LINEAR ALGEBRA **MT(N)-301**

BECHELOR OF SCIENCE (FIFTH SEMESTER)

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
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Course: Linear Algebra **Course Code: MT(N)-301**
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COURSE INFORMATION

The present self-learning material **“Linear Algebra”** course code “**MT(N)-301**”has been designed for B.Sc. (Fifth Semester) learners of Uttarkhand Open University, Haldwani. This self learning material is writing for increase learner access to high-quality learning materials. This course is divided into 14 units of study. The first four units are devoted to vector space & subspace and the application of linear algebra to solve the various types of matrix problem. Unit 5 is focused on mapping between two vector space name as linear transformation. Unit 6 and 7 explained about important application of linear transformation name as homomorphism and isomorphism and dual spaces. The aim of Unit 8, 9 and 10 are to introduce the various applications of polynomial vector space, determinant, eigen vectors and minimal polynomial to solve the linear equations and also brief discussion about the Jordan canonical form to understand the application of nilpotent matrix and use of minimal polynomial. Unit 11 and Unit 12 explain the most essential too in linear algebra name as inner product space, orthogonality and orthonormality. Unit 13 will explain the Gram-Schmidt orthogonalization process. Atlast, Unit 14 explain about the unitary and normal operator. This material also used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

Course Name: Linear Algebra **Course Code: MT(N)-301**

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SYLLABUS

Vector spaces, linearly independent and dependent sets, vector subspace, bases and dimension of vector space.

Linear transformation, Linear functional, Dual spaces and second dual space, Transpose of linear transformation, Algebra of linear transformations, Isomorphism theorems.

Algebras, The algebra of polynomials, Lagrange interpolation, Vandermonde matrix, Polynomial ideals, Taylor's formula, The prime factorization of a polynomial, Algebraically closed fields, Determinant functions, Characteristic values of a linear transformation, Cayley-Hamilton theorem for linear transformations, Annihilating polynomials, Invariant subspaces, Minimal and characteristic polynomials. Diagonalisability of linear transformations, Direct sum decomposition, Invariant direct sums, The primary decomposition theorem, Triangular form, Jordan canonical form, trace and transpose.

Definition and examples of inner product space, orthogonality, orthonormality, Cauchy-Schwarz inequality, Gram-Schmidt orthogonalisation, diagonalisation of symmetric matrices, Hermitian Operator, Unitary and normal operators.

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SUGGESTED READINGS

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BLOCK I
VECTOR SPACE

UNIT 1: -Vector Space

CONTENTS:

1.1 *INTRODUCTION: -*

Linear Algebra is a vital branch of mathematics that focuses on the study of vectors, matrices, and linear transformations. It is primarily concerned with systems of linear equations and the properties of vector spaces and mappings between them. Central to linear algebra are operations such as vector addition, scalar multiplication, and matrix multiplication, which provide a structured way to model and solve real-world problems. The subject also explores key concepts like linear independence, span, basis, dimension, and rank, which help describe the structure and behavior of vector spaces.

In addition to its theoretical importance, linear algebra has extensive practical applications across various disciplines. In computer science, it underpins algorithms in machine learning, computer graphics, and cryptography. In engineering and physics, it is used to model physical systems and solve equations related to circuits, forces, and motion. Linear algebra also plays a crucial role in economics, optimization, and data analysis, where large systems of equations must be managed efficiently. Its blend of theory and application makes linear algebra a foundational tool in modern science and technology.

In this unit we will studied about the A vector space is a group of items, known as vectors, that remain within the same set even after being multiplied (scaled) by numbers, known as scalars, and joined together.

These scalars typically originate from complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$. Associativity, commutativity, the presence of a zero vector, and additive inverses are some of the rules (axioms) that must be followed by operations like vector addition and scalar multiplication in a vector space. $\mathbb{R}^2$ (the two-dimensional plane) and $\mathbb{R}^3$ (the three-dimensional space) are two instances of vector spaces.

1.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- Define Binary Operations on a set.
- Define Group and Field.
- Define Vector space and its properties.

1.3 *BINARY OPERATION: -*

Consider a non-empty set S. A binary operation (also known as a binary composition) in S is any function from $S \times S$ to S.

If $f: S \times S \rightarrow S$ be is a binary composition in S and $xy \in S$ then $f(x, y)$ is the composite of x and y under the composition$f$. It is often indicated by any of the following symbols.

$$*, T, \perp, \oplus, +, \ldots, \text{Juxtaposition}$$

When a binary composition is represented by a set $*$ and $x, y \in S$, the composite of x and y under this composition is represented by $x * y$.

Example1. Binary operation defined on numbers (real numbers $\mathbb{R}$):

- **Addition**: $a + b$ $(e.g., 3 + 5 = 8)$
- **Multiplication**: $a \times b$ $(e.g., 4 \times 6 = 24)$
- **Subtraction**: $a - b(e.g., 7 - 2 = 5)$
- **Maximum**: $max(a, b)(e.g., max(4,9) = 9)$

Example2. Binary operation defined on sets:

- **Union:** $A \cup B(e.g., \{1,2\} \cup \{2,3\} = \{1,2,3\})$
- **Intersection:** $A \cap B$ $(e.g., \{1,2\} \cap \{2,3\} = \{2\})$

1.4 *GROUP: -*

Let G be a non-empty set and $*$ be a binary operation defined on it, then $(G,*)$ is said to be group if it satisfies the following properties:

a) Closure property:

$$a * b \in G \ \forall \ a, b \in G$$

b) Associativity:

$$a * (b * c) = (a * b) * c \ \forall a, b, c \in G$$

c) Existence of identity:

$\exists$ an element $e \in G$ such that

$$a * e = e * a = a \ \forall a \in G$$

Where e is called identity of $*$ in G.

d) Existence of inverse:

For each element $a \in G \ \exists \ b \in G$ such that

$$a * b = b * a = e$$

Where b is called inverse of element a with respect to $*$ and we write

$$b = a^{-1}$$

Abelian or Commutative Group: A group $(G,*)$ is said to be abelian group if

$$a * b = b * a \ \forall \ a, b \in G$$

A group which are not abelian called non-abelian or non-commutative group.

Example3. Show that the set Z of integers (positive or negative including 0 with additive binary operation is an infinite abelian group.

Solution: Let us apply the group-axioms to all integers.

a) **Closure property**: Closure property is satisfied because the sum of any two integers is an integer.
b) **Associativity:** The associative property is satisfied, because of a, b, c are any three integers, then

$$(a + b) + c = a + (b + c)$$

c) **Existence of identity:** The axiom on identity is satisfied, because 0 is the identity element in the set Z such that
d) **Existence of inverse:** The axiom on inverse is satisfied, because the inverse of any integer a is the integer $-a$ such that $a + (-a) = (-a) + a = 0$ the identity element.

e) **Commutativity:** Since, we know that $a+b=b+a\ \forall a,b\in Z$, the commutative law is satisfied.
Also, the number of elements in Z is infinite.
Hence, the set Z is an infinite abelian group with additive binary operation.

Example4. Show that the set $(1,1,i,-i)$ is an abelian finite group of order 4 under multiplication.

Solution:

a) Closure property: Closure property is satisfied as

$$1(-1)-1,1.i=i,i.(-i)\ =\ 1,1.(-i)\ =\ -i\ etc.$$

b) **Associativity:** Associative property is satisfied as

$$(1.i)\ (-i)=1.\{i\ (-i))\ =\ 1,\{1.i).(-1)\ =\ 1,\{i(-1)\}$$
$$=-i\ etc.$$

c) **Existence of identity:** Axioms on identity is satisfied, 1 being the multiplicative identity.

d) **Existence of inverse.** Axiom an inverse in satisfied since the inverse of each element of the set exists

$$1.1=e=1,(-1)\ (-1)\ =\ e\ =\ 1,i(-i)\ =\ e\ =\ 1,(-i)\ (i)$$
$$=\ e\ =\ 1$$

e) **Commutativity:** The commutative law is also satisfied as

$$1(-1)=(-1).1,(-1)\ i\ =\ i\ (-1)\ etc.$$

Since, there are four elements in the given set, hence it is a group of order 4.

Example5. Show that the set of all positive rational numbers forms an abelian group under the composition defined by

$$a*b=\frac{(ab)}{2}$$

Solution: Let Q^+ denote the set of all positive rational numbers to show $(Q^+,*)$ is a group.

a) **Closure property:** For every $a,b\in Q^+,\frac{ab}{2}\in Q^+$
$\Rightarrow Q^+$ is a closed under the composition$*$.

b) **Associativity:** Let $a,b,c\in Q^+$, then

$$(a*b)*c=\frac{[(ab)/2].c}{2}=\frac{a[(bc)/2]}{2}=a*\left(\frac{bc}{2}\right)=a*(b*c)$$

c) **Existence of identity**: An element e will be the identity element if $e \in Q^+$ and if

$$e * a = a = a * e \; \forall a \in Q^+$$

$\Rightarrow$ $$e * a = a\frac{(ea)}{2} = a \Rightarrow \left(\frac{a}{2}\right)(e-2) = 0$$

$\Rightarrow$ $$e = 2$$

$\because$ $$a \in Q^+ \Rightarrow a \neq 0$$

$\because 2 \in Q^+$ and we have $2 * a = \frac{2a}{2} = a = a * 2 \; \forall \; a \in Q^+$

$\Rightarrow$ 2 is the identity element.

d) **Existence of inverse**: Let $a \in Q^+, b$ is the inverse of a, then we must have

$$b * a = e = 2$$

$\Rightarrow$ $$\frac{(ba)}{2} = 2 \Rightarrow b = \frac{4}{a}$$

$\Rightarrow$ $$a \in Q^+ \Rightarrow \frac{4}{a} \in Q^+$$

We have $(4/a) * a = \{(4/a).a\}/2 = 2 = a * (4/a)$

$\Rightarrow$ $4/a$ is the inverse of a

$\Rightarrow$ inverse of each element of Q^+ exist.

e) **Commutativity**: Let $a, b \in Q^+$

$\Rightarrow$ $$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$$

Hence $(Q^+, *)$ is an abelian group.

Example6. Show that the set Q of positive irrational does not form a group with respect to multiplication.

Solution: Here $\sqrt{a}.\sqrt{a} = a \forall a \in Q$, the set of rational numbers, so the set of positive irrationals does not satisfy the closure property with respect to multiplication.

Hence the set of positive irrationals does not form a group with respect to multiplication.

1.5 *FIELD*: -

Let F be a non-empty set that has two binary operations, addition and multiplication, represented by the symbols " $+$ " and " $.$ " respectively. This means that for every $a, b \in F$, we have $a + b \in F$ *and* $a.b \in F$. If the following conditions are satisfied, this algebraic structure $(F, +, .)$ is referred to as a field:

a) Addition is commutative i.e.
$$a + b = b + a \, \forall a, b \in F$$
b) Addition is associative i.e.
$$(a + b) + c = a + (b + c) \forall a, b, c \in F$$

c) $\exists$ an element 0 in F such that
$$a + 0 = a \forall a \in F$$
d) To each element $a \, in \, F$ $\exists$ an element $-a \, in \, F$ such that
$$a + (-a) = 0$$
e) Multiplication is commutative i.e.
$$a.b = b.a \, \forall a, b \in F$$
f) Multiplication is associative i.e.
$$(a.b).c = a.(b.c) \forall a, b, c \in F$$
g) $\exists$ an element 1 in F such that
$$a.1 = a \forall a \in F$$
h) To each element $a \, in \, F$ $\exists$ an element $a^{-1} \, in \, F$ such that
$$aa^{-1} = 1$$
i) Multiplication is distributive with respect to addition. i.e.
$$a.(b + c) = a.b + a.c \, \forall a, b, c \in F$$

Example7. The set Q of all rational numbers is a field the addition and multiplication of rational numbers being the two field compositions. The rational number 0 is the zero element of this field and the rational number 1 is the unity of this field.

Example8. The set R of all real numbers is a field, the addition and multiplication of real numbers being the two field compositions. Since $Q \subset R$, therefore the field of rational numbers is a subfield of the field of rational numbers.

Example9. The set C of all complex numbers is a field, the addition and multiplication of complex numbers being the two field compositions. Since $R \subset C$, therefore the field of real numbers is a subfield of the field of complex numbers.

Example10. The set of numbers of the form $a + b\sqrt{2}$, with $a \, and \, b$ as rational numbers is a field. We can easily show that all the field postulates are satisfied in this case.

1.6 *VECTOR SPACE*: -

Let $(F,+,.)$ be a field. We will refer to the components of F as scalars. Assume that V is a non-empty set, whose elements will be called vectors. If V is a vector space over the field F, then

1. An internal composition known as the addition of vectors, represented by the symbol '$+$', is defined in V. Moreover, V is an abelian group for this composition, i.e.

a) $\alpha+\beta \in V \ \forall \ \alpha,\beta \in V$.
b) $\alpha+\beta=\beta+\alpha \ \forall \ \alpha,\beta \in V$.
c) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma \ \forall \ \alpha,\beta,\gamma \in V$.
d) $\exists$ an element $\mathbf{0} \in V$ such that $\alpha+\mathbf{0}=\alpha \ \forall \alpha \in V$
Where $\mathbf{0}$ is called zero vector.
e) To every vector $\alpha \in V \ \exists$ a vector $-\alpha \in V$ such that

$$\alpha+(-\alpha)=0$$

2. Scalar multiplication is an external composition in V over F i.e., $a\alpha \in V \ \forall \ a \in F$ and $\forall \alpha \in V$. To put it another way, V is closed with respect to scalar multiplication.
3. The two compositions scalar multiplication and vector addition fulfill the following requirements:

a) $a(\alpha+\beta)=a\alpha+a\beta \ \forall a \in F \ and \ \forall \alpha,\beta \in V$
b) $(a+b)\alpha=a\alpha+b\alpha \ \forall \ a,b \in F \ and \ \forall \alpha \in V$
c) $(ab)\alpha=a(b\alpha) \ \forall \ a,b \in F \ and \ \forall \alpha \in V$
d) $1\alpha=\alpha \ \forall \alpha \in V$
Where 1 is unity element of the field F.

When V is a vector space over the field F, we shall say that $V(F)$ is a vector space.

Example11. The vector space of all polynomials over a field F.

Solution: Let $F[x]$ represent the set of all polynomials over a field F in an indeterminate x. Then, $F[x]$ is a vector space over the field F with respect to the product of a polynomial by a constant polynomial (i.e., by an element of F) as scalar multiplication and the addition of two polynomials as vector addition. Let

$$f(x)=\sum a_i x^i=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

$$g(x)=\sum b_i x^i=b_0+b_1x+b_2x^2+b_3x^3+\cdots$$

and $$h(x)=\sum c_i x^i=c_0+c_1x+c_2x^2+c_3x^3+\cdots$$

be an arbitrary member of $F[x]$.

Equality of two polynomials: we define $f(x)=g(x) \Leftrightarrow a_i=b_i \ \forall \ i=0,1,2,3,\ldots$

Addition Composition in $\boldsymbol{F[x]}$: we define

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

$$f(x) + g(x) = \sum(a_i + b_i)x^i$$

$\because$ $a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots$ all are elements of F, therefore $f(x) + g(x) \in F[x]$ and thus $F[x]$ is closed with respect to addition of polynomials.

Scalar multiplication in $\boldsymbol{F[x]}$ over $\boldsymbol{F}$: If k is any scalar i.e., $k \in F$, we define

$$kf(x) = ka_0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3 + \cdots$$

$$= \sum (ka_i)\, x^i.$$

Since $ka_0, ka_1, ka_2, \ldots$ are all elements of F, therefore $kf(x) \in F[x]$ and thus $F[x]$ is closed with respect to scalar multiplication.

Now we shall show that $F[x]$ is a vector space for these two compositions.

Commutativity of addition in $\boldsymbol{F[x]}$: We have

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

$$= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 + \cdots$$

[since addition in the field F is commutative]

$$= g(x) + f(x)$$

Associativity of addition in $\boldsymbol{F[x]}$: We have

$$[f(x) + g(x)] + h(x) = \sum(a_i + b_i)x^i + \sum c_i x^i$$

$$= \sum[(a_i + b_i) + c_i]x^i$$

$$= \sum[a_i + (b_i + c_i)]x^i$$

$$= \sum a_i x^i + \sum(b_i + c_i)x^i$$

$$= f(x) + g(x) + h(x)]$$

Existence of additive identity in $F[x]$: Let 0 denote the zero polynomial over the field F i.e.,

$$0 = 0 + 0x + 0x^2 + 0x^3 + \cdots. Then\ 0 \in F[x] and\ 0 + f(x) = f(x).$$

∴ the zero polynomial 0 is the additive identity.

Existence of additive inverse of each member of $F[x]$: Let $-f(x)$ be the polynomial over the field F defined as

$$-f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \cdots$$

Then $-f(x) \in F[x]$ and we have $-f(x) + f(x) = 0$

i.e., the zero polynomial.

∴ $-f(x)$ is the additive inverse of $f(x)$.

Thus $F[x]$ is an abelian group with respect to addition of poyoomials

Now for the operation of scalar multiplication we make the following observations.

1. If $k \in F$, then

$$k[f(x) + g(x)] = k[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots]$$

$$= k(a_0 + b_0) + k(a_1 + b_1)x + k(a_2 + b_2)x^2 + \cdots$$

$$= (ka_0 + kb_0) + (ka_1 + kb_1)x + (ka_2 + kb_2)x^2 + \cdots$$

$$= [ka_0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3 + \cdots] + [kb_0 + (kb_1)x + (kb_2)x^2 + (kb_3)x^3 + \cdots]$$

$$= k(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) + k(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots)$$

$$= kf(x) + kg(x).$$

2. If $k_1, k_2 \in F$, then

$$(k_1 + k_2)f(x) = (k_1 + k_2)a_0 + [(k_1 + k_2)a_1]x + [(k_1 + k_2)a_2]x^2 + [(k_1 + k_2)a_3]x^3 + \cdots$$

$$= (k_1a_0 + k_2\, a_0) + (k_1a_1 + k_2\, a_1)x + (k_1a_2 + k_2\, a_2)x^2 + \cdots$$

$$= [k_1a_0 + (k_1a_1)x + (k_1a_2)x^2 + (k_1a_3)x^3 + \cdots] + [k_2a_0 + (k_2a_1)x + (k_2a_2)x^2 + (k_2a_3)x^3 + \cdots]$$

$$= k_1(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) + k_2(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$$

$$= k_1f(x) + k_2f(x)$$

1. 3. If $k_1, k_2 \in F$, then

$$(k_1k_2)f(x) = (k_1k_2)a_0 + [(k_1k_2)a_1]x + [(k_1k_2)a_2]x^2 + [(k_1k_2)a_3]x^3 + \cdots$$

$$= k_1(k_2\, a_0) + k_1(k_2\, a_1)x + k_1(k_2\, a_2)x^2 + \cdots$$

$$= k_1[k_2a_0 + (k_2a_1)x + (k_2a_2)x^2 + (k_2a_3)x^3 + \cdots]$$

$$= k_1[k_2f(x)]$$

4. If 1 is the unity element of the field F, then
$$1\, f(x) = 1a_0 + (1a_1)x + (1a_2)x^2 + (1a_3)x^3 + \cdots]$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = f(x).$$

Hence $F[x]$ is a vector space over the field F.

Example12. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c\,(x, y) = (cx, 0)$$

Is V, with these operations, a vector space over the field of real numbers?

Solution: A vector space will not be a vector space if any of its postulates are not met. We will demonstrate that the identity element is absent for the vector addition operation as described in this problem. Assume the identity element for the vector addition operation is the ordered pair (x_1, y_1). In such case, we should have

$$(x, y) + (x_1, y_1) = (x, y) \forall x, y \in R$$

$$\Rightarrow \quad (x + x_1, 0) = (x, y) \forall x, y \in R$$

But if $y \neq 0$, then we cannot have $(x + x_1, 0) = (x, y)$. Thus there exists no element (x_1, y_1) of V such that

$$(x, y) + (x_1, y_1) = (x, y) \forall x, y \in V$$

Therefore, the identity element does not exist and V is not a vector space over the field R.

Example13. How many elements are there in the vector space of polynomials of degree at most n in which the coefficients are the elements of the field $I\ (p)$ over the field $I(p), p$ being a prime number?
Solution: The field $I\ (p)$ is the field $(\{0, 1, 2, \ldots, p-1\}, +_p, \times_p)$. The number of distinct elements in the field I (p) is p.
If $f(x)$ is a polynomial of degree at most n over the field $I\ (p)$,then

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

where $a_0, a_1, a_2, a_3, \ldots, a_n \in I(p)$
Now in the polynomial $f(x)$, the coefficient of each of the $n + 1$ terms $a_0, a_1 x, a_2 x^2, a_3 x^3, \ldots, a_n x^n$ can be filled in p ways because any of the p elements of the field $I\ (p)$ can be filled there.
Thus, we can have $p \times p \times p \times \ldots upto\ (n+1)\ times$ i.e., p^{n+1} dis-tinct polynomials of degree at most n over the field $I\ (p)$. Hence if P. is the vector space of polynomials of degree at most n in which the coefficients are the elements of the field $I\ (p)$ over the field $I\ (p)$, then P_n. has p^{n+1}distinct elements.

1.7 *PROPERTIES OF VECTOR SPACE*:-

Theorem1. Let $V(F)$ be a vector space and **0** be the zero vector of V.
Then

a) $a\mathbf{0} = \mathbf{0}\ \forall a \in F$
b) $\mathbf{0}\alpha = \mathbf{0}\ \forall \alpha \in V$
c) $a(-\alpha) = -(a\alpha)\ \forall a \in F, \forall \alpha \in V$
d) $(-a)\alpha = -(a\alpha)\ \forall a \in F, \forall \alpha \in V$
e) $a(\alpha - \beta) = a\alpha - a\beta\ \forall\ a \in F\ and\ \ \forall \alpha, \beta \in V$
f) $a\alpha = \mathbf{0}\ \Rightarrow a = 0\ or\ \alpha = \mathbf{0}$

Proof:

a) we have

$$a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) \qquad [\because \mathbf{0} = \mathbf{0} + \mathbf{0}]$$

$$= a\mathbf{0} + a\mathbf{0}$$

$$\therefore \quad \mathbf{0} + a\mathbf{0} = a\mathbf{0} + a\mathbf{0} \qquad [\because a\mathbf{0} \in V \text{ and } \mathbf{0} + a\mathbf{0} = a\mathbf{0}]$$

Since V is an abelian group with respect to addition. Therefore by right cancellation law in V, we get

$$\mathbf{0} = a\mathbf{0}$$

b) We have

$$\mathbf{0}\alpha = (\mathbf{0} + \mathbf{0})\alpha \qquad [\because \mathbf{0} = \mathbf{0} + \mathbf{0}]$$

$$= \mathbf{0}\alpha + \mathbf{0}\alpha$$

$$\therefore \quad \mathbf{0} + \mathbf{0}\alpha = \mathbf{0}\alpha + \mathbf{0}\alpha \qquad [\because \mathbf{0}\alpha \in V \text{ and } \mathbf{0} + \mathbf{0}\alpha = \mathbf{0}\alpha]$$

Since V is an abelian group with respect to addition. Therefore by right cancellation law in V, we get

$$\mathbf{0} = \mathbf{0}\alpha$$

c) We have

$$a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$$

$$\Longrightarrow \quad a\mathbf{0} = a\alpha + a(-\alpha)$$

$$\Rightarrow \quad \mathbf{0} = a\alpha + a(-\alpha) \qquad [\because a\mathbf{0} = \mathbf{0}]$$

$\Longrightarrow$ $a(-\alpha)$is the additive inverse of $a\alpha$

$$\Longrightarrow \quad a(-\alpha) = -(a\alpha)$$

d) We have

$$[a + (-a)]\alpha = a\alpha + (-a)\alpha$$

$$\Longrightarrow \quad \mathbf{0}\alpha = a\alpha + (-a)\alpha$$

$$\Rightarrow \quad \mathbf{0} = a\alpha + (-a)\alpha \qquad [\because \mathbf{0}\alpha = \mathbf{0}]$$

$\Longrightarrow$ $(-a)\alpha$ is the additive inverse of $a\alpha$

$$\Rightarrow \qquad (-a)\alpha = -(a\alpha)$$

e) We have

$$a(\alpha - \beta) = a[\alpha + (-\beta)] = a\alpha + a(-\beta)$$
$$= a\alpha + [-(a\beta)] \qquad [\because a(-\beta) = -(a\beta)]$$
$$= a\alpha - a\beta$$

f) Let $a\alpha = \mathbf{0}$ and $a \neq \mathbf{0}$. Then a^{-1} exists because a is a non-zero element of F.

$\therefore a\alpha = \mathbf{0} \Rightarrow a^{-1}(a\alpha) = a^{-1}\mathbf{0} \Rightarrow (a^{-1}a)\alpha = \mathbf{0} \Longrightarrow 1\alpha = \mathbf{0} \Longrightarrow \alpha = \mathbf{0}$

Again let $a\alpha = \mathbf{0}$ and $\alpha \neq \mathbf{0}$.Then to prove that $a = \mathbf{0}.$ Suppose $a \neq \mathbf{0}.$ Then a^{-1}exists.

$\therefore a\alpha = \mathbf{0} \Rightarrow a^{-1}(a\alpha) = a^{-1}\mathbf{0} \Rightarrow (a^{-1}a)\alpha = \mathbf{0} \Longrightarrow 1\alpha = \mathbf{0} \Longrightarrow \alpha = \mathbf{0}$

Thus, we get contradiction that α must be a zero vector. Therefore, a must be equal to zero. Hence

$\alpha \neq \mathbf{0}$ and $a\alpha = \mathbf{0} \Rightarrow a = \mathbf{0}$

Theorem2: Let $V(F)$ be a vector space. Then

a) If $a, b \in F$ and α is a non zero vector of V, we have
$$a\alpha = b\alpha \Rightarrow a = b$$
b) If $\alpha, \beta \in V$ and a is a non zero element of F , we have
$$a\alpha = a\beta \Rightarrow \alpha = \beta$$

Proof:

a) We have $a\alpha = b\alpha$

$$\Rightarrow \qquad a\alpha - b\alpha = \mathbf{0}$$
$$\Rightarrow \qquad (a - b)\alpha = \mathbf{0}$$
$$\Rightarrow \qquad a - b = 0 \qquad \because \alpha \neq 0$$
$$\Rightarrow \qquad a = b$$

b) We have $a\alpha = a\beta$

$$\Rightarrow \qquad a\alpha - a\beta = \mathbf{0}$$
$$\Rightarrow \qquad a(\alpha - \beta) = \mathbf{0}$$
$$\Rightarrow \qquad \alpha - \beta = 0 \qquad \because a \neq 0$$
$$\Rightarrow \qquad \alpha = \beta$$

1.8 *SUMMARY*: -

In this unit we covered important algebraic structures that are essential to both linear and abstract algebra in this course. Combining two members of a set to create another element of the same set is known as a binary operation. A set that possesses a binary operation satisfying closure, associativity, identity, and invertibility is called a group. When two operations (addition and multiplication) are defined and meet the field axioms of commutativity, associativity, distributivity, identities, and inverses for both operations, the structure is called a field. A vector space is a collection of vectors formed over a field that supports vector addition and scalar multiplication, following particular axioms. Additionally, we investigated the generic characteristics of vector spaces, including distributive laws, additive inverses, zero vector uniqueness, and scalar multiplication. A vector space is a collection of vectors constructed over a field that enables vector addition and scalar multiplication, satisfying specified axioms. We also looked into the general properties of vector spaces, such as scalar multiplication, distributive laws, additive inverses, and zero vector uniqueness.

1.9 GLOSSARY: -

- **Vector Space:** A set of vectors along with two operations vector addition and scalar multiplication that satisfies a set of axioms (like associativity, distributivity, existence of a zero vector, etc.
- **Vector:** An element of a vector space. It can be represented as a quantity having both magnitude and direction, but in algebra, it's simply an ordered list of numbers or functions.
- **Scalar:** An element of the field over which the vector space is defined. Scalars are used to multiply vectors (e.g., real numbers in $\mathbb{R}^n$).
- **Field**: A set with two operations (addition and multiplication) where every non-zero element has a multiplicative inverse. Examples: Real numbers ($\mathbb{R}$), Complex numbers ($\mathbb{C}$).
- **Zero Vector**: A unique vector in every vector space that, when added to any other vector, leaves it unchanged. Denoted as **0**.
- **Vector Addition**: An operation that combines two vectors to produce a third vector, following the rules of component-wise addition.
- **Scalar Multiplication**: The operation of multiplying a vector by a scalar (from the field), scaling its magnitude.

- **Linear Combination**: An expression formed by multiplying vectors by scalars and adding the results.
- **Norm:** A function that assigns a length (magnitude) to each vector in a vector space.

1.10 *REFERENCES*: -

- Gilbert Strang(2020), *Linear Algebra for Everyone* (2020).
- Nicholas A. (2024) , Loehr*Advanced Linear Algebra*.
- (2017, Springer), **Vector Spaces"** — *Beilina et al.*

1.11 *SUGGESTED READING*: -

- A.R. Vasishtha, J.N.Sharma and A.K. Vasishtha (52th Edition, 2022), Krishna Publication, Linear Algebra.
- K.P.Gupta (20^{th} Edition,2019), Pragati Publication, Linear Algebra

1.12 *TERMINAL QUESTIONS: -*

(TQ-1) Show that the set Z of all integers form a group with respect to binary operation $*$ defined by

$$a * b = a + b + 1 \forall\, a, b \in Z$$

is an abelian group.

(TQ-2) Show that the complex field C is a vector space over the real field R.

(TQ-3) Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (3y + y_1, -x - x_1)$$
$$c\,(x, y) = (3cy, -cx)$$

Is V, with these operations, a vector space over the field of real numbers?

(TQ-4) The set of all convergent sequences is a vector space over the field of real numbers.

(TQ-5) Define a vector space. State and explain the axioms that must be satisfied for a set to be a vector space.

(TQ-6) Show that the set Q of positive irrational does not form a group with respect to multiplication.

(TQ-7) Show that the set $(1, 1, i, -i)$ is an abelian finite group of order 4 under multiplication.

(TQ-8) Show that any two bases of a finite-dimensional vector space have the same number of elements.

(TQ-9) Define a **group**. List and explain the four group axioms.

(TQ-10) Show that $(\mathbb{Q}, +)$ is a group but $(\mathbb{Q}, \times)$ is not.

UNIT 2:- Subspace

CONTENTS:

2.1 *INTRODUCTION:-*

In this unit, we will study about the important concepts related to vector subspaces and their properties. A vector subspace is a subset of a vector space that is itself a vector space under the same operations. The algebra of subspaces deals with operations like intersection and sum of subspaces. The linear combination of vectors refers to forming new vectors by multiplying given vectors with scalars and adding them. The set of all possible linear combinations of a given set of vectors is called their linear span, which is itself a subspace. Finally, the linear sum of two subspaces is the smallest subspace containing both subspaces, formed by taking all possible sums of vectors from each subspace. Subspaces are important for solving linear equations, defining basis, and understanding vector space structure.

2.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- Define Vector Subspaces.
- Define Linear combinations of vectors.
- Define linear span.

- Define linear sum of two subspaces.

2.3 *VECTOR SUBSPACES: -*

Let $W \subseteq V$ and let V be a vector space over the field F. If W is a vector space over F with respect to the operations of vector addition and scalar multiplication in V, then W is referred to as a subspace of V.

Theorem1: The necessary and sufficient condition for a non-empty subset W of a vector space $V\ (F)$ to be a subspace of V is that W is closed under vector addition and scalar multiplication in V.

Proof: If W is a vector space over F with respect to scalar multiplication in V and vector addition, then W must be closed with respect to these two compositions. Therefore, the condition is necessary.

The condition is sufficient. Now assume that W is closed under vector addition and scalar multiplication in V, and that W is a non-empty subset of V.

Let $\alpha \in W$. If 1 is the unity element of F, then $-1 \in F$. Now W is closed under scalar multiplication. Therefore

$$-1 \in F, \alpha \in W \Rightarrow (-1)\alpha \in W \Rightarrow -(1\alpha) \in W$$

$\Rightarrow \quad -\alpha \in W \qquad [\because \alpha \in W \Rightarrow \alpha \in V \text{ and } 1\alpha = \alpha \text{ in } V].$

Thus, the additive inverse of each element of W is also in W.

Now W is closed under vector addition.

Therefore $\alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W \Rightarrow \mathbf{0} \in W$ where $\mathbf{0}$ is the zero vector of V.

A zero vector of V is also a zero vector of W. Vector addition will be associative and commutative in W as the elements of V are also the elements of W. Thus, in terms of vector addition, W is an abelian group. Moreover, W is closed under scalar multiplication is known. Since W is a subset of V, the other postulates of a vector space will also hold in W. For these two compositions, W itself becomes a vector space.

Theorem2: The necessary and sufficient conditions for a non-empty subset W of a vector space $V\ (F)$ to be a subspace of V are

a) $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$

b) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof: The conditions are necessary: W is an abelian group with respect to vector addition if it is a subspace of V. Consequently, $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$. Also, under scalar multiplication, W must be closed. As a result, condition (b) is also necessary.

The conditions are sufficient: Now suppose W is a non-empty subset of satisfying the two given conditions. From condition (a) we have

$$\alpha \in W, -\alpha \in W \Rightarrow \alpha - \alpha \in W$$

Thus, the zero vector of V belongs to W and it will also be the zero vector of W.

$$Now \qquad 0 \in W, \alpha \in W \Rightarrow 0 - \alpha \in W \Rightarrow -\alpha \in W.$$

Thus, the additive inverse of each element of W is also in W.

$$Again \qquad \alpha \in W, \beta \in W \Rightarrow -\alpha \in W, -\beta \in W$$

$$\Rightarrow \qquad \alpha - (-\beta) \in W, \Rightarrow \alpha + \beta \in W$$

Thus W is closed with respect to vector addition.

Vector addition will be both commutative and associative in W as the components of W are also the elements of V. Therefore, under vector addition, W is an abelian group. W is closed under scalar multiplication as well, according to condition (b). Since W is a subset of V, the other postulates of a vector space will also hold in W. W is therefore a subspace of V.

Theorem3: The necessary and sufficient condition for a non-empty subset W of a vector space $V\ (F)$ to be a subspace of V is

$$a, b \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

Proof: The condition is necessary. If W is a subspace of V, then W must be closed under scalar multiplication and vector addition. Therefore

$$a \in F, \alpha \in W \Rightarrow a\alpha \in W$$

$$and \qquad b \in F, \beta \in W \Rightarrow b\beta \in W$$

$$Now \qquad a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$$

Hence the condition is necessary.

The condition is sufficient. Now suppose W is a non-empty subset of satisfying the given condition i.e.,

$a, b \in F \; and \; \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Taking $a = 1, b = 1$, we see that if $\alpha, \beta \in W$, then $1\alpha + 1\beta \in W \Rightarrow \alpha + \beta \in W$

$[\because \alpha \in W \Rightarrow \alpha \in V and\ 1\alpha = \alpha\ in\ V]$

Thus W is closed under vector addition.

Now taking $a = -1, b = 0$, we see that if $\alpha \in W$ then

$(-1)\ a + 0\alpha \in W$ [In place of β we have taken α]

$\Rightarrow$ $-(1\alpha) + 0 \in W \Rightarrow -\alpha \in W$

Thus, the additive inverse of each element of W is also in W.

Taking$a = 0, b = 0$, we see that if $\alpha \in W$ then

$$0\alpha + 0\alpha \in W \Rightarrow 0 + 0 \in W \Rightarrow 0 \in W$$

Thus, the zero vector of belongs to W. It will also be the zero vector of W.

Since the elements of W are also the elements, of V, therefore vector addition will be associative as well as commutative in V. Thus W is an abelian group with respect to vector addition.

Now taking $\beta = 0$, we see that if $a, b \in Fand \in W$, then

$$a\alpha + b0 \in W\ i.e., a\alpha + 0 \in Wi.e., a\alpha \in W.$$

Thus W is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V of which W is a subset. Hence $W\ (F)$ is a subspace of $V\ (F)$.

Theorem4: A non-empty subset W of a vector space $V\ (F)$ is a subspace of V if and only if for each pair of vectors $\alpha, \beta\ in\ W$ and each scalar a in F the vector $a\alpha + \beta$ is again in W.

Proof: The condition is necessary: W must to be closed with respect to both vector addition and scalar multiplication if W is a subset of V.

Therefore, $a \in F, \alpha, \beta \in W \Rightarrow a\alpha \in W$

$furthermore,$ $a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$

Thus, the condition is necessary.

The condition is sufficient: W is a non-empty subset of V, and $a \in F, \alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$ is given. We are to prove that W is a subspace of V.

(i) Since W is non-empty, therefore there is at least one vector in W, say γ. Now $1 \in F \Rightarrow -1 \in F$. Therefore taking $a = -1, \alpha = \gamma, \beta = \gamma$, we get from the given condition that

$$(-1)\gamma + \gamma = -(1\gamma) + \gamma = -\gamma + \gamma = 0 \; is \; in \; W$$

(ii) Now let $a \in F, \alpha \in W$, Since $0 \; is \; in \; W$, therefore taking $\beta = 0$ in the given condition, we get

$$a\alpha + 0 = a\alpha \; is \; in \; W$$

Thus W is closed with respect to scalar multiplication.

(iii) Let $\alpha \in W$. Since $-1 \in F$ and W is closed with respect to scalar multiplication, therefore,

$$(-1)\alpha = -(1\alpha) = -\alpha \; is \; in \; W.$$

(iv) We have $1 \in F$. If $\alpha, \beta \in W$, then $1\alpha + \beta = \alpha + \beta$. Thus W is closed with respect to vector addition.

The remaining postulates of a vector space will hold in W since they hold in of which W is a subset.

Hence W is a subspace of V.

Example1: The set W of ordered triads $(a_1, a_2, 0)$, where $a_1, a_2 \in F$ is a subspace of $V_3(F)$.

Solution: Let $\alpha = (a_1, a_2, 0)$ and $\beta = (b_1, b_2, 0)$ be any two elements of W. Then $a_1, a_2, b_1, b_2 \in F$ If a, b be any two elements of F, we have

$$a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0) = (aa_1, aa_2, 0) + (bb_1, bb_2, 0) = (aa_1 + bb_1, aa_2 + bb_2) \in W$$

Since $aa_1 + bb_1, aa_2 + bb_2 \in F$ and the last co-ordinate of this triad is zero.

Hence W is a subspace of $V_3(F)$.

Example2: Let V be the vector space of all polynomials in an indeterminate x over a field F. Let W be a subset of V consisting of all polynomials of $degree \leq n$. Then W is a subspace of V.

Solution: Let $\alpha \; and \; \beta$ be any two elements of W. Then α, β are polynomials over F of $degree \leq n$. If a, b are any two elements of F, then $a\alpha + b\beta$ will also be a polynomial of $degree \leq n$. Therefore $a\alpha + b\beta \in W$ Hence W is a subspace of V.

Example3: If a_1, a_2, a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of elements of F, such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

is a subspace of $V_3(F)$.

Solution: Let $\alpha = (x_1, x_2, x_3) \; and \; \beta = (y_1, y_2, y_3)$ be any two elements of W. Then $x_1, x_2, x_3, y_1, y_2, y_3$ are elements of F and are such that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \qquad \text{... (1)}$$

and

$$a_1y_1 + a_2y_2 + a_3y_3 = 0 \qquad \text{... (2)}$$

If $a, b \in F$, we have

$$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$

$$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$$

$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$$

Now $a_1(ax_1 + by_1) + \; a_2(ax_2 + by_2) + \; a_3(ax_3 + by_3)$

$$= a(a_1x_1 + a_2x_2 + a_3x_3) + (a_1y_1 + a_2y_2 + a_3y_3)$$

$$= a0 + b0 = 0$$

$\therefore a\alpha + b\beta = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \in W$

Hence W is a subspace of $V_3(F)$.

Example4: Prove that the set of all solutions (a, b, c) of the equation $a + b + 2c = 0$ is a subspace of the vector space $V_3(R)$.

Solution: Let $W = \{(a, b, c): a, b, c \in R \; and \; a + b + 2c = 0\}$

To prove that W is a subspace of $V_3(R)$ or R^3.

Let $\alpha = (a_1, b_1, c_1) \; and \; \beta = (a_2, b_2, c_2)$ be any two elements of W. Then

$$a_1 + b_1 + 2c_1 = 0 \; and \; a_2 + b_2 + 2c_2 = 0$$

If a, b be any two elements of R, we have

$$a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2)$$

$$= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)$$

$$= (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$$

Now $(aa_1 + ba_2) + (ab_1 + bb_2) + 2(ac_1 + bc_2)$

$$= a(a_1 + b_1 + 2c_1) + b(a_2 + b_2 + 2c_2)$$

$$= a0 + b0 = 0$$

$\therefore \quad a\alpha + b\beta = (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \in W$

Thus $\alpha, \beta \in W \; and \; a, b \in R \Rightarrow a\alpha + b\beta \in W$

Hence W is a subspace of $V_3(R)$.

2.4 *ALGEBRA OF SUBSPACES: -*

Theorem5: The intersection of any two subspaces $W_1 \, and \; W_2$ of vector space $V(F)$ is also a subspace of $V(F)$.

Proof: $\because$ **0** $\in W_1 \; and \; W_2$ both therefore $W_1 \cap W_2$ is not empty.

$Let \quad \alpha, \beta \in W_1 \cap W_2 \; and \; a, b \in F$

$Now \quad \alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1 and \; \alpha \in W_2$

$and \quad \beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1 and \; \beta \in W_2$

$\because W_1$ is a subspace, therefore

$$a, b \in F \; and \; \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$$

$$Similarly \; a, b \in F \; and \; \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$$

Now $a\alpha + b\beta \in W_1 and\ a\alpha + b\beta \in W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

Thus $a, b \in F\ and\ \alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of $V(F)$.

Theorem6: The union of two subspaces is a subspace if and only if one is contained in the other.

Proof: Suppose $W_1\ and\ W_2$ are two subspaces of a vector space V.

Let $W_1 \subseteq W_2\ or\ W_2 \subseteq W_1$. Then $W_1 \cup W_2 = W_2\ or\ W_1$. But W_1, W_2 are subspaces and therefore, $W_1 \cup W_2$ is also a subspace.

Conversely, suppose $W_1 \cup W_2$ is a subspace.

To prove that $W_1 \subseteq W_2\ or\ W_2 \subseteq W_1$.

Let us assume that W_1 is not a subset of W_2 and W_2 is also not a subset of W_1.

Now that W_1 is not a subset of $W^2 \Rightarrow \exists\ \alpha \in W_1\ and\ \alpha \notin W_2$ ... (1)

$and\ W_2$ is not a subset of $W_1 \Rightarrow \exists\ \beta \in W_2\ and\ \beta \notin W_1$ ... (2)

From (1) and (2), we have

$$\alpha \in W_1 \cup W_2\ and\ \beta \in W_1 \cup W_2$$

Since $W_1 \cup W_2$ is a subspace, therefore $\alpha + \beta$ is also in $W_1 \cup W_2$

But $\alpha + \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1 or W_2$.

Suppose $\alpha + \beta \in W_1$. Since $\alpha \in W_1$ and W_1 is a subspace, therefore $(\alpha + \beta) - \alpha = \beta\ is\ in\ W_1$.

But from (2), we have $\beta \notin W_1$. Thus, we get a contradiction. Again, suppose that $\alpha + \beta \in W_2$. Since $\beta \in W_2$ and W_2 is a sub-space, therefore $(\alpha + \beta) - \beta = \alpha$ is in W_2. But from (1), we have $\alpha \notin W_2$. Thus, here also we get a contradiction. Hence either $W_1 \subseteq W_2\ or\ W_2 \subseteq W_1$.

Theorem7: Arbitrary intersection of subspaces i.e., the inner section of any family of subspaces of a vector space is a subspace.

Proof: Let $V(F)$ be a vector space and let $\{W_t : t \in T\}$ be any family of subspaces V. Here T is an index set and is such that $\forall\, t \in T, W_t$ is a subspace of V.

$$Let \qquad U = \bigcap_{t\in T} W_t = \{x \in W_t \forall t \in T\}$$

be the intersection of this family of subspaces of V. Then to prove that U is also a subspace of V.

Obviously $U \neq \phi$, since at least the zero vector $\mathbf{0}\ of\ V$ is in $W_t \forall t \in T$

Now let $a, b \in F$ and α, β be any two elements of $\bigcap_{t\in T} W_t$

Then $\alpha, \beta \in W_t \forall t \in T$. Since each W_t is a subspace of V, therefore $a\alpha + b\beta \in W_t \forall t \in T$.

Thus $a\alpha + b\beta \in \bigcap_{t\in T} W_t$

Thus $a, b \in F\ and\ \alpha, \beta \in \bigcap_{t\in T} W_t \Rightarrow a\alpha + b\beta \in \bigcap_{t\in T} W_t$

Hence $\bigcap_{t\in T} W_t$ is a subspace of $V(F)$.

2.5 *LINEAR COMBINATION OF VECTORS*: -

Let $V(F)$ be a vector space. If $\alpha_1, \alpha_2, \dots, \alpha_n \in V$, then any vector

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n;\ a_1, a_2, \dots, a_n \in F$$

is called a linear combination of vectors $\alpha_1, \alpha_2, \dots, \alpha_n$.

2.6 *LINEAR SPAN*: -

Let $V(F)$ be a vector space and S be any non-empty subset of V. Then the linear span of S is the set of all linear combinations of finite sets of elements of S and is denoted by $L(S)$.

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n;\ \alpha_1, \alpha_2, \dots, \alpha_n \\ is\ any\ arbitrary\ finite\ subset\ of\ S\ and\ a_1, a_2, \dots, a_n$$

$is\ any\ arbitrary\ finite\ subset\ of\ F\}$

Theorem8: The linear span $L(S)$ of any subset S of a vector space $V(F)$ is a subspace of V generated by S i.e.

$$L(S) = \{S\}$$

Proof: Let α, β be any two elements of$L(S)$. Then

$$Then \qquad \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$$

$$and \qquad \beta = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$$

Where the $a's\ and\ b's$ are elements of F and the $\alpha's\ and\ \beta's$ are elements of S.

If a, b be any two elements of F, then

$$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m) + b(b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n)$$

$$= a(a_1\alpha_1) + a(a_2\alpha_2) + \cdots + a(a_m\alpha_m) + b(b_1\beta_1) + b(b_2\beta_2) + \cdots + b(b_n\beta_n)$$

$$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \cdots + (aa_m)\alpha_m + (bb_1)\beta_1 + (bb_2)\beta_2 + \cdots + (bb_n)\beta_n$$

Thus $a\alpha + b\beta$ has been expressed as a linear combination of a finite set $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n$ of the elements of S. Consequently $a\alpha + b\beta \in L(S)$.

Thus $a, b \in F$ and $\alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)$. Hence L(S) is a subspace of $V(F)$.

Also each element of S belongs to $L(S)$, because if $\alpha_r \in L(S)$, then $\alpha_r = 1\alpha_r$ and this implies that$\alpha_r \in L(S)$. Thus $L(S)$ is a subspace of $V\ and\ S$ is contained in $L(S)$.

Now if W is any subspace of V containing S, then each element of $L(S)$ must be in W because W is to be closed under vector addition and scalar multiplication. Therefore $L(S)$ will be contained in W.

Hence $L(S) = \{S\}\ i.e., L(S)$ is the smallest subspace of V containing S.

Example5. The subset containing a single element (1,0,0) of the vector space $V_3(F)$ generates the subspace which is the totality of the elements of the form $(a, 0, 0)$.

Example6. The subset $\{(1, 0, 0), (0, 1, 0)\}$ of $V_3(F)$ generates the subspace which is the totality of the elements of the form $(a, b, 0)$.

Example7. The subset $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $V_3(F)$ generates or spans the entire vector space $V_3(F)$ i.e., $L(S) = V$.
If (a, b, c) be any element of V, then

$$(a, b, c) = a\ (1, 0, 0) + b(0, 1, 0) + c\ (0, 0, 1)$$

Thus $(a, b, c) \in L\ (S)$. Hence $V \subseteq L(S)$. Also $L(S) \subseteq V$.
Hence $L\ (S) = V$.

Example8. Let V be the vector space of all polynomials over the field F. Let S be the subset of V consisting of the polynomials $f_0, f_1, f_2, \ldots$ defined by $f_n = x^n, n = 0, 1, 2, \ldots\ldots$ Then$V = L\ (S)$.

2.7 *LINEAR SUM OF TWO SUBSPACES*:-

Let $V\ (F)$ be a vector space with two subspaces, $W_1 and\ W_2$. $W_1 + W_2$, which represents the linear sum of the subspaces $W_1 and\ W_2$, is the set of all sums$\alpha_1 + \alpha_2$ such that $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$,

$$W_1 + W_2 = \{\alpha_1 + \alpha_2; \alpha_1 \in W_1, \alpha_2 \in W_2\}$$

Theorem9: If $W_1 and\ W_2$ are subspaces of vector space $V(F)$, then

a) $W_1 + W_2$ is subspace of $V(F)$.
b) $W_1 + W_2 = \{W_1 \cup W_2\} i.e. L(W_1 \cup W_2\} = W_1 + W_2$

Proof: a) Let α, β be any two elements of $W_1 and\ W_2$. Then $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ where $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$. If $a, b \in F$, we have

$$a\alpha + b\beta = a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2)$$

$$= (a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2)$$

$\because W_1$ is a subspace of V, therefore $a, b \in F$ and

$$\alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1 + b\beta_1 \in W_1$$

Similarly, $\alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2 + b\beta_2 \in W_2$

Hence $a\alpha + b\beta = (a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) \in W_1 + W_2$

Thus $a, b \in F$ and $\alpha, \beta \ \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of $V(F)$.

b) $\because W_2$ contains the zero vector, therefore if $\alpha_1 \in W_1$, then we can write

$$\alpha_1 = \alpha_1 + 0 \in W_1 + W_2$$

Thus $W_1 \subseteq W_1 + W_2$

Similarly, $W_2 \subseteq W_1 + W_2$

Hence $W_1 \cup W_2 \subseteq W_1 + W_2$

Therefore $W_1 + W_2$ is a subspace of $V(F)$ containig$W_1 \cup W_2$.

Now to prove that $W_1 + W_2 = \{W_1 \cup W_2\}$ we should prove that $W_1 + W_2 \subseteq L(W_1 \cup W_2)$ and$L(W_1 \cup W_2) \subseteq W_1 + W_2$.

Let $\alpha = \alpha_1 + \beta_1$ be any element of $W_1 + W_2$. Then$\alpha_1 \in W_1$ and $\beta_1 \in W_2$. Therefore $\alpha_1, \beta_1 \in W_1 \cup W_2$. We can write

$$\alpha_1 + \beta_1 = 1\alpha_1 + 1\beta_1$$

Thus $\alpha_1 + \beta_1$ is a linear combination of a finite number of elements$\alpha_1, \beta_1 \in W_1 \cup W_2$.

Therefore $\alpha_1, \beta_1 \in L(W_1 \cup W_2)$.

$\therefore$ $W_1 + W_2 \subseteq L(W_1 \cup W_2)$

Also $L(W_1 \cup W_2)$ is the smallest subspace containing $W_1 + W_2$ and $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$. Therefore $L(W_1 \cup W_2)$ must be contained in $W_1 + W_2$. Consequently

$$L(W_1 \cup W_2) \subseteq W_1 + W_2.$$

Hence $W_1 + W_2 = L(W_1 \cup W_2) = \{W_1 \cup W_2\}$.

Example9. If S, T are subset of $V(F)$, then

a) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$
b) $L(S \cup T) = L(S) + L(T)$
c) $S\ is\ a\ subspace\ of\ V \Leftrightarrow L(S) = S$

d) $L(L(S)) = L(S)$

Solution: a) Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$ where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of S. $\because S \subseteq T$, therefore $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is also a finite subset of T. So $\alpha \in L(T)$.

Thus $\alpha \in L(S) \Rightarrow \alpha \in L(T)$

$\therefore L(S) \subseteq L(T)$

b) Let $\alpha \in L(S \cup T)$, then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p$$

Where $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_p\}$is a finite subset of $S \cup T$ such that

$\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq S and \{ \beta_1, \beta_2, \ldots, \beta_p\} \subseteq T$

Now $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m \in L(S)$

And $b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p \in L(T)$

Therefore $\alpha \in L(S) + L(T)$

Hence $L(\beta) \subseteq L(S) + L(T)$

Let γ be any element of $L(S) + L(T)$. Then, $\gamma = \beta + \delta$, where $\beta \in L(S)$ and $\delta \in L(T)$. A linear combination of a finite number of elements from S will now be represented by β, while a linear combination of a limited number of elements from T will be represented by δ. Consequently, a linear combination of a finite number of $S \cup T$ elements will be $\beta + \delta \in L(S \cup T)$ Thus,

$$L(S) + L(T) \subseteq L(S \cup T)$$

Hence $L(S \cup T) = L(S) + L(T)$

c) Suppose S is a subspace of V. Then we are to prove that $L(S) = S$.
Let $\alpha \in L(S)$. Then $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $a_1, a_2, \ldots, a_n \in F$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$. But S is a subspace of V. Therefore, it is closed with respect to scalar multiplication and vector addition. Hence $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in S$. Thus

$$\alpha \in L(S) \Rightarrow \alpha \in S$$

Therefore $L(S) \subseteq S$. Also $S \subseteq L(S)$. Therefore $L(S) = S$.

Converse: Suppose $L(S) = S$. Then to prove that S is a sub-space of V. We know that $L(S)$ is a subspace of V. Since $S = L(S)$, therefore S is also a subspace of V.

d) $L(L(S))$ is the smallest subspace of V containing $L(S)$. But $L(S)$ is a subspace of V. Therefore, the smallest subspace of V containing $L(S)$ is $L(S)$ itself. Hence $L(L(S)) = L(S)$.

2.8 *SUMMARY*:-

The basic ideas of vector spaces have been covered in this section, with an emphasis on vector subspaces and their properties. We looked at operations like intersection and sum of subspaces, which are part of the algebra of subspaces. Additionally, we studied linear vector combinations and how they serve as the foundation for the idea of linear span, which denotes the smallest subspace that contains a specific collection of vectors. Last but not least, we looked at the linear sum of two subspaces, which is the set of all possible vector sums extracted from each subspace. When combined, these ideas offer a more thorough comprehension of the composition and functions of vector spaces.

2.9 *GLOSSARY*: -

- **Vector Subspace** – A subset of a vector space that is itself a vector space under the same operations of vector addition and scalar multiplication.
- **Zero Subspace** – The smallest subspace of any vector space consisting only of the zero vector.
- **Proper Subspace** – A subspace that is strictly smaller than the entire vector space (i.e., not equal to the whole space).
- **Algebra of Subspaces** – The study of operations on subspaces, such as their intersection and sum.
- **Intersection of Subspaces** – The set of all vectors that belong to both subspaces; always forms a subspace.
- **Linear Combination** – An expression formed by multiplying vectors with scalars and adding them together.
- **Linear Span (or Span)** – The set of all linear combinations of a given set of vectors; it is the smallest subspace containing those vectors.
- **Linear Independence in Subspaces** – A set of vectors in a subspace is linearly independent if none of them can be expressed as a linear combination of the others.
- **Basis of a Subspace** – A minimal set of linearly independent vectors that spans the subspace.

- **Dimension of a Subspace** – The number of vectors in a basis of the subspace; it measures the "size" or "degrees of freedom" of the subspace.
- **Linear Sum of Subspaces** – The set of all possible sums of a vector from one subspace and a vector from another; denoted as U+WU + WU+W.
- **Direct Sum of Subspaces** – A special case of linear sum where each element of the sum can be written uniquely as a sum of vectors from the two subspaces.

2.10 *REFERENCES*: -

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- Axler, S. (2015). *Linear Algebra Done Right* (3rd ed.). Springer.
- Lay, D. C., Lay, S. R., & McDonald, J. J. (2015). *Linear Algebra and Its Applications* (5th ed.). Pearson.
- Friedberg, S. H., Insel, A. J., & Spence, L. (2019). *Linear Algebra* (5th ed.). Pearson.

2.11 SUGGESTED *READING*: -

- A.R. Vasishtha, J.N.Sharma and A.K. Vasishtha (52th Edition, 2022), Krishna Publication, Linear Algebra.
- K.P.Gupta (20th Edition,2019), Pragati Publication, Linear Algebra

2.12 *TERMINAL QUESTIONS: -*

(TQ-1) Let V be the set of all pairs (x, y) of real numbers and let F be the field of real numbers. Examine in each of the following cases whether V is a vector space over the fielf of real no. or not?

a) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$
   $c(x + y) = (|c|x, |c|y)$

b) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$
   $c(x + y) = (0, cy)$

c) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$
   $c(x + y) = (c^2x, c^2y)$

(TQ-2) Show that the convergent sequences are a vector space over the field of real numbers.

(TQ-3) Show that the set W of the elements of the vector space $V_3(R)$ of the form $(x+2y, y, -x+3y); x, y \in R$ is a subspace of $V_3(R)$.

(TQ-4) Show that the intersection of any collection of subspaces of a vector space is a subspace. Can you replace 'intersection' by Unton' in this proposition?

(TQ-5) Define a **vector subspace**. State the conditions for a subset of a vector space to be a subspace.

(TQ-6) What is the **zero subspace**? Give an example.

(TQ-7) Explain the difference between a **proper subspace** and the whole vector space.

(TQ-8) What is the **linear span** of a set of vectors? Why is it always a subspace?

(TQ-9) Define the **linear sum of two subspaces**. How is it different from the **direct sum**?

(TQ-10) Show that the set of all polynomials of degree at most nnn forms a subspace of the vector space of all polynomials.

(TQ-11) Prove that the **intersection** of two subspaces is also a subspace.

(TQ-12) Show that the **union** of two subspaces is not necessarily a subspace. Give an example.

(TQ-13) Prove that every subspace of Rn\mathbb{R}^nRn contains the **zero vector**.

(TQ-14) Show that the set of all solutions to a homogeneous linear system forms a subspace.

(TQ-15) Prove that the span of a set of vectors is the **smallest subspace** containing them.

UNIT 3: Linear Dependence and Linear Independence

CONTENTS:

3.1 *INTRODUCTION: -*

In linear algebra, the concepts of **linear independence** and **linear dependence** describe the relationship among a set of vectors in a vector space. A set of vectors is said to be linearly independent if no vector in the set can be expressed as a linear combination of the others.

On the other hand, a set of vectors is linearly dependent if at least one of the vectors can be expressed as a linear combination of the others. In such a case, the above equation admits a non-trivial solution, meaning that some coefficients are not zero. Linear dependence implies redundancy, as one or more vectors do not add any new dimension to the span of the set.

These concepts are fundamental in determining the **basis** of a vector space, since a basis must consist of linearly independent vectors. Moreover, they are directly related to the idea of **dimension**, which counts the maximum number of linearly independent vectors in a space. Thus, the study of linear independence and dependence is essential for understanding vector spaces, solving systems of linear equations, and analyzing linear transformations.

3.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- Define linear dependence.

- Define linear independence.
- Proof theorems related to linear independence and linear dependence.

3.3 *LINEAR DEPENDENCE: -*

Let $V\ (F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V is said to be linearly dependent if there exists scalars $a_1, a_2, \dots, a_n \in F$ not all of them 0 such that

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$$

3.4 *LINEAR INDEPENDENCE: -*

Let $V\ (F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V is said to be linearly independent if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0, a_i \in F, 1 \le i \le n$$

$$\Rightarrow \qquad a_i = 0\ for\ each\ 1 \le i \le n$$

If all of the finite subsets of an infinite set of vectors of V are linearly independent, then the set is said to be linearly independent, if not, it is known as linear dependent.

Theorem1: Let $V(F)$ be a vector space. If $\alpha_1, \alpha_2, \dots, \alpha_n$ are non-zero vectors in V then either they are linearly independent or some $\alpha_k, 2 \le k \le n$, is a linear combination of the preceding ones $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$.

Proof: We don't need to prove anything if $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent. Let us assume that $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly dependent. Then, a relation of the form exists.

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \qquad \dots (1)$$

The scalar coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ are not all zero. Assume that $a_k \ne 0$ for k is the greatest integer, $i.e. a_{k+1} = 0, a_{k+2} = 0, \dots, a_n = 0$. This assumption is safe since, at most, if $a_k \ne 0\ then\ k = n$.

.

Also $2 \le k$. Because if $a_2 = 0, a_3 = 0, \dots, a_n = 0$ then $a_1\alpha_1 = 0\ and\ \alpha_1 \ne 0 \Rightarrow a_1 = 0$. This contradicts the fact that not all the $a's\ are$ 0. Now equation (1) reduces to

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0;\ a_k \ne 0$$
$$a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1}$$
$$a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1})$$

$$\alpha_k = (-a_k{}^{-1}a_1)\alpha_1 + (-a_k{}^{-1}a_2)\alpha_2 + \cdots + (-a_k{}^{-1}a_{k-1})\alpha_{k-1}$$

Thusα_k is a linear combination of its preceding vectors.

Theorem2: The set of non-zero vectors $\alpha_1, \alpha_2, \dots, \alpha_n$of $V(F)$ is linearly dependent if some as, $\alpha_k, 2 \leq k \leq n$,, is a linear combination of the preceding ones.

Proof: If some $\alpha_k, 2 \leq k \leq n$, is a linear combination of the preceding ones $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ then there exists scalars $a_1, a_2, \dots, a_{k-1}$ such that

$$\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{k-1}\alpha_{k-1}$$

$$\Rightarrow \quad 1\alpha_k - a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{k-1}\alpha_{k-1} = 0$$

$$\Rightarrow \quad the\ set\ (\alpha_1, \alpha_2, \dots, \alpha_k)\ is\ linearly\ dependent.$$

Hence the set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$of which $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a subset, must be linearly dependent.

Theorem3: If in a vector space $V(F)$, a vector β is a linear combination of the set of vectors $\alpha_1, \alpha_2, \dots, \alpha_n$, then the set of vectors $\beta, \alpha_1, \alpha_2, \dots, \alpha_n$ is linearly dependent.

Proof: Since β is a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$, therefore there exist scalars $a_1, a_2, \dots, a_n$, such that

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

$$\Rightarrow \quad 1\beta - a_1\alpha_1 - a_2\alpha_2 - \cdots - a_n\alpha_n = 0 \qquad \dots (1)$$

In equation (1) the scalar coefficient of $\beta\ is$ 1 which is $\neq 0$. Hence from equation (1) not all the scalar coefficients are 0. Therefore the set $\beta, \alpha_1, \alpha_2, \dots, \alpha_n$ is linearly dependent.

Theorem4: Let S be a linearly independent subset of a vector Space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is lincarly independent.

Proof: Let $\alpha_1, \alpha_2, \dots, \alpha_n$ are vectors in S and let

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m + b\beta = 0 \qquad \dots (1)$$

Then b must be zero, for otherwise

$$\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \left(-\frac{c_2}{b}\right)\alpha_2 + \cdots + \left(-\frac{c_m}{b}\right)\alpha_m$$

And consequently β is in the subspace spanned by S which is a contradiction.

Putting $b = 0$ in equation (1), we get

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m = 0$$

$$\Rightarrow \quad c_1 = 0, c_2 = 0, \dots, c_m = 0$$

Because the set $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is linearly independent since it is a subset of a linearly independent set S.

Thus equation (1) implies

$$c_1 = 0, c_2 = 0, \dots, c_m = 0, b = 0$$

Therefore, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta\}$ is linearly independent. If S' is the set obtained by adjoining β to S, then we have proved that every finite subset of S' is linearly independent.

Example1: Prove that if two vectors are linearly dependent then one of them is a scalar multiple of the other.

Solution: Let α, β be two linearly dependent vectors of the vector space $V(F)$. Then $a, b \in F$(where a, b not both zero), such that

$$a\alpha + b\beta = 0$$

If $a \neq 0$ then we get

$$a\alpha = -b\beta$$

$$\Rightarrow \qquad \alpha = \left(-\frac{b}{a}\right)\beta \Rightarrow \alpha \text{is a scalar multiple of } \beta.$$

If $b \neq 0$ then we get

$$b\beta = -a\alpha$$

$$\Rightarrow \qquad \beta = \left(-\frac{a}{b}\right)\alpha \Rightarrow \beta \text{is a scalar multiple of } \alpha.$$

Thus one of the vectors $\alpha \ and \ \beta$ is a scalar multiple of the other.

Example2: In the vector space $V_n(F)$, the system of n vectors

$$e_1 = (1,0,0, \ldots ,0), e_2 = (0,1,0, \ldots ,0), \ldots , e_n = (0,0,0, \ldots ,0,1)$$

Is linearly independent where 1 denotes the unity of the field F.

Solution: Let $a_1, a_2, \ldots, a_n$ be any scalars, then

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = 0$$

$$\Rightarrow \qquad a_1(1,0,0, \ldots ,0) + a_2(0,1,0, \ldots ,0) + \cdots + a_n(0,0,0, \ldots ,0,1) = 0$$

$$\Rightarrow \qquad (a_1, a_2, \ldots , a_n) = (0,0, \ldots ,0)$$

$$\Rightarrow \qquad a_1 = 0, a_2 = 0, \ldots , a_n = 0$$

Therefore the given set of n vectors is linearly independent.

In particular $\{(1,0,0), (0,1,0), (00,1)\}$ is a linearly independent subset of $V_3(F)$

Example3: Every superset of a linearly dependent set of vectors is linearly dependent.

Solution: Let $S = \{\alpha_1, \alpha_2, \ldots , \alpha_n\}$ be a linearly dependent set of vectors. Then ∃ scalars $a_1, a_2, \ldots , a_n$ not all zero such that

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \qquad \ldots(1)$$

Now let $S' = \{\alpha_1, \alpha_2, \ldots , \alpha_n, \beta_1, \beta_2, \ldots , \beta_m\}$ be a superset of S. Then from equation (1)

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \cdots + 0\beta_m = 0 \qquad \ldots(2)$$

Since from equation (2) the scalar coefficients are not all zero, therefore S' is linearly dependent.

Example4: A system consisting of a single non-zero vector is always linearly independent.

Solution: Let $S = \{\alpha\}$ be a subset of a vector space V and let $\alpha \neq 0$. If a is any scalar, then

$$a\alpha = 0$$

$$\Rightarrow \qquad a = 0$$

Hence the set S is linearly independent.

Example5: Show that

$$S = \{(1,2,4),(1,0,0),(0,1,0),(0,0,1)\}$$

is a linearly independent subset of $V_3(R)$where R is the field of real numbers.

Solution: We have

$$1(1,2,4) + (-1)(1,0,0) + (-2)(0,1,0) + (-4)(0,0,1)$$
$$= (1,2,4) + (-1,0,0) + (0,-2,0) + (0,0,-4)$$
$$= (0,0,0) i,e. zero\ vector$$

Since in the above relation the scalar coefficients $1,-1,-2,-4$ are not all zero, therefore the given S is linearly independent.

Example6: If F is the field of real numbers, prove that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(F)$ are linearly dependent iff

$$a_1b_2 - a_2b_1 = 0$$

Solution: Let $x, y \in F$. Then

$$x(a_1, a_2) + y(b_1, b_2) = (0,0)$$

$$\Rightarrow \qquad (xa_1 + yb_1, xa_2 + yb_2) = (0,0)$$

Therefore $xa_1 + yb_1 = 0\ and\ xa_2 + yb_2 = 0$

The necessary and sufficient condition for these equations to possess a non-zero solution is that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

$$i.e. \qquad a_1b_2 - a_2b_1 = 0$$

Hence the given system is linearly dependent iff

$$a_1b_2 - a_2b_1 = 0$$

Example7: If $\alpha_1 and\ \alpha_2$ are vectors of $V(F)$, $a, b \in F$, show that the set $\{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\}$ is linearly independent.

Solution: We have

$$(-a)\alpha_1 + (-b)\alpha_2 + 1(a\alpha_1 + b\alpha_2)$$
$$= (-a + a)\alpha_1 + (-b + b)\alpha_2$$
$$= 0\alpha_1 + 0\alpha_2 = 0\ i.e. zero\ vector$$

Whatever may be the scalars $(-a)\ and\ (-b) \because 1 \neq 0$, therefore the given set of vectors is linearly dependent.

Example8: If α, β, γ are linearly dependent vectors of $V(F)$ where F is any subfield of the field of complex numbers then so also are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$.

Solution: Let a, b, c be scalars such that

$$a\,(\alpha+\beta)+b(\beta+\gamma)+c\,(\gamma+\alpha)=0$$

$i.e.$ $$(a+c)\alpha+(a+b)\beta+(b+c)\gamma=0 \qquad \ldots(1)$$

But α, β, γ are linearly independent. Therefore (1) implies

$$a+0b+c=0$$
$$a+b+0c=0$$
$$0a+b+c=0$$

The coefficient matrix A of these equations is

$$A=\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

We have rank $A=3$ i.e., the number of unknowns a, b, c.

Therefore $a=0, b=0, c=0$ is the only solution of the given equations.

Hence $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also linearly independent.

Example9: Show that the set $\{1, x, 1+x+x^2\}$ is a linearly independent set of vectors in the vector space of all polynomials over the real number field.

Solution: Let a, b, c be scalars (real numbers) such that

$$a\,(1)+bx+c(1+x+x^2)=0.$$

We have

$$a\,(1)+bx+c(1+x+x^2)=0$$

$\Rightarrow$ $$(a+c)+(b+c)x+cx^2=0$$

$\Rightarrow$ $$a+c=0, b+c=0, c=0$$

$\Rightarrow$ $$c=0, b=0, a=0$$

Therefore the vectors $1, x, 1+x+x^2$ are linearly independent over the field of real numbers.

Example10: Show that the vectors $(1,1,0,0), (0,1,-1,0), (0,0,0,3)$ in R^4 are linearly independent.

Solution: Let a, b, c be scalars i.e., real numbers such that

$$a(1,1,0,0)+\;b\,(0,1,-1,0)+c\,(0,0,0,3)=(0,0,0,0) \qquad \ldots(1)$$

$Then$ $$a\;+\;0b\;+\;0c\;=\;0$$
$$a\;+\;b\;+\;0c\;=\;0$$
$$0a\;-\;b\;+\;0c\;=\;0$$
$$0a\;+\;0b\;+\;3c\;=\;0$$

The only solution of the above equations is $a=0, b=0\ c=0$

Thus the linear relation (1) among the three given vectors is possible only if $a=0, b=0, c=0$

Hence the three given vectors in R^4 are linearly independent.

3.5 *SUMMARY*: -

In this unit we have examined the ideas of linear dependency and linear independence in this unit. A group of vectors is said to be linearly dependent if at least one of them exhibits redundancy among the others and can be written as a linear combination of the others. A set of vectors is said to be linearly independent if no vector can be expressed as such a combination, indicating that each vector makes a distinct contribution to the span. These principles are essential to comprehending vector space structure because they give rise to the concepts of dimension and basis.

3.6 *GLOSSARY*: -

- **Linear Combination** – An expression formed by multiplying vectors with scalars and adding them together.
- **Linearly Dependent Set** – A set of vectors is linearly dependent if at least one vector can be expressed as a linear combination of the others, or equivalently, if there exist scalars (not all zero) such that their linear combination equals the zero vector.
- **Linearly Independent Set** – A set of vectors is linearly independent if the only solution to their linear combination equaling the zero vector is when all the scalars are zero.
- **Trivial Linear Combination** – The combination of vectors where all scalar coefficients are zero, giving the zero vector.
- **Non-trivial Linear Combination** – A linear combination where at least one scalar is non-zero; if this equals the zero vector, it indicates dependence.
- **Dimension** – The number of vectors in a basis of a vector space; represents the maximum number of linearly independent vectors in the space.
- **Span of a Set** – The collection of all linear combinations of a given set of vectors; forms a subspace.
- **Redundancy in Vectors** – Occurs when a set of vectors is linearly dependent, meaning at least one vector does not add a new "direction" to the span.
- **Unique Representation** – A property of linearly independent vectors, where each vector in the span can be written uniquely as a linear combination of the basis vectors.

3.7 *REFERENCES*: -

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3.8 *SUGGESTED READING*: -

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- K.P.Gupta (20th Edition,2019), Pragati Publication, Linear Algebra
- Kuldeep Singh (2020) — *Linear Algebra: Step by Step* (Oxford Uni. Press, distributed by Dev Publishers, India)

3.9 *TERMINAL QUESTIONS: -*

(TQ-1) Prove that the set of non-zero vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ of $V\ (F)$ is linearly dependent ⇔ one of these vectors is a linear combination of the remaining $(n-1)$ vectors.

(TQ-2) Prove that any subset of a linearly independent set of vectors is also linearly independent.

(TQ-3) In the vector space $F[x]$ of all polynomials over the field F the infinite set $S = \{1, x, x^2, x^3, \dots\}$ is linearly independent.

(TQ-4) If α, β, γ are linearly independent vectors of $V(F)$ $where\ F$ is the field of complex numbers, then so also are

$$\alpha + \beta, \alpha - \beta, \alpha - 2\beta + \gamma$$

(TQ-5) Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors, $(2, -1, 3, 2), (-1, 1, 1, -3)$ and $(1,1,9,-5)$?

(TQ-6) Determine whether the vectors (1,2,3), (2,4,6), and (3,6,9) in $\mathbb{R}^3$are linearly independent.

(TQ-7) Check if the vectors (1,0,1), (0,1,1), and (1,1,2) are linearly independent in $\mathbb{R}^3$.

(TQ-8) Find the values of a for which the vectors (1,a,0), (0,1,a), and (a,0,1) are linearly dependent.

(TQ-9) Show that the set $\{1, x, x^2\}$is linearly independent in the vector space of polynomials P_2.

(TQ-10) Determine whether the vectors (1,2), (3,6) are linearly independent in $\mathbb{R}^3$.

(TQ-11) Show that in $\mathbb{R}^3$, any set of four vectors must be linearly dependent.

(TQ-12) Prove that if a subset S of a vector space is linearly independent, then every subset of S is also linearly independent.

(TQ-13) Show that the set $\{1, x, 1 + x + x^2\}$ is a linearly independent set of vectors in the vector space of all polynomials over the real number field.

(TQ-14) If α, β, γ are linearly dependent vectors of $V(F)$ where F is any subfield of the field of complex numbers then prove that also are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$.

(TQ-15) In the vector space $V_n(F)$, then prove that the system of n vectors

$$e_1 = (1,0,0, \dots ,0), e_2 = (0,1,0, \dots ,0), \dots , e_n = (0,0,0, \dots ,0,1)$$

is linearly independent where 1 denotes the unity of the field F.

(TQ-16) Determine all values of k for which the vectors $(1, k, 1), (2,1,3), (1,1,2)$ are linearly dependent.

UNIT 4: -Basis

CONTENTS:

4.1 *INTRODUCTION: -*

In linear algebra, a **basis** of a vector space is a set of vectors that is both **linearly independent** and **spans the entire space**. This means that every vector in the space can be uniquely expressed as a linear combination of the basis vectors. The concept of a basis provides the simplest and most efficient way to describe a vector space, removing redundancy and ensuring uniqueness of representation.

The number of vectors in a basis is called the **dimension** of the vector space, which serves as a measure of its size or degrees of freedom. Choosing an appropriate basis is fundamental for simplifying problems in vector spaces, such as solving systems of equations, representing linear transformations, and performing computations in geometry and applied mathematics.

4.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- Define Basis of a Vector Space.
- Understand Finite Dimensional Vector Space.
- Solve Dimension of a Vector Space and Dimension of a Subspace.

4.3 *BASIS OF A VECTOR SPACE: -*

A basis of $V\ (F)$ is a subset S of a vector space $V\ (F)$, if

i. The vectors in S are linearly independent.
ii. $V(F)$ is generated by S, $i.e. L(S) = V\ i.e.$, where each vector in V is a linear combination of a finite number of S.

Example1. A system S consisting of n vectors $e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0),\dots, e_n = (0,0,0,\dots,0,1)$ is a basis of $V\ (F)$.

Solution: We should first show that S is a set of vectors that are linearly independent.

Let $a_1, a_2, \dots, a_n$ be any scalars, then

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = 0$$

$$\Rightarrow \quad a_1(1,0,0,\dots,0) + a_2(0,1,0,\dots,0) + \cdots + a_n(0,0,0,\dots,0,1) = 0$$

$$\Rightarrow \quad (a_1, a_2, \dots, a_n) = (0,0,\dots,0)$$

$$\Rightarrow \quad a_1 = 0, a_2 = 0, \dots, a_n = 0$$

Therefore the given set of n vectors is linearly independent.
The next step is to prove that $L(S) = V_n(F)$. Every time, we have $L(S) \subseteq V_n(F)$. We have to prove that $V_n(F) \subseteq L(S)$, that is, that every vector in $V_n(F)$ is a linear combination of $S's$ elements.
Let $\alpha = (a_1, a_2, \dots, a_n)$ be any vector in $V_n(F)$. We can write

$$(a_1, a_2, \dots, a_n) = a_1(1,0,\dots,0) + a_2(0,1,0,\dots,0) + \cdots$$

$$+a_n(0,0,\dots,0,1)$$

$i.e.$ $\qquad \alpha = a_1e_1 + a_2e_2 + \cdots + a_ne_n$

Hence S is a basis of $V(F)$. We shall call this particular basis the standard basis of $V_n(F)$.

4.4 *FINITE DIMENSIONAL VECTOR SPACE: -*

If there is a finite subset $S\ of\ V$ such that $V = L\ (S)$, then the vector space $V(F)$ is said to be finite dimensional or finitely generated.
$V_n(F)$, the vector space of $n - tuples$, is a vector space with finite dimensions.
There are no finite dimensions in the vector space $F\ [x]$ of all polynomials over a field F. It is impossible to find a finite subset $S\ of\ F\ [x]$ that spans $F\ [x]$. It is possible to refer to a vector space that is not finitely created as an infinite dimensional space. For all polynomials over a field, the vector space $F\ [x]$ is hence infinitely dimensional.

Theorem1. Existence of basis of a finite dimensional vector space: There exists a basis for each finite dimensional vector space.

Proof: Let $V(F)$ be a finitely generated vector space. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be a finite subset of such that $L(S) = V$. We can assume that S has no zero members.

If S is linearly independent, then S itself is a basis of V. If S is linearly dependent, then there exists a vector $\alpha_i \in S$ which may be written as a linear combination of the preceding vectors $\alpha_1, \alpha_2, \dots, \alpha_{i-1}$.

If we omit this vector $\alpha_i \in S$, then the remaining set S' of $m-1$ vectors $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m$ also generates V $i.e., V = L(S')$. For if α is any element of V, then $L(S) = V$ implies that α can be written as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_m$ Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + a_i\alpha_i + a_{i+1}\alpha_{i+1} + \cdots + a_n\alpha_n$ But α_i can be expressed as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_{i-1}$. Let $\alpha_i = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1}$. Putting this value of α_i in the expression for α, we get

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + a_i(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1}) +$$

$$a_{i+1}\alpha_{i+1} + \cdots + a_n\alpha_n$$

Thus α has been expressed as a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m$. In this way $\alpha \in V \Rightarrow \alpha$ can be expressed as a linear combination of the vectors belonging to the set S'. Thus S' generates $V i.e., L(S') = V$.

S' is a basis of V if it is linearly independent Following the preceding procedure, we will obtain a new set of $n-2$ vectors that produce V if S' is linearly dependent. If we keep doing this, we will eventually get a linearly independent subset of S that generates and is so a basis of V after a limited number of steps.
The most likely scenario is that we will be left with a subset of S that span V and has just one non-zero vector. A set with a single non-zero vector will constitute a basis of V since we know that it is absolutely linearly independent.

Theorem2. Dimension theorem for vector space. If $V(F)$ is a finite dimensional vector space, then any two bases of V have the same number of elements.

Proof: Assume that $V(F)$ is a finite dimensional vector space. Then V definitely possesses a basis. Let $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2, \dots, \beta_n\}$ be two bases of V. We shall prove that $m = n$.

Since $V = L(S_1)$ and $\beta_1 \in V$, therefore β_1 can be expressed as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$. Consequently, the set $S_3 = \{\beta_1, \alpha_1, \alpha_2, \dots, \alpha_n\}$, which also obviously generates $V(F)$ is linearly dependent. Therefore, there exists a member $\alpha_i \neq \beta_i$, of this set S_3, such that α_i, is a linear combination of the preceding vectors $\beta_1, \alpha_1, \alpha_2, \dots, \alpha_{i-1}$. if we omit the vector α_i from S_3 then V is also generated by the remaining set

$$S_4 = \{\beta_1, \alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m\}$$

Since $V = L(S_4)$ and $\beta_2 \in V$ therefore β_2 can be expressed as a linear combination of the vectors belonging to S_4. Consequently, the set

$$S_5 = \{\beta_2, \beta_1, \alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m\}$$

is linearly dependent. Therefore, there exists a member α_j of this set S_5, such that α_j is a linear combination of the preceding vectors. Obviously α_j will be different from $\beta_1 and\ \beta_2$ since $\{\beta_1, \beta_2\}$ is a linearly independent set. If we exclude the vector α_j from S_5. then the remaining set will generate $V\ (F)$.

We may continue to proceed in this manner. Here each step consists in the exclusion of an α and the inclusion of a β in the set S_1.

Obviously, the set S_1 of $\alpha's$ cannot be exhausted before the set S_2 of $\beta's$ otherwise $V\ (F)$ will be a linear span of a proper subset of S_2, and thus S_2, will become linearly dependent. Therefore, we must have

$$m \nless n$$

Interchanging the role of $S_1\ and\ S_2$, we shall get that

$$n \nless m$$

$Hence$ $$m = n$$

4.5 *DIMENSION OF A FINITELY GENERATED VECTOR SPACE*: -

The dimension of a finite dimensional vector space $V\ (F)$ is the number of elements in any basis of that space, and it is represented by $dim\ V$.

The vector space $V_n(F)$, is of dimension n. The vector space $V_3(F)$, is of dimension 3.

Theorem3. Extension theorem: Every linearly independent subset of a finitely generated vector space $V\ (F)$ forms a part of a basis of V.

Or

Every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis of V or can be extended to form a basis of V.

Proof: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a subset of a finite dimensional vector space $V\ (F)$ that is linearly independent. V has a finite basis, say $\{\beta_1, \beta_2, \ldots, \beta_n\}$, if $dim\, V = n$. Consider the set

$$S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n\}$$

Obviously $L(S_1) = V$. Since the $\alpha's$ can be expressed as linear combinations of the $\beta's$ therefore the set S_1 is linearly dependent.

Consequently, a certain vector of S_1 is a linear combination of its previous vectors. Due to the linear independence of the $\alpha's$, this vector cannot be any of them. Consequently, this vector needs to be some β, let's say β_i. Consider the set after removing the vector β_i. from S_1.

$$S_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_m,\ \beta_1, \beta_2, \ldots,\ \beta_{i-1},\ \beta_{i+1}, \ldots, \beta_n\}$$

Obviously $L(S_2) = V$. S_2 is the necessary extended set that is a basis of V and will be a basis of V if it is linearly independent. The preceding procedure can be repeated a limited number of times to obtain a linearly independent set that contains $\alpha_1, \alpha_2, \ldots, \alpha_m$and spans V if S_2 is not linearly independent. This collection will contain S and will be a basis for V. Exactly $n - m$ elements of the set of $\beta's$ will be adjacent to S to form a basis of V since every basis of V has the same number of elements.

Theorem4. Each set of $(n + 1)$ or more vectors of a finite dimensional vector space $V\ (F)$ of dimension n are linearly dependent.

Proof: Let $V\ (F)$ be a vector space of dimension n that has a finite dimension. Let S be a subset of V that contains $(n + 1)$ or more vectors and is linearly independent. S will then be a component of $V's$ basis. As a result, a basis of V with more than n vectors will be obtained. However, there will be exactly n vectors in each basis of V. Thus, our "assumption" is incorrect. S must therefore be linearly dependent if it consists of $(n + 1)$ or more vectors.

Theorem5. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then any linearly independent set of vectors in V is finite and contains no more than m vectors.

Proof: Let $S = \{\beta_1, \beta_2, \ldots, \beta_m\}$

V has a finite basis and $dim\, V \leq m$ since $L\,(S) = V$. Because of this, all subsets $S'\; of\; V$ that have more than m vectors are linearly dependent. The theorem is so proved.

Theorem6. If $V(F)$ is a finite dimensional vector space of dimension n, then any set of n linearly independent vectors in V forms a basis of V.

Proof: Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a linearly independent subset of a vector space $V(F)$ of dimension n, which has a finite dimension. It is possible to extend S to form a basis of V if it is not a basis of V. As a result, a basis of V with more than n vectors will be obtained. However, there must be exactly n vectors in each basis of V. As a result, S must be a basis of V and our assumption is incorrect.

Theorem7. If a set S of n vectors of a finite dimensional vector space $V(F)$ of dimension n generates $V(F)$, then S is a basis of V.

Proof: Let $V\,(F)$ be a vector space of dimension n that has a finite dimension. Consider a subset of $V, S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, such that $L(S) = V$. S is a basis of V if it is linearly independent. There will be a proper subset of S that forms a basis of $V\; if\; S$ is not linearly independent. Consequently, we will have a basis of V with less than n elements. However, there must be exactly n elements in each basis of $V. S$ must therefore be the basis of V as it cannot be linearly dependent.

4.6 *DIMENSION OF A SUBSPACE*: -

Theorem8. Every subspace W of a finite dimensional vector space $V(F)$ of dimension n is a finite dimensional space with $\dim m \leq n$.

$$Also \qquad V = W\; iff\; \dim V = \dim W$$

Proof: Let $V(F)$ be a vector space of $dim\, n$ with finite dimensions. Assume that V has a subspace W. Any $(n+1)$ vectors in V are linearly dependent, and any subset of W with $(n+1)$ or more vectors is also a subset of V. Consequently, there can be a maximum of n vectors in any linearly independent set of vectors in W. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ be a linearly independent subset of W with maximum number of elements. We claim that S is a basis of W. The proof is as follows:

(i) S is a linearly independent subset of W.

(ii) $L(S) = W$

Let α be any of $W's$ elements. We assume that the biggest independent subset of W contains m vectors, so the $(m+1)$ vectors $\alpha, \alpha_1, \alpha_2, \dots, \alpha_m$ belong to W are linearly dependent. At this point, the set $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is linearly dependent. As a result, it has a vector that can be represented as a linear combination of the vectors that came before it. This vector cannot be any of these m vectors as $\alpha_1, \alpha_2, \dots, \alpha_m$ are linearly independent. It must therefore be α itself. Thus α can be expressed as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_m$. Hence $L(S) = W$.

∴ $$S \text{ is a basis of } W.$$

∴ $$dim\, W = m \text{ and } m \leq n$$

Now if $V = W$, then every basis of $V$is also a basis of W. Hence $dim\, V = dim\, W = n$.

Conversely let $dim\, W = dim\, V = n$. Then to prove that $W = V$.

Let S be one of $W's$ bases. If S includes n vectors, then $L(S) = W$. Given that S is a subset of V and contains n linearly independent vectors, S is also a basis of V, Therefore $L(S) = V$. Hence $W = V$. We thus conclude:

If W is a proper subspace of a finite-dimensional vector space V then W is finite dimensional and $dim\, W < dim\, V$.

Theorem9. If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Proof: Let dim $V = n$. Assume that V has a subspace W. Let S_0 be a subset of W that is linearly independent. Let S be a linearly independent subset of W that has the greatest number of elements and contains S_0. Consequently, S is a linearly independent subset of V as well. Thus, S will have no more than n elements. S is finite as a result, and S_0 is also finite. We are now claiming that S is a basis of W. The following is the proof:

(i) S is a linearly independent subset of W.

(ii) L(S) = W

Because β must be in the linear span of S if it is in W. The subset of W those results from adjoining beta to S will be linearly independent if β is not in the linear span of S. Consequently, S will no longer be the largest linearly independent subset of W that contains S_0. Thus, $\beta \in W \Rightarrow \beta \in L(S)$. Consequently $W \subseteq L\,(S)$. $L(S) \subseteq W$ since S is a subset of the subspace W.

Hence $L(S) = W$

Thus S is finite basis of W and $S_0 \subseteq S$.

Theorem10. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, be a basis of a finite dimensional vector space $V(F)$ of dimension n. Then every element a of V can be uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n;\ a_1, a_2, \dots, a_n \in F$

Proof: Since S is a basis of V , therefore $L(S) = V$ Therefore any vector $\alpha \in W$ can be expressed as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

To show uniqueness let us suppose that

$$\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$$

Then we must show that $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$

We have

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$$

$$\Rightarrow \qquad (a_1 - b_1)\alpha_1 + (a_2 - b_2)\alpha_2 + \cdots + (a_n - b_n)\alpha_n = 0$$

$$\Rightarrow \qquad a_1 - b_1 = 0, a_2 - b_2 = 0, \dots, a_n - b_n = 0$$

$\because$ $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent.

$$\Rightarrow \qquad a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$$

Theorem11. If $W_1\ and\ W_2$ are two subspaces of a finite dimensional vector space $V(F)$, then

$$dim\,(W_1 + W_2) = dim\,W_1 + dim\,W_2 + dim\,(W_1 \cup W_2)$$

Proof: Let $dim\,(W_1 \cap W_2) = k$ and let the set $S = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be a basis of $W_1 \cap W_2$. Then $S \subseteq W_1\ and\ S \subseteq W_2$.

Since S is linearly independent and $S \subseteq W_1$, therefore S can be extended to form a basis of W_1. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_m\}$ be a basis of W_1. Then $dim\, W_1 = k + m$. Similarly let $\{\gamma_1, \gamma_2, \dots, \gamma_k, \beta_1, \beta_2, \dots, \beta_t\}$ be a basis of W_2. Then $dim\, W_2 = k + t$.

$$\therefore \quad \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (m + k) + (k + t) - k$$
$$= k + m + t$$

Therefore, to prove the theorem we must show that

$$\dim(W_1 + W_2) = k + m + t$$

We claim that the set $S_1 = \{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_t\}$ is a basis of $W_1 + W_2$.

Firstly, we have to show that S_1 is linearly independent. Let

$$c_1\gamma_1 + c_2\gamma_2 + \dots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t = 0 \quad \dots (1)$$

$$\Rightarrow b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t = -(c_1\gamma_1 + c_2\gamma_2 + \dots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m) \quad \dots (2)$$

Now $\quad -(c_1\gamma_1 + c_2\gamma_2 + \dots + c_n\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m) \in W_1$ since it is a linear combination of a basis of W_1. Again

$$b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_2$$

Since it is a linear combination of elements belonging to a basis of W_2.

Also, by virtue of the equality (2), $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_1$. Therefore $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_1 \cap W_2$. Therefore it can be expressed as a liner combination of the basis of $W_1 \cap W_2$. Thus we have a relation of the form

$$b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + \dots + d_k\gamma_k$$
$$\Rightarrow \quad b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 - \dots - d_k\gamma_k = 0$$

But $\beta_1, \beta_2, \dots, \beta_t, \gamma_1, \gamma_2, \dots, \gamma_k$ are linearly independent vectors. Therefore we must have

$$b_1 = 0, b_2 = 0, \dots, b_t = 0$$

Putting these values in equation (1), we get

$$c_1\gamma_1 + c_2\gamma_2 + \dots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = 0$$
$$\Rightarrow \quad c_1 = 0, c_2 = 0, \dots, c_k = 0, a_1 = 0, a_2 = 0, \dots, a_m = 0$$

Since the vectors $\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_m$ are linearly independent. Thus, the relation (1) implies

$$c_1 = 0, \dots, c_k = 0, a_1 = 0, \dots, a_m = 0, b_1 = 0, \dots, b_t = 0$$

Therefore, the set S_1 is linearly independent.

Now to show that $L(S_1) = W_1 + W_2$

Since $W_1 + W_2$ is a subspace of V and each elements of $S_1 \in W_1 + W_2$, therefore $L(S_1) \subseteq W_1 + W_2$.

Again let α be any element of $W_1 + W_2$. Then

$$\alpha = some\ element\ of\ W_1 + some\ element\ of\ W_2$$
$$= a\ linear\ combination\ of\ elements\ of\ basis\ of\ W_1 + a\ linear\ combination\ of\ elements\ of\ basis\ of\ W_2$$
$$= a\ linear\ combination\ of\ elements\ of\ basis\ of\ S_1$$

∴ $\alpha \in L(S_1). hence W_1 + W_2 \subseteq L(S_1)$

∴ $L(S_1) = W_1 + W_2$

∴ $S_1\ is\ a\ basis\ of\ W_1 + W_2 and\ consequently$

$$\dim(W_1 + W_2) = k + m + t$$

Hence the theorem proved.

Example2. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

Solution: Let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\ and\ \delta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ four elements of V.

The subset $S = \{\alpha, \beta, \gamma, \delta\}\ of\ V$ is linearly independent because

$$a\alpha + b\beta + c\gamma + d\delta = 0$$

⇒ $$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

⇒ $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

⇒ $$a = 0, b = 0, c = 0, d = 0$$

Also $L(S) = V$ because if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$is any vector in V, then we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\alpha + b\beta + c\gamma + d\delta$$

Therefore S is a basis of V. since no. of elements in $S\ is\ 4$, therefore $dim\ V = 4$.

Example3. If $W_1\ and\ W_2$ are finite-dimensional subspaces with the same dimension, and if $W_1 \subseteq W_2, then\ W_1 = W_2$.

Solution: Since $W_1 \subseteq W_2$, therefore W_1 is also a subspace of W_2. Now $dim\ W_1 = dim\ W_2$. Therefore we must have $W_1 = W_2$.

Example4. Let V be the vector space of ordered pairs of complex numbers over the real field R i.e., let V be the vector space $C(R)$. Show that the set $S = \{(1,0), (i, 0), (0,1), (0, i)\}$ is a basis for V.

Solution: S is linearly independent. We have

$$a\ (1,0) + b\ (i, 0) + c\ (0,1) + d\ (0, i) = (0,0)$$

⇒ $$(a + ib, c + id) = (0,0)$$

⇒ $$a + ib = 0, c + id = 0$$

⇒ $$a = 0, b = 0, c = 0, d = 0.$$

Therefore S is linearly independent. Where $a, b, c, d \in R$

Now we shall show that $L(S) = V$. Let any ordered pair $(a + ib, c + id) \in V\ where\ a, b, c, d \in R$. Then as shown above we can write $(a + ib, c + id) = a\ (1,0) + b\ (i, 0) + c\ (0,1) + d\ (0, i)$. Thus any vector in V is expressible as a linear combination of elements of S. Therefore $L(S) = V$ and so S is a basis for V.

Example5. Show that a system X consisting of the vectors $\alpha_1 = (1,0,0,0), \alpha_2 = (0,1,0,0), \alpha_3 = (0,0,1,0) and\ \alpha_4 = (0,0,0,1)$ is a basis set of $R^4(R)$.

Solution: First we show that the set X is a linearly independent set of vectors.

If a_1, a_2, a_3, a_4 be any scalars i.e, elements of the field R, then

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = \ zero\ vector$$

$$\Rightarrow\ a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) + a_4(0,0,0,1) = (0,0,0,0)$$

$$\Rightarrow\ (a_1, a_2, a_3, a_4) = (0,0,0,0)$$

$$\Rightarrow\ a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$$

Therefore the given set X of four vectors is linearly independent. Now we shall show that X generates R^4 i.e., each vector of R^4 can be expressed as a linear combination of the vectors of X.

Let (a, b, c, d) be any vector inR^4. We can write

$$(a, b, c, d) = a\,(1,0,0,0) + b(0,1,0,0) + c\,(0,0,1,0)\ + d\,(0,0,0,1)$$
$$= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$$

Thus (a, b, c, d) has been expressed as a linear combination of the vectors of X and so X generates R^4.

Since X is a linearly independent subset of R^4 and it also generates R^4, therefore it is a basis of R^4.

4.7 *SUMMARY*: -

In this unit we have studied the vector space's basis is a linearly independent set of vectors that spans the space and offers a minimum and comprehensive representation of all of its elements. We have examined this idea in this unit. We investigated finite-dimensional vector spaces, in which the number of vectors in the basis is finite, and we proved that the dimension of the space is determined by the number of vectors in any basis. We also looked at the dimension of a finitely produced vector space, highlighting how it is dependent on the size of the generating basis. We then applied this concept to the study of a subspace, demonstrating that the subspace inherits a dimension that is smaller than or equal to the parent spaces.

4.8 *GLOSSARY*: -

- **Basis** – A set of linearly independent vectors that spans a vector space; provides the simplest building blocks to represent all vectors in the space.

- **Spanning Set** – A set of vectors whose linear combinations can generate the entire vector space. Every basis is a spanning set but with the extra condition of linear independence**.**
- **Finite Basis** – A basis consisting of a finite number of vectors. A vector space with such a basis is called a finite-dimensional vector space.
- **Infinite Basis** – A basis with infinitely many vectors, associated with infinite-dimensional vector spaces (e.g., function spaces).
- **Cardinality of Basis** – The number of vectors in a basis; defines the **dimension** of the vector space.
- **Standard Basis** – A commonly used basis for Euclidean spaces (e.g., for $\mathbb{R}^n$, the standard basis is {(1, 0…,0), (0,1,0,…,0),…,(0,0,…,1)}.
- **Ordered Basis** – A basis in which the order of vectors matters, important for defining coordinates of vectors uniquely.
- **Change of Basis** – The process of converting coordinates of a vector from one basis to another, often represented using a change-of-basis matrix.
- **Orthonormal Basis** – A basis where vectors are orthogonal (perpendicular) and normalized (length 1). Useful in inner product spaces.
- **Hamel Basis** – A basis for a vector space over a field where every element can be written as a finite linear combination of basic elements.
- **Schauder Basis** – A type of basis for infinite-dimensional spaces where infinite linear combinations (series) may be used.
- **Basis Vector** – An individual vector belonging to a basis set; together with others, it helps in uniquely representing any vector in the space.
- **Coordinate Vector (relative to a basis)** – The unique list of scalars (coefficients) corresponding to the linear combination of basis vectors that represent a given vector.

4.9 *REFERENCES*: -

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4.10 *SUGGESTED READING*: -

- S. **Kumaresan (2000),** *Linear Algebra: A Geometric Approach.* Prentice-Hall of India.
- K. **B. Datta (2002),** *Matrix and Linear Algebra.* Prentice-Hall of India.
- A.R. Vasishtha, J.N.Sharma and A.K. Vasishtha (52th Edition, 2022), Krishna Publication, Linear Algebra.

4.11 *TERMINAL QUESTIONS: -*

(TQ-1) Show that the infinite set $S = \{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis of the vector space $F[x]$ of polynomials over the field F.

(TQ-2) Show that the set $S = \{1, x, x^2, \ldots, x^n\}$ of $n+1$ polynomials in x is a basis of the vector space $P_n(R)$, of all polynomials in x $(of\ degree\ at\ most\ n)$ over the field of real numbers.

(TQ-2) Show that the vectors $\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ form a basis set of R^3.

(TQ-4) Prove that any finite set S of vectors, not all zero vectors, contains a linearly independent subset T which spans the same space as S.

(TQ-5) If $W_1\ and\ W_2$ are finite-dimensional subspaces with the same dimension, and if $W_1 \subseteq W_2, then\ W_1 = W_2$.

(TQ-6) If $W_1\ and\ W_2$ are two subspaces of a finite dimensional vector space $V(F)$, then

$$dim\,(W_1 + W_2) = \ dim\, W_1 + dim\, W_2 + dim\,(W_1 \cup W_2)$$

(TQ-7) Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, be a basis of a finite dimensional vector space $V(F)$ of dimension n. Then every element a of V can be uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n;\ a_1, a_2, \ldots, a_n \in F$

(TQ-8) Show that a system X consisting of the vectors $\alpha_1 = (1,0,0,0), \alpha_2 = (0,1,0,0), \alpha_3 = (0,0,1,0) and\ \alpha_4 = (0,0,0,1)$ is a basis set of $R^4(R)$.

BLOCK II
LINEAR TRANSFORMATION, HOMOMORPHISMAND DUAL SPACE

UNIT 5: -Linear Transformation

CONTENTS:

5.1 *INTRODUCTION*

In linear algebra, a **linear transformation** is a special type of function between two vector spaces that preserves the fundamental operations of vector addition and scalar multiplication.Linear transformations are important because they connect the abstract concept of vector spaces with concrete matrix operations. Every linear transformation can be represented by a **matrix**, and operations such as rotation, reflection, projection, and scaling in geometry are practical examples of linear transformations.

Thus, linear transformation serves as a bridge between algebraic structures and geometric interpretations, making it a fundamental tool in mathematics, physics, computer science, and engineering.

5.2 *OBJECTIVES*

After studying this unit, the learner's will be able to

- Understand the Concept of a Linear Transformation
- Learn The fundamental Properties
- Explore the Relationship.

- Apply Linear Transformations.
- Define Linear Operator and Linear Functional.

5.3 *LINEAR TRANSFORMATION*

Consider the vector spaces $U(F)$ and $V(F)$ over the same field F. A linear transformation is a function T $from$ U $into$ V such that, for all $\alpha, \beta \in U$ and $a, b \in F$,

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad ...(1)$$

The linearity property is another name for condition (1). It is clear that for any $\alpha, \beta \in U$ and for all scalars $a \in F$, the condition (1) is equal to the condition

$$T(a\alpha + \beta) = aT(\alpha) + T(\beta)$$

5.4 *LINEAR OPERATOR*

Consider a vector space $V(F)$. A linear operator on V is a function T from V such that, for all $\alpha, \beta \in U$ and $a, b \in F$,

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

If T is a linear transformation from V $into$ V itself, then T is a linear operator on V.

Example1. The function $T: V_3(R) \to V_2(R)$ defined by $T(a,b,c) = (a,b) \forall\, a,b,c \in R$ is a linear transformation from $V_3(R)$ $into$ $V_2(R)$.

Solution: Let $\alpha = (a_1, b_1, c_1), \beta = (a_2, b_2, c_2) \in V_3(R)$

If $a, b \in R$, then

$$\begin{aligned} T(a\alpha + b\beta) &= T[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)] \\ &= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \\ &= (aa_1, ba_2) + (ab_1, bb_2) \\ &= (aa_1, ab_1) + (ba_2, bb_2) \\ &= a(a_1, b_1) + b.(a\,2\,, b\,2\,) \\ &= a\,T(a_1\,, b_1\,, c_1) + bT(a_2, b_2, c_2) \\ &= aT(\alpha) + bT(\beta) \end{aligned}$$

$\therefore T$ is a linear from$V_3(R)$ $into$ $V_2(R)$.

Example2. Let V (F) be the vector space of all $m \times n$ matrices over the field F. Let P be a fixed $m \times m$ matrix over F, and let Q be a fixed $n \times n$ matrix over F. The correspondence T $from$ V $into$ V defined by $T(A) = PAQ$ $\forall A \in V$ is a linear operator on V.

Solution: If A is a $m \times n$ matrix over the field F, then PAQ is also an $m \times n$ matrix over the field F. Therefore T is a function from V $into$ V. Now let $A, B \in V$ and $a, b \in F$. Then

$$T(aA + bB) = P(aA + bB)Q\ [by\ def.of\ T]$$

$$= (aPA + bPB)Q$$
$$= a\,PAQ + b\,PBQ$$
$$= aT\,(A) + bT(B)$$

$\therefore T$ is a linear transformation from $V into\ V$. Thus T is a linear operator on V.

Example3. Let $V\ (F)$ be the vector space of all polynomials over the field F. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \ \in V$ be a polynomial of degree n in the indeterminate x. Let us define

$$Df(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \ if\ n > 1$$

$and \qquad Df(x) = 0\ if\ f(x)\ is\ a\ constant\ polynomial.$

Then the correspondence $D\ from\ V\ into\ V$ is a linear operator on V.

Solution: If $f(x)$ is a polynomial over the field F, then $Df(x)$ as defined above is also a polynomial over the field F. Thus if $f(x) \in V$, then $Df(x) \in V$. Therefore D is a function from $V\ into\ V$.

Also if $f(x), g(x) \in V$ and $a, b \in F$, then

$$D[af(x) + bg(x)] = a\ Df\ (x) + bDg(x).$$

$\therefore D$ is a linear transformation from $V into\ V$.

The operator $D\ on\ V$ is called the differentiation operator. It should be noted that for polynomials the definition of differentiation can be given purely algebraically, and does not require the usual theory of limiting processes.

Example4. Let $V(R)$ be the vector space of all continuous functions from $R\ into\ R$. If $f \in V$ and we define T by

$$(Tf)(x) = \int_0^x f(t)\,dt\ \forall x \in R$$

Then T is a linear transformation from $V\ into\ V$.

Solution: If f is real valued continuous function, then Tf, as defined above, is also a real valued continuous function. Thus

$$f \in V \Rightarrow Tf \in V.$$

Also the operation of integration satisfies the linearity property. Therefore T is a linear transformation from $V\ into\ V$.

Theorem1. Let T be a linear transformation from a vector space $U(F)$ into a vector space $V(F)$. Then

i. $T(0) = 0$; 0 on the left hand side is zero vector of $U\ and$ 0 on the right hand side is zero vector of V.
ii. $T(-\alpha) = -T(\alpha)\ \forall \alpha \in U$
iii. $T(\alpha - \beta) = T(\alpha) - T(\beta)\ \forall \alpha, \beta \in U$.
iv. $T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n) where \alpha_1, \alpha_2, \ldots, \alpha_n \in U\ and\ a_1, a_2, \ldots, a_n \in F$.

Proof:

i. Let $\alpha \in U$. Then $T(\alpha) \in V$. We have

$$T(\alpha) + 0 = T(\alpha) \qquad [\because 0\ is\ zero\ vector\ of\ V\ and\ T(\alpha) \in V]$$

$= T(\alpha + 0)$ [$\because$ 0 is zero vector of U]

$= T(\alpha) + T(0)$ [$\because T$ *is a linear transformation*]

Now in the vector space V, we have

$\Rightarrow \quad 0 = T(0)$

By left cancellation law for addition in V.

ii. We have

$T\,[\alpha + (-\alpha)] = T(\alpha) + T(-\alpha)$

[$\because T$ *is a linear transformation*]

But $T\,[\alpha + (-\alpha)] = T(0) = 0 \in V$ [*by* (*i*)]

Thus in V, we have

$$T\,[\alpha + (-\alpha)] = 0$$

$$\Rightarrow \quad T(-\alpha) = -T(\alpha)$$

iii. $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$= T(\alpha) + T(-\beta)$ [$\because T$ *is linear*]

$= T(\alpha) + [-T(\beta)]$ [*by* (*ii*)]

$= T(\alpha) - T(\beta)$.

iv. We shall prove the result by induction on n, the number of vectors in the linear combination$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$. Suppose

$T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1})$

$= a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1})$ ... (1)

Then

$T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$= T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1} + a_n\alpha_n)$

$= T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{n-1}\alpha_{n-1}) + a_nT(\alpha_n)$

$= [a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1})] + a_nT(\alpha_n)$ [*by* (1)]

$= a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_{n-1}T(\alpha_{n-1}) + a_nT(\alpha_n)$

Now the proof is complete by induction since the result is true when the number of vectors in the linear combination is 1.

5.5 *RANGE AND NULL SPACE OF LINEAR TRANSFORMATION*

Range of a linear transformation: Let T be the linear transformation from U *into* V, and let $U(F)$ *and* $V(F)$ be two vector spaces. In this case, the set of all vectors β *in* V such that $\beta = T(\alpha)$ for some α *in* U is the range of T, denoted as $R(T)$.

The image set of U *under* T is hence the range of T, i.e.

$$Range(T) = \{T(\alpha) \in V \,;\, \alpha \in U\}$$

Theorem2. If $U(F)$ and $V(F)$ are two vector spaces and T is a linear transformation from U $into$ V, then range of T is a subspace of V.

Proof: Obviously $R(T)$ is a non-empty subset of V.

Let $\beta_1, \beta_2 \in R(T)$. Then there exist vectors α_1, α_2 in U such that

$$T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2.$$

Let a, b be any elements of the field F. We have

$$a\beta_1 + b\,\beta_2 = a\,T(\alpha^1) + b\,T(\alpha_2)$$

$$= T\,(a\alpha_1 + b\alpha_2) \qquad [\because T\ is\ a\ linear\ transformation]$$

Now U is a vector space. Therefore $\alpha_1, \alpha_2 \in U$ and $a, b \in F$

$\Rightarrow$ $a\alpha_1 + b\alpha_2 \in U$.

Consequently $T(a\alpha_1 + b\alpha_2) = a\beta_1 + b\,\beta_2 \in R(T)$.

Thus $a, b \in F$ and $\beta_1, \beta_2 \in R(T)$

$\Rightarrow$ $a\beta_1 + b\,\beta_2 \in R(T)$

Therefore $R(T)$ is a subspace of V.

Null space of a linear transformation: Let T be a linear transformation from U $into$ V, and let $U(F)$ and $V(F)$ be two vector spaces. The set of all vectors α in U such that $T(\alpha) = 0$ $(zero\ vector\ of\ V)$ is then the null space of T, denoted as $N(T)$. Consequently,

$$N(T) = \{\alpha \in U: T(\alpha) = 0 \in V\}.$$

The null space of T is also known as the kernel of T if we consider the linear transformation $T\ from\ U\ into\ V$ to be a vector space homomorphism of U $into$ V.

Theorem3. If $U(F)$ and $V(F)$ are two vector spaces and T is a linear transformation from U $into$ V, then the kernel of T or the null space of T is a subspace of U.

Proof: Let $N(T) = \{\alpha \in U: T(\alpha) = 0 \in V)$.

Since $T(0) = 0 \in V$, therefore at least $0 \in N(T)$

Thus $N(T)$ is a non-empty subset of U.

Let $\alpha_1, \alpha_2 \in N(T)$ Then $T(\alpha_1) = 0$ and $T(\alpha_2) = 0$.

Let $a, b \in F$. Then $a\alpha_1 + b\alpha_2 \in U$ and

$$T\,(a\alpha_1 + b\alpha_2) = a\,T(\alpha_1) + b\,T(\alpha_2)\ [\because T\ is\ a\ linear\ transformation]$$

$$= a0 + b0 = 0 + 0 = 0 \in V.$$

$\therefore$ $a\alpha_1 + b\alpha_2 \in N(T)$

Thus $a, b \in F\ and\ \alpha_1, \alpha_2 \in N(T) \Rightarrow a\alpha_1 + b\alpha_2 \in N(T)$.

Therefore $N(T)$ is a subspace of U.

5.6 *RANK AND NULLITY OF LINEAR TRANSFORMATION*

Considering U to be finite dimensional, let T be a linear transformation from a vector space $U(F)$ onto a vector space $V(F)$. The dimension of the range of T is represented by the rank of $T, \rho(T)$, which is equal to $dim\, R(T)$. i.e.

$$\rho(T) = \dim R(T)$$

The dimension of the null space of T is represented by the nullity of T, represented by $\nu(T)$. i.e.

$$\nu(T) = \dim N(T)$$

Theorem5. Let $U\ and\ V$ be vector spaces over the field F and let T be a linear transformation from $U\ into\ V$. Suppose that U is finite dimensional. Then

$rank\ (T) + nullity\ (T) = dim\ U.$

Proof: Let N be the null space of T. Then N is a subspace of U. Since U is finite dimensional, therefore N is finite dimensional. Let $dim\, N = nullity\ (T) = k$ and let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a basis for N.

Since $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a linearly independent subset of U, therefore we can extend it to form a basis of U. Let $dim\, U = n$ and let $\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}$ be a basis for U.

The vectors$T(\alpha_1), T(\alpha_2), \dots, T(\alpha_k), T(\ \alpha_{k+1}), \dots, T(\alpha_n)$ are in range of T. We claim that $\{T(\ \alpha_{k+1}), \dots, T(\alpha_n)\}$ is a basis for the range of T.

(i) First we shall prove that the vectors

$T(\ \alpha_{k+1}), \dots, T(\alpha_n)$ span the range of T.

Let$\beta \in\ range\ of\ T$. Then there exists $\alpha \in U$ such that $T(\alpha) =\ \beta$.

Now $\quad \alpha \in U \Rightarrow \exists\ a_1, a_2, \dots, a_n \in F\ such\ that\ \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$$\Rightarrow \qquad T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$$

$$\Rightarrow \qquad \beta = a_1T(\alpha_1) + a_2T(\alpha_2) +, \dots, + a_kT(\alpha_k) + a_{k+1}T(\ \alpha_{k+1}) + \cdots + a_nT(\alpha_n)$$

$$\Rightarrow \qquad \beta = a_{k+1}T(\ \alpha_{k+1}) + \cdots + a_nT(\alpha_n)$$

$$\because\ \alpha_1, \alpha_2, \dots, \alpha_k \in N \Rightarrow T(\alpha_1) = 0, T(\alpha_2) = 0, \dots, T(\alpha_k) = 0$$

$\therefore$ The vectors $T(\ \alpha_{k+1}), \dots, T(\alpha_n)$ span the range of T.

(ii) Now we shall show that the vectors $T(\ \alpha_{k+1}), \dots, T(\alpha_n)$ are linearly independent.

Let $c_{k+1}, \dots, c_n \in F$ such that

$$c_{k+1}T(\ \alpha_{k+1}) + \cdots + c_nT(\alpha_n) = 0$$

$$\Rightarrow \qquad T[c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n] = 0$$

$$\Rightarrow \qquad c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in null\ space\ of\ T\ i.e. N$$

$\Rightarrow \qquad c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n = b_1\alpha_1 + \cdots + b_k\alpha_k$

For some$b_1, \ldots, b_k \in F$

[$\because$ each vector in N can be expressed as a linear combination of the vectors$\alpha_1, \alpha_2, \ldots, \alpha_k$ forming a basis of N]

$\Rightarrow$The vectors $T(\alpha_{k+1}), \ldots, T(\alpha_n)$ are linearly independent.

$\therefore$The vectors $T(\alpha_{k+1}), \ldots, T(\alpha_n)$ form a basis of range of T.

$\therefore \qquad rank\ T = \dim of\ range\ of\ T = n - k.$

$\therefore \qquad rank\ (T) + nullity\ (T) = (n-k) + k = n = dim\ U.$

Example5. Show that the mapping $T: V_2(R) \to V_3(R)$ defined as

$$T(a,b) = (a+b, a-b, b)$$

is a linear transformation from$V_2(R) into\ V_3(R)$. Find the range, rank, null-space and nullity of T.

Solution: Let $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in V_2(R)$.

Then $T(\alpha) = (a_1 + b_1, a_1 - b_1, b_1)\ and\ T(\beta) = (a_2 + b_2, a_2 - b_2, b_2)$.

Also let $a, b \in R$. Then $a\alpha + b\beta \in V_2(R)$ and

$$\begin{aligned} T(a\alpha + b\beta) &= T[a\ (a_1, b_1) + b\ (a_2, b_2)] \\ &= T\ (aa_1 + ba_2, ab_1 + bb_2) \\ &= (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 \\ &\quad + bb_2) \\ &= (a(a_1 + b_1) + b(a_2 + b_2), a\ (a_2 + b_2) + b(a_2 - b_2), ab_1 \\ &\quad + bb_2) \\ &= a\ (a_1 + b_1, a_1 - b_1, b_1) + b\ (a_2 + b_2, a_2 - b_2, b_2) \\ &= aT(\alpha) + bT(\beta) \end{aligned}$$

$\therefore T$ is a linear transformation from $V_2(R) into\ V_3(R)$.

Now $\{(1,0), (0,1)\}$is a basis for $V_2(R)$. We have

$$T(1,0) = (1+0, 1-0, 0) = (1,1,0)$$

$$and \qquad T(0,1) = (0+1, 0-1, 0) = (1,-1,1)$$

The vectors $T(1,0), T(0,1)$ span the range of T. Thus the range of T is the subspace of $V_3(R)$ spanned by the vectors $(1,1,0), (1,-1,1)$.

Now the vectors $(1,1,0), (1,-1,1) \in V_3(R)$ are linearly independent because if $x, y \in R$, then

$$x\ (1,1,0) + y(1,-1,1) = (0,0,0)$$

$$(x+y, x-y, y) = (0,0,0)$$

$$\Rightarrow \qquad x + y = 0, x - y = 0, y = 0$$

$$\Rightarrow \qquad x = 0, y = 0.$$

$\therefore$The vectors $(1,1,0), (1,-1,1)$ form a basis for range of T.

Hence $rank\ T = dim\ of\ range\ of\ T = 2$.

Nullity of $T = dim\ of\ V^2(R) - rank\ T = 2 - 2 = 0$.

Null space of T must be the zero subspace of $V_2(R)$.

Otherwise, $(a,b) \in\ null\ space\ of\ T$

$$\Rightarrow \qquad T(a,b) = (0,0,0)$$

$$\Rightarrow \qquad (a+b, a-b, b) = (0,0,0)$$

$$\Rightarrow \qquad a + b = 0, a - b = 0, b = 0$$

$$\Rightarrow \qquad a = 0, b = 0.$$

$\therefore$ $(0, 0)$ is the only element of $V_2(R)$ which belongs to null space of T.
Null space of T is the zero subspace of $V_2(R)$.

Example6. Consider the basis $S = \{\alpha_1, \alpha_2, \alpha_3\}$ of R^3 where $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1,0,0)$. Express $(2, -3, 5)$ in terms of the basis$\alpha_1, \alpha_2, \alpha_3$.

Let $T: R^3 \to R^2$ be defined as
$T(\alpha_1) = (1, 0), T(\alpha_2) = (2, -1), T(\alpha_3) = (4,3)$. Find $T(2, -3, 5)$.
Solution: Let $(2, -3, 5) = a\alpha_1 + b\alpha_2 + c\alpha_3$
$= a\,(1, 1, 1) + b(1, 1, 0) + c(1, 0, 0)$.
Then $a + b + c = 2, a + b = -3, a = 5$.
Solving these equations, we get $a = 5, b = -8, c = 5$.
$\therefore (2, -3, 5) = 5\alpha_1 - 8\alpha_2 + 5\alpha_3$.
Now $T(2, -3, 5) = T(5\alpha_1 - 8\alpha_2 + 5\alpha_3)$
$= 5T(\alpha_1) - 8T(\alpha_2) + 5T\,(\alpha_3)$
$[\because\ T\ is\ a\ linear\ transformation]$
$= 5(1, 0) - 8\,(2, -1) + 5\,(4, 3)$
$= (5,0) - (16, -8) + (20,15)$
$= (9, 23)$.

5.7 *ALGEBRA OF LINEAR TRANSFORMATION*

Let V be a vector space over a field F. The set of all linear transformations from V to itself is denoted by

$$L(V) = \{T: V \to V \mid T\ is\ linear\}.$$

This set becomes an algebra (called the algebra of linear transformations) when we define the following operations:

1. **Addition:**
   For $T_1, T_2 \in L(V) and\ v \in V$

$$(T_1 + T_2)(v) = T_1(v) + T_2(v)$$

2. **Scalar Multiplication:**
   For $a \in F, and\ T \in L(V)$

$$(aT)(v) = a \cdot T(v)$$

3. **Multiplication (Composition):**
   For $T_1, T_2 \in L(V) and\ v \in V$

$$(T_1T_2)(v) = T_1\big(T_2(v)\big)$$

These operations satisfy the properties of algebra over F:

- Under addition and scalar multiplication, $L(V)$ is a vector space.
- Under multiplication (composition), $L(V)$ is closed and associative.
- Multiplication distributes over addition.

5.8 *LINEAR FUNCTIONAL*

Consider a vector space $V\ (F)$. We know that the field F is a vector space over F. This is the vector space $F(F)$. We'll just write it as F. A linear functional on V is a linear transformation from $V into\ F$. We will now define a linear functional independently.
Consider a vector space V (F). A linear functional on V is defined as a function f from V into F if

$$f(a\alpha + b\beta) = af(\alpha) + bf(\beta) \forall\, a, b \in F\ and\ \alpha, \beta \in V$$

for every α in V, $f\ (\alpha)$ is in F if f is a linear functional on $V\ (F)$. A linear function on V is a scalar valued function since $f(\alpha)$ is a scalar.
Example7. Let $V_n(F)$ be the vector space of ordered n-tuples of the elements of the field F.
Let $x_1, x_2, \dots, x_n$, be n field elements of F. If

$$\alpha = (a_1, a_2, \dots, a_n) \in V(F)$$

Let f be a function from $V_n(F)\ into\ F$ defined by

$$f(x) = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

Then prove that f is a functional on $V_n(F)$.
Solution: Let $\beta = (b_1, b_2, \dots, b_n) \in V_n(F)$. If $a, b \in F$, we have

$$\begin{aligned} f\ (a\alpha + b\beta) &= f\ [a\ (a_1, a_2, \dots, a_n) + b\ (b_1, b_2, \dots, b_n)] \\ &= f(aa_1 + bb_1, \dots, aa_n + bb_n) \\ &= x_1(aa_1 + bb_1), \dots, x_n\ (aa_n + bb_n) \\ &= a(x_1a_1 + \cdots + x_na_n) + b\ (x_1b_1 + \cdots + x_nb_n) \\ &= af\ (a_1, a_2, \dots, a_n) + bf\ (b_1, b_2, \dots, b_n) \\ &= af(\alpha) + bf\ (\beta) \end{aligned}$$

$\therefore f\ is\ a\ linear\ functional\ on\ V_n(F).$

Example8. Prove that the trace function is a linear functional on the space of all $n \times n$ matrices over a field F.
Solution: Let n be a positive integer and F be a field. Let $V\ (F)$ be the vector space of all $n \times n$ matrices over F. If $A = [a_{ij}]_{n\times n} \in V$, then the trace of A is the scalar

$$tr\ A = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Thus, the trace of A is the scalar obtained by adding the elements of A lying along the principal diagonal.

The trace function is a linear functional on V because if $a, b \in F$ and $A = [a_{ij}]_{n\times n}, B = [b_{ij}]_{n\times n} \in V$, then

$$tr\ (aA + bB) = tr\ \left(a\left[a_{ij}\right]_{n\times n} + b\left[b_{ij}\right]_{n\times n}\right)$$

$$= tr\ \left(\left[aa_{ij} + bb_{ij}\right]_{n\times n}\right) = \sum_{i=1}^{n}(aa_{ii} + bb_{ii})$$

$$= a\sum_{i=1}^{n} a_{ii} + b\sum_{i=1}^{n} b_{ii}$$

$$= a\ (tr\ A) + b\ (tr\ B).$$

⇒ The trace function is a linear functional.

Theorem6. Let f be a linear functional on a vector space $V\ (F)$. Then

(i) $f(0) = 0$ Where 0 on the left-hand side is zero vector of V, and 0 on the right-hand side is zero element of F.

(ii) $f(-\alpha) = -f(\alpha)\ \forall\ \alpha \in V$.

Proof:(i) Let $\alpha \in V$. Then $f(\alpha) \in F$. We have

$f(\alpha) + 0 = f(\alpha)$ $[\because 0\ is\ zero\ element\ of\ F\ ]$

$= f(\alpha + 0)$ [∵ 0 is zero element of V]

$= f(\alpha) + f(0)$ $[\because f\ is\ a\ linear\ functional]$

Now F is a field, we have

$f(\alpha) + 0 = f(\alpha) + f(0)$

⇒ $f(0) = 0$

By left cancellation law for addition in F.

(ii) We have

$f\ [\alpha + (-\alpha)] = f(\alpha) + f(-\alpha)$

$[\because f\ is\ a\ linear\ functional]$

But $f\ [\alpha + (-\alpha)] = f(0) = 0$ $[by\ (i)]$

Thus in F, we have

$$f(\alpha) + f(-\alpha) = 0$$

⇒ $$f(-\alpha) = -f(\alpha)$$

5.9 *SUMMARY*

In this unit, we studied the concept of **Linear Transformation**, which is a mapping between vector spaces that preserves vector addition and scalar multiplication. We explored **Linear Operators** (transformations from a vector space to itself), the **Range (Image)** and **Null Space (Kernel)** of a transformation, and the important **Rank-Nullity Theorem**, which relates the dimension of the range and null space to the dimension of the domain. We also discussed the **Algebra of Linear Transformations**, covering their addition, scalar multiplication, and composition, as well as the notion of **Linear Functionals**, which are special linear transformations mapping vectors to scalars. Together, these topics provide a foundation for understanding the structure and behavior of linear systems in abstract and applied mathematics.

5.10 *GLOSSARY*

- **Linear Mapping:** Another term for a linear transformation; emphasizes the mapping property from one vector space to another.
- **Linear Operator:** A linear transformation from a vector space V to itself (T:V→V).
- **Endomorphism:** A homomorphism (linear transformation) from a vector space to itself.
- **Automorphism:** A bijective linear transformation from a vector space to itself (an invertible linear operator).
- **Matrix Representation:** Every linear transformation T:V→W can be represented as a matrix once bases for V and W are chosen.
- **Kernel (Null Space):** The set of all vectors in the domain that are mapped to the zero vector.
- **Range (Image):** The set of all possible outputs (values) of a linear transformation.
- **Rank:** The dimension of the range (image) of a linear transformation.
- **Nullity:** The dimension of the kernel (null space) of a linear transformation.
- **Algebra of Linear Transformations:** The study of addition, scalar multiplication, and composition of linear transformations.
- **Linear Functional:** A special linear transformation from a vector space to its underlying field, i.e., f:V→F.
- **Diagonalizable Operator:** A linear operator that can be represented by a diagonal matrix with respect to some basis.
- **Invertible Transformation:** A linear transformation that has an inverse mapping, requiring it to be both injective and surjective.

5.11 *REFERENCES*

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5.12 *SUGGESTED READING*

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- K.P.Gupta (20th Edition,2019), Pragati Publication, Linear Algebra
- **Axler, S. (2015).** *Linear Algebra Done Right* (3rd ed.). Springer.
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5.13 *TERMINAL QUESTIONS*

(TQ-1) The function $T: V_3(R) \rightarrow V_2(R)$ defined by $T(a, b, c) = (3a - 2b + c, a - 3b - 2c) \forall\, a, b, c \in R$ is a linear transformation from $V_3(R)$ *into* $V_2(R)$.

(TQ-2) The function $T: V_3(R) \rightarrow V_2(R)$ defined by $T(a, b, c) = (a - b, a - c)$ is a linear transformation.

(TQ-3) Show that the mapping $T: R^2 \rightarrow R^3$ is defined as $T(a, b) = (a - b, b - a, -a)$ is a linear transformation from R^2 *into* R^3. Find the range, rank, null space and nullity of T.

(TQ-4) Let V be a finite-dimensional vector space over the field F and let B be an ordered basis for V. Prove that the function f_i which assigns to each vector α in V the i^{th} coordinate of α relative to the ordered basis B is a linear functional on V.

(TQ-5) What is the difference between a linear transformation and a general function?

(TQ-6) Define the kernel and range of a linear transformation.

(TQ-7) What do you mean by the rank and nullity of a linear transformation? State the **Rank-Nullity Theorem**.

(TQ-8) Explain the difference between a linear transformation and a linear operator.

(TQ-9) Define a linear functional. Give an example.

(TQ-10) Prove that the composition of two linear transformations is also linear.

(TQ-11) Show that every linear transformation from $\mathbb{R}^n$ tom $\mathbb{R}^m$can be represented by a matrix.

(TQ-12) State and prove the Rank-Nullity Theorem.

(TQ-13) Let $V_n(F)$ be the vector space of ordered n-tuples of the elements of the field F.

Let $x_1, x_2, \dots, x_n$, be n field elements of F. If

$$\alpha = (a_1, a_2, \dots, a_n) \in V(F)$$

Let f be a function from $V_n(F)$ $into$ F defined by

$$f(x) = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Then prove that f is a functional on $V_n(F)$.

(TQ-15) Consider the basis $S = \{\alpha_1, \alpha_2, \alpha_3\}$ of R^3 where $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1,0,0)$. Express $(2, -3, 5)$ in terms of the basis$\alpha_1, \alpha_2, \alpha_3$.

(TQ-16) Let $T: R^3 \to R^2$ be defined as

$T(\alpha_1) = (1, 0), T(\alpha_2) = (2, -1), T(\alpha_3) = (4,3)$. Find $T(2, -3, 5)$.

The function $T: V_3(R) \to V_2(R)$ defined by $T(a, b, c) = (a, b) \forall\, a, b, c \in R$is a linear transformation from$V_3(R) into\ V_2(R)$.

5.14 *ANSWERS*

(TQ-3) Null space of $T = \{0\}$, nullity of $T = 0$, rank of $T = 2$. The set $\{(1, -1, -1), (-1,1,0)\}$ is a basis set for $R(T)$.

UNIT 6: -Homomorphism and Isomorphism

CONTENTS:

6.1 *INTRODUCTION*

This section will cover some key ideas that create more complex relationships and structures in vector spaces. A vector space is said to be homomorphic if linear transformations maintain scalar multiplication and vector addition. A specific bijective homomorphism that demonstrates the structural similarity of two vector spaces is called an isomorphism of a vector space. A subspace of a vector space can be "factored out" to create a new space of cosets, which is the quotient space. In order to ensure that every vector has a distinct decomposition, the direct sum of spaces offers a method for building larger vector spaces from smaller ones. Last but not least, coordinates explain how vectors can be uniquely defined in relation to a selected basis.

In linear algebra, a *homomorphism* is a structure-preserving map between two vector spaces defined over the same field. It is essentially a **linear transformation** that respects the operations of vector addition and scalar multiplication.

An isomorphism is a special kind of homomorphism that is both **one-to-one** (injective) and **onto** (surjective). It establishes a perfect correspondence between two vector spaces, showing that they are essentially the same in terms of structure.

6.2 *OBJECTIVES*

After studying this unit, the learner's will be able to

- Understand Homomorphism of a Vector Space.
- Explore Isomorphism of a Vector Space.
- Study Quotient Space.
- Examine Direct Sum of Spaces and Disjoint Subspaces.

6.3 *HOMOMORPHISM OF A VECTOR SPACE*

We consider two vector spaces $U(F)$ *and* $V(F)$. Then a mapping

$$f: U \to V$$

is called to as a homomorphism or a linear transformation if

i. $f(\alpha + \beta) = f(\alpha) + f(\beta) \forall\ \alpha, \beta \in U$
ii. $f(a\alpha) = af(\alpha) \forall\ a \in F, \forall \alpha \in U$

Then the conditions (i) and (ii) can be combined into a single condition

$$f(a\alpha + b\beta) = af(\alpha) + bf(\beta) \forall\ \alpha, \beta \in U and\ \forall\ a, b \in F$$

If f is a homomorphism of $U\ onto\ V$, then V is called a homomorphic image of U.

Theorem1. If f is a homomorphism of $U(F)\ into\ V(F)$, then

(i) $f(0) = 0'$ where $0\ and\ 0'$ are the zero vectors of $U\ and\ V$ respectively.

(ii) $f(-\alpha) = -f(\alpha)\ \forall \alpha \in U$

Proof: (i) Let $\alpha \in U$. Then $f(\alpha) \in V$. Since $0'$ is the zero vector of V, therefore

$$f(\alpha) + 0' = f(\alpha) = (\alpha + 0) = f(\alpha) + f(0).$$

Now V is an abelian group with respect to addition of vectors.

$\therefore$ $f(\alpha) + 0' = f(\alpha) + f(0)$

$\Rightarrow$ $0' = f(0)$ by left cancellation law.

(ii) If $\alpha \in U$, then $-\alpha \in U$. Also, we have

$$0' = f(0) = f[\alpha + (-\alpha)] = f(\alpha) + f(-\alpha)$$

Now $f(\alpha) + f(-\alpha) = 0' \Rightarrow f(-\alpha) = additive\ inverse\ of\ f(\alpha)$

$$\Rightarrow\ f(-\alpha) = -f(\alpha).$$

6.4 *ISOMORPHISM VECTOR SPACE*

We consider two vector spaces $U(F)\ and\ V(F)$. Then a mapping

$$f: U \to V$$

is called to as a isomorphism of $U\ onto\ V$ if

i. f is one-one.
ii. f is onto.
iii. $f(a\alpha + b\beta) = af(\alpha) + bf(\beta) \forall\ \alpha, \beta \in U and\ \forall\ a, b \in F$

Symbolically, we write $U(F) \cong V(F)$ to indicate that the two vector spaces $U\ and\ V$ are isomorphic.
The vector space $V(F)$ is also called the isomorphic image of the vector space $U(F)$.
If f is a homomorphism of $U(F)\ into\ V(F)$, then f will become an isomorphism of $U\ into\ V$ if f is one-one. Also, in addition if f is onto V, then f will become an isomorphism of $U\ onto\ V$.

Theorem2. Two finite dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension.
Proof: Let $U(F)\ and\ V(F)$ be two vector spaces of dimension n, each of which has a finite dimension. Then prove that $U(F) \cong (F)$. Let the set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_n\} and\ \{\beta_1, \beta_2, \dots, \beta_n\}$ be the basis of $U\ and\ V$ respectively.

Any vector $\alpha \in U$ can be uniquely expressed as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

Let $f: U \to V$ be defined by

$$f(\alpha) = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$$

Since in the expression of α as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$, the scalars$a_1, a_2, \dots, a_n$are unique, therefore the mapping f is well defined

i.e., $f(\alpha)$ is a unique element of V.

***f* is one-one**. We have

$$f(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = f(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$$

$\Rightarrow$ $$a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$$

$\Rightarrow$ $$(a_1 - b_1)\beta_1 + (a_2 - b_2)\beta_2 + \cdots + (a_n - b_n)\beta_n = 0$$

$\Rightarrow$ $$a_1 - b_1 = 0, a_2 - b_2 = 0, \dots, a_n - b_n = 0$$

$\Rightarrow$ $$a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$$

$\Rightarrow$ $$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$$

$\therefore$ f is one-one.

f is onto V. If $a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$ is any element of V, then $\exists$ an element $a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n \in U$ such that

$$f(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$$

$\therefore$ f is onto.

***f* is a linear transformation.** We have

$$f[a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)]$$

$$= f[(aa_1 + bb_1)\alpha_1, (aa_2 + bb_2)\alpha_2, \dots, (aa_n + bb_n)\alpha_n]$$

$$= (aa_1 + bb_1)\beta_1, (aa_2 + bb_2)\beta_2, \dots, (aa_n + bb_n)\beta_n$$

$$= a(a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n) + b(b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n)$$

$$= af(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + bf(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$$

$\therefore$ f is a linear transformation.

Hence f is an isomorphism of $U onto\ V$.

$\therefore$ $$U \cong V.$$

Conversely, let $U(F)\ and\ V(F)$ be two isomorphic finite dimensional vector spaces.

Then to prove that $dim\ U = dim\ V$.

Let $dim\ U = n$. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of U. If f is an isomorphism of $U\ onto\ V$, we shall show that $S' = \{f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)\}$ is a basis of V. Then V will also be of dimension n.

First, we shall show that S' is linearly independent.

$Let \quad a_1 f(\alpha_1) + a_2 f(\alpha_2) + \dots + a_n f(\alpha_n) = 0 (zero\ vector\ of\ V)$

$\Rightarrow \quad f(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) = 0$ [$\because f$ $is\ linear\ transformation.$]

$\Rightarrow \quad a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$ [$\because f\ is\ one - one\ and\ f(0) = 0$]

$\Rightarrow \quad a_1 = 0, a_2 = 0, \dots, a_n = 0$

[$\because \alpha_1, \alpha_2, \dots, \alpha_n$ $are\ linearly\ independent$]

$\therefore S'$ is linearly independent.

Now we have to prove that $L(S') = V$. For this we prove that any vector $\beta \in V$ can be expressed as a linear combination of the vectors of the set S'. Since f is onto V. therefore $\beta \in V \Rightarrow \exists \alpha \in U | f(\alpha) = \beta$.

Let $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$

Then $\beta = f(\alpha) = f(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n)$

$= c_1 f(\alpha_1) + c_2 f(\alpha_2) + \dots + c_n f(\alpha_n)$

Thus β is a linear combination of the vectors of the set S'. Hence

$$L(S') = V$$

$\therefore S'$ is a basis of V. Since S' contains n vectors, thereforedim $V = n$.

Theorem3. Every n dimensional vector space $V(F)$ is isomorphic to $V_n(F)$.

Proof: Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be any basis of $V(F)$. Then every vector $\alpha \in V$ can be uniquely expressed as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n;\ a_i \in F$$

The ordered $n - tuple$ $(a_1, a_2, \dots, a_n) \in V_n(F)$.

Let $f: V(F) \to V_n(F)$ be defined by $f(\alpha) = (a_1, a_2, \dots, a_n)$.

Since in the expression of α as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$ the scalars $a_1, a_2, \dots, a_n$ are unique, therefore $f(\alpha)$ is a unique element of $V_n(F)$ and thus the mapping f is well defined

f **is one-one**. Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ and $\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$ be any two elements of V. We have

$$f(\alpha) = f(\beta)$$

$$\Rightarrow \quad f(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = f(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$$

$$\Rightarrow \quad (a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$$

$$\Rightarrow \quad a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$$

$$\Rightarrow \quad \alpha = \beta$$

$\therefore f$ is one-one.

$\boldsymbol{f\ is\ onto\ V_n(F)}$. Let $(a_1, a_2, \dots, a_n) \in V_n(F)$. Then $\exists$ an element $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in V(F)$ such that

$$f(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = (a_1, a_2, \dots, a_n)$$

$\therefore f$ is onto.

f **is a linear transformation.** If $a, b \in F \ and \ \alpha, \beta \in V(F)$, we have

$f(a\alpha + b\beta)$

$$= f[a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)]$$

$$= f[(aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bb_n)\alpha_n]$$

$$= (aa_1 + bb_1), (aa_2 + bb_2), \dots, (aa_n + bb_n)$$

$$= (aa_1, aa_2, \dots, aa_n) + (bb_1, bb_2, \dots, bb_n)$$

$$= a(a_1, a_2, \dots, a_n) + b(b_1, b_2, \dots, b_n)$$

$$= af(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + bf(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$$

$$= af(\alpha) + bf(\beta)$$

$\therefore f$ is a linear transformation.

Hence f is an isomorphism of $V(F) onto V_n(F)$.

$\therefore$ $$V(F) \cong V_n(F) .$$

Example1. Show that the mapping $f : V_3(F) \to V_2(F)$ defined by $f: (a_1, a_2, a_3) \to (a_1, a_2)$ is a homomorphism of $V_3(F) onto\ V_2(F)$.

Solution: Let $\alpha = (a_1, a_2, a_3)\ and\ \beta = (b_1, b_2, b_3)$ be any two elements of $V_3(F)$. Also let a, b be any two elements of F. We have

$$\begin{aligned} f(a\alpha + b\beta) &= f[a(a_1, a_2, a_3) + b(b_1, b_2, b_3)] \\ &= f[(aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)] \\ &= (aa_1 + bb_1, aa_2 + bb_2) \\ &= a(a_1, a_2) + b(b_1, b_2) \\ &= af(a_1, a_2, a_3) + bf(b_1, b_2, b_3) \end{aligned}$$

$\therefore f$ is a linear transformation.

To show that f is onto $V_2(F)$. Let (a_1, a_2) be any element of $V_2(F)$. Then $(a_1, a_2, 0) \in V_3(F)$ and we have $f(a_1, a_2, 0) = (a_1, a_2)$. Therefore f is onto $V_2(F)$.

Therefore f is a homomorphism of $V_3(F) onto\ V_2(F)$.

Example2. If V is a finite dimensional vector space and f is an isomorphism of $V\ into\ V$, prove that f must map $V\ onto\ V$.

Solution: Let $V\ (F)$ be a vector space of dimension n that has finite dimensions. Let f be a linear transformation, f is one-one, and let f be an isomorphism of $f\ into\ V$. To prove that $f\ is\ onto\ V$.

Let V have a basis $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. First, we will prove that $S' = \{f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)\}$ is also a basis of V. We claim that S' is linearly independent. The proof is as follows:

Let $$a_1 f(\alpha_1) + a_2 f(\alpha_2) + \dots + a_n f(\alpha_n) = 0$$

$\Rightarrow$ $f(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) = 0$ [$\because f\ is\ linear\ transformation.$]

$\Rightarrow$ $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$ [$\because f\ is\ one - one\ and\ f(0) = 0$]

$\Rightarrow$ $a_1 = 0, a_2 = 0, \dots, a_n = 0$

[$\because \alpha_1, \alpha_2, \dots, \alpha_n\ are\ linearly\ independent$]

$\therefore S'$ is linearly independent.

Now V is of dimension n and S' is a linearly independent subset of V containing n vectors. Therefore S' must be a basis of V. Therefore each vector in V can be expressed as a linear combination of the vectors belonging to S'.

Now we shall show that f is onto V. Let α be any element of V. Then there exist scalars $c_1, c_2, \dots, c_n$ such that

$$\alpha = c_1 f(\alpha_1) + c_2 f(\alpha_2) + \cdots + c_n f(\alpha_n)$$

$$= f(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)$$

Now $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n \in V$ and the $f - image$ of this element is α. Therefore $f\ is\ onto\ V$. Hence f is an isomorphism of $V\ onto\ V$.

Example3. If V is finite dimensional and f is a homomorphism of V into itself which is not onto prove that there is some $\alpha \neq 0\ in\ V$ such that $f(\alpha) = 0$.

Solution: If f is a homomorphism of V into itself, then $f(0) = 0$ Suppose there is no non-zero vector $\alpha\ in\ V$ such that $f(\alpha) = 0$. Then f is one-one. Because

$$f(\beta) = f(\gamma)$$

$\Rightarrow \quad f(\beta) - f(\gamma) = 0$

$\Rightarrow \quad f(\beta - \gamma) = 0 \quad [\because f is\ a\ linear\ transformation]$

$\Rightarrow \quad \beta - \gamma = 0 \Rightarrow \beta = \gamma$

Now V is finite dimensional and f is a linear transformation of V into itself. Since f is one-one, therefore f must be onto V. But it is given that f is not onto. Therefore, our assumption is wrong. Hence there will be a non-zero vector $\alpha\ in\ V$ such that $f(\alpha) = 0$.

Example4. Define linear transformation of a vector space $V(F)$ into a vector space $W(F)$. Show that

$$T: (a, b) \to (a + 2, b + 3)$$

the mapping of $V_2(R)$ into itself is not a linear transformation.

Solution: Let $V(F)\ and\ W(F)$ be two vector spaces over the same field F. A mapping $T: V \to W$ is called a linear transformation of V into W if

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \forall\ \alpha, \beta \in V\ and\ \forall\ a, b \in F$$

Now to show that the mapping

$$T: (a, b) \to (a + 2, b + 3)$$

of $V_2(R)$ into itself is not a linear transformation.

Take $\alpha = (1, 2)\ and\ \beta = (1, 3)$ as two vectors of $V_2(R)$ and $a = 1, b = 1$ as two elements of the field R.

Then $a\alpha + b\beta = 1(1, 2) + 1(1, 3) = (2,5)$

By the definition of mapping T, we have

$$T(a\alpha + b\beta) = T\ (2,5) = (2 + 2, 5 + 3) = (4, 8) \qquad \dots (1)$$

$Also \quad T(\alpha) = T\ (1,2) = (1 + 2, 2 + 3) = (3,5)$

$and \quad T(\beta) = T(1,3) = (1 + 2, 3 + 3) = (3, 6).$

$\therefore \quad aT(\alpha) + bT(\beta) = 1\ (3,5) + 1\ (3,6) = (3,5) + (3,6)$

$= (6,11) \quad \ldots (2)$

From equation (1) and (2), we see that

$$T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$$

Hence T is not a linear transformation of $V_2(R)$ into itself.

6.5 *QUOTIENT SPACE*

Let W be any subspace of $V(F)$, which is a vector space. Let α be any of $V's$ elements. Then the set

$$W + \alpha = \{\gamma + \alpha;\ \gamma \in W\}$$

is called a right cost of $W\ in\ V$ generated by α. Similarly, the set

$$\alpha + W = \{\alpha + \gamma;\ \gamma \in W\}$$

is called a left cost of $W\ in\ V$ generated by α.

We will refer to $W + \alpha$ as simply a coset of $W\ in\ V$ created by α. It is obvious that both $W + \alpha\ and\ \alpha + W$ are subsets of V. Since addition in V is commutative, we get $W + \alpha = \alpha + W$.

The following results about costs are both to be remembered:

(i) We have $0 \in V$ and $W + 0 = W$. Therefore W itself is a coset of $W\ in\ V$.

(ii) $\alpha \in W \Rightarrow W + \alpha = W$

6.6 *DIRECT SUM OF SPACES*

Let $W_1, W_2, \ldots, W_m$ be subspaces of a vector space $V(F)$. If each element $\alpha \in V$ can be expressed in one and only one way as $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$, then V is said to be the direct sum of $W_1, W_2, \ldots, W_m$ where $\alpha_1 \in W_1, \alpha_2 \in W_2, \ldots, \alpha_m \in W_m$.

If a vector space $V(F)$ is a direct sum of its two subspaces $W_1\ and\ W_2$ then we should have not only $V = W_1 + W_2$ but also that each vector of V can be uniquely expressed as sum of an element of W_1 and an element of W_2. Symbolically the direct sum is represented by the notation $V = W_1 \oplus W_2$.

Disjoint subspaces: In the vector space $V(F)$, two subspaces $W_1 and\ W_2$ are considered disjoint if their intersection is the zero subspace, i.e. if $W_1 \cap W_2 = \{0\}$.

6.7 *COORDINATES*

Let $V(F)$ be a vector space with finite dimensions. Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V.
When we say that the vectors of B have been enumerated in a well-defined manner, we mean that the vectors that occupy the first, second, nth positions in the set B are fixed.
Let $\alpha \in V$. Then there exists a unique $n - tuple$ $(x_1, x_2, \dots, x_n)$ of scalars such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \sum_{i=1}^{n} x_i\alpha_i$$

The $n - tuple$ $(x_1, x_2, \dots, x_n)$ is called the $n - tuple$ of coordinates ofα relative to the ordered basis B. The scalarx_i, is called i^{th} coordinate of a relative to the ordered basis B. The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Is called the coordinate matrix of a relative to the ordered basis B.
It should be mentioned that the α vector's coordinates are unique for the same basis set B, but only in relation to a specific ordering of B. There are various ways to arrange the basis of set B. The coordinates of α may change with a change in the ordering of B.

6.8 *KERNEL OF A HOMOMORPHISM*

If $T: V \to$W is a linear transformation (homomorphism) between vector spaces V and W, then the **kernel of T** is defined as

$$Ker\ T = \{v \in V : Tv = 0_w\}$$

where 0_w is the zero vector in W
The kernel consists of all vectors in the domain V that are mapped to the zero vector of the codomain W.
Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$be defined by

$$T(x, y, z) = (x + y, y + z)$$

Then

$$Ker\ T = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0, y + z = 0\}$$

Properties:

1. **$Ker\ T$ is always a subspace of V.**

Sol: The $T: V \to$ W is

$$ker(T) = \{v \in V, T(v) = 0\}$$

To be a subspace of V, a set must satisfy three conditions:

Since $T(0) = 0, 0 \in ker(T)$

If $v_1, v_2 \in \ker T$, then $T(v_1 + v_2) = 0 + 0 = 0$, so $v_1 + v_2 \in \ker T$.

If $v \in \ker T, c \in \mathbb{R}$ then $T(cv) = cT(v) = 0 = c.0 = 0$ so $cv \in \ker T$.

2. **If $\boldsymbol{Ker\ T} = \{\mathbf{0}\}$, then T is injective (one-to-one).**

   By definition, T is **injective** if

   $T(v_1) = T(v_2) \Rightarrow v_1 = v_2 \Rightarrow T(v) = 0 \Rightarrow v = 0$

   But this is exactly the definition of $Ker\ T = \{0\}$. So If the kernel only contains the zero vector, then no two distinct vectors in V can map to the same image → T is **injective**.

3. The dimension of the kernel is called the **nullity** of T. **Nullity** is defined as:

$$Nullity(T) = \dim((kerT))$$

It measures how many "degrees of freedom" there are in the solution to $T(v) = 0$. By the **Rank-Nullity Theorem**:

$$dimv = rank(T) + nullity\ (T)$$

where:

- $rank(T) = \dim(Im(T))$ (dimension of the image),
- $Nullity(T) = \dim((kerT))$

6.9 *INJECTIVE HOMOMORPHISM (ONE-TO-ONE)*

An **injective homomorphism** (or **one-to-one linear transformation**) is a linear map $T: V \to W$ between two vector spaces such that **different vectors in V map to different vectors in W**. Formally, T is injective if

$T(v_1) = T(v_2) \Rightarrow v_1 = v_2, \Rightarrow v_1, v_2 \epsilon V$

Equivalently, T is injective if and only if its **kernel is trivial**, i.e.,

$$\boldsymbol{Ker\ T} = \{\mathbf{0}\}$$

This means that the only vector in V that maps to the zero vector in W is the zero vector itself. Injective homomorphisms preserve distinctness of elements and play a key role in establishing isomorphisms between vector spaces.

6.10 *IMAGE OF A HOMOMORPHISM*

The **image of a homomorphism** (also called the **range**) describes the set of all possible outputs of the homomorphism.

If $T: V \rightarrow$W is a linear homomorphism (linear transformation) between vector spaces, then the **image** of T is defined as:

$$Im(T) = \{T(v) \in W\ \ /v\ \in V\}$$

Properties:

1. **Subspace Property**: $Im(T)$ is always a subspace of W.
2. **Relation to Subjectivity**: If $Im(T) = W$, then T is **surjective** (onto).
3. **Rank**: The dimension of $Im(T)$ is called the **rank** of T.

Example5. Show that the set $S = ((1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $R^3\ (R)$ where R is the field of real numbers. Hence find the coordinates of the vector (a,b,c) with respect to the above basis.

Solution: The dimension of the vector space $R^3\ (R)$ is 3. If the set S is linearly independent, then S will form a basis of $R^3\ (R)$ because S contains 3 vectors. Let x,y,z be scalars in R such that

$$x\ (1,0,0) + y\ (1,1,0) + z\ (1,1,1) = 0 = (0,0,0)$$

$\Rightarrow \quad (x+y+z, y+z, z) = (0,0,0)$

$\Rightarrow \quad x+y+z = 0, y+z = 0, z = 0 \ \Rightarrow x = 0, y = 0, z = 0$

$\Rightarrow \quad the\ set\ S\ is\ linearly\ independent.$

$\therefore \quad S\ is\ a\ basis\ of\ R^3\ (R).$

Now to find the coordinates of (a,b,c) with respect to the ordered basis S. Let p,q,r be scalars in R such that

$$(a,b,c) = p\ (1,0,0) + q\ (1,1,0) + \ r(1,1,1)$$

$\Rightarrow \quad (a,b,c) = (p+q+r, q+r, r)$

$\Rightarrow \quad p+q+r = a, q+r = b, r = c$

$\Rightarrow \quad r = c, q = b-c, p = a-b$

Hence the coordinates of the vector (a,b,c) are (p,q,r) i.e.,$(a-b, b-c, c\ )$.

Example6. Construct three subspaces W_1, W_2, W_3 of a vector space V so that $V = W_1 \oplus W_2 = W_1 \oplus W_3\, but\ W_2 \neq W_3$.
Solution: Take the vector space $V = R^2$
Obviously $W_1 = \{(a, 0); a \in R\}, W_2 = \{(0, a); a \in R\} and\ W_3 = \{(a, a);\ a \in R\}$ are three subspaces of R^2.
We have $V = W_1 + W_2$ and $W_1 \cap W_2 = \{(0,0)\}$
∴ $V = W_1 \oplus W_2$
Also, it can be easily shown that
$V = W_1 + W_3$ and $W_1 \cap W_3 = \{(0,0)\}$
∴ $V = W_1 \oplus W_2$
Thus $V = W_1 \oplus W_2 = W_1 \oplus W_3\, but\ W_2 \neq W_3$.

6.11 *SUMMARY*

An important term in the study of vector spaces is homomorphism and isomorphism. A homomorphism is a linear transformation between two vector spaces that respects the algebraic structure of the spaces by preserving the operations of scalar multiplication and vector addition. By transferring vector spaces into one another while preserving their linear characteristics, homomorphisms aid in our understanding of their linkages. In contrast, an isomorphism is a unique kind of homomorphism that is both onto (surjective) and one-to-one (injective). By establishing a perfect correspondence between two vector spaces, an isomorphism demonstrates that they are structurally similar and only differ in the labels of their constituent parts.

6.12 *GLOSSARY*

- **Homomorphism (Linear Transformation):** A function $T: V \to W$ between two vector spaces that preserves vector addition and scalar multiplication.
- **Isomorphism:** A bijective (one-to-one and onto) homomorphism that establishes structural equivalence between two vector spaces.
- **Kernel of a Homomorphism:** The set of all vectors in V that map to the zero vector in W under T.
- **Image of a Homomorphism:** The set of all vectors in W that are outputs of $T(v)$ for some $v \in V$.
- **Injective Homomorphism (One-to-One):** A homomorphism where distinct vectors in V map to distinct vectors in W; equivalently, the kernel contains only the zero vector.
- **Surjective Homomorphism (Onto):** A homomorphism whose image equals the entire codomain W.

- **Linear Isomorphism:** A bijective homomorphism, implying that the two vector spaces have the same dimension.
- **Automorphism:** An isomorphism from a vector space to itself.
- **Endomorphism:** A homomorphism from a vector space to itself.
- **Dimension Preservation:** In isomorphisms, both vector spaces must have the same dimension, ensuring structural equivalence.

6.13 *REFERENCES*

- **Kuldeep Singh (2020)** — *Linear Algebra: Step by Step* (Oxford Uni. Press, distributed by Dev Publishers, India)
- **S. Chand (2023)** —*Mathematical Sciences Linear Algebra 2024*.
- **S. C. Malik & Savita Arora (2010),** *Mathematical Analysis and Linear Algebra.* New Age International Publishers.

6.14 *SUGGESTED READING*

- A.R. Vasishtha, J.N.Sharma and A.K. Vasishtha (52th Edition, 2022), Krishna Publication, Linear Algebra.
- K.P.Gupta (20th Edition,2019), Pragati Publication, Linear Algebra

6.15 *TERMINAL QUESTIONS*

(TQ-1) Let $V(R)$ be the vector space of all complex numbers $a + ib$ over the field of reals R and let T be a mapping from $V(R) to\, V_2(R)$ defined as $T(a + ib) = (a, b)$. Show that T is an isomorphism.

(TQ-2) $V\ (F)$ *and* $W\ (F)$ are two finite dimensional vector spaces such that $dim\, V = dim\, W$. If f is an isomorphism of $V\ into\ W$ prove that f must map $V\ onto\ W$.

(TQ-3) Find the coordinates of the vector $(2, 1, -6)$ of R^3 relative to the basis $\alpha_1 = (1, 1, 2), \alpha_2 = (3, -1, 0), \alpha_3 = (2, 0, -1)$.

(TQ-4) Let $W_1\ and\ W_2$ be two subspaces of a finite dimensional vector space V. If $dim\, V = dim\, W_1 + dim\, W_2\ and\ W_1 \cap W_2 = \{0\}$, prove that $V = W_1 \oplus W_2$.

(TQ-5) Show that the set $S = ((1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $R^3\ (R)$ where R is the field of real numbers. Hence find the coordinates of the vector (a, b, c) with respect to the above basis.

(TQ-6) Define a homomorphism between two vector spaces. Give an example.

(TQ-7) What is meant by the kernel and image of a homomorphism? Prove that both are subspaces.
(TQ-8) Define Rank-Nullity.
(TQ-9) What is an isomorphism? Give necessary and sufficient conditions for a linear transformation to be an isomorphism.
(TQ-10) Prove that if $T: V \to W$ is an isomorphism, then $dimV = dimW$.
(TQ-11) Show that the composition of two isomorphisms is an isomorphism.
(TQ-12) Show that the inverse of an isomorphism is also an isomorphism.
(TQ-13) Give an example of a homomorphism that is **not** an isomorphism.

UNIT 7: -Dual Space

CONTENTS:

7.1 *INTRODUCTION: -*

In linear algebra, the concept of **dual space** plays an important role in extending the study of vector spaces. Given a vector space V over a field F, the dual space of V, denoted by V^*, is defined as the set of all **linear functionals** from V to F. A linear functional is a linear transformation that maps a vector in V to a scalar in the field F. The dual space itself forms a vector space over the same field F. If V is finite-dimensional with $dim(V) = n$, then the dual space V^* also has dimension n. Each basis of V gives rise to a **dual basis** in V^*, which allows us to connect vectors with linear functional in a systematic way.

The concept of dual space is significant because it provides a framework for studying vector spaces from the perspective of linear functionals rather than vectors. It is widely used in advanced topics like differential geometry, functional analysis, optimization, and theoretical physics.

7.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- Define Dual Space.
- Define Dual Bases.
- Define Second Dual Space.

7.3 *DUAL SPACE: -*

Consider the vector space V over the field F. Then the set V', which contains all linear functionals on V, is a vector space over the field F. The dual space of V is the vector space V'.
The dual space of V is also sometimes denoted by the symbols V^* *and* $\hat{V}$. Another name for the dual space of V is the conjugate space of V.

7.4 *DUAL BASES: -*

Theorem1. Let V be an n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis for V. If $\{x_1, x_2, \dots, x_n\}$ is any ordered set of n scalars, then there exists a unique linear functional f *on* V such that $f(\alpha_i) = x_i;\ i = 1, 2, \dots, n$.

Proof: **Existence of f.** Let $\alpha \in V$.

Since $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V, therefore there exist unique scalars$a_1, a_2, \dots, a_n$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

For this vector α, let us define

$$f(\alpha) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Obviously $f(\alpha)$ as defined above is a unique element of F. Therefore f is a well-defined rule for associating with each vector α in V a unique scalar $f(\alpha)$ *in* F. Thus f is a function from V *into* F.

The unique representation of $\alpha_i \in V$ as a linear combination of the vectors belonging to the basis B is

$$\alpha_i = 0\alpha_1 + 0\alpha_2 + \cdots + 1\alpha_i + 0\alpha_{i+1} + \cdots + 0\alpha_n$$

Therefore according to our definition of f, we have

$$f(\alpha_i) = 0x_1 + 0x_2 + \cdots + 1x_i + 0x_{i+1} + \cdots + 0x_n$$

i.e. $\quad f(\alpha_i) = x_i;\ i = 1, 2, \dots, n$

Now to show that f is a linear functional. Let $a, b \in F$ and $\alpha, \beta \in V$. Let

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

and $$\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$$

Then
$f(a\alpha + b\beta)$

$$= f[a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)]$$

$$= f[(aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bb_n)\alpha_n]$$

$$= (aa_1 + bb_1)x_1 + (aa_2 + bb_2)x_2 + \cdots + (aa_n + bb_n)x_n$$

$$= a(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + b(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$$

$$= af(\alpha) + bf(\beta)$$

$\therefore f$ is a linear functional on V. Thus there exists a linear functional f *on* V such that $f(\alpha_i) = x_i;\ i = 1, 2, \ldots, n$.

Uniqueness of f. Let g be the linear functional on Vsuch that

$$g(\alpha_i) = x_i;\ i = 1, 2, \ldots, n.$$

For any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in V$, we have

$$g(\alpha) = g(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$$

$$= a_1g(\alpha_1) + a_2g(\alpha_2) + \cdots + a_ng(\alpha_n) \qquad [\because g \text{ is linear}]$$

$$= a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$= f(\alpha)$$

Thus $g(\alpha) = f(\alpha) \forall\ \alpha \in V$

$\therefore \quad g = f$

The above expression shows the uniqueness of f.

Theorem2. Let V be an n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Then there is a uniquely determined basis $B' = \{f_1, f_2, \ldots, f_n\}$ for V' such that $f_i(\alpha_j) = \delta_{ij}$. Consequently the dual space of an n-dimensional space is n-dimensional

The basis B' is called the dual basis of B.

Proof: Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an ordered basis for V. Therefore by theorem 1, there exists a unique linear functional $f_1 \; on \; V$ such that

$$f_1(\alpha_1) = 1, f_1(\alpha_2) = 0, \ldots, f_1(\alpha_n) = 0$$

Where $\{1, 0, \ldots, 0\}$ is an ordered set of n scalars.

In fact, for each $i = 1, 2, \ldots, n$ there exists a unique linear functional such that

$$f_i(\alpha_j) = \begin{cases} 0 \; if \; i \neq j \\ 1 \; if \; i = j \end{cases}$$

$i.e.$ $$f_i(\alpha_j) = \delta_{ij} \qquad \ldots (1)$$

Where $\delta_{ij} \in F$ is Kronecker delta i.e. $\delta_{ij} = 1 \; if \; i = j$ and $\delta_{ij} = 0 \; if \; i \neq j$.

Let $B' = \{f_1, f_2, \ldots, f_n\}$. Then B' is a subset of V' containing n distinct elements of V'. We shall show that B' is a basis for V'.

First we shall show that B' is linearly independent.

Let $$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = \hat{0}$$

$\Rightarrow$ $$(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)(\alpha) = \hat{0}(\alpha) \; \forall \alpha \in V$$

$\Rightarrow c_1 f_1(\alpha) + c_2 f_2(\alpha) + \cdots + c_n f_n(\alpha) = 0 \forall \alpha \in V \; [\because \hat{0}(\alpha) = 0]$

$\Rightarrow$ $$\sum_{i=1}^{n} c_i f_i(\alpha) = 0 \forall \alpha \in V$$

Putting $\alpha = \alpha_j \; ; j = 1, 2, \ldots, n$

$\Rightarrow$ $$\sum_{i=1}^{n} c_i f_i(\alpha_j) = 0; j = 1, 2, \ldots, n$$

$\Rightarrow$ $$\sum_{i=1}^{n} c_i \delta_{ij} = 0; j = 1, 2, \ldots, n$$

$\Rightarrow$ $$c_j = 0; j = 1, 2, \ldots, n$$

$\Rightarrow f_1, f_2, \ldots, f_n$ are linearly independent.

Also, we have to prove that the linear span of B' is equal to V'.

Let $f \in V'$. The linear functional f will be completely determined if we define it on a basis for V. So let

$$f(\alpha_i) = a_i \,;\, i = 1,2,\dots,n \qquad \dots(2)$$

We have to prove that

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = \sum_{i=1}^{n} a_i f_i$$

We know that two linear functional on V are equal if they agree on a basis of V. Let $\alpha_j \in B \,;\, j = 1,2,\dots,n$, then

$$\left[\sum_{i=1}^{n} a_i f_i\right](\alpha_j) = \sum_{i=1}^{n} a_i f_i\,(\alpha_j)$$

$$= \sum_{i=1}^{n} a_i \delta_{ij} \qquad [from\ equation\ (1)]$$

$$= a_j, on\ summing\ with\ respect$$

$$to\ i\ and\ remembering\ that$$

$$\delta_{ij} = 1\ when\ i = j\ and\ \delta_{ij} = 0$$

$$when\ i \neq j$$

$$= f(\alpha_j)$$

Thus $\left[\sum_{i=1}^{n} a_i f_i\right](\alpha_j) = f(\alpha_j) \forall \alpha_j \in B$

$$Therefore\ f = \sum_{i=1}^{n} a_i f_i$$

Thus every element $f\ in\ V'$ can be expressed as a linear combination of $f_1, f_2, \dots, f_n$.

$$\therefore V' = linear\ span\ of\ B'.$$

Hence B' is a basis for V'.

Theorem3. Let V be an n-dimensional vector space over the field F and let $B = \{a, \ldots, an)$ be a basis for V. Let $B' = \{f_1, f_2, \ldots, f_n\}$ be the dual basis of B. Then for each linear functional $f\ on\ V$, we have

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

and for each vector $\alpha\ in\ V$ we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

Proof: Since B' is dual basis of B, therefore

$$f_i(\alpha_j) = \delta_{ij} \qquad \ldots (1)$$

If f is a linear functional on V, then $f \in V'$ for which B' is basis. Therefore f can be expressed as a linear combination of $f_1, f_2, \ldots, f_n$. Let

$$f = \sum_{i=1}^{n} c_i f_i$$

Then

$$f(\alpha_j) = \left[\sum_{i=1}^{n} c_i f_i\right](\alpha_j)$$

$$= \sum_{i=1}^{n} c_i f_i\,(\alpha_j)$$

$$= \sum_{i=1}^{n} c_i\,\delta_{ij}$$

$$= c_j\,; j = 1,2, \ldots, n$$

$$\therefore \qquad f = \sum_{i=1}^{n} f(\alpha_i) f_i$$

Now let α be any vector in V. Let

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \sum_{j=1}^{n} x_j\alpha_j \qquad \ldots(2)$$

Then

$$f_i(\alpha) = f_i\left(\sum_{j=1}^{n} x_j\alpha_j\right) \qquad [from\ equation(2)]$$

$$= \sum_{j=1}^{n} x_j f_i(\alpha_j) \qquad [\because f_i is\ linear\ functional]$$

$$= \sum_{j=1}^{n} x_j\, \delta_{ij}$$

$$= x_i$$

$$\therefore \alpha = f_1(\alpha)\alpha_1 + f_2(\alpha)\alpha_2 + \cdots + f_n(\alpha)\alpha_n = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

Theorem4. Let V be an n-dimensional vector space over the field F. If α is a non-zero vector in V, there exists a linear functional $f\ on\ V$ such that $f(\alpha) \neq 0$.

Proof: Since $\alpha \neq 0$, therefore $\{\alpha\}$ is a linearly independent subset of V. So it can be extended to form a basis for V. Thus there exists a basis $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\ for\ V$ such that $\alpha_1 = \alpha$.

If $B' = \{f_1, f_2, \ldots, f_n\}$ is the dual basis, then

$$f_1\ (\alpha) = f_1(\alpha_1)\ =\ 1 \neq 0.$$

Thus there exists linear functionals f_1, such that

$$f_1\ (\alpha) \neq 0$$

Corollary: Let V be an n-dimensional vector space over the field F. If

$$f\ (\alpha) = 0 \forall f\ \in V', then\ \alpha = 0.$$

Proof: Suppose $\alpha \neq 0$. Then there is a linear functional $f\ on\ V$ such that $f(\alpha) \neq 0$. This contradicts the hypothesis that

$f\ (\alpha) = 0 \forall f \in V'$. Hence we must have $\alpha = 0$.

Theorem5. Let V be an n-dimensional vector space over the field F. If α, β are any two different vectors in V, then there exists a linear functional $f\ on\ V$ such that $f\ (\alpha) \neq\ f(\beta)$.

Proof: We have $\alpha \neq \beta \Rightarrow \alpha - \beta \neq 0$.

Now $\alpha - \beta$ is a non-zero vector in V. Therefore by theorem 4, there exists, a linear functional $f\ on\ V$ such that

$$f\ (\alpha - \beta) \neq 0$$

$$\Rightarrow \quad f\ (\alpha) - f(\beta) \neq 0$$

$$\Rightarrow \quad f(\alpha) \neq f\ (\beta).$$

Hence the result.

7.5 *SECOND DUAL SPACE*: -

Every vector space V is known to have a dual space V' that contains all of the linear functionals on V.
V' is now a vector space as well. It will therefore also have a dual space $(V')'$ that contains all linear functionals on V'. This dual space of V' is known as the Second Dual Space of V, and we will simply refer to it as V''

$dim\ V = dim\ V' = dim\ V''$ indicates that they are isomorphic to each other if V is finite-dimensional.

Theorem6. Let V be a finite dimensional vector space over the field F. If α is any vector in V, the function $L_\alpha\ on\ V'$ defined by

$$L_\alpha = f(\alpha)\ \forall\ f\ \in\ V'$$

is a linear functional on V i.e.$L_\alpha \in V''$.

Also the mapping $\alpha \to L_\alpha$ is an isomorphism of $V\ onto\ V$".

Proof: If $\alpha \in V\ and\ f\ \in V'$ then $f\ (\alpha)$ is a unique element of F. Therefore the correspondence L_α defined by

$$L_\alpha(f) = f(\alpha) \forall f \in V' \qquad \text{... (1)}$$

is a function from V' *into* F.

Let $a, b \in F$ *and* $f, g \in V'$. Then

$$L_\alpha(af + bg) = (af + bg)(\alpha)$$

$$= (af)(\alpha) + (bg)(\alpha)$$

$$= af(\alpha) + bg(\alpha)$$

[*by scalar multiplication of linear functionals*]

$$= a[L_\alpha(f)] + b[L_\alpha(g)]$$

Therefore L_α is a linear functional on V' and thus $L_\alpha \in V''$. Now let ψ be the function from V *into* V" defined by

$$\psi(\alpha) = L_\alpha \forall \alpha \in V$$

***ψ* is one-one.** If $\alpha, \beta \in V$, then

$$\psi(\alpha) = \psi(\beta)$$

$\Rightarrow \quad L_\alpha = L_\beta \Rightarrow L_\alpha(f) = L_\beta(f) \forall f \in V'$

$\Rightarrow \quad f(\alpha) = f(\beta) \forall f \in V'$

$\Rightarrow \quad f(\alpha) - f(\beta) = 0 \ \forall f \in V'$

$\Rightarrow \quad f(\alpha - \beta) = 0 \ \forall f \in V'$

$\Rightarrow \quad \alpha - \beta = 0 \qquad$ [*by theorem* 4]

$\Rightarrow \quad \alpha = \beta$

$\therefore \psi$ is one − one.

ψ is a linear transformation.

Let $a, b \in F$ *and* $\alpha, \beta \in V$. Then

$$\psi(a\alpha + b\beta) = L_{(a\alpha + b\beta)} \qquad [\textit{by definition of } \psi]$$

For every $f \in V'$, we have

$$L_{(a\alpha+b\beta)}(f) = f(a\alpha + b\beta) = af(\alpha) + bf(\beta) = aL_\alpha(f) + bL_\beta(f)$$
$$= (aL_\alpha)(f) + (bL_\beta)(f) = (aL_\alpha + bL_\beta)(f)$$

$\therefore L_{(a\alpha+b\beta)} = aL_\alpha + bL_\beta = a\psi(\alpha) + b\psi(\beta)$

$\Rightarrow \psi(a\alpha + b\beta) = a\psi(\alpha) + b\psi(\beta)$

$\Rightarrow \psi$ is a linear transformation from V *into* V'. Since we have $dim\, V = dim\, V''$ therefore ψ is one-one$\Rightarrow \psi$ must also be onto.

Hence ψ is an isomorphism of V *onto* V''.

Theorem7. Let V be a finite dimensional vector space over the field F. If L is a linear functional on the dual space V' *of* V, then there is a unique vector $\alpha \in V$ such that $L(f) = f(\alpha)\ \forall f \in V'$.

Proof: This theorem is an immediate corollary of theorem 6. We should first prove theorem 6. Then we should conclude like this:

The correspondence $\alpha \to L_\alpha$ is a one-to-one correspondence between V *and* V''. Therefore if $L \in V''$, there exists a unique vector $\alpha \in V$ such that $L = L_\alpha$. i.e. such that

$$L(f) = f(\alpha)\ \forall f \in V'$$

Theorem8. Let V be a finite dimensional vector space over the field F. Each basis for V' is the dual of some basis for V.

Proof: Let $B' = \{f_1, f_2, \dots, f_n\}$ be a basis for V. Then there exists a dual basis $(B')' = \{L_1, L_2, \dots, L_n\}$ *for* V'' such that

$$L_i(f_j) = \delta_{ij} \qquad \dots(1)$$

By previous theorem, for each i there is a vector $\alpha_i \in V$ such that

$$L_i = L_{\alpha_i};\ L_{\alpha_i}(f) = f(\alpha_i)\forall f \in V' \qquad \dots(2)$$

The correspondence $\alpha \to L_\alpha$, is an isomorphism of V *onto* V''. Under an isomorphism of a basis is mapped onto a basis. Therefore $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V because it is the image set of a basis for V''under the above isomorphism.

Putting $f = f_j$, in (2), we get

$$f_j(\alpha_i) = L_{\alpha_i}(f_j) = L_i(f_j) = \delta_{ij}$$

$\therefore B' = \{f_1, f_2, \dots, f_n\}$ is the dual of the basis B. Hence the theorem.

Theorem9. Let V be a finite dimensional vector space over the field F. Let B be a basis for V and B' be the dual basis of B. Then show that

$$B'' = (B')' = B$$

Proof: Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V, $B' = \{f_1, f_2, \dots, f_n\}$ be the dual basis of B in V' and $B'' = (B')' = \{L_1, L_2, \dots, L_n\}$ be the dual basis of B in V". Then

$$f_i(\alpha_j) = \delta_{ij}$$

and $$L_i(f_j) = \delta_{ij}; i = 1,2, \dots, n \;\&\; j = 1,2, \dots, n$$

If $\alpha \in V$, then there exists $L_\alpha \in V$" such that

$$L_\alpha(f) = f(\alpha) \forall f \in V'$$

Taking α_i, in place of α, we see that for each$j = 1, \dots, n$,

$$L_{\alpha_i}(f_j) = f_j(\alpha_i) = L_i(f_j) = \delta_{ij}$$

Thus L_{α_i}, and L_i, agree on a basis for V'. Therefore

$$L_{\alpha_i} = L_i$$

If we identify V" $with$ V through natural isomorphism $\alpha \leftrightarrow L_\alpha$, then we consider L_α, as the same element as α. So

$$L_{\alpha_i} = L_i = \alpha_i \; ; \; i = 1, 2, \dots, n.$$

Thus $B'' = B$.

Example1. Find the dual basis of the basis set $B = \{(1, 1, 3), (0, 1, -1), (0, 3, -2)\}$ for $V_3(R)$.

Solution: Let $\alpha_1 = (1, -1, 3), \alpha_2 = (0, 1, -1), \alpha_3 = (0, 3, -2)$.

Then $B = \{\alpha_1, \alpha_2, \alpha_3\}$. If $B' = \{f_1, f_2, f_3\}$ is a dual basis of B , then

$$f_1(\alpha_1) = 1, f_1(\alpha_2) = 0, f_1(\alpha_3) = 0,$$

$$f_2(\alpha_1) = 1, f_2(\alpha_2) = 0, f_2(\alpha_3) = 0$$

$$and \qquad f_3(\alpha_1) = 1, f_3(\alpha_2) = 0, f_3(\alpha_3) = 0$$

Now to find explicit expressions for f_1, f_2, f_3. Let $(a, b, c) \in V_3(R)$.

Let $(a, b, c) = x(1, -1, 3) + y(0, 1, -1) + z(0, 3, -2)$ ... (1)

$$= x\alpha_1 + y\,\alpha_2 + z\alpha_3$$

Then$f_1(a, b, c) = x, f_2(a, b, c) = y, f_3(a, b, c) = z$

Now to find the values of x, y, z.

From (1), we have

$$x = a, -x + y + 3z = b, 3x - y - 2z = c$$

Solving these equations, we have

$$x = a, y = 7a - 2b - 3c\,, z = b + c - 2a$$

Hence $f_1(a, b, c) = a,$ $\quad f_2(a, b, c) = 7a - 2b - 3c,$

$f_3(a, b, c) = b + c - 2a$

Therefore$B' = \{f_1, f_2, f_3\}$ is a dual basis of B where f_1, f_2, f_3 are as defined above.

Example2. The vectors $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, -1)\, and\, \alpha_3 = (1, -1, -1)$ form a basis $V_3(C)$ If $\{f_1, f_2, f_3\}$ the dual basis and if $\alpha = (0, 1, 0)$, find $f_1(\alpha), f_2(\alpha) and\ f_3(\alpha)$.

Solution: Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$.Then

$$f_1(\alpha) = a_1, f_2(\alpha) = a_2 and\ f_3(\alpha) = a_3$$

Now $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$\Rightarrow\ (0, 1, 0) = a_1(1, 1, 1) + a_2(1, 1, -1) + a_3(1, -1, -1)$

$\Rightarrow\ (0, 1, 0) = (a_1 + a_2 + a_3, a_1 + a_2 - a_3, a_1 - a_2 - a_3)$

$\Rightarrow a_1 + a_2 + a_3 = 0, a_1 + a_2 - a_3 = 1, a_1 - a_2 - a_3 = 0$

$\Rightarrow a_1 = 0, a_2 = \frac{1}{2}, a_3 = -\frac{1}{2}$

$\therefore f_1(\alpha) = 0, f_2(\alpha) = \frac{1}{2} and \ f_3(\alpha) = -\frac{1}{2}$

Example3. If f is a non-zero linear functional on a vector space V and if x is an arbitrary scalar, does there necessarily exist a vector $\alpha \ in\ V$ such that $f(\alpha) = x$?

Solution: f is a non-zero linear functional on V. Therefore there must be some non-zero vector $\beta \ in\ V$ such that $f(\beta) = y$ where y is a non-zero element of F.

If x is any element of F, then

$$x = (xy^{-1})y = (xy^{-1})f(\beta) = f[(xy^{-1})\beta]$$

$$[since\ f\ is\ linear\ functional]$$

Thus there exists $x = (xy^{-1})\beta \in V$ such that $f(\alpha) = x$.

Example4. Prove that if f is a linear functional on an n-dimen-sional vector space $V(F)$, then the set of all those vectors α for which $f(\alpha) = 0$ is a subspace of V, what is the dimension of that subspace?

Solution: Let $N = \{\alpha \in V : f(\alpha) = 0\}$.

N is not empty because at least $0 \in N\ such\ that\ f(0) = 0$

Let $\alpha, \beta \in N$ Then $f(\alpha) = 0, f(\beta) = 0$

If $a, b \in F$ we have

$$f(a\alpha - b\beta) = af(\alpha) + bf(\beta) = a0 + b0 = 0 \therefore a\alpha + b\beta \in N$$

Thus $a, b \in F\ and\ \alpha, \beta \in N \Rightarrow a\alpha + b\beta \in N$,

$\Rightarrow N$ is a subspace of V . This subspace N is the null space of f. We know that

$$dim\ V = dim\ N + dim\ (range\ of\ f).$$

(i) If f is zero linear functional, then range of f consists of zero element of F alone. Therefore $dim\ (range\ f\ ) = 0$ in this case.
In this case, we have

$$\dim V = \dim N + 0 \ \Rightarrow \ n \ = \ dimN$$

(ii) If f is a non-zero linear functional on V, then $f\ is\ onto\ F$. So range of f consists of all F in this case. The dimension of the vector space $F'\ is$ 1. . In this case we have

$$dim\ V = dim\ N + 1 \ \Rightarrow dimN \ = \ n \ - \ 1$$

7.6 *SUMMARY*:-

In linear algebra, the dual space of a vector space V over a field F, denoted by V^*, is the set of all linear functionals from V to F. A linear functional is a linear transformation that maps a vector to a scalar while preserving linearity. The dual space plays a crucial role in understanding the structure of vector spaces, providing a way to study them through scalar-valued functions. The dimension of $V^*$is equal to the dimension of V, and the dual basis corresponding to a given basis of V provides an important link between the space and its dual. Dual spaces are widely used in functional analysis, quantum mechanics, and differential geometry, as they provide a natural framework to study bilinear forms, adjoint operators, and duality principles in mathematics.

7.7 *GLOSSARY*:-

- **Dual Space** ($\boldsymbol{V^*}$):The set of all linear functionals from a vector space V into its underlying field F.
- **Linear Functional:** A linear transformation f:V→F that maps vectors to scalars while preserving linearity, i.e.,

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v).$$

- **Basis of a Vector Space:**A linearly independent set of vectors in V that spans the entire space.
- **Dual Basis:** For a basis $\{v_1, v_2, \dots \dots, v_n\}$ of V, the corresponding dual basis $\{f_1, f_2, \dots \dots, f_n\}$ in $\boldsymbol{V}^*$is defined such that $f_i(v_j) = \delta_{ij}$.
- **Kronecker Delta ($\boldsymbol{\delta_{ij}}$):** A function defined as $\delta_{ij} = 1$, and $\delta_{ij} = 0$ if $i \neq 0$. Used in defining dual bases.
- **Dimension of Dual Space:** If $dim(V) = n$, then $dim(\boldsymbol{V}^*) = n$.

- **Evaluation Map:** A natural map $\phi: V \to \boldsymbol{V}^{**}$(double dual), defined by $\phi(v)(f) = f(v)$, where $f \in \boldsymbol{V}^{*}$.
- **Double Dual ($\boldsymbol{V}^{**}$):**The dual space of the dual space. There is a canonical isomorphism $V \cong \boldsymbol{V}^{**}$for finite-dimensional spaces.
- **Annihilator:** For a subspace $W \subseteq V$, the annihilator $W_0 \subseteq V*$ is the set of all functionals $f \in \boldsymbol{V}^{*}$such that $f(w) = 0$ for all $w \in W$.
- **Reflexivity:** The property that a finite-dimensional vector space V is naturally isomorphic to its double dual $\boldsymbol{V}^{**}$.

7.8 *REFERENCES*: -

- **Gilbert Strang (2016)** – *Introduction to Linear Algebra*, 5th Edition, Wellesley Cambridge Press.
- **Sheldon Axler(2015)** – *Linear Algebra Done Right*, 3rd Edition, Springer.
- **David C. Lay, Steven R. Lay, Judi J. McDonald** (2016)– *Linear Algebra and Its Applications*, 5th Edition, Pearson.

7.9 SUGGESTED *READING*: -

- A.R. Vasishtha, J.N.Sharma and A.K. Vasishtha (52th Edition, 2022), Krishna Publication, Linear Algebra.
- K.P.Gupta (20th Edition,2019), Pragati Publication, Linear Algebra

7.10 *TERMINAL QUESTIONS: -*

(TQ-1) Let V be a vector space over the field F. Let f be a non-zero linear functional on V and let N be the null space of f. Fix a vector $\alpha_0 \in V$ which is not in N. Prove that for each $\alpha \in V$ there is a scalar c *and* a vector β in N such that $\alpha = c\alpha_0 + \beta$. Prove that c *and* β are unique.

(TQ-2) Prove that every finite dimensional vector space Vis isomor-phic to its second conjugate space v ** under an isomorphism which is independent of the choice of a basis in V.

(TQ-3) Find the dual basis of the basis set $B = \{(1,0,0)(0,1,0),(0,0,1)\}$ for $V_3(R)$.

(TQ-4) Find the dual basis of the basis set $B = \{(1,-2,3)(1,-1,1),(2,-4,7)\}$ of $V_3(R)$.

(TQ-5) Prove that if f is a linear functional on an n-dimen-sional vector space $V(F)$, then the set of all those vectors α for which $f(\alpha) = 0$ is a subspace of V, what is the dimension of that subspace?

(TQ-6) Explain the difference between the dual space and the bidual space (double dual).

(TQ-7) The vectors $\alpha_1 = (1,1,1), \alpha_2 = (1,1,-1)$ and $\alpha_3 = (1,-1,-1)$ form a basis $V_3(C)$ If $\{f_1, f_2, f_3\}$ the dual basis and if $\alpha = (0,1,0)$, find $f_1(\alpha), f_2(\alpha) and$ $f_3(\alpha)$.

(TQ-8) Find the dual basis of the basis set $B = \{(1,1,3),(0,1,-1),(0,3,-2)\}$ for $V_3(R)$.

7.11 *ANSWERS: -*

(TQ-3) $B' = \{f_1, f_2, f_3\}$ where $f_1(a,b,c) = a, f_2(a,b,c) = b$ $f_3(a,b,c) = c$

(TQ-4) $B' = \{f_1, f_2, f_3\}$ where $f_1(a,b,c) = -3a - 5b - 2c, f_2(a,b,c) = 2a + b$ $f_3(a,b,c) = a + 2b + c$

BLOCK - III

ALGEBRA OF POLYNOMIAL AND CANNONICAL FORM

UNIT-8: ALGEBRA OF POLYNOMIAL

CONTENTS

8.1 INTRODUCTION

In this unit, we focused on algebra of polynomial, Vandermonde matrix, eigen avalue and eigen function. Now again, we emphasize on Vector Spaces. After the study of Linear Transformation, we have studied some properties of a linear operator. Here, we shall elaborate these concepts and matrices help us in a great deal. Basis of a matrix and its role to understand eigen values and eigen vectors will be discussed in detail. Besides this, diagonalisation process and required conditions will be discussed thoroughly.

8.2 OBJECTIVE

After the study of this chapter, learner shall understand:

- Linear operators and their properties.
- Polynomial of matrices.
- Vandermonde matrix
- For finite-dimensional vector spaces, T can be represented as a matrix.
- How can we convert square matrix into diagonal matrix?
- Role of basis of a linear transformation in diagonalisation.

8.3 POLYNOMIAL OF MATRICES

In linear algebra, the algebra of polynomials plays a central role, especially when dealing with linear operators and matrices.

Definition: Let $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$

be a polynomial with coefficients from a field F (e.g., R, C).

If A is a square matrix of order $m \times m$, then the polynomial of matrix A is defined as:

$$p(A) = a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n$$

Where, I is the identity matrix of the same size as A.

Example 1 (*i*): If $p(x) = 2 + 3x + x^2$ then for matrix A, $p(A) = 2I + 3A + A^2$.

(*ii*): If $p(x) = x^3 - 5x + 7$ then for matrix A, $p(A) = A^3 - 5A + 7I$.

Example 2: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, then $A^2=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}$. Let $f(x)=2x^2-3x+5$ and $g(x)=x^2-5x-2$ then, $f(A)=2A^2-3A+5I=\begin{bmatrix}14 & 20\\ 30 & 44\end{bmatrix}+\begin{bmatrix}-3 & -6\\ -9 & -12\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}=\begin{bmatrix}16 & 14\\ 21 & 37\end{bmatrix}$

And $g(A)=A^2-5A-2I=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}+\begin{bmatrix}-5 & -10\\ -15 & -20\end{bmatrix}+\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Thus, we can say that A is zero of $g(t)$.

Properties: Let f, g be the polynomials. Then for any square matrix A and scalar k, we have the followings:

(*i*) $(f+g)(A)=f(A)+g(A)$

(*ii*) $(kf)(A)=kf(A)$

(*iii*) $(fg)(A)=f(A)g(A)$

(*iv*) $f(A)g(A)=g(A)f(A)$

Point (*iv*) shows that any two polynomials in *A* commute with each other.

8.4 *FACTORISATION OF POLYNOMIAL*

Let the ring $K[t]$ of polynomials over a field *K*. A polynomial $p\in K[t]$ of positive degree is said to be irreducible if $p=fg$ implies f *or* g is a scalar.

Lemma 1: Suppose $p\in K[t]$ is irreducible. If p divides the product fg of polynomials $f, g\in K[t]$, the p divides g. More, generally, if p divides the product of n polynomials $f_1, f_2, ..., f_n$, then p divides one of them.

Proof: Suppose p divides fg but not f. Because p is irreducible, the polynomials f and p must then be relative prime. Thus, there exist polynomials $m, n\in K[t]$ such that $mf+np=1$. Multiplying this equation by g, we obtain $g=mfg+npg$.

Now suppose p divides $f_1, f_2, ..., f_n$. If p divides f_1, then we are through. If not, then by the above result p divides the product $f_2, ..., f_n$. By induction on n, p divides one of the polynomials $f_2, ..., f_n$. Thus, the lemma is proved.

Unique Factorization Theorem 1: Let f be a non-zero polynomial in $K[t]$. Then f can be written uniquely (except for order) as a product, $f = kp_1p_2...p_n$, where $k \in K$ and p_i are monic irreducible polynomials in $K[t]$.

Proof: We prove the existence of such a product first. If f is irreducible or if $f \in K$, then such a product clearly exists. On the other hand, suppose $f = gh$ where f and g are nonscalars. Then *g* and *h* have degrees less than that of *f*. By induction, we can assume

$g = k_1g_1g_2...g_r$ and $h = k_2h_1h_2...h_s$

Where $k_1, k_2 \in K$ and g_i and h_j are monic irreducible polynomials. Accordingly,

$f = (k_1k_2)g_1g_2...g_r.k_1h_2h_3...h_s$, is our desired representation.

We next prove uniqueness (except for order) of such a product for f. Suppose

$f = (k_1)p_1p_2...p_r. = k^{'}q_1q_2...q_m$

Where $k, k^{'} \in K$ and $p_1, p_2,...p_n, q_1, q_2,..., q_m$ are monic irreducible polynomials. Now p_1 divides $k^{'} q_1, q_2,..., q_m$. Because p_1 is irreducible, it must divide one of the q_i by the lemma. Say p_1 divides q_1. Because p_1 and q_1 are both irreducible and monic, $p_1 = q_1$. Accordingly,

$kp_2...p_n = k^{'}q_2...q_m$

By induction, we have that $n = m$ and $p_2 = q_2,..., p_n = q_m$ for some rearrangement of the q_i. We also have that $k = k^{'}$. Thus, the theorem is prove.

When the field *K* is the complex field *C*, we obtain the result known as the Fundamental Theorem of Algebra, whose proof is beyond the scope of this discussion.

Corollary 1: (Fundamental theorem of algebra): Let $f(t)$ be a non-zero polynomial over the complex field *C*. Then $f(t)$ can be written uniquely (except order) as a product $f(t) = k(t - r_1)(t - r_2)...(t - r_n)$

Where, $k, r_i \in C$ as a product of linear polynomials.

In the case of the real field R we have the following result.

Corollary 2: (Fundamental theorem of algebra): Let $f(t)$ be a non-zero polynomial over the complex field *R*. Then $f(t)$ can be written uniquely (except order) as a product

$$f(t) = kp_1(t)p_2(t)...p_m(t)$$

Where, $k \in R$ and $p_i(t)$ are monic irreducible polynomials of degree one or two.

8.5 *LAGRANGES INTERPOLATION*

Lagrange's Interpolation is a technique to find out a unique polynomial of degree at most *n* that passes through *n*+1 given distinct points $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$.

i.e., if want to find out a unique polynomial $P(x)$ of degree at most n such that:

$P(x_i) = y_i \; for\, i = 0,1,...,n$. Then the Lagrange interpolation polynomial is:

$P(x) = \sum_{i=0}^{n} y_i L_i(x)$, where, the Lagrange basis polynomial are $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$.

Here, each $L_i(x)$ satisfies:

(i) $L_i(x_i) = 1$

(ii) $L_i(x_j) = 0$ for $j \neq i$

Thus, each term $y_i L_i(x)$ contributes only at its own x_i

Example 3: Suppose we want the polynomial that passes through the points $(1,2),(2,3),(4,5)$ then find the interpolating polating polynomial.

Solution: First, compute the basis polynomials:

$$L_0(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{(x-2)(x-4)}{(-1)(-3)} = \frac{(x-2)(x-4)}{3}$$

$$L_1(x) = \frac{(x-1)(x-4)}{(2-1)(2-4)} = \frac{(x-1)(x-4)}{(1)(-2)} = \frac{(x-1)(x-4)}{2}$$

$$L_2(x)=\frac{(x-1)(x-2)}{(4-1)(4-2)}=\frac{(x-1)(x-2)}{(3)(2)}=\frac{(x-1)(x-2)}{6}$$

So, the required interpolating polynomial is $P(x)=2L_0(x)+3L_1(x)+5L_2(x)$

8.6 *VANDERMONDE MATRIX*

A Vandermonde matrix is a special type of matrix where each row is a **geometric progression** of the corresponding x_i. For $n+1$ distnict numbers $x_0, x_1, ..., x_n$, the Vandermonde matrix is:

$$V=\begin{bmatrix} 1 & x_0 & x_0^{\;2} & \cdots & x_0^{\;n} \\ 1 & x_1 & x_1^{\;2} & \cdots & x_1^{\;n} \\ 1 & x_2 & x_2^{\;2} & \cdots & x_2^{\;n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^{\;2} & \cdots & x_n^{\;n} \end{bmatrix}$$

Here, each row corresponds to powers of one interpolation point.

- It is used to set up equations for finding coefficients of the interpolating polynomial.
- Its determinant is, $\det(V)=\prod_{0\le i\le j\le n}(x_j-x_i)$

which is nonzero if all x_i are distinct.

- This ensures that the interpolation problem has a unique solution**.**

Note 1: The Vandermonde matrix links interpolation points to polynomial coefficients and guarantees uniqueness when the x_i values are distinct.

2: If we want to find a polynomial $P(x)=a_0+a_1x+a_2x^2+...+a_nx^n$, that passes through the points (x_i, y_i), we can write the system:

$$V\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}=\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Thus, solving this system gives the coefficients of the interpolating polynomial.

Example 4: Let we have three data points, $(0,1),(1,3),(2,2)$ and we want to find the polynomial,

$P(x) = a_0 + a_1 x + a_2 x^2$.

Now, in the first step, the Vandermonde matrix is,

$$V = \begin{bmatrix} 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

In second step, write the system: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

In third step, solve the equation:

On solving the first row: $a_0 = 1$.

From second row: $1 + a_1 + a_2 = 3 \Rightarrow a_1 + a_2 = 2$.

From third row: $1 + 2a_1 + 4a_2 = 2 \Rightarrow 2a_1 + 4a_2 = 1$.

So, we get the unknown, $a_0 = 1;\ a_1 = \frac{7}{2};\ a_2 = -\frac{3}{2}$

Hence the final polynomial is, $P(x) = 1 + \frac{7}{2}x - \frac{3}{2}x^2$.

8.7 *POLYNOMIAL IDEAL*

A polynomial ideal is a set of polynomials closed under addition and multiplication by any polynomial, usually described by its generators. They are the algebraic foundation for solving systems of polynomial equations.

When the ring is a polynomial ring $R[x_1, x_2, ..., x_n]$(for example $R[x, y]$), we talk about polynomial ideals.

- A **polynomial ideal** is just an ideal in this polynomial ring.
- That means: it is a set of polynomials closed under addition and under multiplication by any polynomial from the ring.

Generators of polynomial ideals

Every polynomial ideal can be described by its **generators**.
If $f_1, f_2, ..., f_k$ are polynomials, then the ideal generated by them is:

$$< f_1, f_2, ..., f_k >= \{g_1 f_1 + g_2 f_2 + ... + g_k f_k \mid g_i \in R[x_1, x_2, ..., x_n]\}$$

This means: all combinations of $f_1, f_2, ..., f_k$ multiplied by arbitrary polynomials.

Example 5: In $R[x] : < x^2 >= \{x^2 h(x) \mid h(x) \in R[x]\}$

i.e., all polynomials divisible by x^2.

In $R[x, y] : < x, y >= \{xh(x, y) + yg(x, y) \mid h, g \in R[x, y]\}$

i.e., all polynomials with no constant term (since every term is at least divisible by *x* or *y*).

Importance: Polynomial ideals are central in algebraic geometry and computational algebra**:**

- They describe sets of polynomial equations.
- The **variety** of an ideal $< f_1, f_2, ..., f_k >$ is the set of all points where all f_i vanish.
- Algorithms like Gröbner bases are used to work with polynomial ideals computationally.

8.8 *TAYLOR'S FORMULA*

For a smooth scalar function $f(x)$, the Taylor expansion around *a* point a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x)$ is the remainder term.

In linear algebra, we often apply Taylor's formula to matrix functions**.**

Suppose A is a square matrix and $f(x)$ is a smooth function (like e^x, $\sin x$, $\cos x$, $In(1+x)$, etc.). Then we can define:

$$f(A) = f(0)I + f^{'}(0)A + \frac{f^{''}(0)}{2!}A^2 + \ldots + \frac{f^{n}(0)}{n!}A^n + \ldots$$

This is just the Taylor series expansion of $f(x)$, with x replaced by the matrix A.

8.9 *ALGEBRICALLY CLOSED FIELD*

Definition: A field F is called algebraically closed if every non-constant polynomial with coefficients in F has at least one root in F. In other words,

If $p(x) \in F[x]$ and degree$(p) \geq 1$ then there exist some $a \in F$ such that $p(a) = 0$

A field F is algebraically closed if:

1. Every polynomial in $F[x]$ factors completely into linear factors over F.

   $p(x) = c(x-a_1)(x-a_2)\ldots(x-a_n); a_i \in F$

2. It has no proper algebraic extensions (it is "maximal" in that sense).

Example 6: The polynomial $x^2+1=0$ has $\pm i \in C$ but not algebraically closed in Real number (R) and also not algebraically closed in rational number (Q)

8.10 *BASICS OF LINEAR OPERATORS*

In this section, we shall discuss linear operators (T) on a finite-dimensional vector space V(**F**). We know that a linear operator T on a vector space V(**F**) is a mapping T: V → V, such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall \; x, y \in V \text{ and } \alpha, \beta \in \mathbf{F}.$$

We have already studied following **important properties** of such a linear operator T:

(i) T is non-singular (i.e. one to one) if and only if ker(T) = {0}.
(ii) T is invertible ⇔ T is non-singular ⇔ T is onto.
(iii) T is singular ⇔ ker T ≠ {0}

8.11 *EIGEN VALUES & EIGEN VECTORS*

Now, we shall define eigen value and eigen vectors of T as:

Eigen Values of T: Let T be linear operator on a vector space V(**F**). A scalar $\alpha \in \mathbf{F}$ is called an eigen value or characteristic value of T, if there exists some $V \neq 0$, $v \in V$ such that, $T(v) = \alpha$ v.

Eigen Vector: If α is an eigen value of T, then $v \in V$ such that $T(v) = \alpha$ v is called an eigen vector or **characteristic vector** belonging to α.

Eigen space: The set of all eigenvectors of T belonging to an eigenvalue α is called an eigenspace of T, belonging to α. It is represented as W_α. Hence

$$W_\alpha = \{ v \in V : T(v) = \alpha v \}.$$

Example 7: Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a linear operator defined be T(x, y) = (2x + y, x + 2y). By trial and error method, we find one eigen value of T and corresponding eigen vector.

We observe that $T(1, 1) = (3, 3) = 3(1, 1)$

Or $T(2, 2) = (6, 6) = 3(2, 2)$

Here 3 is an eigenvalue of T and (1, 1) , (2, 2) $\in \mathbf{R}^2$ are corresponding eigenvectors.

Also T(3, -3) = (3, -3) = 1(3, -3)

So here 1 is eigenvalue of T and (3, -3) $\in \mathbf{R}^2$ is corresponding eigenvector.

Example 8: Let $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a linear operator whose matrix with respect to the standard basis $\{ e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So, $T(e_1) = e_1 = 1e_1$

$T(e_2) = e_2 = 1e_2$

$T(e_3) = 0 = 0\ e_3.$

We observe that 1, 1 and 0 are eigenvalues of T and corresponding eigenvectors are e_1, e_2 and e_3 respectively.

Note: Now we discuss the eigenspace W_1 corresponding to eigenvalue 1. So $W_1 = \{ v \in \mathbf{R}^3 : T(v) = 1.v \}$. Let $v \in \mathbf{R}^3$. then there exist $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$v = \alpha e_1 + \beta e_2 + \gamma e_3$$

So $v \in W_1$ *iff* $T(\alpha e_1 + \beta e_2 + \gamma e_3) = 1(\alpha e_1 + \beta e_2 + \gamma e_3)$

iff $\alpha T(e_1) + \beta T(e_2) + \gamma T(e_3) = \alpha e_1 + \beta e_2 + \gamma e_3$

iff $\alpha e_1 + \beta e_2 + \gamma.0\, e_3 = \alpha e_1 + \beta e_2 + \gamma e_3$

iff $\gamma e_3 = 0$ or $\gamma = 0$

So $W_1 = \{ \alpha e_1 + \beta e_2 : \alpha, \beta \in \mathbf{R} \}$. In the same way, we can show that the eigenspace W_0, corresponding to eigenvalue '0' is

$$W_0 = \{ \gamma e_3 : \gamma \in \mathbf{R} \}$$

Theorem 2: Let T be a linear operator on a vector space V(**F**).

(i) If $0 \neq v \in V$ is an eigenvector of T, then $\alpha \in \mathbf{F}$ satisfying $T(v) = \alpha v$ is **unique**.

(ii) The eigenspace W_α corresponding to an eigen value $\alpha \in \mathbf{F}$ is a subspace of V(**F**).

(iii) $W_\alpha = \ker(T - \alpha I)$.

Proof: (i) As we know, for uniqueness; we always consider two values and show that both are equal *i.e*. value is unique. Suppose, if possible, there exist $\alpha, \beta \in \mathbf{F}$ such that $T(v) = \alpha v$ and $T(v) = \beta v$

$$\Rightarrow \quad \alpha \beta = \beta v \quad \text{or} \quad (\alpha - \beta)v = 0$$

But $v \neq 0$, so

$$\alpha - \beta = 0 \quad \text{or} \quad \alpha = \beta$$

Hence α is unique.

(ii) We know that $W_\alpha = \{ v \in V : T(v) = \alpha v \}$

Claim: W_α is a subspace of V(**F**). As $T(0) = 0 \Rightarrow T(0) = \alpha 0$. So $0 \in W_\alpha$ i.e. W_α is non-empty. Let $v_1, v_2 \in W_\alpha$ and $a, b \in \mathbf{F}$. then

$T(v_1) = \alpha v_1$ and $T(v_2) = \alpha v_2$

Now, $T(av_1 + bv_2) = aT(v_1) + b\,T(v_2)$; as T is linear

$= a(\alpha v_1) + b(\alpha v_2)$

$T(av_1 + bv_2) = a(\alpha v_1) + b(\alpha v_2)$

So, $av_1 + bv_2$ is an eigenvector, corresponding to eigenvalue α.

Hence $av_1 + bv_2 \in W_\alpha$, $\forall\, v_1, v_2 \in W_\alpha$; $a, b \in \mathbf{F}$.

Hence W_α is a subspace of V(**F**).

(iii) By definition,

$$W_\alpha = \{ v \in V : T(v) = \alpha v \}$$

So $W_\alpha = \{ v \in V : T(v) = \alpha Iv \}$, where I is identity operator

$$= \{ v \in V : T(v) = (\alpha I) v \}$$

$$= \{ v \in V : (T - \alpha I) v = 0 \}$$

Hence $W_\alpha = \ker (T - \alpha I)$.

Theorem 3: Let T be a linear operator on a finite-dimensional vector space V(**F**). Then $\alpha \in \mathbf{F}$ is an eigenvalue of T if and only if $T - \alpha I$ is singular.

Proof: Necessary Condition: Let α be an eigenvalue of T. Then there exists some $0 \neq v \in V$, Such that, $T(v) = \alpha v$

$\Rightarrow$ $T(v) = \alpha I(v)$ where I is identity operator.

$\Rightarrow$ $T(v) = (\alpha I)(v)$

$\Rightarrow$ $(T - \alpha I)(v) = \mathbf{0}$, where $v \neq \mathbf{0}$.

So $v \in \ker (T - \alpha I)$. We already know that $\mathbf{0} \in \ker (T - \alpha I)$. So, $\ker (T - \alpha I) \neq \{\mathbf{0}\}$.

Hence $T - \alpha I$ is singular.

Sufficient condition: Let $T - \alpha I$ be singular operator .

$\Rightarrow \ker(T - \alpha I) \neq \{\mathbf{0}\}$,

$\Rightarrow$ there exists some $\mathbf{0} \neq v \in V$, such that $(T - \alpha I)(v) = \mathbf{0}$.

$\Rightarrow T(v) - \alpha I(v) = \mathbf{0}$.

$T(v) = \alpha v$, where $I(v) = v$.

So, α is an eigenvalue of T.

Note: (1) If T is singular, then '**0**' is always an eigenvalue of T. As $T - \mathbf{0}I = T$, it can be obviously observed.

(2) Till now, we have understood that if T is a linear operator on a finite-dimensional vector space, then the following statements are equivalent:

(i) α is an eigenvalue of T.

(ii) The operator $T - \alpha I$ is singular or non-invertible.

(iii) $\det(T - \alpha I) = 0$.

Characteristic values and Characteristic polynomial of a matrix:

Suppose T be a linear operator on a finite dimensional (say dim V = n) vector space V(**F**). Let β be an ordered basis for V and let A be the matrix of T with respect to the basis β i.e. $A = [T]_\beta$. For any scalar $\alpha \in \mathbf{F}$, we have

$$[T - \alpha I]_\beta = [T]_\beta - \alpha [I]_\beta$$

$$= A - \alpha I, \text{ where I is } n \times n \text{ unit matrix.}$$

So $\det(T - \alpha I) = \det[T - \alpha I]_\beta = \det(A - \alpha I)$. Hence α is a characteristic value of T **if and only if** $\det(A - \alpha I) = 0$.

Note: From above discussion, we **conclude** that –

(i) Let $A = [a_{ij}]_{n \times n}$; $a_{ij} \in \mathbf{F}$. A scalar $\alpha \in \mathbf{F}$ is called an **eigen value** of A if $\det(A - \alpha I) = 0$.

(ii) Let $A = [a_{ij}]_{n \times n}$; $a_{ij} \in \mathbf{F}$. Then the polynomial f(x) = det (A – α I) is called the **characteristic polynomial** of the matrix A.

The equation f(x) = 0 is called the **characteristic equation** of the matrix A. Here we observe that $\alpha \in \mathbf{F}$ is an eigen value of the matrix A if and only if f(α) = 0.

Similar Matrices: Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and $P = [c_{ij}]_{n \times n}$ where $a_{ij}, b_{ij}, c_{ij} \in \mathbf{F}$. Then A and B are called similar matrices if, there exist a matrix P such that

$$A = P^{-1} B P$$, where P is non-singular matrix.

You might have studied that **similarity** of matrices is an **equivalence relation** i.e. it is reflexive, symmetric and transitive.

Theorem 4: Similar matrices have the same characteristic polynomial and hence the same characteristic values.

Proof: Let us consider two square matrices A and B of $n \times n$ order. Then A and B are similar i.e. there exists an non-singular matrix P such that

$$B = P^{-1} A P.$$

So, $B - xI = P^{-1} A P - xI = P^{-1} A P - x P^{-1} I P$

$$= P^{-1} (A - xI) P.$$

So, $\det (B - xI) = \det (P^{-1} (A - xI) P)$

$$= \frac{1}{\det P} \det (A - xI) \det P$$

$\det (B - xI) = \det (A - xI)$.

⇒ A and B have the same characteristic polynomials and consequently same eigenvalues.

Note: (1) You have studied in the chapter '**Linear Transformation**' that, if T be linear operator on an n-dimensional vector space. If β , β' are two ordered bases of V such that $A = [T]_\beta$ and $B = [T]_{\beta'}$, then there exists a non-singular matrix P (over **F**) such that $B = P^{-1} A P$.

(2) Let T be a linear operator on a finite-dimensional vector space V(**F**). then the characteristic polynomial of T is det(A – xI) , where A is the matrix of T in any ordered basis for V.

(3) If T is a linear operator on an n-dimensional vector space and if A = $[T]_\beta$ with respect to an ordered basis β for V, then A is an $n \times n$ matrix and so det (A – xI) is a polynomial of degree n. Hence T **cannot** have more than n distinct eigenvalues.

(4) The eigenvalues of a linear operator defined on V(F) **may not** belong to **F**. For example, let T be a linear operator an $R^2(\mathbf{R})$, whose matrix with respect to the standard ordered basis is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic polynomial of A is det (A – xI) = 0

$$\Rightarrow \quad x^2 + 1 = 0$$

This equation has no roots in **R** (though, its roots $x = \pm i \in \mathbf{C}$).

Cayley-Hamilton Theorem 5 (for a linear operator): Every linear operator T on an n-dimensional vector space V(**F**) satisfies its characteristic equation f(x) = 0, i.e. f(T) = 0.

Proof: Let A be the matrix of T with respect to any basis β of V. So, A = $[T]_\beta$

Hence for matrices, Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation. Hence if $f(x) = \det(A - xI) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n = 0$, is the characteristic equation of A, then

$$f(A) = a_0I + a_1A + a_2A^2 + \ldots + a_n A^n = 0$$

$$\Rightarrow \quad a_0[I]_\beta + a_1[T]_\beta + a_2[T^2]_\beta + \ldots + a_n[T^n]_\beta = [0]_\beta$$

$$\Rightarrow \quad [f(T)]_\beta = [0]_\beta$$

Hence $f(T) = a_0 I + a_1T + a_2 T^2 + \ldots + a_n T^n = 0$

Example 9: Find the eigen values, eigen vectors and eigen spaces of A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: Step-I: Characteristic equation of A is |A – xI| = 0

$$\Rightarrow \quad \begin{vmatrix} -x & 1 \\ 1 & -x \end{vmatrix} = 0 \text{ or } x^2 - 1 = 0 \text{ or } x = \pm 1$$

Hence eigenvalues of A are {+1, -1}.

Step-II: An eigenvector X, corresponding to the eigenvalue 1 is given by

$AX = \alpha X$ or $(A - \alpha I)X = 0$.

Here $\alpha = 1$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So, $(A - \alpha I)X = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = x_2$.

We can take any value for solution. Let $x_1 = x_2 = 1$. Then an eigen vector corresponding to α = 1 is

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [1 \;\; 1]^T$.

Again eigenvector for $\alpha = -1$ is

$(A - \alpha I)X = 0$ or $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 + x_2 = 0$

If $x_1 = 1$, then $x_2 = -1$

So an eigenvector corresponding to $\alpha = -1$ is $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Step-III: The two eigenspaces W_1 and W_{-1} are given by

$W_1 = \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbf{R}\}$ and $W_{-1} = \{\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \beta \in \mathbf{R}\}$.

Example 10: Let T : $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a linear operator, where

$T(e_1) = 5e_1 - 6e_2 - 6e_3$; $T(e_2) = -e_1 + 4e_2 + 2e_3$; $T(e_3) = 3e_1 - 6e_2 - 4e_3$

Find the characteristic values of T and compute the corresponding eigenvectors.

Solution: On the basis of given relations, the matrix of T is

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -6 & 4 & -6 \\ -6 & 2 & -4 \end{bmatrix}$$

So the characteristic equation is det (A – xI) = 0.

$$\Rightarrow \quad \begin{vmatrix} 5-x & -1 & 3 \\ -6 & 4-x & -6 \\ -6 & 2 & -4-x \end{vmatrix} = 0.$$

On solving, we get x = 1, 2, 2. So eigenvalues of T are 1, 2, 2.

Case-I: An eigenvector corresponding to $\alpha = 2$ is given by

$$(A - 2I)X = 0$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 3 \\ -6 & 2 & -6 \\ -6 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + 2R_1$

$$\Rightarrow \quad \begin{bmatrix} 3 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since rank of coefficient matrix = number of non-zero rows = 1. So, n – r or 3 – 1 = 2 variables can be given arbitrary values.

So we have $3x_1 - x_2 + 3x_3 = 0$ …..(1)

If we take $x_3 = 0$, we get one arbitrary solution $X = [1 \quad 3 \quad 0]^T$.

If we take $x_2 = 0$, we get $X = [1, \quad 0, \quad -1]^T$.

So, two eigenvectors corresponding to $\alpha = 2$ are

$$X_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Case-II: Now eigenvector corresponding to $\alpha = 1$ is

$$(A - I)X = 0$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & 3 \\ -6 & 3 & -6 \\ -6 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow \frac{R_1}{4}$, we get

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{3}{4} \\ -6 & 3 & -6 \\ -6 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 6R_1$ and $R_3 \to R_3 + 6R_1$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{2}{3} R_2$ and $R_3 \to 2R_3$

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence rank of coefficient matrix is 2. So only 3 – 2 = 1 variable can be given arbitrary value.

Now $x_1 - \frac{x_2}{4} + \frac{3x_3}{4} = 0$ and $0 + x_2 - x_3 = 0$

Let $x_3 = \lambda \in \mathbf{R}$, then $x_2 = \lambda$

So $x_1 = \frac{\lambda}{4} - \frac{3\lambda}{4} = -\frac{\lambda}{2}$

So $X = [x_1 \quad x_2 \quad x_3]^T = [-\frac{\lambda}{2} \quad \lambda \quad \lambda]^T = [-\frac{1}{2} \quad 1 \quad 1]^T = [-1 \quad 2 \quad 2]^T$.

Example 11: Show that the eigen values of a diagonal matrix are exactly the elements in the diagonal. Hence prove that if a matrix B is similar to a diagonal matrix D, then the diagonal elements of D are the eigen values of B.

Solution: Step-I: Let A = $\begin{bmatrix} a_{11} & 0 & 0 & \dots & \dots & 0 \\ 0 & a_{22} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & a_{nn} \end{bmatrix}$

Characteristic equation of A is det (A – xI) = 0. So

$$\begin{vmatrix} a_{11}-x & 0 & 0 & \dots & \dots & 0 \\ 0 & a_{22}-x & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & a_{nn}-x \end{vmatrix} = 0$$

$\Rightarrow \quad (a_{11}-x)(a_{22}-x) \ldots\ldots (a_{nn}-x) = 0$

$\Rightarrow x = a_{11}, a_{22}, \ldots, a_{nn}$.

Hence the eigenvalues of A are its diagonal entries.

Step-II: We have already proved that similar matrices have identical eigenvalues. So both matrices have same eigen values.

Example 12: Let V be the vector space of all real-valued continuous functions. Prove that the linear operator T: V→V defined as (Tf)x = $\int_0^x f(t)dt$ has no eigenvalues.

Solution: Suppose α is an eigenvalue of T. Then there exists some $0 \neq f \in V$ such that $Tf = \alpha f$.

$\Rightarrow \quad (Tf)(x) = (\alpha f)(x)$

$\Rightarrow \quad \int_0^x f(t)dt = \alpha f(x)$, by given condition ….(1)

Differentiating with respect to x, we get

$f(x) = \alpha f'(x)$, or $\frac{f'(x)}{f(x)} = \frac{1}{\alpha}$, considering $\alpha \neq 0$

On integration, $\log_e f(x) = \frac{x}{\alpha} + \log_e a$ or $f(x) = a\, e^{x/\alpha}$ ….(2)

Putting x = 0 in equation (2), we get

$f(0) = ae^0$ or $a = f(0)$

So $f(x) = f(0)\, e^{x/\alpha}$ ….(3)

For equation (3) , we have

$$\int_0^x f(0) e^{t/\alpha}\, dt = \int_0^x f(t)\, dt$$

$f(0)\, (\alpha\, e^{t/\alpha})_0^x = \alpha\, f(x)$, using equation (1)

$\Rightarrow$ $f(0)\, \alpha\, (e^{x/\alpha} - 1) = \alpha\, f(0)\, e^{x/\alpha}$; using (3)

$\Rightarrow$ $\alpha\, e^{x/\alpha} - \alpha = \alpha\, e^{x/\alpha}$

$\Rightarrow$ $\alpha = 0$, contradiction.

So initial assumption was wrong. Hence T has no eigenvalue.

Note: We observed that diagonal matrices are easiest to find eigen values. So it is a natural question, whether we can transform every square matrix into diagonal matrix?

The answer is **NO**. Then there is a need of condition for that. Let us study these basics:

8.12 *DIAGONALIZABLE OPERATOR*

A linear operator T on a finite-dimensional vector space V(F) is called diagonalizable, if there exists an ordered basis β of V such that the matrix of T with respect to the basis β is a diagonal matrix, so

$$[T]_\beta = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_n \end{bmatrix} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n).$$

Diagonalizable matrix: An $n \times n$ matrix A over a field **F** is said to be diagonalizable, if it is **similar** to a diagonal matrix. Also A is diagonalizable if there exists an invertible matrix P such that $P^{-1} A P = D$, where D is a diagonal matrix. The matrix P is our *actual need*.

8.13 *BASIS OF DIAGONALIZABLE OPERATORS*

Theorem 6: A linear operator T on a finite-dimensional vector space V(**F**) is diagonalizable if and only if there exists a basis of V consisting of eigenvectors of T.

Proof: If Part: Let T be diagonalizable, Then ∃ an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of V such that the matrix of T relative to β is $[T]_\beta = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_n \end{bmatrix}$. From above expression, we get,

$T(V_1) = \alpha_1 v_1 + 0v_2 + \ldots + 0v_n$

$T(V_2) = 0v_1 + \alpha_2 v_2 + \ldots + 0v_n$

- -

$T(V_n) = 0 + 0 + \ldots + \alpha_n v_n$

Or , we can write $T(V_i) = \alpha_i v_i$; i = 1, 2, …, n. Hence $v_1, v_2, \ldots, v_n$ are eigenvectors of T i.e. the basis β consists of eigenvectors of T.

Only if part: Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of T. Then, $\exists\ \alpha_i \in$ **F** such that

$T(V_i) = \alpha_i v_i$; i = 1, 2, …, n.

So, $[T]_\beta = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \alpha_n \end{bmatrix}$

Hence T is diagonalizable.

Theorem 7: Let T be a linear operator on a finite-dimensional vector space V(**F**). Then the eigenvectors corresponding to distinct eigenvalues of T are linearly independent.

Proof: Let $\alpha_1, \ldots, \alpha_m$ be m distinct eigen values of T and let $v_1, \ldots, v_m$ be the corresponding eigen vectors of T. Then

$T(V_i) = \alpha_i v_i$; i = 1, 2, …, m ….(1)

Claim: S = {v_1, ….., v_m} is linearly independent. Here we use the **principle of mathematical induction**. If m = 1, then S = { v_1} where $v_1 \neq 0$. We know that a single non-zero vector is always linearly independent. So result is true for = 1. Suppose the set { v_1, ..., v_k} is linearly independent, where k < m. We shall prove that the set { v_1, ..., v_k , v_{k+1}} is also linearly independent .

Let $\beta_1 v_1 + \ldots.. + \beta_k v_k + \beta_{k+1} v_{k+1} = 0$; $\beta_i \in$ **F** ….(2)

$\Rightarrow \quad T(\beta_1 v_1 + \ldots.. + \beta_k v_k + \beta_{k+1} v_{k+1}) = T(0)$

$\Rightarrow \quad \beta_1 T(v_1) + \ldots. + \beta_k T(v_k) + \beta_{k+1} T(v_{k+1}) = 0$

$\Rightarrow \quad \beta_1 (\alpha_1 v_1) + \ldots. + \beta_k (\alpha_k v_k) + \beta_{k+1} (\alpha_{k+1} v_{k+1}) = 0$ ….(3)

Multiplying equation (2) by α_{k+1} and then subtracting from equation (3), we get

$$\beta_1(\alpha_1 - \alpha_{k+1})v_1 + \ldots. + \beta_k(\alpha_k - \alpha_{k+1})v_k = 0.$$

But v_1,.., v_n are linearly independent.

So $\beta_1(\alpha_1 - \alpha_{k+1}) = 0 = \ldots. = \beta_k(\alpha_k - \alpha_{k+1})$

$\Rightarrow \quad \beta_1 = 0 \ldots. = \beta_k$ as $\alpha_1, \ldots.. , \alpha_{k+1}$ are all distinct.

Putting these values in equation (2), we get

$\beta_{k+1} v_{k+1} = 0 \quad \Rightarrow \quad \beta_{k+1} = 0$, as $v_{k+1} \neq 0$.

So {v_1, ..., v_{k+1}} are also linearly independent if { v_1, ..., v_k} are linearly independent. But we have already proved that the result is true for m = 1. Hence by principle of mathematical induction, S = { v_1, ..., v_m}is linearly independent.

Corollary 3: If T is a linear operator on an n-dimensional vector space V(**F**), then T can not have more than n distinct eigenvalues.

Proof: Let us consider that T has m distinct eigenvalues where m > n. From this theorem, the corresponding m eigen vectors of T are linearly independent. But dim V = n, so maximum number of linearly independent vectors in V(F) is n. **Contradiction!**

So T can't have more than n distinct eigen values.

Corollary 4: Let T be a linear operator on an n-dimensional vector space V(F) and suppose that T has n distinct eigenvalues. Then T is diagonalizable.

Proof: Suppose T has n distinct eigenvalues, say c_1, ….., c_n. Let v_1, …., v_n be the corresponding eigenvectors. By using this theorem, v_1, …., v_n are linearly independent over **F**. Since dim V = n, so $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of V which consists of eigenvectors of T. Hence by this theorem, T is diagonalizable.

Corollary 5: Let T be a linear operator on a finite-dimensional vector space. Let c_1, ….., c_m be distinct eigenvalues of T and W_i be the eigenspace of T corresponding to the eigenvalue c_i ; $1 \leq i \leq m$.

So $W = W_1 + W_2 + \ldots + W_m$

If β_i is an ordered basis for W_i , then $\beta = \{ \beta_1, \ldots, \beta_m \}$ is an ordered basis for W. Further dim W = dim W_1 + …. + dim W_m.

Proof: Let $W_1 + W_2 + \ldots + W_m = 0$; where $w_i \in W_i$; ; $1 \leq i \leq m$.

Claim: $w_i = 0$ for each i. Suppose there are some non-zero w_i. If we ignore zero w_i, then,

$w_{i_1} + w_{i_2} + \ldots + w_{i_k} = 0$, each w_{i_k} is non-zero.

$\Rightarrow$ All these vectors are linearly dependent.

But corresponding eigenvalues c_{i_1} , ….., c_{i_k} are all distinct.

Contradiction!

So by this theorem, all $w_i = 0$.

Step II: As β_i is an ordered basis for W_i.

$\Rightarrow$ β_i spans W_i.

$\Rightarrow$ $\beta = \{ \beta_1, \ldots, \beta_m \}$ spans the subspace $W = W_1 + W_2 + \ldots + W_m$

Claim: β is a linearly independent set. Let $x_1 + \ldots + x_m = 0$, where $x_i \in W_i$ is some linear combination of the vectors in β_i. So as proved in Step-I, $x_i = 0$ for each i. As each β_i is linearly independent.

$\Rightarrow$ all the scalars in x_i must be zero.

$\Rightarrow$ β is a linearly independent set.

Hence β is a basis of $W = W_1 + \ldots + W_m$

$\Rightarrow$ $\dim W = \dim W_1 + \ldots + \dim W_m$.

Theorem 8: Let $c_1, \ldots, c_n$ be n distinct eigenvalues of an $n \times n$ matrix A and let $X_1, \ldots, X_n$ be the corresponding eigenvectors of A. If $P = [X_1, \ldots, X_n]$ be $n \times n$ matrix, then A is diagonalizable and $P^{-1} A P = \text{diag}(c_1, \ldots, c_n)$.

Proof: By corollary (3) of previous theorem, it is obvious that A is diagonalizable. Since we know that eigenvectors associated with different eigenvalues are linearly Independent.

$\Rightarrow$ $X_1, \ldots, X_n$ are linearly independent.

So all X_i are non-zero vectors also.

$\Rightarrow$ $\det(P) \neq 0$ i.e. P is invertible.

Given that $AX_i = c_i X_i$, $i = 1, 2, \ldots, n$ ….(1)

Now $AP = A[X_1, \ldots, X_n] = [AX_1, \ldots, AX_n]$

$= [c_1 X_1, \ldots, c_n X_n]$ using(1)

$$= [X_1, \ldots, X_n]\begin{bmatrix} c_1 & 0 & \ldots & \ldots & 0 \\ 0 & c_2 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & c_n \end{bmatrix}$$

So, $AP = P\,\text{diag}(c_1, \ldots, c_n)$

$\Rightarrow$ $P^{-1} A P = \text{diag}(c_1, \ldots, c_n)$.

Example 13: Let $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ then ,

(i) Find eigenvalues of A, corresponding eigenvectors and eigenspaces of A.

(ii) Is a diagonalizable ?

(iii) Find a non-singular matrix P such that $P^{-1} A P$ is a diagonal matrix.

Solution: (i) Characteristic equation of A is

$$|A - xI| = \begin{vmatrix} 5-x & -6 & -6 \\ -1 & 4-x & 2 \\ 3 & -6 & -4-x \end{vmatrix} = 0$$

On solving we get x = 1, 2, 2.

Case (i): Eigenvector corresponding to x = 1 is given by (A – I)x = 0

$$\Rightarrow \quad \begin{bmatrix} 4 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} -1 & 3 & 2 \\ 4 & -6 & -6 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 4R_1$ and $R_3 \rightarrow R_3 + 3R_1$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 6 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{2} R_2$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 3x_2 + 2x_3 = 0$, and $3x_2 + x_3 = 0$

Since rank of coefficient matrix = 2. So only 3 – 2 = 1 variable will take arbitrary value. Let $x_3 = 3$, then $x_2 = -1$ and $x_1 = 3$

So $X_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

Case (ii): Eigen vector, corresponding to x = 2 is (A – 2I)X = 0

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & -6 \\ 3 & -6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + 2x_2 + 2x_3 = 0$

Here rank of coefficient matrix is 1. So 3 – 1 = 2 variables can take arbitrary value. By taking $x_2 = 0$, we get $x_1 = 2$, $x_3 = 1$. By taking $x_3 = 0$, we get $x_1 = 2$, $x_2 = 1$. So two linearly independent eigenvectors corresponding to x = 2 are $X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Case (iii): $W_1 = \{ aX_1 : a \in \mathbf{R} \} = \{ a(3, -1, 3) : a \in \mathbf{R} \}$

$W_2 = \{bX_2 + cX_3 : b, c \in \mathbf{R} \} = \{ b(2, 0, 1) + c(2, 1, 0) : b , c \in \mathbf{R}\}$

(iii) First we show that X_1, X_2 , X_3 are linearly independent over **R**. Let a, b $\in$ **R** such that $aX_1 + bX_2 + c X_3 = 0$. Then

a(3, -1, 3) + b(2, 0, 1) + c(2, 1, 0) =(0, 0, 0)

3a + 2b + 2c = 0

-a + 0b + 0c = 0 $\quad \Rightarrow$ a = 0

3a + b + 0c = 0

So we have b + c = 0 and b = 0

$\Rightarrow$ c = 0

So X_1, X_2 , X_3 are linearly independent. Hence A is diagonalizable.

(iii) Let P = $[X_1 \quad X_2 \quad X_3]$ = $\begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

Now using elementary properties of matrices, we can get P^{-1} . Then it can be easily verified that

$$P^{-1}A\,P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 14: For the matrix, A = $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$, find a matrix P, such that $P^{-1}A\,P$ is a diagonal matrix.

Solution: For given matrix, characteristic equation is |A – xI| = $\begin{vmatrix} 1-x & 2 & 0 \\ 2 & 1-x & -6 \\ 2 & -2 & 3-x \end{vmatrix} = 0$

On solving, we get x = 5, 3, -3. As A is 3×3 matrix having three different eigenvalues. So A is diagonalizable.

Case I: Eigenvector, corresponding to x = 5 is given by (A – 5I)X = 0

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & -6 \\ 2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow \frac{1}{2} R_1$, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 2 & -4 & -6 \\ 2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 + R_1$, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & -6 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow \frac{1}{3} R_2$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2x_1 + x_2 + 0\, x_3 = 0.$

$x_2 + 2x_3 = 0.$

If we take, $x_3 = -1$, then $x_2 = 2$, $x_1 = 1$.

So eigenvector corresponding to x = 5 is, $X_1 = [1 \quad 2 \quad -1]^T$.

Case II: Now eigenvector corresponding to x = 3 is $(A - 3I)X = 0$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & -6 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 + R_1$, we get

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + x_2 = 0$ and $x_3 = 0$

$\Rightarrow \quad x_1 = x_2$ and $x_3 = 0$

So eigenvector corresponding to x = 3 is , $X_2 = [1 \quad 0 \quad 0]^T$.

Case III: eigenvector corresponding to x = -3 is

$(A + 3I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & -6 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow \frac{1}{2} R_1$, we get

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & -6 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -6 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 + x_2 = 0$ and $x_2 - 2x_3 = 0$

If we take $x_3 = 1$, then $x_2 = 2$ and $x_1 = -1$. So eigenvector corresponding to $x = -3$ is $X_3 = [-1 \quad 2 \quad 1]^T$. Here eigen vectors corresponding to distinct eigen values are linearly independent.

So $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$.

Now, we can get P^{-1} such that

$$P^{-1} A P = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Example 15: Find the eigenvalues and bases of the corresponding characteristic spaces of the matrix A $= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

Is **A** diagonalizable? Justify.

Solution: The characteristic equation of A is $\begin{vmatrix} 2-x & 1 & 0 \\ 0 & 1-x & -1 \\ 0 & 2 & 4-x \end{vmatrix} = 0$

On solving, we get $x = 2, 2, 3$.

Case (i): Eigenvector, corresponding to $x = 2$ is given by $(A - 2I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_2 = 0$, $x_2 + x_3 = 0 \quad \Rightarrow x_3 = 0$

x_1 can take any real value . Let $x_1 = 1$

So $\quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case (ii): Eigen vector, corresponding to $x = 3$ is $(A - 3I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + x_2 = 0$, $2x_2 + x_3 = 0$.

If we take $x_3 = -2$, then $x_2 = 1$, $x_1 = 1$. So eigenvector corresponding to x = 3 is

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Bases: The characteristic space W_2, corresponding to the eigenvalue x = 2 is spanned by X_1. Hence $\{X_1\}$is a basis of W_2. Similarly $\{X_2\}$ is a basis of W_3. Thus we have obtained two linearly independent eigen vectors X_1 and X_2, corresponding to eigen values 2, 2, 3 of A. So we can't get a 3×3 invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence A is **not** diagonalizable.

Theorem 9: Let T be a linear operator on a finite-dimensional vector space V(**F**). If $c_1, \ldots, c_k$ are k distinct eigenvalues of T and W_i be the eigenspace of T corresponding to the eigenvalue c_i $(1 \leq i \leq k)$, then the following conditions are equivalent –

(i) T is diagonalizable.

(ii) The characteristic polynomial of T is
$f(x) = (x - c_1)^{d_1} \ldots\ldots (x - c_k)^{d_k}$, where
$d_k = \dim W_i$ $(1 \leq i \leq k)$ and $d_1 + \ldots + d_k = \dim V = n$.

(iii) $\dim V = \dim W_1 + \ldots + \dim W_k$.

Proof: Since we know that $W_i = \{ v_i : T(v_i) = c_i v_i \}$

$\Rightarrow \quad W_i = \{ v_i : (T - c_i I)(v_i) = 0 \}$

Claim: We shall prove **(i) ⇒ (ii)**

Suppose T is diagonalizable. Then there exists an ordered basis $\beta = \{ v_1, \ldots, v_n \}$ of V such that the matrix of T relative to β is

$$[T]_\beta = \begin{bmatrix} c_1 & 0 & \dots & \dots & 0 \\ 0 & c_2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & c_n \end{bmatrix}.$$

Suppose c_1 is repeated d_1 times, …, c_k is repeated d_k times. Then

$$[T]_\beta = \text{diag}\,[c_1 \;\; \dots \;\; c_1 \;\; \dots \;\; \dots \;\; c_k \;\; \dots \;\; c_k].$$

So, characteristic polynomial of T is given by

$$f(x) = (x - c_1)^{d_1} \dots\dots (x - c_k)^{d_k}, \text{ where}$$

$$d_1 + d_2 + \dots + d_k = n = \dim V.$$

Thus $[T - c_i I]_\beta$ has only d_i zeros on the main diagonal for all i = 1, 2, …., k and

$$\text{rank}\,(T - c_i I) = n - d_i\,;\ \forall\, i = 1, 2, \dots, k \qquad \dots(1)$$

Then by **rank-nullity theorem**,

$$\text{Rank}(T - c_i I) + \text{Nullity}(T - c_i I) = \dim V = n \qquad \dots(2)$$

Using equation (1), we have

$$\text{Nullity}\,(T - c_i I) = d_i$$

$$\Rightarrow \quad \dim \ker(T - c_i I) = d_i$$

$$\Rightarrow \quad \dim W_i = d_i \text{ for } i = 1, 2, \dots, k.$$

Claim: Now we shall prove **(ii) ⇒ (iii)**

Here given that, $\dim V = d_1 + d_2 + \dots + d_k$

$$\Rightarrow \quad \dim V = \dim W_1 + \dots + \dim W_k.$$

Claim: Now we shall show **(iii) ⇒ (i)** .

$$\text{Let } \dim V = \dim W_1 + \dots + \dim W_k \qquad \dots(3)$$

Let $W = W_1 + W_2 + \dots + W_k$.

Since c_1, …., c_k are distinct eigenvalues of T and W_1 , …. , W_k are the corresponding eigenspaces of T, so

dim W = dim W_1 + ….. + dim W_k (we have proved this in theorem) ….(4)

Further, if β_i is a basis of W_i , for i = 1, 2, …., k ; where W_i = ker(T – c_iI) ,

Then β = { β_1, …., β_k } is a basis of W. From equations (3) and (4), we conclude that

dim V = dim W and so V = W = $W_1 + W_2 + \ldots + W_k$, since W is a subspace of V.

Hence β = { β_1, …., β_k } is a basis of V consisting of eigenvectors of T and so T is diagonalizable.

Check your progress

Problem 1: Find the characteristic polynomials for the identity operator and zero operator on an n-dimensional vector space.

Problem 2: If $c \neq 0$, is an eigenvalue of an invertible operator T, then prove that c^{-1} is an eigenvalue of T^{-1}.

Problem 3: Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbf{R}^3$, each vector of which is eigen vector of T.

Problem 4: Find the eigenvalues, eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 5: Find the eigenvalues, eigenvectors and eigenspaces of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 6: Find the eigenvalues, eigenvectors and eigenspaces of the matrix $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$.

Also prove that **A** is diagonalizable.

8.14 *SUMMARY*

In this unit, we understood the concept of linear operators and their different applications. One of such applications is invertibility of T. Then we elaborated the role of bases of T and their representations. At last, we ensured some conditions of diagonalisation of square matrices.

8.15 *GLOSSARY*

Eigen Values of T: Let T be linear operator on a vector space V(**F**). A scalar $\alpha \in$ **F** is called an eigen value or characteristic value of T, if there exists some V $\neq$ 0,

$$v \in V \text{ such that, } T(v) = \alpha v.$$

Eigen Vector: If α is an eigen value of T, then $v \in V$ such that $T(v) = \alpha v$ is called an eigen vector or characteristic vector belonging to α.

Eigen space: The set of all eigenvectors of T belonging to an eigenvalue α is called an eigenspace of T, belonging to α. It is represented as W_α. Hence,

$$W_\alpha = \{ v \in V : T(v) = \alpha v \}.$$

Similar Matrices: Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and $P = [c_{ij}]_{n \times n}$ where $a_{ij}, b_{ij}, c_{ij} \in$ **F**. Then A and B are called similar matrices if, there exist a matrix P such that $A = P^{-1} B P$, where P is non-singular matrix.

8.16 *REFERENCES*

- S Kumaresan; Linear Algebra-A Geometric Approach; PHI-2016.
- K Hoffman & Ray Kunze; Linear Algebra; PHI-1971.
- P Halmos; Finite-Dimensional Vector Spaces;
- G Strang; Introduction To Linear Algebra;

8.17 *SUGGESTED READING*

- NPTEL videos.
- Schaum series.
- A R Vashishtha, Krishna Prakashan; Meerut.

- Graduate Text In Mathematics, Springer.

8.18 *TERMINAL QUESTION*

Long answer type question

1: Let T be a linear operator on a vector space V(**F**). Then prove the following:

(iv) If $0 \neq v \in V$ is an eigenvector of T, then $\alpha \in$ **F** satisfying T(v) = αv is **unique**.

(v) The eigenspace W_α corresponding to an eigen value $\alpha \in$ **F** is a subspace of V(**F**).

(vi) W_α = ker (T – α I) .

2: State and prove the Cayley-Hamilton Theorem for a linear operator.

3: Find the eigen values, eigen vectors and eigen spaces of 2×2 identity matrix.

4: Show that the eigen values of a diagonal matrix are exactly the elements in the diagonal. Hence prove that if a matrix B is similar to a diagonal matrix D, then the diagonal elements of D are the eigen values of B.

5: Let *V* be the vector space of all real-valued continuous functions. Then prove that the linear operator T: V→V defined as (Tf)x = $\int_0^x f(t)dt$ has no eigenvalues.

6: For the matrix, A = $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$, find a matrix P, such that P^{-1}A P is a diagonal matrix.

Short answer type question

1: Let T be a linear operator on a finite-dimensional vector space V(**F**). Then prove that $\alpha \in$ **F** is an eigenvalue of T if and only if T– α I is singular.

2: Prove that similar matrices have the same characteristic polynomial and hence the same characteristic values.

3: Prove that A linear operator T on a finite-dimensional vector space V(**F**) is diagonalizable if and only if there exists a basis of V consisting of eigenvectors of T.

4: Let T be a linear operator on a finite-dimensional vector space V(**F**). Then prove that the eigenvectors corresponding to distinct eigenvalues of T are linearly independent.

1: For the matrix A = $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$, prove that there exists a matrix P such that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2: Let A = $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ then

(i) Find eigenvalues of A, corresponding eigenvectors and eigen spaces of A.

(ii) Is a diagonalizable?

(iii) Find a non-singular matrix P such that $P^{-1}A\,P$ is a diagonal matrix.

8.19 *ANSWERS*

Answers of check your progress:

1. $\{(1 - x)^n , (-1)^n x^n\}$.

3. (eigen values are 3, -1, -1, and P = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$).

4. $\{1, k(1,0,0) : k \in \mathbf{R} \}$

5. [1, -1; $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $W_1 = L\{ X_1 , X_2 \}$, $W_{-1} = L \{ X_3 \}$] .

6. $\{1, 2, 5;\ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$

Answer of long question:

3: Eigenvalues of A are { +1, -1}. Eigen vector corresponding to $\alpha = 1$ is

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\,1\ \ 1]^T$ and eigenvector corresponding to $\alpha = -1$ is $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6: $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

Unit-9: DETERMINANTS

CONTENTS:

9.1 *INTRODUCTION*

In this unit we show that how to find the determinant of the matrix, we emphasize that an $n \times n$ array of scalars enclosed by straight lines called determinant of order n, the

determinant function was first discovered during the investigation of system of linear (Homogeneous and Non Homogeneous) Equation.
We solved the determinant of matrix of order $1, 2, 3 \ldots$ and then we define a determinant of general $n \times n$ matrix.

9.2 *OBJECTIVE*

After reading this unit you will be able to:

- Understand minors and cofactors.
- Find determinant value of a square matrix.
- Understand properties of determinant and its uses.
- Find product of the two determinant and its uses.
- Know about singular and nonsingular matrices.
- Find solution system of non-homogeneous linear using Cramer's Rule.
- Find Adjoint of a square matrix.

9.3 *DETERMINANT*

Definition: Each *n*-square matrix is assigned a special scalar is called determinant of A, and it is denoted by |A|

$$|A| = \begin{vmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} \ldots & a_{nn} \end{vmatrix}$$

9.3.1 DETERMINANTS OF ORDERS 1

If $A = [a_{11}]_{1\times1}$ then $|A| = a_{11}$

9.3.2 DETERMINANTS OF ORDERS 2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| =$ (Product of principal diagonal element) – (Product of non-principal diagonal element)

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

Example: - if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then, |A|= (2.5) – (4.3)

= 10 – 12

= -2

9.3.3 DETERMINANTS OF ORDERS 3

Let $A = \lfloor a_{ij} \rfloor_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $|A| = a_{11}.a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

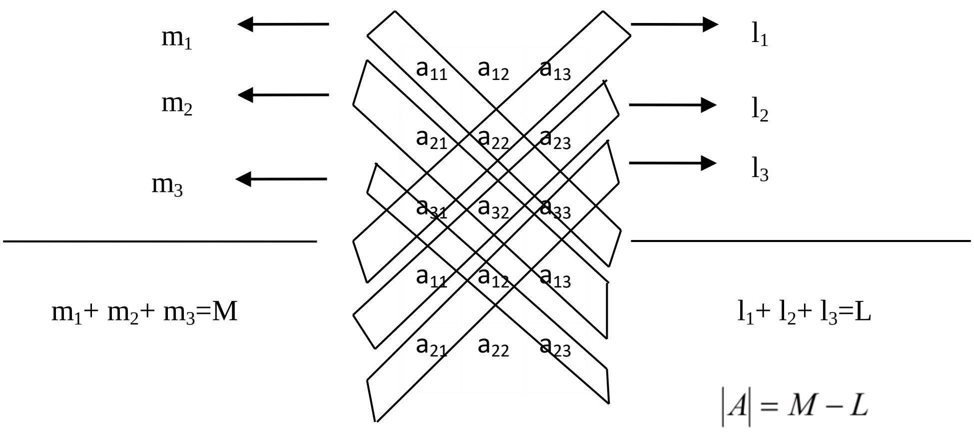
Or

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{a33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Or

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then arrange these number in rows and columns and first two rows again write in last



9.4 *MINORS AND COFACTORS*

Consider the determinant of 3×3 matrix (in general)

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then if we leave the column and the row passing through the elementa_{ij}, then the second order determinant is called minor of the element a_{ij} and it is denoted by M_{ij}

For example: The minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

The minor of the element $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$

Cofactors: The minor M_{ij} multiplied by $(-1)^{i+j}$ is called cofactor of the element a_{ij} and it is denoted by A_{ij}

$$A_{ij} = (-1)^{i+j}.M_{ij}$$

For example: - The cofactor of the element $a_{11} = (-1)^{1+1}M_{11} = M_{11}$

The cofactor of the element $a_{12} = (-1)^{1+2}M_{12} = -M_{12}$

9.5 *DEFINITION OF DETERMINANTS IN TERMS OF COFACTOR*

Let A be any n-row's square matrix then the determinants of A is the sum of the product of the element of any column or any row with their corresponding cofactor

i.e. $|A| = \sum_{i=1 \, or \, j=1}^{n} a_{ij} A_{ij}$ where, either i or j is fixed

Example:

1. If $i = 1$ then

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

2. Write the cofactors and minors of each element of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: The matrix of the element $a_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 6 - 3 = 3 = M_{11}$

The matrix of the element $a_{12} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 + 6 = 9 = M_{12}$

The matrix of the element $a_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5 = M_{13}$

The matrix of the element $a_{21} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3 + 1 = 4 = M_{21}$

The matrix of the element $a_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1 = M_{22}$

The matrix of the element $a_{23} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 = M_{23}$

The matrix of the element $a_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 = M_{31}$

The matrix of the element $a_{32} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4 = M_{32}$

The matrix of the element $a_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 = M_{33}$

The cofactor of the element $a_{11} = (-1)^{1+1} m_{11} = 3 = A_{11}$

The cofactor of the element $a_{12} = (-1)^{1+2} m_{12} = -9 = A_{12}$

The cofactor of the element $a_{13} = (-1)^{1+3} m_{13} = -5 = A_{13}$

The cofactor of the element $a_{21} = (-1)^{2+1} m_{21} = -4 = A_{21}$

The cofactor of the element $a_{22} = (-1)^{2+2} m_{22} = 1 = A_{22}$

The cofactor of the element $a_{23} = (-1)^{2+3} m_{23} = 3 = A_{23}$

The cofactor of the element $a_{31} = (-1)^{3+1} m_{31} = -5 = A_{31}$

The cofactor of the element $a_{32} = (-1)^{3+2} m_{32} = 4 = A_{32}$

The cofactor of the element $a_{33} = (-1)^{3+3} m_{33} = 1 = A_{33}$

9.6 *PROPERTIES OF DETERMINANTS*

Theorem 1: The value of determinant does not change when rows and columns are interchange

Proof: Let A be any square matrix of order *n*

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}_{n*n}$$

Then

$$|A| = \sum_{i=1\, j=1}^{n} a_{ij} A_{ij} \text{ where } A_{ij} \text{ is cofactor of } a_{ij}$$

Let us take a matrix of order 3 for example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Hence the determinants of any matrix A and its transpose matrix $A^T$are equal.

Theorem 2: If any two columns or rows of a determinant are interchanged then the values of determinant is negative multiple of determinant of original matrix.

Proof: - Consider a matrix A of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then } |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \ldots \ldots . . (1)$$

Now interchanging any two rows or columns

$$R_1 \leftrightarrow R_2$$

Then new matrix $A_1 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| \text{ interchange} = a_{21}(a_{12}a_{33} - a_{32}a_{13}) - a_{22}(a_{11}a_{33} - a_{31}a_{13}) + a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

$$= -[a_{32}a_{13}a_{21} - a_{21}a_{33}a_{12} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}]$$
$$- [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})]$$

So, $|A_1| = -|A|$

Note:

1. If any row or column in any matrix is multiplied by any scalar K then determinant of the matrix is K times of the determinants of the original matrix

For example: $\begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = K\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

2. If all the elements of matrix multiplied by constant K then determinant is equal to K^n time of the value of determinant of original matrix, where n is order of matrix.

i.e. $|KA| = K^n|A|$

3. If any two rows or columns are identical of any matrix then determinant is zero.

Theorem 3: If in a determinant each element in any row or column consists of the sum of two terms, then determinant can be written as sum of two determinants

Proof: Let $A = \begin{bmatrix} a_{11} + a & a_{12} & a_{13} \\ a_{21} + b & a_{22} & a_{23} \\ a_{31} + c & a_{32} & a_{33} \end{bmatrix}$

Expanding the determinant along the first column

$$|A| = (a_{11} + a)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{21} + b)\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (a_{31} + c)\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$
$$+ c\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix}$$

Theorem 4: If the element of any row or column added by K time the corresponding element of any other row or column, then determinants of the matrix are same

Proof: -Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} + Ka_{12} & a_{12} & a_{13} \\ a_{21} + Ka_{22} & a_{22} & a_{23} \\ a_{31} + Ka_{32} & a_{32} & a_{33} \end{bmatrix}$

Then $B = \begin{vmatrix} a_{11}+Ka_{12} & a_{12} & a_{13} \\ a_{21}+Ka_{22} & a_{22} & a_{23} \\ a_{31}+Ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} Ka_{12} & a_{12} & a_{13} \\ Ka_{22} & a_{22} & a_{23} \\ Ka_{32} & a_{32} & a_{33} \end{vmatrix}$

$$= |A| + K\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$= |A| + KO$ [$\therefore$ If any two columns are identical then determinant will be zero]

$= |A|$

Example: If $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ then show that

$$|A| = (a-b)(b-c)(c-a)$$

Solution: $|A| = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Applying $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ then we get

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

Expanding the determinant along the first column

$$|A| = 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - a\begin{vmatrix} 0 & b^2-a^2 \\ 0 & c^2-a^2 \end{vmatrix} + a^2\begin{vmatrix} 0 & b-a \\ 0 & c-a \end{vmatrix}$$

$$= (b-a)(c^2-a^2) - (b^2-a^2)(c-a) - 0 + 0$$

$$= (b-a)(c-a)(c+a) - (b-a)(b+a)(c-a)$$

$$= (b-a)(c-a)\{(c+a)-(b+a)\}$$

$= (b-a)(c-a)(c-b)$

$= (a-b)(b-c)(c-a)$

9.7 *VANDERMODE MATRIX*

A matrix is at form

$$A = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_n & \alpha_n^2 & \alpha_n^{n-1} \end{bmatrix}$$ is called vandermode matrix

And its determinant

$$|A| = \prod_{1<i<j\leq n} (\propto_j - \propto_i)$$

Example 4: Find the determinant of

$$\Delta n = \begin{vmatrix} 0 & 0 & \cdots & a \\ 0 & 0 & \cdots & a0 \\ & & \therefore & \\ a & 0 & 0 & 0 \end{vmatrix}$$

Solution: $\Delta_n = (-1)^{n+1} a\, \Delta_{n-1}$

$= (-1)^{n+1} a (-1)^{(n+1)-1} \Delta_{n-2}$

$= (-1)^{n+1} a (-1)^{(n+1)-1} a (-1)^{(n+1)-2} \Delta_{n-3}$

$\Delta_n = (-1)^{(n+1)+n+(n-1)+\,---4} a^{n-2} \Delta_2$

Where $\Delta_2 = (-1)^3 a^2$

Then $\Delta_n = (-1)^{\left[\frac{(n+1))(n+2)}{2} - 1\right]}$

Example 5: Let A be a square matrix of order n, then show that

1. $|\overline{A}| = \overline{|A|}$ **2.** $|A^\theta| = \overline{|A|}$

Solution 1. let $A = [a_{ij}]_{n\times n}$ $A = [a_{ij}]_{n*n}$ then $\overline{A} = [\overline{a_{ij}}]_{n\times n}$

So $|\overline{A}| = |\overline{a_{ij}}| = \overline{|a_{ij}|} = \overline{|A|}$

2. A be a square matrix of order n, and $A^\theta = \overline{A^T}$

So $|A^\theta| = |\overline{A^T}| = |\overline{A}^T| = |\overline{A}| = \overline{|A|}$

$\because$ $|A^T| = |A|$ and $|\overline{A}| = \overline{|A|}$

Example 6: Show that the determinant of Hermitian matrix always a real number

Solution: Let A be a Hermitian matrix

Then $A^{\theta} = A$

$|A^{\theta}| = |A|$

$|\overline{A^T}| = |A|$

$\overline{|A|} = |A|$

Let $x + iy$ is the determinant of A

$|A| = x + iy$

$\overline{|A|} = x - iy$

But $\overline{|A|} = |A|$

$x - iy = x + iy$

$2iy = 0$

$y = 0$

$|A| = x + i0 = x$

Example 7: Show that the determinant of Skew symmetric matrix of odd order is zero.

Solution: Let A be a skew symmetric of odd order

$A^T = -A$

$|A^T| = |-A| = |(-1)A| = (-1)^n|A| \quad \because |KA| = K^n|A|$

$|A| = (-1)^n|A| \quad \because |A^T| = |A|$

Since n is odd so $(-1)^n = -1$

Now $|A| = -|A|$

$2|A| = 0$

$|A| = 0$

9.8 *PRODUCT OF THE TWO DETERMINANT OF THE SAME ORDER*

Example 8: If A and B are two square matrices of same order then prove that

$$|AB| = |A||B|$$

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$A.B = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

Now we know that

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $|B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$

$$|A||B| = \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{vmatrix}$$

Hence $|AB| = |A||B|$

Rule: Let A and B are only two matrices of same order

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $|B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$ then

$$|A||B| = \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{vmatrix}$$

In general this is simply row by column multiplication

Example 9: If A be a square matrix of order n then show that

$$|A^K| = |A|^K$$

Solution: Let A and B are two square matrices of order n

Then we know that $|A.B| = |A||B|$

If we replace B with A then

$$|A.A| = |A||A|$$

$$|A^2| = |A|^2$$

In similar way $|A^K| = |A|^K$

Example 10: Show that the determinant of an idempotent matrix is either 0 or 1

Solution: Let A is an idempotent matrix, then

$$A^2 = A$$

$$|A^2| = |A|$$

$$|A|^2 = |A|$$

$$|A|(|A| - 1) = 0$$

$$|A| = 0 \quad \text{or} \quad |A| - 1 = 0$$

$$|A| = 0 \quad \text{or} |A| = 1$$

Note: It is necessary condition the determinant of idempotent matrix is 0 or 1 but not sufficient.

For Example: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then $|A| = 0$ but $A^2 \neq A$

Example 11: Show that the determinant of orthogonal matrix is either 1 or -1

Solution: Let A is an orthogonal matrix, then

$$A\,A^T = I$$

$$|A\,A^T| = |I|$$

$$|A||A^T| = 1$$

$$|A||A| = 1 \qquad \because |A^T| = |A|$$

$$|A|^2 = 1 \qquad |I| = 1$$

$$|A| = 1 \text{ or } -1$$

Note: Determinant of a diagonal matrix, upper triangular matrix, lower triangular matrix is the product of principal diagonal elements.

Example 12: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ Then $|A| = ?$

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ is upper triangular matrix so its determinant values are the product of principal diagonal matrix

Hence $|A| = 1 \cdot 5 \cdot 9 = 45$

Example 13: Show that the value of determinant of skew Hermitian matrix of order *n*, is either 0 (zero) or purely imaginary if *n* is odd and real, if *n* is even.

Solution: Let A be a skew Hermitian matrix and

$$|A| = x + iy$$

$A^{\theta} = -A$ (By definition of skew Hermitian matrix)

$$|A^{\theta}| = |-A|$$

$$|\overline{A^{T}}| = (-1)^{n}|A|$$

$$\overline{|A|} = (-1)^{n}|A|$$

Case 1: If *n* is even then,

$\overline{|A|} = |A| \quad \Rightarrow \quad x - iy = x + iy \quad \Rightarrow \quad y = 0 \quad$ so $\quad |A| = x$

$\Rightarrow$ $|A|$ is real

Case 2: If *n* is odd then,

$$|\overline{A}| = |A| \Rightarrow x - iy = -(x + iy) \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\overline{|A|} = |A| \quad \Rightarrow \quad x - iy = -(x + iy) \quad \Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0$$

$|A| = iy$ If $y = 0$ then $|A| = 0$

If $y \neq 0$ then $|A|$ is purely imaginary.

9.9 *NON SINGULAR MATRIX AND SINGULAR MATRIX*

Non- Singular Matrix: A matrix '*A*' is said to be non-singular matrix if its determinant is non zero.

Singular Matrix: A matrix '*A*' is said to be singular matrix if its determinant is zero.

9.10 *LINEAR EQUATION (HOMOGENEOUS AND NON-HOMOGENOUS EQUATION)*

Linear homogenous equation: The equation is of the form $ax + by + cz = 0$ is called linear homogenous equation in x, y, z.

Linear non-homogenous equation: The equation is of the form $ax + by + cz = B$ where $B \neq 0$ is called non-homogenous equation in x, y, z.

9.11 *SYSTEM OF NON-HOMOGENOUS LINEAR EQUATION (CRAMER'S RULE)*

If we have n linear simultaneous equation in n variable$x_1, x_2, x_3 \cdots\cdots x_n$

i.e. $a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$

Suppose A_{ij} is the cofactor of element a_{ij} in Δ then multiplying this given equation by $A_{11}, A_{21}, A_{31} \cdots\cdots A_{n1}$ and adding

$$x_1(a_{11}A_{11} + a_{21}A_{21} + \cdots a_{n1}A_{n1}) + x_2(a_{12}A_{11} + a_{22}A_{21} + \cdots a_{n2}a_{n1}) + \cdots + x_n(a_{1n}A_{11} + a_{2n}A_{21} + \cdots A_{nn}a_{nn})$$

$$x_1\Delta + x_2(0) + x_3(0) + \cdots = b_1A_{11} + b_2A_{21} + \cdots A_nA_{n1}$$

$x_1\Delta = \Delta_1$ where, Δ_1 is the determinant obtained by replacing first column element of Δ by $b_1, b_2 \cdots b_n$ then $x_1 = \frac{\Delta_1}{\Delta}$

Again multiplying these equations by $A_{12}, A_{22}, \cdots A_{n2}$ and adding then we get

$$x_2\Delta = \Delta_2 \quad \Rightarrow \quad x_2 = \frac{\Delta_2}{\Delta}$$

Where Δ_2 is determinant obtained by replacing second column element of Δ by $b_1, b_2 \cdots b_n$

In similar way, we get

$$x_3\Delta = \Delta_3 \quad \Rightarrow \quad x_3 = \frac{\Delta_3}{\Delta}$$

… …

$$x_n\Delta = \Delta_n \quad \Rightarrow \quad x_n = \frac{\Delta_n}{\Delta}$$

This method of solving n simulations linear non-homogenous equation provided $|A| \neq 0$ where A is the coefficient matrix. This method is known as Cramer's rule

Example 14: Solve the following system at equation by Cramer's rule

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

Solution: The coefficient matrix of given system at non-homogenous linear equation is

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$\Delta = |A| = 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2(1+1) + 3(-1-1) \quad = 4 - 6 \quad = -2 \quad \neq 0$$

Therefore the system of non-homogenous linear has unique solution

Now using Cramer's rule

$$\Delta_1 = \begin{bmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = -2, \qquad \Delta_2 = \begin{bmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix} = -4, \qquad \Delta_3 = \begin{bmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{bmatrix} = -6$$

Hence the solution is

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1, \qquad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2, \qquad z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

9.12 *ADJOINT OF A SQUARE MATRIX*

Let $A = [a_{ij}]_{n*n}$ be a square matrix of order *n* then the transpose of a matrix $B=[A_{ij}]_{n*n}$ where A_{ij} is the cofactor of the element a_{ij} called Adjoint of matrix A and it is denoted by Adj*A* or adj*A*.

If $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}_{n \times n}$

Then the cofactor matrix $C = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{bmatrix}_{n \times n}$

Then adj*A*= transpose of the matrix C

adj*A*= $C = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots \\ A_{n1} & \dots & A_{nn} \end{bmatrix}_{n \times n}$

Example 17: Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: Let us find the cofactor A_{11} , A_{12} , A_{13} etc at the element of $|A|$ we have

$A_{11} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$, $A_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$, $A_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$,

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3 , A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 , A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1 ,$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 , A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 , A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 ,$$

Therefore the matrix C formed at the cofactor of the element of $|A|$ is

$$C = \begin{vmatrix} -5 & 1 & 3 \\ 3 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now adj*A* is the transpose of the matrix *C*.

adj*A*= 3

Example 18: Prove that at $x = 4$ the values of given determinant

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Solution: We have $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} \text{ or } \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} \text{ by } R_2 \to -\frac{1}{2}R_2, R_3 \to -\frac{1}{6}R_3$$

Solving the determinant along the first row then we get

$$(x-2)(30-24) - (2x-3)(10-6) + (3x-4)(4-3)$$

$6(x-2) - 4(2x-3) + 1.(3x-4)$

Put $x = 4$ then the value of determinant

= 6(4-2) - 4(8-3) + (3.4 - 4)

= 6.2 - 4(5) + 8

=12 - 20 + 8

=0

Example 19: If $\begin{vmatrix} t_1 & t_1^2 & 1+t_1^3 \\ t_2 & t_2^2 & 1+t_2^3 \\ t_3 & t_3^2 & 1+t_3^3 \end{vmatrix} = 0$ then prove that

$$t_1.t_2.t_3 = -1 \text{ where } t_1 \neq t_2 \neq t_3$$

Solution: We have $\begin{vmatrix} t_1 & t_1^2 & 1+t_1^3 \\ t_2 & t_2^2 & 1+t_2^3 \\ t_3 & t_3^2 & 1+t_3^3 \end{vmatrix} = \begin{vmatrix} t_1 & t_1^2 & 1 \\ t_2 & t_2^2 & 1 \\ t_3 & t_3^2 & 1 \end{vmatrix} + \begin{vmatrix} t_1 & t_1^2 & t_1^3 \\ t_2 & t_2^2 & t_2^3 \\ t_3 & t_3^2 & t_3^3 \end{vmatrix}$

(By theorem (properties of determinants))

$$= \begin{vmatrix} t_1 & t_1^2 & 1 \\ t_2 & t_2^2 & 1 \\ t_3 & t_3^2 & 1 \end{vmatrix} + t_1t_2t_3 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

(By taking t_1, t_2, t_3 common from first row, second row and third row of the second determinant)

$$=\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} + t_1t_2t_3 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_2 & t_3^2 \end{vmatrix}$$

(By $C_1 \leftrightarrow C_3$ then $C_3 \leftrightarrow C_2$ of the first determinant so determinant is unchanged)

$$= (1 + t_1.t_2.t_3) \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_2 & t_3^2 \end{vmatrix}$$

By vandermode matrix the value of above determinant is

$$= (1 + t_1.t_2.t_3)(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \quad \text{but} \quad t_1 \neq t_2 \neq t_3$$

So $(t_1 - t_2).(t_2 - t_3)(t_3 - t_1) \neq 0$

So if $\begin{vmatrix} t_1 & t_1^2 & 1 + t_1^3 \\ t_2 & t_2^2 & 1 + t_2^3 \\ t_3 & t_3^2 & 1 + t_3^3 \end{vmatrix} = 0$ then $(1 + t_1.t_2.t_3)$ must be zero.

Hence $t_1.t_2.t_3 = -1$

Example 20: Prove that if $x \neq y \neq z$ and $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = 0$ then $yz + zx + xy = 0$

Solution: We have $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = 0$

Multiplying by x, y, z in first, second and third column of the determinant from left side respectively then we get

$$\frac{1}{x.y.z} \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} = 0$$

Taking xyz common from 3^{rd} row at the above determinant

$\frac{x.y.z}{x.y.z} \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = 0$ $R_1 \leftrightarrow R_3$ after that $R_2 \leftrightarrow R_3$

Then determinant is $(-1)^2$ time the original determinant.

$$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = 0 \qquad (C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$$

$$\begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix} = 0$$

Expanding along first row

$$= \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ (y-x)(y^2+xy+x^2) & (z-x)(z^2+zx+x^2) \end{vmatrix} = 0$$

$$= (y-x)(z-x) \begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix} = 0$$

Taking $(y - x)$ and $(z - x)$ is common from first and second column

$$= (y-x)(z-x)\{(y+x)(z^2+zx+x^2) - (y^2+xy+x^2)(z+x)\} = 0$$

$$=(y-x)(z-x)(z-y)(yz^2 + xz^2 - zy^2 - xy^2) = 0$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx) = 0$$

But $x - y \neq 0, y - z \neq 0, z - x \neq 0$ because x, y, z all are distinct, so $(xy + yz + zx) = 0$.

Example 21: Solve the following system of linear equation by Cramer's rule

$$2x - y + 3z = 8$$

$$-x + 2y + z = 4$$

$$3x + y - 4z = 0$$

Solution: We have $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2(-8-1) + 1(+4-3) + 3(-1-6)$

$$\Delta = -18 + 1 + (-21)\Delta = -38$$

Thus $\Delta \neq 0$ and therefore the system has a unique solution given by

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \qquad \text{i.e.}$$

$$\frac{x}{\begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix}} = -\frac{1}{38}$$

$$\text{Or given by } \frac{x}{-76} = \frac{y}{-76} = \frac{z}{-76} = \frac{1}{-38}$$

Hence $x = 2,\ y = 2,\ z = 2$

9.13 *USEFUL METHOD FOR FINDING THE VALUE OF DETERMINANT OF ORDER 4 OR MORE*

Let *A* be any non zero square matrix of order *n* $A = \lfloor a_{ij} \rfloor_{n\times n}$ with $n > 1$

Step 1: Choose an element in A such that $a_{ij} = 1$ or if nonexistent, $a_{ij} \neq 0$

Step 2: Using a_{ij} as a swivel, apply elementary row or column operations to put 0's in all the other positions in the column or row containing a_{ij}

i.e. if we apply row operation then to put 0 in all the other position in the column and similar for column operation

Step 3: Expand the determinant by the column or row (according to our selection of operation) containing a_{ij}

Example 22: Find the determinant of a matrix A of order 4×4

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

Solution: $|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$

Step 1: Choose an element a_{23} because $a_{23} = 1$

Step 2: Apply row operation and put 0's in all the other positions in third column

Apply $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 + R_2$

$$|A| = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Step 3: Now expanding the determinant by the third column

$$|A| = (-1)^{2+3} \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= -(4 - 18 + 5 - 30 - 3 + 4) \quad = 38$$

9.14 *DETERMINANTS AND VOLUME:*

Let A is any square matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \dots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Let $t_1 = (a_{11}, a_{12}, \cdots a_{1n})$

$$t_2 = (a_{21}, a_{22}, \cdots a_{2n})$$

$$\cdots \qquad \cdots$$

$$t_n = (a_{n1}, a_{n2}, \cdots a_{nn})$$

Then the determinant are related to the notions of area and volume

Let U be the parallelepiped determined by

$$U = \{a_1 t_1 + a_2 t_2 + \cdots a_n t_n : 0 \leq a_i \leq 1 \ \forall \ i = 1, 2, \dots n\}$$

When $n = 2$ then U is parallelogram

Let V denote the volume of U then

V = Absolute volume of determinant of A

Example 23: Let $t_1 = (1,1,1) t_2 = (1,1,0) t_3 = (0,2,3)$

Then find the volume of the parallelepiped in three dimension space

Solution: $t_1 = (1,1,1) t_2 = (1,1,0) t_3 = (0,2,3)$

So the volume is the absolute volume of

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 1(3-0) - 1(3-0) + 1(2-0)$$

$$= 3 - 3 + 2 \quad = 2$$

Hence volume $V = |2| = 2$

Example 24: Find the value of $|A|$ where

$A = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$ Where w is the cube root of unity.

Solution: Cube root of unity in complex number system is solution of the equation$z^3 - 1 = 0$, then the values of z satisfied the above equation is called cube root of unity.

Now $z^3 - 1 = 0$, $\quad z^3 = 1 \because \cos 0 + i\sin 0 = 1$

$z^3 = \cos 0 + i\sin 0 \qquad \because \cos$ and $\sin$ are periodic function

So $\cos(0) = \cos(0 + 2k\pi)$ and $\quad \sin(0) = \sin(0 + 2k\pi)$

So $z^3 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad = \quad \cos 2k\pi + i\sin 2k\pi$

$$z = \left(\cos(2k\pi) + i(\sin 2k\pi)\right)^{1/3} \quad = \quad \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$$

$[\cos\theta + i\sin\theta = e^{i\theta}$, $\qquad (\cos\theta + i\sin\theta)^n = e^{in\theta}$ or $e^{in\theta} = \cos n\theta + i\sin n\theta]$

Put $k = 0$ then $z = 1$

$k = 1$ then $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k = 2$ then $z = \cos\frac{(2\pi)^2}{3} + i\sin\frac{(2\pi)^2}{3} = \omega^2$

Hence cube root of unity are ω and ω^2.

And also $1 + \omega + \omega^2 = 0$

Now the given determinant applying $c_1 \to c_1 + c_2 + c_3$

Then we get

$$|A| = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \text{ or } |A| = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix}$$

$$|A| = 0$$

Example 25: Evaluate $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Solution: Let us denote the given determinant by $\det(A)$

$$\det(A) = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Applying row transformation by using $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$

Then we get

$$\det(A) = \begin{vmatrix} -a^2 & ab & ac \\ ab - a^2 & -b^2 + ab & bc + ac \\ ac - a^2 & bc + ab & -c^2 + ac \end{vmatrix}$$

Taking a, b, c are common from first, second, third columns respectively

Then we get

$$\det(A) = abc \begin{vmatrix} -a & a & a \\ b - a & -b + a & b + a \\ c - a & c + a & -c + a \end{vmatrix}$$

Now applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

Then $$\det(A) = a.b.c \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Taking common a, b, c from first, second and third row respectively

Then $$\det(A) = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Expanding along first row

$$\det(A) = 4a^2b^2c^2$$

9.15 *SUMMARY*

In this unit we learned to find the value of determinant of any matrix, which will help us in solving the linear equation, it will also be helpful to understand the concept of eigen value and rank of matrix,

9.16 *GLOSSARY*

1. **Identical row or column:** Any two row or column are same
2. **Parallelepiped:** A solid body of which each face is a parallelogram,
3. **Absolute:** Free from imperfection.

9.17 *REFERENCES*

1. Linear Algebra, Vivek.sahai & Vikas Bist :Narosa publishing House
2. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media
3. Schaum’s out line (Linear Algebra)

9.18 *SUGGESTED READINGS*

1. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media
2. Schaum’s out line (Linear Algebra)

9.19 *SELF ASSESMENT QUESTIONS*

9.19.1 Multiple choice question:

1. The values of $\det A$, where $A = \begin{vmatrix} l & m & n & p \\ o & t & u & q \\ o & o & v & r \\ o & o & o & s \end{vmatrix}$

(a) l.m (b) l.t

(c) l.m.n.p (d) l.t.v.s

2. The value of t show that $\begin{vmatrix} t-4 & 3 \\ 2 & t-9 \end{vmatrix} = 0$

(a) 3, 10 (b) 5, 7

(c) 8, 9 (d) 1, 2

3. If $A = (a_{ij})_{6*6}$ such that $a_{ii} = 1$ and $a_{ij} = 2$ if $i + j = 7$ otherwise zero then det of A is

(a) 3 (b) 9

(c) -27 (d) 27

4. Determinant of Nilpotent matrix will be

(a) A prime number (b) Multiple of 2

(c) Always 1 (d) None

5. Determinant of Skew symmetric matrix of order 3 is

(a) 3 (b) 5

(c) 1 (d) 0

6. If A is any non singular square matrix f order 3, then determinant of $ad_j(A)$ is

(a) $|A|$ (b) $|A|^2$

(c) $|A|^3$ (d) None

7. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ then cofactor of element a_{13} is

(a) $|A|$ (b) $|A|^2$

(c) $|A|^3$ (d) None

8. If A is any Square matrix of order n and determinant of A^T is

(a) 1 (b) 0

(c) 4 (d) None

9. If $A = \begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$ then determinant of A is

(a) abc (b) $a(b - c)$

(c) $abc(a-b)(b-c)$ (d)$abc(a-b)(b-c)(c-a)$

10. If $\begin{vmatrix} u & -6 & -1 \\ 2 & -3u & u-3 \\ -3 & 2u & u+2 \end{vmatrix} = 0$ then the value of U

(a) $1, 2, -3$ (b) $3, 7, 8$

(c) $5, 6, -1$ (d) $-1, -2, 3$

ANSWERS:

1. (d)	2. (a)	3. (c)	4. (d)	5. (d)
6. (b)	7. (b)	8. (a)	9. (d)	10. (a)

9.19.2 Fill in the blanks:

Fill in the blanks '…….' So that the following statements are complete and correct

1. A is square matrix of order n and $|A^{\theta}| = \cdots$ ...

2. The value of determinant …………. When rows and columns are interchanged

3. $\begin{vmatrix} y+z & x & y \\ z=x & z & x \\ x+y & y & z \end{vmatrix} \neq 0$ then $x+y+z$ is……….and x, z are…………

4. If A and B be two Square matrix of same order then $|A.B| = \cdots$ ...

5. Determinant of hermition matrix is always………….

6. If $4x - 3y = 15$ and $2x + 5y = 1$ then $x = \cdots$ ... and y ... ...

7. A is idempotent matrix of order n and its determinant is ……..or…….

8. If A is non zero and$|A| = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$ is also……

ANSWERS:

1. $\overline{|A|}$ **2.** Does not change **3.** Non zero, distinct **4.** $|A|.|B|$

5. Real **6.** 3, 1 **7.** 1, 0 **8.** Non zero

9.20 *TERMINAL QUESTIONS*

9.20.1 Short answer type questions:

1. Show that if $A = \begin{vmatrix} u^3 & 3u^2 & 3u & 1 \\ u^2 & u^2+2u & 2u+1 & 1 \\ u & 2u+1 & u+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$ then determinant of A is $(u-1)^6$

2. Evaluate $\begin{vmatrix} 1 & yz & x(y+z) \\ 1 & za & y(z+a) \\ 1 & xy & z(x+y) \end{vmatrix}$

3. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ y+z & t+x & t+x & x+y \\ t & x & y & z \end{vmatrix} = 0$

4. Show that $|\text{adj}A| = |A|^{n-1}$, where n is a order of matrix A

5. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 7 & 8 & 0 & 0 \\ 9 & 8 & 7 & 5 \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 77 & 99 & 55 \\ 10 & 20 & 125 \\ 87 & 119 & 180 \end{vmatrix}$

7. If $D_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ and $D_2 = \begin{vmatrix} -x & a & -p \\ y & -b & q \\ z & -c & r \end{vmatrix}$ then show that $D_1 = D_2$

ANWERS: 2. 0

2.20.2 Long answer type questions:

1. Show that at least one real number x, show that det A is zero where

$$A = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}$$

2. Solve the following system of linear equation by Cramer's rule

$$2x - y + 3z = 8$$

$$-2x + 2y + z = 4$$

$$3x + y - 4z = 0$$

3. Solve the following system of linear equation by Cramer's rule

$$x + y + z = 9$$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

4. Solve the following system of linear equation by Cramer's rule

$$x + y + 4z = 6$$

$$3x + 2y - 2z = 9$$

$$5x + y + 2z = 13$$

5. Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$

6. Show that the determinant of any matrix A, whose first row is the sum of other row is zero.

7. Show that $\begin{vmatrix} 4 & 5 & 6 & a \\ 5 & 6 & 7 & b \\ 6 & 7 & 8 & c \\ a & b & c & 0 \end{vmatrix} = (a - 2b + c)^2$

8. Prove that $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+y & z & t \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1+u \end{vmatrix} = xyzu\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u}\right)$

ANSWERS:

2. $x = 2, y = 2, z = 2$

3. $x = 1, y = 3, z = 5$

4. $x = 2, y = 2, z = 1/2$

5. $\begin{bmatrix} 7 & -11 & -5 \\ 8 & -14 & -5 \\ -6 & -13 & -5 \end{bmatrix}$

UNIT-10: ELEMENTARY CANONICAL FORM

CONTENTS

10.1 *INTRODUCTION*

An upper triangular matrix of a specific shape known as a Jordan matrix encoding a linear operator on a finite-dimensional vector space with regard to some basis is called a Jordan normal form, or Jordan canonical form (JCF) in linear algebra. In such a matrix, the diagonal entries to the left and bottom of any non-zero off-diagonal entry equal to 1 are identical, and they are located immediately above the main diagonal (on the superdiagonal).

A vector space *V* over a field *K* is defined. If and only if all of the matrix's eigenvalues fall inside *K*, or, to put it another way, if the operator's characteristic polynomial divides into linear factors over *K*, there will be a basis with regard to which the matrix has the necessary form. If *K* is algebraically closed (that is, if it is the field of complex numbers), then this condition is always met. The eigenvalues (of the operator) are the diagonal entries of the normal form, and the algebraic multiplicity of the eigenvalue is the number of times each eigenvalue appears.

The Jordan normal form of an operator is sometimes known as the Jordan normal form of *M* if the operator was initially given by a square matrix *M*. Any square matrix that has its field of coefficients expanded to include all of the matrix's eigenvalues has a Jordan normal form. While it is customary to group blocks for the same eigenvalue together, no ordering is imposed among the eigenvalues or among the blocks for a given eigenvalue, though the latter could be ordered by weakly decreasing size. Despite its name, the normal form for a given *M* is not entirely unique because it is a block diagonal matrix formed of Jordan blocks, the order of which is not fixed.

In particular, the Jordan–Chevalley decomposition is straightforward when applied to a basis where the operator adopts its Jordan normal form. The Jordan normal form is a specific case of the diagonal form for diagonalizable matrices, such as normal matrices.

The Jordan decomposition theorem was initially proposed by Camille Jordan in 1870, and the Jordan normal form bears his name.

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{pmatrix}$$

A matrix example in Jordan normal form. Every matrix entry that isn't visible is zero. The squares that are delineated are called "Jordan blocks". One number lambda is present on the main diagonal of each Jordan block, whereas ones are present above it. The eigenvalues of the matrix are called lambdas, and they don't have to be unique.

10.2 *OBJECTIVES*

After reading this unit learners will be able to

- Understand the concept of Jordan blocks.
- Implement the application of Jordan canonical form.
- Understand the concept of Jordan decomposition theorem.
- Visualized and understand the concept of nilpotent operator.

10.3 *JORDAN BLOCKS*

Let *V* denote a finite dimensional vector space over a field *F*.

Suppose that the characteristic polynomial of *T* splits in *F* and $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of *T* in *F*. Let $N_{\lambda_1}, N_{\lambda_2}, \ldots, N_{\lambda_k}$ be the distinct eigenspaces of *T*.

We know that the diagonalizability of *T* means the following direct sum decomposition of *V* in terms of distinct eigenspaces of *T* given by

$$V = N_{\lambda_1} \oplus N_{\lambda_2} \oplus \ldots \oplus N_{\lambda_k}.$$

Naively, diagonalizability fails if some N_{λ_i} is "small".

Definition 1: Let $\lambda \in F$. We define a Jordan block J_λ to be the matrix

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{pmatrix}$$

Note that the principal diagonal entries are all λ and the upper diagonal entries are all 1. Every other entry is 0. We often omit 0 from the expression.

Our aim is to select an ordered basis *B* of *V* such that

$$[T]_B = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ & & \ldots & \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & A_k \end{pmatrix}$$

where each 0 is a zero matrix, and each A_i is a square matrix of the form (λ) or a Jordan block J_λ defined above, such that λ is an eigenvalue of *T*.

Definition 2: The matrix $[T]_B$ is called a Jordan canonical form of *T*. We say that the ordered basis *B* is a Jordan canonical basis for *T*.

Jordan block A_i is almost a diagonal matrix. $[T]_B$ is a diagonal matrix if and only if each A_i is of the form (λ).

Example 1: Suppose that T is a linear operator on C^8 , and $B = \{v_1, \cdot\cdot\cdot, v_8\}$ is an ordered basis for C^8 such that

$$J = [T]_B = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & & & \\ & (1) & & \\ & & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} & \\ & & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

is a Jordan canonical form of T.

The characteristic polynomial of T is

$$\det(tI - J) = (t-1)^4 (t-3)^3 t^2,$$

and hence the multiplicity of each eigenvalue is the number of times the eigenvalue appears on the diagonal of J.

Also observe that v_1, v_4, v_5 and v_7 are the only vectors in B that are eigenvectors of T. These are the vectors corresponding to the columns of J with no 1 above the diagonal entry. Note that,

$T(v_2) = v_1 + v_2$ and therefore $(T - I)(v_2) = v_1$ and $(T - I)(v_3) = v_2$, since v_1 and v_4 are eigenvectors of T corresponding to $\lambda = 2$. It follows that $(T - I)^3 (v_i) = 0$ for $i = 1, 2, 3, 4$.

Similarly, $(T - 3I)^2 (v_i) = 0$ for $i = 5, 6$ and $(T - 0I)^2 (v_i) = 0$ for $i = 7, 8$

In view of these observations, we can say that:

If v lies in a Jordan canonical basis for a linear operator T and is associated with a Jordan block with diagonal entry λ, then $(T - \lambda I)^p (v) = 0$ for some large enough p. Eigenvectors satisfy this condition for $p = 1$.

Our aim is to prove that every linear operator whose characteristic polynomial splits has a Jordan canonical form that is unique upto the order of the Jordan blocks. It is not true that Jordan canonical form is completely determined by the characteristic polynomial of the operator.

Example 2: Let T' be the linear operator on C^8 such that $[T']_B = J'$, , where B is the ordered basis of the previous example and

$$J' = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 3 & & & \\ & & & & & 3 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

Then the characterstic polynomial of T' is also $(t-1)^4(t-3)^2t^2$, which is the same as that of T of the previous example, but the Jordan canonical forms are different.

10.4 *GENERALIZED EIGENSPACES*

We now extend the definition of eigenspace to generalized eigenspace of an operator T. Our aim is to select ordered bases for these subspaces such that their union form an ordered basis for V and the Jordan canonical form is achieved.

Definition 3: Let T be a linear operator on a vector space V, and let $\lambda \in F$. A nonzero vector v in V is called a generalized eigenvector of T corresponding to λ if and only if $(T - \lambda I)^p(v) = 0$ for some positive integer p.

Note that if v is a generalized eigenvector of T corresponding to λ, and if p is the smallest positive integer for which $(T - \lambda I)^p(v) = 0$, then $(T - \lambda I)^{p-1}(v)$ is an eigenvector of T corresponding to λ. Therefore, λ is an eigenvalue of T

Definition 4: Let T be a linear operator on V, and let $\lambda \in F$ be an eigenvalue of T. The generalized eigenspace of T corresponding to λ, denoted by K_λ, is the subset of V defined by

$$K_\lambda = \{v \in V \mid (T - \lambda I)^p(v) = 0,\ p \in N\}.$$

K_λ consists of the zero vector and all generalized eigenvectors corresponding to λ.

Theorem 1: Let T be a linear operator on V, and let λ be an eigenvalue of T. Then

(i) K_λ is a T-invariant subspace of V containing the eigenspace

$N_\lambda(= \ker(T - \lambda I))$.

(ii) For any scalar $\mu \neq \lambda$, the restriction of $T - \mu I$ to K_λ is one-one.

Proof (i): It is easy to verify.

(ii) Let $v \in K_\lambda$ and $(T-\mu I)(v)=0$. Suppose that $v \neq 0$. Let p be the smallest integer for which

$(T-\lambda I)^p(v) = 0$, and let $w = (T-\lambda I)^{p-1}(v) \neq 0$. Then $(T-\lambda I)(w) = (T-\lambda I)^p(v) = 0$, and hence $w \in N_\lambda$. Furthermore,

$(T-\mu I)(w) = (T-\mu I)(T-\lambda I)^{p-1}(v) = (T-\lambda I)^{p-1}(T-\mu I)(v) = 0$,

so that $w \in N_\mu$. But $N_\lambda \cap N_\mu = \{0\}$, and thus $w = 0$, contrary to the hypothesis. So $v = 0$ and $(T-\mu I)_{|K_\lambda}$ is one-one.

Theorem 2: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits in V. Suppose that λ is an eigenvalue of T with multiplicity m. Then

(i) $\dim(K\lambda) \leq m$.

(ii) $K_\lambda = \ker((T-\lambda I)^m)$.

Proof (i): Let $W = K_\lambda$, and let $p(t)$ be the characteristic polynomial of $T_W = T_{|W}$. Then $p(t)$ divides the characteristic polynomial of T, and therefore it follows that λ is the only eigenvalue of T_W. Hence $p(t) = (t-\lambda)^d$, where $d = \dim(W)$ and $d \leq m$.

(ii) Clearly $\ker((T-\lambda I)^m) \subset K_\lambda$. Now let W and $p(t)$ be as in (i). Then $p(T_W)$ is 0 by the Cayley-Hamilton theorem. Therefore, $(T-\lambda I)^d(v) = 0$ for all $v \in W$. Since $d \leq m$, we have $K_\lambda \subset \ker((T-\lambda I)^m)$.

10.5 *JORDAN CANONICAL FORM*

Theorem 3: Let T be a linear operator on a finite dimensional vector space V, such that the characteristic polynomial of T splits in F. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct eigenvalues of T. Then, for every $v \in V$, there exist vectors

$v_1 \in K_{\lambda_1}, v_2 \in K_{\lambda_2}, ..., v_k \in K_{\lambda_k}$;

Such that $v = v_1 + v_2 + ... + v_k$

Proof: The natural number k denotes the number of distinct eigenvalues of T. The proof in by induction on the number k.

Let k = 1, and let m be the multiplicity of λ_1. Then, $(T-\lambda_1)^m$ is the characteristic polynomial of T, and hence $(T-\lambda_1 I)^m = 0$ by the Cayley-Hamilton theorem. Thus $V = k\lambda_1$, and the result follows.

Now suppose that for some integer $k > 1$, the result is true whenever T has less than k distinct eigenvalues. We assume that T has k distinct eigenvalues. Let m_k denote the multiplicity of λ_k and $p(t)$ the characteristic polynomial of T. Then $p(t) = (t-\lambda_k)^{mk}\, q(t)$, for some polynomial $q(t)$ not divisible by $(t-\lambda_k)$. Let $W_k = \text{range}(T-\lambda_k I)^{mk}$. Then, W_k is T-invariant.

Observe that $(T-\lambda_k I)^{mk}$ maps $k\lambda_i$ onto itself for $i < k$. For suppose that $i < k$. Since $(T-\lambda_k I)^{mk}$ maps $k\lambda_i$ into itself and since $\lambda_k \neq \lambda_i$,
it follows from a previous theorem that the restriction of $T-\lambda_k I$ to $k\lambda_i$ is one-to-one and hence onto.

One consequence of this observation is that for $i<k$, $k\lambda_i$ is contained in W_K; and hence λ_i is an eigenvalue of T_{W_K} for $i < k$. Next, observe that λ_k is not an eigenvalue of T_{W_K}. For, suppose that $T(v) = \lambda_k v$ for some v $\in W_k$. Then $v = (T - \lambda_k I)^{mk}\,(w)$ for some $w \in V$, and it follows that

$$0 = (T-\lambda_k I)(v) = (T-\lambda_k I)^{m_k+1}(w).$$

Therefore, $W \in K_{\lambda_k}$ and by a previous theorem we get $v = (T-\lambda_k I)^{mk}\,(w) = 0$. This shows that v can not be an eigenvector, hence λ_k is not an eigenvalue of T_{W_K} .

We observe that every eigenvalue of T_{W_K} is an eigenvalue of T and the distinct eigenvalues of T_{W_K} are $\lambda_1, \cdots, \lambda_{k-1}$. Now let $v \in V$. Then $(T-\lambda_k I)^{mk}\,(v) \in W_K$. Since T_{W_K} has $k-1$ distinct eigenvalues $\lambda_1, \cdots, \lambda_{k-1}$, the induction hypothesis applies.

Let K'_{λ_i} be the generalized eigenspace for the operator T_{W_K} with respect to the eigenvalue λ_i, for $i = 1, 2, \ldots, k-1$. Hence, by the induction hypothesis, there exist vectors

$$w_1 \in K'_{\lambda_1}, w_2 \in K'_{\lambda_2}, \ldots, w_{k-1} \in K'_{\lambda_{k-1}},$$

such that

$$(T-\lambda I)^{m_k}(v) = w_1 + w_2 + \ldots + w_{k-1}.$$

We note that

(a) $K'_{\lambda_i} \subset K_{\lambda_i}$ for $i<k$

(b) $(T-\lambda_k I)^{m_k}$ maps K_{λ_i} onto itself for $i<k$

Therefore, it follows that there exist vectors $v_i \in K_{\lambda_i}$ for $i < k$, such that $(T-\lambda_k I)^{m_k}(v_i) = w_i$. Hence, $(T-\lambda_k I)^{m_k}(v) = (T-\lambda_k I)^{m_k}(v_1) + ... + (T-\lambda_k I)^{m_k}(v_{k-1})$,

and it follows that

$v-(v_1+v_2+...+v_{k-1}) \in K_{\lambda_k}$. Therefore, there exists a vector $v_k \in K_{\lambda_k}$ such that $v = v_1+v_2+...+v_k$.

Theorem 4: Let *T* be a linear operator on a finite dimensional vector space *V*, such that the characteristic polynomial of *T* splits in *F*. Let $\lambda_1+\lambda_2+...+\lambda_k$ be the distinct eigenvalues of *T* with multiplicities $m_1+m_2+...+m_k$ respectively. For $1 \le i \le k$, let B_i denote an ordered basis for K_{λ_i}. Then, the following statements are true.

(i) $B_i \cap B_j = \phi$ for $i \ne j$.

(ii) $B = B_1 \cup ... \cup B_k$ is an ordered basis for *V*.

(iii) $\dim(K\lambda_i) = m_i, \; for\, i = 1, ..., k$.

Proof (*i*): Let $v \in B_i \cap B_j \subset K\lambda_i \cap K\lambda_j$, where $i \ne j$. By a previous theorem, $T-\lambda_i I$ is one-one on $K\lambda_j$, and therefore $(T-\lambda_i I)^p(v) \ne 0$ for every positive integer *p*. This contradicts the fact that $v \in K\lambda_i$, and the result follows.

(ii) Let $v \in V$. We know by the previous theorem that, for $1 \le i \le k$, there exist vectors $v_i \in K\lambda_i$ such that $v = v_1 + \cdot \cdot \cdot + v_k$. Therefore *B* spans *V*, since each v_i is a linear combination of the vectors of B_i. Let *q* be the cardinality of *B*. Then dim $V \le q$. For each *i*, let $d_i = \dim(K\lambda_i)$. Then, $q = \sum_{i=1}^k d_i \le \sum_{i=1}^k m_i = \dim(V)$. Hence, $q = \dim(V)$; consequently *B* is a basis for *V*.

(iii) Using (ii) we see that $\sum_{i=1}^k d_i = \sum_{i=1}^k m_i$. But $d_i \le m_i$, and therefore $d_i = m_i$ for all *i*.

Corollary 1: Let *T* be a linear operator on a finite dimensional vector space *V*, such that the characteristic polynomial of *T* splits in *F*. Then *T* is diagonalizable if and only if $N_\lambda = K_\lambda$ for every eigenvalue λ of *T*.

Proof: *T* is diagonalizable over *F* if and only if $\dim(N_\lambda) = \dim(K_\lambda)$ for each eigenvalue λ of *T*. But $\dim(N_\lambda) \le \dim(K_\lambda)$, and hence these subspaces have same dimension if and only if they are equal.

Our aim is to select suitable bases for the generalized eigenspaces of the linear operator *T*, so that we may use the previous theorem and obtain a Jordan canonical form. We will find the following definition useful.

Definition 5: Let *T* be a linear operator on a vector space *V*. Let *v* be a generalized eigenvector of *T* corresponding to the eigenvalue λ. Suppose that *p* is the smallest positive integer for which $(T-\lambda I)^p(v)=0$. Then, the ordered set

$$C=\{(T-\lambda I)^{p-1}(v),(T-\lambda I)^{p-2}(v),...,(T-\lambda I)(v),v\}$$

is called a cycle of length *p* of generalized eigenvectors of *T* corresponding to λ. The vectors $(T-\lambda I)^{p-1}(v)$ and *v* are called the initial vector and the end vector of the cycle, respectively.

Remark: Notice that the initial vector of a cycle of generalized eigenvectors of *T* is the only eigenvector of *T* in the cycle. Also observe that if *v* is an eigenvector of *T* corresponding to the eigenvalue λ, then the set $\{v\}$ is a cycle of generalized eigenvectors of *T* corresponding to λ of length 1.

Let us recall some of the main observations of the first example that we discussed. Suppose that *T* is a linear operator on C^8, and $B = \{v_1, \cdot\cdot\cdot, v_8\}$ is an ordered basis for C^8 such that

$$J=[T]_B=\begin{pmatrix} \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix} & & & \\ & (1) & & \\ & & \begin{pmatrix}3&1\\0&3\end{pmatrix} & \\ & & & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}$$

is a Jordan canonical form of *T*.

(1) The first four vectors of *B* lie in K_1.

(2) The vectors in *B* that determine the first Jordan block of *J* are of the form

$$\{v_1, v_2, v_3\} = \{(T-I)^2(v_3), (T-I)(v_3), v_3\}.$$

(3) $(T-I)^3(v_3) = 0$.

The relation between these vectors is the key to finding Jordan canonical form. We observe that the subset $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_4\}$, $C_3 = \{v_5, v_6\}$, $C_4 = \{v_7, v_8\}$ are the cycles of generalized eigenvectors of T that occur in B. Notice that B is a disjoint union of these cycles. Moreover, if $W_i = \text{span}(C_i)$, for $1 \le i \le 4$, we see that C_i is a basis for W_i and $[T_{W_i}]_{C_i}$ is the i-th Jordan block of the Jordan canonical form of T.

Theorem 5: Let T be a linear operator on a finite dimensional vector space V whose characteristic polynomial splits in F. Suppose that B is a basis for V such that B is a disjoint union of cycles of generalized eigenvectors of T. Then the following statements are true:

(i) For each cycle C of generalized eigenvectors contained in B, the subspace $W = \text{span}(C)$ is T-invariant, and $[T_W]_C$ is a Jordan block.

(ii) B is a Jordan canonical basis for V.

Proof: Suppose that the cycle C corresponding to λ has length p, and v is the end vector of C. Then, $C = \{v_1, \cdot \cdot \cdot , v_p\}$, where $v_i = (T - \lambda I)^{p-i}(v)$ for $i < p$ and $v_p = v$. We have $(T-\lambda I)(v_1) = (T-\lambda I)^{p-(i-1)}(v) = v_{i-1}$. Therefore, T maps W into itself, and we see that $[T_W]_C$ is a Jordan block.

We can repeat the arguments of (*i*) for each cycle in B and finally obtain $[T]_B$.

With the help of following theorems we will see that a Jordan canonical basis is nothing but union of disjoint cycles of generalized eigen vectors corresponding to the eigen values of the operator.

Properties 1: Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Suppose that $C_1, \cdot \cdot \cdot, C_r$ are cycles of generalized eigenvectors of T corresponding to λ, such that the initial vectors of the C_i s are distinct and form a linearly independent set. Then the C_i's are disjoint and $C = \bigcup_{i=1}^{r} C_i$ is linearly independent.

2: Every cycle of generalized eigenvectors of a linear operator is linearly independent.

3: Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T. Then K_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ.

Example 3: Let $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$

The characteristic polynomial of A is $p(t) = (t - 3)(t - 2)^2$, hence $\lambda_1 = 3$, $\lambda_2 = 2$ are the distinct eigenvalues with multipilcites 1 and 2 respectively. Then dim(K_{λ_1}) = 1 and dim(K_{λ_2}) = 2. Clearly,

$N_{\lambda_1} = \ker(T - 3I) = K_{\lambda_1}$ and $(-1, 2, 1) N_{\lambda_1}$. Therefore, $B_1 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for K_{λ_1}.

Since dim(K_{λ_2}) = 2, therefore a generalized eigenspace has a basis consisting of union of cycles of length 1 or a single cycle of length 2. The first case is impossible because the vectors in this case would be eigenvectors contradicting the fact that dim(N_{λ_2}) = 1. Therefore, the desired basis is a cycle of length 2. A vector v is the end vector of such a cycle if and only if $(A - 2I)(v) \neq 0$, but $(A - 2I)^2(v) = 0$. Simple calculation shows that

$$\left\{ \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

is a basis for the solution space of $(A - 2I)^2 x = 0$. Now choose a vector v in this set so that $(A - 2I)v \neq 0$. The vector $v = (-1, 2, 0)$ is a candidate for v. Since $(A - 2I)(v) = (1, -3, -1)$ we obtained the cycle of generalized eigenvectors $B_2 = \{(A - 2I)v, v\} = \left\{ \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$

Then, $B = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$

is a Jordan canonical basis and

$$J = [T]_B = \begin{pmatrix} (3) & \\ & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

Is a Jordan canonical form for A.

10.6 *JORDAN DECOMPOSITION THEOREM*

Definition 6: An operator $T: V \to V$ is called nilpotent if $T^k = 0$ for some positive integer k.

Theorem 6 (Jordan Decomposition): Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits in F. Then $T = S + Z$, where S is a diagonalizable operator, Z is a nilpotent operator and $SZ = ZS$.

Proof: We divide the proof into the following steps.

Step 1: T has only one distinct eigenvalue λ, of multiplicity $n = \dim V$. Then, $V = K_\lambda$. If we take $Z = T - \lambda I$, $S = \lambda I$, then $T = Z + S$ and $ZS = SZ$. Moreover, S is diagonal in every basis and Z is nilpotent, for $V = K_\lambda = \ker(Z^n)$.

Step 2: In the general case, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T with multiplicities $n_1, \ldots, n_k$. Let $T_i = T_{|K_{\lambda_i}}$. Then $T = T_1 \oplus \cdots \oplus T_k$. Since each T_i has only one eigenvalue λ_i, we can apply the previous result.

Thus $T_i = S_i + Z_i$; such that $S_i = \lambda_i I$ is diagonal on K_{λ_i} and $N_i = T_i - S_i$ is nilpotent of order n_i on K_{λ_i}. Then $T = S + N$, where $S = S_1 \oplus \cdots \oplus S_k$ and $Z = Z_1 \oplus \cdots \oplus Z_k$. Clearly $SZ = ZS$. Moreover, Z is nilpotent and S is diagonalizable. For, if $m = \max(n_1, \cdots, n_k)$,

then $Z_m = (Z_1)^m \oplus \cdots \oplus (Z_k)^m = 0$; and S is diagonalized by a basis for V which is made up of bases for the generalized eigenspaces. Hence the proof.

Definition 7 (Uniqueness of *S* and *Z*): Under the hypothesis of the Jordan decomposition theorem, there is only one way of expressing T as $S + Z$, where S is diagnalizable, Z is nilpotent and $SZ = ZS$.

Proof: Let $K\lambda_1, \ldots, K\lambda_k$ be the generalized eigenspaces of T corresponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then, $V = K\lambda_1 \oplus \ldots \oplus K\lambda_k$ and $T = T_1 \oplus \ldots \oplus T_k$, where $T_i = T_{|K\lambda_i}$.

Note that $K\lambda_i$ is invariant under every operator that commute with T. Since S and Z both commute with T, therefore $K\lambda_i$ is invariant under S and Z. Put $S_i = \lambda_i I$ and $Z_i = T_i - S_i$. It suffices to show that $S_{|K\lambda_i} = S_i$, for this $Z_{|K\lambda_i} = Z_i$, proving the uniqueness of S and Z.

Since S is diagonalizable, so is $S_{|K\lambda_i}$. Therefore $S_{|K\lambda_i} - \lambda_i I = S_{|K\lambda_i} - S_i$ is diagonalizable. This operator is the same as $Z_i - Z_{|K\lambda_i}$. Since $Z_{|K\lambda_i}$ commutes with $\lambda_i I$ and with T_i, it also commutes with Z_i. We can use binomial theorem to prove that $Z_i - Z_{|N_i}$ is nilpotent.

Hence, the matrix representation of $S_{|N_i} - S_i$ is nilpotent diagonal matrix, and therefore the zero matrix. Hence the proof.

Computation:

By a previous theorem, each generalized eigenspace $K\lambda_i$ contains an ordered basis B_i consisting of a union of disjoint cycles of generalized eigenvectors corresponding to i . Then $B = \bigcup_{i=1}^{k} B_i$ is a Jordan canonical basis for *T*. For each *i*, let $T_i = T_{|k\lambda_i}$, and let $A_i = [T_i]_{B_i}$. Then A_i is the Jordan canonical form for T_i, and

$$J = [T]_B = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ : & : & : & : \\ 0 & 0 & 0 & A_k \end{pmatrix}$$

is the Jordan canonical form for *T*. We now follow the book by Friedberg et.al. to describe the technique of dot diagrams, followed by some illustrative examples.

The Dot Diagram of $T_i = T_{|K\lambda_i}$ **:** Suppose that B_i is a disjoint union of cycles of generalized eigen vectors $C_1, \cdot \cdot \cdot , C_{n_i}$ with length $p_1 \geq p_2 \geq \cdot \cdot \cdot \geq p_{n_i}$ respectively. The dot diagram of T_i contains one dot for each vector in B_i, and the dots are configured according to the following rules.

- The array consists of n_i columns (one column for each cycle).
- Counting from left to right, the $j^{\text{-th}}$ column consists of the p_j dots that correspond to the vectors of C_j starting with the initial vector at the top and continuing down to the end vector.

$$\begin{array}{llll} .(T-\lambda_i I)^{p_1-1}(v_1) & .(T-\lambda_i I)^{p_2-1}(v_2) & \cdots & .(T-\lambda_i I)^{p_{n_i}-1}(v_{n_i}) \\ .(T-\lambda_i I)^{p_1-2}(v_1) & .(T-\lambda_i I)^{p_2-2}(v_2) & \cdots & .(T-\lambda_i I)^{p_{n_i}-2}(v_{n_i}) \\ : & : & : & : \\ .(T-\lambda_i I)(v_1) & .(T-\lambda_i I)(v_2) & \cdots & .(v_{n_i}) \end{array}$$

$.v_1$ $.v_2$

- The dot diagram of T_i has n_i columns (one for each cycle) and p_1 rows. Since $p_1 \geq p_2 \geq \cdot\cdot\cdot \geq p_{n_i}$, the columns of the dot diagram either become shorter in length or remain the same in length as we move from left to right

(i) $n_i = \dim(N_{\lambda_i})$

(ii) r_i is the number of dots in the $i^{\text{-th}}$ row, given by

$$r_1 = \dim V - \text{rank}(T - \lambda_1 I);$$

$$r_j = \text{rank}((T - \lambda_i I))^{j-1} - \text{rank}((T - \lambda_i I)^{j}) \text{ if } j > 1.$$

Example 3: Let $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$

Then, $p(t) = (t-2)^3\,(t-3)$ is the characteristic polynomial. The distinct eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$ with multiplicities 3 and 1 respectively. Therefore, $\dim(K\lambda_1) = 3$ and $\dim(K\lambda_2) = 1$. Let $T_1 = T_{|K\lambda_1}, T_2 = T_{|K\lambda_2}$.

The dot diagram of T_1: It has 3 dots. The possibilities are

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•••      • •     •
         •       •
                 •
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We now calculate $r_1 = 4 - \text{rank}(A - 2I) = 4 - 2 = 2$. Therefore, $r_2 = 1$ and the dot diagram is

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• •
•
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Therefore, the Jordan canonical form for T_1 is $\begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \\ & (2) \end{pmatrix}$ and the Jordan canonical form for T is

$$J = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & & \\ & (2) & \\ & & (3) \end{pmatrix}$$

We now find a Jordan canonical basis for T. We first find a Jordan canonical basis for T_1.

$$.(T-2I)v_1 \quad .v_2$$
$$.v_1$$

Therefore $v_1 \in \ker((T-2I)^2)$ but $v_1 \notin \ker((T-2I))$. Now

$$(A-2I) = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix};\ (A-2I)^2 = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

It is easy to see that a basis for $\ker((T-2I)^2) = K\lambda_1)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Note that $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ do not belong to N_{λ_1}. Choose $v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

And consider $(T-2I)(v_1) = (A-2I)(v_1) = \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}$

Now choose $v_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ which belongs to N_{λ_1} and which is linearly independent of $\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}$. Then

$B_1 = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}\right\}$ is linearly independent and hence a basis for K_{λ_1}.

Therefore, the Jordan canonical basis $B_1 = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}\right\}$ is associated to the diagram as

$$\cdot\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix} \quad \cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$
$$\cdot\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$$

Since $\lambda_2 = 3$ has multiplicity 1, we have $\dim(K_{\lambda_2}) = \dim(N_{\lambda_2}) = 1$. Hence, any eigenvector constitute a basis B_2. Therefore, we may consider

$$B_2 = \left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}.$$

Thus,

$$B = B_1 \cup B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a Jordan canonical basis for A. If we take $Q = \begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.

then $QJQ^{-1} = A$

Example 4: Let $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$.

The characteristic polynomial is $p(t) = (t-2)^2(t-4)^2$ and the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$. Let $T_1 = K_{\lambda_1}$, $T_2 = K_{\lambda_2}$.

Dot diagram of T_1 :

$\cdot\,\cdot \qquad :$

Now $r_1 = 4 - \text{rank}(A - 2I) = 4 - 2 = 2$. Therefore, the correct dot diagram is

$\cdot\,\cdot$

Hence $A_1 = [T_1]_{B_1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. In this case B_1 is any basis of N_{λ_1}

e.g., $B_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$.

Dot diagram of T_2: We have $r_1 = 4 - \text{rank}(A - 4I) = 4 - 3 = 1$, therefore the correct dot diagram is

:

and the Jordan block $A_2 = [T_2]_{B_2} = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$, where B_2 is any basis for K_{λ_2} corresponding to the dots. In this case B_2 is a cycle of length 2. The end vector of this cycle is a vector $v \in K_{\lambda_2} = \ker((T - 4I)^2)$, such that $v \notin N_{\lambda_1} = \ker((T - 4I))$. It is easy to see that a basis for N_{λ_1} is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Choose v to be any solution of

$$(A - 4I)x = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

for example, $v = (A - 4I)x = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$

Thus $B_2 = \{(A - 4I)v, v\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$. Therefore,

$$B = B_1 \cup B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

is a Jordan canonical basis for A. The corresponding Jordan canonical form is

$$J = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

Where $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

Check your progress

Problem 1: For the characteristic polynomial $(t-1)^4(t-3)^2t^2$ find the Jordan canonical form.

Problem 2: Check the characteristic polynomial for the matrix $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$.

10.7 *SUMMARY*

In this unit, we have learned about the important concept of Jordan blocks, Jordan canonical forms, Jordan decomposition theorem, generalized eigenspaces and nilpotent operator. After completion of this unit learners will be able to:

- Formation of Jordan Canonical form on the basis of characteristic polynomial of any matrix.
- Find out any matrix is nilpotent or not.
- Visualized the concept of Jordan decomposition theorem.

10.8 *GLOSSARY*

- Jordan Blocks
- Jordan canonical form
- Jordan decomposition theorem
- Generalized eigenspaces

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10.11 *TERMINAL QUESTION*

Long Answer Type Question:

1. Let T be a linear operator on a finite dimensional vector space V, such that the characteristic polynomial of T splits in F. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct eigenvalues of T. Then show that , for every $v \in V$, there exist vectors

$v_1 \in K_{\lambda_1}, v_2 \in K_{\lambda_2}, ..., v_k \in K_{\lambda_k}$; Such that $v = v_1 + v_2 + ... + v_k$

2. Let T be a linear operator on a finite dimensional vector space V, such that the characteristic polynomial of T splits in F. Let $\lambda_1 + \lambda_2 + ... + \lambda_k$ be the distinct eigenvalues of T with multiplicities $m_1 + m_2 + ... + m_k$ respectively. For $1 \le i \le k$, let B_i denote an ordered basis for K_{λ_i}. Then prove that the following statements are true.

(a) $B_i \cap B_j = \phi$ for $i \neq j$.

(b) $B = B_1 \cup ... \cup B_k$ is an ordered basis for *V*.

(c) $\dim(K\lambda_i) = m_i,\ for\, i = 1, ..., k$

3. Let *T* be a linear operator on a finite dimensional vector space *V* such that the characteristic polynomial of *T* splits in *F*. Then prove that *T* = *S* + *Z*, where *S* is a diagonalizable operator, *Z* is a nilpotent operator and *SZ* = *ZS*.

Short answer type question:

1. Let *T* be a linear operator on *V*, and let λ be an eigenvalue of *T*. Then prove that

(i) K_λ is a *T*-invariant subspace of *V* containing the eigenspace $N_\lambda(= \ker(T - \lambda I))$.

(ii) For any scalar $\mu \neq \lambda$, the restriction of $T - \mu I$ to K_λ is one-one.

2. Let *T* be a linear operator on a finite dimensional vector space *V* such that the characteristic polynomial of *T* splits in *V*. Suppose that λ is an eigenvalue of *T* with multiplicity *m*. Then

(i) dim(Kλ) ≤ m.

(ii) $K_\lambda = \ker((T - \lambda I)^m)$

3. Under the hypothesis of the Jordan decomposition theorem prove that, there is only one way of expressing *T* as *S* + *Z*, where *S* is diagnalizable, *Z* is nilpotent and *SZ* = *ZS*.

Fill in the blanks:

1. Every cycle of generalized eigenvectors of a linear operator is ……………

2. An operator *T*: *V* → *V* is called nilpotent if …………………for some positive integer *k*

10.12 *ANSWERS*

Answers of check your progress:

1:
$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 3 &&& \\ &&&&& 3 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}$$

2: $p(t) = (t-2)^2(t-4)^2$

Answer of fill in the blanks questions:

1. linearly independent **2.** $T^k = 0$

BLOCK - IV

INNER PRODUCT SPACE

UNIT-11: INNER PRODUCT SPACES

CONTENTS

11.1 *INTRODUCTION*

The concept of an inner product space evolved from the practical needs of Fourier analysis and classical geometry, with early ideas of orthogonality and function expansions in the 18th century laying groundwork. Mathematician Giuseppe Peano formalized the idea in 1898 by defining a vector space with an inner product, which he termed a "linear system". The modern notion of an abstract inner product space, generalized from the Euclidean dot product, became a crucial tool in the development of functional analysis, particularly with the work of David Hilbert and the introduction of Hilbert spaces.

An inner product space is a vector space equipped with an inner product, a binary operation that takes two vectors and returns a scalar. This operation, a generalization of the dot product, allows for formal definitions of geometric concepts like vector length (norm), the distance between vectors, and orthogonality. Inner product spaces satisfy properties including linearity, symmetry, and positive-definiteness, which define the inner product itself and the resulting norm.

11.2 *OBJECTIVE*

After the study of this chapter, we shall understand:

- Inner product space
- Norm of a Vector
- Normed linear space
- Cauchy-Schwarz's inequality

11.3 *INNER PRODUCT SPACES*

In this chapter, we shall consider vector spaces over the field or real numbers (**R**) or complex numbers (**C**) only. In R^3, we define dot product (or scalar product) as follows:

Let $\vec{a} = (x_1, x_2, x_3)$, $\vec{b} = (y_1, y_2, y_3)$ in R^3 where all x_i , $y_j \in \mathbf{R}$

Now $\vec{a} . \vec{b} = x_1y_1 + x_2y_2 + x_3y_3 = \vec{b} . \vec{a}$

We **observe** that dot product satisfies the following properties:

(i) $\vec{a} . \vec{a} \geq 0$ i.e. $x_1^2 + x_2^2 + x_3^2 \geq 0$

Also if $x_1^2 + x_2^2 + x_3^2 = 0$

$\Rightarrow$ $x_1 = x_2 = x_3 = 0$

i.e. $\vec{a} = (0, 0, 0) = \vec{0}$

(ii) $\vec{a} . \vec{b} = \vec{b} . \vec{a}$, as we already know.

(iii) $\vec{a} . (\lambda \vec{b} + \mu \vec{c}) = \lambda (\vec{a} . \vec{b}) + \mu (\vec{a} . \vec{c}) \quad \forall \lambda, \mu \in \mathbf{R}$

Here (ii) and (iii) properties can easily be verified. Similarly we can define dot product on $\mathbf{R}^n$. Sometime $\vec{a} . \vec{b}$ is represented as $< \vec{a} , \vec{b} >$. Now we generalize the concept of dot product as inner product in a vector space.

Inner Product: An inner product on a vector space V is a map $< , > : V \times V \to \mathbf{R}$ satisfying the following properties :

(i) $< x , x > \geq 0$ and $< x , x > = 0$ if and only if $x = 0$.

(ii) $< x , y > = < y , x >$

(iii) $<x + z , y> = < x , y > + < z , y >$ and $< x , y + z > = < x , y > + < x , z >$

(iv) $< ax , y > = a < x , y > \quad \forall x , y , z \in V$ and $a \in \mathbf{R}$

Generally function in analysis is represented by ***f*** ; but here, we represent it by **< , >.**

So **(V, < , >)** is called an inner product space. For brevity, we say V is an inner product space without explicitly mentioning the inner product < , >.

Example 1: The dot product defined above on $\mathbf{R}^n$ (in particular $\mathbf{R}^2$) is an inner product. It can be easily verified. Sometimes it is called **standard inner product**.

Example 2: If we consider inner product on **V(C),** where C represent field of complex numbers, then following properties must be satisfied :

(i) $< x , y > \; = \overline{< y , x >}$, where $\overline{< y , x >}$ is complex conjugate of $< x , y >$.

(ii) $< x , x > \; \geq 0$ and $< x , x > \; = 0 \Leftrightarrow x = 0$.

(iii) $<\alpha x + \beta y , z > \; = \alpha < x , z > + \beta < y , z >$ where $\alpha , \beta \in \mathbf{C}$

(iv) $< x , \alpha y + \beta z > \; = \bar{\alpha} < x , y > + \bar{\beta} < x , z >$

Example 3: Prove that the vector space $\mathbf{C}^n(\mathbf{C}) = \{ (\alpha_1, \ldots\ldots \alpha_n) : \alpha_I \in \mathbf{C} \}$ is an inner product space with respect to the inner product : $< u , v > \; = \alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \ldots\ldots\ldots + \alpha_n\overline{\beta_n}$, where $u = (\alpha_1, \ldots\ldots, \alpha_n)$, $v = (\beta_1, \ldots\ldots, \beta_n) \in \mathbf{C}^n$

Solution: Given that : $< u , v > \; = \alpha_1\overline{\beta_1} + \ldots\ldots\ldots + \alpha_n\overline{\beta_n}$ ………(1)

So we have

(i) $< v , u > \; = \beta_1\overline{\alpha_1} + \ldots\ldots\ldots + \beta_n\overline{\alpha_n}$

$\Rightarrow \overline{< v , u >} = \overline{\beta_1\overline{\alpha_1} + \ldots\ldots\ldots + \beta_n\overline{\alpha_n}} = (\overline{\beta_1\overline{\alpha_1}}) + \ldots\ldots\ldots + (\overline{\beta_n\overline{\alpha_n}})$

$= \overline{\beta_1}\, \alpha_1 + \ldots\ldots\ldots + \overline{\beta_n}\, \alpha_n$ (as $\overline{(\overline{\alpha_n})} = \alpha_n \; \forall \; n$)

$= \alpha_1\overline{\beta_1} + \ldots\ldots\ldots + \alpha_n\overline{\beta_n}$

So, $\overline{< v , u >} = \; < u , v >$

(ii) $< u , u > \; = \alpha_1\overline{\alpha_1} + \ldots\ldots\ldots + \alpha_n\overline{\alpha_n}$

$= |\alpha_1|^2 + \ldots\ldots\ldots + |\alpha_n|^2 \geq 0$

Also $< u , u > \; = 0 \Leftrightarrow |\alpha_1|^2 + \ldots\ldots\ldots + |\alpha_n|^2 = 0$

$\Leftrightarrow \alpha_1 = 0 = \alpha_2 = \ldots\ldots = \alpha_n$

$\Leftrightarrow u = (\alpha_1, \ldots\ldots \alpha_n),= (0, \ldots\ldots , 0) = \bar{0}$

(iii) Let $\alpha , \beta \in \mathbf{C}$ and $w = (\gamma_1 , \ldots\ldots , \gamma_n) \in C^n$, then

$< \alpha u + \beta v , w > \; = \; < \alpha(\alpha_1, \ldots\ldots, \alpha_n) + \beta(\beta_1, \ldots\ldots, \beta_n), (\gamma_1 , \ldots\ldots , \gamma_n) >$

$= \; < (\alpha\alpha_1 + \beta\beta_1 , \ldots\ldots, \alpha\alpha_n + \beta\beta_n), (\gamma_1 , \ldots\ldots , \gamma_n) >$

$= (\alpha\alpha_1 + \beta\beta_1)\, \overline{\gamma_1} + \ldots\ldots + (\alpha\alpha_n + \beta\beta_n)\, \overline{\gamma_n}$

$= (\alpha\alpha_1 \overline{\gamma_1} + \beta\beta_1 \overline{\gamma_1}) + \ldots\ldots + (\alpha\alpha_n \overline{\gamma_n} + \beta\beta_n \overline{\gamma_n})$

$= \alpha\, (\alpha_1 \overline{\gamma_1} + \ldots\ldots + \alpha_n \overline{\gamma_n}) + \beta\, (\beta_1 \overline{\gamma_1} + \ldots\ldots + \beta_n \overline{\gamma_n})$

$= \alpha < u , w > + \beta < v , w >$

Hence $\mathbf{C^n}$ is an inner product space.

Note: The inner product given by equation (1) is called the **standard inner product on $\mathbf{C^n}$.**

Example 4: Prove that the following is an inner product on $\mathbf{R^2}$,

$< u , v > \ = \alpha_1 \beta_1 - 2\, \alpha_1 \beta_2 - 2\, \alpha_2 \beta_1 + 5\, \alpha_2 \beta_2$, where $u = (\alpha_1 , \alpha_2)$ and $v = (\beta_1 , \beta_2) \in \mathbf{R^2}$.

Solution: Here < u , v > will be a real number, so

(i) $< u , v > \ = \ < \overline{v , u} >$, obviously.

(ii) $< u , u > \ = \alpha_1 \alpha_1 - 2\, \alpha_1 \alpha_2 - 2\, \alpha_2 \alpha_1 + 5\, \alpha_2 \alpha_2$

$= \alpha_1^2 - 4\, \alpha_1 \alpha_2 + 5\, \alpha_2^2$

$= \alpha_1^2 - 4\, \alpha_1 \alpha_2 + 4\, \alpha_2^2 + \alpha_2^2$

$= (\alpha_1 - 2\, \alpha_2)^2 \geq 0$

Now, $< u , u > \ = 0$,

$\Leftrightarrow (\alpha_1 - 2\, \alpha_2)^2 + \alpha_2^2 = 0$,

$\Leftrightarrow \alpha_1 - 2\, \alpha_2 = 0$ and $\alpha_2 = 0$.

So $< u , u > \ = 0 \Leftrightarrow u = (\alpha_1 , \alpha_2) = (0 , 0)$

(iii) Let $\alpha , \beta \in \mathbf{R}$ and $w = (\gamma_1 , \gamma_2) \in \mathbf{R^2}$, then

$\alpha u + \beta v = \alpha(\alpha_1 , \alpha_2) + \beta(\beta_1 , \beta_2) = (\alpha\alpha_1 + \beta\beta_1 , \alpha\alpha_2 + \beta\beta_2)$

Now, $< \alpha u + \beta v , w > = < (\alpha\alpha_1 + \beta\beta_1 , \alpha\alpha_2 + \beta\beta_2) , (\gamma_1 , \gamma_n) >$

$= (\alpha\alpha_1 + \beta\beta_1)\gamma_1 - 2(\alpha\alpha_1 + \beta\beta_1)\gamma_2 - 2\, (\alpha\alpha_2 + \beta\beta_2)\gamma_1 + 5\, (\alpha\alpha_1 + \beta\beta_1)\, \gamma_1$

$= \alpha\, (\alpha_1 \gamma_1 - 2\, \alpha_1 \gamma_2 - 2\, \alpha_2 \gamma_1 + 5\, \alpha_2 \gamma_2)$

$+ \beta\, (\beta_1 \gamma_1 - 2\, \beta_1 \gamma_2 - 2\, \beta_2 \gamma_1 + 5\, \beta_2 \gamma_2)$

$= \alpha <u , w > + \ \beta < v , w >$, (using (1))

Hence < u , v > , defined by equation (1), is an inner product on $\mathbf{R^2}$.

Example 5: Let V be the vector space of all real polynomials of degree ≤ 2. Prove that

$< f(x) , g(x) > = \int_0^1 f(x)g(x)dx$, $\forall$ f(x) , g(x) $\in$ V, is an inner product on V.

Solution: (i) Since f(x) and g(x) are real polynomials, so < f(x) , g(x) > $\in \mathbf{R}$

Hence $< f(x) , g(x) > \ = \ < \overline{f(x) , g(x)} > = < \overline{g(x) , f(x)} >$

(ii) Now $< f(x) , f(x) > \ = \ \int_0^1 f(x)f(x)dx = \int_0^1 f(x)^2 dx \ \geq 0$

Also, < f(x) , f(x) > = 0 , if and only if

$\int_0^1 f(x)^2 dx = 0$, if and only if

$\Rightarrow$ f(x) = 0 ,

So, < f(x) , f(x) > = 0 $\Leftrightarrow$ f(x) = 0,

(iii) Let $\alpha, \beta \in$ **R** and f(x), g(x), h(x) $\in$ V. Then

$$< \alpha\, f(x) + \beta\, g(x) , h(x) > = \int_0^1 \big(\alpha f(x) + \beta g(x)\big) h(x) dx$$

$$= \alpha \int_0^1 f(x)h(x)dx + \beta \int_0^1 g(x)h(x)dx$$

$$= \alpha < f(x) , h(x) > + \beta < g(x), h(x) >$$

Hence < f(x) , g(x) > , defined by equation (1) is an inner product on V.

Example 6: Given α_1 = (1, 3), α_2 = (2, 1) $\in$ **R**. Find an $\alpha \in \mathbf{R}^2$ such that < α , α_1 > = 3,

< , α_2 > = -1. Here < , > is the standard inner product on $\mathbf{R}^2$.

Solution: We know that the standard inner product on $\mathbf{R}^2$ is

$$< (a_1 , a_2) ,(b_1 , b_2) > = a_1b_1 + a_2b_2 \qquad …….(1)$$

Let α = (x, y) $\in \mathbf{R}^2$.

So, < , α_1 > = < (x, y), (1, 3) > = x + 3y = 3 ………(2)

< α , α_2 > = < (x, y), (2, 1) > = 2x + y = -1 ……….(3)

On solving equations (2) and (3), we get x = -6/5 , y = 7/5

So, $\alpha = (\frac{-6}{5}, \frac{7}{5})$

Example 7: Let W_1 and W_2 be two subspaces of a vector space V . If W_1 and W_2 are both inner product spaces, then prove that $W_1 + W_2$ is also an inner product space.

Solution: Let x, y $\in W_1 + W_2$, then

$x = x_1 + x_2$, $y = y_1 + y_2$ where $x_1 , y_1 \in W_1$ and $x_2 , y_2 \in W_2$

We define, $< x , y > = < x_1 , y_1 > + < x_2 , y_2 >$ ……..(1)

Here, $<x_1 , y_1 >$ is the inner product on W_1 and $<x_2 , y_2 >$ is the inner product on W_2.

Now from equation (1), we have

(i) $< \overline{y , x} > = \overline{< y_1 , x_1 > + < y_2 , x_2 >} = \overline{< y_1 , x_1 >} + \overline{< y_2 , x_2 >}$
$= < x_1 , y_1 > + < x_2 , y_2 >$ (as W_1 and W_2 are I.P.S.)
$= < x , y >$

(ii) $< x , x > = < x_1 , x_1 > + < x_2 , x_2 >$

Since, $< x_1 , x_1 > \geq 0$ and $< x_2 , x_2 > \geq 0$

$\Rightarrow < x_1 , x_1 > + < x_2 , x_2 > \geq 0$

$\Rightarrow < x , x > \geq 0$

Also, $< x , x > = 0$

$\Leftrightarrow < x_1 , x_1 > = 0$ and $< x_2 , x_2 > = 0$

$\Leftrightarrow x_1 = 0$ and $x_2 = 0$

$\Leftrightarrow x = x_1 + x_2 = 0$

(iii) Let $\alpha , \beta \in$ **F** and $z = z_1 + z_2 \in W_1 + W_2$

Now, $\alpha x + \beta y = \alpha (x_1 + x_2) + \beta (y_1 + y_2) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)$

So, $< \alpha x + \beta y , z > = < (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) , z_1 + z_2 >$
$= < (\alpha x_1 + \beta y_1 , z_1 > + < \alpha x_2 + \beta y_2 , z_2 >$
$= \alpha < x_1 , z_1 > + \beta < y_1 , z_1 > + \alpha < x_2 , z_2 > + \beta < y_2 , z_2 >$
$= \alpha (< x_1 , z_1 > + < x_2 , z_2 >) + \beta (< y_1 , z_1 > + < y_2 , z_2 >)$
$= \alpha < x , z > + \beta < y , z >$, (using eqn. (1))

Hence, $W_1 + W_2$ is also an inner product space.

Theorem 1: Let V be an inner product space and u, v, w $\in$ V ; $\alpha , \beta \in$ **F** (where F = **R** or **C**) then,

(i) $< u, \alpha v > = \bar{\alpha} < u, v >$

(ii) $< 0, v > = < u, 0 > = 0$

(iii) $< u, v > = 0, \ \forall u \in V \Rightarrow v = 0$, and

(iv) $< u, v > = 0, \ \forall v \in V \Rightarrow u = 0$,

(v) $< u, w > = < v, w > , \ \forall w \in V \Leftrightarrow u = v$

Proof: (i) By definition of inner product

$< u, \alpha v > = < \overline{\alpha v , u} > = \bar{\alpha} < \overline{v , u} > = \bar{\alpha} < u, v >$

(ii) We know for any u ∈ V and 0 ∈ **F** , 0u = 0 ∈ V

So, < 0, v > = < 0u, v > = 0 < u, v > = 0

Similarly , < u, 0 > = < u, 0v > = $\bar{0}$ < u, v > = 0 < u, v > = 0

(iii) It is given that < u, v > = 0, ∀ u ∈ V

In particular, we can write,

< u, v > = 0

⟺ u = 0

Similarly, we can prove other part.

(iv) Let < u, w > = < v, w > , ∀ w ∈ V then,

< u – v, w > = < u, w > − < v, w > = 0

< u – v, w > = 0 ∀ w ∈ V

So by previous part, u – v = 0 ⇒ u = v.

Conversely, If we take u = v, then

< u, w > − < v, w > = < u – v, w > = < 0 , w > = 0

Hence , < u, w > = < v, w > ∀ w ∈ V.

Note: If V is an inner product space with standard inner product and say V = $\mathbf{R}^3$, then for a ∈ $\mathbf{R}^3$,

We have, < a, a > = $a_1^2 + a_2^2 + a_3^2$ where a = (a_1 , a_2 , a_3),

Here $\sqrt{a_1^2 + a_2^2 + a_3^2}$ or $\sqrt{< a, a >}$ is defined as norm of vector a. Actually, it is generalization of length of a physical vector.

Norm of a Vector: Let V be an inner product space. The norm function ‖ . ‖ : V → R has the following properties :

(i) ‖ x ‖ ≥ 0 and ‖ x ‖ = 0 if and only if x = 0 ; x ∈ V

(ii) ‖ αx ‖ = $|\alpha|$ ‖ x ‖ , α ∈ **F** , x ∈ V, Norm of a vector v ∈ V is defined as $\| v \| = \sqrt{< v, v >}$.

A vector u in an inner product space V is said to be of unit norm or unit length if

$\| u \| = 1$ or $< u, u > = 1$.

Furthermore, given a non-zero vector v ∈ V, there is a vector u ∈ V such that

$\| u \| = 1$ and $v = \| v \| u$.

This u is called the **unit vector along v**, because $u = \frac{v}{\| v \|}$ and $\| u \| = \frac{\| v \|}{\| v \|} = 1$

Example 8: (i) Find the norm of the vector x = (2, -3, 6) ∈ $\mathbf{R}^3$ **.**

(ii) Prove that $\frac{x}{\| x \|}$ is of unit length.

Solution: (i) Using the concept of standard inner product of $\mathbf{R}^3$, we have

$< x, x > = 2(2) + (-3)(-3) + 6(6) = 49$

Hence, $\| x \| = \sqrt{< x, x >} = \sqrt{49} = 7$ units

(ii) Let $u = \frac{x}{\| x \|} = \frac{1}{7}(2, -3, 6) = (\frac{2}{7}, \frac{-3}{7}, \frac{6}{7})$

$< u, u > = \frac{2}{7}\left(\frac{2}{7}\right) + \left(\frac{-3}{7}\right)\left(\frac{-3}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{6}{7}\right) = \frac{49}{49} = 1$

$\Rightarrow \| u \| = 1 \Rightarrow u = \frac{x}{\| x \|}$ is of unit length.

Example 9: Let V be an inner product space and x, y, z ∈ V. Prove that

$\| x + y \|^2 + \| x - y \|^2 = 2 (\| x \|^2 + \| y \|^2)$. Also interpret it geometrically.

Solution: Some writers say it parallelogram law.

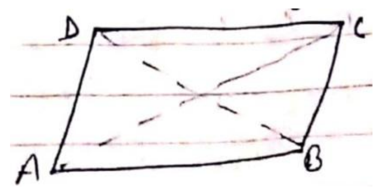
We have $\| x + y \|^2 = < x + y , x + y > = < x , x + y > + < y , x + y >$

$= < x , x > + < x , y > + < y , x > + < y , y >$ ……….(1)

Now, $\| x - y \|^2 = < x - y , x - y > = < x , x - y > - < y , x - y >$

$= < x , x > - < x , y > - < y , x > + < y , y >$ ……….(2)

Adding equation (1) and (2) , we have



$\| x + y \|^2 + \| x - y \|^2 = 2\,(< x , x > + < y , y >) \quad = 2\,(\| x \|^2 + \| y \|^2)$

Geometric interpretation: Let **x** and **y** be two vectors in the vector space V_2**(R)** with standard inner product defined on it. Suppose the vector **x** is represented by the side AB and the vector **y** by the side BC of a parallelogram ABCD. Then the vectors **x + y** and **x – y** represented the diagonals AC and DB of the parallelogram.

So, $AC^2 + DB^2 = 2(AB^2 + BC^2)$ i.e. the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals.

Example 10: Prove that we can **always** define an inner product on a finite-dimensional vector space V(**R**) or V(**C**).

Solution: Let V be a finite- dimensional vector space over the field F = **R** or **C**.

Let B = { α_1 , ….. , α_n } be a basis for V .

Let $\alpha , \beta \in$ V. Then we can write $\alpha = a_1 \alpha_1 + \ldots + a_n \alpha_n$ and $\beta = b_1 \alpha_1 + \ldots + b_n \alpha_n$

Where, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are uniquely determined elements of **F**.

Let us define $< \alpha , \beta > = a_1 \overline{b_2} + \ldots + a_n \overline{b_n}$ ……….(1)

Now it can be easily verified that above expression satisfies all the conditions of inner product. Hence, we can always define an inner product on a finite dimensional vector space V(**C**).

Example 11: If , β are vectors in an inner product space V(**F**) and a, b $\in$ **F**, then prove that

(i) $\| a\alpha + b\beta \|^2 = |a|^2 \| \alpha \|^2 + a\bar{b} < \alpha , \beta > + \bar{a}b < \beta , \alpha > + |b|^2 \| \beta \|^2$

(ii) $\text{Re} < \alpha , \beta > = \frac{1}{4} (\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2)$

Solution: (i) We have,

$\| a\alpha + b\beta \|^2 = < a\alpha + b\beta, a\alpha + b\beta > = < a\alpha , a\alpha + b\beta > + < b\beta , a\alpha + b\beta >$

$= a < \alpha , a\alpha + b\beta > + b < \beta , a\alpha + b\beta >$

$= a < \alpha , a\alpha > + a < \alpha , b\beta > + b < \beta , a\alpha > + b < \beta , b\beta >$

$= a\bar{a} < \alpha, \alpha > + a\bar{b} < \alpha, \beta > + b\bar{a} < \beta, \alpha > + b\bar{b} < \beta, \beta >$

$= |a|^2 \| \alpha \|^2 + a\bar{b} < \alpha, \beta > + \bar{a}b < \beta, \alpha > + |b|^2 \| \beta \|^2$

(ii) Now we can write

$\| \alpha + \beta \|^2 = < \alpha + \beta, \alpha + \beta > = < \alpha, \alpha + \beta > + < \beta, \alpha + \beta >$

$= < \alpha, \alpha > + < \alpha, \beta > + < \beta, \alpha > + < \beta, \beta >$

$\| \alpha + \beta \|^2 = \| \alpha \|^2 + < \alpha, \beta > + < \beta, \alpha > + \| \beta \|^2$ …..(1)

Also, $\| \alpha - \beta \|^2 = < \alpha - \beta, \alpha - \beta > = < \alpha, \alpha - \beta > - < \beta, \alpha - \beta >$

$= < \alpha, \alpha > - < \alpha, \beta > - < \beta, \alpha > + < \beta, \beta >$

$\| \alpha - \beta \|^2 = \| \alpha \|^2 - < \alpha, \beta > - < \beta, \alpha > + \| \beta \|^2$ …..(2)

Now subtracting equation (2) from equation (1), we get

$\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2 = < \alpha, \beta > + < \beta, \alpha > + < \alpha, \beta > + < \beta, \alpha >$

$= 2(< \alpha, \beta > + < \beta, \alpha >)$

$= 2(< \alpha, \beta > + < \overline{\alpha, \beta} >)$

$= 2 (2 \text{ Re} < \alpha, \beta >)$

So, $\text{Re} < \alpha, \beta > = \frac{1}{4} (\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2)$

Note: (1) If F = **R**, then $\text{Re} < \alpha, \beta > = < \alpha, \beta >$

So, $< \alpha, \beta > = \frac{1}{4} (\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2)$

(2) An inner product space V(**R**) is called **Euclidean space** while V(**C**) is called **unitary space**.

Example 12: If α and β are vectors in a unitary space, then prove that –

(i) $4 < \alpha, \beta > = \| \alpha + \beta \|^2 - \| \alpha - \beta \|^2 + i \| \alpha + i\beta \|^2 - i \| \alpha - i\beta \|^2$

(ii) $< \alpha, \beta > = \text{Re} < \alpha, \beta > + i \text{ Re} < \alpha, i\beta >$

Solution: (i) As in previous example, we can write

$$\| \alpha + \beta \|^2 = \| \alpha \|^2 + < \alpha , \beta > + < \beta , \alpha > + \| \beta \|^2$$

and $\| \alpha - \beta \|^2 = \| \alpha \|^2 - < \alpha , \beta > - < \beta , \alpha > + \| \beta \|^2$

So, $\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2 = 2< \alpha , \beta > + 2 < \beta , \alpha >$ ……..(1)

Now $\| \alpha + i \beta \|^2 = < \alpha + i \beta, \alpha + i \beta > = < \alpha , \alpha + i \beta > + < i \beta , \alpha + i \beta >$

$$= < \alpha , \alpha > + < \alpha , i \beta > + < i \beta , \alpha > + < i \beta , i \beta >$$

$$= \| \alpha \|^2 + \bar{i} < \alpha , \beta > + i < \beta , \alpha > + i \bar{i} < \beta , \beta >$$

$$= \| \alpha \|^2 - i < \alpha , \beta > + i < \beta , \alpha > + \| \beta \|^2$$

So $i \| \alpha + i \beta \|^2 = i \| \alpha \|^2 + < \alpha , \beta > - < \beta , \alpha > + i \| \beta \|^2$ ………..(2)

Replacing i by –i , we get

$-i \| \alpha + i \beta \|^2 = - i \| \alpha \|^2 + < \alpha , \beta > - < \beta , \alpha > + i \| \beta \|^2$ ………..(3)

Hence adding equations (1), (2) and (3), we get

$$\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2 + i \| \alpha + i \beta \|^2 - i \| \alpha - i \beta \|^2 = 4 < \alpha , \beta >$$

(**ii**) From the knowledge of complex numbers, we have

$< \alpha , \beta > = Re < \alpha , \beta > + i \, Im < \alpha , i \beta >$ ……….(1)

If z = x + iy, then y = Im z = Re { -i (x + iy) } = Re (-iz)

$\therefore Im < \alpha , \beta > = Re \{ -i < \alpha , \beta > \} = Re \{ \bar{i} < \alpha , \beta > \} = Re \{ < \alpha , i \beta > \}$

So from (1), we have

$$< \alpha , \beta > = Re < \alpha , \beta > + i \, Re < \alpha , i \beta >$$

Note: In the study of physical vectors, we define dot/scalar product as $\vec{a}. \vec{b}$ = ab cos θ, where a = $|\vec{a}|$, b = $|\vec{b}|$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Since we know that $|\cos \theta| \leq 1$. So, $ab|\cos \theta| \leq ab$ as $a \geq 0 , b \geq 0$.

$|\vec{a}.\vec{b}| \leq$ ab or $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$

This is a particular case of Cauchy-Schwarz's inequality, which we shall study for an inner product space.

11.4 *CAUCHY SCHWARZ INEQUALITY*

Theorem 2: Let V be an inner product space. If x , y ∈ V, then

$|< x , y >| \leq \| x \| + \| y \|$. Further, equality holds if and only if x and y are linearly dependent (that is, one is a multiple of other).

Proof: Here we shall give three different proofs of Cauchy-Schwarz's inequality:

(i) It is basically geometric in nature
(ii) Here we shall use basic concepts of calculus
(iii) Here we shall use some results on quadratic equations.

Proof: Case (i): If x = 0 or y = 0,

Then $< x , y > = 0$ and either $< x , x > = 0$ or $< y , y > = 0$,

Hence the result is obviously true.

Case (ii): Now consider the case, when $\| x \| = \| y \| = 1$,

Consider $< x - y , x - y >$, then by definition of inner product

$$< x - y , x - y > \geq 0,$$

$$\Rightarrow < x , x > - 2< x , y > + < y , y > \geq 0$$

$$1 - 2< x , y > + 1 \geq 0$$

$$\Rightarrow \quad < x , y > \leq 1 \quad \text{..........(1)}$$

Similarly, $< x + y , x + y > \geq 0$,

$$\Rightarrow \quad -< x , y > \leq 1 \quad \text{.........(2)}$$

Combining both results, we get

$$|< x , y >| \leq 1 \quad \text{or } |< x , y >| \leq \| x \| \ \| y \| \text{ as } \| x \| = \| y \| = 1$$

Now, we prove the statement concerning the equality

Let $|< x , y >| = 1$, then $< x , y >$ = 1 or -1

If $< x , y >$ = 1, then from the above discussion of inequalities, we deduce that

$$< x - y , x - y > = 0 \quad \text{or } x = y$$

If $< x , y >$ = -1, we can deduce that x = -y.

Thus equality holds if and only if either x + y = 0 or x – y = 0.

i.e. if and only if x = $\pm$ y.

So x and y are **linearly dependent,** when equality holds.

Case (iii): Now suppose x and y be non-zero and not necessarily of unit length.

Then $u = \frac{x}{\| x \|}$ and $v = \frac{y}{\| y \|}$ s.t. $\| u \| = \| v \| = 1$

Then as in last case, we have $|< u , v >| \leq 1$

$$\text{So } |< \frac{x}{\| x \|} , \frac{y}{\| y \|} >| = \frac{|< x , y >|}{\| x \| \| y \|} \leq 1$$

$$\Rightarrow |< x , y >| \leq \| x \| \| y \|,$$

Now, in the case of equality, we have $| < x , y > | \leq \| x \| \| y \|$,

If x and y are non-zero, then $< x , y > = \| x \| \| y \|$ or

$$-< x , y > = \| x \| \| y \|$$

If we assume, $< x , y > = \| x \| \| y \|$

$$\Leftrightarrow < \frac{x}{\| x \|} , \frac{y}{\| y \|} > = 1$$

$$\Leftrightarrow \frac{x}{\| x \|} = \frac{y}{\| y \|}$$

$$\Leftrightarrow x = \left(\frac{\| x \|}{\| y \|} \right) y$$

$\Rightarrow$ x is a scalar multiple of y, or x and y are linearly dependent.

The other case is similar.

Proof 2: Fix x and y in V.

If y = 0, then the result is obviously true.

So, we take $y \neq 0$

Let us consider the real valued function of the real variable

$$f(t) = < x + ty, x + ty >.$$

We want to investigate the extremum points of f.

So, $f(t) = < x , x > + 2t < x , y > + t^2 < y , y >$ ....(1)

So we observe that f(t) is a polynomial in t with real coefficients.

Now $f'(t) = 2 < x , y > + 2t < y , y >$

So t_o will be an extremum point for f if $f'(t_o) = 0$,

i.e. $< x , y > + t_o < y , y > = 0$

So, $t_o = -\frac{<x , y>}{<y , y>}$

Now $f''(t) = 2 < y , y > = 2 \| y \|^2 > 0$ as $y \neq 0$

So f(t) is minimum at $t = t_o$

$\Rightarrow 0 \leq f(t_o) \leq f(t)$ for all t

$\Rightarrow f(t) \geq 0$ for all t

From equation (1) , we get

$$< x , x > + 2t_o < x , y > + t_o^2 < y , y > \geq 0$$

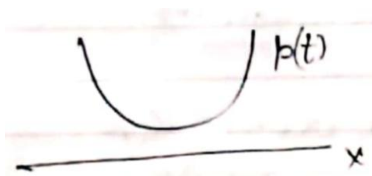
$$< x , x > - \frac{2\,(<x , y>)^2}{<y , y>} + \frac{(<x , y>)^2}{<y , y>} \geq 0$$

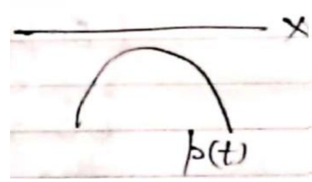
$$< x , x > - \frac{(<x , y>)^2}{<y , y>} \geq 0$$

$$\Rightarrow \| \, x \, \|^2 \geq \frac{(<x , y>)^2}{\| \, y \, \|^2}$$

$$\Rightarrow |< x , y >| \leq \| \, x \, \| \| \, y \, \|$$

Proof 3: Let p(t) = $at^2 + bt + c$ be a quadratic polynomial in t with real coefficient. We know that for imaginary roots, p(t) will always remain +ve or always remain –ve.





For this to happen , $b^2 – 4ac \leq 0$.

Now f(t) as in second proof is a quadratic polynomial in t with real coefficients

$$a = < y , y > , \; b = 2 < x , y > \text{ and } c = < x , x >$$

Also f(t) is always non negative. So we conclude that $b^2 – 4ac \leq 0$. From this, we shall get the required result.

Note: If we consider $\mathbf{R}^{\mathbf{n}}$ with dot(scalar) product, then Cauchy-Schwarz inequality becomes

$$| \sum_{i=1}^{n} x_i y_i | \leq (\sum_i x_i^2)^{1/2} \; (\sum_i y_i^2)^{1/2} , \text{ for all } x_i , y_i \in \mathbf{R}.$$

This concrete inequality is quite useful in analysis.

Theorem 3: (Triangle Inequality) If α, β are vectors in an inner product space V, then

$$\| \, \alpha + \beta \, \| \leq \| \, \alpha \, \| + \| \, \beta \, \|$$

Proof: We have , $\| \, \alpha + \beta \, \|^2 = < \alpha + \beta , \alpha + \beta > = < \alpha, \alpha + \beta > + < \beta , \alpha + \beta >$

$$= < \alpha, \alpha > + < \alpha, \beta > + < \beta , \alpha > + < \beta , \beta >$$

$$= \| \, \alpha \, \|^2 + \| \, \beta \, \|^2 + (< \alpha, \beta > + < \overline{\alpha, \beta} >)$$

$$\| \, \alpha + \beta \, \|^2 = \| \, \alpha \, \|^2 + \| \, \beta \, \|^2 + 2 \, \text{Re} \, (< \alpha, \beta >) \qquad \ldots\ldots(1)$$

But Re $(\bar{z}) \leq |z|$,

So, $\| \alpha + \beta \|^2 \leq \| \alpha \|^2 + \| \beta \|^2 + 2 \, | < \alpha, \beta > |$

$\leq \| \alpha \|^2 + \| \beta \|^2 + 2 \| \alpha \| \| \beta \|$, (by Cauchy Schwarz inequality)

$\Rightarrow \| \alpha + \beta \|^2 \leq (\| \alpha \| + \| \beta \|)^2$

So, $\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$

Geometrical Interpretation:

Suppose the vectors α , β represent the sides AB and BC respectively of a Δ ABC in the Euclidean space.

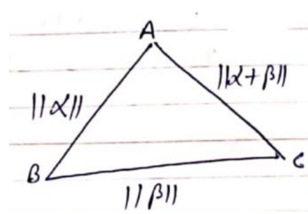
Then $\| \alpha \| = AB$ and $\| \beta \| = BC$.

Also the vector $\alpha + \beta$ represents the side AC of the triangle ABC and $\| \alpha + \beta \| = AC$.

Then from above inequality we know, $\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$

$\Rightarrow AC \leq AB + BC$

If inequality holds, i.e. AC < AB + BC is true for any triangle ABC.



If equality holds, then AC = AB + BC means points A, B, C are collinear.

Example 13: Verify Cauchy Schwarz inequality for $\alpha = (1, 2, -2)$, and $\beta = (2, 3, 6) \in \mathbf{R}^3$.

Solution: With standard inner product, we have

$< \alpha, \beta > = 2 + 6 - 12 = -4$, so $| < \alpha, \beta > | = 4$

Now, $\| \alpha \|^2 = 1 + 4 + 4$, so $\| \alpha \| = 3$

And $\| \beta \|^2 = 4 + 9 + 36$, then $\| \beta \| = 7$

So , $\| \alpha \| \| \beta \| = 21$

Hence, $| < \alpha, \beta > | \leq \| \alpha \| \| \beta \|$ is verified.

Example 14: If in an inner product space V, $\| \alpha + \beta \| = \| \alpha \| + \| \beta \|$, then prove that α and β are linearly dependent. Show by means of an example that the converse may **NOT** be true.

Solution: Given expression is $(\| \alpha + \beta \|)^2 \leq (\| \alpha \| + \| \beta \|)^2$

$$< \alpha + \beta , \alpha + \beta > = \| \alpha \|^2 + \| \beta \|^2 + 2 \| \alpha \| \| \beta \|$$

$$< \alpha, \alpha > + < \alpha, \beta > + < \beta , \alpha > + < \beta , \beta > = \| \alpha \|^2 + \| \beta \|^2 + 2 \| \alpha \| \| \beta \|$$

$$< \alpha, \beta > + < \overline{\alpha, \beta} > = 2 \| \alpha \| \| \beta \|$$

$$2 \text{ Re } (< \alpha, \beta >) = 2 \| \alpha \| \| \beta \| \text{ or } \text{Re } (< \alpha, \beta >) = \| \alpha \| \| \beta \| \quad \ldots\ldots(1)$$

But, $\text{Re } (< \alpha, \beta >) \leq | < \alpha, \beta > |$

So, $| < \alpha, \beta > | \geq \| \alpha \| \| \beta \|$ …….(2)

But, by Cauchy Schwarz inequality

$$| < \alpha, \beta > | \leq \| \alpha \| \| \beta \| \quad \ldots\ldots.(3)$$

From equation (2) and (3) , we have

$$| < \alpha, \beta > | = \| \alpha \| \| \beta \|$$

So from the equality case of Cauchy Schwarz inequality, we conclude that α and β are linearly dependent .

Conversely, let us take,

$$\alpha = (1, -2, 2), \ \beta = (-2, 4, -4) \in R^3$$

Then obviously α and β are linearly dependent as $\beta = -2\alpha$

Now, $\| \alpha \| = \sqrt{1 + 4 + 4} = 3$;

$$\| \beta \| = \sqrt{4 + 16 + 16} = 6$$

$$\alpha + \beta = (-1, 2, -2) \Rightarrow \| \alpha + \beta \| = \sqrt{1 + 4 + 4} = 3$$

So, $\| \alpha + \beta \| \neq \| \alpha \| + \| \beta \|$

but α and β are linearly dependent .

Example 15: If W is a subspace of V and v ∈ V satisfies $< v, w > + < w, v > \leq < w, w >$, for all w ∈ W , then prove that $< v, w > = 0$ for all w ∈ W, where V is an inner product space.

Solution: Since W is a subspace of V(F), therefore

$$\frac{1}{n} . w = \frac{w}{n} \in W, \forall n \in N ; \frac{1}{n} \in \mathbf{F}.$$

Given expression is

$$< v, w > + < w, v > \leq < w, w >, \text{ for all } w \in W \qquad .....(1)$$

Replacing w by $\frac{w}{n}$ in equation (1) , we get

$$< v, \frac{w}{n} > + < \frac{w}{n}, v > \leq < \frac{w}{n}, \frac{w}{n} > \quad \text{or} \quad \frac{1}{n} < v, w > + \frac{1}{n} < w, v > \leq \frac{1}{n^2} < w, w >$$

or $< v, w > + < w, v > \leq \frac{1}{n} < w, w >, \quad \forall n \in N$

Taking lim $n \to \infty$, we get

$$< v, w > + < w, v > \leq 0$$

Thus $< v, w > + < w, v > \leq 0 , \forall w \in W$ .....(2)

Replacing w by –w in equation (2), we get

$$< v, -w > + < -w, v > \leq 0$$

$$-< v, w > - < w, v > \leq 0$$

or $$< v, w > + < w, v > \geq 0 \qquad .....(3)$$

From equations (2) and (3), we conclude that

$$< v, w > + < w, v > = 0 , \forall w \in W \qquad .....(4)$$

Since W is a subspace of V, so $i \in \mathbf{F}$ and $w \in W \Rightarrow iw \in W$

Replacing w by iw in equation (4), we get

$$< v, iw > + < iw, v > = 0$$

$$\bar{\imath} < v, w > + i < w, v > = 0$$

$$-i < v, w > + i < w, v > = 0$$

$$-< v, iw > + < iw, v > = 0 \qquad .....(5)$$

So subtracting equation (5) from equation (4), we get

$$2 < v, w > = 0 \text{ or } < v, w > = 0, \forall w \in W.$$

Definition (Metric): A metric on a set X is a function d : X × X → R with the following

properties:

(i) d(x, y) ≥ 0 for x, y ∈ X and d(x, y) = 0 if and only if x = y.

(ii) d(x, y) = d(y, x), for all x, y ∈ X

(iii) d(x, z) ≤ d(x, y) + d(y, z), for all x, y, z ∈ X

(iv) Property (iii) is called the triangle inequality.

Theorem 4: Let V be an inner product space. If we define d(x, y) = $\| x - y \|$ for x, y ∈ V, Then d is a metric on V.

Proof: (i) By definition of norm, we know

$$\| x - y \| \geq 0$$

$$\Rightarrow d(x, y) \geq 0,$$

Also, d(x, y) = 0 , if and only if

$$\| x - y \| = 0, \text{ if and only if}$$

$$x - y = 0, \text{ if and only if}$$

$$x = y$$

(ii) d(x, y) = $\| x - y \| = \| (-1)(y - x) \|$

$= |-1| \parallel y - x \parallel$ by $\parallel \alpha x \parallel = |\alpha| \parallel x \parallel$; $\alpha \in F$, $x \in V$

$= \parallel y - x \parallel = d(y, x)$

(iii) $d(x, z) = \parallel x - z \parallel$

$= \parallel (x - y) + (y - z) \parallel$

$\leq \parallel x - y \parallel + \parallel y - z \parallel$, by triangle inequality

So , $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in V$.

Hence d is a metric on V.

Problem 1: Let V be a vector space of all real polynomials of degree ≤ 2, with inner product

$$< f(x), g(x) > = \int_{-1}^{1} f(x)g(x)dx \text{ , } \forall f(x), g(x) \in V$$

If $f(x) = x^2 + x - 4$ and $g(x) = x - 1$, then find

(i) $< f(x), g(x) >$ and

(ii) $< g(x), g(x) >$

Check your progress

Which of the following problems are True or False

Problem 1: If u = (-1, 1/4) and v = (4, -1/8) then $\|u\| = \frac{\sqrt{17}}{4}$.

Problem 2: Cauchy Schwarz inequality for $\alpha = (1, 2, -2)$ and $\beta = (2, 3, 6) \in \mathbf{R}^3$ is verified.

Problem 3: If α, β are vectors in an inner product space V, then

$\parallel \alpha + \beta \parallel \leq \parallel \alpha \parallel + \parallel \beta \parallel$ is called triangle inequalities.

Problem 4: If α is non – zero vector then norm of α is always positive.

Problem 5: If $\alpha = \mathbf{0}$ then norm of α is never zero.

11.5 *SUMMARY*

In this chapter we understood the process of generalization from ordinary vectors to vector spaces. So other basic concepts viz angle, length, distance were also generalized respectively as inner product, norm, and metric, An inner product space is a vector space equipped with an inner product, a generalized dot product operation that assigns a scalar to each pair of vectors. This additional structure allows for the formal definition of geometric concepts like the length (norm) of a vector, the distance between vectors, and the angle between them. Key properties of an inner product include linearity, positive-definiteness (a vector's inner product with itself is zero only if it's the zero vector), and symmetry (or Hermitian symmetry for complex spaces).

11.6 *GLOSSARY*

➢ **Inner Product:** An inner product on a vector space V is a map $< \, , \, > : V \times V \rightarrow R$ satisfying the following properties :

(i) $< x , x > \geq 0$ and $< x , x > = 0$ if and only if $x = 0$.

(ii) $< x , y > = < y , x >$

(iii) $<x + z , y > = < x , y > + < z , y >$ and $< x , y + z > = < x , y > + < x , z >$

(iv) $< ax , y > = a < x , y > \;\; \forall \, x , y , z \in V$ and $a \in R$.

➢ **Norm of a Vector:** Let V be an inner product space. The norm function $\| . \| : V \rightarrow R$ has the following properties :

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$; $x \in V$

(ii) $\| \alpha x \| = |\alpha| \, \| x \|$, $\alpha \in \mathbf{F}$, $x \in V$,

Norm of a vector $v \in V$ is defined as $\| v \| = \sqrt{< v, v >}$.

11.7 *REFERENCES*

➢ S Kumaresan; Linear Algebra-A Geometric Approach; PHI-2016.

➢ K Hoffman & Ray Kunze; Linear Algebra; PHI-1971.

➢ P Halmos; Finite-Dimensional Vector Spaces;

➢ G Strang; Introduction To Linear Algebra;

11.8 *SUGGESTED READING*

➢ NPTEL videos.

➢ Schaum series.

➢ A R Vashishtha, Krishna Prakashan; Meerut.

➢ Graduate Text In Mathematics, Springer.

11.9 *TERMINAL QUESTION*

1: Prove that for any $\alpha \in \mathbf{R}^2$, we can write $\alpha = <\alpha, e_1> e_1 + <\alpha, e_2> e_2$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$

2: Let V be a vector space over a field **F**. Let W_1 and W_2 be twp subspaces of **V(F)** such that W_1 and W_2 are two inner product spaces also. Then prove that –

(i) A positive multiple of an inner product is also an inner product.
(ii) Difference of two inner products may **not** be an inner product.

3: Let V (R) be a vector space of polynomials with inner product defined by < f , g > = $\int_0^1 f(x)g(x)dx$. If $f(x) = x^2 + 1$ and $g(x) = x - 1$, then find < f, g > and ‖ g ‖ .

11.10 *ANSWERS*

Answers of check your progress:

1. True **2.** True **3.** True **4.** True **5.** False

Answers of terminal question:

2. (**i**) Let < u, v > be an inner product and $\lambda > 0$, $\lambda \in \mathbf{R}$. Then it can be easily verified that λ< u, v > is also an inner product.

(ii) Difference of two inner products may not be positive. Now do it yourself.

3. < f , g > = $\frac{-7}{12}$ and ‖ g ‖ = $\frac{1}{\sqrt{3}}$.

UNIT-12: ORTHOGONALITY AND ORTHONORMALITY

CONTENTS

12.1 *INTRODUCTION*

Orthogonality and orthonormality are key concepts in linear algebra that describe specific relationships between vectors in a vector space. Two vectors are said to be **orthogonal** if their dot product is zero, meaning they are perpendicular to each other. A set of vectors is called **orthonormal** if the vectors are not only orthogonal but also each have a unit length (norm equal to one). Orthonormal sets simplify many mathematical operations, such as projections, transformations, and decompositions, and are widely used in applications like signal processing, quantum mechanics, and numerical computations. These concepts form the foundation for constructing efficient and stable vector representations in various mathematical and engineering problems.

12.2 *OBJECTIVES*

After the study of this chapter, we shall understand:

- **Understand the concept of orthogonality** in vector spaces and identify orthogonal vectors using the dot product.
- **Define and recognize orthonormal sets** of vectors and understand their significance in forming orthonormal bases.
- **Learn to compute the dot product** and use it to test for orthogonality and normalize vectors.
- **Apply the Gram-Schmidt process** to convert a set of linearly independent vectors into an orthonormal set.
- **Explore the geometric interpretation** of orthogonality and orthonormality in Euclidean spaces.
- **Understand the role of orthonormal vectors** in simplifying vector projections, decompositions, and matrix operations.
- **Use orthonormal bases** to represent vectors and solve problems more efficiently in linear algebra and applied mathematics.
- **Recognize applications** of orthogonality and orthonormality in real-world fields such as signal processing, machine learning, and quantum physics.

12.3 *ORTHOGONALITY*

Orthogonality: Let V be an inner product space. An element $u \in V$ is said to be orthogonal to $v \in V$ if $< u, v > = 0$. Obviously, orthogonality is a symmetric relation i.e. if u is orthogonal to v, then v is also orthogonal to u.

$$< u, v > = 0 \text{ , if and only if } < v , u > = 0$$

Note: (1) Zero vector is orthogonal to each $v \in V$ as $< 0, v > = < v , 0 > = 0$

(**2**) If $u \in V$ is orthogonal to $v \in V$, then every scalar multiple of u is orthogonal to v. Let $k \in \mathbf{F}$ and $< u , v > = 0$ then $< ku , v > = k < u, v > = 0$. So ku is also orthogonal to v , , $\forall k \in \mathbf{F}$.

(**3**) Zero vector is the **only** vector which is orthogonal to itself. If u is orthogonal to u, then

$< u, u > = 0 \Rightarrow u = 0$

(**4**) A vector $u \in V$ is said to be orthogonal to set S if it is orthogonal to each vector in S. That is $< u , v > = 0$, for every $v \in V$.

(**5**) Two subspaces W_1 and W_2 of V(F) are called orthogonal if every vector in each subspace is orthogonal to every vector in the other.

(**6**) Let **S** be a set of vectors in an inner product space *V*. Then **S** is said to be an orthogonal set provided that any two distinct vectors in **S** are orthogonal. So, $< u , v > = 0$, for every distinct u, v ∈ **S.**

Example 1: If $u = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are the vectors. Are these vectors u and v orthogonal?

Solution: To check the orthogonality, first calculate the dot product:

$u.v = (2)(1) + (1)(2) + (-1)(3) = 2 + 2 - 3 = 1$.

Since, $u.v = 1 \neq 0$. Hence the vectors are not orthogonal.

Note: The vectors u and v are **not orthogonal** because their dot product is not zero.

Example 2: Find a vector that is orthogonal to the vector $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Solution: Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be a vetor such that to be orthogonal to the given vector *u*. Now, if the vectors u and v are orthogonal then their dot product will be zero.

So, $u.v = 3x + 4y = 0$. Here, $x = 4, y = -3$ are one of the solution of the equation $3x + 4y = 0$. Therefore, $v = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ are orthogonal to u.

12.4 *ORTHOGONAL BASIS*

An orthogonal basis for a vector space is a set of linearly independent vectors that are mutually orthogonal, meaning the dot product of any two distinct vectors is zero. Unlike an orthonormal basis, the vectors in an orthogonal basis are not necessarily of unit length.

Definition: A set of vectors $\{v_1, v_2, v_3, ..., v_n\}$ in a real inner product space *V* is an orthogonal basis if:

(*i*) The vectors span *V* (i.e. they form a basis)

(*ii*) $v_i.v_j = 0 \;\forall\, i \neq j$

Example 3: Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then,

$v_1.v_2 = (1)(0) + (0)(2) = 0$. So, $\{v_1.v_2\}$ is an orthogonal basis for R^2.

How is it useful?

- Orthogonal bases simplify vector projections.
- They're used in the **Gram-Schmidt process** to construct orthonormal bases.
- Calculations (especially involving decompositions or projections) become much simpler when working with orthogonal vectors.

12.5 *ORTHONORMALITY*

Orthonormality: Let **S** be a set of vectors in an inner product space V. The **S** is said to be an orthonormal set if:

(a) $u \in S \Rightarrow \| u \| = 1$

(b) $u, v \in S$ and $u \neq v$, then $< u , v > = 0$

Thus an orthonormal set is an orthogonal set with the additional property that norm of each vector is 1. So a set **S** consisting of mutually orthogonal unit vectors is called an orthonormal set.

A finite set S = { α_1 , ….. , α_n } is orthonormal if

$$< \alpha_i , \alpha_j > \; = S_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases},$$

12.6 *ORTHONORMAL BASIS*

Orthonormal Basis: If an orthonormal set **S** is a basis of an inner product space V, then the set **S** is called an orthonormal basis of *V*.

To convert an orthogonal basis to an orthonormal basis: If $S = \{v_1, v_2, v_3, ..., v_n\}$ is the set of orthogonal basis of any vector space *V* then we can convert this orthogonal set to orthonormal set by normalizing each vector of *S* to form orthonormal basis.

Normalized the each vector as: $\hat{v}_i = \dfrac{v_i}{\|v_i\|} \; \forall \, i = 1, 2, 3, \ldots, n$

Example 4: Let $v_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix}$ and $v_2 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}$ are two vectors then show that $\{v_1, v_2\}$ forms an orthonormal basis of R^2.

Solution: To show a set is an **orthonormal basis**, we need to verify two things:

1. Vectors are **unit vectors** (norm = 1).
2. Vectors are **orthogonal** (dot product = 0).

Step 1: Check norms: $\|v_1\| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}} = \sqrt{1} = 1$

$$\|v_2\| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(-\dfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}} = \sqrt{1} = 1$$

Step 2: Check orthogonality: $v_1 . v_2 = \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right) + \left(\dfrac{1}{\sqrt{2}}\right)\left(-\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{2} - \dfrac{1}{2} = 0$

Therefore, the set $\{v_1, v_2\}$ is an orthonormal basis of R^2.

Example 5: Given the vectors, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Use normalization to check whether this set forms an orthonormal basis of R^3.

Solution: Step 1: Normalize each vector,

$$\|u_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2} = \frac{u_1}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\|u_2\| = \sqrt{(-1^2) + 1^2 + 0^2} = \sqrt{2} = \frac{u_2}{\sqrt{2}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\|u_3\| = \sqrt{0^2 + 0^2 + 1^2} = 1 = u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2: Check orthogonality

$$u_1.u_2 = 1(-1) + 1.1 + 0.0 = 0$$

$$u_1.u_3 = 1.0 + 1.0 + 0.1 = 0$$

$$u_2.u_3 = (-1).0 + 1.0 + 0.1 = 0$$

All vectors are orthogonal and can be normalized to unit vectors.

So, the normalized vectors form an orthonormal basis of R^3.

Example 6: the set **S** = { (1, 0, 0), (0, 1, 0), (0, 0, 1) } is an orthonormal basis of $\mathbf{R}^3$.

Also, it can be easily verified that the set

$\mathbf{S}'$ = { ($\frac{1}{\sqrt{2}}$, 0, $\frac{1}{\sqrt{2}}$) , ($\frac{1}{\sqrt{2}}$, 0, $\frac{-1}{\sqrt{2}}$), (0, 1, 0) } is another orthonormal basis of $\mathbf{R}^3$.

Theorem 1: (Pythagoras Theorem) Prove that vectors **x** and **y** in a real inner product space (Euclidean space) V are orthogonal if and only if

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

Solution: We have,

$$\| x + y \|^2 = < x + y, x + y > = < x, x + y > + < y, x + y >$$

$$= < x, x > + < x, y > + < y, x > + < y, y >$$

$$= \| x \|^2 + < x, y > + < \overline{x, y} > + \| y \|^2$$

$$\| x + y \|^2 = \| x \|^2 + 2 < x, y > + \| y \|^2 \text{ as V is real I.P.S.} \quad .....(1)$$

But given that , $\| x + y \|^2 = \| x \|^2 + \| y \|^2$ .....(2)

So we have, $\| x \|^2 + \| y \|^2 = \| x \|^2 + 2 < x, y > + \| y \|^2$

$\Rightarrow \quad < x, y > = 0$

$\Rightarrow$ x and y are orthogonal.

Conversely, let x and y be orthogonal

$\Rightarrow \quad < x, y > = 0$

then as done above, it can be observed,

$$\| x + y \|^2 = \| x \|^2 + 2 < x, y > + \| y \|^2$$

By using, $< x, y > = 0$, we get

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

Theorem 2: In a complex inner product space (or unitary space) *V*, if *x* is **orthogonal** to *y*, then $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

However, the converse may **NOT** be true. Justify.

Solution: If x is orthogonal to y, then $< x, y > = 0$

$\Rightarrow \quad < \overline{x, y} > = 0$

$\Rightarrow \quad < y , x > = 0,$

Now,

$$\| x + y \|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$, (by previous example)

So, $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

Conversely, let $V = \boldsymbol{C^2(C)}$ with standard inner product

Let x = (0, i) and y = (0, 1) ∈ V. Then

$$\langle x , y \rangle = 0 + i = i \neq 0$$

So x is not orthogonal to y.

Also, $\| x \|^2 = 0(0) + i(\bar{i}) = i(-i) = 1$

$$\| y \|^2 = 0 + 1 = 1$$

Now, x + y = (0, 1 + i)

$$\| x + y \|^2 = 0 + (1 + i)(1 - i) = 2$$

Hence, $\| x + y \|^2 = \| x \|^2 + \| y \|^2$, though x is not orthogonal to y.

Example 7: Find a vector of unit length which is orthogonal to the vector (3, -2, 2) of $\mathbf{R^3(R)}$ relative to the standard inner product .

Solution: Let x = (3, -2, 2) and y = (a, b, c) ∈ $\mathbf{R^3}$ be orthogonal vectors.

Then $\langle x, y \rangle = 0$

$$\Rightarrow 3a - 2b + 2c = 0$$

This system has infinite (actually uncountable) solutions. Let us take one solution by taking

a = 2, b = -3, c = -6

So, y = (2, -3, -6) is orthogonal to x = (3, -2, 2)

Now, $\| y \|^2 = 4 + 9 + 36 = 49 \Rightarrow \| y \| = 7$

So, $u = \frac{y}{\| y \|} = \frac{1}{7}(2, -3, -6) \Rightarrow u = \left(\frac{2}{7}, \frac{-3}{7}, \frac{-6}{7}\right)$

Theorem 3: An orthogonal set of non-zero vectors in an inner product space V is linearly independent.

Proof: Let **S** be an orthogonal set of non-zero vectors of V. In order to show that **S** is linearly independent, we shall prove that every finite subset of **S** is linearly independent.

Let $\{v_1, v_2, \ldots, v_n\}$ be any finite subset of **S**.

By orthogonality of **S**, we have

$$< v_i , v_j > = 0 , \text{ for } i \neq j \qquad \ldots(1)$$

Let us assume $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$; where $\alpha_i \in \mathbf{F}$,

So, $< \alpha_1 v_1 + \ldots + \alpha_n v_n , \alpha_1 v_1 + \ldots + \alpha_n v_n > = 0$

$$\Rightarrow \quad < \alpha_1 v_1 , \alpha_1 v_1 + \ldots + \alpha_n v_n > + \ldots + < \alpha_n v_n , \alpha_1 v_1 + \ldots + \alpha_n v_n > = 0$$

$$(< \alpha_1 v_1 , \alpha_1 v_1 > + \ldots + < \alpha_1 v_1 , \alpha_n v_n >) + \ldots + (< \alpha_n v_n , \alpha_1 v_1 > + \ldots + < \alpha_n v_n , \alpha_n v_n >) = 0$$

$$\alpha_1 \overline{\alpha_1} < v_1 , v_1 > + \alpha_2 \overline{\alpha_2} < v_2 , v > + \ldots + \alpha_n \overline{\alpha_n} < v_n , v_n > \text{; using equation (1)}$$

$$\| \alpha_1 \|^2 \| v_1 \|^2 + \| \alpha_2 \|^2 \| v_2 \|^2 + \ldots + \| \alpha_n \|^2 \| v_n \|^2 = 0$$

But every term is non-negative and sum is zero.

So, $\| \alpha_i \|^2 \| v_i \|^2 = 0 \ \forall \ i$

But each $v_i \neq 0$, by statement.

So, $| \alpha_i |^2 = 0 \ \forall \ i$

$\Rightarrow \quad \alpha_i = 0 \ \forall \ I = 1, 2, 3, \ldots, n.$

So, $\{v_1, v_2, \ldots, v_n\}$ is linearly independent subset of **S**.

$\Rightarrow$ **every** finite subset of **S** is linearly independent.

$\Rightarrow$ **S** in linearly independent.

Note: In the same way, it can be proved that an orthonormal set S in an inner product space V is linearly independent.

Theorem 4: If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set in V and if $w \in V$, then,

$$u = w - \sum_{i=1}^{n} < w, v_i > v_i \text{ ; is orthogonal to each of } v_1, v_2, \ldots, v_n.$$

Solution: For any i = 1, 2, ….., n. we have

$$< u, v_i > = < w - \sum_{i=1}^{n} \alpha_i v_i , v_i > \text{, where } \alpha_i = < w, v_i >$$

$$= < w - \alpha_1 v_1 - \alpha_2 v_2 - \ldots - \alpha_n v_n , v_i >$$

$$= < w, v_i > - \alpha_1 < v_1 , v_i > - \ldots - \alpha_i < v_i , v_i > - \ldots - \alpha_n < v_n , v_i >$$

$$= < w, v_i > - 0 - \ldots - \alpha_i - 0 - \ldots - 0$$

$$< u, v_i > = < w, v_i > - < w, v_i > = 0$$

So, $< u, v_i > = 0$, for i = 1, 2, ….., n

Hence u is orthogonal to v_i , for i = 1, 2, ….., n.

Complete Orthonormal Set: An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.

Orthonormal dimension: Let V be a finite-dimensional inner product space of dimension n. If S is any orthonormal set in V then S is linearly independent. So S cannot contain more than n distinct vectors. The orthonormal dimension of V is defined as the largest number of vectors an orthonormal set in V can contain.

For finite dimensional inner product spaces, orthonormal dimension is same as linear dimension.

Note: Now we recall some **basics** of vectors in $\mathbf{R}^2$. It will help us to '*visualize*' the geometry behind **Gram-Schmidt orthogonalisation process**.

Example 8: Let us consider two vectors $\vec{a}$ and $\vec{b}$ in R^2. Then

$$|\vec{a}| = a \text{ and } |\vec{b}| = b$$

We have to find:

(i) projection of $\vec{a}$ on $\vec{b}$

(ii) component of $\vec{a}$ along $\vec{b}$.

(iii) component of $\vec{a}$ perpendicular to $\vec{b}$.

Solution: Let us realize these vectors as shown –

So, OA = $|\vec{a}|$ = a , $\angle$ AOB = θ

(i) Projection of $\vec{a}$ on $\vec{b}$

$$= OB = OA\cos\theta$$

$$= \frac{a(\vec{a}.\vec{b})}{a\,b} \text{ as } \vec{a}\,.\,\vec{b} = ab\cos\theta$$

Projection of $\vec{a}$ on $\vec{b}$ = $\frac{(\vec{a}.\vec{b})}{b}$.

(ii) Component of of $\vec{a}$ along $\vec{b}$ = (Projection of $\vec{a}$ on $\vec{b}$)

$\hat{b} = (\frac{(\vec{a}.\vec{b})}{b})\frac{\vec{b}}{b} = \frac{(\vec{a}.\vec{b})\,\vec{b}}{b^2}$,

(iii) From vector law of addition, we have

$$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$$

$$\vec{a} = \frac{(\vec{a}.\vec{b})\,\vec{b}}{b^2} + \overrightarrow{BA}$$

So, $\overrightarrow{BA}$ = component of $\vec{a}$ perpendicular to $\vec{b}$ = $\vec{a} - \frac{(\vec{a}.\vec{b})\,\vec{b}}{b^2}$

These fundamental concepts will help you to understand the next theorem.

Check your progress

Problem 1: Prove that $\|\alpha v\| = |\alpha|\,\|v\|$, for all $\alpha \in$ **F** , x ∈ V

Problem 2: If {$v_1, v_2, \ldots, v_n$ } is an orthonormal set and if w = $\sum_{i=1}^{n} \alpha_i v_i \in V$, Then prove that $\alpha_i = < w, v_i >$ for i = 1, 2, …, *n*.

12.7 *SUMMARY*

Orthogonality and orthonormality are fundamental concepts in vector spaces, especially in inner product spaces. Two vectors are said to be **orthogonal** if their inner product is zero, meaning they are at a right angle to each other. A set of vectors is called **orthonormal** if all the vectors are not only mutually orthogonal but also have unit length (norm equal to one).

Orthonormal sets are especially useful because they simplify many operations, such as projections and coordinate transformations. Any orthogonal set of non-zero vectors is linearly independent, and using processes like **Gram-Schmidt** (we will learn in upcoming units) any linearly independent set can be converted into an orthonormal set, which is very helpful in building orthonormal bases in vector spaces.

12.8 *GLOSSARY*

- **Inner Product:** An inner product on a vector space V is a map $< \, , \, > : V \times V \to R$ satisfying the following properties :

(i) $< x , x > \geq 0$ and $< x , x > = 0$ if and only if $x = 0$.

(ii) $< x , y > = < y , x >$

(iii) $< x + z , y > = < x , y > + < z , y >$ and $< x , y + z > = < x , y > + < x , z >$

(iv) $< ax , y > = a < x , y >$ $\forall x , y , z \in V$ and $a \in R$.

- **Norm of a Vector:** Let V be an inner product space. The norm function $\| . \| : V \to R$ has the following properties :

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$; $x \in V$

(ii) $\| \alpha x \| = |\alpha| \, \| x \|$, $\alpha \in \mathbf{F}$, $x \in V$,

Norm of a vector $v \in V$ is defined as $\| v \| = \sqrt{< v, v >}$.

- **Complete Orthonormal Set:** An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.
- **Gram-Schmidt orthogonlisation Process:** Every finite-dimensional inner product space has an orthonormal basis.

12.9 *REFERENCES*

- S Kumaresan; Linear Algebra-A Geometric Approach; PHI-2016.
- K Hoffman & Ray Kunze; Linear Algebra; PHI-1971.
- P Halmos; Finite-Dimensional Vector Spaces;
- G Strang; Introduction To Linear Algebra;

12.10 *SUGGESTED READING*

- NPTEL videos.
- Schaum series.
- A R Vashishtha, Krishna Prakashan; Meerut.
- Graduate Text In Mathematics, Springer.

12.11 *TERMINAL QUESTION*

Long answer type question

1. Explain the concept of orthogonality and orthonormality in inner product spaces. Illustrate with suitable examples.

2. State and explain the Gram-Schmidt orthogonalization process. Apply it to the vectors $\vec{v}_1 = (1,1,0)$ and $\vec{v}_2 = (1,0,1)$ to obtain an orthonormal set.

3. Prove that any set of non-zero orthogonal vectors in an inner product space is linearly independent.

4. Describe how to express a given vector as a linear combination of an orthonormal basis. Give an example in R^3.

5. Discuss the role of orthogonal and orthonormal vectors in projection of vectors. How does an orthonormal basis simplify vector projection?

Short answer type questions:

1. Define orthogonality in the context of vector spaces.
2. What is an orthonormal set of vectors?
3. How can you check if two vectors are orthonormal?
4. What is the significance of orthonormal basis in vector spaces?
5. Can a set of orthogonal vectors be linearly dependent? Justify.
6. Write the condition for two vectors $\vec{u}$ and $\vec{u}$ to be orthogonal in terms of dot product.
7. Give an example of an orthonormal set in R^2
8. What is the norm of each vector in an orthonormal set?
9. What is the angle between two orthogonal vectors?

Objective type questions:

1: Let $\vec{u}$ and $\vec{v}$ be two vectors in an inner product space. If $\langle \vec{u}, \vec{v} \rangle$=0, then the vectors are:

A) Linearly dependent
B) Equal
C) Orthogonal
D) Orthonormal

2: Which of the following is true for an orthonormal set of vectors $\{v_1, v_2, ..., v_n\}$?

A) $\langle v_i, v_j \rangle = 1 \forall i \neq j$

B) $\langle v_i, v_j \rangle = 1 \forall i \neq j$ and $\|v_i\| = 1 \forall i$

C) $\langle v_i, v_j \rangle = 0 \forall i = j$

D) $\langle v_i, v_j \rangle = 0 \ \forall i, j$

3: Which of the following statements is not true about an orthonormal basis in a vector space?

A) All vectors are orthogonal to each other
B) All vectors have unit length
C) Any vector in the space can be written as a linear combination of the basis vectors
D) The inner product of any two basis vectors is always 1

4: If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then the angle between them is:

A) 0°

B) 90°

C) 180°

D) Cannot be determined

5: In R^3, which of the following sets is orthonormal?

A) {(1,0,0),(0,1,0),(0,0,1)}
B) {(1,1,0),(0,1,1),(1,0,1)}
C) {(2,0,0),(0,2,0),(0,0,2)}
D) $\{(1,1,1),(-1,-1,1),(1,-1,-1)\}$

6: If a set of non-zero vectors are pairwise orthogonal, then the set is:

A) Linearly dependent
B) Orthonormal
C) Linearly independent
D) A basis

7: Which of the following is always true for an orthonormal set $\{\vec{u}, \vec{v}\}$

A) $\vec{v}_1 + \vec{v}_2$ is orthogonal $\vec{v}_1 - \vec{v}_2$

B) $\left\| \vec{v}_1 + \vec{v}_2 \right\| = 2$

C) $\left\|\vec{v}_1, \vec{v}_2\right\| = 1$

D) $\vec{v}_1 = \vec{v}_2$

8: The Gram-Schmidt process is used to:

A) Find the determinant of a matrix
B) Find eigenvalues of a matrix
C) Convert a linearly independent set to an orthonormal set
D) Solve a system of linear equations

9: Let $\vec{v}_1 = (1,2)$ and $\vec{v}_2 = (-2,1)$. Then $\vec{v}_1$ and $\vec{v}_2$ are:

A) Orthogonal
B) Orthonormal
C) Linearly dependent
D) Not orthogonal

10: If $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis in R^3 and $\vec{v} \in R^3$ then the projection of $\vec{v}$ onto the subspace spanned by the basis is:

A) $\vec{v}$

B) $\sum_{i=1}^{3} \langle \vec{v}, \vec{u}_i \rangle \vec{u}_i$

C) $\sum_{i=1}^{3} \vec{u}_i \vec{v}$

D) $\langle \vec{v}, \vec{v} \rangle$

True and False questions:

1: If two vectors are orthogonal, then their dot product is zero.
2: An orthonormal set is always linearly dependent.
3: All orthonormal vectors are orthogonal, but not all orthogonal vectors are orthonormal.
4: If a vector has unit length, it is automatically orthogonal to other vectors.
5: The zero vector is orthogonal to every vector in a vector space.
6: Every orthogonal set of vectors in R^n can be converted into an orthonormal set.
7: If $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ must both be the zero vector.
8: In an orthonormal basis, the coordinates of a vector are given by inner products with the

basis vectors.

9: Orthonormal vectors must all lie on the same line.

10: The Gram-Schmidt process can be applied only to orthogonal vectors.

Fill in the blanks questions:

1: Two vectors are said to be __________ if their inner product is zero.

2: A set of vectors is called __________ if the vectors are mutually orthogonal and each has unit length.

3: The process used to convert a linearly independent set into an orthonormal set is called the __________ process.

4: If $\vec{u}.\vec{v} = 0$, then vectors $\vec{u}$ and $\vec{v}$ are __________.

5: In an orthonormal basis, the length (or norm) of each vector is equal to __________.

6: If a set of non-zero vectors is orthogonal, then it is always __________.

7: The dot product of any two different vectors in an orthonormal set is __________.

8: The projection of a vector $\vec{v}$ onto a unit vector $\vec{u}$ is given by __________.

9: The standard basis vectors in R^n form an __________ basis.

10: To normalize a vector $\vec{v}$, divide it by its __________.

12.12 *ANSWERS*

Answers of check your progress:

2. Given that $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set. So

$$< v_i , v_j > = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \qquad \ldots..(1)$$

We have , $< w, v_i > = (\alpha_1 v_1 + \ldots.. + \alpha_n v_n , v_i)$

$$= \alpha_1 < v_1 , v_i > + \ldots. + \alpha_i < v_i , v_i > + \ldots. + \alpha_n < v_n , v_i >$$

$$= 0 + \ldots. + \alpha_i + 0 + \ldots. + 0$$

$$< w, v_i > = \alpha_i \text{, for } i = 1, 2, \ldots., n.$$

Answer of Short answer type questions:

4: An orthonormal basis simplifies computations, such as projections and coordinate transformations, because the inner products directly give the components of any vector.

5: No, a set of non-zero orthogonal vectors is always linearly independent.

6:	$\vec{u}.\vec{v}=0$	**7:**	{(1,0),(0,1)}
8:	The norm is 1.	**9:**	90 degrees

Answer of objective type questions:

1:	C	**2:**	B	**3:**	D	**4:**	B
5:	A	**6:**	C	**7:**	A	**8:**	C
9:	A	**10:**	B				

Answer of True and False:

1:	True	**2:**	False	**3:**	True	**4:**	False
5:	True	**6:**	True	**7:**	False	**8:**	True
9:	False	**10:**	False				

Answer of fill in the blanks

1:	Orthogonal	**2:**	Orthonormal	**3:**	Gram-Schmidt
4:	Orthogonal	**5:**	1	**6:**	Linearly independent
7:	0	**8:**	$(\vec{v}.\vec{u})\vec{u}$	**9:**	Orthonormal
10:	magnitude (or norm)				

UNIT-13: GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

CONTENTS

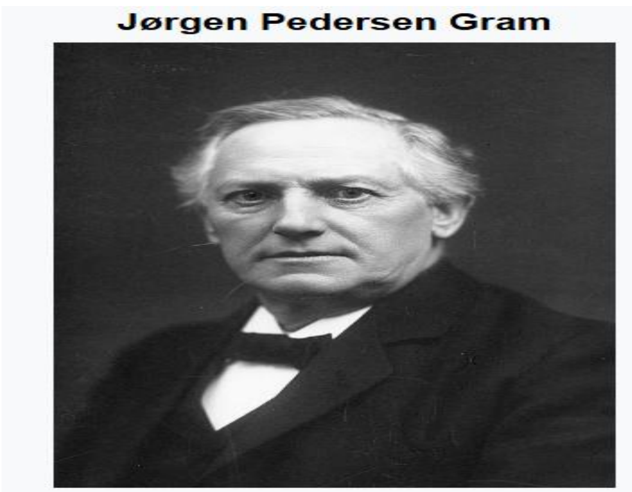
13.1 *INTRODUCTION*

The Gram-Schmidt orthogonalization process is a method used in linear algebra to convert a set of linearly independent vectors into an orthogonal (or orthonormal) set spanning the same subspace. It works by iteratively subtracting the projections of each vector onto the previously obtained orthogonal vectors, ensuring that each new vector added to the set is orthogonal to those already processed. This technique is fundamental in many applications, such as simplifying computations in vector spaces, constructing orthonormal bases in Hilbert spaces, and performing QR decomposition in numerical analysis.

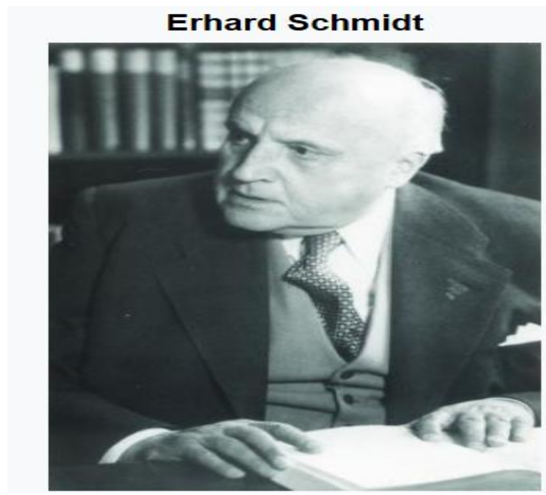
The **Gram-Schmidt orthogonalization process** is named after two mathematicians: **Jørgen Pedersen Gram**, a Danish mathematician, and **Erhard Schmidt**, a German mathematician.

- **Jørgen P. Gram** introduced concepts related to orthogonality and projections in the late 19th century (around 1883), particularly in the context of statistics.
- **Erhard Schmidt** later formalized and extended Gram's ideas in 1907, particularly in the setting of Hilbert spaces.

Although both contributed independently, the process as we know it today became widely recognized due to Schmidt's work, and the combined name acknowledges both of their contributions.



27 June 1850-29 April 1916 (aged 65)

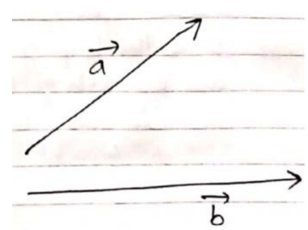


13 January 1876-6 December 1959 (aged 83)

13.2 *OBJECTIVES*

After the study of this chapter, we shall understand:

- Orthogonalisation and Gram-Schmidt process.
- Cauchy Schwarz and Bessel inequalities.
- Riesz representation theorem.



13.3 *GRAM-SCHMIDT ORTHOGONALISATION PROCESS*

Theorem 1: Every finite-dimensional inner product space has an orthonormal basis.

Proof: Let V(F) be an n-dimensional inner product space and let $S = \{v_1, \ldots, v_n\}$ be a basis of V. Firstly, we shall **construct** an orthogonal set in V with the help of elements of S. Since S is a basis, so all elements of S are non-zero.

Let us take,

$$w_1 = v_1,\ w_2 = v_2 - \frac{<v_2, w_1> w_1}{\|w_1\|^2} \text{ or } w_2 = v_2 - \frac{<v_2, v_1> v_1}{\|v_1\|^2} \quad \ldots..(1)$$

Since $v_1 \neq 0$, so $\| v_1 \| \neq 0$,

We have, $< w_2, w_1 > = < v_2 - \alpha v_1, v_1 >$ where $\alpha = \frac{<v_2, v_1>}{\|v_1\|^2}$

So, $< w_2, w_1 > = < v_2, v_1 > - \alpha < v_1, v_1 >$

$$= < v_2, v_1 > - \frac{<v_2, v_1>}{\|v_1\|^2} \| v_1 \|^2 = < v_2, v_1 > - < v_2, v_1 > = 0$$

$\therefore\ < w_2, w_1 > = 0$ and $v_2 = \alpha v_1 + w_1 = \alpha w_1 + w_2$,

We observe that $w_2 \neq 0$, for otherwise, $v_2 = \alpha v_1$

$\Rightarrow v_1, v_2$ are linearly dependent.

This is contradictory, as S is a basis , so every subset of S will be linearly independent.

$$\text{Let } w_3 = v_3 - \frac{<v_3, w_2> w_2}{\|w_2\|^2} - \frac{<v_3, w_1> w_1}{\|w_1\|^2} \quad \ldots..(2)$$

where $\| w_2 \| \neq 0$, $\| w_1 \| \neq 0$

We can write, $w_3 = v_3 - \alpha_1 w_1 - \alpha_2 w_2$, where

$$\alpha_1 = \frac{<v_3, w_1>}{\|w_1\|^2} \text{ and } \alpha_2 = \frac{<v_3, w_2>}{\|w_2\|^2} \quad \ldots..(3)$$

Now, $< w_3, w_2 > = < v_3 - \alpha_1 w_1 - \alpha_2 w_2 , w_2 >$

$$= < v_3, w_2 > - \alpha_1 < w_1, w_2 > - \alpha_2 < w_2, w_2 >$$

$$= < v_3, w_2 > - \frac{<v_3, w_1>}{\|w_1\|^2} < w_1, w_2 > - \frac{<v_3, w_2>}{\|w_2\|^2} \| w_2 \|^2$$

$$= < v_3, w_2 > - 0 - < v_3, w_2 > \quad (\text{as } < w_1, w_2 > = 0)$$

$< w_3, w_2 > = 0$

Similarly, $< w_3, w_1 > = 0$,

Also, $v_3 = \alpha_1 w_1 + \alpha_2 w_2 + w_3$

If follows that { w_1, w_2, w_3 } is an orthogonal set. Further $w_3 \neq 0$, for otherwise,{w_1, w_2, w_3 } is linearly dependent, which is again a contradiction. Here you should note that { w_1, w_2, v_3 }= { v_1, $v_2 - \alpha_1 v_1$, v_3 } is linearly independent as { v_1, v_2, v_3 }are linearly independent. Proceeding in a similar manner, if we take

$w_n = v_n - \frac{< v_n, w_{n-1} > w_{n-1}}{\| w_{n-1} \|^2} - \ldots - \frac{< v_n, w_1 > w_1}{\| w_1 \|^2}$, then it can be verified that {w_1, …., w_n} is an orthogonal set . Consequently, T = { $\frac{w_1}{\| w_1 \|}$, …. , $\frac{w_n}{\| w_n \|}$ } is an orthogonal set. Since an orthonormal set is linearly independent and so T forms basis of V as dim V = n.

Hence T is an orthonormal basis of v.

Note: (1) To obtain an orthonormal basis of V, where V = $\mathbf{R}^3$ i.e. dim V = 3, we proceed as follows:

(i) Let {v_1, v_2, v_3} be a basis of V.

(ii) Find {w_1, w_2, w_3} where $w_1 = v_1$

$$w_2 = v_2 - \frac{< v_2, w_1 > w_1}{\| w_1 \|^2}$$

$$w_3 = v_3 - \frac{< v_3, w_2 > w_2}{\| w_2 \|^2} - \frac{< v_3, w_1 > w_1}{\| w_1 \|^2}$$

(iii) { $\frac{w_1}{\| w_1 \|}$, $\frac{w_2}{\| w_2 \|}$, $\frac{w_3}{\| w_3 \|}$ } is an orthogonal basis of V.

(2) Generally existence theorem in analysis are non-constructive i.e. you prove the theorem, but there is no formula or general method to solve numerical questions. But Gram-Schmidt process is constructive in nature. It provides a method to solve numerical.

Example 1: Apply the Gram-Schmidt process to the vectors given below to obtain an orthonormal basis for R^3**(R)** with the standard inner product:

(i) S_1 = { (1, 1, 0) , (1, 0, 1) , (0, 1, 1) }

(ii) S_2 = { (1, 1, 0) , (1, 0, -1) , (0, 3, 4) }

Solution: (i) Let v_1 = (1, 1, 0) , v_2 = (1, 0, 1) , v_3 = (0, 1, 1)

Let $w_1 = v_1$ = (1, 1, 0) , $\Rightarrow$ $\| w_1 \|^2 = < w_1 , w_1 > = 1^2 + 1^2 + 0 = 2$

$$\therefore \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$$

Let $w_2 = v_2 - \frac{<v_2, w_1> w_1}{\|w_1\|^2}$ .....(1)

$$< v_2, w_1 > = < v_2, v_1 > = 1^2 + 0 + 0 = 1$$

So, $w_2 = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = (\frac{1}{2}, -\frac{1}{2}, 1)$.

$$\| w_2 \|^2 = < w_2, w_2 > = \frac{3}{2}$$

So, $\frac{w_2}{\|w_2\|} = \sqrt{\frac{2}{3}}\,(\frac{1}{2}, -\frac{1}{2}, 1) = = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$

Again, let $w_3 = v_3 - \frac{<v_3, w_1> w_1}{\|w_1\|^2} - \frac{<v_3, w_2> w_2}{\|w_2\|^2}$ .....(2)

So we obtain, $< v_3, w_1 > = < v_3, v_1 > = 0 + 1 + 0 = 1$ and $< v_3, w_2 > = \frac{1}{2}$

$$\| w_1 \|^2 = 2, \quad \| w_2 \|^2 = \frac{3}{2}$$

So form equation (2) , we have

$$w_3 = (0, 1, 1) - \frac{1}{2}(1, 1, 0) - \frac{2}{3}(\frac{1}{2}, -\frac{1}{2}, 1)\frac{1}{2} = (-\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$$

$$\| w_3 \|^2 = \frac{4}{3} \Rightarrow \frac{w_3}{\|w_3\|} = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

Hence orthonormal basis is $\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \}$

(ii) Do it yourself.

$$S_1 = \{ (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (0, 1, 0) \}$$

Example 2: Let V be a set of real functions satisfying $\frac{d^2y}{dx^2} + 9y = 0$,

(i) Prove that V is a two-dimensional real vector space.

(ii) In V, inner product is defined by

$$< y, z > = \int_0^{\pi} yz\, dx$$

Find an orthonormal basis for V.

Solution: (i) Suppose V is a collection of solutions of

$$\frac{d^2y}{dx^2} + 9y = 0$$

Let $\frac{d}{dx} \equiv D$

$\Rightarrow$ $(D^2 + 9)\, y = 0$

Auxiliary equation is $m^2 + 9 = 0$ or $m = \pm 3i$

So, solution is $y = c_1 \cos 3x + c_2 \sin 3x$

Let $V = \{c_1 \cos 3x + c_2 \sin 3x : c_1, c_2 \in R\}$ .....(1)

Let $S = \{\cos 3x, \sin 3x\}$

The Wronskian of $v_1 = \cos 3x$ and $v_2 = \sin 3x$ is

$$W(x) = \begin{vmatrix} v_1 & v_2 \\ \frac{dv_1}{dx} & \frac{dv_2}{dx} \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3 \neq 0$$

So S is linearly independent subset of V and by equation (1) , L(S)=V.

Hence S is a basis of V.

Thus, $\dim V = 2$

(ii) Let $v_1 = \cos 3x$, $v_2 = \sin 3x$

Now $w_1 = v_1$, So $\|w_1\|^2 = \langle w_1, w_1 \rangle = \int_0^\pi \cos^2(3x)\, dx$

$$= \int_0^\pi \frac{\cos 6x + 1}{2}\, dx = \frac{\pi}{2} \text{, on solving}$$

$$\therefore \frac{w_1}{\|w_1\|} = \sqrt{\frac{2}{\pi}} . \cos 3x$$

Let $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle w_1}{\|w_1\|^2}$ .....(2)

$$\therefore \langle v_2, w_1 \rangle = \langle v_2, v_1 \rangle = \int_0^\pi \sin 3x \cos x\, dx = \frac{1}{2}\int_0^\pi \sin 6x\, dx = 0,$$

$\therefore \quad w_2 = v_2 = \sin 3x$

Now, $\| w_2 \|^2 = < w_2 , w_2 > = \int_0^{\pi} \sin^2(3x)\, dx = \int_0^{\pi} \left(\frac{1-\cos 6x}{2}\right) dx = \frac{\pi}{2}$

$\therefore \quad \frac{w_2}{\|w_2\|} = \sqrt{\frac{2}{\pi}} \sin 3x$

Hence an orthonormal basis of V is $\left\{ \sqrt{\frac{2}{\pi}} \cos 3x , \sqrt{\frac{2}{\pi}} \sin 3x \right\}$

Example 3: Obtain an orthonormal basis for V, the space of all real polynomials of degree at most 2, the inner product being defined by

$$< f, g > = \int_0^1 f(x)g(x)\, dx$$

Solution: We have, $V = \{ a_0 + a_1 x + a_2 x^2 ; a_i \in \mathbf{R} \}$

Let $S = \{1, x, x^2\}$. Then obviously, S is a basis of V

Let $v_1 = 1$, $v_2 = x$ and $v_3 = x^2$

So, $w_1 = v_1 = 1$

Now $\|w_1\|^2 = < w_1 , w_1 > = \int_0^1 1.1 .\, dx = 1$

Let $\quad w_2 = v_2 - \frac{< v_2, w_1 > w_1}{\|w_1\|^2}$ …..(1)

Now $< v_2 , w_1 > = < v_2 , v_1 > = \int_0^1 x\, dx = \frac{1}{2}$

$\therefore w_2 = x - \frac{1}{2}$

Hence , $\|w_2\|^2 = < w_2 , w_2 > = \int_0^1 (x - \frac{1}{2})^2\, dx = \frac{1}{12}$

So, $\frac{w_2}{\|w_2\|} = \sqrt{12}\,(x - \frac{1}{2}) = 2\sqrt{3}\,(x - \frac{1}{2})$

Let $w_3 = v_3 - \frac{< v_3, w_1 > w_1}{\|w_1\|^2} - \frac{< v_3, w_2 > w_2}{\|w_2\|^2}$ …..(2)

Since, $< v_3, w_1 > = \int_0^1 x^2\, dx = \frac{1}{3}$

$$< v_3, w_2 > = \int_0^1 x^2\left(x - \frac{1}{2}\right)dx = \frac{1}{12}$$

$$w_3 = x^2 - \frac{1}{3}.\,1 - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6}$$

$$\|w_3\|^2 = < w_3 , w_3 > = \int_0^1 (x^2 - x + \frac{1}{6})^2\, dx = \frac{1}{180}$$

$$\frac{w_3}{\|w_3\|} = \sqrt{180}\,(x^2 - x + \frac{1}{6}) = 6\sqrt{5}\,(x^2 - x + \frac{1}{6})$$

Hence an orthonormal basis of V is

$$\{ 1, 2\sqrt{3}\,(x - \frac{1}{2}) , 6\sqrt{5}\,(x^2 - x + \frac{1}{6}) \}$$

13.4 *BESSEL'S INEQUALITY*

Theorem 2: If V is an inner product space and if $\{w_1 , \ldots ,w_n\}$ is an orthonormal set in V , then

$\sum_{i=1}^{n} |\, w_i , v\, |^2 \leq \|v\|^2$, for all $v \in V$

Furthermore, equality holds if and only if V is in subspace spanned by $w_1 , \ldots ,w_n$.

Proof: Let $v \in V$ be arbitrary.

Consider the vector

$$x = v - \sum_{i=1}^{n} \alpha_i w_i \text{ ; where } \alpha_i = < v , w_i > \qquad \ldots..(1)$$

Then, $< x, x > = < v - \sum_{i=1}^{n} \alpha_i w_i , v - \sum_{j=1}^{n} \alpha_j w_j$

$$= < v, v > - < v , \sum_{j=1}^{n} \alpha_j w_j > - < \sum_{i=1}^{n} \alpha_i w_i , v > + < \sum_{i=1}^{n} \alpha_i w_i , \sum_{j=1}^{n} \alpha_j w_j >$$

$$= \|v\|^2 - \sum_{j=1}^{n} \overline{\alpha_j} < v , w_j > - \sum_{i=1}^{n} \alpha_i < w_i , v > + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j < w_i , w_j >$$

$= \|v\|^2 - \sum_{j=1}^{n} < \overline{v , w_j} > < v , w_j > - \sum_{i=1}^{n} < v, w_i > < \overline{v , w_i} > + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_i}.\, 1$ (as $< w_i , w_j > = 1$ only if i = j)

So, $< x, x > = \|v\|^2 - \sum_{i=1}^{n} | < v, w_i > |^2 - \sum_{i=1}^{n} | < v, w_i > |^2 + \sum_{i=1}^{n} | < v, w_i > |^2$

$< x, x > = \|v\|^2 - \sum_{i=1}^{n} | < v, w_i > |^2 = \|v\|^2 - \sum_{i=1}^{n} | < \overline{v , w_i} > |^2$ as $|z| = |\bar{z}|$

$\therefore \ \|x\|^2 = \|v\|^2 - \sum_{i=1}^{n} |<w_i, v>|^2$ .....(2)

Since $\|x\|^2 \geq 0$, so by equation (2), we have

$$\|v\|^2 - \sum_{i=1}^{n} |<w_i, v>|^2 \geq 0 \quad \text{or} \quad \sum_{i=1}^{n} |<w_i, v>|^2 \leq \|v\|^2 \text{ for each } v \in V$$

If the equality holds i.e. if $\sum_{i=1}^{n} |<w_i, v>|^2 = \|v\|^2$, then from equation (2), we have

$$\|x\|^2 = 0 \text{ or } \|x\| = 0$$

$$\Rightarrow \ x = 0$$

So, $v = \sum_{i=1}^{n} \alpha_i w_i = \sum_{i=1}^{n} <v, w_i> w_i$

Thus, if the equality holds, then v is linear combination of $\{ w_1, \ldots, w_n \}$.

Conversely, if v is a linear combination of $\{ w_1, \ldots, w_n \}$, then we can write

$v = \sum_{i=1}^{n} \alpha_i w_i$ where $\alpha_i = <v, w_i>$

So, $x = 0 \Rightarrow \|x\|^2 = 0$

Hence from equation (2), we have

$$\|v\|^2 = \sum_{i=1}^{n} |<w_i, v>|^2 \quad \text{i.e. equality holds.}$$

13.5 *ORTHOGONAL COMPLEMENT*

Let V be an inner product space, and let S be any set of vectors in V. The orthogonal complement of S (written as $S^{\perp}$ and read as S perpendicular or S perp.) is defined by

$$\mathbf{S^{\perp} = \{ v \in V : <u, v> = 0 \ \forall \ u \in S \}}$$

Thus $S^{\perp}$ is the set of all those vectors in V which are orthogonal to every vector in S.

Theorem 3: Let S be any set of vectors in an inner product space V. Then $S^{\perp}$ is a subspace of V.

Proof: By definition, $S^{\perp} = \{ v \in V : <u, v> = 0 \ \forall \ u \in S\}$

Since $<0, u> = 0 \ \forall \ u \in S$

So, $0 \in S^{\perp}$ and thus $S^{\perp}$ is not empty.

Let x, y ∈ F and w_1 , w_2 ∈ $S^\perp$

Then < w_1 , u > = 0 ∀ u ∈ S and

< w_2 , u > = 0 ∀ u ∈ S

So, < $xw_1 + yw_2$, u > = x < w_1 , u > + y < w_2 , u >

= x.0 + y.0 = 0 ∀ u ∈ S

So, $xw_1 + yw_2 \in S^\perp$ ∀ w_1 , w_2 ∈ $S^\perp$ and x, y ∈ **F**

Hence $S^\perp$ is a subspace of V.

Note: (1) Here we should note that S MAY NOT be a subspace of V while $S^\perp$ is always a subspace of V.

(2) Obviously, it can be observed that $V^\perp = \{ \bar{0} \}$ and $\{ \bar{0} \}^\perp = V$.

Orthogonal Complement of an orthogonal complement: Let S be any subset of an inner product space V. the $S^\perp$ is a subset of B.

We define $(S^\perp)^\perp$, written as $S^{\perp\perp}$, by

$S^{\perp\perp} = \{ v \in V : < v, u > = 0 , \forall u \in S^\perp \}$

Obviously $S^{\perp\perp}$ is a subspace of V.

Note: It is very easy to show that $S \subset S^{\perp\perp}$

Let u ∈ S , then < u, v > = 0 ∀ v ∈ $S^\perp$.

So by definition of $S^{\perp\perp}$, we conclude that u ∈ $S^{\perp\perp}$. So ⊂ $S^{\perp\perp}$

Theorem 4: (Projection Theorem) Let W be any subspace of a finite dimensional inner product

space V. Then **(i)** $V = W \oplus W^\perp$ **(ii)** $W^{\perp\perp} = W$

Proof: (**i**) By definition , $W^\perp = \{ v \in V : < v, u > = 0 , \forall u \in W \}$, and $W^\perp$ is a subspace of V. By the given hypothesis, W is also a finite dimensional inner product space and so W has an orthonormal basis.

Let $S = \{ w_1, \ldots, w_m \}$ be an orthonormal basis of W.

$$\therefore \; < w_i, w_j > = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \qquad \ldots..(1)$$

Let $v \in V$ be arbitrary,

Let $w = \sum_{i=1}^{m} \alpha_i w_i$, where $\alpha_i = < v, w_i >$ …..(2)

Now we assume $x = v - w$ …..(3)

Then,

$$< x, w_i > = < v - w, w_i > = < v, w_i > - < w, w_i >$$

$$= < v, w_i > - < \alpha_1 w_1 + \ldots + \alpha_m w_m , w_i >$$

$$= < v, w_i > - \alpha_1 < w_1, w_i > - \ldots - \alpha_i < w_i, w_i > - \ldots - \alpha_m < w_m, w_i >$$

$$= < v, w_i > - 0 - \ldots - \alpha_i$$

$$= < v, w_i > - < v, w_i >$$

So, $< x, w_i > = 0$, for $i = 1, 2, \ldots, m$. …..(4)

Since S is a basis of W, each $u \in W$ is expressible as

$$u = \beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_m w_m ; \quad \beta_i \in F$$

We have , $< x, u > = < x, \beta_1 w_1 + \ldots + \beta_m w_m >$

$$= \overline{\beta_1} < x, w_1 > + \ldots + \overline{\beta_m} < x, w_m >$$

$$= \overline{\beta_1} .0 + \ldots + \overline{\beta_m} . 0 = 0, \qquad \text{(using eqn. 4)}$$

So $< x, u > = 0$, $\forall u \in W$

$$\Rightarrow x \in W^{\perp} .$$

From equation (3) , $v = w + x$ where $w \in W$ and $x \in W^{\perp}$

$$\therefore V = W + W^{\perp} \qquad \ldots..(5)$$

Now we shall prove that $W \cap W^{\perp} = \{0\}$

Let $y \in W \cap W^{\perp}$ be arbitrary,

$\Rightarrow y \in W$ and $y \in W^{\perp}$

Now $y \in W^{\perp} \Rightarrow \langle y, u \rangle = 0 \ \forall \ u \in W$

In particular, $\langle y, y \rangle = 0$ as $y \in W$

$\Rightarrow y = 0$ and $W \cap W^{\perp} = \{0\}$ …..(6)

From equation (5) and (6) , we get

$$V = W \oplus W^{\perp}$$

(ii) From part (i) , we have

$$V = W \oplus W^{\perp} \qquad \text{…..(7)}$$

Since $W^{\perp}$ is a subspace of V, on replacing W by $W^{\perp}$ in eq^n (7), we get,

$$V = W^{\perp} \oplus W^{\perp\perp} \qquad \text{…..(8)}$$

As V is finite-dimensional, so from eqns (7) & (8), we get

$$\dim V = \dim W + \dim W^{\perp} \qquad \text{…..(9)}$$

and $\dim V = \dim W^{\perp} + \dim W^{\perp\perp}$

$$\Rightarrow \dim W = \dim W^{\perp\perp} \qquad \text{…..(10)}$$

But we already know that $W \subset W^{\perp\perp}$.

So from equation (10), we have

$$W = W^{\perp\perp}$$

Example 4: If S_1 and S_2 are subsets of an inner product space V, then show that

$$S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$$

Solution: Let $x \in S_2^{\perp}$, then $< x, y > = 0$, for each $y \in S_2$.

In particular, $< x, z > = 0$, $\forall\, z \in S_1$ as $S_1 \subset S_2$

$$\Rightarrow x \in S_1^{\perp}$$

Hence $S_2^{\perp} \subset S_1^{\perp}$

Example 5: If W_1 and W_2 are subspaces of a finite-dimensional inner product space V, then prove that –

(i) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

Solution: Since we know that

$$W_1 \subset W_1 + W_2 \text{ and } W_2 \subset W_1 + W_2$$

So by previous example, we have

$$(W_1 + W_2)^{\perp} \subset W_1^{\perp} \text{ and } (W_1 + W_2)^{\perp} \subset W_2^{\perp}$$

So, $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ …..(1)

Now, suppose $z \in W_1^{\perp} \cap W_2^{\perp}$ be arbitrary

$$\Rightarrow \quad z \in W_1^{\perp} \text{ and } z \in W_2^{\perp}$$

$$\Rightarrow \; < z , x > = 0 , \forall\, x \in W_1 \text{ and } < z , y > = 0 , \forall\, y \in W_2 \qquad \text{…..(2)}$$

Now any $t \in W_1^{\perp} \cap W_2^{\perp}$ can be written as

$$t = x + y \text{ for some } x \in W_1 , y \in W_2$$

So $< z, t > = < z , x + y > = < z , x > + < z , y >$

$= 0$, (using eq^n (2))

So, $z \in (W_1 + W_2)^{\perp}$ and hence

$$W_1^{\perp} \cap W_2^{\perp} \subset (W_1 + W_2)^{\perp} \qquad \text{….(3)}$$

From equation (1) and (3) , we get

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp} \qquad \text{….(4)}$$

(ii) Since $W_1^{\perp}$ and $W_2^{\perp}$ are subspaces of V, so on taking $W_1^{\perp}$ in place of W_1 and $W_2^{\perp}$ in place of W_2 in eq^n (4), we get

$$(W_1^{\perp} + W_2^{\perp})^{\perp} = (W_1^{\perp})^{\perp} \cap (W_2^{\perp})^{\perp}$$

So $(W_1^{\perp} + W_2^{\perp})^{\perp} = W_1^{\perp\perp} \cap W_2^{\perp\perp}$

$$= W_1 \cap W_2 \qquad \text{as } W^{\perp\perp} = W$$

$$\Rightarrow (W_1^{\perp} + W_2^{\perp})^{\perp\perp} = (W_1 \cap W_2)^{\perp}$$

$$\Rightarrow W_1^{\perp} + W_2^{\perp} = (W_1 \cap W_2)^{\perp}$$

Example 6: Let W be a finite-dimensional proper subspace of an inner product space V. Let $\alpha \in V$ and $\alpha \notin W$. Show that there is a vector $\beta \in V$ such that $\alpha - \beta$ is orthogonal to W.

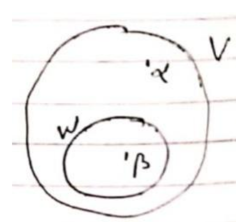
Solution: We know that every finite-dimensional inner product space has an orthonormal basis. Let $\{ \alpha_1 , …., \alpha_n \}$ be an orthonormal basis of W.

Let $\beta = \sum_{i=1}^{n} < \alpha , \alpha_i > \alpha_i$ where $< \alpha , \alpha_i > \in \mathbf{F}$

Then $\beta \in W$, For each j, $1 \leq j \leq n$ we have

$$< \alpha - \beta , \alpha_j > = < \alpha - \sum_{i=1}^{n} < \alpha , \alpha_i > \alpha_i , \alpha_j >$$

$$= < \alpha , \alpha_j > - \sum_{i=1}^{n} < \alpha , \alpha_i > < \alpha_i , \alpha_j >$$



$$= < \alpha , \alpha_j > - < \alpha , \alpha_j > \quad \text{as } < \alpha_i , \alpha_j > = \delta_{ij}$$

$$= 0$$

$$< \alpha - \beta , \alpha_j > = 0, \text{ for all } j = 1, 2, …., n. \qquad \text{…..(1)}$$

Let $w \in W$ be arbitrary, we can write

$$w = \sum_{i=1}^{n} a_i \alpha_i \quad \text{where } a_i \in \mathbf{F}$$

We have $< \alpha - \beta , w > \; = \; < \alpha - \beta , \sum_{i=1}^{n} a_i \alpha_i >$

$= \sum_{i=1}^{n} \overline{a_i} < \alpha - \beta , \alpha_i > \; = 0$, by eq^n (1)

$\therefore \; < \alpha - \beta , w > \; = 0$, for each $w \in W$

Hence $\alpha - \beta$ is orthogonal to W.

13.6 *RIESZ REPRESENTATION THEOREM*

Theorem 5: Let V(R) be a finite-dimensional linear functional $f : V \rightarrow \mathbf{R}$. Then there exists a unique $y \in V$ such that $f(x) = < x , y >$, $\forall \, x \in V$.

Proof: Suppose there exists $y \in V$ such that

$$f(x) = < x , y > , \text{ for all } x \in V.$$

Let us choose an orthonormal basis $\{ e_1 , \ldots, e_n \}$ of V

Then $y = \sum_{i=1}^{n} \alpha_i e_i$ for some $\alpha_i \in \mathbf{R}$

Now $f \in L(V, \mathbf{R})$ and f is completely determined if we know $f(e_i)$ for $1 \leq i \leq n$

Now $f(e_i) = < e_i , y > = \alpha_i$ for $1 \leq i \leq n$

This suggest that we take $y = \sum_{i=1}^{n} f(e_i) \, e_i$

It is easy to check that $f(x) = < x , y >$ for all $x \in V$

For if $x = \sum \alpha_i e_i$, then $f(x) = \sum \alpha_i f(e_i)$ …..(1)

Also $< x , y > \; = \; < x , \sum f(e_i) \, e_i >$

$$= < \sum \alpha_j e_j , \sum f(e_i) \, e_i >$$

$$= \sum_{i,j} f(e_i) \, \alpha_j < e_i , e_j > \qquad \text{…..(2)}$$

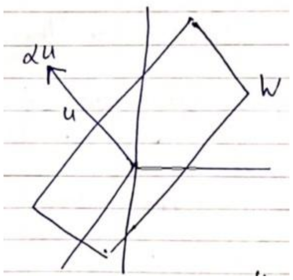
$$= \sum f(e_i) \, \alpha_i \text{ as } < e_i , e_j > \; = \; \delta_{ij} = \begin{cases} 1 , i = j \\ 0 , i \neq j \end{cases}$$

From equations (1) and (2) , we conclude that

$f(x) = \langle x , y \rangle$ for all $x \in R^n$

Uniqueness: Now, suppose z is such that,

$f(x) = \langle x , z \rangle$ for all $x \in V$



then, $f(x) = \langle x , z \rangle = \langle x , y \rangle$

$\Rightarrow \langle x , z - y \rangle = 0$ for all x.

In particular, for $x = z - y$, we obtain

$$\langle z - y , z - y \rangle = 0$$

$$z - y = 0$$

$$z = y$$

So y is unique.

Geometric Interpretation:

If $f = 0$, then the obvious choice is $y = 0$.

If $f \neq 0$, then f is a linear form and $W = \ker f$ is of

dimension $n - 1$, where $n = \dim V$.

Thus there is a unit vector u perpendicular to W, for

$V = W \oplus W^{\perp}$ (that is , u is a unit normal to the "plane"). y must therefore be a multiple αu of u. The choice of α is determined by the equation

$f(u) = \langle u , y \rangle = \langle u , \alpha u \rangle = \alpha$

Thus we take $y = \alpha u$ where $\alpha = f(u)$

For $x \in V$, we have $x = w + tu$, where $w \in W$ and $t \in \mathbf{R}$

Then $f(x) = f(w + tu) = f(w) + t\, f(u) = t\, f(u)$

Also $\langle x, y \rangle = \langle w + tu, \alpha u \rangle = \alpha \langle w, u \rangle + t\, \alpha \langle u , u \rangle = t\, \alpha = t\, f(u)$

Hence the result.

Theorem 6: For any linear operator T on a finite-dimensional inner product space V, there exists a unique linear operator T^* on V such that

$$< T\alpha , \beta > = < , T^* \beta > \text{ for all } \alpha , \beta \in V.$$

Proof: Let T be a linear operator on a finite dimensional inner product space V over the field **F**. Let $\beta \in$ V and f be a functional from V into **F** defined by

$$f(\alpha) = < T\alpha , \beta > \ \forall \alpha \in V \qquad \text{.....(1)}$$

Here $T\alpha$ stands for $T(\alpha)$

Claim: f is a linear functional on V.

Let a , b $\in$ **F** and $\alpha_1 , \alpha_2 \in V$, then

$$f (a \alpha_1 + b \alpha_2) = < T(a \alpha_1 + b \alpha_2) , \beta >$$

$$= < (aT\alpha_1 + bT\alpha_2) , \beta > \quad \text{as T is linear}$$

$$= a < T\alpha_1 , \beta > + b < T\alpha_2 , \beta >$$

$$= a f (\alpha_1) + b f (\alpha_2) , \text{ using equation (1)}$$

Hence f is a linear functional on V.

So by **Riesz representation theorem**, there exists a unique $\beta' \in V$ such that

$$f (\alpha) = < \alpha , \beta' > \ \forall \alpha \in V \qquad \text{.....(2)}$$

From equations (1) and (2), we observe that if T is a linear operator on V, then corresponding to every vector β in V, there is a uniquely determined vector β' in V such that

$$< T\alpha , \beta > = < \alpha , \beta' > \ \forall \alpha \in V$$

Let us denote by T^* the rule which associates β with β' i.e. let $T^* \beta = \beta'$

Then T^* is a function from V in to V and is such that

$$< T\alpha , \beta > = < \alpha , T^* \beta > \ \forall \alpha, \beta \in V \qquad \text{.....(3)}$$

Claim: T^* is a linear operator on V.

Let a, b ∈ **F** and $\beta_1, \beta_2 \in V$. Then $\forall\, \alpha \in V$, we have

$$< \alpha, T^* (a\beta_1 + b\beta_2) > \; = \; < T\alpha, a\beta_1 + b\beta_2 > \qquad \text{using equation (3)}$$

$$= \bar{a} < T\alpha, \beta_1 > + \bar{b} < T\alpha, \beta_2 >$$

$$= \bar{a} < \alpha, T^* \beta_1 > + \bar{b} < \alpha, T^* \beta_2 > \qquad \text{again by (3)}$$

$$= \; < \alpha, a\,T^* \beta_1 > + < \alpha, b\,T^* \beta_2 >$$

$$= \; < \alpha, a\,T^* \beta_1 + b\,T^* \beta_2 >$$

Hence $T^* (a\beta_1 + b\beta_2) = a\,T^* \beta_1 + b\,T^* \beta_2$

Thus T^* is a linear operator on V

Hence corresponding to a linear operator T on V, there exists a linear operator T^* on V

such that , $< T\alpha, \beta > \; = \; < \alpha, T^* \beta > \quad \forall\, \alpha, \beta \in V$

Uniqueness: Let S be a linear operator on V such that

$$< T\alpha, \beta > \; = \; < \alpha, S\beta > \quad \forall\, \alpha, \beta \in V$$

Then $< \alpha, T^* \beta > \; = \; < \alpha, S\beta > \quad \forall\, \alpha, \beta \in V$

$$\Rightarrow \qquad T^* \beta = S\beta \qquad \forall\, \beta \in V$$

$$\Rightarrow \qquad T^* = S$$

So T^* is unique.

Check your progress

Problem 1: Apply the Gram–Schmidt orthogonalisation process to the vectors $v_1 = (1,1,0), v_2 = (1,0,1)$ and $v_3 = (0,1,1) \in R^3$ (standard dot product) to produce an orthonormal basis $\{e_1, e_2, e_3\}$ for span $\{v_1, v_2, v_3\}$. Show your working.

13.7 *SUMMARY*

In this chapter we understood the process of generalization from ordinary vectors to vector spaces. So other basic concepts viz angle, length, distance were also generalized respectively as inner product, norm, and metric. As we have studied orthogonal compotent of ordinary vectors, we studied here Gram-Schmidt orthogonalisation process. Besides this, we learned various concepts and applications of inner product.

13.8 *GLOSSARY*

➢ **Inner Product:** An inner product on a vector space V is a map $< \, , \, > : V \times V \rightarrow R$ satisfying the following properties :

(i) $< x , x > \geq 0$ and $< x , x > = 0$ if and only if x = 0.

(ii) $< x , y > = < y , x >$

(iii) $< x + z , y > = < x , y > + < z , y >$ and $< x , y + z > = < x , y > + < x , z >$

(iv) $< ax , y > = a < x , y > \; \forall x , y , z \in V$ and $a \in R$.

➢ **Norm of a Vector:** Let V be an inner product space. The norm function $\| . \| : V \rightarrow R$ has the following properties :

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if x = 0 ; $x \in V$

(ii) $\| \alpha x \| = |\alpha| \, \| x \|$, $\alpha \in \mathbf{F}$, $x \in V$,

Norm of a vector $v \in V$ is defined as $\| v \| = \sqrt{< v, v >}$.

➢ **Complete Orthonormal Set:** An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.

➢ **Gram-Schmidt orthogonlisation Process:** Every finite-dimensional inner product space has an orthonormal basis.

13.9 *REFERENCES*

➢ S Kumaresan; Linear Algebra-A Geometric Approach; PHI-2016.

➢ K Hoffman & Ray Kunze; Linear Algebra; PHI-1971.

➢ P Halmos; Finite-Dimensional Vector Spaces;

➢ G Strang; Introduction To Linear Algebra;

13.10 *SUGGESTED READING*

➢ NPTEL videos.

➢ Schaum series.

➢ A R Vashishtha, Krishna Prakashan; Meerut.

- Graduate Text In Mathematics, Springer.

13.11 *TERMINAL QUESTION*

1. Explain the Gram-Schmidt Orthogonalization process. Derive the general formula used to construct an orthogonal basis from a linearly independent set of vectors.

2. Apply the Gram-Schmidt process to the vectors $v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)$ in R^3 and find an orthonormal basis. Show all necessary steps.

Short answer type question

1. What is the Gram-Schmidt orthogonalization process?

2. State the necessary condition for applying the Gram-Schmidt process to a set of vectors.

3. What is the difference between an orthogonal set and an orthonormal set?

4. How does the Gram-Schmidt process ensure that vectors are orthogonal?

5. Why is it necessary to normalize vectors after applying the Gram-Schmidt process?

6. What is the role of projection in the Gram-Schmidt process?

7. Can the Gram-Schmidt process be applied to complex vector spaces? Justify your answer.

8. Does the Gram-Schmidt process change the dimension of the subspace spanned by the vectors? Explain.

9. Write the formula for the projection of a vector vvv onto a vector uuu.

10. What type of product (operation) is used in the Gram-Schmidt process to calculate projections?

Objective type questions:

1. The Gram-Schmidt process is used to:

A) Solve linear equations
B) Diagonalize a matrix
C) Convert a set of linearly dependent vectors into orthogonal vectors
D) Convert a set of linearly independent vectors into an orthogonal basis

2. What property does the set of vectors produced by the Gram-Schmidt process satisfy?

A) They are linearly dependent
B) They are orthogonal
C) They are not normalized
D) They form a skew-symmetric set

3. In the Gram-Schmidt process, after orthogonalizing the vectors, what step is required to get an orthonormal basis?

A) Take the determinant
B) Normalize each vector
C) Take the inverse of the vectors
D) Apply Gaussian elimination

4. The Gram-Schmidt process requires that the initial set of vectors be:

A) Linearly dependent
B) Orthonormal
C) Linearly independent
D) Eigenvectors

5. In an inner product space, the Gram-Schmidt process can be applied to:

A) Only real vector spaces
B) Only complex vector spaces
C) Both real and complex inner product spaces
D) Only Euclidean spaces

6. Which of the following is true about the vectors produced by the Gram-Schmidt process?

A) They span a different subspace than the original vectors
B) They span the same subspace as the original vectors
C) They are always eigenvectors
D) They are linearly dependent

7. In the Gram-Schmidt process, which operation is repeatedly used to make vectors orthogonal?

A) Cross product
B) Matrix multiplication
C) Projection
D) Inverse computation

8. What is the purpose of subtracting projections in the Gram-Schmidt process?

A) To increase vector length
B) To ensure orthogonality

C) To normalize the vector
D) To find eigenvalues

9. If the Gram-Schmidt process is applied to a set of vectors in R^n, how many orthogonal vectors are obtained?

A) $n + 1$
B) Same as the number of original vectors
C) Always 1
D) Depends on the dimension of the space only

10. Which of the following is a necessary condition for applying the Gram-Schmidt process successfully?

A) The matrix must be symmetric
B) Vectors must be orthogonal
C) Vectors must be linearly independent
D) Vectors must be unit vectors

Fill in the blanks questions:

1. The Gram-Schmidt process converts a set of linearly independent vectors into an _______ set.

2. The vectors obtained from the Gram-Schmidt process span the _______ subspace as the original set of vectors.

3. In the Gram-Schmidt process, each new vector is made orthogonal by subtracting its _______ onto the previous vectors.

4. After applying the Gram-Schmidt process, we can obtain an _______ basis by normalizing each orthogonal vector.

5. The inner product used in the Gram-Schmidt process depends on the _______ structure of the vector space.

6. The Gram-Schmidt process can be applied to vectors in both _______ and _______ inner product spaces.

7. The projection of vector v onto vector u is given by $\frac{< v, u >}{< u, u >}.u$, where $\langle \cdot, \cdot \rangle$ denotes the _______.

8. The orthogonal vectors produced by the Gram-Schmidt process are not necessarily of _______ length.

True and false questions:

1. The Gram-Schmidt process can be applied to any set of vectors, whether linearly independent or not.

2. The vectors generated by the Gram-Schmidt process are orthogonal to each other.

3. After applying the Gram-Schmidt process, the resulting vectors always form an orthonormal set.

4. The Gram-Schmidt process changes the span of the original set of vectors.

5. The Gram-Schmidt process can be used in both real and complex inner product spaces.

6. The projection of a vector onto another is required to ensure orthogonality in the Gram-Schmidt process.

7. You must apply matrix inversion during the Gram-Schmidt orthogonalization process.

8. The Gram-Schmidt process can be used to create an orthogonal basis for any subspace of R^n.

13.12 *ANSWERS*

Answers of check your progress: $e_1 = \frac{1}{\sqrt{2}}(1,1,0)$ **;** $e_2 = \frac{1}{\sqrt{6}}(1,-1,2)$ and $e_3 = \frac{1}{\sqrt{3}}(-1,1,1)$

Answer of objective type question:

1:	D	**2:**	B	**3:**	B	**4:**	C
5:	C	**6:**	B	**7:**	C	**8:**	B
9:	B	**10:**	C				

Answer of fill in the blanks:

1:	Orthogonal	**2:**	Same	**3:**	Projection
4:	Orthonormal	**5:**	Vector Space	**6:**	Real, Complex
7:	Inner product	**8:**	Unit		

Answer of TRUE and FALSE:

1:	False	**2:**	True	**3:**	False	**4:**	False
5:	True	**6:**	True	**7:**	False	**8:**	True

UNIT-14: UNITARY AND NORMAL OPERATOR

CONTENTS

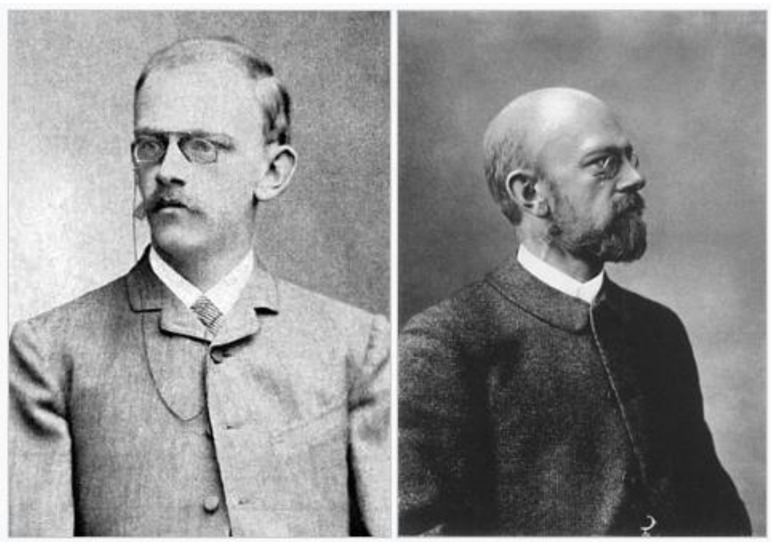
14.1 *INTRODUCTION*

German mathematician David Hilbert, who lived from January 23, 1862, to February 14, 1943, was a very influential mathematician of the late 19th and early 20th centuries. The foundations of geometry, the spectral theory of operators and its application to integral equations, the calculus of variations, commutative algebra, algebraic number theory, mathematical physics, and the foundations of mathematics (especially proof theory) are just a few of the many fundamental concepts that Hilbert discovered and developed.

Hilbert embraced and upheld the transfinite numbers and set theory of Georg Cantor. He introduced a set of issues in 1900 that paved the way for 20th-century mathematical research.

Important tools utilized in modern mathematical physics were invented by Hilbert and his pupils, who also helped to establish rigor in the field. Hilbert was a pioneer in the fields of mathematical logic and proof theory.

An inner product structure on a C-vector spaces induces a "mirrored" twin for every linear transformation, called the adjoint. Linear operators equal their own adjoints have many important properties.



Hilbert in 1886 Hilbert in 1907

14.2 *OBJECTIVES*

After reading this unit learners will be able to

- Understand the basic concept of unitary operator and normal operator.
- Understand the basic concept of adjoint operator and self-adjoint operator.
- Understand the concept of skew-symmetric and skew-Hermitian operator.
- Understand the concept of positive and non-negative operator.

14.3 *ADJOINT OPERATORS*

Let T be a linear operator on an inner product space V (here V **need not** be finite dimensional). We say that T has an adjoint T^* if there exists a linear operation T^* in V such that $< T\alpha , \beta > = < \alpha , T^* \beta > \quad \forall \alpha, \beta \in V$

Note: In previous unit, we have proved that every linear operator on a finite-dimensional inner product space posses an adjoint. But it should be noted that if V is **not** finite-dimensional, then some linear operator on V may possess an adjoint while the other may not. In any case if T possesses an adjoint T^*, then it must be unique. Also observe that the adjoint of T depends not only upon T, but also on the inner product on V.

Theorem 1: Let V be a finite-dimensional inner product space and let B = { α_1 , …., α_n } be an ordered orthonormal basis for V . Let T be a linear operator on V and let $A = [a_{ij}]_{m \times n}$ be the matrix of T with respect to the ordered basis B. Then $a_{ij} = < T \alpha_j, \alpha_i >$

Proof: As B is an orthonormal basis for V, so for any $\beta \in V$,

$$\beta = \sum_{i=1}^{n} < \beta, \alpha_i > \alpha_i$$

Replacing β by $T \alpha_j$, we get

$$T\alpha_j = \sum_{i=1}^{n} < T \alpha_j , \alpha_i > \alpha_i \; ; \quad j = 1, 2, \ldots, n \qquad \ldots(1)$$

Now if $A = [a_{ij}]_{m \times n}$ be the matrix of T in the ordered basis B, then we have

$$T\alpha_j = \sum_{i=1}^{n} a_{ij} \; \alpha_i \; ; \quad j = 1, 2, \ldots, n \qquad \ldots(2)$$

Since the expression for $T\alpha_j$ as a linear combination of vectors in B is unique, so from equations (1) and (2), we have

$$a_{ij} = < T \alpha_j , \alpha_i > \qquad ; 1 \leq i \leq n, 1 \leq j \leq n$$

Corollary 1: Let V be a finite dimensional inner product space and let T be a linear operator on V. In any orthonormal basis for V, the matrix of T^* is the conjugate transpose of the matrix of T.

Proof: Let B = { α_1 , …., α_n } be an orthonormal basis for V. Let $A = [a_{ij}]_{m \times n}$ be the matrix of T in ordered basis B.

Then $\qquad a_{ij} = < T \alpha_j , \alpha_i > \qquad \ldots(1)$

Now T^* is also a linear operator on V.

Let $C = [c_{ij}]_{n \times n}$ be the matrix of T^* in the ordered basis B.

Then $c_{ij} = < T^* \alpha_j, \alpha_i >$ …..(2)

We have $c_{ij} = < T^* \alpha_j, \alpha_i > = < \overline{\alpha_\imath,\ T^*\alpha_\jmath} >$

$= < \overline{T\alpha_\imath,\ \alpha_j} >$ by definition of T^*

$= \overline{a_{\jmath\imath}}$

So $C = [\overline{a_{\jmath\imath}}]_{n \times n}$ and hence $C = A^*$, where A^* is the conjugate transpose of A.

Note: It should be remembered that in this corollary the basis B is an orthonormal basis and **not** an ordinary basis.

Theorem 2: Let S and T be linear operators on an inner product space V and $c \in$ **F**. If S and T possess adjoints, the operators $S + T$, cT, ST, T^* will possess adjoints.

Also (**i**) $(S + T)^* = S^* + T^*$

(**ii**) $(cT)^* = \bar{c}\, T^*$

(**iii**) $(S\,T)^* = T^* S^*$

(**iv**) $(T^*)^* = T$

Proof: (**i**) As S and T are linear operators on V, so S + T is also a linear operator on V.

Now for every $\alpha, \beta \in V$, we have

$< (S + T)\,\alpha\,,\ \beta > = < S\alpha + T\alpha\,,\ \beta > = < S\alpha\,,\ \beta > + < T\alpha\,,\ \beta >$

$= < \alpha\,, S^*\beta > + < \alpha\,, T^*\beta >$, by definition of adjoint

$= < \alpha\,, S^*\beta + T^*\beta >$

$= < \alpha\,, (S^* + T^*)\,\beta >$

Thus for the linear operator S + T on V there exists a linear operator $S^* + T^*$ on V such that

$< (S + T)\,\alpha\,,\ \beta > = < \alpha\,, (S^* + T^*)\,\beta >$ for all $\alpha, \beta \in V$

Therefore, the linear operator S + T has an adjoint. By the definition and by the uniqueness of adjoint, we get

$$(S + T)^* = S^* + T^*$$

(ii) Since T is a linear operator on V, therefore cT is also a linear operator on V. For every $\alpha, \beta \in V$, we have

$$< (cT)\,\alpha\,,\ \beta > = < cT\,\alpha\,,\ \beta > = c < T\,\alpha\,,\ \beta > = c < \alpha\,, T^*\,\beta >$$

$$= < \alpha\,, \bar{c}\,T^*\,\beta > = < \alpha\,, (\,\bar{c}\,T^*\,)\,\beta >$$

$$< (cT)\,\alpha\,,\ \beta > = < \alpha\,, (cT\,)^*\,\beta >$$

Thus for the linear operator cT on V , $\exists$ a linear operator $(cT\,)^*$ or $\bar{c}\,T^*$ on V such that

$$< (cT)\,\alpha\,,\ \beta > = < \alpha\,, (cT\,)^*\,\beta > \forall\, \alpha, \beta \in V.$$

Hence the linear operator cT possesses an adjoint. By the definition and by the uniqueness of adjoint, we get

$$(cT)^* = \bar{c}\,T^*$$

(iii) We observe that ST is a linear operator on V

Now $\forall\, \alpha, \beta \in V$, we have

$$< (ST)\,\alpha\,,\ \beta > = < ST\,\alpha\,,\ \beta >$$

$$= < T\,\alpha\,, S^*\,\beta > \qquad \text{by definition of adjoint}$$

$$= < \alpha\,, T^*\,S^*\,\beta >$$

$$= < \alpha\,, (\,T^*\,S^*\,)\,\beta >$$

Thus for the linear operator ST on V $\exists$ a linear operator $T^*\,S^*$ on V such that

$$< (ST)\,\alpha\,,\ \beta > = < \alpha\,, (\,T^*\,S^*\,)\,\beta > \ \forall\, \alpha, \beta \in V$$

Therefore, the linear operator ST has an adjoint. By the definition and by the uniqueness of adjoint, we get $(S\,T)^* = T^*\,S^*$

(iv) The adjoint of T i.e. T^* is a linear operator on V. For every $\alpha, \beta \in V$, we have

$$< T^*\,\alpha\,, \beta > = < \overline{\beta,\ T^*\alpha} >$$

$= < \overline{T\beta, \ \alpha} >$

$= < \alpha, T\beta >$

Thus for the linear operator T^* on V, there **exists** a linear operator T on V such that

$$< T^* \alpha, \beta > = < \alpha, T\beta > \text{ for all } \alpha, \beta \in V$$

Therefore, the linear operator T^* has an adjoint. By the definition and by the uniqueness of adjoint, we have $(T^*)^* = T$

Note: (1) If V is a finite-dimensional inner product space, then the result is true for arbitrary linear operators S and T. In a finite-dimensional inner product space, each linear operator possesses and adjoint.

(2) The operation of adjoint behaves like the operation of conjugation on complex numbers.

14.4 *SELF-ADJOINT OPERATORS*

Self-adjoint transformation: A linear operator T on an inner product space V is said to be self-adjoint if $T^* = T$

A self-adjoint linear operator on a real inner product space is called **symmetric** while a self-adjoint linear operator on a complex inner product space is called **Hermitian**.

e.g. the zero operator $\hat{0}$ and the identity operator I on *any* inner product space V are self-adjoint. For every $\alpha, \beta \in V$, we have

$$< \hat{0}\alpha, \beta > = < 0, \beta > = 0 = < \alpha, 0 > = < \alpha, \hat{0}\beta >$$

So $\hat{0}^* = \hat{0}$

Similarly, $< I\alpha, \beta > = < \alpha, \beta > = < \alpha, I\beta >$

So $I^* = I$

14.5 *SKEW-SYMMETRIC/ SKEW-HERMITION OPERATORS*

Skew-symmetric / skew-Hermitian operator: If a linear operator T on an inner product space V is such that $T^* = -T$

then T is called *skew-symmetric* or *skew-Hermitian* according as the vector space V is real or complex.

Theorem 3: Every linear operator T on a finite dimensional complex inner product space V can be **uniquely** expressed as

$$T = T_1 + iT_2, \text{ where } T_1 \ \&\ T_2 \text{ are self-adjoint linear operators on V.}$$

Proof: Let $T = \frac{1}{2}(T + T^*) + i\left(\frac{T - T^*}{2i}\right)$

Suppose $T_1 = \frac{T + T^*}{2}$ and $T_2 = \frac{T - T^*}{2i}$

So, $T = T_1 + iT_2$ .....(1)

Now $T_1^* = (\frac{T + T^*}{2})^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = T_1$

So T_1 is self-adjoint

Again $T_2^* = [\frac{1}{2i}(T - T^*)]^* = \left(\overline{\frac{1}{2i}}\right)(T - T^*)^* = \frac{1}{(-2i)}(T^* - T)$

$$T_2^* = \frac{1}{2i}(T - T^*)$$

So T_2 is also self-adjoint. Thus T can be expressed as a sum of two self-adjoint operators.

Uniqueness: Let $T = U_1 + iU_2$ where U_1 and U_2 are both self-adjoint linear operators.

So, $T^* = (U_1 + iU_2)^* = U_1^* + \bar{i}\, U_2^* = U_1^* - i\, U_2^* = U_1 - i\, U_2$

So $T + T^* = 2U_1$ or $U_1 = \frac{1}{2}(T + T^*) = T_1$

Similarly, $T - T^* = 2i\, U_2$ or $U_2 = \frac{1}{2i}(T - T^*) = T_2$

So $T = T_1 + iT_2 = U_1 + iU_2$ i.e. representation is unique.

Note: If T is linear operator on a complex inner product space V which is **Not** finite dimensional, then the above result will be still true *provided*, it is given that T possesses adjoint.

Theorem 4: Every linear operator T on a finite-dimensional inner product space V can be uniquely expressed as $T = T_1 + T_2$, where T_1 is self-adjoint and T_2 is skew.

Proof: Let $T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*)$

where $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2}(T - T^*)$

then $T = T_1 + T_2$ …..(1)

Now $T_1^* = [\frac{1}{2}(T + T^*)]^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + T) = T_1$

So T_1 is self-adjoint.

Similarly $T_2^* = [\frac{1}{2}(T - T^*)]^* = \frac{1}{2}(T - T^*)^* = \frac{1}{2}(T^* - T)$

$$T_2^* = -\frac{1}{2}(T - T^*) = -T_2$$

So T_2 is skew.

Hence T can be expressed as a sum of two linear operators where T_1 in self-adjoint and T_2 in skew.

Uniqueness: Let $T = U_1 + U_2$, where U_1 is self-adjoint and U_2 in skew.

Then $T^* = (U_1 + U_2)^* = U_1^* + U_2^* = U_1 - U_2$

So $T + T^* = 2U_1$ or $U_1 = \frac{1}{2}(T + T^*) = T_1$

and $T - T^* = 2U_2$ or $U_2 = \frac{1}{2}(T - T^*) = T_2$

Hence $T = T_1 + T_2 = U_1 + U_2$

$\Rightarrow$ The expression (1) for T is unique.

Note: If T is a linear operator on an inner product space V which is **NOT** finite-dimensional, then the above result will be still true *provided* T possesses adjoint.

Theorem 5: A necessary and sufficient condition that a linear transformation T on an inner product space V be $\hat{0}$ is that $< T\alpha, \beta > = 0, \forall \alpha, \beta \in V$

Proof: Necessary condition: Let T = 0, then $\forall \alpha, \beta \in V$, we have

$$< T\alpha, \beta > = < \hat{0}\alpha, \beta > = < 0, \beta > = 0$$

So the condition is necessary.

Sufficient condition: Let T be a linear operator such that

$$< T\alpha, \beta > = 0, \forall \alpha, \beta \in V$$

Taking $\beta = T\alpha$, we get

$$< T\alpha, T\alpha > = 0 \quad \forall \alpha \in V$$

$$\Rightarrow \quad T\alpha = 0 \quad \forall \alpha \in V$$

$$\Rightarrow \quad T = \hat{0}$$

Hence the condition is sufficient.

Theorem 6: A necessary and sufficient condition that a linear transformation T on a unitary space be $\hat{0}$ is that $< T\alpha, \alpha > = 0 \quad \forall \alpha \in V$

Proof: Necessary condition: Let $T = \hat{0}$, then $\forall \alpha \in V$

$$< T\alpha, \alpha > = < \hat{0}\alpha, \alpha > = < 0, \alpha > = 0$$

Hence the condition is necessary.

Sufficient condition: Let T be a linear operator satisfying

$$< T\alpha, \alpha > = 0 \quad \forall \alpha \in V \qquad \text{.....(1)}$$

Replacing α by $\alpha + \beta$, we get

$$< T(\alpha + \beta), \alpha + \beta > = 0$$

$$\Rightarrow \quad < T\alpha + T\beta, \alpha + \beta > = 0$$

$$\Rightarrow \quad < T\alpha, \alpha > + < T\alpha, \beta > + < T\beta, \alpha > + < T\beta, \beta > = 0$$

$$\Rightarrow \quad < T\alpha, \beta > + < T\beta, \alpha > = 0, \qquad \text{using (1)}$$

So $\forall \alpha, \beta \in V$, we have

$$< T\alpha, \beta > + < T\beta, \alpha > = 0 \qquad \text{.....(1)}$$

Since above result is true $\forall\ \beta \in V$, so by replacing β and $i\beta$, we get

$$< T\alpha , i\beta > + < T\, i\beta , \alpha > = 0$$

$$\bar{i} < T\alpha , \beta > + i < T\beta , \alpha > = 0$$

$$-i < T\alpha , \beta > + i < T\beta , \alpha > = 0$$

$$\Rightarrow \quad - < T\alpha , \beta > + < T\beta , \alpha > = 0 \qquad \ldots.(2)$$

Adding equation (1) and (2), we get

$$2 < T\beta , \alpha > = 0$$

$$< T\beta , \alpha > = 0 \ \ \forall\ \alpha, \beta \in V$$

Let $\alpha = T\beta$, then

$$< T\beta, T\beta > = 0 \quad \forall\ \beta \in V$$

$$\Rightarrow \quad T\beta = 0 \quad \forall\ \beta \in V$$

$$\Rightarrow \quad T = \hat{0}$$

Hence the condition in sufficient.

Note: (**1**) Above result **may fail** for Eulidean space, e.g., let us consider $V_2(R)$ with standard inner product space. Let T be a linear operator on $V_2(R)$ defined as

$$T(a, b) = (b, -a) \qquad \forall\ (a, b) \in V_2(R)$$

Then obviously $T \neq \hat{0}$. But

$$< T(a, b), (a, b) > = < (b, -a), (a, b) >$$

$$= ba - ab = 0$$

So $< T\,\alpha, \alpha > = 0 \quad \forall\ \alpha \in V_2(R)$, through $T \neq \hat{0}$.

(**2**) However if T is self-adjoint then the above theorem is true for Euclidean spaces also. Finally, we have the following theorem –

Theorem 7: A necessary and sufficient condition that a self-adjoint linear transformation T on an inner product space V be $\hat{0}$ is that

$$< T\alpha, \alpha > = 0 \text{ , for all } \alpha \in V$$

Proof: Necessary part is same as in previous theorem.

Sufficient condition: Let $< T\alpha, \alpha > = 0 \quad \forall \alpha \in V$

So $< T(\alpha + \beta), \alpha + \beta > = 0 \quad \forall \alpha, \beta \in V$

$$\Rightarrow \quad < T\alpha + T\beta, \alpha + \beta > = 0$$

$$< T\alpha, \alpha > + < T\alpha, \beta > + < T\beta, \alpha > + < T\beta, \beta > = 0$$

$$\Rightarrow \quad < T\alpha, \beta > + < T\beta, \alpha > = 0$$

$$\Rightarrow \quad < T\alpha, \beta > + < \beta, T^* \alpha > = 0$$

$$\Rightarrow \quad < T\alpha, \beta > + < \beta, T\alpha > = 0 \text{ , as given } T = T^* \qquad \text{.....(1)}$$

Now two cases may arise –

Case I: If V is a complex inner product space. Then do as in previous theorem.

Case II: If V is a real inner product space.

Then $< \beta, T\alpha > = < T\alpha, \beta >$ as $< \alpha, \beta > = < \overline{\beta, \alpha} > = < \beta, \alpha >$

So from equation (1), we have

$$2 < T\alpha, \beta > = 0 \text{ or } < T\alpha, \beta > = 0 \quad \forall \alpha, \beta \in V$$

Let us put $\beta = T\alpha$

$$\Rightarrow < T\alpha, T\alpha > = 0 \; \forall \; \alpha \in V$$

$$\Rightarrow \quad T\alpha = 0 \; \forall \; \alpha \in V$$

$$\Rightarrow \quad T = \hat{0}$$

Theorem 8: A necessary and sufficient condition that a linear transformation T on a unitary space (of any dimension) be self-adjoint (Hermitian) is that,

$$< T\alpha , \alpha > \text{ be real } \forall \ \alpha \in V$$

Proof: Necessary condition: Let T be self-adjoint operator on a unitary space V i.e. $T^* = T$.

Then for every $\alpha \in V$, we have

$$< T\alpha , \alpha > \ = < \alpha , T^* \alpha > \ = < \alpha , T\alpha > \ = < \overline{T\alpha , \alpha} >$$

$$\Rightarrow \quad < T\alpha , \alpha > \text{ is real } \forall \ \alpha \in V$$

Sufficient condition: Let $< T\alpha , \alpha >$ be real $\forall \ \alpha \in V$. We have to prove that $T^* = T$. For every $\alpha, \ \beta \in V$, we have

$$< T(\alpha + \beta), \alpha + \beta > = < T\alpha + T\beta, \alpha + \beta >$$

$$< T(\alpha + \beta), \alpha + \beta > = < T\alpha , \alpha > + < T\alpha , \beta > + < T\beta , \alpha > + < T\beta, \beta > \qquad ....(1)$$

Since $< T(\alpha + \beta), \alpha + \beta >$, $< T\alpha , \alpha >$ and $< T\beta , \beta >$ are real.

$$\Rightarrow \quad < T\alpha , \beta > + < T\beta , \alpha > \text{ must be real}$$

So $\quad < T\alpha , \beta > + < T\beta , \alpha > \ = \overline{< T\alpha , \beta > + < T\beta , \alpha >}$

$$= < \overline{T\alpha , \beta} > + < \overline{T\beta , \alpha} >$$

$$= < \beta , T\alpha > + < \alpha , T\beta >$$

So $\forall \ \alpha, \ \beta \in V$, we have

$$< T\alpha , \beta > + < T\beta , \alpha > \ = < \beta , T\alpha > + < \alpha , T\beta > \qquad ....(2)$$

Replacing β by $i\beta$ in equation (2), we get

$$< T\alpha , i\beta > + < T i\beta , \alpha > \ = < i\beta , T\alpha > + < \alpha , T i\beta >$$

$$\bar{i} < T\alpha , \beta > + i < T\beta , \alpha > \ = i < \beta , T\alpha > + \bar{i} < \alpha , T\beta >$$

$$-i < T\alpha , \beta > + i < T\beta , \alpha > \ = i < \beta , T\alpha > - i < \alpha , T\beta >$$

$-<T\alpha,\beta> + <T\beta,\alpha> = <\beta,T\alpha> - <\alpha,T\beta>$ …(3)

on equation(2) – equation(3), we get

$<T\alpha,\beta> = <\alpha,T\beta>$

$<T\alpha,\beta> = <\alpha,T\beta>$

$<T\alpha,\beta> = <T^*\alpha,\beta> \ \forall \alpha, \beta \in V$

$\Rightarrow \quad T = T^*$

Note: If V is finite-dimensional, then we can take *advantage* of the fact that T must possess adjoint. So in this case, the **converse** part of the theorem can be easily proved as:

Since $<T\alpha,\alpha>$ is real $\forall\ \alpha \in V$

So, $<T\alpha,\alpha> = <\overline{T\alpha,\alpha}> = <\overline{\alpha,T^*\alpha}> = <T^*\alpha,\alpha>$

$\Rightarrow \quad <T\alpha - T^*\alpha, \alpha> = 0 \ \forall\ \alpha$

$\Rightarrow \quad <(T - T^*)\alpha,\alpha> = 0 \ \forall\ \alpha \in V$ (by previous theorem)

$\Rightarrow \quad T - T^* = \hat{0}$ or $T = T^*$

Example 1: Let V = $V_2(C)$ with standard inner product. Let T be the linear operator defined by

T(1, 0) = (1, -2) and T(0, 1) = (i, -1)

If α = (a, b) ∈ $V_2(C)$, then find $T^*\alpha$

Solution: Obviously B = {(1,0), (0, 1)} is an orthonormal basis of V. Let us find $[T]_B$ i.e.

T(1, 0) = (1, -2) = 1(1, 0) – 2(0, 1)

T(0, 1) = (i, -1) = i(1, 0) – 1(0, 1)

$\therefore \quad [T]_B = \begin{bmatrix} 1 & i \\ -2 & -1 \end{bmatrix} \Rightarrow [T^*]_B = \begin{bmatrix} 1 & -2 \\ -i & -1 \end{bmatrix}$

Now, (a, b) = a(1, 0) + b(0, 1). So coordinate matrix of T^* (a, b) in B is

$= \begin{bmatrix} 1 & -2 \\ -i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a & -2b \\ -ia & -b \end{bmatrix},$

$T^*(a, b) = (a – 2b)(1, 0) + (-ia – b)(0, 1) = (a – 2b, -ia – b)$

Example 2: A linear operator on R^2 is defined by

$$T(x, y) = (x + 2y, x – y)$$

Find the adjoint T^*, if the inner product is standard one.

Solution: Let B = {(1, 0) , (0, 1)} be an orthonormal basis of V, We find $[T]_B$. By given rule.

$$T(1, 0) = (1, 1) \text{ and } t(0, 1) = (2, -1).$$

So $[T]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

The matrix of T^* in the ordered basis B is the transpose of the matrix $[T]_{B.}$

So $[T^*]_B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

The coordinate matrix of $T^*(x, y)$ in the basis B

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2y - y \end{bmatrix}$$

So $T^*(x, y) = (x + y, 2x – y)$

Example 3: Let T be a linear operator on $V_2(C)$ defined by

$$T(1, 0) = (1+ i, 2) ; T(0, 1) = (i, i)$$

Using the standard inner product –

(i) Find the matrix of T^* in the standard ordered basis
(ii) Does T commute with T^* ?

Solution: (i) T(1, 0) = (1 + i, 2) = (1 + i) (1, 0) + 2(0, 1)

$$T(0, 1) = (i, i) = i(1, 0) + i(0, 1)$$

So $[T]_B = \begin{bmatrix} 1 + i & i \\ 2 & i \end{bmatrix}$

Then $[T^*]_B = \begin{bmatrix} 1 - i & 2 \\ -i & -i \end{bmatrix}$

(ii) $[T]_B [T^*]_B = \begin{bmatrix} 1+i & i \\ 2 & i \end{bmatrix} \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix} = \begin{bmatrix} 3 & 3+2i \\ 3-2i & 5 \end{bmatrix}$

$[T^*]_B [T]_B = \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix} \begin{bmatrix} 1+i & i \\ 2 & i \end{bmatrix} = \begin{bmatrix} 6 & 3i+1 \\ -3i+1 & 2 \end{bmatrix}$

Since $[T]_B [T^*]_B \neq [T^*]_B [T]_B$

$\Rightarrow \quad [T T^*]_B \neq [T^* T]_B$

So $\quad T T^* \neq T^* T$

Example 4: Prove that the product of two self-adjoint operators on an inner product space is self-adjoint iff the two operators commute.

Solution: Let T and S be two self-adjoint operators s.t. $\quad T^* = T$ and $S^* = S$

IF PART: Let T and S commute i.e. TS = ST

Now, $\quad (TS)^* = S^* T^*$

$= S T$

$= T S$

So TS is also self-adjoint.

ONLY IF PART: Let ST be self-adjoint

$(ST)^* = ST$

$\Rightarrow \quad T^* S^* = ST$

$\Rightarrow \quad T S = S T$

i.e. S and T commute

Example 5: Let $\forall\ \alpha, \beta \in V$ and T is a linear transformation on V. Also if $f(\alpha) = <\overline{\beta, T\alpha}>, \forall\ \alpha \in V$, then prove that f is a linear functional. Also find a vector β' such that $f(\alpha) = <\alpha, \beta'> \quad \forall\ \alpha \in V$

Solution: (**i**) Given that $\quad f(\alpha) = <\overline{\beta, T\alpha}>, \forall\ \alpha \in V$

So f is a function from V into F. Let a, b $\in$ V and $\alpha_1, \alpha_2 \in V$. Then

$$f(a\alpha_1 + b\,\alpha_2) = < \overline{\beta, T(a\alpha_1 + b\,\alpha_2)} > = < T(a\alpha_1 + b\,\alpha_2), \beta >$$

$$= a < T\alpha_1, \beta > + b < T\alpha_2, \beta >$$

$$= a < \overline{\beta, T\alpha_1} > + < \overline{\beta, T\alpha_2} > = a\, f(\alpha_1) + b\, f(\alpha_2)$$

So f is a linear functional on V.

(**ii**) If V is finite dimensional , then there exists a unique vector β' such that

$$f(\alpha) = < \alpha, \beta' > \quad \forall\, \alpha \in V$$

We have $f(\alpha) = < \overline{\beta, T\alpha} > = < T\alpha, \beta > = < \alpha, T^* \beta > \ \forall\, \alpha$

$\therefore$ if $f(\alpha) = < \alpha, \beta' > \ \forall\, \alpha$, then

$$< \alpha, T^* \beta > = < \alpha, \beta' > \quad \forall\, \alpha$$

Hence $\beta = T^* \beta$

Example 6: Let V be a finite-dimensional inner product space and T be a linear operator on V. If T is invertible, then prove that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Solution: Suppose T is invertible. Then

$$T\,T^{-1} = I$$

$\Rightarrow$ $(T\,T^{-1})^* = I^*$

$\Rightarrow$ $(T^{-1})^*\, T^* = I$ as $I^* = I$

$\Rightarrow$ T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$.

Example 7: Let T be a linear operator on a finite-dimensional inner product space V. Then T is self-adjoint iff its matrix in every orthonormal basis is a self-adjoint matrix.

Solution: Let B be any orthonormal basis for T. Then

$$[T^*]_B = [T]_B^* \qquad \text{.....(1)}$$

IF PART: Let T be self-adjoint i.e. $T = T^*$. Then from (1), $[T]_B = [T]_B^*$ i.e. $[T]_B$ is a self-adjoint matrix.

ONLY IF PART: Let $[T]_B$ be a self-adjoint matrix. Then $[T]_B = [T]_B^*$

$= [T^*]_B$; using eq^n (1)

$\therefore \quad T = T^*$

Example 8: If T is a self-adjoint linear operator on a finite dimensional inner product Space V, then det(T) is real.

Solution: Let B be any orthonormal basis for V. Then

$$[T^*]_B = [T]_B^*$$

But $\quad T^* = T \Rightarrow [T]_B = [T]_B^*$ .....(1)

Let $\quad [T]_B = A \Rightarrow A = A^*$

$\det A = \det(A^*) = \overline{\det(A)} = \det(A)$ is real.

Example 9: If T is self-adjoint, then S^*TS is self-adjoint $\forall$ S. Conversely if S is invertible and S^*TS is self-adjoint, then T is self-adjoint. Prove both results.

Solution: Given that T is self-adjoint, so $T^* = T$. Now $(S^*TS)^* = S^*T^*(S^*)^* = S^*TS$

So S^*TS is self-adjoint. Now, conversely, let S be invertible, then S^* in also invertible. If S^*TS is self-adjoint , then

$$(S^*TS)^* = S^*TS$$

$$\Rightarrow \quad S^*T^*S = S^*TS$$

So $\quad (S^*)^{-1}(S^*T^*S)S^{-1} = (S^*)^{-1}(S^*TS)S^{-1}$

$$\Rightarrow \quad ((S^*)^{-1}S^*)\ T^*(SS^{-1}) = ((S^*)^{-1}S^*)\ T(SS^{-1})$$

$$\Rightarrow \quad I T^* I = I T I$$

$$\Rightarrow \quad T^* = T$$

or T is self-adjoint.

Example 10: Let V be a finite-dimensional inner product space, and T be any linear operator on V. Suppose W is a subspace of V which is invariant under T. Then prove that the orthogonal complement of W is invariant under T^*.

Solution: Given that W is invariant under T.

Claim: $W^\perp$ is invariant under T^*.Let $\beta \in W^\perp$ be arbitrary. Then we shall prove that T^* B is in $W^\perp$ i.e. T^* B is orthogonal to every vector in W. Let $\alpha \in W$. Then

$$< \alpha , T^* \beta > \; = < T\alpha , \beta >$$

$$= 0 \;, \text{ since } \alpha \in W \Rightarrow T\alpha \in W \text{ and } \beta \text{ is orthogonal to every vector in W.}$$

So $T^* \beta$ is orthogonal to every vector $\alpha \in W$

So $T^* \beta$ is in $W^\perp$.

$\Rightarrow$ $W^\perp$ is invariant under T^*.

14.6 *POSITIVE OPERATOR*

Positive operator: A linear operator T on an inner product space V is called positive (in symbols, T > 0), if –

(i) T is self adjoint i.e. $T^* = T$, and
(ii) $< T\alpha , \alpha > \; > 0 \; \forall \; \alpha \neq 0$

If $\alpha = 0$, then $< T\alpha , \alpha > = 0$. Hence if T is positive, then $< T\alpha , \alpha > \; \geq \; 0 \;\; \forall \; \alpha \in V$ and $< T\alpha , \alpha > = 0 \Rightarrow \alpha = 0$.

14.7 *NON-NEGATIVE OPERATOR*

Non-Negative operator: A linear operator T on an inner product space V is called non-negative, if –

(i) It is self-adjoint , and
(ii) $< T\alpha , \alpha > \; \geq \; 0 \;\; \forall \; \alpha \in V$

Note: (1) Every positive operator is also a non-negative operator.

(2) If T is a non-negative operator, then $< T\alpha , \alpha > = 0$, is possible even if $\alpha \neq 0$. So a non-negative operator **may not** be a positive operator

(3) If S and T are two linear operators on an inner product space V, then we define

$$S > T \text{ if } S - T > 0$$

(4) Some authors say a positive operator as '**positive definite**'.

Theorem 9: Let V be an inner product space and T be a linear operator on V. Let 'p' be the function defined on ordered pairs of $\alpha, \beta \in V$ by

$$p(\alpha, \beta) = < T\alpha , \beta >$$

Then the function p is an inner product on V iff T is a positive operator.

Proof: Step I: Let a, b $\in$ F and $\alpha_1 , \alpha_2 \in V$. Then

$$p\,(a\alpha_1 + b\,\alpha_2, \beta) \;=\; < T(a\alpha_1 + b\,\alpha_2) , \beta > = < Ta\alpha_1 + Tb\alpha_2) , \beta >$$

$$= a < T\alpha_1 , \beta > + b < T\alpha_2 , \beta >$$

$$= a\, p(\alpha_1 , \beta) + b\, p(\alpha_2 , \beta)$$

So the function p satisfies linearity property.

Step II: Now the function p will be an inner product on V if and only if

$$p(\alpha, \beta) = \overline{p(\beta,\ \alpha)} \text{ and } p(\alpha, \alpha) > 0, \alpha \neq 0$$

So we have $p(\alpha, \beta) = < T\alpha , \beta >$

$$\Rightarrow \overline{p(\beta,\ \alpha)} = < \overline{T\beta , \alpha} > = < \alpha , T \beta >$$

Also $p(\alpha, \alpha) = < T\alpha , \alpha >$.

Hence the function p will be an inner product on iff

(i) $< T\alpha , \beta > \; = < \alpha , T \beta > \forall\, \alpha, \beta \in V$ i.e. T is self-adjoint.

(ii) $< T\alpha , \alpha > \; > 0$ if $\alpha \neq 0$

Hence the function p will be an inner product on V iff the linear operator T is positive.

Note: Now we shall show that if V is finite-dimensional, then every inner product on V is of the type as discussed in next theorem –

Theorem 10: Let V(F) be a finite-dimensional inner product space with inner product < , > . If p is any inner product on V, there is a unique positive linear operator T on V such that $p(\alpha, \beta) = \langle T\alpha, \beta \rangle \ \forall \ \alpha, \beta \in V$.

Proof: Let $\beta \in V$ be a fixed vector and $f : V \to F$ such that

$$f(\alpha) = p(\alpha, \beta) \ \forall \ \alpha \in V$$

As we have seen, p satisfies linearity property, so f is a linear functional on V. Hence by Riesz representation theorem, there exists a unique vector β' in such that

$$f(\alpha) = \langle \alpha, \beta' \rangle \ \forall \ \alpha \text{ in } V$$

$$\Rightarrow \quad p(\alpha, \beta) = \langle \alpha, \beta' \rangle \ \forall \ \alpha \text{ in } V$$

Let us define $T : V \to V$ such that $T\beta = \beta'$.

So $\quad p(\alpha, \beta) = \langle \alpha, T\beta \rangle \ \forall \ \alpha, \beta \in V \qquad \ldots..(1)$

We also have, $p(\alpha, \beta) = \langle \alpha, T\beta \rangle$

$$p(\alpha, \beta) = \overline{p(\beta, \alpha)}, \text{ by conjugacy property of inner product p}$$

$$= \langle \overline{\beta, T\alpha} \rangle = \langle T\alpha, \beta \rangle$$

Thus, we have, $p(\alpha, \beta) = \langle T\alpha, \beta \rangle \ \forall \ \alpha, \beta \in V \qquad \ldots..(2)$

Linearity of T: Let $\alpha_1, \alpha_2 \in V$ and $a_1, a_2 \in F$. Then for all $r \in V$, we have

$$\langle T(a_1\alpha_1 + a_2\alpha_2), r \rangle = p(a_1\alpha_1 + a_2\alpha_2, r)$$

$$= a_1 p(\alpha_1, r) + a_2 p(\alpha_2, r) \quad \text{, by linearity of p}$$

$$= \langle (a_1T\alpha_1 + a_2T\alpha_2), r \rangle, \text{ by linearity of inner product } \langle , \rangle$$

So, we have, $T(a_1\alpha_1 + a_2\alpha_2) = a_1T\alpha_1 + a_2T\alpha_2$

Hence T is a linear operator. Thus, we have proved the existence of a linear operator T with p(α, β) = < T α , β >. Since p is an inner product, so by previous theorem, T is positive.

Uniqueness: Suppose there are two linear operators T and U such that

$$p(\alpha, \beta) = < T\alpha , \beta > = < U\alpha , \beta > \ \forall\, \alpha, \beta \in V$$

Then $< T\alpha - U\alpha , \beta > = 0 \ \forall\, \alpha, \beta \in V$ …..(3)

Let us keep α fixed. Then from equation (3), we see that the vector T α − U α is orthogonal to every vector β in V.

Therefore $T\alpha - U\alpha = 0, \forall\, \alpha \in V$

$\Rightarrow \quad T\alpha = U\alpha , \forall\, \alpha \in V$

Hence T is unique.

Theorem 11: Let V be a finite-dimensional inner product space and T a linear operator on V. Then t is positive if and only if there is an invertible linear operator U on V such that $T = U^* U$.

Proof: Let $T = U^* U$, where U in an invertible linear operator on V.

Since $T^* = (U^*U)^* = U^*(U^*)^* = U^*U = T$

So T is self-adjoint. Also,

$< T\alpha , \alpha > = < U^*U\alpha , \alpha > = < U\alpha , U^{**} \alpha > = < U\alpha , U\alpha > \geq 0$

Also $< T\alpha , \alpha > = 0 \Rightarrow < U\alpha , U\alpha > = 0 \Rightarrow U\alpha = 0$

$\Rightarrow \quad \alpha = 0$, as U in invertible and V is finite-dimensional, so U is non-singular.

So if $\alpha \neq 0$, then $< T\alpha , \alpha > > 0$

Hence T is positive.

Conversely, suppose T is positive.Then p(α, β) = < T α , β > is an inner product on V. Suppose $\{ \alpha_1 , \ldots, \alpha_n \}$ be a basis for V which is orthonormal with respect to the inner product < , > and let $\{ \beta_1 , \ldots, \beta_n\}$ be a basis orthonormal with respect to the inner product p. So,

$$p(\beta_i , \beta_j) = \delta_{ij} = < \alpha_i , \alpha_j >$$

Now, let U be the unique linear operator on V such that U $\beta_i = \alpha_i$; i = 1, 2, …., n. Obviously U is invertible, because it carries a basis onto a basis. We have

$$p(\beta_i , \beta_j) = < \alpha_i , \alpha_j > = < U\beta_i , U\beta_j >$$

Now let $\alpha, \beta \in V$; such that

$$\alpha = \sum_{i=1}^{n} x_i\beta_i \text{ and } \beta = \sum_{j=1}^{n} y_j\beta_j \text{ . Then}$$

$$< T\alpha , \beta > = p(\alpha, \beta)$$

$$< T\alpha , \beta > = p\left(\sum_{i=1}^{n} x_i\beta_i , \sum_{j=1}^{n} y_j\beta_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j}\ p(\beta_i\beta_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{y_j} < U\beta_i, U\beta_j > = < \sum_{i=1}^{n} x_iU\beta_i , \sum_{j=1}^{n} y_jU\beta_j >$$

$$= < U\sum_{i=1}^{n} x_i\beta_i , U\sum_{j=1}^{n} y_j\beta_j > = < U\alpha, U\beta > = < U^* U\alpha, \beta >$$

Thus $\forall \alpha, \beta \in V$, we have

$$< T\alpha , \beta > = < U^* U\alpha, \beta >$$

$\Rightarrow$ $T = U^* U$

Positive Matrix: Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n over the field of R or C, then A is said to be positive if :

(i) $A^* = A$, and
(ii) $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,\overline{x_i}\ x_j > 0$, where x_1 , …., $x_n \in F$ and not all zero

Principal Minors of a Matrix: Let $A = [a_{ij}]_{n \times n}$ be a an arbitrary field **F**. The principal minors of A are the n scalars defined as –

der $A^{(K)}$ = det $\begin{pmatrix} a_{11} & \cdots & a_{1K} \\ \vdots & \ddots & \vdots \\ a_{K1} & \cdots & a_{KK} \end{pmatrix}$, where K = 1, 2, …, n.

Suppose $A = [a_{ij}]_{n \times n}$ over **R** or **C**. Then A is positive if the principal minors of A are all positive. (Its converse is also true).

Example 1: Which of the following matrices are positive –

(i) $\begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

Solution: (i) Here obviously $A^* = A$. So A is self-adjoint. Now principal minors of A are 1 and

$\begin{vmatrix} 1 & 1+i \\ 1-i & 3 \end{vmatrix}$ i.e. 1 and 1.

So both the principal minors of A are +ve . Hence A is a +ve matrix.

(ii) It is not self-adjoint. Hence it is not positive.

(iii) Here $A^* = A$. Also all the principal minors viz 1,

$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$ and $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{vmatrix}$ are positive (**verify**). Hence A is positive.

Example 2: Prove that every entry on the main diagonal of a positive matrix is positive.

Solution: Let $A = [a_{ij}]_{n \times n}$ be a positive matrix. So

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \overline{x_i}\, x_j > 0, \qquad \text{….(1)}$$

where x_1 , …., x_n are any n scalars (not all zero). Now suppose that out of n scalars x_1 , …., x_n , we take $x_i = 1$ and each of the remaining (n – 1) scalars is taken as zero. Then from equation (1), we conclude that $a_{ii} > 0$ $\forall$ i. Hence each entry on the main diagonal of a +ve matrix is positive.

14.8 *UNITARY OPERATOR*

Definition: In a inner product space *V*, let *T* be a linear operator. Then the operator *T* is called unitary operator if adjoint T^* of T exist and $TT^* = T^*T = I$

Note 1: In a finite dimensional inner product space T is unitary *iff* $T^*T = I$

2: A linear operator T on a finite dimensional inner product space V is unitary iff T preserve inner product.

14.9 *NORMAL OPERATOR*

In this section we will learn about the important topic in inner product space.

Definition: Let in a inner product space *V*, *T* be a linear operator. Then the operator *T* is called normal operator or normal if it commutes with its adjoint *i.e.*, $TT^* = T^*T$.

Note 1: If vector space is of finite dimensional then T^* will definitely exist.

2: If vector space is not of finite dimensional then definition will make sense only if T possesses adjoint.

Theorem 12: Every self-adjoint operator is normal.

Proof: Let we consider *T* be a self-adjoint operator then obviously, $T^* = T$.

Therefore, we can say that $TT^* = T^*T$,

Hence *T* is normal

Theorem 13: Every unitary operator is normal.

Proof: Let we consider *T* be a unitary operator then obviously, $TT^* = T^*T = I$

Therefore, we can say that $TT^* = T^*T$,

Hence *T* is normal.

Theorem 14: Let in a inner product space *V*, *T* be a normal operator. Then a necessary and sufficient condition that α be a characteristic vector of *T* is that it be a characteristic vector of T^*.

Proof: Let us consider T be a normal operator on an inner product space V. If $\alpha \in V$, then we have,

$$\|T(\alpha)\|^2 = (T\alpha, T\alpha) = (\alpha, T^*T\alpha) = (\alpha, TT^*\alpha)$$

$$= (T^*\alpha, T^*\alpha) = \|T^*(\alpha)\|$$

Since *T* is normal and if $\alpha \in V$,

$$\|T\alpha\| = \|T^*\alpha\| \qquad \ldots\ldots\ldots.. (1)$$

If *c* be scalar, then (1) can be written as

$$(T - cI)^* = T^* - \bar{c}I^* = T^* - \bar{c}I$$

Now we have to show, $T - cI$ is normal.

We have, $(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c}I)$

$$= TT^* - \bar{c}T - cT^* + c\bar{c}I$$

Also $(T - cI)^*(T - cI) = (T^* - \bar{c}I)(T - cI)$

$$= T^*T - \bar{c}T - cT^* + c\bar{c}I$$

As we know *T* is normal. So,

$$(T - cI)(T - cI)^* = (T - cI)^*(T - cI)$$

Thus, $(T - cI)$ is normal. Now from (1),

$$\|(T - cI)(\alpha)\| = \|(T - cI)^*(\alpha)\| \ \forall \alpha \in V$$

$\Rightarrow \|(T-cI)\alpha\| = \|(T^* - \bar{c}I)^*(\alpha)\| \ \forall \alpha \in V$ ………. (2)

By equation (2) we can say that,

$\Rightarrow (T-cI)\alpha = 0 \text{ iff } (T^* - \bar{c}I)\alpha = 0$

i.e., $T(\alpha) = c\alpha \text{ iff } T^*\alpha = \bar{c}\alpha$

Thus, we can say that α is a eigen vector of T corresponding to the eigen value c if and only if it is a characteristic vector of T^* corresponding to the eigen value $\bar{c}$.

Remark 1: The characteristic vector for *T* belonging to distinct characteristic values is orthogonal if *T* is a normal operator on an inner product space *V*.

2: In a normal operator's characteristic spaces are pairwise orthogonal to each other.

Definition (Normal matrix): A square order complex matrix A is called normal if,

$$AA^* = A^*A.$$

If matrix is diagonal matrix D, then obviously

$$DD^* = D^*D$$

Remark 1: A unitarily equivalent to a diagonal matrix *iff* matrix is normal.

Solved example

Example 1: If in a inner product space *V*, *T* be a normal operator. Then *cT* is also a normal operator for any scalar *c*.

Proof: We have given that *T* be a normal operator i.e., $TT^* = T^*T$

Since, $(cT)^* = \bar{c}T^*$

Now, $(cT)(cT)^* = (cT)(\bar{c}T^*) = c\bar{c}(TT^*)$

Again, $(cT)^*(cT) = (\bar{c}T^*)(cT) = (\bar{c}c)(T^*T)$

Thus we can say, $(cT)(cT)^* = (cT)^*(cT)$

Hence, cT is normal.

Example 2: In a inner product space *V*, if T_1, T_2 are normal operator with the property that either commutes with the adjoint of other, then prove that $T_1 + T_2$ and T_1T_2 are also normal operator.

Solution: We have given T_1, T_2 are normal. Therefore,

$T_1T_1^* = T_1^*T_1$ and $T_2T_2^* = T_2^*T_2$

According to question it is given that,

$T_1T_2^* = T_2^*T_1$ and $T_2T_1^* = T_1^*T_2$

Now, $(T_1 + T_2)(T_1 + T_2)^* = (T_1 + T_2)(T_1^* + T_2^*)$

$$= T_1T_1^* + T_1T_2^* + T_2T_1^* + T_2T_2^*$$

$$= T_1T_1^* + T_2^*T_1 + T_1^*T_2 + T_2T_2^*$$

$$= T_1^*(T_1 + T_2) + T_2^*(T_1 + T_2) = (T_1^* + T_2^*)(T_1 + T_2)$$

$$= (T_1 + T_2)^*(T_1 + T_2)$$

Thus, $T_1 + T_2$ is normal.

Now, $(T_1T_2)(T_1T_2)^* = T_1T_2T_2^*T_1^* = T_1(T_2T_2^*)T_1^*$

$$= T_1(T_2^*T_2)T_1^*$$

$$= (T_1T_2^*)(T_2T_1^*)$$

$$= (T_2^*T_1)(T_1^*T_2)$$

$$= T_2^*(T_1T_1^*)T_2$$

$$= T_2^*(T_1^*T_1)T_2$$

$$= (T_2^*T_1^*)(T_1T_2) = (T_1T_2)^*(T_1T_2)$$

Thus, T_1T_2 is normal.

Example 3: In a finite dimensional complex inner product space let T be the linear operator. Show that T is normal if and only if its real and imaginary parts commute.

Solution: Let $T = T_1 + iT_2$. Then $T_1^* = T_1$ and $T_2^* = T_2$. Let we assume that $T_1T_2 = T_2T_1$ then we have to prove that T is normal.

We have, $T^* = (T_1 + iT_2)^* = T_1^* + \bar{i}\,T_2^* = T_1 - iT_2$

$\therefore TT^* = (T_1 + iT_2)(T_1 - iT_2) = T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 = T_1^2 + T_2^2$ $\quad [\because T_1T_2 = T_2T_1]$

Also, $T^*T = (T_1 - iT_2)(T_1 + iT_2) = T_1^2 + iT_1T_2 - iT_2T_1 + T_2^2 = T_1^2 + T_2^2$

$\therefore TT^* = T^*T$. Hence T is normal.

Conversely, we assume that T is normal then we have to prove that $TT^* = T^*T$.

$\Rightarrow T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 = T_1^2 + iT_1T_2 - iT_2T_1 + T_2^2$

$\Rightarrow 2i(T_1T_2 - T_2T_1) = 0$

$\Rightarrow T_1T_2 - T_2T_1 = 0 \quad [\because 2i \neq 0]$

$\Rightarrow T_1T_2 = T_2T_1$

Check your progress

Problem 1: In a finite dimensional complex inner product space let T be the linear operator. Show that T is normal if and only if its real and imaginary parts commute.

Solution: Let $T = T_1 + iT_2$. Then $T_1^* = T_1$ and $T_2^* = T_2$. Let we assume that $T_1T_2 = T_2T_1$ then we have to prove that T is normal.

We have, $T^* = (T_1 + iT_2)^* = T_1^* + \bar{i}\,T_2^* = T_1 - iT_2$

$\therefore TT^* = (T_1 + iT_2)(T_1 - iT_2) = T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 = T_1^2 + T_2^2$ $\quad [\because T_1T_2 = T_2T_1]$

Also, $T^*T = (T_1 - iT_2)(T_1 + iT_2) = T_1^2 + iT_1T_2 - iT_2T_1 + T_2^2 = T_1^2 + T_2^2$

$\therefore TT^* = T^*T$. Hence T is normal.

Conversely, we assume that T is normal then we have to prove that $TT^* = T^*T$.

$\Rightarrow T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 = T_1^2 + iT_1T_2 - iT_2T_1 + T_2^2$

$\Rightarrow 2i(T_1T_2 - T_2T_1) = 0$

$\Rightarrow T_1T_2 - T_2T_1 = 0 \qquad [\because 2i \neq 0]$

$\Rightarrow T_1T_2 = T_2T_1$

Problem 2: Let S and T be two positive linear operators on an inner product space V. Then prove that S + T is also positive operator.

Solution: Given $S^* = S$ and $T^* = T$

So $(S + T)^* = S^* + T^* = S + T$

So S+ T is self adjoint.

Also, if $\alpha \in V$, then

$< (S+T)\alpha, \alpha > = < S\alpha + T\alpha, \alpha > = < S\alpha, \alpha > + < T\alpha, \alpha >$

But S and T are positive. So $< S\alpha, \alpha > > 0$ and $< T\alpha, \alpha > > 0$.

$\Rightarrow < (S+T)\alpha, \alpha > > 0$.

Hence S + T is positive.

Problem 3: Let V be a finite-dimensional inner product space and T be a self- adjoint linear operator on V. Prove that the range of T is the orthogonal complement of the null space of T i.e. $R(T) = [N(T)]^{\perp}$.

Solution: Let $\alpha \in R(T)$. Then $\exists$ a vector $\beta \in V$ such that $\alpha = T\beta$. Let r be an arbitrary vector of $[N(T)]^{\perp}$. Then Tr = 0

We have

$< \alpha, r > = < T\beta, r > = < \beta, T^*r > = < \beta, Tr >$ as $T^* = T$

$= < \beta, 0 > = 0$

Thus $<\alpha, r> = 0 \quad \forall \; r \in N(T)$

So, $\alpha \in [N(T)]^{\perp} \Rightarrow R(T) \subseteq [N(T)]^{\perp}$ …..(1)

Since $V = N(T) \oplus [N(T)]^{\perp}$

$\Rightarrow \quad \dim V = \dim N(T) + \dim [N(T)]^{\perp}$ …..(2)

By **Rank- nullity theorem**, we have

$$\dim V = \dim R(T) + \dim N(T) \qquad …..(3)$$

So we conclude that $\dim R(T) = \dim [N(T)]^{\perp}$ …..(4)

From equation (1) and (4) , we conclude that

$$R(T) = [N(T)]^{\perp}$$

14.10 *SUMMARY*

In this unit we have learned about the most essential tool name as operators used in inner product space like adjoint operator, self-adjoint operator, skew-symmetric operator, positive operator, unitary operator and normal operator. Mostly, the uses of these operators to solve out the matrix problems. Other important concepts introduced in this unit were:

- Every self-adjoint operator is normal.
- Every unitary operator is normal
- The operation of adjoint behaves like the operation of conjugation on complex numbers
- Every positive operator is also a non-negative operator

14.11 *GLOSSARY*

- Unitary operator
- Normal operator
- Adjoint operator
- Self-adjoint operator
- Skew-symmetric or Hermitian operator.

14.12 *REFERENCES*

- S Kumaresan; Linear Algebra-A Geometric Approach; PHI-2016.
- K Hoffman & Ray Kunze; Linear Algebra; PHI-1971.

- P Halmos; Finite-Dimensional Vector Spaces;
- G Strang; Introduction To Linear Algebra;

14.13 *SUGGESTED READING*

- NPTEL videos.
- Schaum series.
- A R Vashishtha, Krishna Prakashan; Meerut.
- Graduate Text In Mathematics, Springer.

14.14 *TERMINAL QUESTION*

Long answer type question

1: Let S and T be linear operators on an inner product space V and c $\in$ **F**. If S and T possess adjoints, then prove that the operators S + T, cT, ST, T^* will possess adjoints.

2: Prove that Every linear operator T on a finite dimensional complex inner product space V can be uniquely expressed as

$T = T_1 + iT_2$, where T_1 & T_2 are self-adjoint linear operators on V.

3: Prove that every linear operator T on a finite-dimensional inner product space V can be uniquely expressed as $T = T_1 + T_2$, where T_1 is self-adjoint and T_2 is skew.

4: Prove that the necessary and sufficient condition that a linear transformation T on a unitary space (of any dimension) be self-adjoint (Hermitian) is that,

$$< T\alpha , \alpha > \text{ be real } \forall \alpha \in V$$

Short answer type question

1: Let V be a finite-dimensional inner product space and let B = { α_1 , …., α_n } be an ordered orthonormal basis for V . Let T be a linear operator on V and let $A = [a_{ij}]_{m \times n}$ be the matrix of T with respect to the ordered basis B. Then prove that $a_{ij} = < T\alpha_j, \alpha_i >$.

2: In any orthonormal basis for *V* and *T* be the linear operator on *V*, then prove that the matrix of T^* is the conjugate transpose of the matrix of *T*.

3: Prove that the necessary and sufficient condition that a linear transformation T on an inner product space V be $\hat{0}$ is that $< T\alpha, \beta > = 0, \forall \alpha, \beta \in V$

4: Prove that the necessary and sufficient condition that a linear transformation T on a unitary space be $\hat{0}$ is that $< T\alpha, \alpha > = 0 \quad \forall \alpha \in V$

5: A linear operator on R^2 is defined by

$$T(x_1, y_1) = (x_1 + 2y_1, x_1 - y_1)$$

Find the adjoint T^*, if the inner product is standard one.

6: Prove that the product of two self-adjoint operators on an inner product space is self-adjoint iff the two operators commute.

7: If T is self-adjoint, then $S^* TS$ is self-adjoint $\forall$ S. Conversely if S is invertible and $S^* TS$ is self-adjoint, then T is self-adjoint. Prove both results.

8: Prove that characteristic of normal operator are pair-wise orthogonal.

9: Prove that each self-adjoint and unitary operaor are normal operator

10: If in a inner product space *V*, *T* be a normal operator. Then prove that *cT* is also a normal operator for any scalar *c*.

11: If in a finite dimensional vector space *V*, *T* be a linear operator. If $\|T\alpha\| = \|T^*\alpha\| \ \forall \alpha \in V$

Fill in the blanks

1: $(S + T)^*$ = …………..

2: A linear operator T on an inner product space V is said to be self-adjoint if ………..

3: A linear T is called *skew-symmetric* or *skew-Hermitian* according as the vector space V is ……………………

4: A necessary and sufficient condition that a linear transformation T on a unitary space be $\hat{0}$ is that ………………..

14.15 *ANSWERS*

Answer of short question

5: $[T^*]_B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

Answer of fill in the blanks

1: $S^* + T^*$ **2:** $T^* = T$ **3:** Real or Complex

4: $< T\alpha, \alpha > = 0 \quad \forall \ \alpha \in V$

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