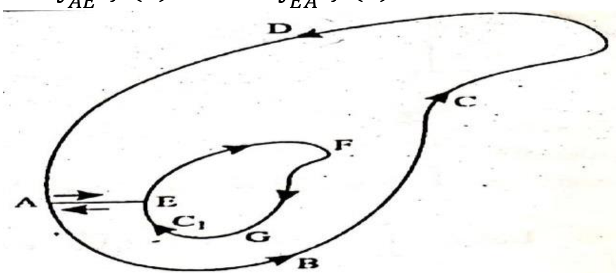

Fig. 7(e)

and so $\int_{AE} f(z)dz + \int_{EA} f(z)dz = 0$

Hence $\int_{ABCDA} f(z)dz + \int_{EFGE} f(z)dz = 0$

or

$$\int_C f(z)dz - \int_{EGFE} f(z)dz = 0$$

or

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

CDeduction. If the contour C contains non-intersecting contours $C_1, C_2 \ldots C_n$, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

Theorem. An upper bound for a complex integral. If a function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, then $\left|\int_C f(z)dz\right| \leq Ml$.

Proof. Divide the contour C into n parts by means of points $z_0, z_1, z_2, \ldots z_n$. We choose a point ξ_r on each arc joining z_{r-1} to z_r. Form the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

Also suppose maximum value of $(z_r - z_{r-1}) \to 0$ as $n \to \infty$. We define $\int_C f(z)dz = \lim_{n\to\infty} S_n$

$$|S_n| = \left|\sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})\right| \leq \sum_{r=1}^{n} |f(\xi_r)| \cdot |z_r - z_{r-1}| \leq \sum_{r=1}^{n} M|z_r - z_{r-1}| \quad \ldots\ldots (1)$$

Making $n \to \infty$ and noting (1), we get

$$\left|\int_C f(z)dz\right| \leq \lim_{n\to\infty} M \sum_{r=1}^{n} |z_r - z_{r-1}| \qquad \ldots\ldots\ldots (2)$$

But $\lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}|$

$$= \lim_{n\to\infty} [|z_1 - z_0| + |z_2 - z_1| + \cdots + |z_n - z_{n-1}|]$$
$$= \lim_{n\to\infty} [\text{chord} z_1 z_0 + \text{chord} z_2 z_1 + \cdots + \text{chord} z_n z_{n-1}]$$
$$= \lim_{n\to\infty} [\text{arc} z_1 z_0 + \text{arc} z_2 z_1 + \cdots + \text{arc} z_n z_{n-1}]$$

= arc length of contour $C = l$.

Using this in (2), $\left|\int_C f(z)dz\right| \leq Ml$

Question. Find $\int_c \frac{z^2}{z-5}$ dz where c is a circle $|z| = 2$.

Solution. Since z = 5 is only singularity and lies outside the circle, the function is analytic inside and onto the contour, so the value of integral is zero.

Question. Which of the following is the statement of Cauchy's Integral Theorem?

a) If a function f (z) is analytic and its derivative f' (z) is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$
b) If a function f (z) is non-analytic and its derivative f' (z) is continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$
c) If a function f (z) is analytic and its derivative f' (z) is discontinuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$
d) If a function f (z) is non-analytic and its derivative f' (z) is discontinuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$
Solution. Answer: **a**
Explanation: Cauchy's Integral Theorem states that 'If a function f (z) is analytic and its derivative f' (z) is continuous at all points inside and on a simple closed curve C, then
$\int_C f(z)\,dz = 0$.

Question. Which of the following theorems can be applied in the function of Cauchy's Integral Theorem?
a) Green's Theorem
b) Stokes Theorem
c) Gauss Divergence Theorem
d) Taylors Theorem
Solution. Answer (**a**) is correct.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Integration of Entire function is always zero on any contour.
Problem 2. The value of $\int_{|z|=1} \frac{z}{z-2}\,dz$ is equal to.
Problem 3. The value of $\int_c \sin z\,dz$, where c is a circle z = 2 .
Problem 4. If $f(z)$ is analytic within and on whole C, $\int_c f(z)dz$ is equal to.
Problem 5. If f has no singularities inside the curve, then the complex line integral around that closed curve is zero.

7.7 *SUMMARY:-*

1. WEIERSTRASS M-TEST

If $|u_n(z)| \leq M_n$ where M_n is independent of z in a domain R and ΣM_n, the series of positive constants is convergent, then the series $\Sigma u_n(z)$ is uniformly convergent. A uniformly convergent series of continuous functions can be integrated by term.

2. Green's Theorem. If $P(x, y), Q(x, y), \frac{\partial Q}{\partial \dot{x}}, \frac{\partial P}{\partial y}$ all are continuous functions of x and y in a closed contour C, then $\int_C (Pdx + Qdy) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \ldots *$

3. Cauchy's Theorem. If $f(z)$ is analytic function of z and if $f'(z)$ is continuous at each point within and on a closed contour C, then $\int_C f(z)dz = 0$.

4. If f has no singularities inside the curve, then the complex line integral around that closed curve is zero.

5. The domain must be simply connected or the curve must at least be deformable to a point without crossing outside the domain.

6. In complex analysis, Cauchy's theorem states that if a function is holomorphic on a simply connected domain, then the integral of the function over any closed curve in that domain is zero. There are several *important extensions* (or generalizations) of Cauchy's theorem that relax its assumptions or broaden its applicability.

7. Simply connected domain has no "holes" in it, A domain that is not simply connected is called a *multiply connected* domain; that is, a multiply connected domain has "holes" in it.

8. Application of Cauchy's Integral Theorem

- Solving complex integrals quickly
- Verifying analyticity of a function
- Evaluating real definite integrals using contour integration
- Deriving Taylor and Laurent series
- Computing residues in advanced complex analysis

7.8 GLOSSARY:-

1. **Analytic function:** An analytic function (also called a holomorphic function) is a complex function that can be locally represented by a convergent power series.
2. **continuous function**: in complex analysis, the idea of a continuous function is almost the same as in real analysis, but with the function defined on complex numbers instead of real numbers.

3. Double integration: Double integration (also called a double integral) is a way to integrate a function of two variables over a region in the plane. It extends the idea of single-variable integration (area under a curve) to two dimensions (volume under a surface).

7.9 *REFERENCES:-*

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7.10 *SUGGESTED READING:-*

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7.11 *TERMINAL AND MODEL QUESTIONS:-*

Q 1. States and prove Cauchy theorem.

Q 2. If a function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, then prove that $\left|\int_C f(z)dz\right| \leq Ml$.

Q 3. States and prove Extension of Cauchy theorem.

Q 4. Define Complex line integral.

Q 5. Compute $\int_\gamma z^2 dz$ along the unit circle.

Q 6. Prove that Integrals $\int_\gamma e^z dz = \int_\gamma Sinz\, dz = \int_\gamma Cosz\, dz = 0$, where γ is any circle.

7.12 *ANSWERS:-*

TQ5. 0

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. True

CQY 5. True

UNIT 8: Complex Integration-II

CONTENTS:

8.1 *INTRODUCTION:-*

The Cauchy integral formula is rooted in the 19th-century work of French mathematician Augustin-Louis Cauchy, who developed it as a central result of complex analysis. Building on his Cauchy's Integral Theorem, the formula, originally published in a less general form, establishes that a holomorphic function on a disk is fully determined by its values on the disk's boundary. This breakthrough provided powerful new tools for evaluating complex integrals and understanding function behavior.

8.2 *OBJECTIVES:-*

After studying this unit, learner will be able to

(i) Cauchy's integral formula

(ii) Poisson's Integral formula

(iii) Morera's theorem

(iv) Leoville's theorem

8.3 *CAUCHY'S INTEGRAL FORMULA:-*

Cauchy's integral formula. If $f(z)$ is analytic within and on a closed contour C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}$$

Proof. Suppose $f(z)$ is analytic within and on a closed contour C and a is an interior point of C. To prove that $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}$. Describe a circle γ about the centre $z = a$ of small radius r s.t. this circle $|z - \ddot{u}| = r$ does not intersect the curve C. The function $\frac{f(z)}{z-a}$ is analytic in the annulus bounded by C and γ. Hence by,

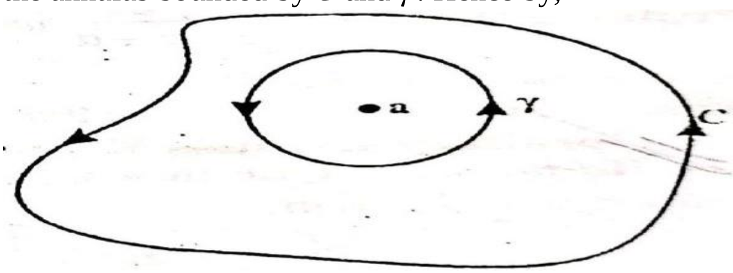

Fig. 8(a)

corollary to Cauchy's theorem,

$$\int_C \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a} \qquad \dots\dots(1)$$

or

$$\int_C \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(z)-f(a)}{z-a}dz + \int_\gamma \frac{f(a)dz}{z-a} \qquad \dots\dots (2)$$

Since $f(z)$ is analytic within C and so it is continuous at $z = a$ so that given $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(z) - f(a)| < \varepsilon$ ... **(3)** for $|z - a| < \delta$ ... **(4).** Since r is at our choice and so we can take $r < g/$ so that (4) is satisfied $\forall z$ on the circle γ. For any point z on $\gamma, z - a = re^{i0}$.

$$\int_\gamma \frac{f(a)}{z-a}dz = \int_0^{2\pi} \frac{f(a)re^{i(0}id(0)}{re^{i0}} = 2\pi i f(a).$$

Hence, by (2), $\left|\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a)\right| = \left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right|$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}\cdot|dz| < \frac{\varepsilon}{r}\int_\gamma |dz| = \frac{\varepsilon}{r}\cdot 2\pi r$$

or

$$\left|\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a)\right| < 2\pi i$$

Since ε is arbitrary and so making $\varepsilon \to 0$, we get

$$\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a) = 0 \text{ or } f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} \qquad \dots\dots (5)$$

Remarks. (1) $|\boldsymbol{a}-\boldsymbol{b}|<\varepsilon \Rightarrow \boldsymbol{a}-\boldsymbol{b}=0$
This result is of vital importance for further study.
(2) $\int_\gamma |dz| =$ circumference of the circle $\gamma = 2\pi$. radius
(3) Prom the equations (1) and (5),

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Corollary 1. Gauss Mean Value Theorem.
If $f(z)$ is analytic in a domain D and if the circular domain $|z-z_0| \le \rho$ is contained in D, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right)d\theta$$

That is to say, the value of $f(z)$ at z_0 is equal to the average of its value of the boundary of the circle $|z-z_0| = \rho$.
Proof. By (1) and (5),

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-z_0}$$

$$|z-z_0| = \rho \Rightarrow z - z_0 = \rho e^{i\theta} \Rightarrow dz = \rho i e^{i\theta}d0$$

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} f\left(z_0+\rho e^{i\theta}\right)\frac{\rho i e^{i0}}{\rho e^{i0}}d0$$

or

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i0}\right)d\theta$$

8.4 *EXTENSION OF CAUCHY'S INTEGRAL FORMULA:-*

Extension of Cauchy's integral formula to multiply connected regions. If $f(z)$ is analytic in a ring-shaped region bounded by two closed curves C_1 and C_2 and a is a point in the region between C_1 and C_2.

$$f(a) = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-a}dz$$

where C_2 is outer curve.
Proof. Describe a circle γ about the point $z=a$ of radius r such that the circle lies in the ring shapped region. The function $\frac{f(z)}{z-a}$ is analytic in the region bounded by three closed curves C_1, C_2, and γ. By corollary 2 to Cauchy's theorem $\int_{C_2}\frac{f(z)}{z-a}dz = \int_{C_1}\frac{f(z)}{z-a}dz + \int_\gamma \frac{f(z)}{z-a}dz$ where the integral along each curve is taken in anti-clockwise direction. Using Cauchy's integral formula,

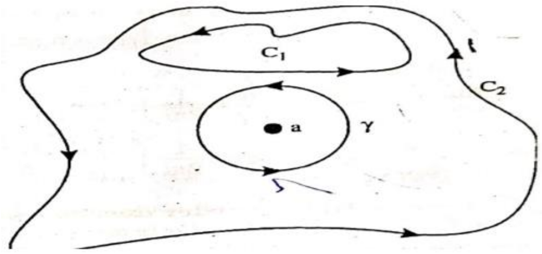

Fig. 8(b)

$$\int_{C_2} \frac{f(z)}{z-a} dz = \int_{C_1} \frac{f(z)}{z-a} dz + 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-a} dz$$

Note. In the ***Fig. 8(b)*** γ is a closed circle.

8.5 *CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVE OF ANALYTIC FUNCTION:-*

Cauchy integral formula for the derivative of an analytic function.

If a function $f(z)$ is analytic within and on a closed contour C and a is ary' point lying in it, then

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^2}$$

Or, Using Cauchy's integral formula to find the first derivative of an analytic function $f(z)$ at $z = z_0$.

Proof. Let $a + h$ be a point in the neighbourhoods of a point a, then by Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

and

$$f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z-(a+h)}$$

From which we get

$$\frac{f(a+h)-f(a)}{h}=\frac{1}{2\pi i}\int_C \frac{f(z)}{h}\left[\frac{1}{z-a-h}-\frac{1}{z-a}\right]dz$$
$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\left(1-\frac{h}{z-a}\right)^{-1}-1\right]dz$$
$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\frac{h}{z-a}+\left(\frac{h}{z-a}\right)^2+\left(\frac{h}{z-a}\right)^3\dots\right]dz$$
$$=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\left[\frac{1}{z-a'}+\frac{h}{(z-a)^2}+\frac{h^2}{(z-a)^3}+\cdots\right]dz$$

Taking limit as $h\to 0$, we get or

$$\lim_{h\to 0}\frac{f(a+h)-f\left(\frac{d}{d}\right)}{h}=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\left[\frac{1}{z-a}+0+0\dots\right]dz$$

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$$

Theorem. Higher order derivatives.

If a function $f(z)$ is analytic within and on a dlosed contour C and a is any point within C then derivatives of all orders are analytic and are given by

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

Proof. Prove as in previous theorem 5, that

$$f^{(1)}(a)=f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$$

This proves that the required result is true for $n=1$. Let us suppose that the required result is true for $n=m$ so that

$$f^{(m)}(a)=\frac{m!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{m+1}}$$

Let $a+h$ be a point in the neighbourhood of a. Observe that

$$\frac{f^{(m)}(a+h)-f^{(m)}(a)}{h}$$
$$=\frac{m!}{2\pi ih}\int_C f(z)\left\{\frac{1}{(z-a-h)^{m+1}}-\frac{1}{(z-a)^{m+1}}\right\}dz$$

$$=\frac{m!}{2\pi ih}\int_C \frac{f(z)}{(z-a)^{m+T}}\left[\left(1-\frac{\dot{h}}{z-a}\right)^{-(m+1)}-1\right]dz$$
$$=\frac{m!}{2\pi ih}\int_C \frac{f(z)}{(z-a)^{m+1}}\left[\frac{h(m+1)}{z-a}+\frac{(m+1)(m+2)}{2!}\left(\frac{h}{z-a}\right)^2+\cdots\right]dz$$
$$=\frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\left[\left(\frac{m+1}{z-a}\right)+\frac{(m+1)(m+2)}{2!}\frac{h}{(z-a)^2}+\cdots\right]dz$$

Taking limit as $h \to 0$

$$\lim_{h\to 0}\frac{f^{(m)}(a+h)-f^{(m)}(a)}{h}$$
$$=\frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\left[\left(\frac{m+1}{z-a}\right)+0+0+\cdots\right]dz'''$$

or

$$f^{(m+1)}(a)=\frac{(m+1)m!}{2\pi i}\int_C \frac{f(z)ds}{(z-a)^{m+2}}$$

or

$$f^{(m+1)}(a)=\frac{(m+1)!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{m+2}}$$

This proves that the required result is true for $n=m+1$ if it is true for $n=m$. But we have already seen that it is true for $n=1$ and so it is true for $n=2$ and so on. It follows that the required result is true for any positive integral value of n.

Since $f^{(1)}(a), f^{(2)}(a), f^{(3)}(a), \ldots$ all exist. Consequently $f^{(1)}(a), f^{(2)}(a), \ldots$ all aré analytic within C. Problem. Prove that $f'''(a)=\frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$ where C is a contour containing $z=\alpha$.

Corollary. If C be a closed contour containing the origin inside it, prove that

$$\frac{a^n}{n!}=\frac{1}{2\pi i}\int_C \frac{e^{az}dz}{z^{n+1}}$$

Solution. We have

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$
$$f^{(n)}(0)=\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}}$$

Taking $f(z)=e^{az}$ so that $f^{(n)}(z)=a^n e^{az}$, we obtain or

$$f^{(n)}(0)=a^n=\frac{n!}{2\pi i'}\int_C \frac{e^{az}dz}{z^{n+1}}$$
$$\frac{a^n}{n!}=\frac{1}{2\pi i}\int_C \frac{e^{az}dz}{z^{n+1}}$$

Question 1. Evaluate $\int_C \frac{e^{2z}dz}{(z+1)^4}$ where c is $|z|=3$

Solution. We know that

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

Put $a=-1, n=3$

$$f^{(3)}(-1)=\frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z+1)^4} \qquad (1)$$

Take $f(z)=e^{2z}$, then $f^{(n)}(z)=2^n e^{2z}$

$$f^{(3)}(-1)=2^3e^{-2}=\frac{8}{e^2}$$

$\therefore$
$$\frac{8}{e^2}=\frac{3!}{2\pi i}\int_C \frac{e^{2z}dz}{(z+1)^4} \text{ or } \frac{8\pi i}{3e^2}=\int_C \frac{e^{2z}dz}{(z+1)^4}$$

Question 2. Using Cauchy integral formula, calculate the following integrals :

(i) $\int_C \frac{zdz}{(9-z^2)(z+i)}$, where C is the circle $|z|=2$ described in positive sense.

(ii) $\int_C \frac{dz}{z(z+\pi i)}$, where C-is $|z+3i|=1$

(iii) $\int_C \frac{\cosh(\pi z)dz}{z(z^2+1)}$, where C is circle $|\cdot z|=2.$

(iv) $\int_C \frac{e^{az}dz}{(z-\pi i)}$, where C is the ellipse $|z-2|+|z+2|=6.$

(v) Evaluate $\int_C \frac{dz}{z-2}$, where C is $|z|=3$.

Solution. By Cauchy's integral formula, $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}$ or,

$$\int_C \frac{f(z)}{z-a}dz=2\pi i f(a) \qquad (1)$$

where $z=a$ is a point inside contour C and $f(z)$ is analytic within and upon C.

Step I. Let $I=\int_C \frac{zdz}{(9-z^2)(z+i)}$

Take

$$f(z)=\frac{z}{9-z^2}$$

Then

$$I=\int_C \frac{f(z)}{[z-(-i)]}$$
$$=2\pi i f(-i), \text{ by } (1)$$
$$=2\pi i\left[\frac{-i}{9-(-i)^2}\right]=\frac{2\pi}{9+1}=\frac{\pi}{5}$$

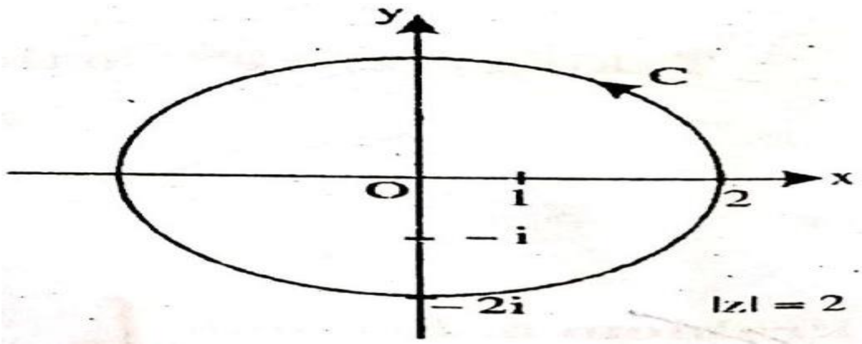

Fig. 8(c)

Here $f(z)$ is analytic within ani upon C s.t. $|z| = 2$ and $z = -i$ lies inside C.

Step II. Let $I = \int_C \frac{dz}{z(z+\pi i)}$.

Take $f(z) = \frac{1}{z}$, then

$$I = \int_C \frac{f(z)}{[z-(-\pi i)]}$$
$$= 2\pi i f(-\pi i), \text{ by } (1)$$
$$= 2\pi i \left(\frac{1}{-\pi i}\right) = -2$$

Here $z = -\pi i$ lies inside C and $f(z)$ is analytic within C.

Step III. Let $I = \int_C \frac{\cosh(\pi z)}{z(z^2+1)} iz.$

Take $f'(z) = \cosh(\pi z) = \cos(i\pi z)$ and C is $|z| = 2$.

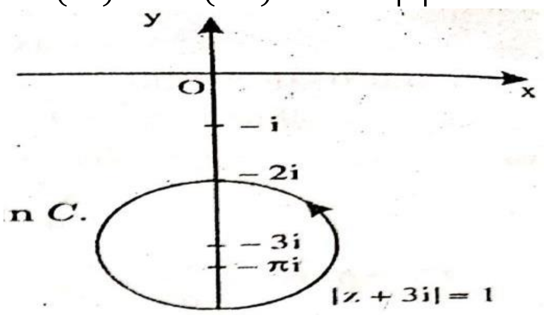

Fig. 8(d)

$$\boldsymbol{I} = \int_C \frac{f(z)}{z(z^2+1)}$$
$$\mathrm{I} = \int_C \left[\frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}\right] f(z)dz$$

…….. *(2)*

$$\frac{1}{z(z-i)(z+i)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$
$$A = \frac{1}{(z-i)(z+i)} = 1 \text{ at } z = 0$$
$$\mathrm{B} = \frac{1}{z(z+i)} = -\frac{1}{2} \text{ at } z = i$$
$$\mathrm{C} = \frac{1}{z(z-i)} = -\frac{1}{2} \text{ at } z = -i$$

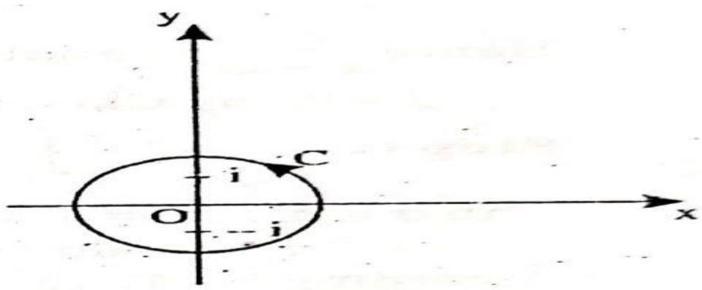

Fig. 8(e)

Here $z = 0, i, -i$ are points inside C.
According to (1), (2) gives

$$I = 2\pi i[Af(0) + Bf(i) + Cf(-i)]$$

$$= 2\pi i\left[f(0) - \frac{1}{2}f(i) - \frac{1}{2}f(-i)\right]$$

$$= 2\pi i\left[\cos(0) - \frac{1}{2}\cos(i^2\pi) - \frac{1}{2}\cos(-i^2\pi)\right]$$

$$= 2\pi i\left[1 + \frac{1}{2} + \frac{1}{2}\right] = 4\pi i \qquad \ldots\ldots\ldots\ldots.. (1)$$

Step IV. Let $I = \int_C \frac{e^{az}dz}{z-\pi i}$

C is ellipse $|z-2| + |z+2| = 6$

or

$$[(x-2)^2 + y^2]^{1/2} = 6 - [(x+2)^2 + y^2]^{1/2}$$

Squaring, we get

$$x^2 + y^2 + 4 - 4x = 36 + (x^2 + y^2 + 4 + 4x) - 12[(x+2)^2 + y^2]^{1/2}$$

or, $$12(x^2 + y^2 + 4 + 4x)^{1/2} = 36 + 8x$$

or, $$3(x^2 + y^2 + 4 + 4x)^{1/2} = 9 + 2x$$

Again squaring,

$$9(x^2 + y^2 + 4 + 4x) = 81 + 4x^2 + 36x$$

or,

$$5x^2 + 9y^2 = 45$$

or,

$$\frac{x^2}{9} + \frac{y^2}{5} = 1.$$

Comparing, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get $a^2 = 9, b^2 = 5$

$$a = 3, b = \sqrt{5} = 2.2 \text{ approx.}$$

Evidently $z = \pi i = 3.14i$ lies outside C.

$\therefore I = 0$, by Cauchy's theorem.

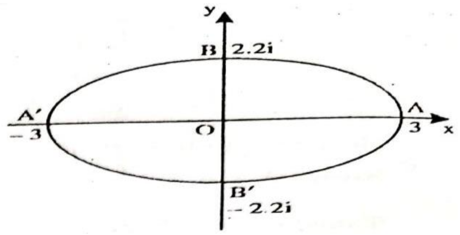

Fig: 8(f)

Step (v). $I = \int_C \frac{dz}{z-2} = \int_C \frac{f(z)dz}{z-a}$

Then $a = 2, f(z) = 1. C$ is circle $|z| = 3$ whose centre is at $z = 0$ and radius $= 3.$

$\therefore\ a = 2$ lies inside $C.$

According to (1), (2) gives

For

$$I = 2\pi i f(\alpha) = 2\pi i f(2) = 2\pi i(1) = 2\pi i$$
$$f(z) = 1 \Rightarrow f(2) = 1.$$

Question 3. (i) Evaluate $\int_C \frac{\tan(z/2)dz}{(z-x_0)^2}$ where C is the boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$ and it is described in positive sense, where $|x_0| < 2.$

(ii) Evaluate $\int_C \frac{dz}{z^2+2z+2}$, where C is the square having vertices at $(0,0), (-2,0), (-2,-2), (0,-2)$ oriented in anticlockwise direction. (Kanpur 1999)

(iii) Evaluate $\int_C \frac{\sin z dz}{\left(z-\frac{\pi}{4}\right)^3}$ where C is $\left|z - \frac{\pi}{4}\right| = \frac{1}{2}$

(iv) If C is unit circle about the origin, described in positive sense, show that

$$\int_C \frac{e^{-z}}{z^2} dz = -2\pi i \text{ and } \int_C \left(\frac{\sin z}{z}\right) dz = 0$$

Solution. By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)}$$

and

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \quad (1)$$

This $\Rightarrow \int_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

and

$$\int_C \frac{f(z)dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(a) \quad (2)$$

where $z = a$ lies inside C and $f(z)$ is analytic within and upon $C.$
Step I. Let

$$I=\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz$$

where C is rectangle $ABCD$ and point $z=x_0$ lies within C. Take $f(z)=\tan(z/2)$.

Then $I=\int_C \frac{f(z)dz}{(z-x_0)^2}$

$=\frac{2\pi i}{1!}f^{(1)}(x_0)$, according to (2)

$=2\pi i\left\{\frac{d}{dz}[\tan(z/2)]\right\}_{z=x_0}$

$=2\pi i\frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)=\pi i\sec^2\left(\frac{x_0}{2}\right)$

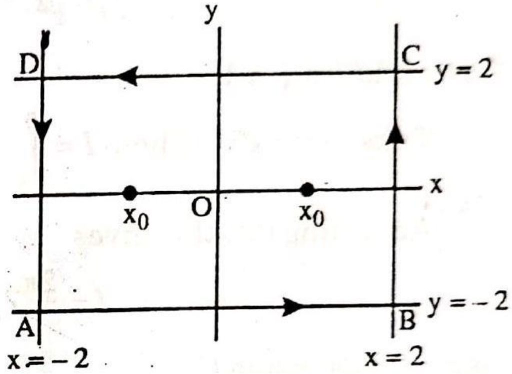

Fig. 8(g)

Ans.

Step II. Let $I=\int_C \frac{dz}{(z^2+2z+2)}$

C is square $OABC$.

$$z^2+2z+2=0 \text{ gives } (z+1)^2+1=0$$

or,

$$(z+1)^2=-1=i^2 \text{ or } z+1=\pm i$$

$z=\beta$ lies inside C as shown in the figure.

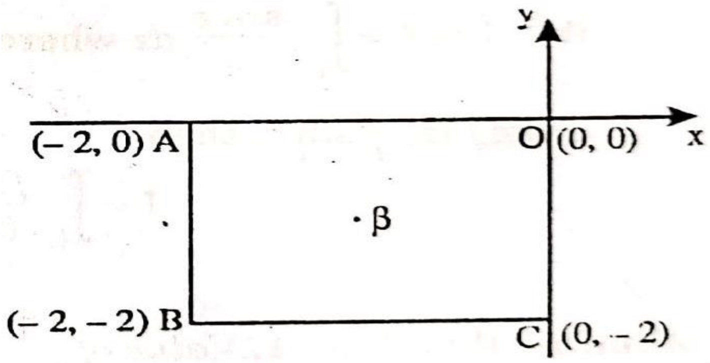

Fig. 8(h)

where $f(z)=\frac{1}{z-\alpha}$

Ans.

$$\therefore\ I=-\pi$$

Step III. Let

$$I = \int_C \frac{\sin z dz}{\left(z - \frac{\pi}{4}\right)^3}$$

where C is $\left|z - \frac{\pi}{4}\right| = \frac{1}{2}$.

Take $f(z) = \sin z$ and so $f'(z) = \cos z$,

$$f''(z) = -\sin z, f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Then $I = \int_C \frac{f(z)dz}{\left(z-\frac{\pi}{4}\right)^3} = \frac{2\pi i}{2!} f''\left(\frac{\pi}{4}\right)$, according to (2).

$$= \pi i \left(-\frac{1}{\sqrt{2}}\right) = \frac{-\pi i}{\sqrt{2}}$$

Step IV. (a) Let $I = \int_C \frac{e^{-z}}{z^2} dz$ C is circle $|z| = 1$.

Take $f(z) = e^{-z}$. Then $I = \int_C \frac{f(z)dz}{(z-0)^2}$

According (2), this gives

$$I = \frac{2\pi i}{1!} f'(0)$$

as $z = 0$ lies inside C.

$$f(z) = e^{-z} \Rightarrow f'(z) = -e^{-z} \Rightarrow f'(0) = -e^{-0} = -1$$

$$\therefore I = \frac{2\pi i}{1!}(-1) = -2\pi i$$

(b) Let $I = \int_C \frac{\sin z}{z} dz$ where $|z| = 1$.

Take $f(z) = \sin z$, then

$I = \int_C \frac{f(z)dz}{(z-0)} = 2\pi i f(0) = 2\pi i \sin(0) = 0.$

Question 4. The value of $\int \frac{1}{z} dz$ where C is circle $z = e^{i\theta}, 0 \leq 0 \leq \pi$ is:

(a)πi

(b)$-\pi i$

(c)$2\pi i$

(d)0

Solution. Ans.(a).$z = e^{i0} \Rightarrow dz = e^{i\theta} i d\theta$

$\Rightarrow dz = izd0 \Rightarrow \frac{dz}{z} = id0$

$$\int \frac{1}{z} dz = \int_0^{\pi} i d\theta = \pi i$$

Question 5. Evaluate $\int_c |z| dz$, where is upper half part of corde $|z| = 1$.

Solution. Here $|z| = 1, z = e^{i\theta}, dz = e^{i0} i d\theta$

$$\int_c |z| dz = \int_c dz = \int_0^{\pi} e^{i\theta} i d\theta = \left(i^{i\theta}\right)_0^{\pi} = e^{i\pi} - 1 = -1 - 1 = -2$$

Question 6. Evaluate the following integrals by using Cauchy's integral formula:

(i) $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)} dz$ where c is circle $|z| = 3$

(ii) $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1} dz \forall t > 0$ where c is $|z| = 3$

(iii) $\int_C \frac{(z-1)dz}{(z+1)^2(z-2)}$, where c is $|z - i| = 2$

(iv) $\int_C \frac{(\sin z)^6}{\left(z-\frac{\pi}{6}\right)^3} dz$, where c is ircle $|z| = 1.$

(v) $\int_C \frac{e^{3z}dz}{z+i}$ if c is circle $|z + 1 + i| = 2$

(vi) $\int_C \frac{e^{az}}{z^2+1} dz$ if c is circle $|z| = 2.$

Solution. Step I. Here we use two results:
(R_1) If $f(z)$ is analytic within and on a closed contour C, then $\int_C f(z)dz = 0.$

(R_2) If $z = a$ is a point inside a closed contour C and $f(z)$ is analytic within and on C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} = f(a)$$

(R_3) $f^{(n)}(a) = \frac{!n}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$ for $n = 1,2,3, ...$
where $z = a$ is inside C.

To evaluate (i). $I = \int_C \frac{\{\sin(\pi z^2)+\cos(\pi z^2)\}}{(z-1)(z-2)} dz$
where c is circle $|z| = 3.$

$$z = 1, z = 2 \text{ both lic inside } c$$

Take $f(z) = \sin(\pi z^2) + \cos(\pi z^2)$

Then $I = \int_C \frac{f(z)dz}{(z-1)(z-2)} = \int_C f(z)dz\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

$$= \int_C \frac{f(z)dz}{z-2} - \int_C \frac{f(z)dz}{z-1} = 2\pi i[f(2) - f(1)], \text{ by } (R_2)$$

$$= 2\pi i[\{\sin(\pi 2^2) + \cos(\pi 2^2)\} - \{\sin(\pi \cdot 1) + \cos(\pi \cdot 1)\}]$$

$$= 2\pi i[(0 + 1) - (0 - 1)] = 4\pi i$$

To evaluate (ii). $I = \frac{1}{2\pi i}\int_C \frac{e^{zt}dz}{(z^2+1)} \forall t > 0$
where C is $|z| = 3$. Take $f(z) = e^{zt}$ "

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

Here $z = i, z = -i$ both lie inside C.

$$I = \frac{1}{2\pi i}\int_C \frac{f(z)}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

This $\Rightarrow 2iI = f(i) - f(-i)$, by (R_2)

$$= e^{it} - e^{-it} = 2i\sin(t)$$

$$\Rightarrow I = \sin(t)$$

To evaluate(iii).

$I=\int_C \frac{(z-1)dz}{(z+1)^2(z-2)}$

where C is $|z-i|=2$

$$(z+1)^2(z-2)=0 \Rightarrow z=-1, z=2$$

If $z=-1$, then $|z-i|=|-1-i|=|1+i|=\sqrt{2}<2=R$

If $z=2$, then $|z-i|=|2-i|=\sqrt{5}>2=R$

$\therefore\ z=-1$ lies inside C and $z=2$ lies outside C.

Take $f(z)=\frac{(z-1)}{z-2}$, then $I=\int_C \frac{f(z)}{[z-(-1)]^2}$

$\Rightarrow\ I=2\pi i\frac{f'(-1)}{1!}$, by $(R_3)=2\pi i\left(-\frac{1}{9}\right)=-\frac{2\pi i}{9}$

For $f(z)=\frac{z-1}{z-2}=1+\frac{1}{z-2}\Rightarrow f'(z)=-\frac{1}{(z-2)^2}f'(-1)=-\frac{1}{(3)^2}$

To evaluate (iv). Let $I=\int_C \frac{(\sin z)^6 dz}{\left(z-\frac{\pi}{6}\right)^3}$, where C is $|z|=1$.

$z=\frac{\pi}{6}=\frac{3.14}{6}=0.52$ lies inside C. Take $f(z)=(\sin z)^6$

Then $I=\int_C \frac{f(z)dz}{\left(z-\frac{\pi}{6}\right)^3}=\frac{2\pi i}{2!}f''\left(\frac{\pi}{6}\right)=\pi i f''\left(\frac{\pi}{6}\right)$, by (R_3)

$$f(z)=(\sin z)^6\Rightarrow f'(z)=6(\sin z)^5\cos(z)$$
$$\Rightarrow f''(z)=6[5(\sin z)^4(\cos z)^2-(\sin z)^6]$$
$$\Rightarrow f''\left(\frac{\pi}{4}\right)=6\left[5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^6\right]=\frac{21}{16}$$
$$I=\pi i f''\left(\frac{\pi}{6}\right)=\pi i\left(\frac{21}{16}\right)$$

To evaluate (v). Let $I=\int_C \frac{e^{3z}dz}{z+i}$, where c is $|z+1+i|=2$

Take $f(z)=e^{3z}\&z+i=0\Rightarrow z=-i$

If $z=-i$, then $|z+1+i|=|-i+1+i|=1<2=R$.

$\therefore\ z=-i$ lies inside C.

By $(R_2), I=2\pi i f(-i)=2\pi i e^{-3i}$

To evaluate (vi). Let $I=\int_C \frac{e^{az}}{z^2+1}dz, c$ is $|z|=2$.

Same as (ii) and answer is $I=2\pi i\sin(a)$.

Question 9. Evaluate the following:

(i) $\int_C \frac{(z^2-4)dz}{z(z^2+9)}$, where c is $|z|=1$.

(ii) $\int_C \frac{e^z}{z-2}dz$, where c is (a) $|z| = 3$, (b) $|z| = 1$.

(iii) $\int_C \frac{(z-3)dz}{z^2+2z+5}$, where c is (a) $|z| = 1$, (b) $|z+1-i| = 2$.

Solution.(i). Let

$$I = \int_C \frac{(z^2-4)dz}{z(z^2+9)},$$

where c is circle $|z| = 1. z(z^2 + 9) = 0 \Rightarrow z = 0, 3i, -3i$

If $z = 0$, then $|z| = |0| = 0 < 1$,

If $z = \pm 3i$, then $|z| = |\pm 3i| = 3 > 1$.

$\therefore$ $z = 0$ lies inside C. $z = 3i, z = -3i$

both lie outside C. Take $f(z) = \frac{z^2-4}{z^2+9}$, then

$$I = \int_C \frac{f(z)}{z-0} = 2\pi i f(0), \text{by } (R_2) = 2\pi i\left(\frac{0-1}{0+9}\right) = -\frac{8\pi i}{9}$$

(ii). Let

$$I = \int_C \frac{c^n}{z-2}dz$$

(a) C is circle $|z| = 3$. Evidently $z = 2$ lies inside c.

(b) Cis circle $|z| = 1$, in this case $z = 2$ lies outside c. $f(z) = \frac{e^z}{z-2}$ is analytic inside $C. \therefore I = 0$, by (R_1)

(iii). Let

$$I = \int_C \frac{(z-3)dz}{z^2+2z+5}$$

(a) Where C is $|z| = 1$, $z^2 + 2z + 5 = 0 \Rightarrow z = -\frac{2\pm\sqrt{(4-20)}}{2}$

$\Rightarrow$ $z = -1 \pm 2i$. Take $\alpha = -1 + 2i, \beta = -1 - 2i$.

$|\alpha| = |\beta| = \sqrt{5} > 1. \therefore$ $z = \alpha, z = \beta$ lie outside c.

$\therefore$ $I = 0$, by(R_1).

(b) Here C is $|z + 1 - i| = 2$

If $z = \alpha = -1 + 2i$, then $|z + 1 - i| = |(-1 + 2i) + 1 - i| = |i| = 1 < 2$

If $z = \beta = -1 - 2i$, then $|z + 1 - i| = |-1 - 2i + 1 - i| = |-3i| =$

$3 > 2.$

$\therefore\ z = \alpha$ lies inside C and $z = \beta$ lies outside C.

$$I = \int_C \frac{(z-3)dz}{(z^2+2z+5)} = \int_C \frac{(z-3)dz}{(z-\alpha)(z-\beta)}$$

Take $f(z) = \frac{z-3}{z-\beta}$, then $I = \int_C \frac{f(z)dz}{z-\alpha}$

$\therefore\ I = 2\pi i f(\alpha) = 2\pi i\left(\frac{z-3}{z-\beta}\right)$ at $z = \alpha$

$$\Rightarrow\ I = \frac{2\pi i(\alpha-3)}{\alpha-\beta} = 2\pi i\frac{(-1+2i-3)}{4i} = \pi(-2+i)$$

Question 10. Evaluate the following integrals:

(i) $\int_C \frac{dz}{z^2-1}$ where C is $x^2+y^2=4$

(ii) $\int_C \frac{e^{3iz}dz}{(z+\pi)^3}$, where C is $|z-\pi| = 3.2$

(iii) $\int_C \frac{(4-3z)dz}{z(z-1)(z-2)}$, where C is $|z| = \frac{3}{2}$

Solution. **(i).**

Let $I = \int_C \frac{dz}{(z-1)(z+1)}$ where C is $|z| = 2.$

Evidently $z = 1, z = -1$ both lie inside C.

$I = \frac{1}{2}\int_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right) d{::} = 2\pi i[f(1) - f(-1)] = 2\pi i(1-1) = 0$

(ii). Let $I = \int_C \frac{e^{3iz}dz}{(z+\pi)^3}$, where C is $|z-\pi| = 3.2z+\pi = 0 \Rightarrow z = -\pi$.

Now if $\dot{z} = -\pi$, then $|z-\pi| = |-\pi-\pi| = 2\pi = 2\times 3.14 > 3.2$

$\therefore\ z = -\pi$ lies outside C.

$$\Rightarrow\ \int_C \frac{e^{3iz}dz}{(z+\pi)^3} = 0$$

(iii) $I = \int_C \frac{(1-3z)dz}{z(z-1)(z-2)}$

Here C is $|z| = \frac{3}{2}$. Evidently $z = 0, z = 1$ both inside C and $z = 2$ lies outside c.

Take $f(z) = \frac{4-3z}{z-2}$

Then $I = \int_C \frac{f(z)dz}{z(z-1)} = \int_C f(z)\left(\frac{1}{z-1} - \frac{1}{z}\right) dz$

By $(R_2), I = 2\pi i[f(1) - f(0)]$

$$\Rightarrow\ I = 2\pi i\left[\left(\frac{4-3z}{z-2}\right)_{\text{at } z=1} - \left(\frac{4-3z}{z-2}\right)_{\text{at } z=0}\right]$$

$= 2\pi i[-1-(-2)] = 2\pi i.$

Question 13. (i) Integrate $\frac{1}{(z^3-1)^2}$, the counter clockwise around the circle $|z-1|=1$

(ii) Evaluate $\int_C \frac{zdz}{z^2+1}$, where c is (a) $\left|z+\frac{1}{z}\right|=2$, (b) $|z+i|=1$.

Solution. **(i)** Let $I=\int_C \frac{dz}{(z^3-1)^2}, c$ is $|z-1|=1$

$(z^3-1)=0 \Rightarrow (z-1)(z^2+z+1)=0 \Rightarrow z=1$ and $z=-\frac{1+i\sqrt{3}}{2}$

If $z=1$ lies inside C.

If $z=-\frac{1+i\sqrt{3}}{2}$, then $|z-1|=\left|\frac{-1+i\sqrt{3}}{2}-1\right|=\left|\frac{-3+i\sqrt{3}}{2}\right|$

$$=\frac{1}{2}\sqrt{12}=\frac{2\sqrt{3}}{2}=\sqrt{3}>1$$

$\therefore\ z=-\frac{1+i\sqrt{3}}{2}$ lies outside C. Similarly, $z=-\frac{1-i\sqrt{3}}{2}$ lies outside C. Take $f(z)=\frac{1}{(z^2+z+1)^2}$, then

$$I=\int_C \frac{f(z)}{(z-1)^2}=2\pi i f'(1), \text{ by } (R_3) \text{ of Problem } (8)$$

$$\Rightarrow\ I=2\pi i\left\{\frac{-2(2z+1)}{(z^2+z+1)^3}\right\}_{z=1}=-\frac{4\pi i}{9}$$

(ii) (a) Let $I=\int_C \frac{zdz}{z^2+1}$, where C is (a) $\left|z+\frac{1}{z}\right|=2$

$$\left|z+\frac{1}{z}\right|=2 \Rightarrow |z^2+1|=|2z| \Rightarrow |x^2-y^2+2ixy+1|=2|x+iy|$$

$\Rightarrow\ (x^2-y^2+1)^2+4x^2y^2=4(x^2+y^2)$

$\Rightarrow\ (x^2-y^2)^2+1+2(x^2-y^2)+4x^2y^2=4(x^2+y^2)$

$\Rightarrow\ (x^2+y^2)^2+1=2x^2+6y^2$

$\Rightarrow\ (x^2+y^2-1)^2+2(x^2+y^2)=2x^2+6y^2$

$\Rightarrow\ (x^2+y^2-1)^2=4y^2 \Rightarrow x^2+y^2-1=\pm 2y$

$\Rightarrow\ (x-0)^2+(y\pm 1)^2=2$

$\Rightarrow$ Circle c_1, centre $(0,1), r_2=\sqrt{2}$ and circle c_2, centre $(0,-1), r_2=\sqrt{2}$

$\Rightarrow c_1, |z-i|=\sqrt{2}$ and c_2: $|z+i|=\sqrt{2}$

$$I=\int_C \frac{zdz}{(z+i)(z-i)}=\int_{C_1} \frac{zdz}{(z+i)(z-i)}+\int_{C_2} \frac{zdz}{(z+i)(z-i)}$$

Take $f=\frac{z}{z+i}$ for c_1 and $g=\frac{z}{z-i}$ for c_2

Then $I=\int_{C_1} \frac{f(z)dz}{z-i}+\int_{C_2} \frac{g(z)dz}{z+i}$

$=2\pi i[f(i)+g(-i)]$, by Cauchy's integral formula.

$$=2\pi i\left[\left(\frac{z}{z+i}\right)_{\text{at } z=i}+\left(\frac{z}{z-i}\right)_{\text{at } z=-i}\right]$$

$$=2\pi i\left[\frac{1}{2}+\frac{1}{2}\right]=2\pi i$$

(ii) (b) Let $I = \int_C \frac{zdz}{z^2+1}$, where C is $|z+i| = 1$.
Centre of circle C is at $z = -i$ and radius 1 .
Take $f = \frac{z}{z-i}$, then $I = \int_C \frac{f(z)dz}{z+i}$
$\therefore\ l = 2\pi i f(-i) = 2\pi i\left(\frac{z}{z-i}\right)$ at $z = -i$ or $I = \pi i$

Question 12. Let $P(z) = a + bz + cz^2$ and

$$\int_C \frac{P(z)}{z}dz = \int_C \frac{P(z)}{z^2}dz = \int_C \frac{P(z)}{z^3}dz = 2\pi i$$

where C is circle $|z| = 1$. Evaluate $P(z)$.
Solution. Here we use the formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \ldots (^*), \text{ where } z = a \text{ lies inside } c$$

In view of this we evaluate $P(z)$.
Give $1 = \frac{1}{2\pi i}\int_C \frac{P(z)dz}{(z-0)} = \frac{1}{2\pi i}\int_C \frac{P(z)dz}{(z-0)^2} = \frac{1}{2\pi i}\int_C \frac{P(z)dz}{(z-0)^3}$

$$\Rightarrow\ 1 = P(0) = P'(0) = \frac{P''(0)}{!2} \qquad (1)$$

Given $P(z) = a + bz + cz^2$
Then $P'(z) = b + 2cz, P''(z) = 2c$

$$\Rightarrow\ P(0) = a, Pp'(0) = b, P''(0) = 2c \qquad (2)$$

Putting this in (1), $1 = a = b = c$
Now (2) ⇒ $P(z) = 1 + z + z^2$.

8.6 *POISSON'S INTEGRAL FORMULA:-*

Poisson's Integral formula. If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2 - a\bar{a})f(z)dz}{(z-a)(R^2 - z\bar{a})}$$

Hence deduce the Poisson's formula

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f\left(Re^{i\phi}\right)d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

where $a = re^{i\theta}$ is any point inside the circle $|z| = R$

Proof. Suppose $f(z)$ is analytic within and on the circle C defined $|z| = R$ and $a = re^{i0}$ is any point A inside C so that

$$0 < r < R.$$

The inverse $A'(a')$ of $A(a)$ w.r.t. the circle C is given by $a' = \frac{R^2}{\bar{a}}$ which lies outside the circle C. By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} \tag{1}$$

Since $f(z)$ is analytic within and upon the circle C and so $\frac{f(z)}{z-a'}$ is analytic within and on

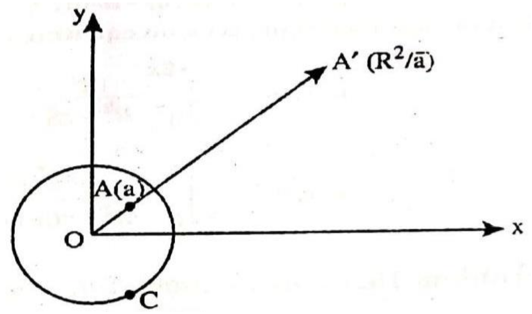

Fig. 8(i)

$$\int_C \frac{f(z)dz}{z-a'} = 0 \qquad \dots\dots\dots\dots\ (\mathbf{2})$$

[Note that $\frac{f(z)}{z-a}$ is not analytic within C] (1) - (2) gives or

$$f(a) - 0 = \frac{1}{2\pi i}\int_C \left[\frac{f(z)}{z-a} - \frac{f(z)}{z-a'}\right] dz$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(a-a')f(z)dz}{(z-a)(z-a')}$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(a - \frac{R^2}{\bar{a}}\right) f(z)dz}{(z-a)\left(z - \frac{R^2}{\bar{a}}\right)} = \frac{1}{2\pi i}\int_C \frac{(a\bar{a} - R^2)f(z)dz}{(z-a)(\bar{a}z - R^2)}$$

or

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2 - a\bar{a})f(z)dz}{(z-a)(R^2 - z\ddot{a})} \qquad \dots\dots..(3)$$

This proves the first required result.
Any point z on $|z| = R$ is expressible as $z = Re^{i\phi}$.
Also $a = re^{i0}$ so that $\bar{a} = re^{-i0}$.
Now

$$R^2 - a\bar{a} = R^2 - re^{i0}\cdot re^{-i0} = R^2 - r^2 \tag{4}$$

$$(z-a)(R^2 - z\bar{a}) = \left(Re^{i\phi} - re^{i0}\right)\left(R^2 - Re^{i\phi}re^{-i\theta}\right) \tag{4}$$

$$= Re^{i\phi}\left(R - be^{i(\theta-\phi)}\right)\left(R - re^{-i(\theta-\phi)}\right)^C \tag{4}$$

$$= Re^{i\phi}\left[R^2 + r^2 - rR\left(e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}\right)\right] \tag{5}$$

$$= Re^{i\phi}[R^2 + r^2 - 2rR\cos(\theta-\phi)] \tag{6}$$

$$dz = d\left(Re^{i\phi}\right) = Rie^{i\phi}d\phi \tag{6}$$

Writing (3) with the help of (4), (5) and (6),

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2 - r^2)f(z)Re^{i\phi}id\phi}{[R^2 - 2Rr\cos(\phi - \theta) + r^2]\mathrm{Re}^{i\phi}}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f\left(Re^{i\phi}\right)d\phi}{[R^2 - 2Rr\cos(\theta - \phi) + r^2]}$$

This proves the second required result.

Remark. If we assume $\boldsymbol{f(a) = u(r, \theta) + iv(r, \theta)}$ and $f\left(Re^{i\phi}\right) = u(R, \phi) + iv(R, \phi)$

then the last equation gives, on equating real and imaginary parts,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(R, \phi)d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$
$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)v(R, \phi)d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

Theorem. Using Poisson's integral formula for a circle, show that
$$\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)d\phi}{5 - 4\cos(\theta - \phi)} = \frac{2\pi}{3}e^{\cos 0}\cos(\sin\theta)$$

Solution. By Poisson's integral formula,
$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f\left(Re^{i\phi}\right)d\phi}{R^2 + r^2 - 2rR\cos(\theta - \phi)} \qquad (1)$$

If we compare R.H.S. of (1) with the given integral, then we find
$$R^2 + r^2 = 5 \qquad (2)$$
$$rR = 2 \qquad (3)$$
$$f\left(Re^{i\phi}\right) = e^{i\cos\phi}\cos(\sin\phi) \qquad (4)$$

(2) & (3) ⇒ $R = 2, r = 1$ and so $R^2 - r^2 = 4 - 1 = 3$

Now (4) ⇒ $f\left(re^{i\theta}\right) = e^{\cos\theta}\cos(\sin\theta)$

Putting values from (2), (3), (5) and (6) in (1), we get
$$e^{\cos\theta}\cos(\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{3e^{\cos\phi}\cos(\sin\phi)d\phi}{5 - 4\cos(\theta - \phi)}$$
$$\Rightarrow \frac{2\pi}{3}e^{\cos\theta}\cdot\cos(\sin\theta) = \int_0^{2\pi} \frac{e^{\cos\theta}\cos(\sin\phi)d\phi}{5 - 4\cos(\theta - \phi)}$$

8.7 *MORERA'S THEOREM:-*

Morera's theorem. If $f(z)$ is a continuous function in a domain D and if for every closed contour C in the domain D,
$$\int_C f(z)dz = 0$$
then $f(z)$ is analytic within D. (It is a sort of converse of Cauchy's theorem).

Proof. Let z_0 be a fixed point and z a variable point inside the domain D.

The value of the intergal $\int_{z_0}^{z} f(t)dt$ is independent of the curve joining z_0 to z and depends on z only.
Write

$$F(z) = \int_{z_0}^{z} f(t)dt$$

Let $z + h$ be a point in the neighbourhood of z.

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(t)dt - \int_{z_0}^{z} f(t)dt$$

$$= \int_{z_0}^{z+h} f(t)dt + \int_{z}^{z_0} f(t)dt = \int_{z}^{z+h} f(t)dt$$

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{z}^{z+h} f(t)dt - f(z)\right| = \frac{1}{|h|}\left|\int_{z}^{z+h} [f(t) - f(z)]dt\right|$$

$$\le \frac{1}{|h|}\int_{z}^{z+h} |f(t) - f(z)||dt| < \frac{\varepsilon}{|h|}|h|$$

or $\left[\dfrac{|f(t) - f(z)| < \varepsilon \text{ for } |t - z| < \delta \text{ because of continuity of } f(z)]}{\left.\dfrac{F(z+h) - F(z)}{h} - f(z)\right| < \varepsilon \text{ which tends to 0 as } \varepsilon \to 0}\right.$

Thus $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} - f(z) = 0,$ or $F'(z) = f(z).$

Thus the derivative of $F(z)$ exists and so $F(z)$ is analytic in D. But we know that the derivative of analytic function is analytic. (Refer Theorem 6).

Therefore $F'(z)$. i.e. $f(z)$ is analytic in D.

Remark. The above theorem can also be restated as follows: If (z) is analytic in a simply connected region D of the complex plane, show that there exists a function $F'(z)$, analytic in D, and such that $F'(z) = f(z)$ for z in D.

Theorem. A necessary and sufficient condition for a function $f(z)$ to possess an indefinite integral in a simply connected domain D is that the function is analytic in D. Further, any two indefinite integrals differ by a constant.

Proof. Let $f(z)$ possess an indefinite integral $F(z)$ so that

$$F(z) = \int_{a}^{z} f(t)dt \qquad \dots\dots \quad (1)$$

To prove that $f(z)$ is analytic.
By (1), $F'(z) = f(z)$, showing thereby $F(z)$ possess a derivative $f(z)$.

Also, the derivative of an and function is analytic. It follows that $f(z)$ is analytic.

Conversely suppose that $f(z)$ is analytic in a domain D. To prove that $f(z)$ possesses an indefinite integral. Let z_0 be any fixed point and z an arbitrary point in D.

$$\text{Write } 1\ F(z) = \int_{z_0}^{z} f(t)dt \qquad \dots\dots..(2)$$

For the integral of $f(z)$ along any curve in D joining z_0 to z is the same.
Prove as in Theorem 8 that $F'(z) = f(z)$. This proves that $f(z)$ possesses indefinite integral, given by (2).

Second Part. Let $F(G)$ and $G(Z)$ be two indefinite integrals of the same function $f(z)$. Then

$$F'(z) = f(z) = G'(z)$$

$$\text{This } \Rightarrow F'(z)' = G'(z) \Rightarrow \frac{d}{dz}(F - G) = 0$$

Integrating, we get $F - G = c$.
This completes the proof.

Fundamental theorem of Integral Calculus. Let $f(z)$ be single palued analytic function in a simple connected domain D. If $a, b \in D$, then

$\int_a^b f(z)dz = F(b) - F(a),$ where $F(z)$ is an indefinite integral of $f(z)$.

Proof. By definition of indefinite integral,

$$F(z) = \int_{z_0}^{z} f(t)dt$$

$$F(b) - F(a) = \int_{z_0}^{b} f(t)dt - \int_{z_0}^{a} f(t)dt = \int_{z_0}^{b} f(t)dt + \int_{a}^{z_0} f(t)dt$$

$$= \int_a^b f(t)dt$$

or $F(b) - F(a) = \int_a^b f(z)dz$.

Cauchy's inequality. If $f(z)$ is analytic within and on a circle C, given by $|z - a| = R$ and if $|f(z)| \le M$ for every z on C, then $\left|f^{(n)}(a)\right| \le \frac{Mn!}{R^n}$.

Proof. $|z - a| = R \Rightarrow z - a = \mathrm{Re}^{i0} \Rightarrow dz = i\mathrm{Re}^{i0}d\theta \Rightarrow |dz| = Rd\theta$

We know that $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$

or

$$\left|f^{(n)}(a)\right| \le \frac{n!}{2\pi}\int_c \frac{|f(z)| \cdot |dz|}{|z - a'|^{n+1}} \le \frac{Mn!}{2\pi R^{n+1}}\int_0^{2\pi} Rd\theta$$

$$= \frac{Mn!}{2\pi R^{n+1}} 2\pi R$$

$$\left|f^{(n)}(a)\right| \le \frac{Mn!}{R^n}$$

Remark 1. If we take $a_n = \frac{f^{(n)}(a)}{n!}$, then $|a_n| \leq \frac{M}{R^n}$.

Remark 2. A function $f(z)$ is called an integral function or entire function if it is analytic in every finite region.

8.8 *LIOVILLE'S THEOREM:-*

If an entire function $f(z)$ is bounded for all values of z, then it is constant.

Or, If a function $f(z)$ is analytic for finite values of z, and is bounded, then $f(z)$ is constant.

Or, If f is regular in whole z-plane and if $|f(z)| < k\ \forall z$, then $f(z)$ must be constant.

Proof. Let a and b be arbitrary distinct points in z-plane and let C be a large circle with centre $z = 0$ and radius R such that C encloses a and b. Equation of C is $|z| = Rsothatz = Re^{i\theta}, dz = iRe^{i\theta}_{d\theta}|dz| = Rd\theta.$

$f(z)$ is bounded $\forall z \Rightarrow |f(z)| \leq M\ \ \forall\ z$ where $M > 0$.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}, f(b) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-b}$$

$$f(a) - f(b) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) f(z)dz$$

$$= \left(\frac{a-b}{2\pi i}\right)\int_C \frac{f(z)dz}{(z-a)(z-b)}$$

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi}\int_C \frac{|f(z)|\cdot|dz|}{(|z|-|a|)(|z|-|b|)}$$

$$\leq \frac{M|a-b|\cdot 2\pi R}{2\pi(R-|a|)\cdot R-|b|)}$$

or

$|f(a) - f(b)| \leq \frac{MR|a-b|}{(R-|a|)(R-|b|)} \to 0$ as $R \to \infty$.

$\therefore\ f(a) - f(b) = 0$ or $f(a) = f(b)$, showing thereby $f(z)$ is constant.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. If any entire function is bounded then it is constant.
Problem 2. If $f(z)$ is a continuous function in a domain D and if for every

closed contour C in the domain D, $\int_C f(z)dz = 0$ then $f(z)$ is analytic within D.

Problem 3. $\int_\gamma |dz| =$ circumference of the circle $\gamma = 2\pi$. radius

Problem 4. If $f(z)$ is analytic within and on a circle C, given by
$|z-a| = R$ and if $|f(z)| \leq M$ for every z on C, then
$\left|f^{(n)}(a)\right| \leq \frac{Mn!}{R^n}$.

8.9 *SUMMARY:*

1. Cauchy's integral formula. If $f(z)$ is analytic within and on a closed contour C, and if a is any point within C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}$
2. Poisson's Integral formula. If $f(z)$ is analytic within and on a circle C defined by $|z| = R$ and if a is any point within C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2-a\bar{a})f(z)dz}{(z-a)(R^2-z\bar{a})}$
3. If an entire function $f(z)$ is bounded for all values of z, then it is constant.

4. A function $f(z)$ is called an integral function or entire function if it is analytic in every finite region.

8.10 GLOSSARY :-

integration

Analytic function

8.11 *REFERENCES:-*

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8.12 *SUGGESTED READING:-*

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2. J.B. Conway, (2000), Functions of One Complex Variable, Narosa Publishing House.
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8.13 *TERMINAL AND MODEL QUESTIONS:-*

Q 1. States and prove Fundamental theorem of Integral Calculus.

Q 2. States and prove Cauchy integral formula for higher order derivative.

Q 3. States and prove Morera's theorem.

Q 4. Define Cauchy’s Roots Test.

Q 5. States and prove Poisson's Integral formula.

8.14 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. True

BLOCK III
POWER SERIES

UNIT-9: Power Series

CONTENTS

9.1 *INTRODUCTION:-*

A power series is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where a_n are complex coefficients and z_0 is a fixed complex number called the center of the series. In complex analysis, power series play a fundamental role because they provide a powerful way to represent complex functions locally with high precision and smoothness. This unit introduces the concept of power series, their convergence properties, and the radius and region of convergence, which determine where the series defines an analytic function. It also explores how term-by-term differentiation and integration preserve convergence within this region, allowing power series to serve as a bridge between algebraic expressions and analytic behavior of complex functions. Through these concepts, power series become essential tools for the study of analytic continuation, Taylor and Laurent expansions, and solving differential equations in the complex plane.

9.2 *OBJECTIVES:-*

The main objectives of the unit “power series” presented in point form:

- To define power series and understand their general structure and notation.
- To study the concepts of radius of convergence and interval/region of convergence, and learn methods for determining them.
- To explore the relationship between power series and analytic functions, including conditions under which a power series represents an analytic function.
- To understand the properties of power series such as term-by-term differentiation and integration, and how these operations affect convergence.
- To learn how power series can be used to express complex functions through Taylor and Maclaurin series expansions.
- To apply power series techniques in problem-solving, including function approximation and solving complex differential equations.
- To introduce the concept of uniqueness of power series representation and analytic continuation

9.3 *POWER SERIES:-*

Definition: A series of the form, $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-a)^n$

is called a power series, where,

a_n, a = complex constant

$z =$ complex variable.

The second form $\sum a_n (z-a)^n$ can be reduced to the first form by substitution $z = \zeta + a$ so that,

$\sum a_n (z-a)^n = \sum a_n \zeta^n$.

The first form is easier to work with than the second, so we focus only on the first.

$\sum a_n z^n$ *or* simply $\sum_{n=0}^{\infty} a_n z^n$.

9.3.1 *ABSOLUTE CONVERGENCE OF* $\sum a_n z^n$ *:-*

The power series $\sum a_n z^n$ is said to be absolute convergent if the series $\sum |a_n||z|^n$ is not convergent.

The power series $\sum a_n z^n$ is said to be conditionally convergent if $\sum a_n z^n$ is convergent but $\sum |a_n||z|^n$ is not convergent.

9.4 *TEST FOR CONVERGENCE OF SERIES* $\sum a_n z^n$:-

Here we have provided some list of tests and results which are helpful to find out the convergence of the series.

1. If $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

2. If $\lim_{n\to\infty} \frac{u_n}{v_n}$ =finite non-zero quantity, then the two series $\sum u_n$ and $\sum v_n$ have identical in nature.

3. **Comparison test:** $\sum u_n$ is absolutely convergent if $|u_n| \le |v_n|$ and $\sum v_n$ is convergent.

4. **Root test:** Let $\sum |u_n|^{1/n} = l$. Then series $\sum u_n$ is convergent (absolutely) or divergent according as $l < 1$ or $l > 1$. The test will fail if $l = 1$.

5. **Ratio test:** The series $\sum u_n$ is convergent or divergent according as, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ or > 1.

6: The series $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

7: **Dirichlet's test:** The series $\sum a_n u_n$ is convergent if

(i) $|s_n| = \left|\sum_{i=1}^{n} a_i\right| \le k \forall n, k$ being a finite number.

(ii) $\lim_{n\to\infty} u_n = 0$

(iii) $\sum (u_n - u_{n+1})$ is convergent.

9.5 *RADIUS OF CONVERGENCE OF POWER SERIES:-*

The radius of convergence of a power series is a fundamental concept in complex analysis that determines the region in which the series converges to an analytic function. Given a power series of the form $\sum a_n (z-z_0)^n$, the radius of convergence R specifies the distance from the center z_0 within which the series converges absolutely and uniformly on compact sets, and beyond which it diverges. This radius can be computed using tests such as the ratio or root test, and it reflects how the behavior of the coefficients a_n influences convergence. Understanding the radius of convergence is essential for analyzing where a power series represents a valid analytic function and how its domain of analyticity is determined by singularities in the complex plane.

Consider the power series $\sum a_n z^n = \sum u_n(z)$, say, $\sum u_n$ is convergent if,

$$\lim_{n\to\infty} |u_n|^{1/n} < 1.$$

This implies, $\lim_{n\to\infty} |a_n z^n|^{1/n} < 1 \Rightarrow \lim_{n\to\infty} |a_n|^{1/n} .|z| < 1$

Taking $\lim_{n\to\infty} |a_n|^{1/n} = \dfrac{1}{R}$, we get

$$\frac{|z|}{R} < 1 \qquad \text{or} \qquad |z| < R.$$

Hence, $\sum a_n z^n$ is convergent or divergent according as, $|z| < R$ or $|z| > R$.So, corresponding to every power series $\sum a_n z^n$, $\exists$ a non-negative number R such that $|z| < R$ if the series is convergent and $|z| > R$ if the series is divergent.

Now if we draw a circle of radius R with centre at origin, then

(*i*) The power series $\sum a_n z^n$ is convergent for every z within this circle ($|z| < R$).

(*ii*) The power series $\sum a_n z^n$ is divergent for every z outside the circle ($|z| > R$).

Such a circle is known as the **circle of convergence**, and its radius R is referred to as the **radius of convergence** of the power series $\sum a_n z^n$.

There are three possibilities for R.

(*i*) $R = 0$, in this situation, the series converges only at $z = 0$.

(*ii*) R is finite, in this situation, the series is convergent at each point within this circle and diverges at each point outside of the circle.

(*iii*) R is infinite, in this situation the series is convergent $\forall z$.

Note: $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

9.6 *SOME FUNCTION OF A POWER SERIES:-*

A **power series** defines a function, often called the *some function* of the series, within the region where the series converges. Given a series of the form $\sum a_n (z - z_0)^n$, this function is obtained by assigning to each point z in its interval or disk of convergence the value of the infinite sum. Within this region, the sum function is analytic, meaning it possesses derivatives of all orders and can itself be differentiated or integrated term-by-term. The behavior of this function is closely tied to the radius of convergence, beyond which the series no longer represents a meaningful value. Thus, the function of a power series provides a powerful tool for expressing and studying analytic functions in complex analysis.

If $f(z) = \sum a_n z^n$, then $f(z)$ is called some function of power series.

Theorem 1: The power series $\sum a_n z^n$ either,

(*i*) Converges for every z. or **(*ii*)** Converges only for $z = 0$

(*iii*) converges for some z .

Proof: It is enough to provide one example for each case.

(i) Consider the power series $\sum \frac{z^n}{n!}$. Now, comparing this with $\sum u_n(z)$, we find that

$$\frac{u_{n+1}}{u_n} = \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} = \frac{z}{(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{1}{(n+1)} \cdot |z| = 0 < 1$$

Hence the power series, $\sum \frac{z^n}{n!}$ is convergent for every *z*.

(ii) Consider the power series $\sum z^n n! = \sum u_n$, say

Then, $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} n!.|z|^n = \begin{cases} 0, \text{if } z = 0 \\ \infty, \text{if } z \neq 0 \end{cases}$

$\because \sum u_n$, i.e., $\because \sum n! z^n$ is convergent if $z = 0$ and divergent if $z \neq 0$.

(iii) The power series $\sum z^n$ is convergent if $|z| < 1$ and is not convergent if $|z| \geq 1$.

Theorem 2: If the power series $\sum a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for every *z* for which $|z| < |z_0|$.

Proof: Suppose the power series $\sum a_n z^n$ is convergent for $z = z_0$, so that $\sum a_n {z_0}^n$ is convergent. Consequently, $\lim_{n\to\infty} a_n z_0^n = 0$. … (1)

Our aim is to prove that $\sum a_n z^n$ is convergent for every *z* for which $|z| < |z_0|$.

(1) $\Rightarrow \exists$ a real positive constant $M > 0$ s.t., $\left|a_n z^n\right| \leq M \; \forall \, n$.

Now, $\left|a_n z^n\right| \leq \frac{M}{|z_0|^n} \cdot |z^n| = M\left|\frac{z}{z_0}\right|^n$

Or $\left|a_n z^n\right| \leq M\left|\frac{z}{z_0}\right|^n$. But the geometric series $\sum \frac{|z|^n}{|z_0|^n}$ is convergent $\forall \, z$ s.t.

$\frac{|z|}{|z_0|} < 1$ *i.e.*, $|z| < |z_0|$.

$\therefore$ By comparison test $\sum |a_n z^n|$ is convergent $\forall z$ s.t., $|z| < |z_0|$. Consequently, $\sum a_n z^n$ is absolutely convergent $\forall z$ s.t., $|z| < |z_0|$.

Remarks: For every power series $\sum a_n z^n$, there exists a number R such that $0 \le R \le \infty$ with the following properties.

(i) The series converges absolutely for every z such that $|z| < R$.

(ii) The series divergent if $|z| > R$

This statement is called Cauchy Hadamard theorem.

Theorem 3: To show that the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$, obtained by differentiating the power series $\sum_{n=0}^{\infty} a_n z^n$, has the same radius of convergence as the original series $\sum_{n=0}^{\infty} a_n z^n$.

Proof: Let us consider R and R' are the radius of convergence of the series.

$\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ respectively.

Then, $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}, \lim_{n\to\infty} |n a_n|^{1/n} = \frac{1}{R'}$.

Now, we have only to show that $R = R'$.

For this we have only to prove that $\lim_{n\to\infty} n^{1/n} = 1$. Now. By Cauchy theorem on limit

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+!}}{a_n}$$

So, $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$

Theorem 4: (Analyticity of the power series): The sum function $f(z)$ of the power series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside the circle of convergence.

Or

If the radius of convergence of power series $\sum_{n=0}^{\infty} a_n z^n$ is a positive real number R, then prove that the function $f(z)$ defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $|z| < R$.

Proof: Suppose, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

If R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ then it will also be the power series of $\sum_{n=0}^{\infty} n a_n z^{n-1}$ which is an derived series of $\sum_{n=0}^{\infty} a_n z^n$ (By theorem 3).

Let us consider a point z within the circle $|z| = R$ so that $|z| < R$. Also $\exists$ a real number $r > 0$ s.t. $|z| < r < R$. Then the series $\sum_{n=0}^{\infty} a_n z^n$ is convergent for $|z| < R$ so that $\sum_{n=0}^{\infty} a_n r^n$ is bounded.

This implies a finite real number $M > 0$ s.t. $|a_n r^n| < M$.

For the sake of convenience, we write $|z| = \rho, |h| = \in$

h is chosen so that, $|z| + |h| < r$.

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| = \left| \sum a_n \left[\frac{(z+h)^n - z^n}{h} - n z^{n-1} \right] \right|$$

$$\leq \sum_{n=0}^{\infty} |a_n|.\left[\frac{(\rho+\in)^n-\rho^n}{\varepsilon}-n\rho^{n-1}\right]$$

$$< \sum_{n=0}^{\infty} \frac{M}{\in r^n}.\left[(\rho+\in)^n-\rho^n-n\in\rho^{n-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{M}{\in}.\left[(\rho+\in)^n-\rho^n-n\in\rho^{n-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{M}{\in}.\left[\left(\frac{\rho+\in}{r}\right)^n-\left(\frac{\rho}{r}\right)^n-\frac{n\in}{\rho}\left(\frac{\rho}{r}\right)^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{M}{\in}.\left[\frac{1}{1-\left(\frac{\rho+\in}{r}\right)}-\frac{1}{1-\left(\frac{\rho}{r}\right)}-\frac{\in}{\rho}\sum_{n=0}^{\infty} n\left(\frac{\rho}{r}\right)^n\right] \qquad \ldots (1)$$

But $\dfrac{1}{1-\left(\frac{\rho+\in}{r}\right)}=\dfrac{r}{r-\rho-\in}$ … (2)

Let $A=\sum_{n=0}^{\infty} n\left(\frac{\rho}{r}\right)^n$. Then

$$A=\frac{\rho}{r}+2\left(\frac{\rho}{r}\right)^2+3\left(\frac{\rho}{r}\right)^3+\ldots$$

$$A\frac{\rho}{r}=\left(\frac{\rho}{r}\right)^2+2\left(\frac{\rho}{r}\right)^3+\ldots$$

Subtracting, $A\left(1-\frac{\rho}{r}\right)=\frac{\rho}{r}+\left(\frac{\rho}{r}\right)^2+\left(\frac{\rho}{r}\right)^3=\ldots=\dfrac{\frac{\rho}{r}}{1-\frac{\rho}{r}}$ (By sum infinite series of geometric progression)

$$A = \frac{\frac{\rho}{r}}{\left(1-\frac{\rho}{r}\right)^2} = \frac{r\rho}{(r-\rho)^2} \qquad \ldots (3)$$

Writing (1) with the help of (3)

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| < \frac{M}{\varepsilon}\left[\frac{r}{r-\rho-\varepsilon} - \frac{r}{r-\rho} - \frac{\varepsilon r}{(r-\rho)^2}\right]$$

$$= \frac{Mr}{\varepsilon}\left[\frac{1}{r-\rho-\varepsilon} - \frac{r-\rho+\varepsilon}{(r-\rho)^2}\right]$$

$$= \frac{Mr.\varepsilon^2}{\varepsilon(r-\rho-\varepsilon)(r-\rho)^2}$$

Or $\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| < \frac{Mr.\varepsilon}{(r-\rho-\varepsilon)(r-\rho)^2}$

Making $h \to 0$, so that $|h| = \varepsilon \to 0$, we get

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \le 0.$$

Consequently, $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = g(z)$. For modulus of any quantity ≥ 0

Or $f'(z) = g(z)$.

But $g(z)$ exists so that $f'(z)$ exists $\forall\, z$ s.t. $|z| < R$.

It means that $f(z)$ is analytic for $|z| < R$ and has $g(z)$ as its derivative. Since $g(z)$ is itself a power series with the same radius of convergence, we may differentiate it to obtain the second derivative of the original series. Using similar reasoning, one can show that this second derivative also remains analytic within the same circle of radius $|z| = R$. Moreover the derivatives can be found by term by term differentiation. Thus the sum of function of a power series represents an analytic function inside the circle of convergence.

Corollary 1: The function defined by a power series is continuous throughout any domain completely contained within its circle of convergence.

Corollary 2: A power series $\sum_{n=0}^{\infty} a_n z^n$ may be integrated term by term along any contour Γ that lies entirely inside its circle of convergence.

Corollary 3: A power series $\sum_{n=0}^{\infty} a_n z^n$ may be differentiated term by term in every region located within its circle of convergence.

Example 1: Evaluate the radius of convergence of the following power series.

(*i*) $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$ **(*ii*)** $\sum_{n=0}^{\infty} \frac{2^{-n} z^n}{1+in^2}$ **(*iii*)** $\sum_{n=0}^{\infty} \frac{(n!)^2 z^n}{(2n)!}$ **(*iv*)** $\sum_{n=0}^{\infty} \frac{n! z^n}{n^n}$

Solution (*i*): We have $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$.

Comparing this series with $\sum_{n=0}^{\infty} a_n z^n$, we get $a_n = \frac{1}{n^n}$.

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$\frac{1}{R} = 0$ so that, $R = \infty$.

(ii) We have here, $a_n = \frac{2^{-n}}{1+in^2}$.

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left[\frac{2^{-n}}{\sqrt{(1+n^4)}}\right]^{1/n} = \lim_{n\to\infty} \frac{1}{2(1+n^4)^{1/2n}}$$

$$= \lim_{n\to\infty} \frac{(1+n^4)^{-1/2n}}{2} = \lim_{n\to\infty} \frac{1}{2(n^2)^{1/n}}\left[1 - \frac{1}{2n^5} + ...\right]$$

$$= \frac{1}{2}.\lim_{n\to\infty} \frac{1}{(n^{1/n})^2}\left[1 - \frac{1}{2n^5} + ...\right]$$

$$= \frac{1}{2}\times 1\times 1 = \frac{1}{2}.\text{ For } \lim_{n\to\infty} n^{1/n} = 1$$

$\therefore R = 2$.

(iii) Since we have, $a_n = \dfrac{(n!)^2}{2n!}$. Then

$$a_{n+1} = \frac{((n+1)!)^2}{\{2(n+1)\}!} = \frac{[(n+1)n!]^2}{(2n+2)(2n+1)2n!}$$

Or $\dfrac{a_{n+1}}{a_n} = \dfrac{((n+1)!)^2}{\{2(n+1)\}!} = \dfrac{(n+1)^2}{(2n+2)(2n+1)} = \dfrac{n+1}{2(2n+1)} = \dfrac{1+1/n}{4(1+1/2n)}$

$1/R = \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{1+0}{4(1+0)} = 1/4$. Hence radius of convergence is $R = 4$

(*iv*) Here, $a_n = \dfrac{n!^2}{n^n}$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$$

So, $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{(n+1)^{n+1}} = \dfrac{n^n}{(n+1)^n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n}$

$1/R = \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n} = 1/e$. Hence, $R = e$.

Example 2: Compute the radius of convergence of the power series listed below.

(*i*) $\sum_{n=0}^{\infty}(3+4i)^n z^n$ **(*ii*)** $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$ **(*iii*)** $\sum_{n=0}^{\infty}\frac{1}{n^p}z^n$

Solution (*i*): We have given, $\sum a_n z^n = \sum_{n=0}^{\infty}(3+4i)^n z^n$.

$\Rightarrow a_n = (3+4i)^n$. Then $|a_n| = |(3+4i)^n| = \sqrt{[3^2+4^2]^n} = 5^n$.

$$\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}(5^n)^{1/n} = 5 \text{ Or } R = \frac{1}{5}$$

(*ii*) We have $\sum a_n (z-a)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$ *i.e.*, $a_n = \frac{(-1)^n}{n}$ and $a = 2i$

Now, $a_{n+1} = \frac{(-1)^{n+1}}{n+1} \Rightarrow \frac{a_{n+1}}{a_n} = -\left(\frac{n}{n+1}\right)$.

So, $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+!} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = 1$. Hence, $R = 1$

(*iii*) We have given $\sum a_n z^n = \sum_{n=0}^{\infty}\frac{1}{n^p}z^n$. After comparing we get $a_n = \frac{1}{n^p}$ so that,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} = \left(\frac{1}{1+\frac{1}{n}}\right)^p = \left(\frac{1}{1+0}\right)^p = 1. \text{ Hence, } R = 1$$

Example 3: Prove that the series, $1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+....$ has unit radius of convergence.

Solution: Neglecting the first term,

$$a_n=\frac{a(a+1)...(a+n-1)\,b(b+1)...(b+n-1)}{1.2...n.c(c+1)...(c+n-1)}$$

$$a_{n+1}=\frac{a(a+1)...(a+n-1)\,(a+n)b(b+1)...(b+n-1)(b+n)}{1.2...n.(n+1)c(c+1)...(c+n-1)(c+n)}$$

Dividing, $\frac{a_{n+1}}{a_n}=\frac{(a+n)(b+n)}{(n+1)(c+n)}=\frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}$

$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{(1+0)(1+0)}{(1+0)(1+0)}=1$. Hence, $R=1$.

Example 4: Find the radius of convergence of the series $\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+....$

Solution: The coefficient of z^n of the given power series is given by,

$a_n=\frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$ then $a_{n+1}=\frac{1.3.5...(2n+1)}{2.5.8...(3n+2)}$

Now, $\frac{a_{n+1}}{a_n}=\frac{(2n+1)}{(3n+2)}=\frac{2}{3}.\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$. Then, $\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{2.(1+0)}{3.(1+0)}=\frac{2}{3}$.

Hence, $R=\frac{3}{2}$.

Theorem 5 (Abel's theorem on limit): If $\sum a_n z^n$ converges, then $f(z)=\sum a_n z^n$ approaches to $f(1)$ as $z \to 1$ in such a manner that $\frac{|1-z|}{1-|z|}$ remains bounded.

Proof: We 1have already given that $\sum a_n z^n$ is convergent and $\frac{|1-z|}{1-|z|}$ is bounded. Now we have to prove that for $f(z) \to f(1)$ as $z \to 1$.

Without loss of generality, we may assume $\sum a_n = 0$ … (1)

Since this can be obtained by adding suitable constant to a_0. According to equation (3) of theorem 4, $\exists K > 0$ s.t. $\frac{|1-z|}{1-|z|} \le K$ … (2)

Write, $s_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ … (3)

Then $s_n(1) = a_0 + a_1 + a_2 + ... + a_n = s_n$, say

Making $n \to \infty$ then we obtaion, $\lim_{n \to \infty} s_n = 0$ … (4)

Applying definition of limit, given $\varepsilon > 0, \exists$ a positive integer m s.t.,

$\forall n \ge m \Rightarrow |s_n - 0| < \varepsilon$

Or, $|s_n| < \varepsilon \,\forall n \ge m$ … (5)

By (2), $f(1) = \sum_{n=0}^{\infty} a_n (1)^n = \sum_{n=0}^{\infty} a_n = 0$

Or, $f(1) = 0$ … (6)

then by (3)

$s_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$

$= s_0 + (s_1 - s_0)z + (s_2 - s_1)z^2 + ... + (s_n - s_{n-1})z^n$

$$= s_0(1-z) + s_1(z-z^2) + ... + s_{n-1}(z^{n-1} - z^n) + s_n z^n ,$$

Making $n \to \infty$, $\lim_{n\to\infty} s_n(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n + \lim_{n\to\infty} s_n z^n$

Using (4), we get

$$\lim_{n\to\infty} s_n(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n + \sum_{n=0}^{\infty} s_n z^n$$

Using (4), we get.

$$\lim_{n\to\infty} s_n(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$$

Or $f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$

$$= (1-z)\left[\sum_{n=0}^{m-1} s_n z^n + \sum_{n=m}^{\infty} s_n z^n\right]$$

$$\therefore |f(z)| \le |1-z| \left[\left|\sum_{n=0}^{m-1} s_n z^n\right| + \sum_{n=m}^{\infty} |s_n| . |z^n|\right]$$

$$\le |1-z| \left[\left|\sum_{n=0}^{m-1} s_n z^n\right| + \varepsilon \sum_{n=m}^{\infty} |z|^n\right]$$ [On summing the geometric progression]

$$\le |1-z| \left[\left|\sum_{n=0}^{m-1} s_n z^n\right| + \varepsilon \frac{|1-z| . |z|^m}{1-|z|}\right]$$

$$\le |1-z| \left[\left|\sum_{n=0}^{m-1} s_n z^n\right| + K\varepsilon |z|^m\right]$$

The first term on the right-hand side can be made arbitrarily small by choosing sufficiently close to 1. This implies that $f(z) \to 0$ as $z \to 1$. But $f(1) = 0$. $f(z) \to f(1)$ as $z \to 1$.

Example 5: Find the domain of convergence of the power series $\sum\left(\frac{2i}{z+i+1}\right)^n$

Solution: Since we have $u_n = \left(\frac{2i}{z+i+1}\right)^n$ then $u_{n+1} = \left(\frac{2i}{z+i+1}\right)^{n+1}$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2i}{z+i+1}\right| = \frac{2}{|z+1+i|}$

As we know that the given series is convergent if $\frac{2}{|z+1+i|} < 1$ i.e., $|z+1+i| > 2$

Hence, the series converges at every point lying outside the circle with center at $z = -(1+i)$ and radius 2.

Check your progress

Problem 1: Find the radius of convergence of the power series $\sum_{n=1}^{\infty}(\log)^n z^n$.

Answer: $R = 0$

Problem 2: Find the radius of convergence of the power series $\sum_{n=1}^{\infty}\left(\frac{n\sqrt{2}+i}{1+2in}\right)z^n$. R^1

Answer: $R = 1$

9.7*SUMMARY:-*

The unit "Power Series" in complex analysis explores infinite series of the form $\sum a_n(z-z_0)^n$, emphasizing their fundamental role in representing analytic functions within a certain region. It introduces the concepts of the radius and circle of convergence**,** which determine where the series converges or diverges, and explains how tests such as the ratio and root tests are used to compute convergence regions. This uni highlights that power series converge absolutely and uniformly inside their circle of convergence and can be differentiated or integrated term-by-term, preserving analyticity. It also discusses boundary behavior, noting that convergence at the boundary is not guaranteed and must be checked separately. Overall, the unit provides a foundational understanding of how power series serve as powerful tools for expanding, analyzing, and approximating analytic functions in the complex plane.

9.8 *GLOSSARY: -*

- Power series
- Absolute convergence of power series
- Convergence of power series
- Radius of convergence of power series
- Some function of power series

9.9 *REFERENCES:-*

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9.10 *SUGGESTED READING:-*

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- J.B. Conway, (2000), Functions of One Complex Variable, Narosa Publishing House,
- E.T. Copson, (1970), Introduction to Theory of Functions of Complex Variable, Oxford University Press.

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9.11 *TERMINAL QUESTIONS:-*

Long Answer Type Question:

1: Find the domain of convergence of following power series

(a) $\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$ **(b)** $\sum_{n=1}^{\infty}\left(\frac{iz-1}{2+i}\right)^n$

2: Determine the behaviour of $\sum \frac{z^n}{n}$ on the circle of convergence.

3: Determine the behaviour of $\sum \frac{z^{4n}}{1+4n}$ on the circle of convergence.

4: Which value of z the series $\sum \frac{1}{(1+z^2)^n}$ is converges and also find its sum.

5: State and prove the Abel's theorem.

Short answer type question:

1: Define the power series and radius of convergence of power series

2: Write down some test to check the convergence of power series.

3: Find the domain of convergence of following power series.

(***i***) $\sum_{n=1}^{\infty} n!z^n$ (***ii***) $\sum_{n=1}^{\infty}\left({}^{2i}\!/_{z+i+1}\right)^n$

4: Determine the region of convergence of the following series

(***i***) $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$ (***ii***) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$

5: Show that the radius of convergence of the power series

$\sum_{n=1}^{\infty}\left(\frac{iz}{2+i}\right)^n$ is given by $|z+i|<\sqrt{5}$.

Objective type questions:

1. A power series $\sum a_n (z-z_0)^n$ converges absolutely for:

a) All z

b) $|z-z_0|<R$

c) $|z-z_0|>R$

d) Only at $|z=z_0$

2. **The radius of convergence *R* of a power series can be found using:**

a) Cauchy–Riemannequations

b) Ratiotest

c) Green'stheorem

d) Stokes' theorem

3. If $\sum a_n z^n$ has radius of convergence *R*, then the series diverges for:

a) $|z|<R$

b) $|z|=R$

c) $|z|>R$

d) Both (b) and (c)

4. For the power series $\sum \frac{z^n}{n!}$, the radius of convergence is:

a) 0

b) 1

c) ∞

d) Undefined

5. The function represented by a power series is:

a) Always analytic inside its circle of convergence

b) Neveranalytic

c) Analytic only at the center

d) Analytic only if the coefficients are real

6. If R_1 and R_2 are radii of convergence for two series, the product's radius is:

a) R_1+R_2

b) $\min(R_1, R_2)$
c) $\max(R_1, R_2)$
d) $R_1 R_2$

7. A power series $\sum a_n (z-2)^n$ has center at:
a) 0
b) 1
c) 2
d) -2

8. The interval or region of convergence of a power series is usually:
a) A line segment
b) Entire plane
c) A disk centered at the expansion point
d) A rectangle in the plane

9. The series $\sum nz^n$ has radius of convergence:
a) 0
b) 1
c) 2
d) ∞

10. At the boundary $|z - z_0| = R$
a) The series always converges
b) The series always diverges
c) Convergence depends on individual terms
d) The series becomes a polynomial

Fill in the blanks:

1: A power series is generally written in the form $\sum a_n (z-z_0)^n$, where z_0 is called the ________.

2: The set of points where a power series converges forms a ________ in the complex plane.

3: The radius of convergence R of $\sum a_n z^n$ can be found using the ratio test $R = \dfrac{1}{\limsup |a_{n+1}/a_n|}$ provided the limit exists, which is known as the ________ formula.

4: Inside the circle of convergence, a power series represents an ________ function.

5: If the radius of convergence of a series is R, then the series converges absolutely for all points satisfying $|z - z_0| <$ ________.

6: The power series $\sum \frac{z^n}{n!}$ has a radius of convergence equal to ________.

7: At the boundary $|z - z_0| = R$, the convergence of a power series is ________.

8: The largest disk in which a power series converges is called the ________ of convergence.

9: A power series $\sum a_n z^n$ converges only at $z = 0$ if its radius of convergence is ________.

10: If a power series converges at a point z_1, then it converges for all z such that $|z - z_0| < |z_1 - z_0|$. This is known as the ________ property of convergence.

9.12 *ANSWERS*

Answer of long answer type questions:

Answer 1(a): This shows that the series is convergent inside the circle of radius 2/3 and centre at $z = 4/3$**.**

(b): This series is convergent for set of values of z which lie inside the circle of radius $\sqrt{5}$ and center at $z = -i$

Answer 2: The series $\sum \frac{z^n}{n}$ is convergent for every value of z other than 1 on the circle of convergence.

Answer 3: The series $\sum \frac{z^{4n}}{1+4n}$ is convergent for all values of z on the circle except $z = \pm 1, \pm i$.

Answer 4: The series is convergent for $|z^2+1|>1$ and $\sum \frac{z^{4n}}{1+4n} = \frac{1}{z^2}$ as $\frac{1}{|z^2+1|}<1.$

Answer of short answer type questions:

Answer 3(*i*): The series is not convergent $\forall\, z$ except $z=0$

(*ii*) The series converges for all points located outside the circle of radius 2 centered at $z=-(1+i)$.

Answer 4(i): Series converges absolutely, for $|z+2|<4$

(*ii*): For all values of *z*, series converges absolutely.

Answer of objective type question:

1: b **2:** b **3:** c **4:** c

5: a **6:** b **7:** c **8:** c

9: b **10:** c

Answer of fill in the blanks

1: Center **2:** disk **3:** ratio **4:** analytic

5: R **6:** ∞ **7:** uncertain **8:** circle

9: 0 **10:** monotonic

UNIT-10:Expansion of Analytic Functions

CONTENTS

10.1 *INTRODUCTION:-*

In complex analysis, the expansion of analytic functions provides powerful tools for representing complex functions in series form around a point. If a function is analytic within a neighbourhood of a point z_0, it can be expressed as a Taylor series, which is a power series involving non-negative integer powers of $(z - z_0)$. This representation not only simplifies computation but also reveals important local properties of the function. However, when a function is analytic in an annular region (a ring-shaped domain) around z_0 but not necessarily at z_0 itself, it can be expressed in a more general form known as the Laurent series, which includes both positive and negative powers of $(z - z_0)$. The Laurent series thus extends the idea of the Taylor series to functions with isolated singularities, enabling deeper analysis of their behaviour near such points and forming the foundation for concepts like residues and contour integration in complex analysis.

10.2 *OBJECTIVES:-*

The objectives of the "Expansion of Analytic Function" unit in complex analysis are to study how analytic functions can be represented in the form of infinite series and to understand the importance of such expansions in complex function theory. This chapter aims to:

- Explain the concept and derivation of the Taylor series for functions that are analytic within a certain region;
- Introduce the Laurent series for functions that are analytic in an annular region, allowing for the inclusion of negative powers;
- Distinguish between the regions of convergence for both series
- Apply these series expansions to identify and classify singularities and to solve problems involving residues and contour integration. Through these objectives, students develop a deeper insight into the behaviour and structure of analytic functions in the complex plane.

10.3 *TAYLOR'S THEOREM:-*

In the complex analysis, Taylor's theorem provides a way to express an analytic function as an infinite power series around a given point within its region of analyticity.

Theorem 1: If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then at every point z inside C,

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(z-a)^n}{n!} \text{ or } f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

Where,

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Proof: Let $f(t)$ be analytic within a circle C whose equation is $|t-a| = R$. Let z be any point within C s.t. $|z-a| = r < R$.

By Cauchy's integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} \\ &= \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)-(z-a)} \\ &= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1} dt \end{aligned}$$

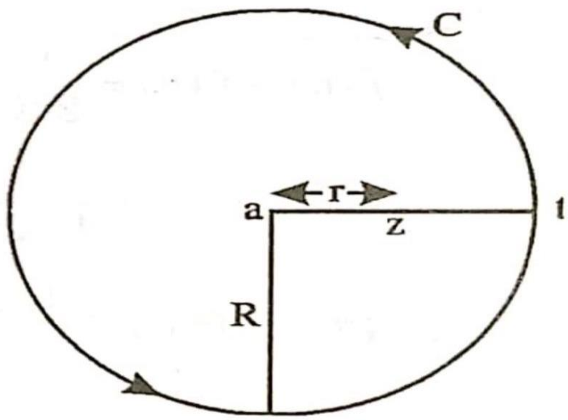

Figure: 1

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \cdots + \left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1}\left(1 - \frac{z-a}{t-a}\right)^{-1}\right]dt$$

$$\left[\text{For } \frac{1}{1-b} = (1-b)^{-1} = 1 + b + b^2 + \cdots + b^n + \frac{b^{n+1}}{1-b}\right]$$

Using the formula, $\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}$, we get

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \ldots + (z-a)^n\frac{f^{(n)}(a)}{n!} + U_{n+1} \quad \ldots (1)$$

where $U_{n+1} = \frac{(z-a)^{1+n}}{2\pi i}\int_C \frac{f(t)dt}{(t-z)(t-a)^{n+1}}$

$$\therefore\ |U_{n+1}| \le \frac{|z-a|^{n+1}}{2\pi}\int_C \frac{|f(t)|\cdot|dt|}{(|t-a| - |z-a|)|t-a|^{n+1}}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\cdot\frac{1}{(R-r)}\cdot 2\pi R \text{ where } M = \max. |f(t)| \text{ on } C.$$

or $|U_{n+1}| \le M.\left(\frac{r}{R}\right)^{n+1}\cdot\frac{1}{1-\left(\frac{r}{R}\right)} \to 0$ as $n \to \infty$.

For $\lim_{n\to\infty}\left(\frac{r}{R}\right)^{n+1} = 0$ as $\frac{r}{R} < 1$.

$\therefore\ \lim_{n\to\infty} U_{n+1} = 0.$

$$f(z) = \lim_{n\to\infty}\left[f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a)\right]$$

or

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^{(n)}(a) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad \dots (2)$$

$$\text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

Note: The above theorem can also be restated as: Let $f(z)$ be analytic at all points within a circle C_0 with its centre z_0 and radius R. Let z be any point inside C_0. Then prove that

$$f(z) = f(z_0) + \sum_{n=1}^{\infty}\frac{(z-z_0)^n}{n!}f^{(n)}(z_0).$$

Deduction: Since z is a point within the circle $|t-a| = R$ s.t. $|z-a| = r < R$ so that we can take $z = a+h, h = z-a$.

Putting in (2), $f(a+h) = \sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(a)$

or

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

This is alternative form to Taylor's series.
(ii) If we write $a = 0$ in (2), then we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \; f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)z^n}{n!}$$

This is also known as Maclaurin's series.
(iii) The domain of convergence of the series (2) is given by $|z-a| < R$, where the radius R of convergence is the distance from a to the nearest singularity of the function $f(z)$. On the circle $|z-a| = R$, the series may or may not converge.

Taylor's theorem is significant because it allows an analytic function to be expressed as a polynomial-like series, making it easier to study its local behaviour, compute function values, and analyse properties such as differentiation and integration in the complex plane.

10.4 *LAURENT'S THEOREM:-*

In the complex analysis, Laurent's theorem extends the idea of Taylor's Theorem to functions that are analytic not in a full disk but in an annular region (a ring-shaped domain) around a point.

Theorem 2: Laurent's Theorem. Suppose a function $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R', $(R' < R)$. If z is any point of the annulus, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}, b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(t)dt}{(t-a)^{-n+1}}$$

Proof: Let $f(z)$ be analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and $R', (R' < R)$. Then if z is any point within the ring space, then

$$R' < |z-a| = r < R.$$

Here we shall make use of the following facts :

(i) $\frac{1}{1-b} = (1-b)^{-1} = 1 + b + b^2 + \cdots + b^n + \frac{b^{n+1}}{1-b}$

(ii) $\left[1 - \frac{t-a}{z-a}\right]^{-1} = \frac{1}{1-[(t-a)/(z-a)]} = \frac{z-a}{z-t}$

(iii) $\lim_{n\to\infty} \left(\frac{r}{R}\right)^n = 0 = \lim_{n\to\infty} \left(\frac{R'}{r}\right)^n$ as $\frac{r}{R} < 1, \frac{R'}{r} < 1.$

(iv) $\int_C |dt| = 2\pi.$ radius of circle $C =$ circumference.

By extension to Cauchy's integral formula,

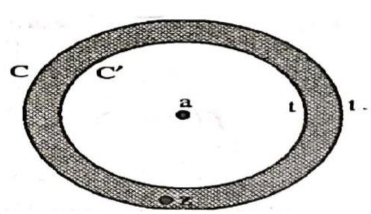

Figure: 2

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}-\frac{1}{2\pi i}\int_{C'}\frac{f(t)dt}{t-z}$$
$$=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)-(z-a)}+\frac{1}{2\pi i}\int_{C'}\frac{f(t)dt}{(z-a)-(t-a)}$$
$$=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1}dt+\frac{1}{2\pi i}\int_C \frac{f(t)}{z-a}\left[1-\left(\frac{t-a}{z-a}\right)\right]^{-1}dt$$

$$=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots+\left(\frac{z-a}{t-a}\right)^n+\left(\frac{z-a}{t-a}\right)^{n+1}\left\{1-\left(\frac{z-a}{t-a}\right)\right\}^{-1}\right]dt$$
$$+\frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left[1+\left(\frac{t-a}{z-a}\right)+\left(\frac{t-a}{z-a}\right)^2+\cdots+\left(\frac{t-a}{z-a}\right)^n+\left(\frac{t-a}{z-a}\right)^{n+1}\left\{1-\frac{t-a}{z-a}\right\}^{-1}\right]dt$$

Taking $a_n=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}$,

$$b_n=\frac{1}{2\pi i}\int_{C'}\frac{f(t)dt}{(t-a)^{-n+1}}=a_{-n}$$

$$f(z)=[a_0+(z-a)a_1+(z-a)^2a_2+\cdots+a_n(z-a)^n+U_{n+1}]$$
$$+\left[\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_n}{(z-a)^n}+V_{n+1}\right] \quad ...(1)$$

where $U_{n+1}=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}dt$,

$$V_{n+1}=\frac{1}{2\pi i}\int_{C'}\frac{f(t)}{z-t}\left(\frac{t-a}{z-a}\right)^{n+1}dt$$

Let $M=\max.|f(t)| \text{on} C, M'=\max.|f(t)| \text{ on } C'$.

$$|U_{n+1}|\le\frac{1}{2\pi}\int_C |f(t)|\cdot\left|\frac{z-a}{t-a}\right|^{n+1}\frac{|dt|}{(|t-a|-|z-a|)}$$
$$\le\frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{2\pi R}{(R-r)}$$

or $|U_{n+1}| \leq M\left(\frac{r}{R}\right)^{n+1} \cdot \frac{1}{1-(r/R)} \to 0$ as $n \to \infty$

Hence $\lim_{n\to\infty} U_{n+1} = 0$

$$|V_{n+1}| \leq \frac{1}{2\pi}\int_{C'} |f(t)| \cdot \left|\frac{t-a}{z-a}\right|^{n+1} \cdot \frac{|dt|}{(|z-a|-|t-a|)}$$
$$\leq \frac{M'}{2\pi}\left(\frac{R'}{r}\right)^{n+1} \frac{2\pi R'}{(r-R')}$$

or $|V_{n+1}| \leq M'\left(\frac{R'}{r}\right)^{n+1} \cdot \frac{1}{(r/R')-1} \to 0$ as $n \to \infty$

Hence $\lim_{n\to\infty} V_{n+1} = 0$

Making $n \to \infty$ in (1) and noting the above facts,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \quad \ldots (2)$$

Deduction. Take C_0 a circle whose equation is

$$R' < |t-a| = R_0 < R.$$

Then

$$a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{n+1}}, b_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{-n+1}} = a_{-n}$$

In this event (2) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} (z-a)^{-n} a_{-n}$$
$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=-1}^{\infty} (z-a)^n a_n = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \text{ with } a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}$$

10.5 *UNIQUENESS OF LAURENT'S EXPANSION:-*

The uniqueness of Laurent's expansion is an important property established in the expansion of analytic function chapter in complex analysis.

Theorem 3: (Uniqueness of Laurent expansion) Suppose that we have obtained in any manner or as the definition of $f(z)$, the formula

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n, R' < |z-a| < R$$

Is the series necessarily identical with the Laurent's series?
Proof: Let $f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$

To prove that (1) is identical with the Laurent's expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \text{ ... (2) with } a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{n+1}}$$

If we show that $A_n = a_n$, the result will be proved. Equation to C_0 is $|t - a| = r$, i.e. $t - a = re^{i\theta}, R' < r < R$.

$$a_n = \frac{1}{2\pi i}\int_{C_0} \sum_{m=-\infty}^{\infty} A_m (t-a)^m \frac{dt}{(t-a)^{n+1}}$$

$$= \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} A_m \int_{C_0} (t-a)^{m-n-1} dt$$

$$= \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} A_m \int_0^{2\pi} r^{m-n-1} e^{i(m-n-1)\theta} ire^{i\theta} d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} A_m r^{m-n} \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

$$\text{If } m \neq n, \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

$$= \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = 0 \text{ as } e^{2p\pi i = 1}$$

$$\text{If } m = n, \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \int_0^{2\pi} e^0 d\theta = 2\pi$$

$$\therefore\ a_n = \frac{1}{2\pi} A_n \cdot r^{n-n} \cdot 2\pi = A_n$$

Example 1: Obtain the Taylor and Laurent's series which represents the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions

(*i*) $|z| < 2$,
(*ii*) $2 < |z| < 3$,
(*iii*) $|z| > 3$.

Solution: Suppose $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$

or

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \quad \dots (1)$$

(*i*) When, $|z| < 2$, then $\frac{|z|}{2} < 1$

$$\begin{aligned} f(z) &= 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1} \\ &= 1 + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right] \\ &= 1 + \frac{3}{2}\sum_0^\infty (-1)^n \frac{z^n}{2^n} - \frac{8}{3}\sum_0^\infty (-1)^n \frac{z^n}{3^n} \\ &= 1 + \sum_0^\infty (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n \ \text{Ans.} \end{aligned}$$

This is Taylor's series valid for $|z| < 2$

(*ii*) When, $2 < |z| < 3$.

Then $\frac{2}{|z|} < 1, \frac{|z|}{3} < 1$.

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$
$$= 1 + \frac{3}{z}\sum_0^\infty (-1)^n \frac{2^n}{z^n} - \frac{8}{3}\sum_0^\infty (-1)^n \frac{z^n}{3^n}$$
$$= 1 + \sum_0^\infty (-1)^n \left[\frac{3.2^n}{z^{n+1}} - \frac{8z^n}{3^{n+1}}\right]$$

This is Laurent's series in the annulus $2 < |z| < 3$.

(***iii***) When, $|z| > 3$, then $\frac{3}{|z|} < 1, \frac{2}{|z|} < \frac{2}{3} < 1$.

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$
$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$
$$= 1 + \frac{3}{z}\sum_0^\infty (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_0^\infty (-1)^n \left(\frac{3}{z}\right)^n$$

$= 1 + \sum_0^\infty \frac{(-1)^n}{z^{n+1}} [3 \cdot 2^n - 3^n \cdot 8]$ Ans.

Example 2: Expand $\frac{1}{z^2-3z+2}$ for

(***i***) $0 < |z| < 1$,

(***ii***) $1 < |z| < 2$,

(***iii***) $|z| > 2$.

Solution: Let $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} \Rightarrow f(z) = \frac{1}{z-2} - \frac{1}{z-1}$ ... (1)

Then

(***i***) When $0 < |z| < 1$.

From (1),

$$f(z) = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= \sum_0^\infty z^n - \frac{1}{2}\sum_0^\infty \left(\frac{z}{2}\right)^n$$
$$= \sum_{n=0}^\infty \left(1 - \frac{1}{2^{n+1}}\right) z^n$$

This is **Maclaurin's expansion** in case $0 < |z| < 1$.

(*ii*) When $1 < |z| < 2$.

Then $\frac{1}{|z|} < 1, \frac{|z|}{2} < 1$.

Now (1) is expressible as

$$f(z) = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\sum_{0}^{\infty} \left(\frac{z}{2}\right)^{n} - \frac{1}{z}\sum_{0}^{\infty} z^{-n}.$$

This is Laurent's expansion in the annulus $1 < |z| < 2$.

(iii) When $|z| > 2$.

Then $\frac{2}{|z|} < 1$, so that $\frac{1}{|z|} < \frac{1}{2} < 1$.

Now (1) is expressible as

$$f(z) = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{0}^{\infty} \left(\frac{2}{z}\right)^{n} - \frac{1}{z}\sum_{0}^{\infty} \left(\frac{1}{z}\right)^{n}$$

$$= \sum_{n=0}^{\infty} (-1 + 2^{n})\frac{1}{z^{n+1}}$$

This is Laurent's expansion in the annulus $2 < |z| < R$.

Example 3: Obtain the expression for $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid when

(i) $|z| < 1$ **(ii)** $1 < |z| < 4$ **(iii)** $|z| > 4$.

Solution: $f(z) = \frac{z^2-4}{z^2+5z+4} = 1 - \frac{(5z+8)}{(z+4)(z+1)}$

$$f(z) = 1 - \frac{1}{1+z} - \frac{4}{z+4} \qquad \dots (1)$$

(*i*) When $|z| < 1$.

$$f(z) = 1 - (1+z)^{-1} - \left(1 + \frac{z}{4}\right)^{-1}$$

$$= 1 - [1 - z + z^2 - \cdots + (-1)^n z^n + \cdots]$$

$$-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\cdots+(-1)^n\left(\frac{z}{4}\right)^n+\cdots\right]$$

$$=-1+[z-z^2+\cdots+(-1)^{n+1}z^n+\cdots]$$

$$+\left[\left(\frac{z}{4}\right)-\left(\frac{z}{4}\right)^2+\cdots+(-1)^{n+1}\left(\frac{z}{4}\right)^n+\cdots\right]$$

$$=-1+\sum_{0}^{\infty}(-1)^{n+1}\cdot[1+4^{-n}]z^n$$

This is Maclaurin's series.

(*ii*) When $1<|z|<4$. Then $\frac{1}{|z|}<1,\frac{|z|}{4}<1$.

Now (1) is expressible as

$$f(z)=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right]-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\cdots\right]$$

$$=\left[-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]-\left[\frac{-z}{4}+\left(\frac{z}{4}\right)^2+\cdots\right]$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{z^n}-\left(\frac{z}{4}\right)^n\right]$$

This is Laurent's series.

(*iii*) When $|z|>4$, then $\frac{4}{|z|}<1$, and $|z|>4>1\Rightarrow|z|>1\Rightarrow\frac{1}{|z|}<1$.

Now (1) is expressible as,

$$f(z)=1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$=1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}+\cdots\right]-\frac{4}{z}\left[1-\frac{4}{z}+\left(\frac{4}{z}\right)^2-\cdots\right]$$

$$=1-\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n}-\frac{4}{z}\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$$

$$= 1 - \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{z^{n+1}} + \left(\frac{4}{z}\right)^{n+1}\right] = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{z^{n+1}}(1 + 4^{n+1})$$

$$= 1 + \sum^{n} (-1)^n(1 + 4^n) \cdot \frac{1}{z^n}$$

Example 4: If $0 < |z-1| < 2$, then express $f(z) = \frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $(z-1)$.
Solution: Let $u = z - 1$. Then, by what is given,

$0 < |u| < 2,$ so that $\frac{|u|}{2} < 1.$

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$A = \left[\frac{z}{z-3}\right]_{z=1} = \frac{1}{1-3} = -\frac{1}{2}, B = \left[\frac{z}{z-1}\right]_{z=3} = \frac{3}{3-1} = \frac{3}{2}$$

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)} = \frac{3}{2(u-2)} - \frac{1}{2u}$$

$$= -\frac{3}{4}\left(1 - \frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

$$= -\frac{3}{4}\sum_{n=0}^{\infty} \left(\frac{u}{2}\right)^n - \frac{1}{2u} = -\frac{3}{4}\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} - \frac{1}{2(z-1)}$$

Example 5: Prove that $\log z = (z-1) - \frac{(z-1)^2}{2!} + \cdots \ldots, |z-1| < 1.$
Solution: Let $f(z) = \log z$. By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{f^{(n)}(a)}{n!}$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

Taking $a = 1$, we get $f(1) = \log 1 = 0$,

$$f'(z) = \frac{1}{z}, f''(z) = \frac{-1}{z^2}$$

$\therefore\ f(1) = 0, f'(1) = 1, f''(1) = -1$ etc.

$$f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!} f''(1) + \cdots = 0 + (z-1) - \frac{(z-1)^2}{2!} + \cdots$$

Example 6: Expand $\log(1+z)$ in a Taylor's series about $z = 0$ and determine the region of eonvergence for the resulting series.

Solution: Let $f(z) = \log(1+z)$. Then

$$f'(z) = \frac{1}{1+z} f''(z) = -\frac{1}{(1+z)^2}, f'''(z) = \frac{2!}{(1+z)^3}$$
$$f^{iv}(z) = -\frac{3!}{(1+z)^4} \dots, f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n} \text{ etc.}$$

Hence $f(0) = \log 1 = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2\,!,$

$$f^{iv}(0) = -3!, \dots, f^{(n)}(0) = (-1)^{n-1}(n-1)! \text{ etc.}$$

Therefore

$$f(z) = \log(1+z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0)$$
$$+\frac{z^4}{4!}f^{iv}(0) + \dots + \frac{z^n}{n!}f^{(n)}(0) + \cdots$$
$$= 0 + z - \frac{z^2}{2!} + \frac{z^3}{3!}\cdot 2! - \frac{z^4}{4!}\cdot 3! + \dots + \frac{z^n}{n!}(-1)^{n-1}(n-1)! + \cdots$$
$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots + (-1)^{n-1}\frac{z^n}{n} + \cdots$$

Let u_n denote the nth term of the series. Then

$$u_n = \frac{(-1)^{n-1}z^n}{n}, u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}.$$
$$\therefore \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{nz}\right| = \frac{1}{|z|}$$

Hence by D' Alembert's ratio test, the series converges for

$$|z| < 1$$

It can be easily shown that the series converges for $|z| = 1$ except for $z = -1$. It may be noted that $z = -1$ is the singularity of $\log(1+z)$ nearest the point $z = 0$. Thus the series converges for all values of z within the circle $|z| = 1$.

Example 7: Prove that $\log z = (z-1) - \frac{(z-1)^2}{2!} + \cdots \dots, |z-1| < 1$.
Solution: Let $f(z) = \log z$. By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{f^{(n)}(a)}{n!}$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

Taking $a = 1$, we get $f(1) = \log 1 = 0$,

$$f''(z) = \frac{1}{z}, f''(z) = \frac{-1}{z^2}$$

$\therefore$ $f(1) = 0, f'(1) = 1, f''(1) = -1$ etc.

$$f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \cdots = 0 + (z-1) - \frac{(z-1)^2}{2!} + \cdots$$

Example 8: Expand $\log(1+z)$ in a Taylor's series about $z = 0$ and determine the region of convergence for the resulting series.

Solution: Let $f(z) = \log(1+z)$. Then,

$$f''(z) = \frac{1}{1+z}f''(z) = -\frac{1}{(1+z)^2}, f'''(z) = \frac{2!}{(1+z)^3}$$

$$f^{2v}(z) = -\frac{3!}{(1+z)^4} \ldots, f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n} \text{ etc.}$$

Hence $f(0) = \log 1 = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2!$,

$$f^{iv}(0) = -3!, \ldots, f^{(n)}(0) = (-1)^{n-1}(n-1)! \text{ etc.}$$

Therefore,

$$f(z) = \log(1+z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0)$$

$$+ \frac{z^4}{4!}f^{iv}(0) + \cdots + \frac{z^n}{n!}f^{(n)}(0) + \cdots$$

$$= 0 + z - \frac{z^2}{2!} + \frac{z^3}{3!} \cdot 2! - \frac{z^4}{4!} \cdot 3! + \cdots + \frac{z^n}{n!}(-1)^{n-1}(n-1)! + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots$$

Let u_n denote the nth term of the series. Then

$$u_n = \frac{(-1)^{n-1}z^n}{n}, u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}.$$

$$\therefore \qquad \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{nz}\right| = \frac{1}{|z|}.$$

Hence by D' Alombert's ratio test, the series converges for

$$|z| < 1.$$

It can be easily shown that the series converges for $|z| = 1$ except for $z = -1$.
If may be noted that $z = -1$ is the singularity of $\log(1 + z)$ nearest the point $z = 0$. Thus thes seties converges for all values of z within the circle $|z| = 1$.

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \bar{a}_m r^{m+n} e^{i(n-m)\theta}$$

$$\int_0^{2\pi} |f(z)|^2 d\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \bar{a}_m r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

$$= 2\pi \sum_{n=0}^{\infty} a_n \bar{a}_n r^{2n}, \text{ according to } (*)$$

$$\text{or } \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

This proves the first required result. By (1),

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 d\theta \leq \frac{M^2}{2\pi} \int_0^{2\pi} d\theta = M^2$$

This proves the second required result.

Example 9: If $f(z) = \sum_0^\infty a_n z^n (|z| < R)$ and $M(r)$ is the upper bound of $|f(z)|$ on the circle $|z| = r, (r < R)$, then prove that $|a_n| r^n \leq M(r) \forall n$.

Solution: Given $|f(z)| \leq M \forall z$ on $|z| = r$. Also $|z| = r \Rightarrow z = re^{i\theta}$.

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - 0)^{n+1}}$$

$$|a_n| \leq \frac{1}{2\pi} \int_C \frac{|f(z)|}{|z|^{n+1}} \cdot |dz| \leq \frac{M}{2\pi r^{n+1}} \cdot 2\pi r$$

10.6 *MAXIMUM MODULUS PRINCIPLE:-*

The maximum modulus principle is a fundamental result in the **e**xpansion of Analytic Function chapter and, more broadly, in complex analysis. It describes how the magnitude (or modulus) of an analytic function behaves within a region of the complex plane.

Theorem 4: Maximum modulus principle. Suppose $f(z)$ is analytic within and on a simple closed contour C and $f(z)$ is not constant. Then $|f(z)|$ reaches its maximum value on C (and not in'side), that is to.say, if M is the maximum value of $|f(z)|$ on and within C, then $|f(z)| < M$ for every z inside C.

Proof: We prove this theorem by the method of contradiction. Analyticity of $f(z)$ declares that $f(z)$ is continuous within and on C. Consequently $[f(z)\ |$ attains its maximum value M at some point within or on C. We want to show that $|f(z)|$ attains the value M at a point lying on the boundary of C (and not inside C). Suppose, if possible, this value is not attained on the boundary of C but is attained at a point $z = a$ within C so that

$$\text{max.}\,|f(z)| = |f(a)| = M \quad \ldots (1)$$

and

$$|f(z)| \leq M \forall z \text{ within } C \quad \ldots (2)$$

Describe a circle Γ with a as centre lying within C. Now $f(z)$ is not constant and its continuity implies the existence of a point $z = b$ inside Γ s.t. $|f(b)| < M$.

Example 10: Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for:
(***a***) $|z| > 3$ (***b***) $0 < |z+1| < 2$

Solution: $f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$ ... (1)

Case I: When $|z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \frac{3}{|z|} < 1 \Rightarrow \frac{1}{|z|} < \frac{1}{3} < 1$.

$$\therefore\ f(z) = \frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1}\right] = \frac{1}{2z}\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{z^n} - \frac{3^n}{z^n}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1}\frac{1}{z^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{z^n}$$

Case II: $0 < |z+1| < 2$. Put $z + 1 = t$, then, by (1)

$$f(z)=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{t+2}\right], |t|<1 \Rightarrow \left|\frac{t}{2}\right|<\frac{1}{2}<1$$

$$=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{2}\left(1+\frac{t}{2}\right)^{-1}\right]=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{t}{2}\right)^n\right]$$

$$=\frac{1}{2}\left[\frac{1}{z+1}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z+1)^n}{2^n}\right]$$

Example 11: If the function $f(z)$ is analytic when $|z|<R$ and has the Taylor's expansion $\sum_0^\infty a_n z^n$. Show that if $r<R$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta=\sum_0^{\infty}|a_n|^2 r^{2n}$$

Hence prove that if

$$|f(z)|\le M \text{ when } |z|<R, \sum_0^{\infty}|a_n|^2 r^{2n}\le M^2$$

Solution: Since $f(z)$ is analytic for $|z|<R$, then $f(z)$ is analytic within and on a closed curve C defined by

$$|z|=r, r<R$$

So $f(z)$ can be expanded in a Taylor's series within $|z|=r$ so that

$$f(z)=\sum_0^{\infty}a_n z^n=\sum_0^{\infty}a_n r^n e^{in\theta}, z=re^{i\theta}$$

Note that if k is any integer, then

$$\int_0^{2\pi}e^{ik\theta}d\theta=\begin{cases}2\pi & if\ k=0\\ 0 & if\ k\neq 0\end{cases} \quad \text{..... (*)}$$

$$|f(z)|^2=f(z)\overline{f(z)}=\sum_{n=0}^{\infty}a_n r^n e^{in\theta}\sum_{m=0}^{\infty}\bar{a}_m r^m e^{-im\theta}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n\bar{a}_m r^{m+n}e^{i(n-m)\theta}$$

$$\int_0^{2\pi} |f(z)|^2 d\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \bar{a}_m r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

$$= 2\pi \sum_{n=0}^{\infty} a_n \bar{a}_n r^{2n}, \text{according to } (*)$$

or $$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

This proves the first required result. By (1),

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 d\theta \le \frac{M^2}{2\pi}\int_0^{2\pi} d\theta = M^2$$

This proves the second required result.

Example 12: If $f(z) = \sum_0^{\infty} a_n z^n (|z| < R)$ and $M(r)$ is the upper bound of $|f(z)|$ on the circle $|z| = r, (r < R)$, then prove that $|a_n| r^n \le M(r) \forall n$.

Solution: Given $|f(z)| \le M \ \forall z$ on $|z| = r$. Also $|z| = r \Rightarrow z = re^{i\theta}$.

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-0)^{n+1}}$$

$$\therefore \ |a_n| \le \frac{1}{2\pi}\int_C \frac{|f(z)|}{|z|^{n+1}} \cdot |dz| \le \frac{M}{2\pi r^{n+1}} \cdot 2\pi r$$

or $|a_n| r^n \le M = M(r)$.

<u>Check your progress</u>

Problem 1: Find Taylor's series expansion of the function $f(z) = \dfrac{z}{z^4+9}$ around $z = 0$.

Problem 2: Find the Laurent's series of the function $f(z) = \dfrac{1}{(z^2-4)(z+1)}$ valid in the region $1 < |z| < 2$.

10.7 *SUMMARY:-*

The “Expansion of Analytic Function” unit in complex analysis focuses on representing analytic functions as infinite series and understanding their behaviour in different regions of the complex plane. It introduces the Taylor series, which expresses a function as a power series in terms of (z-a) when the function is analytic within a neighborhood of a point a, allowing local approximation and analysis. For functions that are analytic in an annular region, the chapter presents the Laurent series, a more general expansion that includes both positive and negative powers of (z-a), enabling the study of functions with isolated singularities. The chapter also discusses the maximum modulus principle, which states that a non-constant analytic function cannot attain its maximum modulus inside a domain but only on its boundary. Together, these concepts provide powerful tools for analyzing, approximating, and classifying analytic functions and understanding their properties in the complex plane.

10.8*GLOSSARY: -*

- Taylor’s theorem
- Laurent’s theorem
- Uniqueness of Laurent’s expansion
- Maximum modulus principal

10.9 *REFERENCES: -*

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10.10*SUGGESTED READING: -*

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10.11 *TERMINAL QUESTION: -*

Long answer type question

1: Find the Laurent series of the function $f(z) = \frac{1}{z^2(1-z)}$ about $z = 0$.

Hint: $f(z) = \frac{1}{z^2}(1-z)^{-1} = \frac{1}{z^2}(1 + z + z^2 + \cdots) = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n$

$f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + \sum_{n=1}^{\infty} z^n$

2: Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence and nature of singularity at $z = -2$.

3: **(*a*)** Expand $\frac{1}{z}$ as Taylor's series about $z = 1$.

(*b*) Determine Laurent's expansion of the function $f(z) = \frac{\sin z}{\left(z - \frac{\pi}{4}\right)^3}$ in the annulus $0 < \left|z - \frac{\pi}{4}\right| < 1$

4: Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series about $(i)\ z = 0\ (ii)\ z = 1\ (iii)$ its Laurent's series for the domain $1 < |z| < \infty$.

5: Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

(i) $0<|z|<1$ **(ii)** $0<|z|<1$ **(iii)** $|z|>2$

Short answer type question

1: Expand $\sin z$ in a Taylor's series about $z=\frac{\pi}{4}$.

2: Expand $\frac{1}{(z+2)(z+1)}, |z|<1$ in the form of Laurent's series.

3: Expand $\frac{1}{(z-1)(z-2)}$ for the regions

(i) $0<|z|<1$ **(ii)** $0<|z|<1$ **(iii)** $|z|>2$

4: Find the expansion of $\frac{1}{(z^2+1)(z^2+2)}$ in powers of z when,

(i) $|z|<1$ **(ii)** $1<|z|<\sqrt{2}$ **(iii)** $|z|>\sqrt{2}$

Objective type question:

1: If $f(z)$ is analytic at $z=a$, its Taylor series expansion about $z=a$ is

A) $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$

B) $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

C) $\sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

D) $\sum_{n=-1}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

2: The Laurent series of a function $f(z)$ is valid in which region?

A) Entire complex plane

B) A disk $|z-a|<R$

C) An annulus $r_1 < |z-a| < r_2$
D) A line segment on the real axis

3: The principal part of a Laurent series consists of

A) Positive powers of $(z-a)$
B) Negative powers of $(z-a)$
C) Constant terms only
D) Both positive and negative powers

4. The Taylor series is a special case of the Laurent series when

A) The principal part is zero
B) The function has an essential singularity
C) The region is an annulus
D) The coefficients are all zero

5. The coefficients a_n and b_n in the Laurent series are given by

A) Real derivatives of $f(z)$
B) Cauchy's integral formulas
C) Fourier transforms
D) Laplace transforms

6. According to the Maximum Modulus Principle, if $f(z)$ is non-constant and

analytic in a domain *D*, the maximum of $|f(z)|$ occurs

A) At the center of D
B) At a critical point of $f(z)$
C) On the boundary of D
D) At any interior point of D

7. If $f(z)$ is constant in a region, then

A) The modulus $|f(z)|$ has both maximum and minimum inside the region
B) The Maximum Modulus Principle does not apply
C) The modulus $|f(z)|$ is the same everywhere
D) $f(z)$ has a singularity

8. The Laurent series is particularly useful for studying

A) Continuous functions
B) Functions with isolated singularities
C) Harmonic functions
D) Real-valued functions

10.12 *ANSWERS:-*

Answer of check your answer

Answer of problem 1: $f(z)=\sum_{n=0}^{\infty}(-1)^n \frac{z^{4n+1}}{3^{2n+2}}$

Answer of problem 2:

$$f(z)=-\frac{1}{24}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{1}{3z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

Answer of long answer type question:

2: $f(z)=\frac{2}{z+2}+\sum_{n=0}^{\infty}(z+2)^n$, Radius of convergence = 1, $z=-2$ is a pole of order 1 and singularity at $z=-2$ is a pole.

3 (*a*): $f(z)=\sum_{n=0}^{\infty}(-1)^n(z-1)^n$;

3(*b*):

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\left[\sin\phi.\cosh(\sin\theta).\cos(m\theta)+\cos\phi.\sinh(\sin\theta).\sin(m\theta)\right]d\theta$$

Where $\phi=\frac{\pi}{4}+\cos\theta; m=n+3$ and $b_n=a_{(-n)}$. So the Laurent's series given by

$$f(z)=\sum_{n=0}^{\infty}a_n\left(z-\frac{\pi}{4}\right)^n+\sum_{n=0}^{\infty}\frac{b_n}{\left(z-\frac{\pi}{4}\right)^n}$$

4: (i) $f(z)=1-2\sum_{n=0}^{\infty}(-1)^n z^n$ **(ii)** $f(z)=1-\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{2^n}$

(iii) $f(z)=1-\frac{2}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n}$

5: (i) $\frac{1}{2z}+\sum_{n=0}^{\infty} z^n - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$ **(ii)** $\frac{1}{2z}-\frac{1}{z}\sum_{n=0}^{\infty} z^n + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

(iii) $\frac{1}{2z}-\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

Answer of short answer type question:

1: $f(z)=\sum_{n=0}^{\infty}\sin\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)\left(\frac{\left(z-\frac{\pi}{4}\right)^n}{n!}\right)$ **2:** $\sum_{n=0}^{\infty}(-1)^n z^n\left[1-\frac{1}{2^{n+1}}\right]$

3: (i) $\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$ **(ii)** $-\sum_{n=0}^{\infty} z^2\left(\frac{1}{2^{n+1}}+\frac{1}{z}\right)$

(iii) $\sum_{n=0}^{\infty}(-1+2^n)\frac{1}{2^{n+1}}$

4: (i) $\sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right)(z^2)^n$ **(ii)**

$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n}}{2^{n+1}}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+2}}$

(iii) $\sum_{n=0}^{\infty}(-1)^n z^{-2n-2}\,|(1-2^n)|$

Answer of objective questions

1:	B	**2:**	C	**3:**	B	**4:**	A
5:	B	**6:**	C	**7:**	C	**8:**	B

BLOCK IV
SINGULAR POINTS AND ANALYTIC CONTINUATION

UNIT-11: Singular Points

CONTENTS

11.1 *INTRODUCTION: -*

The Singularity unit in complex analysis deals with the study of points at which a complex function fails to be analytic or differentiable. These points, known as singular points or singularities, play a crucial role in understanding the behavior of complex functions near regions where they break down. The chapter classifies singularities into different types removable, poles, and essential singularities based on the nature of the function's behavior around them. By analyzing these singular points using tools such as Laurent series expansion and residue calculation, one can evaluate complex integrals and study the local and global properties of analytic functions. This topic forms the foundation for advanced concepts like the residue theorem and evaluation of contour integrals in complex analysis.

11.2 *OBJECTIVE:-*

The main objectives of the chapter "Singularity" in complex analysis are as follows:

1. To understand the concept of analytic functions and identify points where they fail to be analytic**,** known as singularities.

2. To learn the classification of singular points such as removable singularities, poles, and essential singularities.

3. To study the behavior of complex functions in the neighborhood of different types of singularities.
4. To understand how to represent functions near singular points using the Laurent series expansion.
5. To develop the ability to determine residues at singular points, which are crucial for evaluating contour integrals.
6. To apply the knowledge of singularities in solving problems related to the Residue Theorem and complex integration.
7. To build a strong conceptual foundation for advanced topics in analytic continuation**,** meromorphic functions, and complex mapping.

11.3 *ZERO OF AN ANALYTIC FUNCTION:-*

A zero of an analytic function is a point in the complex plane where the function's value becomes zero.

Definition: *A zero of an analytic function f(z) is a value of z such that f(z) = 0.*

Suppose *f(z)* is analytic in a domain *D* and *a* is any point *D*. Then *f(z)* can be expanded as a Taylor's series about *z* = *a* in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad ...(1),$$

$$a_n = \frac{f^{(n)}(a)}{n\,!} \quad ...(2)$$

Suppose $a_0 = a_1 = ... = a_{m-1} = 0$ and $a_m \neq 0$

so that, $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0, f^{(m)}(a) \neq 0.$

In this case we way say that *f(z)* has a zero of order *m* at $z = a$.

Now (1) takes the from

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_{n+m} (z-a)^{n+m}$$

$$= (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n.$$

Taking $\sum_{n=0}^{\infty} a_{n+m} (z-a)^n = \phi(z)$ ...(3),

Since, $f(z) = (z - a)^m \phi(z)$

By (3), $\phi(a) = [\sum_{n=0}^{\infty} a_{n+m} (z-a)^n]_{z=a}$

$$= [a_m + \sum_{n=1}^{\infty} a_{n+m}(z-a)^n]_{z=a} = a_m.$$

But $a_m \neq 0$ so that $\phi(a) \neq 0$.

Thus we define

Definition: An analytic function *f*(*z*) is said to have a zero of order *m* if *f*(*z*) is expressible as $f(z) = (z - a)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$ f(z) is said to have a simple zero at $z = a$ if $z = a$ is a zero of order one.

11.4 *SINGULAR POINTS:-*

Definitions: *A singularity (or singular point)* of a function is the point at which the function ceases be analytic. For example if $f(z) = \frac{1}{z-2}$, then z = 2 is a singularity of *f (z)*.

There re various types of singularities exist.

1. Isolated singularity: A point $z = a$ is said to be isolated singularity of *f(z)* if

(i) *f(z)* is not analytic at $z = a$.

(ii) *f(z)* is analytic in the deleted neighbourhood of $z = a$, i.e. there exists a

neighbourhood of $z = a$ containing no other singularity.

$z = a$ is called a non-isolatied singularity of *f(z)* if $z = a$ is a singularity and every deleted neighborhood of $z = a$ contains at least one-singularity of *f(z)*.

Examples 1: The function *f(z)* = 1/z is analytic everywhere except at $z = 0$, therefore $z = 0$ is an isolated singularity.

2: The function $f(z) = \frac{z+2}{(z-1)z-2)\,(z-3)}$ has three isolated singularities at z = 1, 2 and 3.

3: The function $f(z) = \frac{2}{\sin\left(\frac{\pi}{z}\right)}$ has an infinite number of isolated singularities all lying on the segment of real axis from z = - 1 to z = 1. These singularities are at $z = \pm\frac{1}{n}$, where *n* = 1, 2, 3, ... The origin $z = 0$

is also a singularity, but it is not isolated. Since every neighborhood of $z = 0$ contains other singularities of the function.

4: f(z) = log z has non-isolated singularity at $z = 0$, (every point on negative real axis including $z = 0$ is a non-isolated singularity of $f(z) = \log z$).

5: $f(z) = \frac{1}{\sin(\pi z)}$ has isolated singularity for all *z* such that *z* = *n* and *n* = 0, $\pm$ 2, $\pm$ 3, … .

Definition: *Let* $z = a$ *be an isolated singularity of a function f (z), then by definition, there exists a deleted neighborhood* 0 < |z - a| < r, in which f(z) analytic. Hence, if z be any point this neighborhood, then by Laurent's expansion, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$.

$\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is *called the* **principal part** of the expansion of *f (z).* Now three cases arise.

I Removable singularity: If the principal part of *f(z)* contains no term, i.e. if $b_n = 0$ ∀ n then the singularity $z = a$ is called *removable singularity* of *f (z)*. In this case $f(z) = \sum_{0}^{\infty} a_n (z-a)^n$

An Alternate definition: A singularity $z = a$ is called a removable singularity of f(z) if $\lim_{z \to a} f(z)$ exists finitely.

Examples 6: Suppose $f(z) = \frac{\sin z}{z}$. Then $\lim_{z \to 0} \frac{\sin z}{z} = 1$.

∴ z = 0 is a removable singularity.

Again $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

Since no negative power of z occurs. Hence z = 0 is a removable singularity of f(z). [by the first def.]

II Pole: If $b_n = 0 \; \forall \; n \; s.t. \, n > m$, *i.e.*, if the principal part contains a finite number of terms, say *m*, then the singularity $z = a$ is called a *pole of order m* of *f (z)*. A pole of order one is called a **simple pole.**

Thus if $z = a$ is a pole of order *n* of the function *f(z)* then *f(z)* will have the expansion of the form $f(z) \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}$

An Alternate definition: A function *f(z)* is said to have pole of order *n* if it is expressible as $f(z) = \frac{\phi\,(z)}{(z-a)^n}$ where $\phi\,(z)$ is analytic and $\phi\,(a) \neq 0$.

Residue of a function *f(z)* at a simple pole $z = a$ is defined as $\lim_{z\to a}(z-a)f(z) = Res\,(z=a)$

Or

$Res\,(z=a) = \lim_{z\to a}\frac{\phi\,(z)}{\Psi'(z)}$, where $f(z) = \frac{\phi\,(z)}{\Psi\,(z)}$.

Example 7: If $f(z) = \frac{1}{(z-5)^3(z-4)^2}$, then $z=5$ is a pole of order 3 and $z=4$ is a pole of order 2.

Solution: Res $(z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\left(\frac{z+1}{z-2}\right) = \frac{2}{-1} = -2.$

III Essential Singularity: If $b_n \neq 0$ for indefinitely many values of *n*, i.e., the principal part contains an infinite number of terms, i.e., the series $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ contains an infinite number of terms, then the singularity z = a is called an essential singularity.

An Alternate definition. If there exists no finite value of *n* s.t.

$\lim_{z\to a}(z-a)^n f(z) = c =$ finite non-zero constant, then $z=a$ is called essential singularity.

Example 8: $z=0$ *is an essential singularity of* $e^{\frac{1}{z}}$*, since the expansion of* $e^{\frac{1}{z}}$.

$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^2 3!} + \cdots$ is an infinite series of negative powers of z.

Remark: $z=a$ is called removal singularity, or pole or essential singularity of *f(z)* according as expansion of *f(z)* contains no negative powers of $z-a$ or contains finite number of negative powers of $z-a$ or contains an infinite number of negative powers of $z-a$.

Theorem 1: *If f(z) has a pole at z = a, then |f (z)| → ∞ as z → a.*

Proof: Suppose *f(z)* has a pole of order m *at z = a.*

To prove that *|f (z)| → ∞ as z → a.*

By assumption, the principal part of the expansion of *f(z)* contains only *m* terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^n} + \frac{b_m}{(z-a)^m}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{1}{(z-a)^m}[b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m]$$

The expression within the square bracket on R.H.S. $\to b_m$ as $z \to a$ so that the whole R.H.S. expression $\to \infty$ as $z \to a$.

Consequently, $|f\ (z)| \to \infty$ as $z \to a$.

Theorem 2: *If an analytic function f(z) has a pole of order m at z = a, then* $\frac{1}{f(z)}$ *has a zero of order m at z = a and conversely.*

Proof (i): Suppose an analytic function *f(z)* has a pole of order *m* at *z = a* so that $f(z) = \frac{\phi\ (z)}{(z-a)^m}$ ...(1)

Where, $\phi(a) \neq 0$ and $\phi(z)$ is analytic.

To prove that $\frac{1}{f(z)}$ has a zero of order *m*.

$$\frac{1}{f(z)} = \frac{(z-a)^m}{\phi\ (z)} = (z-a)^m\ \Psi(z), \quad \text{where} \quad \frac{1}{\phi} = \Psi \text{ and } \Psi(a) \neq 0.$$

This ⇒ $\frac{1}{f}$ has a zero of order *m* at z = a.

(ii): Suppose $\frac{1}{f(z)}$ has a zero of order *m* at *z = a* so that $\frac{1}{f(z)} = (z-a)^m g(z)$ where *g(z)* is analytic and *g(a) ≠ 0.*

This ⇒ $f(z) = \frac{1}{(z-a)^m g(z)} = \frac{h\ (z)}{(z-a)^m}$ where $\frac{1}{g(z)} = h\ (z)$.

Now $f(z) = \frac{h\ (z)}{(z-a)^m}$ where *h (z)* as analytic and *h (a)* ≠ 0.

This ⇒ *f (z)* has a pole of order *m* at *z = a*.

Theorem 3: (Zeros are isolated) *Let f (z) be analytic is a domain D. Then unless f (z) is identically zero, there exists a nbd of each point in D through out which the function has no zero, except possibly at the point itself. In order words the zeroes of an analytic function are isolated.*

Proof: Let z = a be a zero of order *m* of an analytic function *f(z)*. Then we may write $f(z) = (z - a)^m \phi\ (z)$, where $\phi\ (z)$ is analytic and $\phi\ (a) \neq 0$.

Evidently $(z - a)^m \neq 0$ at $z \neq a$.

Now there exists no other point in the deleted neighbouhood $|z - a| < r$ at which $f(z) = 0$

Hence the zero $z = a$ is an isolated singularity.

This is true for every zero of $f(z)$. Therefore the zeroes of $f(z)$ are isolated singularity.

Poles are isolated : Let $z = a$ be a pole of order m of an analytic function $f(z)$, then $\frac{1}{f(z)}$ is analytic and has a zero of order m at $z = a$. Since zeroes are isolated and hence poles are isolated.

The point at infinity: The behaviour of the point $z = \infty$ is studied by the substitution $z = \frac{1}{t}$ in $f(z)$. The behaviour of $f(z)$ at $z = \infty$ determined by the behaviour of $f\left(\frac{1}{t}\right)$ at $t = 0$.

It follows that $f(z)$has a zero or pole, at $z = \infty$ is according as $f\left(\frac{1}{t}\right)$ has the corresponding property at $t = 0$.

Theorem 4 (Limiting point of zeros): *Let f(z) be an analytic function in a simply connected region D. Let* $a_1, a_2, \ldots, a_n \ldots$ *be a sequence of zeros having a as its limit point, a being the interior point of D. Then either f(z) vanishes identically or else has an isolated essential singularity at z = a.*

Proof: Suppose $f(z)$ is analytic in a connected domain D so that it is continuous in D. So $f(a) = 0$. So $f(z)$ has zeros at $a_1, a_2, \ldots a_n, \ldots$ as near as we please to a. Consequently $f(a) = 0$. Further a cannot be a zero of $f(z)$ on account of the fact that zeros are isolated. Hence $f(z) = 0$ for every z in the domain D.

Next we consider the case in which $f(z) \neq 0$ for every z inside D. In this case $f(z)$ must have a singularity at $z = a$. This singularity is isolated but it is not a pole. For $|f(z)|$does not tend to ∞ a $z \to a$ in any manner. Therefore $z = a$, which is limit point of zeros, must be an isolated essential singularity.

Remark: (Remember the result) *Limit point of zeros is an isolated essential singularity.*

Limit point of poles: Suppose $z = a$ is a limit point of the sequence of poles of an analytic function $f(z)$. Then every neighbourhood of the point $z = a$ containing poles of the given function. Therefore the point $z = a$ is a singularity of $f(z)$. This singularity cannot be a pole, since it is not isolated. *Such a singularity is called non-isolated essential singularity or essential singularity simply.*

Theorem 5: *If f (z) and g(z) are analytic in Ω and if f(z) = g (z) on a set which has a limit point in Ω, then f (z) is identically equal to g (z).*

Proof: Let, $\Psi(z) = f(z) - g(z)$. Analyticity of the functions *f (z)* and *g (z)* implies the analyticity of $\Psi(z)$. Also zeros of $\Psi(z)$ are isolated and the limit point of zeros of $\Psi(z)$ belongs to the interior of the domain *D*. Therefore this limit point is an isolated essential singularity, i.e., this limit point does not belong to the domain of regularity. But *f (z)* is analytic everywhere. Consequently $\Psi(z) = 0$ so that *f (z) = g (z)*.

Theorem 6: *If a single valued function f (z) has no singularities other than poles in the finite part of the plane or at the infinity, then f (z) is a rational function.*

Proof: A function *f (z)* is said to be polynomial if it is expressible in the form $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$

A function *f*(z) is said to be a rational function if it is expressible in the form $f(z) = \frac{\phi(z)}{\Psi(z)}$ where $\phi(z)$ and $\Psi(z)$ both are polynomials.

Now we come to the proof of the theorem. Let *f(z)* be a single valued function such that *f(z)* has no singularity except poles at any point (including ∞). So let us suppose that *f (z)* has poles at $z_1, z_2, \ldots, z_k$ of order $m_1, m_2, \ldots, m_k$ in the finite part of the z-plane. Then *f(z)* is expressible as

$$f(z) = \frac{\phi(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}\ldots(z-z_k)^{m_k}} \quad \ldots(1)$$

where $\phi(z)$ is analytic for all finite values of z. By (1),

$$\phi(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\ldots(z-z_k)^{m_k} \quad \ldots(2)$$

Consequently $\phi(z)$ has Maclaurin's expression as

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for all finite values of z.} \quad \ldots(3)$$

To discuss the behaviour of $\phi(z)$ $at\ z = \infty$, where it many have a pole of order *m*.

Putting $z = \frac{1}{z'}$ in (3), $\phi\left(\frac{1}{z'}\right) = \sum_{n=0}^{\infty} a_n (z')^{-n}$ ...(4)

The behaviour of $\phi(z)$ at $z = \infty$ is the same as behaviour of $\phi\left(\frac{1}{z'}\right)$ at z' = 0.

Since $\phi\ (z)$ has a pole of order m at $z = \infty$ so that $\phi\left(\frac{1}{z'}\right)$ also has a pole at z' = 0 of order m. It follows that the series (4) for $\phi\left(\frac{1}{z'}\right)$ must contain finite number of terms.

Now (4) becomes $\phi\left(\frac{1}{z'}\right) = \sum_{n=0}^{\infty} a_n (z')^{-n}$ ...(5)

This ⇒ $\phi\ (z) = \sum_{n=0}^{\infty} a_n z^n \Rightarrow \phi\ (z)$ is a polynomial ...(6)

∴ In either case, $\phi\ (z)$ is a polynomial as obvious from (2) and (5). In this event (1) becomes $f(z) = polynomial\ /[(z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_k)^{m_k}]$

This ⇒ *f(z)* is a rational function.

Example 9: *A rational function has no singularities other than poles.*

Solution: Suppose *f(z)* is a rational function so that it is expressible as $f(z) = \frac{\phi\ (z)}{\Psi\ (z)}$ where $\phi\ (z)$ and $\Psi\ (z)$ both are polynomial having no factor in common. Singularities of *f (z)* are given by $\Psi\ (z) = 0$, i.e., by zeros of $\Psi\ (z)$.

But zeros of $\Psi\ (z)$ are the poles of $\frac{1}{\Psi\ (z)}$. Finally singularities of *f (z)* are poles at the zeros of Ψ (z).

Theorem 7: *A function which has no singularity in the finite part of the planes or at infinity is constant.*

Proof: Suppose the function *f (z),*

(i) has no singularity in the finite part of z-plane.

or

(ii) has no singularity at $z = \infty$.

(i) ⇒ $f(z)$ can be expanded in Taylor's series about z = 0 in the form $f(z) = \sum_{n=0}^{\infty} a_n z^n, \ldots$ (1) where z is a point inside or on |z| = R, where R is arbitrary large positive number.

(ii) ⇒ $f\left(\frac{1}{z}\right)$ is analytic at z = 0.

Hence, $f\left(\frac{1}{z}\right)$ can be expanded by Taylor's theorem as

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} b_n z^n$$

This $\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n}$ ...(2)

By (1) and (2), we get $\sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty} \frac{b_n}{z^n}$ ...(3)

(3) can hold if $a_n = 0 = b_n$ for n = 1, 2, 3, ... and $a_0 = b_0$

If follows that $f(z) = a_0 = b_0$

$\therefore f(z) =$ constant.

Theorem 8: *To show that a function which has no singularity in the finite part of the plane and has a pole of order n at infinite is a polynomial of degree n.*

Or

A function f (z), which is regular everywhere except at infinity where it has a pole of order n, is a polynomial of degree n.

Proof: Suppose,

(i) *f (z)* is regular in the finite part of the z-plane.

(ii) *f (z)* has a pole of order *n* at $z = \infty$.

To prove that *f (z)* is a polynomial of degree *n*.

(i) $\Rightarrow f(z)$ can be expanded in Taylor's series about the point $z = 0$ as

$f(z) = \sum_{m=0}^{\infty} a_m z^m$, *then* $f\left(\frac{1}{z}\right) = \sum_{m=0}^{\infty} a_m z^{-m}$ ...(1)

(ii) $\Rightarrow f\left(\frac{1}{z}\right)$ has a pole of order *n* at $z = 0$

$\Rightarrow$ Principal part of Laurent's expansion of $f\left(\frac{1}{z}\right)$ contains only *n* terms

$\Rightarrow \quad f\left(\frac{1}{z}\right) = \sum_{m=0}^{\infty} a_m z^{-m}$

$\Rightarrow \quad f(z) = \sum_{m=0}^{\infty} a_m z^m$

$\Rightarrow \quad f(z)$ is a polynomial of degree *n*.

This proves that *f(z)* is a polynomial of degree *n*.

Theorem 9(a): *A polynomial of degree n has no singularities in the finite part of the plane but has a pole of order n at infinity.*

Proof: Consider a polynomial $P(z)$ of degree n given by,

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n,\ a_n \neq 0.$$

Then $P\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a^2}{z^2} + \cdots + \frac{a_n}{z_n}$ which has pole of order n at $z = 0$. Consequently $P(z)$ has a pole of order n at $z = \infty$. Also it is obvious that $P(z)$ has no singularities in the finite part of the plane.

Characterization of polynomials

Theorem 9 (b): *The order of a zero of a zero of a polynomial equals the order of its first non-vanishing derivative.*

Proof: Suppose $z = a$ is a zero of order m of a polynomial $P(z)$.

Then $P(z) = (z - a)^m Q(z),\ Q(a) \neq 0.$

Differentiating both sides successively m times, we get

$P'(z) = m(z - a)^{m-1} Q(z) + (z - a)^m Q'(z)$

$P''(z) = m(m - 1)(z - a)^{m-2} Q(z) + 2m(z - a)^{m-1} Q'(z) + (z - a)^m Q^n(z)$

....

....

$P^m(z) = m!\, Q(z) + {}^mC_1\, m!\, (z - a)\, Q'(Z) + ... + (z - a)^m Q^m(z).$

Putting $z = a$ in above relations, we get

$P(a) = P'(a) = P''(a) = ... = P^{m-1}(a) = 0$

And, $P^m(a) = m!\, Q(a) \neq 0.$

Hence the order of a zero of a polynomial equals the order of its non-vanishing derivative.

Theorem 10 (Due to Riemann) *If $z = a$ is an isolated singularity of $f(z)$ and if $f(z)$ is bounded on some deleted neighborhood of a, then a is a removable singularity.*

Proof: Let *$f(z)$ be bounded on some deleted neighbourhood $N(a)$ of a.* Let M be the maximum value of $|f(z)|$ on a circle C defined by $|z - a| = r$, where the radius r is chosen so small that C lies entirely within N (a). Laurent's expansion for *f (z)* gives

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^n$$
...(1)

Where, $b_n = \frac{1}{2\pi i}\int_C (z-a)^{n-1} f(z)dz$

$\therefore |b_n| \leq \frac{M}{2\pi}\int_C |z-a|^{n-1}|dz| = \frac{Mr^{n-1}}{2\pi}. 2\pi r = Mr^n$

$\therefore |b_n| \leq Mr^n$ which $\rightarrow 0$ as $r \rightarrow 0$.

$\therefore b_n = 0 \ \forall \ n.$

This ⇒ the principal part of Laurent's expansion for *f(z)* contains no term.

By definition, this proves that $z = a$ is a removable singularity.

Theorem 11 (Weierstrass Theorem): *If $z = z_0$ is an essential singularity of f(z), prove that given any positive numbers, r, ε and any number c, there is a point in the circle $|z - z_0| < r$ at which $|f(z) - c| < \varepsilon$.*

Or

In other words, in every arbitrary neighbourhood of an essential singularity, there exists a point (and therefore an infinite number of points) at which the function differs as little as we please from any pre-assigned number.

Proof: Suppose the theorem is false. Then given $\varepsilon, r > 0$ and a number c, there exists a point in the circle $|z - z_0| < r$ *at which* $|f(z) - c| < \varepsilon$ so that $\frac{1}{|f(z)-c|} < \varepsilon$ whenever $|z - z_0| < r$.

Making use of Riemann's theorem (Theorem 10), we see that the function $\frac{1}{|f(z)-c|}$ has a removable singularity so that principal part of Laurent's expansion $\frac{1}{f(z)-c}$ about the point z_0 does not contain any term so that $\frac{1}{f(z)-c} = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ ...(1)

If $a_0 \neq 0$, we define $\frac{1}{f(z_0)-c} = a_0$ so that $f(z_0) = c + \left(\frac{1}{a_0}\right)$

It means that $\frac{1}{f(z)-c}$ is analytic and non-zero at z_0.

∴ As a result of which *f(z)* itself is analytic at z_0. Contrary to the initial assumption that z_0 is an essential singularity of *f (z)*.

Again if $a_n = 0$ for $n = 0, 1, 2, ..., m - 1$ then (1) becomes

$$\frac{1}{f(z)-c} = \sum_{n=m}^{\infty} a_n (z - z_0)^n$$

$$= a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \ldots$$

$$= (z - z_0)^m [a_m + a_{m+1} (z - z_0) + \ldots]$$

$$= (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n$$

This shows that the point z_0 is a zero of order m of $\frac{1}{f(z)-c}$ so that $f(z) - c$ has a pole of order m at $z - z_0$. Moreover c is merely a constant. Therefore $f(z)$ also has a pole of order m at z_0. Again we get a contradiction. Hence we have the theorem as stated.

Remark: The above theorem can also be expressed as: Show that an analytic function comes arbitrary close to any complex value in every neighbourhood of an essential singularity.

Example 10: *Find the singularities of the function* $\frac{e^{\frac{c}{(z-a)}}}{e^{\frac{z}{a}}-1}$, *indicating the character of each singularity.*

Solution: Let $f(z) = \frac{e^{\frac{c}{(z-a)}}}{e^{\frac{z}{a}}-1}$

(i) We write $\exp\left(\frac{z}{a}\right)$ in place of $e^{\frac{z}{a}}$.

Then, $f(z) = \frac{\exp\left(\frac{c}{z-a}\right)}{e^{\frac{z}{a}}-1} = \frac{\exp\left(\frac{c}{z-a}\right)}{\exp\left(1+\frac{z-a}{a}\right)-1}$

$$= \frac{\exp\left(\frac{c}{z-a}\right)}{\exp(1)\exp\left(\frac{z-a}{a}\right)-1} = \frac{e^{\frac{c}{(z-a)}}}{e.e^{\frac{z-a}{a}}-1}$$

$$= -e^{\frac{c}{(z-a)}}\left[1 - e.e^{\frac{(z-a)}{a}}\right]^{-1}$$

$$= -e^{\frac{c}{(z-a)}}.\left[1 - e.\left\{1 + \frac{z-a}{a} + \left(\frac{(z-a)}{a}\right)^2.\frac{1}{2!} + \cdots\right\}\right]^{-1}$$

$$= -\left[1 + \frac{c}{z-a} + \left(\frac{c}{z-a}\right)^2.\frac{1}{2!} + \cdots\right]$$

$$\times\left[1+e\left\{1+\frac{z-a}{a}+\left(\frac{z-a}{a}\right)^2.\frac{1}{2!}+\cdots\right\}+\right.$$
$$\left.e^2\left\{1+\left(\frac{z-a}{a}\right)+\cdots\right\}^2+\cdots\right]$$

Clearly this expansion contains positive and negative powers of z - a. In particular, terms containing negative powers of *z* - *a* are infinite in number. Hence by definition, $z = a$ is an *essential singularity.*

(ii) $f(z)=\frac{\exp\left(\frac{c}{z-a}\right)}{\exp\left(\frac{z}{a}\right)-1}$

Evidently denominator has zero of order 1 at

$$e^{\frac{z}{a}}=1=e^{2n\,\pi i}, i.e., z=2n\pi ia.$$

Consequently *f (z)* has a *pole of order one* at each point $z = 2n\pi ia$ (where n = 0, $\pm$ 1, $\pm$ 2, ...).

Example 12: *Show that the function* e^z *has an isolated essential singularity at* $z=\infty$.

Solution: Let $f(z) = e^z$ ...(1)

The behaviour of *f (z)* at $z=\infty$ is the same as the behaviour of $f\left(\frac{1}{z}\right)$ at $z = 0$.

(i) $\Rightarrow f\left(\frac{1}{z}\right)= e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{z^2}.\frac{1}{2\,!}+\cdots=1+\sum_{n=1}^{\infty}\frac{1}{z^n\,n!}$

Or

$$f\left(\frac{1}{z}\right)=1+\sum_{n=1}^{\infty}\frac{1}{(z-0)^n\,n\,!}$$

This is Laurent's expansion of $f\left(\frac{1}{z}\right)$ about the point $z = 0$. This expansion contains an infinite number of terms in the negative power of *z*. Hence, by def., $z = 0$ is an essential singularity of $f\left(\frac{1}{z}\right)$. Consequently *f(z)* has essential singularity at $z=\infty$.

Example 13: *Show that the function* $e^{\frac{1}{z}}$ *actually takes every value except zero an infinite number of times in the neighbourhood of* $z = 0$.

Solution: If we show that the function $f(z)=e^{\frac{1}{z}}$ has an essential singularity at $z = 0$, the result will follow.

Evidently $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2\,2\,!} = 1 + \sum_{n=1}^{\infty} \frac{z^{-n}}{n\,!}$

This is Laurent's expansion of $f(z)$ about the point $z = 0$. The principal part of $f(z)$ is $\sum_{n=1}^{\infty} \frac{z^{-n}}{n\,!}$ which contains an infinite number of terms. Hence, by def., $z = 0$ is an essential singularity.

Example 14: *Find kind of singularities of the following:*

(***i***) $\frac{\cot \pi z}{(z-a)^2}$ *at* $z = a$ *and* $z = \infty$ (***ii***) $\tan\left(\frac{1}{z}\right)$ *at* $z = 0$.

(***iii***) *cosec* $\left(\frac{1}{z}\right)$ *at* $z = 0$. (***iv***) $\sin\left[\frac{1}{(1-z)}\right]$ *at* $z = 1$.

Solution: Recall that the limit point of poles is a non-isolated essential singularity, whereas limit point of zeros is an isolated essential singularity.

(***i***) $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{(\sin \pi z)(z-a)^2}$

Poles of $f(z)$ are given by $(\sin \pi\, z)(z-a)^2 = 0$

This ⇒ $\sin \pi\, z = 0, (z-a)^2 = 0.$

Now $\sin \pi\, z = 0$ gives $\pi z = n\pi$ or $z = n$, where $n = 0, \pm 1, \pm 2, \ldots$. Obviously $z = \infty$ is a limit of these poles. Hence $z = \infty$ is non-isolated essential singularity.

$(z - a)^2 = 0$ gives $z = a, a$

Hence $z = a$ is a double pole.

(***ii***) $f(z) \tan\left(\frac{1}{z}\right) = \frac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$

Poles of $f(z)$ are given by $\cos\left(\frac{1}{z}\right) = 0.$

This ⇒ $\frac{1}{z} = 2n\pi \pm \frac{\pi}{2}$

⇒ $z = \frac{1}{\left(2n \pm \frac{1}{2}\right)\pi}$, where $n = \pm 1, \pm 2, \ldots,$

Obviously $z = 0$ is the limit of these poles. Hence $z = 0$ is non-isolated essential singularity.

(***iii***) $f(z) = cosec\left(\frac{1}{z}\right) = \frac{1}{\sin\left(\frac{1}{z}\right)}.$

Poles of *f(z)* are given by $\sin\left(\frac{1}{z}\right) = 0$.

∴ $\frac{1}{z} = n\pi$ or $z = \frac{1}{n\pi}$, where $n = \pm 1, \pm 2, ...$

Evidently $z = 0$ is a limit point of these poles. Hence $z = 0$ a non-isolated essential singularity.

(*iv*) $f(z) = \sin\left(\frac{1}{1-z}\right)$

Zeros of (z) are given by $\sin\left(\frac{1}{1-z}\right) = 0$.

∴ $\frac{1}{1-z} = n\pi$ or $1 - z = \frac{1}{n\pi}$

or

$z = 1 - \frac{1}{n\pi}$ where $n = \pm 1, \pm 2, ...$

Evidently $z=1$ is a limit point of these zeros. Hence $z = 1$ is isolated essential singularity.

Example 15: *Find residue of* $\phi(z) = \cot z$ *at the points* $z_n = n\pi$ *for* $n = 1, 2, ...$. *What is the nature of singularity at* $= \infty$ *? Justify your answer.*

Solution: $\phi(z) = \cot z = \frac{\cos z}{\sin z} = \frac{f\ (z)}{g\ \ (z)}$, say ... (1)

Poles of $\phi(z)$ are given by putting denominator equal to zero.

∴ Poles are given by $z = 0 = \sin 0$

General value is given by $z = n\pi + (-1)^n\ (0) = n\pi = z_n$, say

∴ $z = z_n = n\pi$ for $n = 0., 1, 2, 3, ...$.

These are the poles. Evidently $z = \infty$ is a limit of these poles.

Hence $z = \infty$ is a non-isolated essential singularity of $\phi(z)$.

Residue of $\phi(z)$ at $z = z_n$ is

$$= \lim_{z \to n\pi} \frac{f\ (z)}{g'\ (z)} = \lim_{z \to n\pi} \frac{\cos z}{\cos z} = 1., \text{By (1)}.$$

Residue $(z = z_n) = 1$.

Example 16: *Specify the nature of singularity at* $z = -2$ of $f(z) = (z - 3) \sin\left(\frac{1}{z+2}\right)$.

Solution: Zero of *f (z)* are given by *f (z) = 0* or $(z-3)\sin\left(\frac{1}{z+2}\right)=0$

This implies *z = 3* and $\sin\left(\frac{1}{z+2}\right)=0=\sin 0$

$\Rightarrow \frac{1}{z+2}=n\pi+(-1)^n(0)=n\pi$

$\Rightarrow z+2=\frac{1}{n\pi}\Rightarrow z=-2+\frac{1}{n\pi}$, or $z=-2+\frac{1}{n\pi}$ for *n = 1, 2, 3, ...* .

Limit points of zeros is *z = -2*.

∴ z = - 2 is isolated essential singularity.

Example 17: *What kind of singularity has the function*

(*i*) $f(z)=\frac{1}{\cos\left(\frac{1}{z}\right)}$ *at z = 0 ?*

(*ii*) *and cot z at z = ∞ ?*

Solution (*i*): $f(z)=\frac{1}{\cos\left(\frac{1}{z}\right)}$

Poles of *f (z)* are given by $\cos\left(\frac{1}{z}\right)=0=\cos\left(\frac{\pi}{2}\right)$

This $\Rightarrow \frac{1}{z}=2n\pi\pm\frac{\pi}{2}=\left(2n\pm\frac{1}{2}\right)\pi$

or

$z=\frac{1}{\left(2n\pm\frac{1}{2}\right)\pi}$ where $n=0,1,2,3,\ldots$

Evidently z = 0 is the limit of these poles.

Hence z = 0 is non-isolated essential singularity.

(*ii*) *f (z) = cot z* (See Problem 14)

Example 18: *Find zeros and poles of* $\left(\frac{z+1}{z^2+1}\right)^2$.

Solution: Let $f(z)=\frac{(z+1)^2}{(z^2+1)^2}$

I. Zeros of *f (z)* are given by $(z+1)^2=0$

or, z = -1, -1.

∴ z = - *1* is a zero of order 2.

II. Poles of *f (z)* are given by $(z^2+1)^2=0$ or $(z-i)^2(z-i)^2=0$

or, $z = -i, -i, i, i$.

∴ $z = -1$ and $z = i$ both are poles of order 2.

Example 19: *What kind of singularities have the following.*

(***i***) $\frac{1}{\sin z - \cos z}\ at\ z = \frac{\pi}{4}$.

(***ii***) $\sin z - \cos z$ at $z = \infty$

(***iii***) $\frac{e^z}{z^2+4}$

(***iv***) $\frac{1-e^z}{1+e^z}\ at\ z = \infty$

(***v***) $z\ cosec\ z\ at\ z = \infty$.

Solution: Recall that the limit point of the poles is a non-isolated essential singularity whereas limit point of zeros is an isolated essential singularity.

(***i***) Suppose $f(z) = \frac{1}{\sin z - \cos z}$.

Poles of *f(z)* are obtained by putting the denominator equal to zero, *i.e*., sin z - cos z = 0 which gives tan z = 1.

∴ $z = n\pi + \frac{\pi}{4}$ where $n = 0, \pm 1, \pm 2, \pm 3,$ ...

Obviously $z = \frac{\pi}{4}$ is a simple pole.

(***ii***) $f(z) = \sin z - \cos z$.

Zeros of *f (z) are given by*

$\sin z - \cos z = 0$ or $\tan z = 1$.

Hence $z = n\pi + \frac{\pi}{4}$, where $n = 0, \pm 1, \pm 2. \pm 3, ...$

Evidently $z = \infty$ is the limit point of these zeros and hence $z = \infty$ is isolated essential singularity.

(***iii***) Suppose $f(z) = \frac{e^z}{z^2+4}$.

Poles of *f(z)* are given by putting denominator equal to zero, i.e.,

$$(z^2 + 4) = 0 \quad \text{or} \quad z = \pm 2i$$

Hence $z = 2i, -2i$ are simple poles.

(***iv***) Let $f(z) = \frac{1-e^z}{1+e^z}$.

Poles of *f(z)* are given by $1 + e^z = 0$, i.e. $e^z = -1 = e^{\pi i} = e^{(2n\pi+\pi)\, i}$

This $\Rightarrow$ z = (2n + 1) πi, where n = 0, $\pm$ 1, $\pm$ 2, ...

Evidently $z = \infty$ is the limit of these poles.

Hence $z = \infty$ is non-isolated essential singularity.

(*v*) Let $f(z) = z\, cosec\, z = \frac{z}{\sin z}$.

Poles of f (z) are given by sin z = 0.

$\therefore$ $z = n\pi\ (n = 0, \pm 1, \pm 2, \dots)$.

Evidently $z = \infty$ is the limit of these poles.

Hence $z = \infty$ is non-isolated essential singularity.

Example 20 (a). *Discuss the nature of singularities of the following functions:*

(*i*) *tan z* **(*ii*)** $\frac{1}{z(1-z^2)}$

(*iii*) $\frac{z}{1+z^4}$ **(*iv*)** $\frac{\sin z}{(z-\pi)^2}$.

Solution: (*i*) Let $f(z) = \tan z = \frac{\sin z}{\cos z}$.

To obtain the singularities of *f (z)* equating to zero the denominator of *f (z)*, we get

$\cos z = 0$ or $z = 2n\pi \pm \frac{\pi}{2}, n \in I$

or $z = (4n \pm 1)\frac{\pi}{2}, n \in I$

or $z = (2n + 1)\frac{\pi}{2}, n \in I$.

Hence $z = (2n + 1)\frac{\pi}{2}, (n \in I)$ give the simple poles of *f (z).*

(*ii*) Let $f(z) = \frac{1}{z(1-z^2)}$.

Singularities of *f (z)* are given by $z(1 - z^2) = 0$

or z = 0, -1, 1, which are the simple poles.

(*iii*) Let $f(z) = \frac{z}{1+z^4}$.

Singularities of *f (z)* are given by $1 + z^4 = 0$ or $z = (-1)^{\frac{1}{4}}$

Or $z = (\cos\pi + i\sin\pi)^{\frac{1}{4}} = \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{\frac{1}{4}}$

$$= \cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4} = e^{i\,(2n+1)\,\frac{\pi}{4}}.$$

Putting n = 0, 1, 2, 3, we get $z = e^{\frac{i\pi}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi}{4}}$ which are the simple poles of *f (z).*

(*iv*) Let $f(z) = \frac{\sin z}{(z-\pi)^2}$.

Singularities of *f (z)* are given by $(z - \pi)^2 = 0$.

Thus z = π is a pole of order two of *f* (z).

Example 20(b). *Show that the function* $f(z) = \frac{z^2+4}{e^z}$ *has an isolated singularity at* $z = \infty$

Solution: Take $z = \frac{1}{y}$

$$\therefore f\left(\frac{1}{y}\right) = \left(4 + \frac{1}{y^2}\right) e^{-\frac{1}{y}}$$

$$= \left(4 + \frac{1}{y^2}\right)\left\{1 - \frac{1}{y} + \frac{1}{y^2\,3!} - \frac{1}{y^3.\ 3!} \dots\right\}$$

$$= 4 - \frac{4}{y} + (1+2)\frac{1}{y^2} + \left(-1 - \frac{2}{3}\right)\frac{1}{y^3} + \left(\frac{1}{2} + \frac{1}{6}\right) 1/\frac{1}{y^4} + \cdots$$

$$= 4 - \frac{4}{y} + \frac{3}{y^2} - \frac{5}{3y^3} + \frac{2}{3y^4} \dots\dots$$

This contains an infinite terms of negative powers of y. Hence $f\left(\frac{1}{y}\right)$ has isolated essential singularity at y = 0.

This ⇒ *f (z)* has an isolated essential singularity at $z = \infty$.

Example 20(c): *Find the nature and location of the singularities of the function* $f(z) = \frac{1}{z(e^z-1)}$. *Prove that f (z) can be expanded in the form* $\frac{1}{z^2} - \frac{1}{2z} + a_0 + a_2 z^2 + a_4 z^4 + \cdots$ *where 0 < |z| < 2π and find the values of* a_0 *and* a_2.

Solution: The singularities of *f* (z) are given by z.(e^z - 1) = 0

$\therefore$ z = 0 and $e^z = 1 = e^{2n\pi i}$ (n = 0, $\pm$ 1, $\pm$ 2, ...)

Hence the singularities of *f* are at z = 0 and z = $2n\pi i$ (*n* = 0, $\pm$ 1, $\pm$ 2, ...). It follows that z occurs as a factor of e^z - 1 too. Hence z = 0 is a double pole of (z).

The other singularities $\pm$ 2π, $\pm 4\pi$, $\pm$ 6π, ... are simple pole.

Hence *f (z)* can be expanded as a Laurent's series in the region $0 < |z| < 2\pi$ in powers of *z*. As $z = 0$ is a double pole, the principal part of *f*(*z*) consists of two terms only.

Therefore the expansion of f(z) will be of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{b_1}{z} + \frac{b_2}{z^2}$. ...(1)

Now, $f(z) = \frac{1}{z(e^z-1)}$

$$= \frac{1}{z\left[\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots\right)-1\right]}$$

$$= \frac{1}{z^2}\left[1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\cdots\right]^{-1}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\cdots\right)+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\cdots\right)^2 - \left(\frac{z}{2}+\frac{z^2}{6}+\cdots\right)^3 + \left(\frac{z}{2}+\cdots\right)^4 - \cdots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2}+\frac{1}{12}z^2+\frac{1}{360}z^4+\cdots\right]$$

$$= \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}+\frac{1}{720}z^2+\cdots$$ ...(2)

Comparing (1) and (2), we obtain

$$a_0 = \frac{1}{12} \;\; and \; a_2 = \frac{1}{720}.$$

Example 21: *Find zeros and discuss of singularity of the function* $f(z) = \frac{(z-2)}{z^2}\sin\left(\frac{1}{z-1}\right)$.

Solution: Poles of *f*(*z*) are obtained by putting denominator equal to zero, i.e., $z^2 = 0$.

This ⇒ $z = 0, 0$.

⇒ $z = 0$ is a pole of order two.

Zeros of (*f*(*z*) are given by $(z-2)\sin\left(\frac{1}{z-1}\right) = 0$

This ⇒ $z = 2$ and $\frac{1}{z-1} = n\pi \Rightarrow z = 2, z = \frac{1}{n\pi} + 1$.

Thus z = 2 is a simple zero. The limit point of the zeros given by $z = 1 + \frac{1}{n\pi}$, where $n = \pm 1, \pm 2, \ldots$ is z = 1. Therefore z = 1 is isolated essential singularity.

Example 22: *Find Laurent series of* $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ *about singularity z = - 2 and indicate nature of singularity.*

Solution: $sin\,\theta = \theta = \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} \ldots \ldots \ldots ..$

$$\therefore \qquad f(z) = (z-3)\left[\frac{1}{(z+2)} - \frac{1}{3\,!}.\frac{1}{(z+2)^3} + \frac{1}{5\,!(z+2)^5} \ldots \ldots .\right]$$

which represents Laurent's series.

Second Part: Zeros of f (z) are given by $(z-3)\sin\left(\frac{1}{z+2}\right) = 0 \Rightarrow z - 3 = 0, \sin\left(\frac{1}{z+2}\right) = 0$

$$\Rightarrow \sin\left(\frac{1}{z+2}\right) = 0 = \sin(n\pi) \Rightarrow \frac{1}{z+2} = n\pi$$

$\Rightarrow z + 2 = \frac{1}{n\pi} \Rightarrow z = \frac{1}{n\pi} - 2$ for n = 1, 2, 3, 4, ...

$z = \frac{1}{\infty} - 2 = -2$ is limit of zeros.

∴ z = -2 is isolated essential singularity.

Example 23: *Show that the function* $e^{-\frac{1}{z^2}}$ *has ni singularities.*

Solution: Let $f(z) = e^{-\frac{1}{z^2}} = \frac{1}{e^{-\frac{1}{z^2}}}$

Poles of $f(z)$ are given by $e^{-\frac{1}{z^2}} = 0$, which is not possible for any value of z, real or complex.

Zeros of $f(z)$ are given by $e^{-\frac{1}{z^2}} = 0 = e^{-\infty}$ so that $\frac{1}{z^2} = \infty$.

This ⇒ $z = 0$, (repeated twice).

Hence z = 0 is a zero of order two so that there is no limit of the zero. Consequently there is no singularity. Finally, $f(z)$ is free from any singularity.

Check your progress

Problem 1: Locate and name all the singularities of $f(z) = \frac{z^8+z^4+2}{(z-1)^3\,(3z+2)^2}$

Solution: Poles of $f(z)$ are given by $(z-1)^3\,(3z+2)^2 = 0$

This ⇒ $z = 1$, pole of order 3 and $z = -\frac{2}{3}$, pole of order 2.

Problem 2: Write the principal part of the function $f(z) = \frac{1}{z}\exp\left(\frac{1}{z}\right)$ at its singular point.

Solution: $f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1} n!}$

11.5 *SUMMARY:-*

The chapter on singular points classifies points where a complex function fails to be analytic, focusing primarily on isolated singularities. These are categorized into three types based on the function's Laurent series expansion in a punctured disk around the point: a removable singularity if the principal part (negative powers) is zero, allowing the function to be redefined as analytic; a pole of order m if the principal part has finitely many terms, causing the function's magnitude to approach infinity; and an essential singularity if the principal part has infinitely many terms, leading to the chaotic behavior described by Picard's Theorem, where the function attains every complex value (with one possible exception) infinitely often in every neighborhood. This classification is fundamental for applying the Residue Theorem, which uses the Laurent series coefficient b_1 (the residue) to evaluate complex integrals, and extends to infinity via the substitution $w=1/z$, with functions analytic everywhere except for poles being termed meromorphic.

11.6 *GLOSSARY:-*

1. Zero of an analytic function
2. Singular point
3. Isolated singularity
4. Non-isolated singularity
5. Removable singularity
6. Pole
7. Residue
8. Essential singularity

9. The point at infinity**.**

11.7 *REFERENCES:-*

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11.9 *TERMINAL QUESTION:-*

Long answer type question

1: Classify the singularities of a function of a complex variable. Show that the only singularities of $f(z) = \dfrac{\cot \pi z}{(z-a)^2}$ are poles.

2: Determine the nature of the pole at the origin of the function $f(z) = \dfrac{e^z}{z \sin mz}$.

3: What kind of singularity has the following functions:

(i) $\dfrac{1}{\cos(1/z)}$ at $z = 0$ **(ii)** $\cot z$ at $z = \infty$

4: Determines nature and investigate the behavior of the functions at infinity

(i) $\dfrac{1}{\sin z - \sin a}$ **(ii)** $\dfrac{1}{\cos z + \cos a}$

(iii) $\dfrac{\cot z}{z^2}$ **(iv)** $\dfrac{\cos z}{z^2}$

(v) $\dfrac{1}{z^3(2-\cos z)}$ **(vi)** $\dfrac{e^z}{1+z^2}$

5: Classify the singularities of a function $f(z)$. Locate the singularities of $\log(z-1)$ and classify them.

Short answer type question

1: Distinguish between pole and essential singularity.

2: Define **(*i*)** a removable singularity **(*ii*)** a pole **(*iii*)** an isolated singularity of *f*(*z*). Give one example in each case.

Objective type question:

1. The function $f(z) = \dfrac{\sin z}{z}$ has:

A) A removable singularity at z = 0
B) A simple pole at z = 0
C) An essential singularity at z = 0
D) A non-isolated singularity at z = 0

2. The function $f(z) = \dfrac{1}{\sin(1/z)}$ has:

A) A simple pole at z = 0
B) A pole of order 2 at z = 0
C) An essential singularity at z = 0
D) A non-isolated singularity at z = 0

3. For the function $f(z)=\frac{e^z}{(z-1)^3}$ the point $z=1$ is:

A) A removable singularity
B) A pole of order 1
C) A pole of order 3
D) An essential singularity

4. The nature of the singularity of $f(z)=e^{1/z}$ at z=0 is:

A) Removable
B) Simple Pole
C) Essential Singularity
D) Pole of order 2

5. A function $f(z)$ has a pole of order m at $z=a$. Which of the following is TRUE?

A) $\lim_{z\to a} f(z)$ exists and is finite.
B) $\lim_{z\to a} |f(z)|=\infty$.
C) $\lim_{z\to a}(z-a)^{m-1} f(z)$ exists and is non-zero.
D) $\lim_{z\to a}(z-a)^{m} f(z)$ does not exist.

6. The function $f(z)=\frac{1}{z(z^2+4)}$ has:

A) Three simple poles
B) One simple pole and one pole of order 2
C) Three poles, one of which is of order 2
D) A removable singularity at z=0

7. According to Picard's Theorem, in every neighborhood of an essential singularity, a function:

A) Is bounded.
B) Takes all complex values, with at most one exception.
C) Takes only real values.
D) Becomes zero.

8. The point z=0 for the function $f(z)=\frac{1-\cos z}{z^2}$ is:

A) A simple pole
B) A pole of order 2
C) An essential singularity
D) A removable singularity

9. The singularity of $f(z)=\dfrac{\log(z+1)}{z}$ at z=0 is:

A) A removable singularity
B) A simple pole
C) An essential singularity
D) A branch point (not an isolated singularity)

10. If the principal part of the Laurent series of a function about an isolated singular point contains an infinite number of terms, the point is called:

A) A Removable Singularity
B) A Pole
C) An Essential Singularity
D) A Regular Point

11. *The function f(z)* = $(z-3)^{\frac{1}{2}}$ *has the following singularity at z* = 3:

A) Pole

B) Branch point

C) Removable singularity

D) Essential singularity

12. Which of the following is correct:

A) Zeros of $f(z)$ is a singular point

B) Poles are not isolated

C) Limit points of poles of $f(z)$ are not isolated.

D) Limit points of zeros is an isolated essential singularity.

13. Which one is correct: For the function $f(z) = \tan\left(\frac{1}{z}\right), z = 0$ is

A) an isolated essential singularity

B) a simple pole

C) a non-isolated essential singularity

D) none of these

14. The number of roots of $z^3 e^{-z} = 0$ inside $|z| = 1$ is

A) one

B) three

C) *six*

D) *two.*

15. The function $f(z) = \sin z$:

A) is bounded

B) is not analytic

C) has only real zeros

D) none of these.

Fill in the blanks:

1. $f(z) = \sin\left(\frac{1}{z}\right)$ has an ...... singularity at $z = 0$.

2. $f(z) = \frac{z}{z^2-1}$ has a pole of order .... at $z = 1$.

3. The points where $f'(z) = 0$ or $f'(z) = \infty$, called .... points.

11.10 *ANSWERS*

Answer of long answer type question:

1: $z = 0$ is a pole of order 2

$z = n$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Are simple poles $z = \infty$ is non-isolated essential singularity.

2: $z = 0$ is simple pole.

The function has simple poles at each of these points.

$z = \frac{n\pi}{m}$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$z = \infty$ is non-isolated essential singularity.

3 (*i*): The function has simple poles at each of the point $z = \frac{1}{(2n \pm 1/2)\pi}$, where $n = 0, \pm 1,$ $z = 0$ is a limit of these poles and hence $z = 0$ is non-isolated essential singularity.

3 (*ii*): The function has single poles at each of the points $z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

and $z = \infty$ is a limit of these poles and so $z = \infty$ is non-isolated essential singularity.

4 (*i*): If $\alpha \neq m\pi + \dfrac{\pi}{2}$, $(m = 0, \pm 1, \pm 2, \ldots)$

then $z = 2k\pi + \alpha$ and $z = (2k+1)\pi - \alpha, (k = 0, \pm 1, \pm 2, \ldots)$ are simple poles, if $\alpha = m\pi + \dfrac{\pi}{2}$

then for even m, $\alpha = 2k\pi + \dfrac{\pi}{2}$ and, for odd m, $z = (2k+1)\pi + \dfrac{\pi}{2}$

are poles of second order: $z = \infty$ is a limit of poles in all cases.

Answer of objective questions:

1: A **2:** D **3:** C **4:** C

5: B **6:** A **7:** B **8:** D

9: B **10:** C **11:** C **12:** D

13: A **14:** B **15:** A

Answer of fill in the blanks:

1. Essential **2.** One **3.** Singular.

UNIT-12: Residue Theorem

CONTENTS

12.1 *INTRODUCTION:-*

The residue theorem is one of the central results in complex analysis, providing a powerful method for evaluating complex integrals, especially those around closed contours. It establishes a deep connection between the values of a complex function inside a contour and the integral of that function around the contour, by relating the integral to the sum of residues of the function's singularities enclosed by the path. Essentially, if a function is analytic except for isolated singular points, the integral of the function around a simple closed contour is $2\pi i$ times the sum of its residues at those singularities. This theorem not only simplifies the computation of many real and complex integrals but also has broad applications in mathematics, physics, and engineering, including in evaluating improper integrals, solving differential equations, and analyzing physical systems.

12.2 *OBJECTIVE:-*

The objectives of the chapter residue theorem in complex analysis are:

1. To understand the concept of singularities and residues of complex functions.
2. To learn methods for finding residues at different types of singular points.
3. To state and prove the residue theorem and understand its significance in complex integration.
4. To apply the Residue Theorem in evaluating complex contour integrals.
5. To use the theorem to evaluate definite and improper real integrals that are otherwise difficult to compute.
6. To develop problem-solving skills and deepen the understanding of the relationship between analytic functions and their singularities

12.3 *RESIDUE AT A POLE:-*

The residue at a pole in complex analysis focuses on understanding how to determine the residue of a complex function at its poles, which are specific types of isolated singularities where the function approaches infinity. A pole of a function is a point at which the function can be expressed as a ratio of two analytic functions, with the denominator vanishing at that point.

Definition: Suppose a single valued function *f (z)* has a pole of order *m* at z = a, then y definition of pole the principal part of Laurent's expansion of *f* (z) contains only *m* terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n} \qquad ...(1)$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}, b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{-n+1}}$$

C being a circle $|z - a| = r$.

Evidently
$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz \qquad ...(2)$$

The coefficient b_1 is called the residue of $f(z)$ at the pole z = a and is denoted by the symbol $Res(z = a)$. Thus $Res\ (z = a) = b_1$.

Evidently the value of b_1, given by (2), does not depend upon the order of the pole and hence it represents a general definition of the residue at a pole.

Consider the case in which z = a is a simple pole. *i.e.*, $z = a$ is a pole of order 1.

Then (1) becomes $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{(z-a)}$

This $\Rightarrow \lim_{z\to a}(z-a)f(z) = b_1$

Using (2), we get $Res\ (z=a) = \lim_{z\to a}(z-a)f(z) = b_1$

$$= \frac{1}{2\pi i}\int_C f(z)dz$$

Here the circle *C* may be replaced by any closed contour containing within it no other singularity except $z = a$.

Example 1: If $f(z)$ is a function such that for positive integer m, a value $\phi\ (z_0) \neq 0$ and $\phi\ (z) = (z - z_0)^m f(z)$ is analytic at z_0, then prove that $f(z)$ has a pole of order m at z_0.

Solution: *f(z)* has a pole of order m at z_0

$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-a)^n}$ and $b_m \neq 0$.

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Multiplying by $(z - z_0)^m$, we get

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} a_n\,(z-z_0)^{m+n} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \cdots + b_m.$$

$$= \phi(z), \text{ say}$$

$$\Rightarrow \phi(z) = (z-z_0)^m f(z)$$

Also $b_m \neq 0 \Rightarrow \phi(z_0) \neq 0$. This proves the example.

12.4 *RESIDUE AT INFINITY:-*

The chapter residue at infinity in complex analysis introduces the concept of evaluating the residue of a complex function at the point at infinity, which is treated as an extended point on the complex plane in the context of the Riemann sphere.

Definition: If *f(z)* has an isolated singularity at $z = \infty$ or is analytic there, then the residue at $z = \infty$ is defined, as

$$Res(z = \infty) = -\frac{1}{2\pi i}\int_C f(z)dz$$

where *C* is any closed contour which encloses all the finite singularities of *f (z).* The integral is taken in positive direction (anticlockwise direction).

Remark (i): The function mat be regular at infinity, yet has a residue there. Consider the function $f(z) = \frac{b}{(z-a)}$.

For this function

$$Res(z = \infty) = -\frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_C \frac{b}{(z-a)}dz$$

$$= -\frac{b}{2\pi i}\int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}d\theta =$$

$$-\frac{b}{2\pi}\int_C^{2\pi} d\theta = -b$$

Where, *C* is the circle |*z* - a| = *r*

$\therefore \quad Res(z = \infty) = -b$

Also *z* = *a* is a simple pole of *f (z)* and its residue there is $\frac{1}{2\pi i}\int_C f(z)dz = b$

Therefore *Res*(*z* = *a*) = *b* = -*Res* (*z* = ∞)

(ii) If a function is analytic at a point *z* = *a*, then its residue at *z* = *a* will be zero, but not so at infinity.

(iii) If $f(z) = \frac{(z^2+2z+7)\,e^z}{(z-5)^2\,(z-2)^7(z-9)^3}$, then *f*(*z*) has poles at z = 5, 2, 9 of orders 2, 7, 3 respectively.

12.5 *CAUCHY RESIDUE THEOREM:-*

The "Cauchy residue theorem" in complex analysis presents one of the most powerful and elegant results for evaluating contour integrals of analytic functions. The Cauchy residue theorem states that if a function is analytic inside and on a closed contour except for a finite number of isolated singularities, then the integral of the function around the contour is equal to 2πi times the sum of the residues of the function at those singularities. This theorem serves as a generalization of Cauchy's Integral Theorem and provides a direct link between the local behavior of a function near its singular points and its global integral properties. The chapter focuses on understanding the statement and proof of the theorem, computing residues at different types of singularities, and applying the theorem to evaluate complex contour integrals and real definite integrals that are difficult to solve using elementary methods.

Theorem 1(Cauchy's Residue Theorem): If $f(z)$ is analytic within and on a closed contour C, except at a finite number of poles z_1, z_2, z_3 ......, z_n within C, then

$$\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res\ (z = z_r)$$

where R.H.S. denotes sum of residues of $f(z)$ at its poles lying within C.

Proof: Suppose $\gamma_1, \gamma_2, \ldots\ldots\ \gamma_n$ are the circles with centres at z_1, z_2, ..., z_n, respectively and radii so small that they lie within closed curve C and do not overlap $f(z)$ is analytic within the annulus bounded by these circles and the curve C, By corollary to Cauchy's theorem.

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz$$

...(1)

But, $\int_C f(z)dz$ = residue of $f(z)$ at $z = z_1$.

$$= Res(z = z_1)$$

$\therefore \int_{\gamma_1} f(z)dz = 2\pi i$ Res $(z = z_1)$.

Using this in (1), we get

$$\int_C f(z)dz = 2\pi i \, \text{Res}\,(z = z_1) + \ldots + 2\pi i \; Res\,(z = z_n)$$

$$= 2\pi\, i \sum_{r=1}^{n} Res\,(z = z_r)$$

Theorem 2: If a function *f*(*z*) is analytic except at finite number of singularities including that at infinity), then the sum of residue of these singularities is zero.

Proof: Let *C* be a closed contour which encloses all the singularities of *f*(z), except that at infinity. The sum ΣR of residue at all these singularities within *C* is given by

$$\int_C f(z)dz = 2\pi i \;\; \Sigma R$$

[This follows from Cauchy's residue theorem]

This $\Rightarrow \frac{1}{2\pi i}\int_C f(z)dz = \sum R$

Also $-\frac{1}{2\pi i}\int_C f(z)dz = \text{Re}\, s(z = \infty)$

Adding these two equations, we get $Res(z = \infty) + \Sigma R = 0$.

This proves the required result.

Example 2: Evaluate the residues of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at 1, 2, 3, and infinity and show that their sum is zero.

Solution: Let $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

$$Res\,(z = 1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z^2}{(z-2)(z-3)}$$

$$= \frac{1}{(1-2)(1-3)} = \frac{1}{2}$$

$$Res(z = 2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z^2}{(z-1)(z-3)} = -4$$

$$Res(z = 3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$$

$$Res(z = +\infty) = \lim_{z\to\infty} -z\, f(z) = \lim_{z\to\infty} \frac{-z^3}{(z-1)(z-2)(z-3)} = -1$$

Sum of residues $= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0.$

Example 3: Evaluate the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at z = ∞.

Solution: We expand the function in the neighbourhood of z = ∞ as follows:

$$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \left(1 - \frac{2}{z}\right)^{-1} \left(1 - \frac{3}{z}\right)^{-1}$$

$$= \left(1 + \frac{1}{z} + \cdots\right)\left(1 + \frac{2}{z} + \cdots\right)\left(1 + \frac{3}{z} + \cdots\right)$$

$$= 1 + \frac{6}{z} + \textit{ higher powers of } \frac{1}{z}.$$

Hence the residue at infinity = - 6 = negative coefficient of $\frac{1}{z}$.

Example 4: Evaluate the residues of *f*(z) where *f*(z) = $\frac{e^z}{z^2(z^2+9)}$ at z = 0, -3*i*, + 3*i*.

Solution: Here, $f(z) = \frac{e^z}{z^2(z^2+9)}$.

It's poles are *z* = 0, -3*i*, + 3*i*.

Since *z* = 0 is the pole of second order, so

$$Res\ f(z) = \frac{1}{1\,!} . \frac{d}{dz}\left(\frac{e^z}{z^2+9}\right) \text{ at } z = 0$$

$$= \frac{e^z.(z^2+9) - e^z.2z}{(z^2+9)^2} \text{ at } = 0$$

$$= \frac{1}{9}.$$

z = -3*i* is a simple pole,

$$\operatorname*{Res}_{z=-3i} f(z) = \lim_{z=-3i} \frac{(z+3i)e^z}{z^2(z^2+9)} = \lim_{z=-3i} \frac{e^z}{z^2(z^2-3i)}$$

$$= \frac{e^{-3i}}{(-3i)^2(-6i)} = -\frac{ie^{-3i}}{54}.$$

Similarly, $\underset{z=3i}{\text{Res}} f(z) = \frac{ie^{3i}}{54}$.

Example 5: Evaluate buy the method of calculus of residues $\int_C \frac{dz}{(z-1)(z+1)}$ where, C is circle $|z|=3$.

Solution: Let $f(z) = \frac{1}{(z-1)\,(z+1)}$. Poles of $f(z)$ are given by (z - 1) (z + 1) = 0 or, z = 1, -1. These are simple poles and lie within C.

$$\text{Res } (z=1) = \underset{z\to 1}{\text{Lt}}\,(z-1)f(z) = \underset{z\to 1}{\text{Lt}} \frac{(z-1)}{(z-1)\,(z+1)}$$

$$= \underset{z\to 1}{\text{Lt}} \frac{1}{z+1} = \frac{1}{2}$$

$$\text{Res } (z=-1) = \underset{z\to -1}{\text{Lt}}\,(z+1)f(z) = \underset{z\to -1}{\text{Lt}} \frac{1}{(z-1)} = \frac{-1}{2}$$

Res (z = 1) + Res (z = - 1) = $\frac{1}{2} - \frac{1}{2} = 0$

So, $\int_C f(z)dz = 2\pi i$ (sum of residues) = $2\pi i\,(0) = 0$.

Example 6: Evaluate by method of calculus of residues:

$$\int_C \frac{dz}{(z^2+1)(z-4)}, \text{ where } c \text{ is a circle } |z| = 3.$$

Solution: For poles $(z^2 + 1)(z - 4) = 0 \Rightarrow z = i, -i, 4$, $z = 4$ lies outside c.

$$\text{Res } (z=i) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)(z-4)} = \frac{1}{2i\,(i-4)}$$

$$\text{Res } (z=-i) = \lim_{z\to -i}(z-i)\frac{1}{(z-i)(z+i)(z-4)} = -\frac{1}{2i\,(-i-4)}$$

$$= \frac{1}{2i\,(i+4)}$$

$$\text{Sum of residues} = \frac{1}{2i}\left[\frac{1}{i-4} + \frac{1}{i+4}\right] = \frac{2i}{2i\,(i^2-17)} = -\frac{1}{17}$$

$$\text{Value of integral} = 2\pi i\left(-\frac{1}{17}\right) = -\frac{2\pi i}{17}.$$

Example 7: Using residue theorem, evaluate $\int_C \frac{e^z dz}{z(z-1)^2}$, where C is circle |z| = 2

Solution. Let $I = \int_C \frac{e^z dz}{z(z-1)^2}$

where C is circle |z| = 2.

Here centre is z = 0 and radius = 2.

$\therefore$ z = 0, 1 are poles lying within C.

z = 0 is a simple pole.

Res $(z = 0) = \lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} z \left[\frac{e^z}{z\,(z-1)^2}\right]$

$$= \lim_{z\to 0} \frac{e^z}{z\,(z-1)^2} = \frac{e^0}{(0-1)^2} = 1$$

z = 1 is a pole of order 2. Take $(z) = \frac{\phi\,(z)}{(z-1)^2}$, where $\phi\,(z) = \frac{e^z}{z}$.

Res(z = 1) $= \frac{\phi'(1)}{1!}, \phi'(z) = \frac{e^z\,.\,z-1.\,e^z}{e^2}$

$$\phi'(1) = \frac{e^1\,.\,1-e^1}{1^2} = 0$$

$\therefore$ Res (z = 1) = 0.

By Cauchy's residue theorem, $I = 2\pi i$ (sum of residues within c)

$= 2\pi i$ [Res (z = 0) + Res (z = 1)]

$= 2\pi i$ [1 + 0] $= 2\pi i$

Theorem 3: If an analytic function $f(z)$ has a pole at $z = \infty$, then the residue of $f(z)$ at infinity is the negative of the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ for the values of z in the neighbourhood of $z = \infty$.

Proof: Suppose $f(z)$ has pole of order m at $z = \infty$. Then $f\left(\frac{1}{z'}\right)$ has a pole of order m at z' = 0. $f\left(\frac{1}{z'}\right)$ has Laurent's expansion in the neighbourhood of z' = 0 in the form

$$f\left(\frac{1}{z'}\right) = \sum_{n=0}^{\infty} a_n z'^n + \sum_{n=1}^{\infty} b_n z'^{-n}$$
...(1)

Putting $\frac{1}{z'} = z$, we get $f(z) = \sum_{n=0}^{\infty} a_n z^{-n} + \sum_{n=1}^{m} b_n z^n$
...(2)

$$\int_C f(z)dz = \int_C \left(\sum_{n=0}^{\infty} a_n z^{-n}\right)dz + \int_C \left(\sum_{n=1}^{m} b_n z^n\right)dz$$

$$= \sum_{n=0}^{\infty} a_n \int_C z^{-n} dz + \sum_{n=1}^{m} b_n \int_C z^n dz$$

$$= a_1 \int_C z^{-1} dz \text{ , For, all the other integrals vanish.}$$

$$= a_1 \int_0^{2\pi} (re^{i\theta})^{-1} ire^{i\theta} d\theta = a_1 i \int_0^{2\pi} d\theta = 2\pi\alpha_1 i$$

Or $-\frac{1}{2\pi i}\int_C f(z)dz = -a_1$ But $-\frac{1}{2\pi i}\int_C f(z)dz = \text{Re}\, s(z=\infty)$

$\therefore$ Res $(z = \infty) = -a_1$. ...(3)

From (2), it is clear that $-a_1$ is the negative of the coefficient of $\frac{1}{z}$ in the neighbourhood of $z = \infty$. In view of this, (3) proves the required result.

Theorem 4: To prove that $\lim_{z\to\infty} - zf(z) = \text{Re}\, s(z=\infty)$ provided $f(z)$ is analytic at $z = \infty$.

Proof: Let $f(z)$ be analytic at $z = \infty$ so that $b_n = 0 \forall n\ s.t. 1 \le n \le m$, as obvious from (2).

Now (2) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

or $\quad f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$

or $\quad z\,f(z) = a_1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots$ if $a_0 = 0$.

Taking limit as $z = \infty$, we get $\lim_{z\to\infty} zf(z) = a_1$

or $$\lim_{z\to\infty}[-z\, f(z)] = -a_1 = Res\ (z=\infty)$$

if $a_0 = 0$ and the limit has a definite value.

Example 8: Find residue of $\frac{z^3}{z^2-1}\ at\ z=\infty$.

Solution: $f(z) = \frac{z^3}{z^2-1} = \frac{z^3}{z^2}\left(1-\frac{1}{z^2}\right)^{-1}$

$\Rightarrow \quad f(z) = z\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right] = z+\frac{1}{z}+\frac{1}{z^3}+\cdots$

Res $(z=\infty) = -coeff.of\,\frac{1}{z} = -1.$

12.6 *COMPUTATION OF RESIDUE AT A FINITE POLE:-*

Now we are going to discuss the computation of residue at a finite pole.

1. Residue of *f*(*z*) at a simple pole *z* = *a*.

(***i***) Res $(z=\infty) = \lim_{z\to a}(z-a)f(z)$.

(***ii***) Let $f(z) = \frac{\phi(z)}{\Psi(z)}$ have a simple pole at z = a,

where $\Psi(z) = (z-a)F(z)$ and $F(a) \neq 0$.

Then, residue of f(z) at z = a

$$= \lim_{z\to a}(z-a)f(z)$$

$$= \lim_{z\to a}(z-a)\frac{\phi(z)}{\Psi(z)}$$
$\left[\text{form}\,\frac{0}{0}\right]$

$$= \lim_{z\to a}\frac{(z-a)\left[\phi(a)+(z-a)\phi'(a)+\frac{(z-a)^2}{2!}\phi''(a)+\cdots\right]}{\Psi(a)+(z-a)\Psi'(a)+\frac{(z-a)^2}{2!}\Psi''(a)+\cdots}$$ (by Taylor's theorem)

$$= \lim_{z\to a}\frac{\phi(a)+(z-a)\Psi'(a)+\frac{(z-a)^2}{2!}\phi''(a)+\cdots}{\Psi'(a)+\frac{(z-a)}{2!}\Psi''(a)+\cdots} = \frac{\phi(z)}{\Psi'(z)}$$

For, $\Psi(a) = 0$

∴ Res $(z = a)\frac{\phi(z)}{\Psi'(z)}$

2. ***Residue at a pole of order m.***

Theorem 5: If $f(z)$ has a pole of order m at $z = 0$, then show that the residue at a is the limit of

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)] \text{as } z \to a.$$

Or

To prove that the residue of $\frac{\phi(z)}{(z-a)^m}$ at z = a is $\frac{\phi^{(m-1)}(a)}{(m-1)!}$.

Proof: Suppose *f (z)* has a pole of order *m* at z = *a* so that *f (z)* is expressible

$$f(z) = \frac{\phi(z)}{(z-a)^m}.$$

...(1)

where $\phi(a) \neq 0$ *and* $\phi(z)$ is analytic.

Residue of *f (z)* at z = a is b_1, where b_1 is given by

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_C \frac{\phi(z)dz}{(z-a)^m}$$

$$= \frac{1}{(m-1)!}\cdot\frac{(m-1)!}{2\pi i}\int_C \frac{\phi(z)dz}{(z-a)^{m-1+1}}$$

$$= \frac{1}{(m-1)!}\cdot\phi^{(m-1)}(a), \text{ by Cauchy's integral formula.}$$

Using (1), we get

$$\text{Res } (z = a) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f\, z)]\; as\; z \to a.$$

Proved.

Theorem 6 (Liouville's Theorem): If function is analytic at every point and finite at infinity, then it must be constant.

Proof: Let *f (z)* be the given function. Let *a* and *b* be any two distinct points then the only singularities of the function

$F(z) = \frac{f(z)}{(z-a)(z-b)}$ are z = a and z = b, and possibly at infinity.

But, $Res\ (z = \infty) = \lim_{z\to\infty} - z\ F\ (z)$.

Or, $Res\ (z = \infty) = \left[\lim_{z\to\infty} \frac{-z}{(z-a)(z-b)}\right] \times \left[\lim_{z\to\infty} f(z)\right] = 0$. finite number

Or, Res $(z = \infty) = 0$.

Since the sum of all the residues is zero and so

Res (z = a) + Res (z = b) + Res (z = ∞) = 0

Or, $\lim_{z\to a} (z-a)F(z) + \lim_{z\to b} (z-b)F(z) + \ 0 = 0$

Or, $\frac{f\ (a)}{a-b} + \frac{f\ (b)}{b-a} = 0$ or f (a) = f(b),

showing there by $f(z)$ is constant.

Remark: This theorem has been already proved in previous unit.

3. ***Residue at a pole z = a of any order.***

We have seen that the residue of $f(z)$ at z = a is the coefficient of $\frac{1}{z-a}$ in Laurent's expansion of f(z) and therefore coefficient of $\frac{1}{t}$ in the expansion of $f(a + t)$ as a power series.

Working rule (for computing the residue)

(1) Res $(z = a) = \lim_{z\to a} (z-a)f\ (z)$ for simple pole.

(2) Res $(z = a) = \frac{\phi^{(m-1)}\ (a)}{(m-1)\ !}$ for pole of order m, if $f(z) \frac{\phi\ (z)}{(z-a)^m}$

(3) Res $(z = a) = \frac{1}{2\pi i}\int_C\ f(z)\ dz$ for pole of any order.

(4) Res $(z = \infty) = \frac{1}{2\pi i}\int_C\ f(z)\ , Res\ (z = \infty) = \lim_{z\to\infty} - zf\ (z)$ if limit exists.

(5) Res $(z = \infty)$ = negative of the coefficient of $\frac{1}{z}$ in the expansion of f(z) in the neighbourhood of $z = \infty$.

(6) $\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res\ (z = z_r)$

(7) If $f(z) = \frac{\phi\,(z)}{\Psi\,(z)}$ has a simple pole at z = a, then Res $(z = a) = \frac{\phi\,(a)}{\Psi'(a)}$.

This formula is applied at those places, where $\Psi\,(z)$ can not be factored.

these rules are illustrated by the following examples.

Example 9: Find the residue of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ *at z* = 1, 2, 3.

Solution: Let $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$

(*i*) Take $\phi\,(z) = \frac{z^3}{(z-2)(z-3)}$. Then $f(z) = \frac{\phi\,(z)}{(z-1)^4}$

$$\text{Res}\,(z = 1) = \frac{\phi^{(3)}\,(1)}{3!} \qquad ...(1)$$

Breaking $\phi\,(z)$ into partial fractions

$$\phi\,(z) = z + 5 = \frac{8}{z-2} + \frac{27}{z-3}$$

$$\phi'(z) = 1 + \frac{8}{(z-2)^2} + \frac{27}{(z-3)^2}$$

$$\phi''(z) = -\frac{16}{(z-2)^3} + \frac{54}{(z-3)^3}$$

$$\phi^{(3)}(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\phi^{(3)}(1) = 48 - \frac{162}{16} = \frac{303}{8}$$

Using this in (1|), we get

$$\text{Res}\,(z = 1) = \frac{303}{8\times 6} = \frac{101}{16}$$

Ans.

$$\text{Res}\,(z = 2) = \lim_{z\to 2}\,(z-2)f(z)$$

$$= \lim_{z\to 2}\,\frac{z^3}{(z-1)^4(z-3)} = \frac{8}{1\times(-1)} = -8$$

Ans.

$$\text{Res}\,(z = 3) = \lim_{z\to 3}\,(z-3)f(z)$$

$$= \lim_{z \to 3} \frac{z^3}{(z-1)^4(z-2)}$$

$$= \frac{27}{(3-1)^4(3-2)} = \frac{27}{16}.$$

Ans.

Example 10: Find residue of $\frac{1}{(z^2+1)^3}$ *at z = i.*

Solution: $f(z) = \frac{1}{(z^2+1)^3} = \frac{\phi\,(z)}{(z-i)^3}$, where $\phi\,(z) = \frac{\phi\,(z)}{(z+i)^3}$

$\therefore \quad \phi'(z) = \frac{-3}{(z+i)^4}, \phi''(z) = \frac{12}{(z+i)^5}$

$\phi''(i) = \frac{12}{(i+i)^5} = \frac{12}{(2i)^5} = \frac{3}{8i}$

Res $(z = i) = \frac{\phi''(i)}{2\,!} = \frac{3}{16i}$ z = i is a pole of order 3.

Example 11: Find residue of $f(z) = \frac{1}{(z^2+a^2)^2}$ at z = *ia*.

Solution: $f(z) = \frac{1}{(z+ia)^2\,(z-ia)^2} = \frac{\phi\,(z)}{(z-ia)^2}, \phi(z) = \frac{1}{(z-ia)^2}$

⇒ f (z) has a pole of order 2 at z = ia.

Res (z = ia) = $\lim_{z+ia} \frac{\phi'(z)}{(1)} = \lim_{z \to ia} \frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{i}{4a^3}.$

Example 12: Determine the order of poles and values of residues of the function

(*i*) $cosec\ z$ **(*ii*)** $\frac{z+3}{z^2-2z}.$

Solution: (*i*) $f(z) = cosec\ z = \frac{1}{\sin z}$

Poles are given by $\sin z = 0 = \sin 0$

∴ z = 0 is simple pole of f (z).

Write $f(z) = \frac{\phi\,(z)}{\Psi\,(z)}$, then $\phi\,(z) = 1, \Psi\,(z) = \sin z.$

$Res(z = 0) = \lim_{z \to 0} \frac{\phi\,(z)}{\Psi'\,(z)} = \lim_{z \to 0} \frac{1}{\cos z} = \frac{1}{\cos(0)} = 1.$

∴ Res (z = 0) = 1.

(*ii*) $f(z) = \frac{(z+3)}{z^2-2z} = \frac{z+3}{z\,(z-2)}$

Poles of *f*(z) are z (z - 2) = 0

or, z = 0, z = 2 both are simple poles.

Res (z = 0) $= \lim_{z\to 0} (z-0)f(z)$

$$= \lim_{z\to 0} z.\left[\frac{(z+3)}{z\,(z-2)}\right] = \lim_{z\to 0} \left(\frac{z+3}{z-2}\right) = \frac{0+3}{0-2} = \frac{3}{-2}$$

Res (z = 2) $= \lim_{z\to 0} (z-2)f(z) = \lim_{z\to 2} \frac{(z-2)(z+3)}{z\,(z-2)} = \lim_{z\to 2} \frac{z+3}{z} = \frac{2+3}{2} = \frac{5}{2}.$

∴ Res $(z=0) = -\frac{3}{2}$

Res $(z=2) = \frac{5}{2}$

Example 13: Find residues of $\frac{z+1}{z^2(z-3)}$.

Solution: Let $f(z) = \frac{z+1}{z^2(z-3)}$.

Poles of f(z) are given by

z^2 (z - 3) = 0,

z = 0 is a pole of order 2

z = 3 is a simple pole.

Res $(z=3) = \lim_{z\to 3} (z-3)f(z)$

$$= \lim_{z\to 3}(z-3)\frac{(z+1)}{z^2(z-3)} = \lim_{z\to 3}\frac{z+1}{z^2} = \frac{3+1}{3^2} = \frac{4}{9}$$

For z = 0, $f(z) = \frac{\phi(z)}{z^2}, \phi(z) = \frac{z+1}{z-3}$

$\phi'(z) = \frac{1.(z-3)-1.(z+1)}{(z-3)^2}$

∴ $\phi'(0) = \frac{(0-3)-(0+1)}{(0-3)^2} = \frac{-4}{9}$

Res $(z = 0) = -\frac{4}{9}$, Res $(z = 3) = \frac{4}{9}$.

Example 14: Find the residue of $\frac{z^3}{z^2-1}$ at $z = \infty$.

Solution: Let $f(z) = \frac{z^3}{z^2-1}$.

Then, $f(z) = \frac{z^3}{z^2}\left(1 - \frac{1}{z^2}\right)^{-1}$

$$= z\left[1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right]$$

Or $f(z) = z + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \cdots$

Res $(z = \infty) = -\left(coefficient\ of\ \frac{1}{z}\right) = -(1) = -1.$

Example 15: If $\phi(\text{z})$and $\Psi(z)$ are two regular functions and *z* = a is once repeated root of $\Psi(\text{z}) = 0$ and $\phi(\text{z}) \neq 0$, then prove that residue of $\frac{\phi(\text{z})}{\Psi(\text{z})}$ at *z* = a is

$$\frac{6\phi'(\text{a})\Psi''(\text{a}) - 2\phi(\text{a})\Psi'''(\text{a})}{3[\Psi''(\text{a})]^2}$$

Solution: Given (1) *z* = a is once repeated root of $\Psi(\text{z}) = 0$.

(2) $\phi(\text{a}) \neq 0$

(3) ϕ (z) and $\Psi(z)$ are analytic functions.

(1) $\Rightarrow \Psi(z) = (z-a)^2 f(z)$ ...(4) & $f(a) \neq 0$

Then $\frac{\phi(\text{z})}{\Psi(z)} = \frac{\phi(\text{z})}{(z-a)^2 f(z)}$ ...(5) has pole at z = a of order 2.

We know that residue of $\frac{F\ (z)}{(z-a)^n}$ at z = a is $\frac{F^{(n-1)}(a)}{(n-1)}$

According to this, residue of $\frac{\phi}{\Psi} = \frac{\phi}{(z-a)^2 f}$ at z = a

is $\frac{d}{dz}\left\{\frac{\phi(\text{z})}{f(z)}\right\}$ at z = a.

So our aim is to show that $\frac{d}{dz}\left\{\frac{\phi}{f}\right\}_{at\ z=a} = \frac{6\phi'(\text{a})\Psi''(\text{a}) - 2\phi(\text{a})\Psi'''(\text{a})}{3[\Psi''(\text{a})]^2}$ ...(6)

At $z = a, \frac{d}{dz}\left\{\frac{\phi(z)}{f(z)}\right\} = \frac{\phi'(a)f(a) - f'(a)\phi(a)}{[f(a)]^2}$ ...(7)

we remove f, f' from (7).

By (4), $\Psi(z) = (z-a)^2\, f(z)$

$\Rightarrow \Psi'(z) = 2(z-a)f(z) + (z-a)^2 f'(z)$ ...(8)

Again differentiating w.r.t. z,

$$\Psi''(z) = 2f(z) + 4\,(z-a)f'(z) + (z-a)^2 f''(z)$$ ...(9)

$\Rightarrow \Psi'''(z) = 2f'(z) + 4f'(z) + 4(z-a)f''(z) + 2(z-a)f''(z) + (z-a)^2 f''(z)$ ...(10)

Putting z = a in equations. (8), (9) and (10), we get

$\Psi'(a) = 0$ ...(8), $\Psi''(a) = 2f(a)$ ...(9), $\Psi'''(a) = 6f'(a)$ ...(10)

Putting equations (9') & (10') in (7),

At $a = z, \frac{d}{dz}\left\{\frac{\phi}{dz}\right\} = \frac{\frac{1}{2}\phi'(a)\Psi''(a) - \frac{1}{6}\Psi'''(a).\phi(a)}{\left\{\frac{1}{2}\Psi''(a)\right\}^2} = \frac{6\phi'(a)\Psi''(a) - 2\Psi''(a).\phi(a)}{3\left\{\frac{1}{2}\Psi''(a)\right\}^2}$

General Problems on Calculus of Residues

Example 16: Evaluate $\int_C f(z)dz = ze^{\frac{1}{z}}\,dz$ around the unit circle.

Solution. Let $f(z) = ze^{\frac{1}{z}}\, f(z)$ lhas only one singularity (pole) at z = 0.

$$f(z) = ze^{\frac{1}{z}} = z\left[1 + \frac{1}{z} + \frac{1}{z^2\, 2!} + \frac{1}{z^3 3!} + \cdots\right]$$

$$= z + 1 + \frac{1}{z\, 2!} + \frac{1}{z^2 3!} + \cdots$$

Res (z = 0) = coeff of $\frac{1}{z}$ in the expansion of (z) = $\frac{1}{2!} = \frac{1}{2}$

By Cauchy's residue theorem,

$$\int_C f(z)dz = \int_C ze^{\frac{1}{2}}dz = 2\pi i. Res\ (z=0) = \frac{2\pi i}{2} = \pi i$$

Example 17: Evaluate $\int_C \frac{1}{\cosh(z)} dz$ where c is |z| = 2.

Solution: Write $f(z) = \frac{1}{\cosh(z)}$

Poles of *f*(z) are given by cosh (z) = 0 ⇒ $\cos(iz) = \cos\left(\frac{\pi}{2}\right)$

⇒ $iz = 2n\pi \pm \frac{\pi}{2} \Rightarrow z = -2n\pi i \mp \frac{i\pi}{2} n = 0, 1, 2, ...$

$z = \frac{i\pi}{2}, z = -\frac{i\pi}{2}$ are simple poles inside c.

Formula for Res of $\frac{f(z)}{g(z)}$ for simple pole z = a is $\frac{f(z)}{g(z)}$ when $z \to a$.

$$\text{Res}\left(z = \frac{i\pi}{2}\right) = \lim_{z\to\frac{i\pi}{2}} \frac{1}{\frac{d}{dz}(\cosh z)} = \lim_{z\to\frac{i\pi}{2}} \frac{1}{\sinh(z)}$$

$$= \lim_{z\to\frac{i\pi}{2}} \frac{1}{\frac{1}{i}\sin(iz)} = \frac{i}{\sin\left(i\frac{i\pi}{2}\right)} = \frac{i}{\sin\left(\frac{-\pi}{2}\right)}$$

Similarly, Res $\left(z = -\frac{i\pi}{2}\right) = i$

$$\int_C f(z)dz = 2\pi i\left[Res\ \left(z = \frac{i\pi}{2}\right) + \ Res\ \left(z = -\frac{i\pi}{2}\right)\right] =$$

$2\pi i[(-i) + i] = 0$

Example 18: Obtain Laurent's expansion of the function $f(z) = \frac{1}{z^2 \sinh(z)}$ at the isolated singularity and hence evaluate $\int_C \frac{1}{z^2 \sinh(z)} dz$, where *c* is circle |*z* - 1| = 2.

Solution: $f(z) = \frac{dz}{z^2 \sinh(z)} = \frac{1}{z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)}$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{6} + \frac{z^4}{120}...\right) + \left(\frac{z^2}{6} + \frac{z^4}{120}...\right)^2 + \cdots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36}...\right] = \frac{1}{z^3} - \frac{1}{6_z} + z\left(\frac{1}{36} - \frac{1}{120}\right)...$$

or, $f(z) = \frac{1}{z^3} - \frac{1}{6_z} + \frac{7z}{360} \ldots$
...(1)

which is the required Laurent's expansion.

z = 0 is a pole inside circle c given by |z - 1| = 2

Res (z = 0) = coeff. of $\frac{1}{z-0}$ in the expansion (1) = $-\frac{1}{6}$

By Cauchy's residue theorem,

$$\int_C \frac{1}{z^2 \sinh(z)} dz = 2\pi i.\text{Re}\, s(z=0) = -\frac{2\pi i}{6} = -\frac{i\pi}{3}$$

Example 19: *Evaluate* $\int_C z^4 e^{1/z} dz$ *where c is circle* |*z*| = 1.

Solution: Let $f(z) = z^4\, e^{\frac{1}{z}}$, then z = 0 is the only pole of it which is inside c.

$$f(z) = z^4\, e^{\frac{1}{z}} = z^4 \left(1 + \frac{1}{z} + \frac{1}{z^2\, 2!} + \frac{1}{z^3 3!} + \frac{1}{z^4 4!} + \frac{1}{z^5 5!} \ldots\right)$$

Res (z = 0) = coeff. of $\frac{1}{z}$ in this expansion = $\frac{1}{5!} = \frac{1}{120}$

$$\int_C f(z)dz = \int_C z^4\, e^{\frac{1}{z}}\, dz = 2\pi i. Res\ (z = 0) = \frac{2\pi i}{120} = \frac{\pi i}{60}$$

Theorem 7: If AB is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle

|z - a| = r and if

$$\lim_{z\to a} (z-a)f(z) = k \text{ (constant)},$$

then $$\lim_{r\to 0} \int_{AB} f(z)dz = i\,(\theta_2 - \theta_1)k.$$

Proof: Since**,** $\lim_{z\to a} (z-a)f(z) = k$

∴ given $\varepsilon > 0, \exists\ \delta$ depending upon ε s.t.

|(z - a) f (z) - k| < ε. For |z - a| < δ

But |z - a| = r. Therefore if we take r < δ.

then $|(z - a) f(z) - k| < \varepsilon$ on the arc AB

This $\Rightarrow (z-a)f(z) - k = \eta$ where $|\eta| < \varepsilon$

$$\Rightarrow f(z) = \frac{k+\eta}{z-a}$$

$$\therefore \quad \int_{AB} f(z)dz = \int_{AB} \left(\frac{k+\eta}{z-a}\right) dz$$

$$= \int_{\theta_1}^{\theta_2} \left(\frac{k+\eta}{re^{i\theta}}\right) re^{i\theta} i d\theta, \; z-a = re^{i\theta}$$

$$= ki \int_{\theta_1}^{\theta_2} d\theta + i\int_{\theta_1}^{\theta_2} \eta \, d\theta$$

Check your progress

Problem 1: Apply the calculus of residues to prove that $\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}$.

Problem 2: Find the residue of $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z = 1, 2, 3$.

12.7 *SUMMARY:-*

The Residue Theorem unit establishes a profoundly powerful tool in complex analysis, stating that if a function is analytic inside and on a simple closed contour *C*, except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside *C*, then the contour integral of the function around *C* is simply $2\pi i$ times the sum of the residues of the function at those singularities, formally expressed as . This theorem elegantly reduces the often-difficult evaluation of complex contour integrals to the algebraic computation of residues at the enclosed poles, providing an essential method for solving definite real integrals, evaluating infinite series, and has far-reaching applications in physics and engineering.

12.8 *GLOSSARY:-*

- Residue at a pole
- Residue at infinity
- Cauchy residue theorem.

12.9 *REFERENCES:-*

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12.10*SUGGESTED READING: -*

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- J.B. Conway, (2000), Functions of One Complex Variable, Narosa Publishing House,

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12.11*TERMINAL QUESTION:-*

Long answer type question

1: State and prove Cauchy's residue theorem.

2: Find poles and resides of the function $f(z) = \frac{2z+1}{z^2-z-2}$.

3: Evaluate $\int_C \frac{dz}{\cosh(z)}$ where C is $|z| = 2$

4: Using residue theorem evaluate $\int_C \frac{e^z}{z(z-1)^2}$, where C is circle $|z| = 2$

5: Find resides of the function $\frac{z^3}{(z-1)(z-2)(z-3)}$ at the point $z=1,2,3$ respectively.

Short answer type question

1: Evaluate the integral $\int_{|z|=2} \frac{e^z}{z(z-1)}dz$ using residue theorem.

2: Evaluate the integral $\int_{|z|=3} \frac{\cos z}{z^2+1}dz$ using residue theorem

3: $\int_{|z|=4} \frac{1}{(z^2+9)(z+2)}dz$

4: Evaluate the integral $\int_{|z|=1} \frac{e^z}{z^3}dz$ using residue theorem.

5: Evaluate the integral $\int_{|z|=1} e^{1/z}dz$ using residue theorem.

6: Evaluate the integral $\int_{|z|=2} \frac{\sin z}{z^4}dz$ using residue theorem.

7: Evaluate the integral $\int_{|z|=2} \frac{z^2+1}{z(z^2+4)}dz$ using residue theorem.

Objective type question:

1: The residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at z = 1, 2, 3 are respectively.

a) $\frac{1}{2}, -8, \frac{27}{2}$

b) $1, -8, \frac{27}{2}$

c) $\frac{1}{2}, 0, \frac{27}{2}$

d) None of these

2: The number of poles of $f(z) = \frac{1}{z(z^2+3)(z^2+2)^3}$ inside the circle $|z| = 1$ are:

a) 1

b) 9

c) 5

d) 2

3: The Residue Theorem states that for a simply connected domain D and a function *f*(z) analytic on D except for finitely many isolated singularities inside a simple closed contour C, the integral $\oint_C f(z)dz$ is equal to:

a) The sum of all singularities of *f*(z) in *D*.

b) 2πi times the sum of the residues of *f*(z) at the singularities inside *C*.

c) The product of the residues of *f*(z) at the singularities inside *C*.

d) 2π times the sum of the residues of *f*(z) at the singularities inside *C*.

4: The residue of $f(z) = 1/(z^2 + 1)$ at $z = i$ is:

a) -i/2

b) i/2

c) 1/(2i)

d) 1/(2i)

5: Consider the function $f(z) = \frac{e^z}{z^2}$. Which of the following is the value of $\oint_{|Z|=1} f(z)\,dz$?

a) 0

b) 2πi

c) πi

d) -2πi

6: What is the residue of $f(z)=\dfrac{\cos(z)}{z^5}$ at z = 0?

a) 1/24

b) -1/24

c) 1/4!

d) 0

7: To evaluate the real integral $\int_0^{2\pi}\dfrac{d\theta}{5+3\cos\theta}$ using the Residue Theorem, we substitute $z=e^{i\theta}$. The resulting contour integral is:

a) $\oint_{|z|=1}\dfrac{dz}{3z^2+10z+3}$

b) $\oint_{|z|=1}\dfrac{2dz}{i(3z^2+10z+3)}$

c) $\oint_{|z|=1}\dfrac{2dz}{i(z^2+1)}$

d) $\oint_{|z|=1}\dfrac{dz}{z(5+3\cos z)}$

Fill in the blanks:

1: The coefficient a_{-1} in the Laurent series expansion of *f*(z) about an isolated singularity z_0 is called the ……. of *f*(z) at z_0.

2: According to Cauchy's residue theorem, $\oint_C f(z)dz=$..................., where the sum is over all singularities inside *C*.

3: To evaluate the real integral $\int_{-\infty}^{\infty}\dfrac{dx}{1+x^2}$ using the residue theorem, one typically uses a contour consisting of the real axis from -*R* to *R* and a large _________ in the upper half-plane.
Answer: Semicircle (or semi-circle)

4: The function $f(z)=\cot z$ has singularities at $z=n\pi$. These are all _________ poles.

Match the Following:

Column A: Function

1. $f(z) = 1 / (z^2 + a^2)$
2. $f(z) = e^z / z^n$
3. $f(z) = \sin(z)/z^3$
4. $f(z) = 1 / (z - z_0)^3$

Column B: Residue at the specified singularity
A. 1/((n-1)!)
B. 0
C. 1/(2ai)
D. 0 (Residue for a pole of order >1 where the relevant Laurent coefficient is zero)

12.12*ANSWERS:-*

Answer of check your progress

Problem 2: $\operatorname{Re} s(z=1) = \frac{101}{16}$; $\operatorname{Re} s(z=2) = -8$; $\operatorname{Re} s(z=3) = \frac{27}{16}$

Answer of long answer type question

2: Poles are z = 2, -1, Resides = $\frac{5}{3}, \frac{1}{3}$. **3:** 0

4: $2\pi i$ **5:** $\frac{1}{2}, -8, \frac{27}{2}$

Answer of short answer type question:

1: $2\pi i(e-1)$ **2:** 0 **3:** $\frac{2\pi}{39}(2+3i)$

4: $i\pi$ **5:** $2\pi i$ **6:** $-\frac{i\pi}{3}$

7: $\frac{i\pi}{2}$

Answer of objective question:

1: a **2:** a **3:** b **4:** c

5: b **6:** b **7:** b

Answer of fill in the blanks:

1: Residue **2:** $2\pi i \times \Sigma \text{Res}(f, z_k)$ **3:** Semicircle

4: Simple

Answers of match the following:

1 ⇒ C (Residue at z = ai is 1/(2ai))
2 ⇒ A (Residue at z = 0 is 1/(n-1)!)
3 ⇒ B (The singularity at z = 0 is removable, so residue is 0)
4 ⇒ D (The Laurent series has a coefficient a_{-3}=1, but a_{-1}=0)

UNIT-13: Application of Residue Theorems

CONTENTS

13.1 *INTRODUCTION:-*

The Residue Theorem, a crowning achievement of complex integration, elegantly reduces the global problem of evaluating a contour integral to the local, often straightforward, algebraic computation of residues at enclosed singularities. While its initial power is demonstrated through the evaluation of standard real integrals, its true versatility is unlocked when confronting more sophisticated problems, such as those involving integrals over the entire real line of functions combined with trigonometric expressions like e^{imx} or $\sin(mx)$. For these, the successful application often hinges on establishing that the integral over an auxiliary contour, typically a large semicircle, vanishes in the limit; a step crucially supported by Jordan's Inequality, which provides the necessary bound for the decay of the exponential $e^{-m\text{Im}(z)}$ along the arc. This foundational strategy extends the theorem's reach far beyond real calculus, enabling the summation of infinite series via contour integration of functions like $\pi\cot(\pi z)$, the efficient computation of inverse Laplace and Fourier transforms, and even finding applications in number theory to prove results like the Prime Number Theorem, solidifying the Residue Theorem as an indispensable tool across pure and applied mathematics.

13.2 *OBJECTIVE:-*

After the study of this chapter, learner shall understand:

- **To understand the concept and statement of the Residue Theorem:** Explain how the theorem relates contour integrals of analytic functions to the sum of residues enclosed within a closed contour.
- **To learn methods for calculating residues:** Develop proficiency in determining residues at simple, multiple, and essential poles using various techniques such as limits and Laurent series expansion.
- **To apply the residue theorem to evaluate complex integrals:** Demonstrate how to compute contour integrals around closed paths in the complex plane using residues.
- **To evaluate real definite integrals using contour integration:** Illustrate how real improper integrals—especially those involving rational, trigonometric, and exponential functions—can be solved efficiently with the help of the residue theorem.
- **To explore the role of singularities in integration:** Understand how the location and type of singular points affect the value of contour integrals.
- **To strengthen problem-solving skills in complex integration:** Apply theoretical knowledge to practical examples and develop strategies for selecting suitable contours in various integration problems.
- **To appreciate the theoretical and practical importance of residues:** Recognize the significance of the residue theorem in mathematics, physics, and engineering—particularly in evaluating integrals, solving differential equations, and studying signal processing.

13.3 *JORDAN'S INEQUALITY:-*

If $0 \le \theta \le \pi/2$, then the inequality $\frac{20}{\pi} \le \sin 0 \le 0$ is known as Jordan's inequality.

Theorem 1 (Jordan's Lemma): If $f(z)$ is analytic except at finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)dz = 0, m > 0$$

where Γ denotes the semi-circle $|z| = R, I(z) > 0$. Here R is taken so large that all the singularities of $f(z)$ lie within the semi

circle Γ. (No singularitiy lies on the boundary of the semi circle).
Proof: $\because f(z) \to 0$ uniformly as $|z| \to \infty$

$$\therefore \ \exists \varepsilon > 0 \text{ s.t. } |f(z)| \le \varepsilon \forall z \text{ on } \Gamma. \qquad (1)$$

$$|z| = R \Rightarrow z = Re^{i\theta} \Rightarrow dz = iRe^{i\theta}d\theta \Rightarrow |dz| = Rd\theta \qquad (2)$$

$$e^{imz} = \exp\left(imRe^{i\theta}\right) = \exp[imR(\cos\theta + i\sin\theta)]$$

$$e^{imz} = e^{imR\cos\theta} \cdot e^{-mR\sin\theta}$$

or

Hence $\left|e^{imz}\right| = e^{-mR\sin\theta}$ ... (3)

as $\left|e^{ip}\right| = 1$ for every real p.

$$\left|\int_\Gamma e^{imz}f(z)dz\right| \le \int_\Gamma \left|e^{imz}\right| \cdot |f(z)| \cdot |dz|$$

$$< \int_0^\pi e^{-mR\sin\theta}\varepsilon \cdot Rd\theta$$

$$\le \int_0^\pi \varepsilon R \cdot e^{-mR(2\theta/\pi)}d\theta \text{ as } \frac{20}{\pi} \le \sin\theta \le \theta$$

$$= \frac{\pi\varepsilon R}{-2mR}\left[e^{-2mR\theta/\pi}\right]_0^\pi = \frac{\pi\varepsilon}{2m}\left(1 - e^{-2mR}\right) \to 0 \text{ as } R \to \infty$$

$$\therefore \lim_{R\to\infty} \int_\Gamma e^{imz}f(z)dz = 0, m > 0.$$

13.4 *EVALUATION OF REAL DEFINITE INTEGRALS:-*

This section is mainly devoted to the evaluation of real definite integrals. We evaluate these integrals by the method of contour integration. The contour may be a circle, semi-circle or quadrant of circle

Method for writing the function $\boldsymbol{f(z)}$ of a given integral

Integral		**Function $\boldsymbol{f(z)}$ and Contour**	**Contour**
1.	$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{1 + 2p\cos\theta + p^2}$	$f(z) = \frac{1 + e^{i6\theta}}{1 + 2p\cos\theta + p^2}$ as $2\cos^2 3\theta = 1 + \cos 6\theta$	Unit circle
2.	$\int_0^{2\pi} e^{\cos\theta} \cdot \cos$ $(\sin\theta - n\theta)d\theta$	$f(z) = e^{\cos\theta}$ $\exp i(\sin\theta - n\theta), z = e^{i\theta}$	Unit circle

3.	$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta}$	$f(z)=\frac{e^{i2\theta}}{5+4\cos\theta}, z = e^{i\theta}$	Unit circle
4.	$\int_0^{2\pi} \frac{\sin 2\theta d\theta}{5+4\cos\theta}$	$f(z)=\frac{e^{i2\theta}}{5+4\cos\theta}, z = e^{i\theta}$	Unit circle
5.	$\int_0^{\infty} \frac{\sin 2x dx}{5+4\cos x}$	$f(z)=\frac{e^{i2z}}{5+4\cos z}$	Semi circle
6.	$\int_0^{\infty} \frac{\sin^2 mx dx}{x^2(x^2+a^2)}$	$f(z)=\frac{1-e^{i2mz}}{z^2(z^2+a^2)}$ as $2\sin^2 mx = 1-\cos 2mx$	Semi circle idented at $z=0$
7.	$\int_0^{\infty} \frac{\cos^2 mx dx}{x^2(x^2+a^2)}$	$f(z)=\frac{1+e^{i2mz}}{z^2(z^2+a^2)}$ as $2\cos^2 mx = 1+\cos 2mx$	Semi circle idented at $z=0$
8.	$\int_{-\infty}^{\infty} \frac{\cos mx dx}{x^2+x+1}$	$f(z)=\frac{e^{imz}}{z^2+z+1}$	Semi circle
9.	$\int_{-\infty}^{\infty} \frac{x\sin mx dx}{x^2+a^2}$	$f(z)=\frac{ze^{imz}}{z^2+a^2}$	
10.	$\int_0^{\infty} \frac{(\log x)^2 dx}{x^2+a^2}$	$f(z)=\frac{(\log z)^2}{z^2+a^2}$	Semi circle idented at $z=0$
11.	$\int_0^{\infty} \frac{\log x dx}{x^2+1}$	$f(z)=\frac{\log z}{z^2+1}$	Semi circle idented at $z=0$
12.	$\int_0^{\infty} \frac{\log x dx}{x+1}$	$f(z)=\frac{(\log z)^2}{z+1}$	Double circle
13.	$\int_0^{\infty} \frac{\log x dx}{x^2+x+1}$	$f(z)=\frac{(\log z)^2}{z^2+z+1}$	Double circle
14.	$\int_0^{\infty} \frac{\log x dx}{x^4+x^2+1}$	$f(z)=\frac{\log z}{z^4+z^2+1}$	Semi circle idented at $z=0$

15.	$\int_0^\infty \frac{(\log x)^2 dx}{x+1}$	$f(z) = \frac{(\log z)^3}{z+1}$	Double circle idented at $z = 0$

Remark 1: Note the denominators of integrals $10,11,14$ are even functions of x whereas denominators of $12,13,15$ are not even functions of x.

2: If $\log z$ or $(\log z)^n$ occurs in the integrand, then the contour will be idented at $z = 0$, whereas if integrand contains $\log(x^2+1)$ or $\log(x+1)$, then it will not be idented at $z = 0$. Observe integrals 11 and 14.

13.5 *INTEGRATION ROUND THE UNIT CIRCLE:-*

We proceed to evaluate the integrals of the type

$$\int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta$$

If we take $z = e^{i\theta}$, then the above takes the form

$$\int_C \phi(z) dz. \text{For } \frac{z+z^{-1}}{2} = \cos\theta, \frac{z-z^{-1}}{2i} = \sin\theta$$

where C is the unit circle $|z| = 1$.

Example 1: Evaluate $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$

Solution: Take C as unit circle $|z| = 1$ and so $z = e^{i\theta}$,

$$dz = e^{i\theta} i d\theta = iz d\theta \Rightarrow \frac{dz}{iz} = d\theta$$

Let$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_C \left(\frac{di}{iz}\right) \cdot \frac{1}{a + \frac{b}{2}(e^{i0} + e^{-i\theta})}$

$$= \int_C \left(\frac{dz}{iz}\right) \cdot \frac{1}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{2}{ib} \int_C \frac{dz}{\left(z^2 + \frac{2a}{b} z + 1\right)}$$

or,

$$I = \frac{2}{ib}\int_C f(z)dz, \text{ where } f(z) = \frac{1}{z^2 + \frac{2a}{b}z + 1}$$

For poles : $z^2 + \frac{2a}{b}z + 1 = 0 \Rightarrow z = -\frac{a \pm \sqrt{(a^2-b^2)}}{b}$

Take $\alpha = -\frac{a+\sqrt{(a^2-b^2)}}{b}, \beta = -\frac{a-\sqrt{(a^2-b^2)}}{b}$

Then $\alpha\beta = 1$ & $|\beta| > 1. \therefore\ |\alpha| < 1$

Only Pole $z = \alpha$ lies inside c.

$$\text{Res}(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\alpha - \beta} = \frac{b}{2\sqrt{(a^2-b^2)}}$$

$$\int_C f(z)dz = 2\pi i \cdot \frac{b}{2\sqrt{(a^2-b^2)}} = \frac{\pi i b}{\sqrt{(a^2-b^2)}}$$

$$I = \frac{2}{ib}\int_C f(z)dz = \left(\frac{2}{ib}\right)\frac{\pi i b}{\sqrt{(a^2-b^2)}} = \frac{2\pi}{\sqrt{(a^2-b^2)}}$$

Example 2: Prove that,

$$\int_0^{2\pi}\frac{\cos^2 3\theta d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p}, 0 < p < 1$$

Solution: Let C denote unit circle $|z| = 1$. and

$$I = \int_0^{2\pi}\frac{\cos^2 30d0}{1-\rho\cos 20 + p^2} \text{ s.t. } 0 < p < 1$$

Then

$$I = \frac{1}{2}\int_0^{2\pi}\frac{(1+\cos 60)d\theta}{1-2p\cos 2\theta + p^2}$$

$$= \text{R.P. } \frac{1}{2}\int_0^{2\pi}\frac{\left(1+e^{i60}\right)d\theta}{1-p(e^{i2\theta}+e^{-20})+p^2}$$

Putting $z = e^{i\theta}$ so that $dz = ie^{i\theta} d\theta, \frac{dz}{iz} = d\theta,$ we get

$$I = \text{R.P} \cdot \frac{1}{2} \int_C \frac{(1+z^6)}{1 - p(z^2 + z^{-2}) + p^2} \frac{dz}{iz}$$

$$= \text{R.P.} \left(\frac{-1}{2ip}\right) \int_C \frac{z(1+z^6)dz}{z^4 - \left(\frac{1+p^2}{p}\right) z^2 + 1}$$

or $I =$ R.P. $\frac{-1}{2ip} \int_C f(z)dz$ ... (1), $f(z) = \frac{z(1+z^6)}{z^4 - \left(\frac{1+p^2}{p}\right) z^2 + 1}$

Poles of $f(z)$ are given by $z^4 - \left(\frac{1+p^2}{p}\right) z^2 + 1 = 0$

or $pz^4 - (1+p^2)z^2 + p = 0$ or $z^2 = \frac{(1+p^2) \pm \left[(1+p^2)^2 - 4p^2\right]^{1/2}}{2p}$

or $z^2 = \frac{(1+p^2) \pm (1-p^2)}{2p} = \frac{1}{p}, p$ so that $z = \pm \frac{1}{\sqrt{p}}, \pm\sqrt{p}$

The poles lying within the unit circle C are $\pm\sqrt{p}$ as $0 < p < 1$

$\text{Res}(z = \sqrt{p}) + \text{Res}(z = -\sqrt{p})$

$$= \lim_{z \to \sqrt{p}} (z - \sqrt{p}) f(z) + \lim_{z \to -\sqrt{p}} (z + \sqrt{p}) f(z)$$

$$= \lim_{z \to \sqrt{p}} \frac{(z - \sqrt{p}) z (1+z^6)}{(z^2 - p)(z^2 - 1/p)} + \lim_{z \to -\sqrt{p}} \frac{(z + \sqrt{p}) z (1+z^6)}{(z^2 - p)(z^2 - 1/p)}$$

$$= \frac{(1+p^3)\sqrt{p}}{(\sqrt{p} + \sqrt{p})(p - 1/p)} + \frac{(-\sqrt{p})(1+p^3)}{(-\sqrt{p} - \sqrt{p})(p - 1/p)}$$

as $z^2 - p = (z - \sqrt{p})(z + \sqrt{p})$

$$= \frac{p(1+p^3)}{p^2 - 1}$$

$\int_C f(z)dz = 2\pi i($ Sum of residues within $C) = \frac{2\pi i p(1+p^3)}{p^2-1}$

Now by (1), $I =$ R.P. $\left(-\frac{1}{2ip}\right) \frac{2\pi i p(1+p^3)}{p^2-1} =$ R.P. $\left(\frac{1+p^2-p}{1-p}\right)\pi$

or

$$I = \left(\frac{1+p^2-p}{1-p}\right)\pi$$

Example 3: Evaluate $\int_0^\pi \frac{a d\theta}{a^2 + \sin^2\theta}$ where, $a > 0$.

Solution: Let $I = \int_0^\pi \frac{a d\theta}{a^2 + \sin^2\theta}$

Then

$$I=\int_0^{\pi}\frac{2ad\theta}{2a^2+2\sin^2\theta}=\int_0^{\pi}\frac{2ad0}{2a^2+1-\cos 20}$$

$$=\int_0^{2\pi}\frac{adt}{2a^2+1-\cos t}, \text{putting } 2\theta=t$$

$$=\int_0^{2\pi}\frac{adt}{2a^2+1-\frac{1}{2}(e^{it}+e^{-it})}$$

Putting $z=e^{it}$ so that $dz=ie^{it}dt,$ we get $I=\int_C\frac{2a}{2(2a^2+1)-(z+z^{-1})}\cdot\frac{dz}{iz},$ where C is unit circle $|z|=1$

or $I=\frac{2a}{i}\int_C\frac{dz}{2(2a^2+1)z-z^2-1}=2ai\int_C\frac{dz}{z^2-2(2a^2+1)z+1}$

or

$I=2ai\int_C f(z)dz$

$...(1), f(z)=\dfrac{1}{z^2-2(2a^2+1)z+1}$

Poles of $f(z)$ are given by

$$z^2-2(2a^2+1)z+1=0$$

or

$$z=\frac{2(2a^2+1)\pm\sqrt{[4(2a^2+1)^2-4]}}{2}$$
$$=2a^2+1\pm\sqrt{[(2a^2+1)^2-1]}=2a^2+1\pm 2a\sqrt{(a^2+1)}$$

Taking $\alpha=2a^2+1+2a\sqrt{(a^2+1)}$

$$\beta=2\alpha^2+1-2a\sqrt{(a^2+1)}$$

we get $z=\alpha,\beta.$ Evidently. $|\alpha|>1$ and $|\beta|<1.$ $f(z)$ has only one simple pole $z=\beta$ lying within C.

$$\operatorname{Res}(z=\beta)=\lim_{z\to\beta}(z-\beta)f(z)=\lim_{z\to\beta}\frac{(z-\beta)\cdot 1}{(z-\alpha)(z-\beta)}$$
$$=\frac{1}{\beta-\alpha}=\frac{1}{-4a\sqrt{(a^2+1)}}$$

By Cauchy's residue theorem

$$\int_C f(z)dz=2\pi i(\text{ sum of residues within } C)=\frac{2\pi i}{-4a(a^2+1)^{1/2}}$$

Using this in (1), we get $I = \frac{2ia.2\pi i}{-4a(a^2+1)^{1/2}} = \frac{\pi}{(1+a^2)^{1/2}}$

Example 4: Prove that

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta d\theta}{3+2\cos\theta} = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

n being positive integer.

Solution: Let $I = \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta d\theta}{3+2\cos\theta}$

Then

$$I = \text{ R.P. } \int_0^{2\pi} \frac{\left(1+e^{i\theta}+e^{-i\theta}\right)^n e^{in\theta} d\theta}{3+e^{i\theta}+e^{-i\theta}}$$

Putting $z = e^{i\theta}$ so that $dz = ie^{i\theta}d\theta$, we get

$$I = \text{ R.P. } \int_C \frac{(1+z+z^{-1})^n z^n}{(3+z+z^{-1})} \cdot \frac{dz}{iz}$$

where C is unit circle.

$$I = \text{ R.P. } \frac{1}{i}\int_C \frac{(z^2+z+1)^n dz}{z^2+3z+1} \text{ so } I = \text{ R.P. } \frac{1}{i}\int_C f(z)dz \qquad (1)$$

where

$$f(z) = \frac{(z^2+z+1)^n}{z^2+3z+1}$$

Poles of $f(z)$ are given by $z^2+3z+1=0$. This gives $z = \frac{-3\pm\sqrt{5}}{2}$. Take $\alpha = \frac{-3+\sqrt{5}}{2}, \beta = \frac{-3-\sqrt{5}}{2}$

Then $\alpha\beta = 1, |\alpha| < 1, |\beta| > 1$.

$\therefore$ $f(z)$ has only one simple pole $z = \alpha$ lying within C.

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)(z^2+z+1)^n}{(z-\alpha)(z-\beta)}$$

$$= \frac{(\alpha^2+\alpha+1)^n}{\alpha-\beta} = \frac{1}{5}\left[\left(\frac{-3+\sqrt{5}}{2}\right)^2 + \frac{-3+\sqrt{5}}{2} + 1\right]^n$$

$$= \frac{1}{\sqrt{5}}\left[\frac{9-6\sqrt{5}+5-6+2\sqrt{5}+4}{4}\right]^n = \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ Sum of residues within } C) = \frac{2\pi i(3-\sqrt{5})^n}{\sqrt{5}}$$

Using this in (1), $I = \text{R.P.} \frac{1}{i} \cdot \frac{2\pi i}{\sqrt{5}} (3-\sqrt{5})^n$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

Example 5: Prove that $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a+b\cos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{(a^2-b^2)}\right]$, where $a > b > 0$

Solution: Let $I = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{(1-\cos 2\theta)d\theta}{2a+2b\cos\theta}$

Then

$$I = \text{ R.P. } \int_0^{2\pi} \frac{\left(1-e^{i2\theta}\right)d\theta}{2a + b(e^{i\theta}+e^{-i\theta})}$$

Putting $z = e^{i\theta}$ so that $dz = ie^{i\theta}d\theta$, we get

$$I = \text{ R.P. } \int_C \frac{(1-z^2)}{2a+b(z+z^{-1})}\frac{dz}{iz}$$

where C is unit circle $|z| = 1$.
or

$$I = \text{ R.P. } \frac{1}{i}\int_C \frac{(1-z^2)dz}{2az+bz^2+b} = \text{ R.P. } \frac{1}{ib}\int_C \frac{(1-z^2)dz}{z^2+(2a/b)z+1}$$

$$I = \text{ R.P. } \frac{1}{ib}\int_C f(z)dz \text{ ... (1)}, f(z) = \frac{1-z^2}{z^2+(2a/b)z+1}$$

Poles of $f(z)$ are given by $z^2 + \frac{2a}{b}z + 1 = 0$.
or

$$z = \frac{-(2a/b) \pm [(4a^2/b^2)-4]^{1/2}}{2} = \frac{-a \pm (a^2-b^2)^{1/2}}{b}$$

Take

$$\alpha = \frac{-a+(a^2-b^2)^{1/2}}{b}, \beta = \frac{-a-(a^2-b^2)^{1/2}}{b}$$

Then $\alpha\beta = 1, |\beta| > 1, |\alpha| < 1.$
Hence $f(z)$ has a simple pole a $z = \alpha$ within C.

$$\text{Res}(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{1-z^2}{z-\beta}$$
$$= \frac{1-\alpha^2}{\alpha-\beta} = \frac{\alpha(1/\alpha - \alpha)}{\alpha-\beta}$$
$$= \frac{\alpha(\beta-\alpha)}{\alpha-\beta} = -\alpha \text{ as } \alpha\beta = 1$$
$$\int_C f(z)dz = 2\pi i\cdot\text{Res}(z=\alpha) = -2\pi i\alpha$$

Now (1) becomes
or

$$I = \text{ R.P. } \frac{1}{ib}(-2\pi i\cdots) = \text{ R.P. }\left(\frac{-2\pi\alpha}{b}\right)$$
$$I = \text{ R.P. } \frac{2\pi}{b^2}\left[a-(a^2-b^2)^{1/2}\right] = \frac{2\pi}{b^2}\left[a-(a^2-b^2)^{1/2}\right]$$

Example 6: Evaluate $\int_0^\pi \frac{\sin^4\theta d\theta}{a+b\cos\theta}$, where $a > b > 0.$
Solution: $a > b > 0 \Rightarrow a^2 - b^2 > 0 = \sqrt{a^2-b^2} = \text{real}$

Let

$$I = \int_0^\pi \frac{\sin^4\theta d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{\sin^4\theta d\theta}{a+b\cos\theta} \qquad (1)$$

Take C as $|z| = 1$ or, $z = e^{i\theta}, dz = e^{i\theta}id\theta, \frac{dz}{iz} = d0$
By eqs. (1),

$$I = \int_C\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4\frac{d\theta}{2a+b(e^{i\theta}+e^{-i\theta})} = \frac{1}{16}\int_C\left(z-\frac{1}{z}\right)^4\cdot\frac{dz}{iz}\cdot\frac{1}{2a+(z+z^{-1})b} \,(2)$$

or, $I = \frac{1}{16i}\int_C\frac{(z^2-1)^4dz}{z^4[2az+b(z^2+1)]}$ or $I = \frac{1}{16ib}\int_C f(z)dz$
where $f(z) = \frac{(z^2-1)^4}{z^4\left[z^2+\frac{2az}{b}+1\right]}$

For poles of $f(z): z^4\left[z^2+\frac{2az}{b}+1\right] = 0$
$\Rightarrow z = 0,$ (pole of order 4) and $z^2+\frac{2az}{b}+1 = 0$
or

$$bz^2 + 2az + b = 0 \text{ or, } z = -\frac{a \pm \sqrt{a^2 - b^2}}{b}$$

Take $\alpha = -\frac{a+\sqrt{a^2-b^2}}{b}, \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$. Then $\alpha\beta = 1$ and $\alpha < 1$ and so $\beta > 1$. Thus $z = 0$ (pole of order 4) and $z = \alpha$, (pole of order one) lie within C.

$$\text{Res}(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)(z^2-1)^4}{z^4(z-\alpha)(z-\beta)}$$

$$= \frac{(\alpha^2-1)^4}{\alpha^4(\alpha-\beta)} = \left(\alpha - \frac{1}{\alpha}\right)^4 \cdot \frac{b}{2\sqrt{a^2-b^2}} = (\alpha-\beta)^4 \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \left(\frac{2\sqrt{a^2-b^2}}{b}\right)^4 \cdot \frac{b}{2\sqrt{a^2-b^2}} = \frac{8}{b^3}(a^2-b^2)^{3/2}$$

$$\text{Res }(z = 0) = \text{ coeff. of } \frac{1}{z} \text{ in expansion of } \frac{(z^2-1)^4}{z^4\left(z^2 + \frac{2az}{b} + 1\right)}$$

$$= \text{ coeff. of } \frac{1}{z} \text{ in } (1-z^2)^4\left[1 + \left(z^2 + \frac{2az}{b}\right)\right]^{-1}$$

$$= \text{ coeff of } \frac{1}{z} \text{ in}$$

$$(1 - {}^4C_1 z^2 + {}^4C_2 z^4 \dots)\left[1 - \left(z^2 + \frac{2az}{b}\right) + \left(z^2 + \frac{2az}{b}\right)^2 - \left(z^2 + \frac{2az}{b}\right)^3 + \cdots\right]$$

$$= \left[2\left(\frac{2a}{b}\right) - \left(\frac{2a}{b}\right)^3\right] + {}^4C_{:}\left(\frac{2a}{b}\right) = -\frac{8a^3}{b^3} + \frac{12a}{b}$$

$$\text{Res}(z = 0) + \text{Res}(z = \alpha) = \frac{8}{b^3}(a^2-b^2)^{3/2} + \left(\frac{12a}{b} - \frac{8a^3}{b^3}\right)$$

$$\int_C f(z)dz = 2\pi i[\text{Res}(z = 0) + \text{Res}(z = \alpha)]$$

$$= 2\pi i\left[\frac{8}{b^3}(a^2-b^2)^{3/2} + \frac{12a}{b} - \frac{8a^3}{b^3}\right]$$

Putting this in eq. (2), we get

$$I = \frac{\pi}{b^4}\left[(a^2-b^2)^{3/2} - a^3 + \frac{3}{2}ab^2\right]$$

Example 7: Prove that $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

Solution: Let $I = \int_0^{2\pi} \frac{\cos 20 d\theta}{5+4\cos\theta} =$ R.P. $\int_0^{2\pi} \frac{e^{i2\theta} d\theta}{5+2(e^{i\theta}+e^{-i\theta})}$

Putting $z = e^{i\theta}, dz = ie^{i\theta} d\theta, \frac{dz}{iz} = d\theta.$

$I =$ R.P. $\frac{1}{i}\int_C \frac{z^2 dz}{5+2(z+z^{-1})} \cdot \frac{1}{z}$, where C is the circle $|z| = 1.$

or, $I =$ R.P. $\frac{1}{i}\int_C \frac{z^2 dz}{2z^2+5z+2} =$ R.P. $\frac{1}{2i}\int_C f(z)dz$

where $f(z) = \frac{z^2}{z^2+\frac{5}{2}z+1}$. For poles : $z^2 + \frac{5}{2}z + 1 = 0$

or, $2z^2 + 5z + 2 = 0$ or, $z = -2, -\frac{1}{2}$.

$\therefore$ only $z = -\frac{1}{2}$ lies within C.

$$\text{Res}\left(z = -\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z\to-\frac{1}{2}} \frac{\left(z + \frac{1}{2}\right) z^2}{(z + 2)(z + 2)}$$

$$= \lim_{z\to-\frac{1}{2}} \frac{z^2}{z + 2} = \frac{1/4}{2 - \frac{1}{2}} = \frac{1}{6}$$

$$\int_C f(z)dz = 2\pi i. \text{Res}(z = i) = 2\pi i. \left(\frac{1}{6}\right)$$

Put in (1),

$$I = \text{ R.P. } \frac{1}{2i} \cdot \frac{2\pi i}{6} = \frac{\pi}{6}$$

Example 8: Evaluate by contour integration : $\int_0^{\pi} \left(\frac{1+2\cos\theta}{5+4\cos\theta}\right) d\theta$

Solution: Let $I = \int_0^{\pi} \left(\frac{1+2\cos\theta}{5+4\cos\theta}\right) d\theta$. Then $I = \frac{1}{2}\int_0^{2\pi} \left(\frac{1+2\cos\theta}{5+4\cos\theta}\right) d\theta$

Take circle c as $|z| = 1, z = e^{i\theta}, dz = e^{i\theta}, z = izd0$

$$I = \frac{1}{2} R.P. \int_c \frac{\left(1 + 2e^{i0}\right)}{5 + 2\left(z + \frac{1}{z}\right)} \left(\frac{dz}{iz}\right) = \frac{R.P.}{2i} \int_c \frac{(1 + 2z)dz}{5z + 2z^2 + 2}$$

$$= \frac{R.P.}{4i} \int_c \frac{(1 + 2z)dz}{z^2 + \frac{5z}{2} + 1} \text{ or } I = \frac{R.P.}{4i} \int_c f(z)dz (1)$$

where $f(z) = \frac{1+2z}{z^2+\frac{5z}{2}+1}$. Poles are given by $z^2 + \frac{5z}{2} + 1 = 0$

$\Rightarrow 2z^2 + 5z + 2 = 0 \Rightarrow z = -\frac{5\pm3}{4} = -2, -\frac{1}{2} = \alpha, \beta,$ say.

$z = \alpha$ lies outside c as $|z| = 2 > 1. z = \beta$ lies inside c.

$$\text{Res}(z=\beta) = \lim_{z\to\beta}(z-\beta)f(z) = \lim_{z\to\beta}\frac{(z-\beta)(1+2z)}{(z-\alpha)(z-\beta)}$$
$$= \lim_{z\to\beta}\left(\frac{1+2z}{z-\alpha}\right) = \frac{1+(-1)}{\beta-\alpha} = \frac{0}{\beta-\alpha} = 0$$
$$\int_c f(z)dz = 2\pi i.\text{Res}(z=\beta) = 2\pi i(0) = 0$$

Now(1)

$$\Rightarrow I = 0.$$

Example9: Evaluate$\int_0^{\pi}\left(\frac{1+2\cos\theta}{4+5\cos\theta}\right)d\theta$.

Solution: Let $I = \int_0^{\pi}\frac{(1+2\cos J)}{4+5\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{1+2\cos\theta}{4+4\cos\theta}\right)d\theta$

or, $I = \frac{1}{2}$ R.P. $\int_0^{2\pi}\frac{(1+2e^{i\theta})d\theta}{4+\frac{5}{2}(e^{i\theta}+e^{-i\theta})}$. Take c as unit circle $|z| = 1$ or $z = e^{i\theta}, dz = e^{i\theta}id\theta = izd\theta$

we get $I =$ R.P. $\frac{1}{2}\int_c\frac{2(1+2z)}{8+5(z+z^{-1})}\left(\frac{dz}{iz}\right) =$ R.P. $\frac{1}{i}\int_c\frac{(1+2z)dz}{5z^2+8z+5}$

or

$$I = \text{ R.P. }\frac{1}{5i}\int_c\frac{(1+2z)dz}{z^2+\left(\frac{8}{5}\right)z+1} = \text{ R.P. }\frac{1}{5i}\int_c f(z)dz \qquad (1)$$

where $f(z) = \frac{1+2z}{z^2+\frac{8z}{5}+1}$. Poles of $f(z)$ are given by

$$5z^2+8z+5 = 0 \Rightarrow z = -\frac{8\pm 6i}{10} = -\frac{4\pm 3i}{5}$$

Take $\alpha = \frac{-4+3i}{5}, \beta = \frac{-4-3i}{5}$. Here $|\alpha| = 1 = |\beta|$. $z = \alpha, \beta,$ the simple poles lie inside $c. f(z) = \frac{(1+2z)}{(z-\alpha)(z-\beta)}$

Sum of residues $= \frac{1+2\alpha}{\alpha-\beta} + \frac{1+2\beta}{\beta-\alpha} = \frac{2(\alpha-\beta)}{\alpha-\beta} = 2$

$$\int_c f(z)dz = 2\pi i, \text{Residues } = 2\pi i(2) = 4\pi i$$

$\Rightarrow I =$ R. $\frac{1}{5i}\int_c f(z)dz =$ R.P. $\left(\frac{1}{5i}\right)(4\pi i) = \frac{4\pi}{5}$

Example 10: By the method of contour integration, prove that

$$\int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta - n\theta)d\theta = \frac{2\pi}{n!}$$

where n is a positive integer.

Or

Prove that $\int_0^{2\pi} e^{\cos\theta} \cdot \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$

Solution: Let $|z| = 1$ denote the circle C and

$$I = \int_0^{2\pi} e^{\cos\theta} \cdot \cos(\sin\theta - n\theta) d\theta$$

$$= \text{ R.P. } \int_0^{2\pi} e^{\cos\theta} \cdot e^{i(\sin\theta - n\theta)} d\theta$$

or $I = \text{R.P.} \int_0^{2\pi} \exp[\cos\theta + i(\sin\theta - n\theta)] d\theta$

$$= \text{ R.P. } \int_0^{2\pi} \exp.\left(e^{i\theta} - in\theta\right) d\theta = \text{ R.P. } \int_C \exp(z) e^{-in\theta} \frac{dz}{iz}, z = e^{i\theta}$$

or $I = \text{ R.P. } \frac{1}{i} \int_C \frac{e^z dz}{z^{n+1}} = \text{ R.P. } \frac{1}{i} \int_C f(z) dz$ ... (1)

where $f(z) = \frac{e^z}{z^{n+1}}$. This has pole at $z = 0$ of order $n + 1$.

$\text{Res}(z = 0)' = \frac{1}{n!} \frac{d^n}{dz^n} [e^z]_{z=0} = \frac{1}{n!} \cdot e^0 = \frac{1}{n!}$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i \text{ (sum of residues within } C \text{)} = \frac{2\pi i}{n!}$$

Now (1) takes the form $I = \text{ R.P. } \frac{1}{i} \frac{2\pi i}{n!} = \frac{2\pi}{n!}$

Example 11: Prove that $\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + \sin\theta) d\theta = \frac{2\pi(-1)^n}{n!}$

where n is a positive integer.

Solution: Let $I = \int_0^{2\pi} e^{-\cos\theta} \cdot \cos(n\theta + \sin\theta) d\theta$

Then

$$I = \text{ R.P. } \int_0^{2\pi} e^{-\cos\theta} \cdot e^{-(\sin\theta + n\theta)i} d\theta$$

$$= \text{R.P. } \int_0^{2\pi} \exp[-\cos\theta - i\sin\theta - in\theta] d\theta$$

or

$$I = \text{R.P.} \int_0^{2\pi} \exp\left(-e^{i\theta} - in\theta\right) d\theta = \text{R.P.} \int_0^{2\pi} \exp\left(-e^{i\theta}\right) \cdot e^{-in\theta} d\theta$$

Putting $z = e^{i0}$ so that $dz = ie^{i\theta} d\theta$, we obtain or

$$I = \text{R.P.} \int_0^{2\pi} \exp.(-z) \cdot (z)^{-n} \frac{dz}{iz} \text{ R.P. } \frac{1}{i} \int_C \frac{e^{-z}}{z^{n+1}} dz$$

where C is unit circle $|z| = 1$. or

$$I = \text{R.P. } \frac{1}{i} \int_C f(z) dz \ \ldots (1) \ f(z) = \frac{e^{-z}}{z^{n+1}}$$

$f(z)$ has a pole of order $n + 1$ at $z = 0$ which lies within C,

$$\text{Res}(z = 0) = \frac{1}{n!} \left[\frac{d^n}{dz^n} e^{-z}\right]_{z=0} = \frac{1}{n!} [(-1)^n e^{-z}]_{z=0} = \frac{(-1)^n}{n!}$$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i (\text{ sum of residues within } C) = \frac{2\pi i (-1)^n}{n!}$$

$$\therefore \text{By}(1), I = \text{R} \cdot \text{P} \cdot \frac{1}{i} \frac{2\pi i (-1)^n}{n!} = \frac{2\pi (-1)^n}{n!}$$

Example 12: Prove that $\int_{-\pi}^{\pi} \frac{a\cos d\theta}{a + \cos\theta} = 2\pi a \left[1 - \frac{a}{\sqrt{(a^2-1)}}\right], a > 1$

Solution: Let $I = \int_{-\pi}^{\pi} \frac{a\cos\theta d\theta}{a+\cos\theta} = 2 \int_0^{\pi} \frac{a\cos 0 d\theta}{a+\cos 0}$ as $f(-0) = f(0)$ or

$$\text{I=R.P.} \int_0^{2\pi} \frac{ae^{i\theta} d\theta}{a + \frac{1}{2}(e^{i\theta} + e^{-i\theta})}. \text{ Put } z = e^{i0}, dz = ie^{i\theta} d\theta$$

$\text{I} = \text{ R.P.} \int_C \frac{2az}{2a + (z + z^{-1})} \frac{dz}{iz} = \text{ R.P.} \frac{2a}{i} \int_C f(z) dz$ ... (1)

where $f(z) = \frac{z}{z^2 + 2az + 1}$ and C is the unit circle $|z| = 1$. Poles are given by $z^2 + 2az + 1 = 0$ so that $z = \alpha, \beta$ where $\alpha = -a + \sqrt{(a^2 - 1)}, \beta = -a - (a^2 - 1)^{1/2}$. Then $\alpha\beta = 1, |\alpha| < 1, |\beta| > 1$. Hence $f(z)'$ has one simple pole at $z = \alpha$ within C.

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{z}{z-\beta} = \frac{\alpha}{\alpha-\beta}$$
$$= \frac{-a+(a^2-1)^{1/2}}{2(a^2-1)^{1/2}}$$

By Cauchy's residues theorem,

$$\int_C f(z)dz = 2\pi i\cdot\text{Res}(z=\alpha) = \frac{2\pi i\left[-a+(a^2-1)^{1/2}\right]}{2(a^2-1)^{1/2}}$$

$\therefore$ By (1), $I =$ R.P. $\dfrac{2a}{i}\cdot\pi i\left[\dfrac{-a}{(a^2-1)^{1/2}}+1\right] = 2\pi a\left[1-\dfrac{a}{(a^2-1)^{1/2}}\right]$

Example 13: Show that

$$\int_0^\pi \tan(\theta+ia)d\theta = i\pi, \text{ where } R(a)>0$$

Solution: Let $I = \int_0^\pi \tan(\theta+ia)d\theta$

Then

$$I = \int_0^\pi \frac{2\sin(\theta+ia)\cdot\cos(\theta-ia)}{2\cos(\theta+ia)\cos(\theta-ia)}d\theta$$
$$= \int_0^\pi \left[\frac{\sin 2\theta+\sin(2ia)}{\cos 2\theta+\cos(2ia)}\right]d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\frac{\sin t+i\sinh 2a}{\cos t+\cosh 2a}dt, \text{ where } 2\theta = t$$

Putting $z=e^{it}$ so that $dz = ie^{it}dt, \frac{dz}{iz} = dt$, we get

$$= \frac{1}{2}\int_C\left[\frac{\frac{z-z^{-1}}{2i}+i\sinh 2\cdot}{\frac{z+z^{-1}}{2}+\cosh 2a}\right]\frac{dz}{iz}, \text{where}$$

I$= -\frac{1}{2}\int_C f(z)dz$ ... (1) where $f(z) = \frac{z^2-1-2z\sinh 2a}{z(z^2+2z\cosh 2a+1)}$(i)

Poles of $f(z)$ are given by

$$z(z^2+2z\cosh 2a+1) = 0$$

This

$$\Rightarrow z = 0, z^2 + 2z\cosh 2a + 1 = 0$$
$$\Rightarrow z = 0, z = -\cosh 2a \pm \sqrt{(\cosh^2 2a - 1)}$$
$$\Rightarrow z = 0, z = -\cosh 2a \pm \sinh 2a \Rightarrow z = 0, \alpha, \beta$$

where

$$\alpha = -\cosh 2a + \sinh 2a$$
$$\beta = -\cosh 2a - \sinh 2a$$

Evidently $\alpha\beta = 1, |\alpha| < 1$ so that $|\beta| > 1$.. $f(z)$ has two simple poles at $z = 0$, α within C.

$$\operatorname{Res}(z = 0) = \lim_{z \to 0}(z - 0) \cdot f(z) = \lim_{z \to 0} \frac{z^2 - 2z\sinh 2a - 1}{z^2 + 2z\cosh 2a + 1}$$
$$= \frac{0 - 0 - 1}{0 + 0 + 1} = -1$$

$$\operatorname{Res}(z = \alpha) = \lim_{z \to \alpha}(z - \alpha)f(z)$$
$$= \lim_{\to \alpha} \frac{(z - \alpha)(z^2 - v2z\sinh 2i - 1)}{/} z(z - \alpha)(z - \beta)u$$
$$= \frac{\alpha^2 - 2\alpha\sinh 2a - 1}{\alpha(\alpha - \beta)} = \frac{\alpha - 2\sinh 2a - \beta}{\alpha - \beta} \text{ For } \alpha\beta = 1.$$
$$= \frac{2\sinh 2a - 2\sinh 2a}{2\sinh 2a} = 0.$$

$\operatorname{Res}(z = \alpha) + \operatorname{Res}(z = 0) = 0 + (-1) = -1.$
By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ sum of residues within } C) = 2\pi i(-1)$$

Now, by (1), $I = -\frac{1}{2}(-2\pi i) = i\pi.$

Example 14: Apply the calculus of residues to evaluate **(*i*)** $\int_0^{2\pi} \frac{\sin n\theta d\theta}{1+2a\cos\theta+a^2}$, **(*ii*)** $\int_0^{2\pi} \frac{\cos n\theta d\theta}{1+2a\cos\theta+a^2}$ where $a^2 < 1$ and n is a positive integer.

Solution: Let $I = \int_0^{2\pi} \frac{e^{in\theta} d\theta}{1+2a\cos\theta+a^2}$

Then

$$I=\int_0^{2\pi}\frac{\left(e^{i\theta}\right)^n d\theta}{1+a^2+a(e^{i\theta}+e^{-i0})}$$

Putting $z=e^{i\theta}$ so that $dz=ie^{i\theta}d\theta$, we get

$$I=\int_C \frac{z^n}{1+a^2+a(z+z^{-1})}\frac{dz}{iz}$$

$$=\frac{1}{i}\int_C \frac{z^n dz}{(1+a^2)z+az^2+a}, \text{ where } C \text{ is circle } |z|=1$$

or

$$I=\frac{1}{ai}\int_C \frac{z^n dz}{z^2+az+\frac{z}{a}+1}=\frac{1}{ia}\int_C f(z)\cdot dz \qquad \ldots(1)$$

where

$$f(z)=\frac{z^n}{(z+a)\left(z+\frac{1}{a}\right)}$$

Simple poles of $f(z)$ are $z=-a$ and $z=-\frac{1}{a}$. But $a^2<1$. Hence $z=-a$ lies within C and $z=-1/a$ lies outside C.
$\text{Res}(z=-a)=\lim_{z\to -a}(z+a)f(z)=\lim_{z\to -a}\dfrac{z^n}{z+(1/a)}$

$$=\frac{(-a)^n}{-a+(1/a)}=\frac{(-1)^n a^{n+1}}{1-a^2}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz=2\pi i(\text{ Sum of residues within } C)=\frac{2\pi i(-1)^n a^{n+1}}{1-a^2}$$

$\therefore$ By (1), $I=\frac{1}{ai}\cdot\frac{2\pi i(-1)^n a^{n+1}}{1-a^2}=\frac{2\pi(-1)^n a^n}{1-a^2}$

or

$$\int_0^{2\pi}\frac{e^{in\theta}d\theta}{1+2a\cos\theta+a^2}=\frac{2\pi(-1)^n a^n}{-1-\dot{a}^2}$$

Equating real and imaginary parts we get

$$\int_0^{2\pi} \frac{\cos(n\theta)d\theta}{1+2a\cos\theta+a^2} = \frac{2\pi(-1)^n a^n}{1-a^2}$$

and

$$\int_0^{2\pi} \frac{\sin(n0)d\theta}{1+2a\cos\theta+a^2} = 0$$

13.6 *EVALUATION OF INTEGRALS OF THE TYPE* $\int_{-\infty}^{\infty} f(z)dz$:-

$\int_{-\infty}^{\infty} f(z)dz$ where, the function $f(z)$ is s.t. no pole of $f(z)$ lies on the real line, but poles lie in the upper half of z-plane. We evaluate the above integrals by considering them along a closed contour C consisting of
(*i*) semi circle Γ s.t. $|z| = R$ in the upper half plane.
(*ii*) real axis from $-R$ to R.

Then we try to show that integral along Γ vanishes as $|z| \to \infty$. Thus $\int_C f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(z)dz$. Taking limit as $R \to \infty, \int_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz$. By Cauchy's residues theorem, this becomes

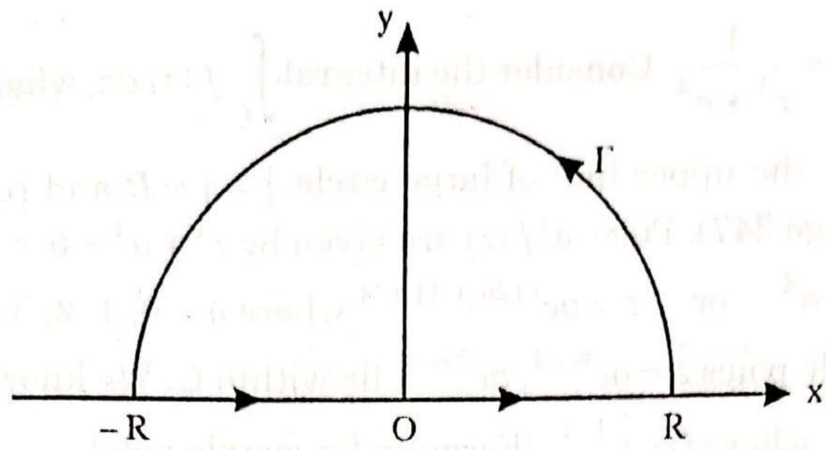

Figure: 1

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i(\text{ Sum of residues within } C)$$

Example 15: Provethat$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.
Solution: Consider the integral,

$$\int_C f(z)dz, \text{ where } f(z) = \frac{1}{1+z^2}$$

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz (1)$$

Here C is the closed contour consisting of Γ, the upper half of the large circle $|z| = R$ and the real axis from $-R$ to R. Poles of $f(z)$ are $z = \pm i$. $f(z)$ has only one simple pole at $z = i$ inside C.

$$\text{Res}(z=i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}.$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ sum of residues within } C) = \frac{2\pi i}{2i} = \pi$$

$$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty}\frac{1}{1+z^2} = 0. \text{ Hence, by Theorem 7,}$$

$$\lim_{R\to\infty}\int_\Gamma f(z)dz = i(\pi-0)(0) = 0$$

Making $R\to\infty$ in (1) and noting this, we get

$$\pi = \int_{-\infty}^{\infty} f(x)d + 0 \text{ or } \int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\int_0^\infty \frac{dx}{1+x^2} = \pi \text{ or } \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Example 16: Prove that if $a > 0$, then $\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$

Solution: Let $f(z) = \frac{1}{z^4+a^4}$. Consider the integral $\int_C f(z)dz$, where C is closed contour consisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R (see Fig. 1). Poles of $f(z)$ are given by $z^4 + a^4 = 0$.

or $z^4 = -a^4 = e^{2n\pi i}e^{\pi i}a^4$ or $z = ae^{i(2n_i+1)\pi/4}$ where $n = 0,1,2,3$. But only two simple poles $z = ae^{\pi i/4}, ae^{3\pi i/4}$ lie within C. We know that $\text{Res}(z=\alpha) = \lim_{z\to\alpha}\frac{\phi(z)}{\psi'(z)}$ where $f = \frac{\phi(z)}{\psi'(z)}$, [Formula for simple pole] or

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha} \frac{1}{4z^3} = \lim_{z\to\alpha} \frac{z}{4z^4} = \lim_{z\to\alpha} \frac{-z}{4a^4} \text{ as } z^4 + a^4 = 0$$

$\therefore$ $\text{Res}\left(z = ae^{\pi i/4}\right) + \text{Res}\left(z = ae^{3\pi i/4}\right)$

$$= \frac{1}{-4a^4}\left[ae^{\pi i/4} + ae^{3\pi i/4}\right] = \frac{-1}{4a^3}\left[e^{\pi i/4} + e^{\pi i}\cdot e^{-i\pi/4}\right]$$

$$= -\frac{1}{4a^3}\left[e^{i\pi/4} - e^{-i\pi/4}\right] = -\frac{1}{4a^3} 2i\sin\frac{\pi}{4} = \frac{-i}{2a^3\sqrt{2}}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ sum of residues within } C)$$

or

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(z)dz = \frac{2\pi i(-i)}{2a^3\sqrt{2}}, = \frac{\pi\sqrt{2}}{2a^3}$$

Making $R \to \infty$, we get $\lim_{R\to\infty} \int_\Gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{\pi\sqrt{2}}{2a^3}$.

$$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{z}{z^4 + a^4} = 0 \qquad \ldots (1)$$

$$\lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)(0) = 0 \qquad \ldots (1)$$

By previous theorems, Eq. (1) becomes $\int_{-\infty}^{\infty} f(x)dx = \frac{\pi\sqrt{2}}{2a^3}$

or

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\int_0^\infty \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{2a^3} \text{ or } \int_0^\infty \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$$

Similar Problem. Prove by contour integration

$$\int_0^\infty \frac{dx}{x^4 + 1} = \frac{\pi}{2}\sqrt{2}$$

Example 17: By the method of contour integration, prove that $\int_0^\infty \frac{\cos mx dx}{x^2+a^2} = \frac{\pi e^{-ma}}{2a}$ where $m \geq 0, a > 0$ and deduce that $\int_0^\infty \frac{x\sin(mx)}{x^2+a^2} dx = \frac{\pi}{2} e^{-ma}$

Solution: Let $f(z) = \frac{e^{\text{inz}}}{z^2+a^2}$. Consider $\int_C f(z)dz$ where C is closed contour as shown in Fig. 1. Poles of $f(z)$ are given by $z^2 + a^2 = 0$ or $z = \pm ia$. $f(z)$ has only simple one pole $z = ia$ inside C.

or

$$\text{Res}(z = ia) = \lim_{z \to ia} (z - ia) f(z) = \lim_{z \to ia} \frac{(z-ia)e^{imz}}{(z+ia)(z-ia)} \quad \ldots (1)$$

$$\text{Res}(z = ia) = \frac{1}{2ia} e^{-ma}$$

or $\lim_{|z| \to \infty} \frac{1}{z^2+a^2} = 0.$ Hence by Jordan's lemma

$$\lim_{R \to \infty} \int_\Gamma \frac{e^{imz}}{z^2 + a^2} dz = \lim_{R \to \infty} \int_\Gamma f(z) dz = 0 \quad \ldots (2)$$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i (\text{ sum of residues with } C) = \frac{2\pi i e^{-m\alpha}}{2ia}$$

or

$$\int_\Gamma f(z) dz + \int_{-R}^{R} f(x) dx = \frac{\pi}{a} e^{-ma}$$

Making $R \to \infty$ and noting (2), we get

$$\int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{a} e^{-ma} \text{ or } \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma} \quad \ldots (3)$$

or

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma} \quad \ldots (4)$$

Deductions. **(i)** Taking $m = a = 1$ in (4), we get

$$\int_0^{\infty} \frac{\cos x}{x^2 + 1} dx = \frac{\pi}{2} e^{-1} = \frac{\pi}{2e}$$

(ii) Taking $m = 1$ in equation (3), we get $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \frac{\pi}{a} e^{-a}$
(iii) Taking $a = 2, m = 1$ in (4), we get $\int_0^{\infty} \frac{\cos x}{x^2+4} dx = \frac{\pi}{4} e^{-2} = \frac{\pi}{4e^2}$
(iv) Taking $m = 1$ in (4)

$$\int_0^{\pi/2} \frac{\cos x}{x^2+a^2} dx = \frac{\pi}{2a} e^{-2}$$

(ii) Differentiating equation (4) w.r.t. m,

$$\int_0^{m} -\frac{x\sin(mx)dx}{x^2+a^2} = \frac{\pi}{2a} e^{-ma}, (-a)$$
$$\Rightarrow \quad \int_0^{m} \frac{x\sin(mx)}{x^2+a^2} dx = \frac{\pi}{2} e^{-ma}$$

$$\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx = \frac{\pi}{2} e^{-a}$$

Example 18: If $a > 0, m > 0$, then $\int_0^{\infty} \frac{\cos mxdx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}(1+ma)e^{-ma}$

Solution: $f(z) = \frac{e^{imz}}{(a^2+z^2)^2}$. Consider the integral $\int_C f(z)dz$, where C is the closed contour c isisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R (See diagram 1). Evidently

$$\lim_{|z|\to\infty} \frac{1}{(z^2+a^2)^2} = 0.$$

Hence, by Jordan's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{imz}dz}{(z^2+a^2)^2} = 0 \text{ or } \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0 \qquad ...(1)$$

Poles of $f(z)$ are $z = \pm ia$. (Repeated two times), $f(z)$ has only one pole order two at $z = ai$ within C.

$$f(z) = \frac{\phi(z)}{(z-ai)^2}, \text{ where } \phi(z) = \frac{e^{imz}}{(z+ai)^2}$$
$$\text{Res}(z = ia) = \lim_{z\to ia} \phi'(x)$$
$$= \lim_{z\to ia} \frac{e^{imz}[im(z+ia)^2 - 2(z+ia)]}{(z+ia)^4}$$
$$= \lim_{z\to ia} \frac{e^{imz}[im(z+ia) - 2]}{(z+ia)^3}$$

$$= \frac{e^{-ma}[im \cdot 2ia - 2]}{(2ia)^3} = \frac{e^{-ma}(1 + ma)}{4a^3 i}.$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[\text{Res}(z = ia)] = 2\pi i e^{-ma} \frac{(1 + ma)}{4a^3 i}$$

or

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi(1 + ma)}{2a^3} e^{-ma}$$

Making $R \to \infty$ and noting (1), we get
or

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2 + a^2)^2} dx = \frac{\pi}{2a^3}(1 + ma)e^{-ma}$$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\cos mx dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^3}(1 + ma)e^{-ma}$$

But

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2 + a^2)^2} dx = 2\int_0^{\infty} \frac{\cos mx}{(x^2 + a^2)^2} dx$$

Hence the last gives $\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}(1 + ma)e^{-ma}$.
Deductions. **(i)** Putting $m = 1$, we get

$$\int_0^{\infty} \frac{\cos x dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}(1 + a)e^{-a}$$

(ii) Taking $m = a = 1$, we obtain

$$\int_0^{\infty} \frac{\cos x}{(x^2 + 1)^2} dx = \frac{\pi}{4} \cdot 2e^{-1} = \frac{\pi}{2e}$$

Example 19: Apply the method of $c$alculus of residues to prove that,

$$\int_0^{\infty} \frac{\log(1 + x^2)}{1 + x^2} dx = \pi \log 2$$

Solution: Let $f(z) = \frac{\log(z+i)}{1+z^2}$. Consider the integral $\int_C f(z)dz$ where C is a closed contour consisting of Γ, the upper half of large circle $|z| = R$ real axis from - R to R as shown in Figure 1.

$$\lim_{|z|\to\infty} zf(z) = \left[\lim_{|z|\to\infty} \frac{z}{z-i}\right]\left[\lim_{|z|\to\infty} \frac{\log(z+i)}{z+i}\right] = 1(0) = 0 \qquad \ldots (1)$$

$\therefore$ By Theorem, $\lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)(0) = 0$

Poles of $f(z)$ are $z = \pm i$.
$f(z)$ has only one simple pole at $z = i$ within C.

$$\text{Res}(z = i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{\log(z+i)}{z+i}$$
$$= \frac{\log 2i}{2i} = \frac{\log\left(2e^{i\pi/2}\right)}{2i} = \frac{\log 2 + (\pi i/2)}{2i}$$

By Cauchy's residue theorem,
or

$$\int_C f(z)dz = 2\pi i[\text{Res}(z = i)] = \frac{2\pi i}{2i}[\log 2 + (i\pi/2)]$$
$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \pi\left[\log 2 + \frac{\pi i}{2}\right]$$

Making $R \to \infty$ and noting (1),

$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1} dx = \pi\left[\log 2 + \frac{\pi i}{2}\right]$$

Equating real parts from both sides, $\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1} dx = \pi \log 2$
or

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(1+x^2)}{x^2+1} dx = \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2$$

Example 20: Apply the calculus of residues to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)}, (a > b > 0)$.

Solution: Let $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$. Consider the integral $\int_C f(z)dz$, wh C is a closed contour as shown in Figure 1, Evidently $\lim_{|z|\to\infty} \frac{1}{(z^2+a^2)(z^2+b^2)} = 0.$

Hence, by Jordan's lemma, $\lim_{R\to\infty} \int_\Gamma \frac{e^{iz}dz}{(z^2+a^2)(z^2+b^2)} = \lim_{R\to\infty} \int_\Gamma f(z)dz = 0$

Poles of $f(z)$ are $z = \pm ia, \pm ib$.
Only two simple poles at $z = ia, ib$ lie within C.

$$\text{Res}(z = ia) = \lim_{z\to ia} (z - ia)f(z) = \lim_{z\to ia} \frac{(z - ia)e^{iz}}{(z - ia)(z + ia)(z^2 + b^2)}$$
$$= \frac{e^{-a}}{2ia(-a^2 + b^2)}$$

$$\text{Res}(z = ib) = \lim_{z\to ib} (z - ib)f(z) = \lim_{z\to ib} \frac{(z - ib)e^{iz}}{(z - ib)(z + ib)(z^2 + a^2)}$$
$$= \frac{e^{-b}}{2ib(-b^2 + a^2)} = \frac{e^{-b}}{2ib(a^2 - b^2)}$$
$$\text{Res}(z = ia) + \text{Res}(z - ib) = \frac{1}{2i(a^2 - b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

By Cauchy's residues theorem,

$$\int_C f(z)dz = 2\pi i(\text{ Sum of residues within } C)$$

or

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i}{2i(a^2 - b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Making $R \to \infty$ and noting (1), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating real parts from both sides

$$\int_{-\infty}^{\infty} \frac{\cos x \cdot dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Note. The last also gives
or

$$2\int_0^\infty \frac{\cos x \cdot dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\int_0^\infty \frac{\cos x \cdot dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Example 21: Show that $\int_0^\infty \frac{x\sin ax}{x^2+k^2}dx = \frac{\pi}{2}e^{-ak}$, (where $a > 0, k > 0$).

Solution: Let $f(z) = \frac{ze^{iaz}}{z^2+k^2}$. Consider the integral $\int_C f(z)(dz)$, where C is a closed contour as shown in figure 1. $\lim_{|z|\to\infty} \frac{z}{z^2+k^2} = 0$. Hence by Jordan's lemma,

$$\lim_{R\to\infty} \int_\Gamma \frac{z}{z^2+k^2} e^{iaz} dz = \lim_{R\to\infty} \int_\Gamma f(z)dz = 0 \qquad \text{... (1)}$$

Poles of $f(z)$ are given by $z^2 + k^2 = 0$ or $z = \pm ik$. Now $z = ik$ is the only simple pole within C.

$$\text{Res}(z = ik) = \lim_{z\to ik} (z - ik)f(z) = \lim_{z\to ik} \frac{ze^{iaz}}{z + ik} = \frac{e^{-ak}}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ Sum of residues within } C) = \pi i e^{-ak}$$

$$\int_\Gamma f(z)dz + \int_{-R}^R f(x)dx = \pi i e^{-ak}$$

Making $R \to \infty$ and noting (1), we get

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^\infty \frac{xe^{iax}}{x^2+k^2}dx = \pi i e^{-ak}$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^\infty \frac{x'\sin ax}{x^2+k^2}dx = 2\int_0^\infty \frac{x\sin ax}{x^2+k^2}dx = \pi e^{-ak}$$

or

$$\int_0^\infty \frac{x\sin axdx}{x^2+k^2}dx = \frac{\pi}{2}e^{-ak} \qquad \text{... (2)}$$

Deductions. **(i)** Putting $k = 1$ in (2), $\int_0^\infty \frac{x\sin ax}{x^2+1}dx = \frac{\pi}{2}e^{-a}$

(ii) Putting $a = 1$ in (2), $\int_0^\infty \frac{x\sin x}{x^2+k^2}dx = \frac{\pi}{2}e^{-k}$

(iii) Putting $a = k = 1$ in (2), $\int_0^\infty \frac{x\sin x}{x^2+1}dx = \frac{\pi}{2}e^{-1} = \frac{\pi}{2e}$.

Example 22: Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^2(b+c)^2}$ where $b > 0, c > 0$.

Solution: Let $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2}$. Consider the integral $\int_C f(z)dz$ where C is the closed contour consisting of $\underset{\sim}{\Gamma}$, the upper half of large circle $|z| = R$ and real axis from $-R$ to R. See Figure 1.

$$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{z}{(z^2+b^2)(z^2+c^2)^2} = 0$$

Hence, by theorem,

$$\lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)(0) = 0 \qquad \dots (1)$$

Poles of $f(z)$ are $z = \pm ib, \pm ic, f(z)$ has only two poles within C, namely at $z = ib$ (order 1) and $z = ic$ (order 2)

$$\text{Res}(z = ib) = \lim_{z\to ib} (z - ib)f(z)$$

$$= \lim_{z\to ib} \frac{(z - ib)}{(z - ib)(z + ib)(z^2 + c^2)^2} = \frac{1}{2ib(-b^2 + c^2)^2}$$

$$f(z) = \frac{\phi(z)}{(z - ic)^2}, \text{ where } \phi(z) = \frac{1}{(z^2 + b^2)(z + ic)^2}$$

$$\phi'(z) = -2z(z^2 + b^2)^{-2}(z + ic)^{-2} - 2(z + ic)^{-3}(z^2 + b^2)^{-1}$$

$$\phi'(ic) = \frac{-2ic}{(-c^2 + b^2)^2(-4c^2)} - \frac{2}{-8ic^3(-c^2 + b^2)}$$

$$= \frac{i}{2c(b^2 - c^2)^2} - \frac{i}{4c^3(b^2 - c^2)} = \frac{(3c^2 - b^2)i}{4c^3(b^2 - c^2)^2}$$

$$\text{Res}(z = ic) = \frac{\phi'(ic)}{1!} = \phi'(ic)$$

$\text{Res}(z = ib) + \text{Res}(z = ic)$

$$= \frac{1}{2ib(b^2 - c^2)^2} + \frac{(3c^2 - b^2)i}{4c^3(b^2 - c^2)^2}$$
$$= \frac{i}{4(b^2 - c^2)^2}\left[\frac{-2}{b} + \frac{3c^2 - b^2}{c^3}\right] = \frac{i[-2c^3 + b(3c^2 - b^2)]}{4bc^3(b^2 - c^2)^2}$$
$$= \frac{i[(c^3 - b^3) - 3c^2(c - b)]}{4bc^3(b^2 - c^2)^2} = \frac{i(c - b)[c^2 + b^2 + cb - 3c^2]}{4bc^3(b^2 - c^2)^2}$$
$$= \frac{i(c - b)(b - c)(b + 2c)}{4bc^3(b^2 - c^2)^2} = -\frac{i(b + 2c)}{4bc^3(b + c)^2}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(\text{ sum of residues within } C)$$

or

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i[-i(b + 2c)]}{4bc^3(b + c)^2}$$

Making $R \to \infty$ and noting (1),

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)(x^2 + c^2)^3} = \frac{\pi(b + 2c)}{2bc^3(b + c)^2} \qquad \dots (2)$$

Deduction. Putting $b = 1, c = 2$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)^2} = \frac{\pi(1 + 4)}{2 \times 8 \times 9} = \frac{5\pi}{144}$$

or

$$\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)^2} = \frac{5\pi}{288}$$

Example 23: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(x^2+9)^2}$.

Solution: Prove as in equation (2) of the above example 22 that,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)(x^2 + c^2)^2} = \frac{\pi(b + 2c)}{2bc^3(b + c)^2}$$

Putting $b = 2, c = 3$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(x^2+9)^2} = \frac{\pi(2+6)}{2 \times 2 \times 3^3 (2+3)^2} = \frac{2\pi}{675}.$$

Example 24: Prove that $\int_0^{\infty} \frac{\cos mx dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}} e^{-\frac{m}{2}\sqrt{3}} \cdot \sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$.

Solution: Let $f(z) = \frac{e^{imz}}{z^4+z^2+1}$

Consider the integral $\int_C f(z)dz$ where C is closed contour as shown in Figure 1.

Evidently $\lim_{|z|\to\infty} \frac{1}{z^4+z^2+1} = 0$. Hence, by Jordan's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{imz} dz}{z^4+z^2+1} = \lim_{R\to\infty} \int_{\Gamma} f(z)dz = 0 \qquad \dots (1)$$

Poles of $f(z)$ are given by

$$z^4 + z^2 + 1 = 0 \text{ or } (z^2-1)(z^4+z^2+1) = 0$$

This $\Rightarrow z^6 - 1 = 0 \Rightarrow z^6 = 1 = e^{2n\pi i} \Rightarrow z = e^{2n\pi i/6}, \Rightarrow z = e^{n\pi i/3}, (n = 0,1,2,3,4,5)$

The values $e^{i\pi/3}, e^{i2\pi/3}, e^{i4\pi/3}, e^{i5\pi/3}$ are the roots of $z^4 + z^2 + 1 = 0$.
The poles lying within C are $e^{\pi i/3}, e^{2\pi i/3}$.
Let $\alpha = e^{\pi i/3}$, then $\alpha^2 = e^{2\pi i/3}$.
Thus $f(z)$ has two simple poles at $z = \alpha, \alpha^2$ within C.
$\text{Res}(z=\alpha) = \lim_{z\to\alpha} \frac{\phi(z)}{\psi'(z)}$ for simple pole where $f(z) = \frac{\phi(z)}{\Psi(z)}$

$$= \lim_{z\to\alpha} \frac{e^{imz}}{4z^3+2z}$$

$\text{Res}(z=\alpha) + \text{Res}(z=\alpha^2)$

$$= \frac{e^{im\alpha}}{4\alpha^3+2\alpha} + \frac{e^{im\alpha^2}}{4\alpha^6+2\alpha^2}$$

$$= \frac{\exp\left(ime^{\pi i/3}\right)}{4e^{\pi i}+2e^{\pi i/3}} + \frac{\exp\left(ime^{2\pi i/3}\right)}{x + 2\exp\left(\frac{2\pi i}{3}\right)} \quad \text{For } \alpha^6 - 1 = 0$$

$$= \frac{\exp\left\{im\left(\frac{1+i\sqrt{3}}{2}\right)\right\}}{-4+2\left(\frac{1+i\sqrt{3}}{2}\right)} + \frac{\exp\left\{im\left(\frac{-1+i\sqrt{3}}{2}\right)\right\}}{4+2\left(\frac{-1+i\sqrt{3}}{2}\right)}$$

$$= \frac{\exp\left\{\frac{m}{2}(i-\sqrt{3})\right\}}{-3+i\sqrt{3}} + \frac{\exp\left\{\frac{m}{2}(-i-\sqrt{3})\right\}}{3+i\sqrt{3}}$$

$$= \exp\left(\frac{-m\sqrt{3}}{2}\right)\left[\frac{\exp(im/2)\cdot(-3-i\sqrt{3})}{12}\right.$$

$$\left.+\frac{(3-i\sqrt{3})\exp(-im/2)}{12}\right]$$

$$= \frac{1}{12}\exp\left(\frac{-m\sqrt{3}}{2}\right)\left[-3\left(e^{im/2}-e^{-im/2}\right)-i\sqrt{3}\left(e^{im/2}+e^{-im/2}\right)\right]$$

$$= \frac{1}{12}\exp\left(\frac{-m\sqrt{3}}{2}\right)\left[-6i\sin\frac{m}{2}-2i\sqrt{3}\cos\frac{m}{2}\right]$$

$$= -\frac{4i\sqrt{3}}{12}\exp\left(\frac{-m\sqrt{3}}{2}\right)\left[\frac{\sqrt{3}}{2}\sin\frac{m}{2}+\frac{1}{2}\cos\frac{m}{2}\right]$$

$$= \frac{-i}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$$

By Cauchy's residues theorem, $\int_C f(z)dz = 2\pi i$ (Sum of residues within).

or $\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i(-i)}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$

Making $R \to \infty$ and noting (1), we get

$$\int_{-\infty}^{\infty}\frac{e^{imx}dx}{x^4+x^2+1} = \frac{2\pi}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$$

Equating real parts from both sides

$$\int_{-\infty}^{\infty}\frac{\cos mx dx}{x^4+x^2+1} = \frac{2\pi}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$$

or

$$\int_{0}^{\infty}\frac{\cos mx dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$$

Example 25: Apply the calculus of residues to prove that

$$\int_0^\infty \frac{x^6dx}{(x^4+a^4)^2} = \frac{3\pi\sqrt{2}}{16a}, a > 0$$

Solution: Let $f(z) = \frac{z^6}{(z^4+a^4)^2}$. Consider the integral $\int_C f(z)dz$ where C is a closed contour as shown in Figure 1. Poles of $f(z)$ are given by

$$z^4 + a^4 = 0 \text{ or } z^4 = -a^4 = -e^{2n\pi i} \cdot e^{\pi i} \cdot a^4$$

$z = ae^{(2n+1)\pi i/4}$ where $n = 0,1,2,3.$ $f(z)$ has two poles each of order 2 at $z = \alpha = \alpha e^{\pi i/4}, z = \beta = e^{3\pi i/4}$ within C and so $\alpha^4 + a^4 = 0 = \beta^4 + a^4$.

$$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{z^7}{(z^4+a^4)^2} = 0 \qquad \text{... (1)}$$

Hence, by theorem, $\lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)(0) = 0$

To find the residue of $f(z)$ at the double pole $z = \alpha$, we put $z = \alpha + t$ in $f(z)$ and equate the coefficient of $\frac{1}{t}$.

$$f(z) = \frac{(\alpha+t)^6}{[(\alpha+t)^4 + a^4]^2} = \frac{(\alpha+t)^6}{(a^4 + \alpha^4 + 4\alpha^3 t + 6\alpha^2 t^2 + \cdots)^2}$$

$$= \frac{(\alpha+t)^6}{(4\alpha^3 t + 6\alpha^2 t^2 + \cdots)^2} \text{ For } a^4 + \alpha^4 = 0.$$

$$= \frac{(\alpha+t)^6}{16\alpha^6 t^2}\left(1 + \frac{3t}{2\alpha}\right) - 2$$

$$= \frac{(\alpha^6 + 6\alpha^5 t + \cdots)}{16\alpha^6 t^2}\left(1 - \frac{3t}{\alpha} + \cdots\right)$$

$$= \frac{(\alpha^6 + 6\alpha^5 t + \cdots)}{16\alpha^6}\left(\frac{1}{t^2} - \frac{3}{t\alpha} + \cdots\right)$$

Here coefficient of $\frac{1}{t}$ is

$$\frac{1}{16\alpha^6}(-3\alpha^5 + 6\alpha^5) = \frac{3}{16\alpha}$$

$$\text{Res}(z=\alpha) + \text{Res}(z=\beta) = \frac{3}{16}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{3}{16a}\left(e^{-\pi i/4} + e^{-3\pi i/4}\right)$$

$$= \frac{3}{16a}\left[\frac{1-i}{\sqrt{2}} + \cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$$

$$= \frac{3}{16a}\left[\frac{1-i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \frac{-3 \times 2i}{16a\sqrt{2}} = \frac{-3i\sqrt{2}}{16a}$$

By Cauchy's residues theorem,

$$\int_C f(z)dz = 2\pi i(\text{ Sum residues within } C) = \frac{2\pi i}{16a}(-3i\sqrt{2})$$

or

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(z)dz = \frac{3\pi\sqrt{2}}{8a}$$

Making $R \to \infty$ and noting (1), we get

$$\int_{-\infty}^{\infty} \frac{x^6 dx}{(x^4 + a^4)^2} = 2\int_0^{\infty} \frac{x^6 dx}{(x^4 + a^4)^2} = \frac{3\pi\sqrt{2}}{8a}$$

or

$$\int_0^{\infty} \frac{x^6 dx}{(x^4 + a^4)^2} = \frac{3\pi\sqrt{2}}{16a}$$

Example 26: By contour integration, prove that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)^3} = \frac{\pi}{8a^3}$$

Solution: Let$f(z) = \frac{z^2}{(z^2+a^2)^3}$. Consider $\int_C f(z)dz$ where C is a closed contour as shown in figure 1. Poles of $f(z)$ are given by $(z^2 + a^2)^3 = 0$ or $z = \pm ia$. Evidently $z = ia$ is the only pole of order 3 within C. Putting $z = ia + t$ in the value of $f(z)$,

$$\begin{aligned} f(ia + t) &= \frac{(ia + t)^2}{[(ia + t)^2 + a^2]^3} = \frac{t^2 - a^2 + 2iat}{(t^2 + 2iat)^3} \\ &= \frac{(t^2 - a^2 + 2iat)}{(2iat)^3}\left[1 + \frac{t}{2ia}\right]^3 \\ &= \left(\frac{t^2 - a^2 + 2iat}{-8ia^3t^3}\right)\left[1 - \frac{3t}{2ia} - \frac{3t^2}{2a^2} \dots\dots\right] \\ &= \frac{(t^2 - a^2 + 2iat)}{-8ia^3}\left[\frac{1}{t^3} - \frac{3}{2iat^2} - \frac{3}{2a^2t}\dots\right] \end{aligned}$$

Here coefficient of $\frac{1}{t}$ is $\frac{-1}{8ia^3}\left[1 + \frac{3}{2} - 3\right] = \frac{1}{16ia^3}$
$\int_C f(z)dz = 2\pi i($ sum of residue within $C) = 2\pi i\text{Res}(z = ia)$
or

$$\int_{\Gamma} f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i}{16ia^3} \quad ...(1)$$

$$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{z^3}{(z^2+a^2)^3} = 0 \quad ...(2)$$

$\therefore$ By Theorem 7, $\lim_{R\to\infty} \int_{\Gamma} f(z)dz = i(\pi - 0)(0) = 0$
Making $R \to \infty$ in (1) and noting (2), we get

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^3} = \frac{\pi}{8a^3}$$

13.7 *POLES LIE ON THE REAL AXIS:-*

In the previous article we have supposed that the function $f(z)$ has no pole on the real line. Now we drop this condition. In the present case the function $f(z)$ has poles within the semicircle Γ as well as on the real line. We exclude the poles on the real line by enclosing them with semi circles of small radii. This procedure is called "identing at a point". This process is illustrated by the following solved problems.

Example 27: Prove that $\int_0^{\infty} \frac{\log x dx}{(1+x^2)^2} = -\frac{\pi}{4}$.

Solution: Let $f(z) = \frac{\log z}{(1+z^2)^2}$. Consider the integral $\int_C f(z)dz$, where C is the closed contour consisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R idented at $z = 0$ by a small semi-circle γ of radius r. (See Fig. 2) $z = 0$ is a branch point of $f(z)$. Poles of $f(z)$ are given by

$$(1+z^2)^2 = 0, \text{ or } z = i, -i$$

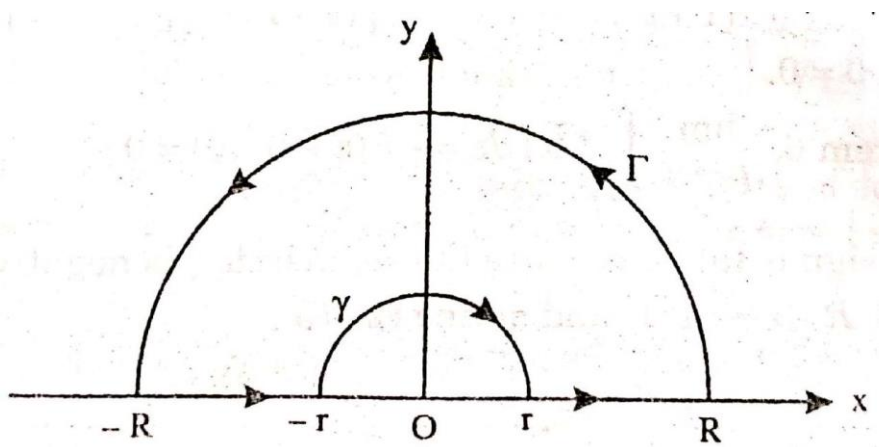

Figure: 2

$f(z)$ has only one pole within C of order 2 at $z = i$.

$$\text{Res}(z=i) = \frac{\phi'(i)}{1!} \text{ where } \phi(z) = \frac{\log z}{(z+i)^2} \text{ so that}$$

$$f(z) = \frac{\phi(z)}{(z-i)^2}$$

$$\phi'(z) = \frac{[z^{-1}(z+i)^2 - 2(z+i)\log z]}{(z+i)^4} = \frac{z^{-1}(z+i) - 2\log z}{(z+i)^3}$$

$$\phi'(i) = \frac{-i2i - 2\log i}{-8i} = -\frac{1}{4i}\left[1 - \log e^{i\pi/2}\right]$$

$$= \frac{i}{4}\left[1 - i\frac{\pi}{2}\right] = \frac{1}{4}\left[i + \frac{\pi}{2}\right]$$

$$\text{Res}(z=i) = \frac{1}{4}\left(\frac{\pi}{2} + i\right)$$

By Cauchy's residue theorem,

$$\int_{-R}^{-r} f(x)dx + \int_{\gamma} f(z)dz + \int_{r}^{R} f(x)dx + \int_{\Gamma} f(z)dz = 2\pi i.\text{Res}(z=i)$$

$$= 2\pi i \cdot \tfrac{1}{4}\left(\tfrac{\pi}{2} + i\right) \quad \text{... (1)}$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\left[\frac{z^2}{(1+z^2)^2}\right]\left[\frac{\log z}{z}\right] = (0)(0) = 0 \quad \text{... (2)}$$

$\therefore$ By theorem, $\lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)(0) = 0$ ... (3)

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z\log z}{(1+z^2)^2} = \lim_{t\to\infty} \frac{(1/t)\log(1/t)t^4}{(1+t^2)^2}$$

$$= \lim_{t\to\infty} \frac{t^3(\log 1 - \log t)}{(1+t^2)^2} = \lim_{t\to\infty}\left[\frac{t^4}{(1+t^2)^2}\right]\left[\frac{-\log t}{t}\right]$$

$$= (1)\cdot 0 = 0.$$

$\therefore$ By theorem, $\lim_{r\to 0} \int_\gamma f(z)dz = -i(\pi - 0)(0) = 0$

The negative sign is taken because the ‘semicircle γ is negatively oriented. Making $r \to 0, R \to \infty$ in (1) and noting (2), (3),

$$\int_{-\infty}^{0} f(x)dx + 0 + \int_{0}^{\infty} f(x)dx + 0 = \frac{\pi}{2}\left[i\frac{\pi}{2} - 1\right]$$

$\int_{-\infty}^{0} \frac{\log x}{(1+x^2)^2} dx + \int_{0}^{\infty} \frac{\log x}{(1+x^2)^2} dx = \frac{\pi}{2}\left[i\frac{\pi}{2} - 1\right]$ ... (4)

$$\int_{-\infty}^{0} \frac{\log x dx}{(1+x^2)^2} = \int_{\infty}^{0} \frac{\log(-y)(-dy)}{(1+y^2)^2}, \text{where } -y = x$$
$$= \int_{0}^{\infty} \frac{\log(ye^{i\pi})dy}{(1+y^2)^2}$$
$$= \int_{0}^{\infty} (\log y + i\pi)\frac{dy}{(1+y^2)^2}$$
$$= \int_{0}^{\infty} (\log x + i\pi)\frac{dx}{(1+x^2)^2}$$

Using this in (4),

$$\int_{0}^{\infty} (\log x + i\pi)\frac{dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{\log x dx}{(1+x^2)^2} = \frac{\pi}{2}\left(\frac{i\pi}{2} - 1\right)$$

Equating real parts,

$$\int_{0}^{\infty} \frac{\log x dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{\log x dx}{(1+x^2)^2} = -\frac{\pi}{2}$$
$$\int_{0}^{\infty} \frac{\log x dx}{(1+x^2)^2} = -\frac{\pi}{4}$$

Example 28: Evaluate $\int_{0}^{\infty} \frac{x^{a-1}dx}{1-x}$ and $\int_{0}^{\infty} \frac{x^{a-1}dx}{1+x}$, where $0 < a < 1$.

Solution: Let $f(z) = \frac{z^{a-1}}{1-z}$, where $0 < a < 1$. Then $1 - a > 0$. For poles : $(1-z)z^{1-a} = 0 \Rightarrow z = 0,1$.

All these are simple poles and lie on x-axis. Consider $\int_C f(z)dz$ where C is a closed contour consisting of upper half part of large circle Γ s.t. $|z| = R$ and real axis from $-R$ to R idented at $z = 0,1$ by small semi circles γ_1, γ_2 of radii r_1, r_2 respectively. Evidently no pole lies writhin C.

$$\therefore \int_C f(z)dz = 0$$

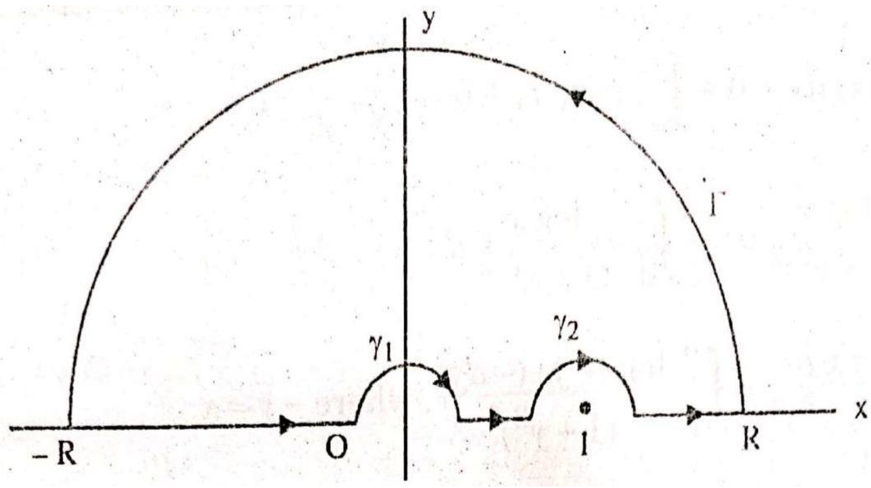

Figure: 3

or, $\int_C f(z)dz + \int_{-R}^{-r_1} f(x)dx + \int_{\gamma_1} f(z)dz + \int_{r_1}^{1-r_2} f(x)dx + \int_{\gamma_2} f(z)dz + \int_{1+r_2}^{R} f(x)dx = 0$

Making $R \to \infty$ and $r_1, r_2 \to 0$, ? get

$$\lim_{R\to\infty} \int_\Gamma f(z)dz + \int_{-\infty}^{0} f(x)dx + \int_{\gamma_1} f(z)dz + \int_0^1 f(x)dx + \int_{\gamma_2} f(z)dz$$

$$+ \int_1^\infty f(x)dx = 0 \text{ where } r_1, r_2 \to 0$$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x)dx + \sum_{i=1}^{2} \lim_{i\to 0} \int_{\gamma_i} f(z)dz$$

$$= -\lim_{R\to\infty} \int_\Gamma f(z)dz \qquad \dots (1)$$

Now $\int_{-\infty}^{0} f(x)dx = \int_{-\infty}^{0} \frac{x^{a-1}}{1-x} dx$, put $x = -t$

$$= \int_\infty^0 (-t)^{a-1} \frac{(-dt)}{1+t} = \int_0^\infty \frac{\left(te^{i\pi}\right)^{a-1} dt}{1+t}$$

$$= \int_0^\infty t^{a-1} e^{i\pi t} \cdot \frac{(-1)dt}{1+t}$$

or,

$$\int_{-\infty}^{0} f(x)dx = -\int_0^\infty \frac{x^{a-1} e^{i\pi a}}{1+x} dx \qquad \dots (2)$$

and

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{x^{a-1}dx}{1-x} \qquad \dots(3)$$

If $K_1 = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{z^a}{1-z} = \frac{0}{1} = 0$

$$\therefore \lim_{r_1\to 0}\int_{\gamma_1} f(z)dz = -i(\theta_2-\theta_1)K_1 = -i(\pi-0)(0) = 0 \qquad \dots(4)$$

If $K_2 = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z^{a-1}}{1-z} = \lim_{z\to 1} - z^{a-1} = -1.$

$$\therefore \lim_{r_2\to 0}\int_{\gamma_1} f(z)dz = -i(\theta_2-\theta_1)K_2 = -i(\pi-0)(-1) = i\pi \qquad \dots.(5)$$

Adding the last two results,

$$\sum_{i=1}^{2}\lim_{r_i\to 0}\int_{\gamma_i} f(z)dz = 0 + i\pi = i\pi \qquad \dots(6)$$

By theorem, $\lim_{R\to\infty}\int_\Gamma f(z)dz = i(\theta_2-\theta_1)K_3 = 0$ ... (7)
as

$$K_3 = \lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty}\frac{z^a}{1-z} = 0 \text{ as } 0 < a < 1$$

Putting values from (2), (3), (6) & (7) in (1), we get

$$-\int_0^\infty \frac{x^{a-1}e^{i\pi a}dx}{1+x} + \int_0^\infty \frac{x^{a-1}dx}{1-x} + i\pi = 0$$

Equating real and imaginary parts,
and

$$-\int_0^\infty \frac{x^{a-1}\cos(\pi a)\,dx}{1+x} + \int_0^\infty \frac{x^{a-1}dx}{1-x} = 0 \quad \dots(8)$$

$$-\int_0^\infty \frac{x^{a-1}\sin(\pi a)\,dx}{1+x} + \pi = 0 \qquad \dots(9)$$

$$(9) \Rightarrow \int_0^\infty \frac{x^{a-1}dx}{1+x} = \frac{\pi}{\sin(\pi a)} \qquad \dots(10)$$

Put eq. (10) in (8), $\int_0^\infty \frac{x^{a-1}dx}{1-x} = \pi\cot(\pi a)$
From eq. (10) & (11), we get the required results.

Check your progress

Problem 1: Apply the calculus of residues to prove that $\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}$.

Problem 2: Prove that $\int_0^\infty \frac{\sin \pi x}{x(1-x^2)} dx = \pi$

13.8 *SUMMARY:-*

The chapter on the application of the residue theorem demonstrates its power as a pivotal tool in complex analysis for evaluating a wide range of definite real integrals that are often intractable by standard calculus techniques. The core strategy involves selecting an appropriate complex contour integral whose value relates directly to the desired real integral. By applying the residue theorem, the value of this contour integral is computed as 2πi times the sum of the residues of the integrand's singularities enclosed within the contour. This approach is systematically tailored to different classes of integrals, including trigonometric integrals over [0, 2π] via substitution on the unit circle, improper rational integrals over (-∞, ∞) using semicircular contours, integrals involving trigonometric functions combined with rational functions, and those requiring indented contours to bypass singularities on the real axis. Ultimately, the method transforms the challenging problem of real integration into the more algebraic and often simpler task of calculating complex residues.

13.9 *GLOSSARY:-*

1. **Jordan's Lemma:** A key result used to prove that the integral along a large semicircular arc in the upper or lower half-plane vanishes for certain integrands, typically those involving e^{iaz}.
2. **Pole:** A type of isolated singularity. A pole of order n at z_0 is one where the function can be written as $f(z) = \frac{g(z)}{(z-z_0)^n}$, with $g(z)$ analytic and non-zero at z_0. A pole of order 1 is called a **simple pole**.
3. **Principal Value (P.V.):** A method for assigning a finite value to an improper integral that is not convergent in the usual sense, often

used when the limits of integration approach infinity symmetrically or when approaching a singularity symmetrically.

4. **Residue:** The coefficient a_{-1} of the $(z-z_0)^{-1}$ term in the Laurent series expansion of a function around an isolated singularity z_0. It is denoted as Res(f, z_0).

5. **Residue Theorem:** The central theorem stating that if a function is analytic inside and on a simple closed contour C, except for a finite number of isolated singularities, then the contour integral is $\oint_C f(z)dz = 2\pi i \sum \text{Res}(f, z_k)$, where the sum is over all residues inside C.

13.10*REFERENCES:-*

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- Goyal, H. S., & Gupta, C. P. (2015). *Function of a Complex Variable and Its Applications* (Revised Edition). Krishna Prakashan Media (P) Ltd., Meerut, India.

13.11*SUGGESTED READING:-*

- L. V. Ahlfors,(1966), Complex Analysis, Second edition, McGraw-Hill, New York.
- J.B. Conway, (2000), Functions of One Complex Variable, Narosa Publishing House,
- E.T. Copson, (1970), Introduction to Theory of Functions of Complex Variable, Oxford University Press.
- Theodore W. Gamelin,(2001) Complex Analysis, Springer-Verlag, 2001.

13.12*TERMINAL QUESTIONS:-*

Long answer type question

1: Evaluate. $\sqrt{\frac{2\pi}{1+a\cos\theta}}, a^2 < 1.$

2: prove that $\int_0^{\pi} \frac{a d\theta}{a^2+\cos^2\theta} = \frac{\pi}{(1+a^2)^{1/2}}$

3: Evaluate, $\int_0^{\pi} \frac{d\theta}{2+\sin^2\theta}$

4: Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, where $0 < a < 1$.

5: If $a \geq 0$, then evaluate $\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx$.

Hint: First proves that $\int_0^{\infty} \frac{\cos(mx)}{x^2+a^2} dx = \frac{\pi}{2a} e^{-ma}$, where $m \geq 0, a > 0$. Then putt $m = a$ and $a = 1$ in the above equation we get the required result.

Short answer type question

1: Prove that $\int_0^{\infty} \frac{\cos mx dx}{x^4+x^2+1} = \frac{\pi}{6}\left(3\sin\frac{m}{2} + \sqrt{3}\cos\frac{m}{2}\right)\exp\left(\frac{-m\sqrt{3}}{2}\right)$

Hint: First prove that,

$$\int_{\infty}^{\infty} \frac{\cos mx dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)$$

$$= \frac{\pi}{\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\left[\left(\sin\frac{m}{2}\right)\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cos\frac{m}{2}\right]$$

$$= \frac{\pi}{2\sqrt{3}\cdot\sqrt{3}}\exp\left(\frac{-m\sqrt{3}}{2}\right)\left[3\sin\frac{m}{2} + \sqrt{3}\cos\frac{m}{2}\right]$$

2: Classify the singularities of a function of a complex variable. Show that the only singularities of $\frac{\cot \pi z}{(z-a)^2}$ are poles. Find the residues of the function at these poles.

3: Determine the nature of the pole at the origin of the function $\frac{e^z}{z\sin mz}$ and find the residue.

Objective type question:

1. The Residue Theorem is used to evaluate:

A. Real integrals

B. Line integrals of analytic functions

C. Improper integrals

D. All of the above

2. If $f(z)$ has isolated singularities inside a simple closed contour C, then according to the Residue theorem: $\oint_C f(z)dz = ?$

A. 0

B. $2\pi i$ Residues of $f(z)$ inside C

C. πi Residues of $f(z)$ inside C

D. None of these

3. The residue of $f(z) = \frac{1}{(z-a)^2}$ at $z = a$ is:

A. 0

B. 1

C. ∞

D. Undefined

4. The integral $\oint_C \frac{dz}{z}$, where C is the unit circle |z|=1, is:

A. 0

B. 2πi

C. πi

D. 1

5. The Residue Theorem helps in evaluating real definite integrals of which form?

A. $\int_{-\infty}^{\infty} f(x)dx$

B. $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

C. Both A and B

D. Neither A nor B

6. The residue at a simple pole $z = a$ is given by:

A. $\lim_{z\to a} f(z)(z-a)^2$

B. $\lim_{z\to a} f(z)(z-a)$

C. $\lim_{z\to a} \frac{d}{dz}[f(z)(z-a)]$

D. None of these

7. Which of the following integrals can be evaluated using the residue theorem?

A. $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

B. $\int_0^{\infty} \frac{\sin x}{x} dx$

C. $\int_{0}^{2\pi} \frac{1}{5-4\cos\theta} d\theta$

D. All of these

8. The residue of $f(z) = \frac{e^z}{(z-1)^3}$ at $z=1$ is:

A. $e/4$

B. e

C. $e/3!$

D. $e/2!$

9. For $f(z) = \frac{1}{(z^2+1)}$, the poles are:

A. $z = \pm 1$

B. $z = \pm i$

C. $z = 0, \infty$

D. None of these

10. The contour integral $\oint_{|z|=2} \frac{dz}{z^2(z-1)}$ is equal to:

A. 0

B. 2πi

C. 2πi×(sum of residues at z=0, z=1)

D. Undefined

13.13*ANSWERS*

Answer of long answer type question:

1. $\frac{2\pi}{\sqrt{1-a^2}}$ **3.** $\frac{\pi}{\sqrt{6}}$. **4.** $2\pi/(1-a^2)$

Answer of short answer type question:

3: $I = -2\pi i(-2/3) = \frac{4\pi i}{4}$

Answer of objective questions

1:	D	**2:**	B	**3:**	A	**4:**	B
5:	C	**6:**	B	**7:**	D	**8:**	D
9:	B	**10:**	C				

UNIT-14: Analytic Continuation

CONTENTS

14.1 *INTRODUCTION:-*

Analytic continuation is a fundamental concept in complex analysis that allows us to extend the domain of a given analytic function beyond its initial region of definition while preserving analyticity. Often, a function represented by a power series converges only within a certain disk, but through analytic continuation, we can construct a larger analytic function that coincides with the original one on the overlap of their domains. This process reveals the deeper structure of complex functions, helps identify natural boundaries and singularities, and enables the study of functions such as the Riemann zeta function, the logarithm, and many special functions in broader domains. Analytic continuation thus serves as an essential tool for expanding the reach of analytic functions and understanding their global behavior in the complex plane.

The unit on meromorphic functions, Rouché's theorem, and analytic continuation introduces fundamental ideas that extend the study of analytic functions to a broader and deeper context within complex analysis. It begins by exploring meromorphic functions, which generalize analytic functions by allowing isolated poles while preserving analyticity elsewhere, providing a rich framework for understanding complex behavior. The unit then presents Rouché's theorem, a key result used to compare analytic functions on a closed contour and determine the number of zeros within it, offering powerful techniques for root-counting and function comparison. Finally, it examines analytic continuation, the method of extending an analytic function beyond its initial domain of definition using overlapping regions where the function remains analytic. Together, these topics build essential tools for analyzing and extending complex functions across wider regions of the complex plane.

14.2 *OBJECTIVES:-*

After the study of this unit, learner shall understand:

1. To understand the concept of meromorphic functions, their definition, properties, and how they differ from analytic and entire functions.
2. To identify and classify poles as the only singularities of meromorphic functions and to study their behavior in the complex plane.
3. To learn the statement, conditions, and applications of Rouché's theorem for determining the number of zeros of analytic functions within a closed contour.
4. To apply Rouché's theorem in solving problems involving root-location, polynomial analysis, and comparison of complex functions.
5. To introduce the idea of analytic continuation and explain how analytic functions can be extended beyond their initial domains.
6. To understand the uniqueness principle of analytic continuation, based on the identity theorem.
7. To develop the ability to use these concepts to analyze the global behavior of complex functions, particularly how functions behave near singularities and across extended regions.

14.3 *MEROMORPHIC FUNCTIONS:-*

The study of meromorphic functions and Rouché's Theorem forms a crucial part of complex analysis, providing powerful tools for understanding the behavior of analytic functions in the complex plane. A meromorphic function, defined as a function that is analytic everywhere except at isolated poles, serves as a natural extension of rational functions and plays a central role in many areas of mathematics and mathematical physics. Rouché's Theorem, on the other hand, is a key result in complex function theory that allows us to compare two analytic functions on a closed contour to determine how many zeros lie inside it. By analyzing the dominance of one function over another on the boundary, the theorem provides a practical method for counting zeros without explicitly solving equations. Together, the concepts of meromorphic functions and Rouché's Theorem offer deep insights into the structure, zeros, and singularities of complex functions, and form essential tools for advanced problem-solving in complex analysis.

Definition (Meromorphic function): A function is called meromorphic if, in the finite complex plane, its only singularities are isolated poles.

Definition (Entire function): A function that is free of singularities throughout the entire finite complex plane is known as an entire function.

Theorem 1: (Mittag Leffler's expansion theorem) Suppose that the only singularities of $f(z)$ in the finite part of the z-plane are simple poles at $a_1, a_2, \dots a_n$ arranged in the order of increasing absolute values. Also suppose that

(*i*) Residues of $f(z)$ at $a_1, a_2, \dots, a_n$ be $b_1, b_2, \dots, b_n$,

(*ii*) $< C_n >$ is a sequence of circles (or rectangles or squares) of radii R_n or R_n is the minimum distance of C_n from the origin) C_n encloses $a_1, a_2, \dots, a_n$ and no other poles. On the circle $C_n, |f(z)| < M$, where M is independent of n and $R_n \to \infty$ as $n \to \infty$.

Then for all values of z except poles,

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

Proof. Consider the integral, $I = \int_{C_n} \frac{f(t)dt}{t-z}$

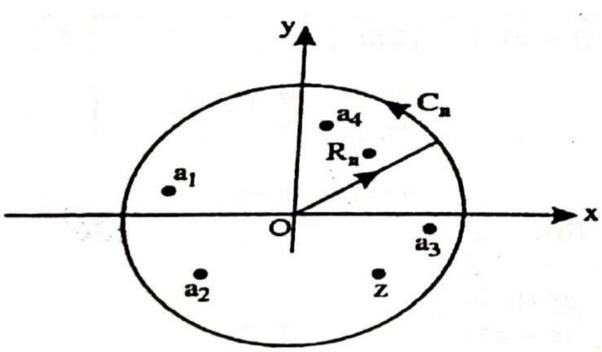

Figure 1

z being any point (except pole) inside the circle C_n. The function $\frac{f(t)}{t-z}$ has simple pole at $t = z$.

Evidently any pole of $f(t)$ is a pole of $f(t)/(t-z)$. But $f(t)$ has simple poles at $t = a_1, a_2, \ldots, a_n$. Hence, $\frac{f(t)}{t-z}$ has simple poles at $z, a_m (m = 1,2, \ldots, n)$.
Residuc of $\frac{f(t)}{t-z}$ at z is $\lim_{t \to z} (t-z) \frac{f(t)}{(t-z)} = f(z)$. Residue of $\frac{f(t)}{t-z}$ at a_m is

$$\lim_{t \to a_m} (t - a_m) \frac{f(t)}{(t-z)} = \left[\lim_{t \to i_m} (t - a_m) f(t)\right] \cdot \left[\lim_{t \to a_m} \frac{1}{t-z}\right]$$

$$= \frac{b_m}{a_m - z}.$$ For residuc of $f(t)$ at a_m is b_m

By Cauchy's residue theorem

$$\int_{C_n} \frac{f(t)}{t-z} dt = 2\pi i \left(\text{Sum of residues of } \frac{f(t)}{t-z} \text{ within } C_n \right)$$

$$= 2\pi i \, [\text{ Residues at } z, a_1, a_2, \ldots, a_n]$$

$$= 2\pi i \left[f(z) + \sum_{m=1}^{n} b_m/(a_m - z) \right]$$

or $\frac{1}{2\pi i} \int_{C_n} \frac{f(t)dt}{t-z} = f(z) + \sum_{m=1}^{n} \frac{b_m}{a_m - z}$ ... (1)

Suppose $f(t)$ is analytic at $t = 0$. Putting $z = 0$ in (1),

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(t)}{t} dt = f(0) + \sum_{m=1}^{n} \frac{b_m}{a_m} \quad \text{... (2)}$$

Subtracting (2) from (1),

$$\frac{1}{2\pi \hat{\imath}} \int_{C_n} \frac{z f(t) dt}{(t-z)t} = f(z) - f(0) + \sum_{m=1}^{n} \frac{z b_m}{a_m (a_m - z)} \quad \text{... (3)}$$

But $\left| \frac{1}{2\pi i} \int_{C_n} \frac{z f(t) dt}{t(t-z)} \right| \le \left| \frac{1}{2\pi i} \right| \int_{C_n} \frac{|z| \cdot |f(t)| \cdot |dt|}{|t|(|t| - |z|)}$

$$\le \frac{1}{9\pi} \frac{M|z|}{R_n (R_n - |z|)} \int_{C_n} |dt| = \frac{M|z|}{R_n - |z|}$$

$\to 0$ as $n \to \infty$.

For $\lim_{n\to\infty} R_n = \infty$

$\therefore \lim_{n\to\infty} \int_{C'_n} \frac{zf(t)dt}{(t-z)t} = 0$

Making $n \to \infty$ in (3) and noting this,
or

$$f(z) - f(0) + \sum_{m=1}^{\infty} \frac{zb_m}{a_m(a_m - z)} = 0$$
$$f(z) = f(0) + \sum_{n=1}^{\infty} \left(\frac{b_n}{z - a_n} + \frac{b_n}{a_n}\right)$$

Finally

$$f(z) = f(0) + \sum_{n=1}^{\infty} v_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

Example 1: Prove that

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}$$

Solution: Let $f(z) = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z\sin z}$

Poles of $f(z)$ are given by $\sin z = 0$,
or

$$z = n\pi \text{ where } n \pm 1, \pm 2, \ldots$$
$$z = 0 \text{ is not a pole of } f(z).$$

For $\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z\cos z - \sin z}{z\sin z}$, $\left[\text{ form } \frac{0}{0}\right]$

$$= \lim_{z\to 0} \frac{-z\sin z + \cos z - \cos z}{\sin z + z\cos z}, \left[\text{ form } \frac{0}{0}\right]$$
$$= \lim_{z\to 0} \frac{-\sin z - z\cos z}{\cos z + \cos z - z\sin z} = \frac{0}{2} = 0$$

or $\lim_{z\to 0} f(z) = 0$, so that $f(z)$ has a removable singularity at $z = 0$
So we can define $f(0) = 0$.

$$\text{Res}(z = n\pi) = \lim_{z\to n\pi} (z - n\pi) f(z)$$

$$= \lim_{z \to n\pi} \frac{(z - n\pi)(z\cos z - \sin z)}{z\sin z}, \text{ form } \frac{0}{0}$$
$$= \lim_{z \to n\pi} \frac{(z - n\pi)(-z\sin z + \cos z - \cos z) + 1.(z\cos z - \sin z)}{\sin z + z\cos z}$$

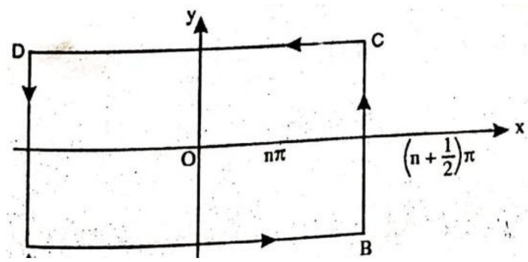

Figure 2

$$= \frac{n\pi\cos n\pi}{n\pi\cos n\pi} = -i$$

Let Res $(z = n\pi) = b_n$ and $z = n\pi = a_n$.
Then Res $(z = a_n) = b_n = 1$ if $a_n = n\pi$
Similarly $b_n = 1$ if $a_n = -n\pi$ for $n = 1,2,3, \dots \dots \dots$
Let the contour c_n be the square $ABCD$ with centre at the origin and each side of length $(2n + 1)\pi$ so that its vertices are at points $\left(n + \frac{1}{2}\right)\pi(\pm 1 \pm i)$.

Among all these poles, there are two poles $z = n\pi, z = -n\pi$ whose absolute values are greater than other Poles.

The minimum distance R_n of C_n from origin is

$$\left(n + \frac{1}{2}\right)\pi \to \infty \text{ as } n \to \infty$$

The length l_n (perimeter of square $ABCD$) is

$$4 \times 2\left(n + \frac{1}{2}\right)\pi = 8\left(n + \frac{1}{2}\right)\pi \ \therefore \ l_n = 8R_n, \text{ which is constant.}$$

$$|\cot z| = \left|i\left(\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\right)\right| = |i|\left|\frac{e^{i2z} + 1}{e^{i2z} - 1}\right| = \left|\frac{e^{i2(x+iy)} + 1}{e^{i2(x+iy)} - 1}\right|$$
$$\leq \frac{|e^{i2x}| \cdot |e^{-2y}| + 1}{|e^{i2x}| \cdot |e^{-2y}| - 1} = \frac{e^{-2y} + 1}{e^{-2y} - 1} \text{ as } |e^{i2x}| = 1$$
$$|\cot z| \leq \begin{cases} \dfrac{e^{-2y} + 1}{e^{-2y} - 1} = \dfrac{1 + e^{2y}}{1 - e^{2y}} \text{ if } y \text{ is negative} \\ \text{or } \dfrac{e^{-2y} + 1}{1 - e^{-2y}} = \dfrac{1}{e^{2y}} - 1 \end{cases} \quad \text{if } y > 0$$

$\to 1$ if $y \to \infty$ or $y \to -\infty$.
$\therefore$ $\cot z$ is bounded
$\Rightarrow$ $f(z) = \cot z - \frac{1}{z}$ is bounded in C_n as $\frac{1}{z} = \frac{1}{\left(n+\frac{1}{2}\right)\pi} \to 0$ as $n \to \infty$.
On $AB, y = \left(n + \frac{1}{2}\right)\pi$ and on $CD, y = -\left(n + \frac{1}{2}\right)\pi$
Now applying Mittag Leffler's theorem,

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

$$=0 + \sum_{n=1}^{\infty} 1\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{\infty} 1 \cdot \left(\frac{1}{z + n\pi} - \frac{1}{n\pi}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

or,

$$\cot z - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

or,

$$\cos z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2} \quad \ldots..(2)$$

Example 2: Prove that, $\pi\cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$

Solution: Replacing z by πz in equation (2) in previous example,

$$\cot \pi z = \frac{1}{z\pi} + \sum_{n=1}^{\infty} \frac{2\pi z}{\pi^2 z^2 - n^2\pi^2}$$

or

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad \ldots (3)$$

Example 3: Prove that, $z\cot \pi z = \frac{1}{\pi} + \frac{z}{\pi}\Sigma'\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$

where accent indicates that $n = 0$ is omitted.
Solution. $\frac{2z}{z^2-n^2} = \frac{1}{z+n} + \frac{1}{z-n}$
Now (3) in previous example is expressible as,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right)$$

Multiplying by $\frac{z}{\pi}$,

$$z \cot \pi z = \frac{1}{\pi} + \frac{z}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right)$$

Example 4: Prove that, $\text{cosec} z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - z^2}$

Solution: Let $f(z) = \text{cosec} z - \frac{1}{z} = \frac{z - \sin z}{z \sin z}, z \neq 0.$
Poles of $f(z)$ are given by $\sin z = 0$

or

$$z = n\pi \text{ where } n = \pm 1, \pm 2, \pm 3, \ldots$$

$z = 0$ is not a pole of $f(z)$, but it is a removable singularity. For
$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z - \sin z}{z \sin z}, \text{ form } \left.\frac{0}{0}\right]$

$$= \lim_{z \to 0} \frac{1 - \cos z}{\sin z + z \cos z}; \text{ by L. Hospital's rule.}$$

$$= \lim_{z \to 0} \frac{\sin z}{\cos z - z \sin z + \cos \dot{z}} = \frac{0}{2} = 0$$

However, we may define $f(0) = 0$.

$$\text{Res}(z = n\pi) = \lim_{z \to n\pi} \frac{\phi(z)}{\psi'(z)}, \text{ where } f(z) = \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z \to n\pi} \frac{(z - \sin z)}{\frac{d}{dz}(z \sin z)} = \lim_{z \to n\pi} \frac{z - \sin z}{z \cos z + \sin z}$$

$$= \frac{n\pi}{n\pi \cos n\pi} = \frac{1}{\cos n\pi} = (-1)^{-n} = (-1)^n$$

Let, $n\pi = a_n, (-1)^n = b_n$

Similarly Res $(z = -n\pi) = (-1)^{in}$
Thus Res $(z = a_n) = b_n = (-1)^n$ if $a_n = -n\pi$ or $n\pi$. See Figure of Problem No. (1) and discuss the same thing here.

$$|\text{cosec} z| = \left|\frac{1}{\sin z}\right| = \left|\frac{2i}{e^{iz} - e^{-iz}}\right| = \frac{2}{|e^{i(x+iy)} - e^{-i(x+iy)}|}$$
$$\leq \frac{2}{|e^{ix}| \cdot |e^{-y}| - |e^{-ix}| \cdot |e^{y}|}$$

As, $|e^{ix}| = 1$,

and $\frac{1}{|a \pm b|} \leq \frac{1}{|a| - |b|}$

$= \frac{2}{e^{-y} - e^{y}}$

$\therefore\ |\text{cosec} z| \leq \begin{cases} \frac{2}{e^{-y}-e^{y}}, \text{ if } y < 0 \\ \frac{2}{e^{y}-e^{-y}}, \text{ if } y > 0 \end{cases}$; $\rightarrow 0$ as $y \rightarrow \infty$ or $y \rightarrow -\infty$.

$\therefore$ cosec$z$ is bounded in c_n.
Cosequently, cosec$(z - \frac{1}{z})$ is bounded in c_n
as

$$\frac{1}{z} = \frac{1}{\left(n + \frac{1}{2}\right)\pi} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now applying Mittag Leffler's theorem,

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

or,

$$f(z) = 0 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$$
$$+ \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{z + n\pi} - \frac{1}{n\pi}\right)$$
$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right] = \sum_{n=1}^{\infty} \frac{2z(-1)^n}{z^2 - n^2\pi^2}$$

or,

$$\text{cosec}z - \frac{1}{z} = 2z\sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2\pi^2}$$

or,

$$\text{cosec}z = \frac{1}{z} + 2z\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - z^2}$$

Example 5: Prove that $\tan z = \sum_{n=1}^{\infty} \frac{2z}{\left(n+\frac{1}{2}\right)^2\pi^2 - z^2}$.

Solution: Let $f(z) = \tan z = \frac{\sin z}{\cos z}$. Poles of $f(z)$ are given by

$$\cos z = 0 = \cos\frac{\pi}{2} \Rightarrow z = m\pi \pm \frac{\pi}{2} \text{ for } m = 0, \pm 1, \pm 2, \ldots$$

or,

$$z = \pm\left(m\pi + \frac{\pi}{2}\right) \text{ for } m = 0,1,2,3,\ldots,n.$$

Let b_n = Residue of $f(z) = \tan z$ at $z = \dot{a}_n = \pm\left(n + \frac{1}{2}\right)\pi$

Then $b_n = \text{Res}(z = a_n) = \lim_{z \to a_n} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \lim_{z \to a_n} \frac{-\sin z}{\sin z} = -1$

Let the contour c_n be the square $ABCD$ with centre at the origin and and length of each side be $2(n+1)\pi$ so that its vertices are points $(n + 1)\pi(\pm 1 \pm i)$

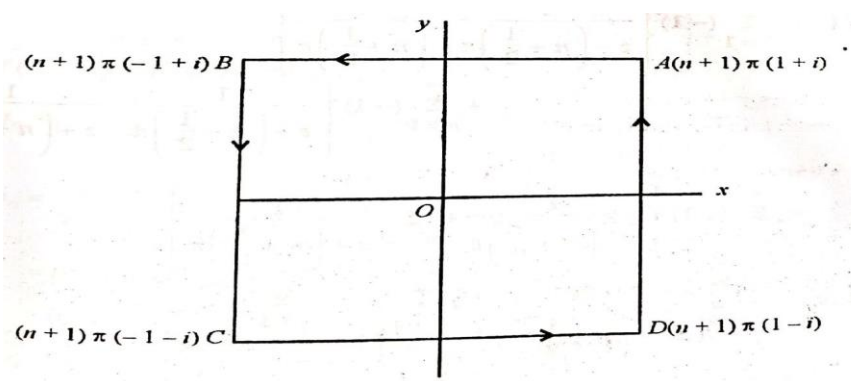

Figure 3

The poles $z = a_m$ for $m = 0,1,2,\ldots,n$ lie within c_n and no other poles lie inside c_n. In this sequence of poles, there are two poles $z = \left(n + \frac{1}{2}\right)\pi$ and $z = -\left(n + \frac{1}{2}\right)\pi$ whose absolute value is greatest compared to those of the other poles. The minimum distance R_n of c_n from the origin is $(n + 1)\pi$.

The length l_n (perimeter of square $ABCD$ of c_n is $4 \times 2(n+1)\pi = 8(n+1)\pi$. Hence $l_n = 8R_n$.

$$|\tan z| = \left|\frac{\sin z}{\cos z}\right| = \left|\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)\frac{1}{i}\right| = \left|\frac{e^{i(x+iy)} - e^{-i(x+iy)}}{e^{i(x+iy)} - e^{-i(x+iy)}}\right|$$

$$\leq \frac{|e^{ix}||e^{-y}| + |e^{-ix}| \cdot |e^{y}|}{|e^{ix}| \cdot |e^{-y}| - |e^{-ix}| \cdot |e^{y}|} \text{ as } \left|\frac{a \pm b}{c \pm d}\right| \leq \frac{|a| + |b|}{|c| - |d|} \text{ if } |c| > |d|$$

$$= \frac{e^{-y} + e^{y}}{e^{-y} - e^{y}} \text{ as } |e^{\pm ix}| = 1$$

$$= \frac{1 + e^{2y}}{1 - e^{2y}}$$

This $\Rightarrow |\tan z| \leq \begin{cases} \frac{1+e^{2y}}{1-e^{2y}} \text{ if } y < 0 \\ \frac{1+e^{-2y}}{1-e^{-2y}} \text{ if } y > 0 \end{cases}$

$\to 1$ as $y \to \infty$ or $y \to -\infty$

$\therefore$ $\tan z$ is bounded in c_n.

On $AB, y = (n+1)\pi$, and on $CD, y = -(n+1)\pi$

Now applying Mittag Leffler's theorem, we get

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

But $f(0) = \tan 0 = 0$.

$$\therefore\ f(z = \sum_{n=1}^{\infty} (-1)\left\{\frac{1}{z - \left(n + \frac{1}{2}\right)\pi} + \frac{1}{\left(n + \frac{1}{2}\right)\pi}\right\}$$

$$+ \sum_{n=1}^{\infty} (-1)\left\{\frac{1}{z + \left(n + \frac{1}{2}\right)\pi} - \frac{1}{z + \left(n + \frac{1}{2}\right)\pi}\right\}$$

$$= \sum_{n=1}^{\infty} (-1)\left\{\frac{1}{z - \left(n + \frac{1}{2}\right)\pi} + \sum_{n=1}^{\infty} \cdot \frac{1}{z + \left(n + \frac{1}{2}\right)\pi}\right\}$$

or,

$$\tan z = -\sum_{n=1}^{\infty} \frac{2z}{z^2 - \left(n + \frac{1}{2}\right)^2 \pi^2} = \sum_{n=1}^{\infty} \frac{2z}{\left(n + \frac{1}{2}\right)^2 \pi^2 - z^2}$$

Example 6: Prove that, $\sec z = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 - 4z^2}$.

Solution. Let $f(z) = \sec z = \frac{1}{\cos z}$

Poles of $f(z)$ are given by $\cos z = 0 = \cos\left(\frac{\pi}{2}\right)$

or,

$$z = m\pi \pm \frac{\pi}{2} \text{ for } m = 0, \pm 1, \pm 2, \ldots, \pm n$$

or,

$$z = \pm\left(m\pi + \frac{\pi}{2}\right) \text{ for } m = 0,1,2,3, \ldots, n.$$

Take $a_m = \left(m + \frac{1}{2}\right)\pi$ or $-\left(m + \frac{1}{2}\right)\pi$ for $m = 0,1,2, \ldots, n$.

Let the contour be c_n, the square $ABCD$ with centre at the origin and length of each side is $2(n+1)\pi$ so that its vertices are at points $(n+1)\pi(\pm 1 \pm i)$ (see Figure of Problem 5).

The contour c_n contains all the above poles and no other Poles. The minimum distance R_n of c_n from the origin is $(n+1)\pi$. The length l_n (perimeter of square $ABCD$) is $4 \times 2(n+1)\pi = 8(n+1)\pi$. Hence $l_n = 8R_n$

$$|\sec z| = \left|\frac{2}{e^{iz} + e^{-iz}}\right| = \left|\frac{2}{e^{i(x+iy)} + e^{-i(x+iy)}}\right|$$
$$= \frac{2}{|e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}|} \leq \left|\frac{2}{|e^{ix}| \cdot |e^{-y}| - |e^{-ix}| \cdot |e^{y}|}\right|$$
$$= \frac{2}{e^{-y} - e^{y}} \text{ as } \left|\frac{1}{a \pm b}\right| \leq \frac{1}{|a| - |b|} \text{ if } |a| > |b|$$

and

$$\left|e^{\pm ix}\right| = 1$$

Thus $|\sec z| = |f(z)| = \begin{cases} \frac{2}{e^{-y} - e^{y}} & \text{if } y < 0 \\ \frac{2}{e^{y} - e^{-y}} & \text{if } y > 0 \end{cases}$

$\rightarrow 0$ as $y \rightarrow \infty$ or $y \rightarrow -\infty$.

$\therefore f(z) = \sec z$ is bounded in c_n.

If $a_n = \left(n + \frac{1}{2}\right)\pi$, then $b_n = \text{Pes } (z = a_n) = \lim_{z \to a_n} \frac{1}{\frac{d}{dz}(\cos z)}$

or,

$$b_n = \lim_{z \to a_n} \frac{-1}{\sin z} = \frac{-1}{\sin\left(n\pi + \frac{\pi}{2}\right)} = -\frac{1}{\cos(n\pi)} = \frac{-1}{(-1)^n}$$
$$= (-1)(-1)^n \text{ as } (-1)^{-n} = (-1)^n$$

or, $b_n = (-1)^{n+1}$. Similarly $b_n = (-1)^n$ if $a_n = -\left(n + \frac{1}{2}\right)\pi f(0) = \sec(0) = 1$

Applying Mittag Leffler's theorem,

$$f(z) = f(0) + \sum_{n=0}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$$

$$\Rightarrow f(z) = 1 + \sum_{n=0}^{\infty} (-1)^{n+1} \left[\frac{1}{z - \left(n + \frac{1}{2}\right)\pi} + \frac{1}{\left(n + \frac{1}{2}\right)\pi}\right]$$
$$+ \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{z + \left(n + \frac{1}{2}\right)\pi} - \frac{1}{\left(n + \frac{1}{2}\right)\pi}\right]$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{z + \left(n + \frac{1}{2}\right)\pi - \frac{1}{z - \left(n + \frac{1}{2}\right)\pi}}\right]$$
$$- \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{\left(n + \frac{1}{2}\right)\pi} + \frac{1}{\left(n + \frac{1}{2}\right)\pi}\right]$$

$$= 1 + \sum_{n=0}^{\infty} (-1)^n \frac{2\left(n + \frac{1}{2}\right)\pi}{\left(n + \frac{1}{2}\right)^2 \pi^2 - z^2} - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2}{\left(n + \frac{1}{2}\right)\pi}$$

or,

$$\sec z = 1 + 4\pi \sum_{n=0}^{\infty} \frac{(2n+1)\pi(-1)^n}{(2n+1)^2\pi^2 - 4z^2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \ldots (1)$$

But $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}$
This $\Rightarrow 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 0$. Putting this in (1),

$$\sec z = 4\pi \sum_{n=0}^{\infty} \frac{(2n+1)\pi(-1)^n}{(2n+1)^2\pi^2 - 4z^2}$$

Note: The above problem can also be expressed as

$$\sec z = 4\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2\pi^2 - 4z^2}$$

or,

$$\sec z = 4\pi \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{4z^2 - (2n-1)^2\pi^2}$$

Example 7: Prove that if $-\pi < \alpha < \pi$, then

$$\frac{\cos(\alpha z)}{\sin(\pi z)} = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=:}^{\infty} (-1)^n \frac{\cos(n\alpha)}{z^2 - n^2}$$

Solution: Let $f(z) = \frac{\cos(\alpha z)}{\sin(\pi z)}$.
Poles of $f(z)$ are given by $\sin(\pi z) = 0 = \sin(n\pi)$
$\Rightarrow \ \pi z = n\pi$ or $z = n$ for $n = 0, \pm 1, \pm 2', \dots$
Take $a_n = n$ or $-n$.

$$b_n = \text{Res}(z = a_n) = \lim_{z \to a_n} \frac{\cos(\alpha z)}{\frac{d}{dz}(\sin \pi z)} = \lim_{z \to a_n} \frac{\cos(\alpha z)}{\pi \cos(\pi z)}$$

$$= \frac{\cos(n\alpha)}{\pi \cos(n\pi)} \text{ if } a_n = n \text{ or } -n$$

$$= \frac{(-1)^n}{\pi} \cos(n\alpha) \text{ as } (-1)^n = (-1)^{-n}, \cos(n\pi) = (-1)^n$$

Let square $ABCD$ denote the closed contour c_n with centre at the origin and each side $= 2\left(n + \frac{1}{2}\right) = 2n + 1$ and so its vertices are at points $\left(n + \frac{1}{2}\right)(\pm 1 \pm i)$.

The contour c_n encloses all poles $a_n (n = 0,1,2, \dots, n)$. In this sequence of poles, there are two pols $z = n$ and $z = -n$ whose absolute value is greater than those of other poles. The minimum distance R_n of c_n from the centre is $\left(n + \frac{1}{2}\right) \Rightarrow \infty$ as $n \to \infty$. The length l_n (perimeter of $ABCD$) is $4 \times 2\left(n + \frac{1}{2}\right) = 8n + 4$ so that $l_n = 8R_n$

Note that $e^{i\alpha z} = e^{i\alpha(x+iy)} = e^{i\alpha x}e^{-\alpha y}$
$\Rightarrow \ |e^{i\alpha z}| = e^{-\alpha y}$ as $|e^{\pm i\alpha x}| = 1$.
Similarly $|e^{-i\alpha z}| = e^{\alpha y}, |e^{i\pi z}| = e^{-\pi y}, |e^{-i\pi z}| = e^{\pi y}$
and

$$\left|\frac{a \pm b}{c \pm d}\right| \leq \frac{|a| + |b|}{|c| - |d|} \text{ if } |c| > |d|$$

In view of this, we get

$$\left|\frac{\cos(\alpha z)}{\sin(\pi z)}\right| = \left|\frac{(e^{i\alpha z} + e^{-i\alpha z})i}{(e^{i\pi z} - e^{-i\pi z})}\right| \leq \frac{|e^{i\alpha z}| + |e^{-i\alpha z}|}{|e^{i\pi z}| - |e^{-i\pi z}|}$$
$$\leq \frac{e^{-\alpha} + e^{\alpha y}}{e^{-\pi y} - e^{\pi y}} \text{ or } \frac{e^{\alpha y} + e^{-\alpha y}}{e^{\pi y} - e^{-\pi y}}$$
$$= \frac{e^{-\alpha y}(1 + e^{2\alpha y})}{e^{-\pi y}(1 - e^{2\pi y})} \text{ or } \frac{e^{\alpha y}(1 + e^{-2\alpha y})}{e^{\pi y}(1 - e^{-2\pi y})}$$
$$< \frac{1 + e^{2\alpha y}}{1 - e^{2\pi y}} \text{ or } \frac{1 + e^{-2\alpha y}}{1 - e^{-2\pi y}} \to 1 \text{ as } y \to -\infty \text{ or } y \to +\infty$$

according as $y < 0$ or $y > 0$.
On $AB, y = n + \frac{1}{2}$, and on $CD, y = -\left(n + \frac{1}{2}\right)$
$\therefore f(z)$ is bounded in c_n.

$$f(0) = \lim_{z \to 0} \frac{\cos(\alpha z)}{\sin(\pi z)} = \left\{\lim_{z \to 0} \frac{\pi z}{\sin(\pi z)}\right\}\left\{\lim_{z \to 0} \cos(\alpha z)\right\} \cdot \frac{1}{\pi z}$$
$$= (1)(1) \cdot \frac{1}{\pi z} \text{ or, } f\left(0; = \frac{1}{\pi z}\right.$$

By Mittag Leffler's theorem,

$\text{f}(z) = f(0) + \sum_{n=1}^{\infty} \ b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right)$

$= \frac{1}{\pi z} + \sum_{n=1}^{\infty} \ (-1)^n \frac{\cos(n\alpha)}{\pi}\left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=1}^{\infty} \ (-1)^n \frac{\cos(n\alpha)}{\pi}\left[\frac{1}{z+n} - \frac{1}{n}\right]$

$$= \frac{1}{\pi z} + \sum_{n=1}^{\infty} \ (-1)^n \frac{\cos(n\alpha)}{\pi} \cdot \left(\frac{2z}{z^2 - n^2}\right)$$

or, $\frac{\cos(\alpha z)}{\sin(\pi z)} = \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty} \ (-1)^n \frac{\cos(n\alpha)}{z^2 - n^2}$.

Example 8: Prove that $\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + \sum_{n=1}^{\infty} \ \frac{2z}{z^2 + 4\pi^2 n^2}$

Solution: Let $f(z) = \frac{1}{2} + \frac{1}{e^z - 1} = \frac{e^z + 1}{2(e^z - 1)}$. Then

$$f(0) = \lim_{z\to 0} \frac{e^z+1}{2(e^z-1)} = \lim_{z\to 0} \frac{2+z+\frac{z^2}{2!}+\cdots}{2\left(z+\frac{z^2}{2!}+\cdots\right)}$$

$$= \left\{\lim_{z\to 0} \frac{2+z+\frac{z^2}{2!}+\cdots}{2\left(z+\frac{z^2}{2!}+\cdots\right)}\right\}\cdot\frac{1}{z} = \left(\frac{2}{2}\right)\left(\frac{1}{z}\right) = \frac{1}{2}$$

or, $f(0) = \frac{1}{z}$

Poles of $f(z)$ are given by $2(e^z-1)=0$

$\Rightarrow\ e^z = 1 = e^{2n\pi i} \Rightarrow z = 2n\pi i$. for $n = 0, \pm 1, \pm 2, \ldots$

Take $a_m = 2m\pi i$ or $-2m\pi i$, for $m = 0,1,2,\ldots,n$.

$$b_n = \text{Res}(z=a_n) = \lim_{z\to a_n} \frac{\frac{1}{2}(e^z+1)}{\frac{d}{dz}(e^z-1)} = \lim_{z\to a_n} \frac{\frac{1}{2}(e^z+1)}{e^z}$$

$$= \frac{\frac{1}{2}(1+1)}{1} = 1 \text{ if } a_n = 2n\pi i \text{ or } -2n\pi i$$

Let the centour c_n be square $ABCD$ with centre at the origin and length of each side $= 2\times 2\left(n+\frac{1}{2}\right)\pi = (4n+2)\pi$

and its vertices are at points $2\left(n+\frac{1}{2}\right)\pi(\pm 1 \pm i) = (2n+1)\pi(\pm 1 \pm i)$

The contour c_n encloses all poles $a_m = 0,1,2,\ldots,n$. See figure of previous example

$$|f(z)| = \frac{1}{2}\left|\left(\frac{e^z+1}{e^z-1}\right)\right| = \frac{1}{2}\left|\frac{e^x\cdot e^{iy}+1}{e^x\cdot e^{iy}-1}\right|$$

$$\leq \frac{1}{2}\left(\frac{e^x+1}{e^x-1}\right) \text{ or } \frac{1}{2}\left(\frac{e^x+1}{1-e^x}\right) \text{ according as } x>0 \text{ or } x<0$$

$$\to \frac{1}{2} \text{ as } x\to\infty \text{ or } x\to -\infty.$$

$\therefore\ f(z)$ is bounded in c_n.

Applying Mittag Leffler theorem,

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{\dot{a}_n}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} 1\left(\frac{1}{z - 2n\pi i} + \frac{1}{2n\pi i}\right) + \sum_{n=1}^{\infty} 1\left(\frac{1}{z + 2n\pi i} - \frac{1}{2n\pi i}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4n^2\pi^2}$$

or, $\frac{1}{2} + \frac{1}{e^z - 1} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4\pi^2 n^2}$
or, $\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4\pi^2 n^2}$

Example 9: Show that $\frac{1}{2\pi i}\int_C \frac{dz}{z-a}$ is an integer.
where, c is circle $|z - a| = r$

Or,

If C is a circle $|z - a| = r$, then

$$\frac{1}{2\pi i}\int_C \frac{dz}{z - a} = 1$$

Solution: On circle $|z - a| = r$, we can take $z - a = re^{i0}$ so that

$$dz = re^{i\theta} i d\theta$$

$$\frac{1}{2\pi i}\int_C \frac{dz}{z - a} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{re^{i\theta} i d0}{re^{i\theta}} = \frac{1}{2\pi i}\int_0^{2\pi} i d0 = 1$$

Theorem 2 (Number of poles and zeros of a meromorphic function): Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C, and let $f(z) \neq 0$ on C. Prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)dz}{f(z)} = N - \beta$$

where N and P are respectively the number of zeros and the number of poles of $f(z)$ inside C. A pole or zero of order n is counted n times.

Proof: Suppose that $f(z)$ is analytic within and on a simple closed curve C except at a pole $z = a$ of order p inside C. Also suppose that $f(z)$ has a zero of order n at $z = b$ inside C. Then we wish to prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = n - p$$

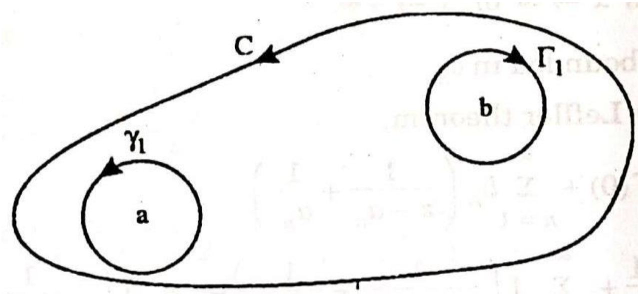

Figure 4

Let γ_1 and Γ_1 be non-overlapping circles inside C with their centres at $z = a$ and $z = b$ respectively.

Then, by Corollary to Cauchy's theorem,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f'(z)dz}{f(z)} \qquad \dots(1)$$

$f(z)$ has a pole of order p at $z = a$

$$\Rightarrow f(z) = \frac{g(z)}{(z-a)^p} \qquad \dots(2)$$

where $g(z)$ is analytic and non zero within and on γ_1.
Taking log of (2)

$$\log f(z) = \log g(z) - p\log(z-a)$$

Differentiating, this w.r.t. z,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{p}{z-a}$$

or

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}dz = \int_{\gamma_1} \frac{g'(z)}{g(z)}dz - p\int_{\gamma_1} \frac{dz}{z-a}$$

$$= \int_{\gamma_1} \frac{g'(z)}{g(z)}dz - 2\pi i p \qquad \dots(3)$$

Since $g(z)$ is analytic and so $g'(z)$ is analytic and hence $g'(z)/g(z)$ is analytic within and on γ_1. Hence, by Cauchy's theorem

$$\int_{\gamma_1} \frac{g'(z)}{g(z)}dz = 0$$

Then

$$\int_{\gamma_1} \frac{f'(z)}{f(z)} dz = 0 - 2\pi i p, \text{ by } (3) \quad \ldots (4)$$

$f(z)$ has a zero of order n at $z = b$

$$\Rightarrow f(z) = (z-b)^n \phi(z) \quad \ldots (5)$$

where $\phi(z)$ is analytic and non-zero within and on Γ_1. Consequently $\phi'(z)$ and so $\phi'(z)/\phi(z)$ is analytic within and on Γ_1.

Hence, by Cauchy's theorem,

$$\int_{\Gamma_1} \frac{\phi'(z)}{\phi(z)} dz = 0 \quad \ldots (6)$$

Taking log of (5),

$$\log f(z) = n\log(z-b) + \log \phi(z)$$

Differentiating this

$$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

Integrating along Γ_1 and noting (6),

$$\int_{\Gamma_1} \frac{f'(z)}{f(z)} dz = n \int_{\Gamma_1} \frac{dz}{z-b} = 2\pi i n$$ (See previous example)

or

$$\int_{\Gamma_1} \frac{f'(z)}{f(z)} dz = 2\pi i n \quad \ldots (7)$$

Writing (1) with the help of (4) and (7),

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -p + n \quad \ldots (8)$$

Now we suppose that $f(z)$ has poles of order p_m at $z = a_m$ for $m = 1,2, \ldots, r$ and $f(z)$ has zero of order n_m at $z = b_m$ for $m = 1,2, \ldots, s$ within C. Enclose each pole and zero by non-overlapping circles $\gamma_1, \gamma_2, \ldots, \gamma_r$ and $\Gamma_1, \ldots, \Gamma_s$. This type of construction is always possible. Since poles and zeros are isolated. Now (8) becomes

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = -\sum_{m=1}^{r} p_m + \sum_{m=1}^{s} n_m$$

Taking $\sum_{m=1}^{r} p_m = P, \sum_{m=1}^{s} n_m = N$, we obtain

$$\frac{1}{2\pi i}\int_C \frac{f'(z)dz}{f(z)} = N - P$$

Example 10: If $f(z) = z^5 - 3iz^2 + 2z + i - 1$, then evaluate $\int_C \frac{f'(z)}{f(z)}dz$, where C encloses zero of $f(z)$?

Solution: Given $f(z)$ has 5 zeros. Since, $N = 5$, it has no poles.

$\therefore\ P = 0$. By Theorem 2,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = N - P = 5 - 0 = 5$$

$$\int_C \frac{f'(z)}{f(z)}dz = 10\pi i$$

Theorem 3: Principle of argument. If $f(z)$ is analytic inside and on C, then

$$N = \frac{1}{2\pi} \cdot \Delta_C \arg f(z)$$

Proof: We know that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P \qquad \dots (1)$$

where $P =$ number of poles inside $C, N =$ number of zeros inside C.
In the present case $f(z)$ has no poles inside C and hence $P = 0$. Then (1) takes the form

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - 0 = N$$

or

$$2\pi i N = [\log f(z)]_C = \Delta_C \log f(z) \qquad \dots (2)$$

where Δ_C stands for the variation of $\log f(z)$ as z moves once round C. But

$$\log f(z) = \log|f(z)| + i\arg f(z)$$

For

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

or

$$\Delta_C \log f(z) = \Delta_C \log|f(z)| + i\Delta_C \arg f(z)$$

But $\Delta_C \log|f(z)| = 0$ as $\log|f(z)|$ is single valued.
Hence $\Delta_C \log f(z) = i \cdot \Delta_C \arg f(z)$. Using this in (2), $2\pi i N = i \cdot \Delta_C \arg f(z)$
or

$$N = \frac{1}{2\pi}\Delta_C \arg f(z)$$

14.4 *ROUCHE'S THEOREM:-*

Rouché's theorem is a powerful result in complex analysis that helps determine the number of zeros of analytic functions within a closed contour without having to solve the function explicitly. The theorem compares two analytic functions on a simple closed curve and states that if one function dominates the other on the boundary, then both functions have the same number of zeros inside that region. This comparison principle makes Rouché's Theorem particularly useful for locating zeros of complex polynomials, analyzing stability in differential equations, and simplifying root-finding problems. By providing a geometric approach to counting zeros based on boundary behavior, the theorem becomes an essential tool in both theoretical and applied aspects of complex function theory.

Theorem 4 (Rouche's theorem): If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ both have the same number of zeros inside C.

Proof: Suppose $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and

$$|g(z)| < |f(z)| \text{ on } C.$$

(*i*) Firstly we shall prove that neither $f(z)$ nor $f(z) + g(z)$ has zeros on C.

If $f(z)$ has a zero at $z = a$ on C, then $f(a) = 0$.
Also $|g(a)| < |f(a)| = 0$.
This

$$\Rightarrow |g(a)| < 0 \Rightarrow |g(a)| = 0$$

$$f(a) = 0. |g(a)| = 0 \Rightarrow |f(a)| = 0| = g(a)| \Rightarrow |f(a) = |g(a)|$$
$$\Rightarrow |g(z)| = |f(z)| \text{ at } z = a \text{ on } C.$$

Contrary to the assumption $|g(z)| < |f(z)|$ on C.Again if $f(z) + g(z)$ has a zero at $z = a$ on C, then $f(a) + g(a) = 0$ so that $f(a) = -g(a)$ or $|g(a)| = |f(a)|$.
Again we get a contradiction. Hence the result (i) is established.
(ii) Let N_1 and N_2 be number of zeros of f and $f + g$ respectively inside C.

If we show that $N_1 = N_2$, the result will be proved. Since the functions f and $f + g$ both are analytic within and on C and have no poles inside C. Therefore, by the usual formula
gives

$$\frac{1}{2\pi i}\int_C f' dz = N - P$$
$$\frac{1}{2\pi i}\int_C f' dz = N_1 \text{ and } \frac{1}{2\pi i}\int_C \frac{f' + g'}{f + g} dz = N_2$$

Subtracting we get

$$\frac{1}{2\pi i}\int_C \left[\frac{f' + g'}{f + g} - \frac{f'}{f}\right] dz = N_2 - N_1 \qquad \dots (1)$$

Take $g/f = \phi$ so that $g = \phi f$

or

$$|g| < |f| \Rightarrow |g/f| < 1 = |\phi| < 1.$$
$$\frac{f' + g'}{f + g} = \frac{f' + f'\phi + \phi' f}{f + \phi f} = \frac{f'(1 + \phi) + \phi' f}{f(1 + \phi)}$$

or

$$\frac{f' + g'}{f + g} - \frac{f'}{f} = \frac{\phi'}{1 + \phi}$$

Using this in (1),

or

$$N_2 - N_1 = \frac{1}{2\pi i}\int_C \frac{\phi'}{1+\phi}dz$$

$$N_2 - N_1 = \frac{1}{2\pi i}\int_C \phi'(1+\phi)^{-1}dz \qquad ...(2)$$

Since we have seen that $|\phi| < 1$ and so binomial expansion of $(1+\phi)^{-1}$ is possible and the binomial expansion thus obtained is uniformly convergent and hence term by term integration is permissible. Hence

$$\int_C \phi'(1+\phi)^{-1}dz = \int_C \phi'[1-\phi+\phi^2-\phi^3+\cdots]dz$$
$$= \int_C \phi' dz - \int_C \phi'\phi dz + \int_C \phi^2\phi' dz - \int_C \phi^2\phi' dz + \cdots$$

The functions f and g both are analytic within and on C and $g(z) \neq 0$ for any point on C. Hence $g/f = \phi$ is analytic and non-zero for any point on C. Therefore ϕ and its all derivatives are analytic. By Cauchy's integral theorem, each integral on R.H.S. of (3) vanishes. Consequently

$$\int_C \phi'(1+\phi)^{-1}dz = 0$$

In this event, (2) takes the form

$$N_2 - N_1 = 0 \text{ or } N_1 = N_2.$$

Example 11: Consider the function $z^6 - 5z^4 + 7$

(*i*) If $f(z) = 7, g(z) = z^6 - 5z^4$, then $f + g =$ given polynomial and $\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^6-5z^4}{7}\right| \leq \frac{|z|^6+5|z|^4}{7} = \frac{1+5}{7} < 1$ and so given polynomial has no zero in $|z| < 1$.

(*ii*) If $f(z) = -5z^4, g(z) = z^6 + 7$, then $f + g =$ given polynomial has 4 zeros in $|z| < 3$.

(*iii*) If $f(z) = z^6, g(z) = -5z^4 + 7$, then $f + g =$ given polynomial has 6 zeros in
$|z| < 3$.

Theorem 5: Fundamental Theorem of Algebra. Every Polynomial of degree n has exactly' n zeros.

Or

Prove that the polynomial equation $P(z) = a_0 + a_1z + a_2z^2 + \cdots \ldots\ldots + a_nz^n, a_n \neq 0, n \geq 1$ has exactly n roots.

Proof: Consider the Polynomial, $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ s.t. $a_n \neq 0$

Take $f(z) = a_nz^n, g(z) = a_0 + a_1z + a_2z^2 + ... + a_{n-1}z^{n-1}$

Let C be a circle $|z| = r$ where, $r > 1$.

$$|g(z)| \leq |a_0| + |a_1|r + |a_2|r^2 + \cdots + |a_{n-1}|r^{n-1}$$
$$\leq |a_0|r^{n-1} + |a_1|r^{n-1}|+|a_2|r^{n-1} + \cdots|a_{n-1}|r^{n-1}$$
$$= (|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|)r^{n-1}$$

But $|f(z)| = |a_nz^n| = |a_n|r^n$

$$f(z) = a_nz^n, g(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}$$

$$\therefore \left|\frac{g(z)}{f(z)}\right| \leq \frac{||a_0|+|a_1| + |a_2| + \cdots + |a_{n-1}|]r^{n-1}}{|a_n|r^n}$$
$$= \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|r}$$

Now if $|g(z)| < |f(z)|$ so that $|g(z)/f(z)| < 1$, then

$$\frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{r|a_n|} < 1$$

This $\Rightarrow r > \frac{|a_0|+|a_1|+|a_2|+\cdots|a_{n-1}|}{|a_n|}$.

Since r is arbitrary and hence by choosing r large enough, the last condition can be satisfied so that $|g(z)| < |f(z)|$. Now applying Rouche's theorem, we find that the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z)$. But $f(z) = a_nz^n$ has exactly n zeros all located at $z = 0$. Consequently $f(z) + g(z)$ has exactly n zeros. Consequently the given polynomial has exactly n zeros.

Theorem 6 (Inverse Function Theorem): Suppose a function $w = f(z)$ be analytio at a point $z = z_0$ where $f'(z_{0)} \neq 0$. Also let $w_0 = f(z_0)$. Then there exists a neighbourhood G of the point w_0 in w-plane in which the function $w = f(z)$ has a unique inverse $F(w) = z$, where $F = f^{-1}$ in the sense that the function F is analytic and single valued in the neighbourhood G and $F(w_0) = z_0$ and $F'(w) = \frac{1}{f'(z)}$.

Proof: Given (1) $\dot{w_0} = f(z_0), f'(z_0) \neq 0$
Write $\phi(z) = f(z) - w_0$ ... (1)

Then, $\phi(z_0) = f(z_0) - w_0 = 0$, by (1)
or,

$$\phi(z_0) = 0, \phi'(z) = f'(z)$$
$$f'(z_0) \neq 0 \Rightarrow f(z) \text{ is not a constant function}$$

$\Rightarrow \phi(z) \neq 0$ and $f'(z) \neq 0$.

It is given that $f(z)$ is analytic at $z = z_0$. Hence the function $\phi(z)$ is analytic in same neighbourhood of z_0. Since zeros are isolated and therefore $\phi(z)$ and $f'(z)$ do not have any zero in some deleted neighbourhood of z_0. Therefore given $\varepsilon > 0, \exists nbd 0 < |z - z_0'| < \varepsilon$ such that $\phi(z)$ is analytic and $\phi(z) \neq 0, f'(z) \neq 0$. Suppose $D = \{z: |z - z_0| < \varepsilon\}$ and $C = \{z: |z - z_0| = \varepsilon\}$. Then D represents open disc and C represents its boundary. Since $\phi(z) \neq 0\ \forall z$ such that $|z - z_0| \leq \varepsilon$, we conclude that $|\phi(z)|$ alttains its minimum value m on the circle C.

Choose $\delta > 0$ such that $0 < \delta < m$. Now we want to prove that $f(z)$ assumes exactly once every value w_1 in the open disc $D_1 = \{w: |w - w_0| < \delta\}$. For this we apply Rouche's theorem to the functions $w_0 - w_1$ and $\phi(z)$. Evidently $|w_0 - w_1| < \delta < m = \min_{z \varepsilon C} |\phi(z)| \leq |\phi(z)|$
$\therefore \because |w_0 - w_1| < |\phi(z)|$ on C

$$\frac{|w_0 - w_1|}{|\phi(z)|} < 1 \text{ on } C$$

By Rouche's theorem the function $\phi(z)$ and the function $\phi(z) + w_0 - w_1 = f(z) - w_0 + w_0 - w_1 = f(z) - w_1$ have the same number of zeros in D. But the function $\phi(z)$ has only one zero z_0 in D as

$$\phi(z_0) = f(z_0) - w_0 = w_0 - w_0 = 0$$

Consequently the function $f(z) - w_1$ has only one zero z_1 in D. It means that $f(z)$ assumes value w_1 exactly once in D.

Consequently $w = f(z)$ has a unique inverse function F and so we assume $z = F(w)$ in D. Now we want to show that F is analytic.

$$y\ \frac{F(w) - F(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \left\{\frac{f(z) - f(z_1)}{z - z_1}\right\}^{-1} = \{f' z_1\}^{-1}$$

as $z \to z_1$ and so $w \to w_1$.

$\therefore\ F'(w_1) = \dfrac{1}{f'(z_1)}$

This $\Rightarrow F'(w)$ exists in the $nbdD_1$ of w_0 so that F is analytic. This completes the proof of the theorem.

Example 12: Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution I: Consider the circle C_1 defined by $|z| = 1$. Suppose $f(z) = 12$ and $g(z) = z^7 - 5z^3$. Then f and g both are analytic within and on C_1.

$$\left|\frac{g}{f}\right| = \left|\frac{z^7-5z^3}{12}\right| \le \frac{|z|^7+5|z|^3}{12} = \frac{(1)^7+5(1)^3}{12} = \frac{6}{12} < 1.$$

$\left|\frac{g}{f}\right| < 1$ or $|g| < |f|$.

Applying Rouche's theorem, we find that $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside C_1 as $f(z) = 12$. But $f(z)$ has no zeros inside C_1. It means that $z^7 - 5z^3 + 12$ has no zeros inside C_1.

II: Consider the circle C_2 defined by $|z| = 2$.

Let $f(z) = z^7, g(z) = -5z^3 - 12$. Then f and g both are analytic within and on C_2.

$$\left|\frac{g}{f}\right| = \frac{|-5z^3+12|}{||z^7|} \le \frac{5|z|^3+12}{|z|^7} = \frac{5(2)^3+12}{2^7} = \frac{52}{128} < 1$$

or $|g/f| < 1$ or $|g| < |f|$

Hence, by Rouche's theorem, $f + g = z^7 - 5z^3 + 12$ has the same number of zeros inside C_2 as $f(z) = z^7$.

But $f(z) = z^7$ has seven zeros inside C_2, all located at the origin. It follows that $z^7 - 5z^3 + 12$ has seven zeros inside C_2.

Thus we have shown that the given equation has no root inside $|z| = 1$, but has seven roots inside $|z| = 2$. From this we can conclude the required result.

Example 13: Using Rouche's theorem determine the number of zeros of the polynomial

$$P(z) = z^{10} - 6z^7 + 3z^3 + 1 \text{ in } |z| < 1$$

Solution: Let $P(z) = z^{10} - 6z^7 + 3z^3 + 1$,

$$f(z) = -6z^7, g(z) = z^{10} + 3z^3 + 1$$

Then $P(z) = f(z) + g(z)$

Consider the circle C defined by $|z| = 1$. Then $f(z)$ and $g(z)$ both are analytic within and upon C.

$$\left|\frac{g}{f}\right| = \left|\frac{z^{10} + 3z^3 + 1}{-6z^7}\right| \leq \frac{|z|^{10} + 3|z|^3 + 1}{6|z|^7} = \frac{1^{10} + 3(1)^3 + 1}{6(1)^7}$$

$$= \frac{5}{6} < 1$$

Or $\left|\frac{g}{f}\right| < 1$ or $|g| < |f|$

Applying Rouche's theorem, we find that, $f + g = P(z)$ has the same number of zeros inside C as $f(z) = -6z^7$. But $f(z)$ has seven zeros inside C. Hence $P(z)$ has seven zeros inside C.

Example 14: Use Rouche's theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < 3/2$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

Solution I: Let $|z| = 2$ represent the circle C_1. We have $z^5 + 15z + 1 = 0$. Take $f(z) = z^5$ and $g(z) = 15z + 1$.

Then, $\left|\frac{g}{f}\right| = \left|\frac{15z+1}{z^5}\right| = \frac{15|z|+1}{|z|^5} = \frac{15.2+1}{2^5} = \frac{31}{32} < 1.$

$\therefore$ $|g| < |f|$. Applying. Rouche's theorem, we find that $f(z) + g(z) = z^5 + 15z + 1$ has the same number of zeros as $f(z)$ inside C_1. But $f(z)$ has five zeros inside C_1, all located at $z = 0$. It follows that $z^5 + 15z + 1 = 0$ has five roots inside $|z| = 2$.

II: Consider the circle C_2 defined by $|z| = 3/2$.

Take $f(z) = 15z, g(z) = z^5 + 1$.

Then, $\left|\frac{g}{f}\right| = \left|\frac{z^5 + 1}{15z}\right| \leq \frac{|z|^5 + 1}{15|z|} = \frac{(3/2)^5 + 1}{15(3/2)} = \frac{275}{720} < 1$

Or $|g| < |f|$

Use Applying Rouche's theorem, we find that $f + g = z^5 + 15z + 1$ has the same number of zeroes inside C_2 as $f(z) = 15z$. But $f(z)$ has one zero, located at $z = 0$. It follows $z^5 + 15z + 1 = 0$ has one root inside C_2. As a result of which four zeroes of $z^5 + 15z + 1$ must lie in the ring $3/2 < |z| < 2$.

Example 15: Show that the equation $z^4 + 2z^3 + 3z^2 + 4z + 5 = 0$ has no real or purely imaginary roots and that it has one complex root in each quadrant.

Solution: Let $f(z) = z^4 + 2z^3 + 3z^2 + 4z + 5 = 0$ and $f(z) = u + iv$.

I: ***To prove that the given equation has no real root.***

Because all the coefficients of the equation are real and positive, it cannot be satisfied by any positive value of the variable z, indicating that the equation has no positive real roots.

Putting $z = -x$

$f(-x) = x^4 - 2x^3 + 3x^2 - 4x + 5$

$= x^2(x^2 - 2x + 1) + 2x^2 - 4x + 5$

$= x^2(x^2 - 1)^2 + 2(x-1)^2 + 3 > 0 \, \forall x \quad \ldots (1)$

Hence, the equation has no negative real root.

II: ***The equation has purely imaginary roots.***

Taking $z = iy, y^4 - 2iy^3 - 3y^2 + 4iy + 5 = 0$

Or $(y^4 - 3y^2 + 5) - 2i(y^3 - 2y) = 0$

This $\Rightarrow y^4 - 3y^2 + 5 = 0, y^3 - 2y = 0$

There is no single value of yyy that satisfies both equations simultaneously, making them inconsistent. Therefore, the given equation does not possess any purely imaginary roots.

III: ***To show that the equation has one root in the first quadrant.***

For this let $z = \text{Re}^{i\theta}, 0 \le \theta \le 2\pi, R \to \infty$ OABO. Let c denote the complete boundary of this quadrant (see figure 5).

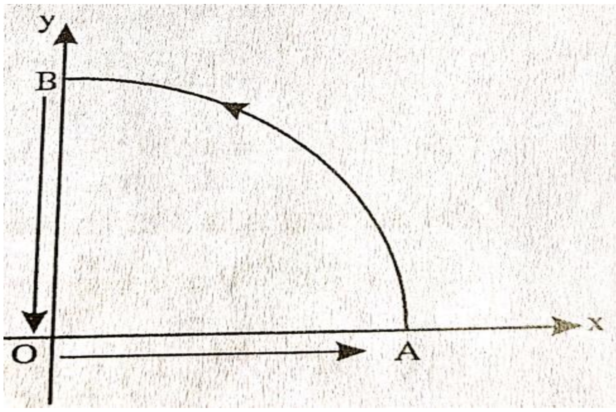

Figure 5

(*a*) **Along OA:** On this line $z = x$ and x varies from 0 to ∞.

$$f(z) = f(x) = u + iv = x^4 + 2x^3 + 3x^2 + 4x + 5$$

$$\arg f = \tan^{-1}\frac{u}{v} = \tan^{-1}\left(\frac{0}{x^4 + 2x^3 + 3x^2 + 4x + 5}\right) = \tan^{-1} 0 = 0$$

$$\Delta_{OA} \arg f = 0$$

(*b*) **Along arc AB:** $z = \mathrm{Re}^{i\theta}$ where $R \to \infty, 0 \le \theta \le \frac{\pi}{2}$.

$$f(z) = u + iv = R^4 e^{4i\theta}\left[1 + \frac{2}{Re^{i\theta}} + \frac{3}{R^2 e^{2i\theta}} + \frac{4}{R^3 e^{3i\theta}} + \frac{5}{R^4 e^{4i\theta}}\right]$$

$$\to R^4 e^{4i\theta} \text{ as } R \to \infty$$

$$\arg f = \tan^{-1}\left(\frac{R^4 \sin 4\theta}{R^4 \cos 4\theta}\right) = \tan^{-1}(\tan 4\theta) = 4\theta$$

$$\Delta_{AB} \arg f = 4[\theta]_0^{\pi/2} = 4\left(\frac{\pi}{2} - 0\right) = 2\pi$$

(*c*) **Along BO:** On this line $z = iy$ and y varies from ∞ to 0.

$$f(z) = f(iy) = u + iv = y^4 - 2iy^3 - 3y^2 + 4iy + 5$$

$$= (y^4 - 3y^2 + 5) + i(-2y^3 + 4y)$$

$$u = y^4 - 3y^2 + 5, v = -2y^3 + 4y = 2y(2 - y^2)$$

$$\arg f \equiv \tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}\frac{2y(2 - y^2)}{(y^4 - 3y^2 + 5)}$$

$\Delta_{BO}\arg f = \left[\tan^{-1}\frac{v}{u}\right]_{y=\infty}^{0}$ which is zero for both limits.

It means that as y moves from ∞ to 0 along BO, the point $w = u + iv$ starting from any point on u-axis comes back to some point on u-axis. The manner in which w moves is shown below :

y	∞	$\sqrt{2}$	0
(u, v)	$(\infty, -\infty)$	$(3,0)$	$(5,0)$
$\tan^{-1}(v/u)$	0	0	0

Also

$$\sqrt{2} < y < \infty \Rightarrow u > 0, v < 0$$
$$0 < y < \sqrt{2} \Rightarrow u > 0, v > 0.$$

From the chart it is clear that as y moves from ∞ to 0 along y-axis, the point (u, v) starting from third quadrant reaches in the first quadrant. But the curve is parallel to u-axis through out this journey and so Δ_{BO} arg $f = 0$.

Finally

$$\Delta_c \arg f = \Delta_{OA}\arg f + \Delta_{AB}\arg f + \Delta_{BO}\arg f$$
$$= 0 + 2\pi + 0 = 2\pi$$
$$N = \frac{1}{2\pi}\cdot \Delta_c \text{arc} f = \frac{1}{2\pi}\cdot 2\pi = 1$$

This shows that the equation has one root in the first quadrant. Since complex roots occur in pairs.

Hence the second root (conjugate to the first) will lie in the fourth quadrant.

The equation is of degree four and so it will have four roots. Out of the remaining two roots one lies in the second quadrant and the other conjugate to it lies the third quadrant.

Example 16: In which quadrant do the roots of the equation

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0 \text{ lie ?}$$

Solution: Let $f(z) = z^4 + z^3 + 4z^2 + 2z + 3 = 0, f(z) = u + iv$

I: To prove that the given equation has no real root. Evidently $f(x) = x^4 + x^3 + 4x^2 + 2x + 3$. Since all the coefficients of this equation are all real and positive and so it is not satisfied by any positive value of x, showing thereby it has no positive real root.

Putting, $z = -x$,

$$\begin{aligned} f(-x) &= x^4 - x^3 + 4x^2 - 2x + 3 \\ &= x^2\left(x^2 - x + \frac{1}{4}\right) + \frac{15}{4}x^2 - 2x + 3 \\ &= x^2\left(x - \frac{1}{2}\right) + 2(1 - x) + 1 + \frac{15}{4}x^2 > 0 \text{ if } \frac{1}{2} < x < 1 \end{aligned}$$

Again, $f(-x) = x^3(x-1) + 4x\left(x - \frac{1}{2}\right) + 3 > 0$ if $x > 1$

Thus, $f(-x) > 0$ if $\frac{1}{2} < x < 1$ or if $x > 1$.

This proves that the equation has no negative real root. Finally, the equation has no real root.

II: The equation has no purely imaginary root.

Putting $z = iy$, or

$$y^4 - iy^3 - 4y^2 + 2iy + 3 = 0$$
$$(y^4 - 4y^2 + 3) + iy(2 - y^2) = 0$$

This implies, $y^4 - 4y^2 + 3 = 0, y(2 - y^2) = 0$

$\Rightarrow (y^2 - 1)(y^2 - 3) = 0, y(2 - y^2) = 0.$

These two equations are not satisfied by any common value of y. Hence the result II follows.

III: To determine the number of complex roots in the first quadrant. For this let $z = Re^{i\theta}, 0 \le \theta \le \frac{\pi}{2}, R \to \infty$ define the first quadrant OABO. Let c denote the complete boundary of this quadrant. See Fig. 82 on Page 469.

(*a*) Along OA. $z = x$ and x varies from 0 to ∞.

$$f(z) = u + iv = f(x) = x^4 + x^3 + 4x^2 + 2x + 3$$
$$\arg f = \tan^{-1}\frac{v}{u} = \tan^{-1}\left(\frac{0}{x^4 + x^3 + 4x^2 + 2x + 3}\right) = 0 \forall x \ge 0.$$
$\therefore\ \Delta_{OA} \arg f = 0$

(*b*) Along arc **AB**, $z = R^{i\theta}, R \to \infty, 0 \le \theta \le \pi/2$

$$f(z) = R^4 e^{4i\theta}\left[1 + \frac{1}{Re^{i\theta}} + \frac{4}{R^2 e^{2i\theta}} + \frac{2}{R^3 e^{3i\theta}} + \frac{3}{R^4 e^{4i\theta}}\right]$$
$$\to R^4 \iota^{4\theta} \text{ as } R \to \infty$$

$$\Delta_{AB} \arg f = [4\theta]^{\pi/2} = 4\left(\frac{\pi}{2} - 0\right) = 2\pi$$

(*c*) Along BO. $z = iy$ and y varies from ∞ to 0 .

This, $u + iv = f(z) = y^4 - iy^3 - 4y^2 + 2iy + 3$

And it implies, $u = y^4 - 4y^2 + 3 = (y^2 - 3)(y^2 - 1)$

$$v = -y^3 + 2y = y(2 - y^2)$$
$$\Delta_{BO} \arg f = \left[\tan^{-1}\frac{v}{u}\right]_{y=\infty}^{0} = \left[\tan^{-1}\left(\frac{-y^3 + 2y}{y^4 - 4y^2 + 3}\right)\right]_{\infty}^{0}$$

which is zero for both the limits.

It means that as y changes from ∞ to 0 , the point $w = u + iv$ starting from any point on u-axis comes back to some point on u-axis. The manner in which w moves is shown below:

y	∞	$\sqrt{3}$	$\sqrt{2}$	1	0
(u, v)	$(\infty, -\infty)$	$(0, -\sqrt{3})$	$(-1,0)$	$(0,1)$	$(3,0)$
$\tan^{-1}\left(\frac{v}{u}\right)$	0	$-\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0

$$\arg f = \tan^{-1}\frac{v}{u}\tan^{-1}\frac{y(2 - y^2)}{(y^2 - 3)(y^2 - 1)}$$

From the chart and diagram, it is clear that as y moves from ∞ to 0 along positive y-axis, the point (u, v) takes one complete round around the origin in clockwise direction. Hence

$$\Delta \arg f(z) = -2\pi$$

Thus the total change in $\arg f(z)$ is given by

$$\Delta \arg f(z) = 0 + 2\pi - 2\pi = 0$$

Now the principle of argument declares that the number of zeros in the first quadrant is

$$\frac{1}{2\pi}\Delta\arg f(z) = \frac{1}{2\pi}\cdot(0) = 0$$

It means that the equation has no complex root in the first quadrant. IV. Since the coefficients of the given equation are all real the conjugate complex roots occur in pairs. It follows that there is no complex root in the fourth quadrant.

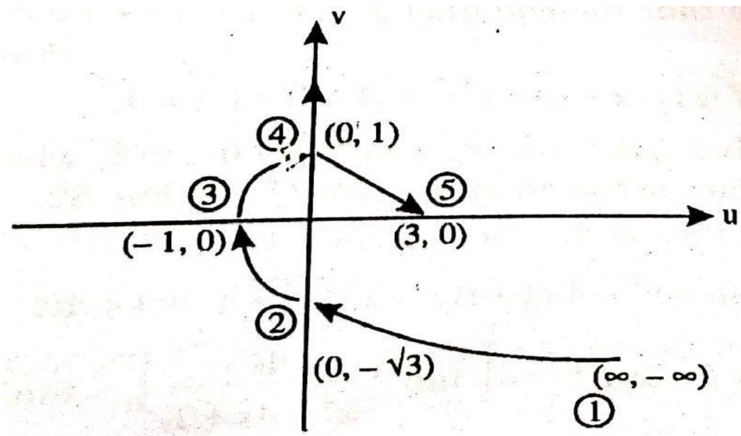

Figure 6

14.5 *ANALYTIC CONTINUATION:-*

Analytic continuation is a process of extending the definition of a domain of an analytic function in which it is originally defined. This process is not possible in case of functions of a real variable.

Definition: Suppose a function $f_1(z)$ is analytic in the domain D_1. If there exists a function $f_2(z)$ analytic in a domain D_2 such that

(*i*) D_2 has a part D_{12} common with D_1.

(*ii*) $f_1(z) = f_2(z)$ for every z in D_{12}.

then the function $f_2(z)$ is known as analytic continuation of $f_1(z)$ from D_1 into D_2 via D_{12}. Of cource we may equivalently say that f_1 is analytic continuation of f_2 from D_1 to D_2 via. D_{12}.

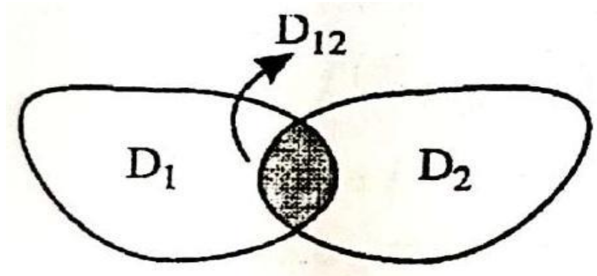

Figure 7

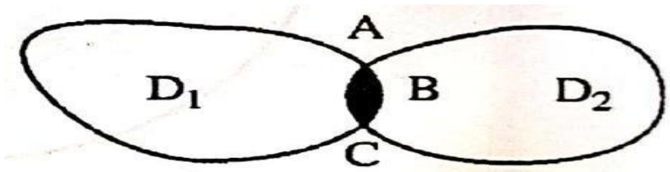

Figure 8

For analytic continuation, it is sufficient that D_1 and D_2 have only a small arc in common for example the arcABC is common in D_1 and D_2.

An alternate definition: If $f(z)$ is analytic in a domain S_1 and if $f(z)$ is also analytic in a domain S_2 containing S_1 and if $\phi(z) = f(z) \forall z \in S_2$, then $\phi(z)$ is said to give the analytic continuation of $f(z)$ in the domain S_2.

Example 17: Let $f(z) = \sum_{n=0}^{\infty} z^n, \phi(z) = \frac{1}{1-z}$.

Then $f(z)$ is analytic at all points within the circle $|z| = 1$ and $\phi(z)$ is analytic at all points except $z = 1$. Also $\phi(z) = f(z)$ within $|z| = 1$. Hence $\phi(z)$ gives the continuation of $f(z)$ over the rest of the plane.

Example 18: Let $f_1(z) = \sum_{n=0}^{\infty} z^n, f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$
The first power series $f(z)$ is convergent inside the circle R_1 defined by $|z| = 1$ and has the sum $= \frac{1}{1-z}$. The second power series is in G.P. geometrical (progression) with first term $\frac{1}{2}$ and common ratio $= \frac{1+z}{2}$ and hence it is convergent for $\left|\frac{1+z}{2}\right| < 1$ or $|z+1| < 2$.

The sum function of the second power series is

$$\frac{1}{2} \cdot \frac{1}{1-(1+z)/2} = \frac{1}{1-z}$$

Thus $f_1(z)$ is analytic inside the circle R_1 s.t. $|z| = 1$ and $f_2(z)$ is analytic inside the circle R_2 s.t. $|z + 1| = 2$. Also $f_1(z) = f_2(z)$ in a region common to the interiors of R_1 and R_2.

Thus, it is obvious that $f_2(z)$ extends the domain of the analytic function $f_1(z)$ to a larger domain R_2. Here f_2 is the analytic continuation of f_1 from R_1 into R_2.

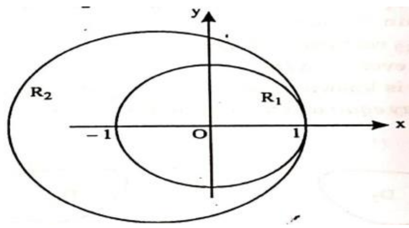

Figure 9

14.6 *COMPLETE ANALYTIC FUNCTION:-*

An analytic function f with domain D is called a function element and is denoted by (f, D).

Definition: Suppose $f(z)$ is analytic in a domain D. Let us form all possible analytic continuations of (f, D) and then all possible analytic continuations of (f_1, D_1), $(f_2, D_2), (f_3, D_3), \dots, (f_n, D_n)$ and so on. At some stage we arrive at a function $F(z)$ such that for any $v, F(v)$ denotes the value of values obtained for v by all possible continuation to v, that is to say.

$$F(z) = \begin{cases} f_1(z) \text{ if } z \in D_1 \\ f_2(z) \text{ if } z \in D_2 \\ \dots\dots\dots\dots\dots\dots \\ f_n(z) \text{ if } z \in D_n \end{cases}$$

Such a function $F(z)$ is called complete analytic function. In this process of continuation, we may arrive at a closed curve beyond which it is not possible to take analytic continuation. Such a closed curve is called the natural boundary of the compelte analytic function. A point outside the natural boundary is called the singularity of complete analytic function.

Theorem 7: If $f(z)$ is analytic in a domain R and $f(z) = 0$ at all points on arc PQ inside R, then $f(z) = 0$ throughout R.

Proof: Suppose $f(z)$ is analytic within a domain R. Let PQ be an arc inside R.s.t.

$$f(z) = 0 \forall z \text{ on } PQ \qquad \dots(1)$$

To prove that $f(z) = 0$ through out R. Take an

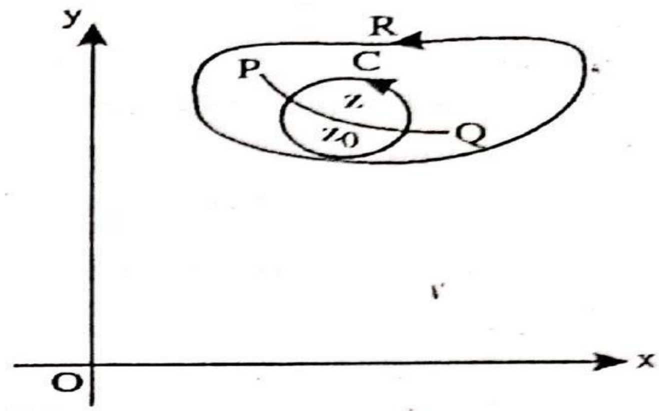

Figure 10

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad ...(2)$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$
Since z_0 lies on PQ and hence $f(z_0) = 0$; by (1).
This $\Rightarrow f(z) = 0$ at $z = z_0$

$$\Rightarrow f(z), f'(z), f''(z), \dots, f^{(n)}(z) = 0 \text{ at } z = z_0$$
$$\Rightarrow f^{(n)}(z_0) = 0 \text{ when } n = 0,1,2,3, \dots$$

Here $f^{(0)}(z) = f(z)$

$$\Rightarrow a_n = 0, \text{ for } n = 0,1,2,3, \dots$$

In this event (2) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} 0(z + z_0)^n = 0$$
$$\therefore f(z) = 0 \text{ for any point inside } C$$

By considering another arc inside R, we can repeat the same process. In this way, we can prove that $f(z) = 0$ throughout R.

Deductions (i) If $f(z)$ is analytic in a domain R and if $f(z)$ vanishes at any point of R_0, where R_0 is a part of R, then $f(z) = 0$ throughout R.

Solution: Take an arcPQ inside R_0. Then $f(z) = 0 \forall z$ on PQ. Now prove this as in Theorem 7.
(ii) If $f_1(z)$ and $f_2(z)$ are analytic in the same domain D and are such that

$f_1(z) = f_2(z)$ in a domain D_0 which is a part of D, then $f_1(z) = f_2(z)$ i.h roughout D.

Or,

Show that if two functions $f_1(z)$ and $f_2(z)$ are equal at all points of a line L in a region D in which they are holomorphic, the functions are equal at all points of D.

Solution: Suppose $f_1(z)$ and $f_2(z)$ are holomorphic (analytic) in a region D. Let D_0 be a part of D s.t.

$$f_1(z) = f_2(z) \forall z \in D_0 \qquad \dots (1)$$

To prove that $f_1(z) = f_2(z)$ throughout D.
Write

$$f(z) = f_1(z) - f_2(z)$$

Now

$$(1) \Rightarrow f(z) = 0 \ \forall z \in D_0, \qquad \dots (2)$$

Take an arc(or line)PQ in D_0. Then we have

$$f(z) = 0 \forall z \text{ on } PQ. \qquad \dots (3)$$

This follows from (2).
Now we shall prove a lemma.
Lemma: If $f(z)$ is analytic in a domain R and if $f(z) = 0$ along an arcPQ inside R, then $f(z) = 0$ throughout R.

The proof of the lemma starts. Prove as in Theorem 7.
Going back to the actual problem, we have $f(z) = 0 \forall z \in D$.
(This follows from the lemma).
or, $f_1(z) - f_2(z) = 0$ throughout D

$f_1(z) = f_2(z)$ throughout D

An alternative proof: If z_1, z_2, lie on the line L, then

$$\lim_{z_2 \to z_1} \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \lim_{z_2 \to z_1} \frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1}$$

Thus the first derivatives of f and ϕ are equal at all points of L. Similarly, all the other derivatives of f and ϕ can be shown to be equal at all points of L; and therefore, the functions are equal at all points of D.

Theorem 8: If a function $f(z)$ and all its derivatives vanish at point a, then $f(z)$ and all its derivatives will vanish at all points in the domain of a.

Proof: By Taylor's theorem, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ where $\frac{f^n(a)}{n!} = a_n$

By assumption, $a_0 = a_1 = a_2 = a_3 = \cdots = 0$.
Hence $f(z), f'(z), f''(z)$ all vanish at all points of the domain.

Theorem 9: Uniqueness of analytic continuation. There cannot be more than one continuation of analytic continuation $f_2(z)$ into the same domain

Proof: Let $f_1(z)$ be analytic in a domain D_1 and let $f_2'(z)$ and $g_2(z)$ be analytic continuations of the same function $f_1(z)$ from D_1 into the domain D_2 via D_{12} which is common to both D_1 and D_2 (See fig. 7 & 8).

If we show that $f_2(z) = g_2(z)$ throughout D_2, the result will follow.

By the definition of analytic continuation.

(i) $f_1(z) = f_2(z) \forall z \in D_{12}$

and $f_2(z)$ is analytic in D_2

(ii) $f_1(z) = g_2(z) \forall z \in D_{12}$

and $g_2(z)$ is analytic in D_2.

From (i) and (ii), it follows that, $f_2(z) = f_1(z) = g_2(z) \forall z \in D_{12}$

Or, $f_2(z) = g_2(z) \forall z \in D_{12}$

Or, $(f_2 - g_2)(z) = 0 \forall z \in D_{12}$

f_2 and g_2 are analytic in $D_2 \Rightarrow f_2 - g_2$ is analytic in D_2.
Thus we see that $(f_2 - g_2)(z)$ vanishes in D_{12} which is a part of D_2. Also, the function is analytic in D_2. Hence, we must have

$(f_2 - g_2)(z) = 0 \forall z \in D_2$. (See Theorem 7)
or, $f_2(z) = g_2(z) \forall z \in D_2$.

Check your progress

Problem 1: Show that one root of the equation $z^4 + z + 1 = 0$ lies in the first quadrant.

Problem 2: Using Rouche's theorem to show that three out of the four zeroes of $z^4 + 6z + 3 = 0$ lie in $1 < |z| < 2$.

14.7 *SUMMARY:-*

This unit cover the topic on meromorphic functions, Rouché's theorem, and analytic continuation explores key concepts of complex analysis that describe how analytic functions behave, extend, and relate within the complex plane. It begins with meromorphic functions, which are analytic everywhere in a domain except at isolated poles, highlighting their structure, properties, and representation as ratios of analytic functions. The chapter then introduces Rouché's theorem**,** a powerful result used to compare two analytic functions on a closed contour to determine whether they have the same number of zeros inside it, making it especially useful in locating zeros of complex functions and polynomials. Finally, it discusses analytic continuation**,** the process through which an analytic function can be extended beyond its initial radius or region of convergence by using overlapping analytic segments, emphasizing the principle of uniqueness based on the identity theorem. Overall, the chapter explains how these tools collectively deepen our understanding of the global behavior of analytic functions across the complex plane.

14.8 *GLOSSARY:-*

- Meromorphic function
- Rouche's theorem
- Analytic continuation
- Complete analytic function

14.9 *REFERENCES:-*

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14.10 *SUGGESTED READING:-*

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- J.B. Conway, (2000), Functions of One Complex Variable, Narosa Publishing House,
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14.11 *TERMINAL QUESTIONS:-*

Long answer type question

1: State and prove Mittag Leffler's expansion theorem.

2: State and prove Rouche's theorem.

3: Define analytic continuation by making figure on answer sheet.

4: Show that roots of the equation $z^6 - 9z^2 + 11 = 0$ all lie between the circles $|z| = 1$ and $|z| = 3$.

5: Prove that all the roots of $z^5 + z - 16i = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Short answer type question

1: Find the number of roots of the equation $z^5 + 5z + 1 = 0$.

2: Prove that $z^8 + 3z^3 + 7z + 5$ has exactly two zeroes in first quadrant.

3: Show that the equation $e^{-z} = z - (1+i)$ has one root in the first quadrant.

4: Find out the number of zeroes of the polynomial $F(z) = z^8 - 4z^5 + z^2 - 1$ that lie inside the circle $|z| = 1$.

5: Find the number of zeros of the polynomial $2z^4 - 2z^3 + 2z^2 + 2z + 11$ inside the circle $|z| = 1$.

Objective type question:

1. A meromorphic function on a domain *D* is analytic everywhere except at:

A. Essential singularities
B. Poles
C. Branch points
D. Removable singularities

2. Which of the following is a meromorphic function on *C*?

A. e^z
B. $\sin z$
C. $\frac{1}{z^2+1}$
D. $\log z$

3. A function is meromorphic in the finite complex plane if and only if its

singularities (in that plane) are:

A. Removable
B. Poles only
C. Poles or essential
D. Poles or branch points

4. Rouché's Theorem helps in determining:

A. The order of a pole
B. The number of zeros inside a closed contour
C. Radius of convergence of a Taylor series
D. Existence of Laurent expansion

5. According to Rouché's Theorem, if $|f(z)-g(z)|<|f(z)|$ on a simple closed contour *C*, then:

A. $g(z)$ has no zeros inside C
B. $f(z)$ and $g(z)$ have the same number of zeros inside C
C. $f(z)$ must be constant inside C
D. $f(z)$ and $g(z)$ have no poles

6. To apply Rouché's Theorem, both functions must be:

A. Meromorphic on C
B. Continuous on C
C. Analytic inside and on C
D. Analytic only on C

7. Analytic continuation allows extending an analytic function:

A. To regions where the function is multivalued
B. Beyond its radius of convergence
C. Only within its original circle of convergence
D. To infinity

8. Analytic continuation is unique if:

A. The function is periodic
B. The function is bounded
C. The continuation is performed along two different paths
D. The domain is simply connected and the initial analytic function is fixed

9. The process of analytic continuation is based on which principle?

A. Cauchy Integral Formula
B. Maximum Modulus Principle
C. Identity Theorem
D. Liouville's Theorem

10. Which of the following is NOT true for a meromorphic function?

A. It can be expressed as the ratio of two analytic functions
B. It may have essential singularities
C. It may have poles
D. It is analytic except at isolated singularities

11. Rouché's Theorem is typically applied to determine:

A. Behavior near essential singularities
B. The number of zeros for polynomials
C. Whether a function is entire
D. The sum of residues

12. Analytic continuation fails when:

A. A singularity blocks the extension
B. The function is entire
C. The domain is simply connected
D. The function is bounded

14.12*ANSWERS:-*

Answer of short answer type question:

1: One zero inside $|z|=1$ (here take $f=5z$, $g=z^4+1$)

3 zero inside $|z|=2$ (here take $f=z^4$, $g=5z+1$)

5: Take $f=11, g=$ remaining terms, then

$\left|\frac{f}{g}\right|=\frac{2+2+2+2}{11}=\frac{8}{11}<1$. Polynomial does not have any zero inside *c*.

Answer of objective questions

1:	B	**2:**	C	**3:**	B	**4:**	B
5:	B	**6:**	C	**7:**	B	**8:**	D
9:	C	**10:**	B	**11:**	B	**12:**	A

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