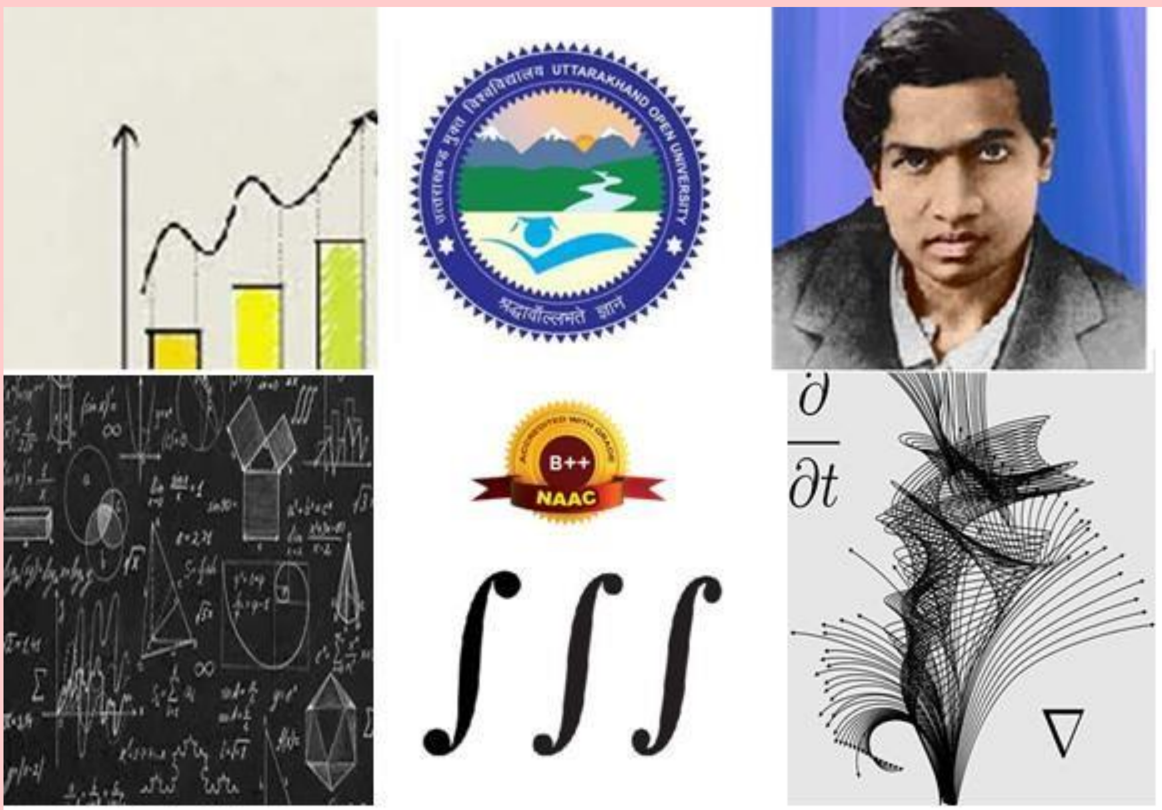
Bachelor of Science
(SIXTH SEMESTER)

MT(N)-223
Linear Programming and Game Theory



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: LINER PROGRAMMING AND GAME THEORY

COURSE CODE:MT(N)-223

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India, 263139

Linear Programming and game Theory **Course Code -MT(N)-223**

Editor

Dr. Deepa Bisht
Assistant Professor (A.C)
Department of Mathematics
Uttarakhand Open University
Haldwani, Uttarakhand

S.NO	Name of unit writer	Block	Unit
1	Dr. Kamlesh Bisht Assistant Professor (A.C) Department of Mathematics Uttarakhand Open University Haldwani	I,II	2,3,4,5
2	Dr. Deepa Bisht Assistant Professor (A.C) Department of Mathematics Uttarakhand Open University Haldwani	I, IV	1,10,11
3	Dr. Deepak Kumar Sharma Assistant Professor (A.C) Department of Mathematics Uttarakhand Open University Haldwani	II, III	6,7,8,9
4	Dr. Jyoti Rani Assistant Professor Department of Mathematics Uttarakhand Open University Haldwani	V	12,13,14

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CONTENTS

Linear Programming and Game theory

COURSE INFORMATION

The present self-learning material **"Linear Programming and Game Theory"** has been designed for B.Sc. (Sixth Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Three Blocks.

First block is Linear Programming Problem. In **t**he **first unit** of this block, we will study Introduction and formulation of Linear Programming Problems, Components of an LPP: decision variables, objective function, constraints, non-negativity, Types of LPP: maximization, minimization, Real-life applications in industry, economics, management, Feasible, infeasible, unbounded and optimal solutions

In **unit second,** solving two-variable LPP using graphical approach, plotting constraints and identifying feasible region, Corner point principle and extreme points. **Third unit** focuses on the Definition of convex sets and convex combinations, Polyhedral sets: intersections of half-spaces, Hyperplanes, half-spaces and separation theorem, Extreme points and their significance, Basic solutions and Basic Feasible Solutions (BFS), Correspondence between BFS and extreme points.

Second block is Simplex Method: **Fourth unit** of this block we will study the Introduction to Simplex algorithm. Tableau format and pivot operations, Improving a BFS and optimality condition. **Fifth unit is** Need for artificial variables in ≥ and = constraints, Big-M method (penalty method), Two-phase method (Phase I: feasibility, Phase II: optimality), Comparison of Big-M vs Two-Phase, Solving LPP containing equality and ≥ type constraints. **Sixth unit** will examine causes of degeneracy, Effects of degeneracy on simplex iterations, Cycling phenomenon. Seventh unit is Motivation for revised simplex in large LPPs, Basis matrix, inverse of basis.

Third block is Duality**. The Eigth unit** of this block is Formulation of the dual problem Primal–dual relationships, Weak and strong duality theorems. **Ninth unit** of this block will Concept and need for Dual Simplex Method, Conditions for feasibility and optimality in dual-simplex. **Fouth Block** is Sensitivity Analysis, Linear and Integer Programming. The **tenth** unit of this block is post-optimality analysis of LPP, Changes in objective function coefficients (cost vector), Changes in right-hand side vector (resource availability). **Eleventh unit is** concept of parametric programming, Variations in cost coefficients and RHS parameters, Graphical and simplex-based parametric analysis. **Fifth block is** Application. **Twelve unit** of this block is Definition and mathematical formulation, Balanced and unbalanced assignment models, Hungarian method for optimal assignment, Maximization and minimization cases. **In thirteen** Unit we define the structure and formulation of transportation models. In **Fourteen and last unit** of this book is Game theory.

Course Name: Linear Programming and Game theory **Credi**t-04
Course Code: MT(N)- 223

SYLLABUS

Linear Programming and Game theory: Linear Programming Problem, Convexity and Basic Feasible Solutions Formulation, Canonical and standard forms,

Graphical method; Convex and polyhedral sets, Hyperplanes, Extreme points; Basic solutions, Basic Feasible Solutions, Reduction of feasible solution to basic feasible solution, Correspondence between basic feasible solutions and extreme points.

Simplex Method: Optimality criterion, Improving a basic feasible solution, Unboundedness, Unique and alternate optimal solutions; Simplex algorithm and its tableau format; Artificial variables, Two-phase method, Big-M method.

Duality: Formulation of the dual problem, Duality theorems, Complimentary slackness theorem, Economic interpretation of the dual, Dual-simplex method.

Sensitivity Analysis: Changes in the cost vector, right-hand side vector and the constraint matrix of the linear programming problem.

Applications: Transportation Problem: Definition and formulation, Methods of finding initial basic feasible solutions: Northwest-corner rule, Least- cost method, Vogel approximation method; Algorithm for obtaining optimal solution. Assignment Problem: Mathematical formulation and Hungarian method.

BLOCK I: LINEAR PROGRAMMING PROBLEM, CONVEXITY AND BASIC FEASIBLE SOLUTIONS

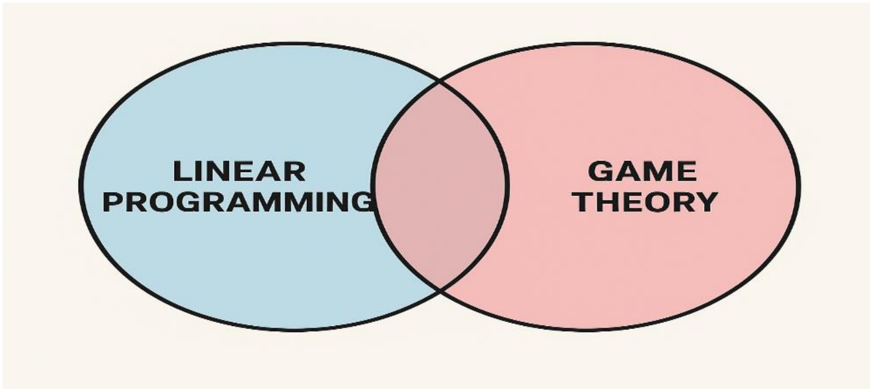
UNIT I- LINEAR PROGRAMMING PROBLEM AND GAME THEORY

CONTENTS:

1.1 INTRODUCTION

In today's era of limited resources and increasing competition, decision making has become an important part of every business, industry and economic system. Two powerful mathematical tools that aid scientific and rational decision-making are linear programming (LP) and game theory. Both aim to provide optimal solutions, but they differ in the types of decision problems they solve. Linear Programming focuses on optimization of resources, whereas Game Theory deals with competitive situations involving two or more decision-makers.

Linear programming (LP) is widely used for optimizing specific types of problems. In 1947, George Bernard Dantzig developed the simplex algorithm, a highly effective method for solving linear programming



The diagram shows that although Linear Programming and Game Theory are different fields, some game-theoretic problems can be solved using linear programming techniques.

problems (LPP). Since then, LP has been applied in diverse industries such as banking, education, forestry, petroleum, manufacturing, and trucking. The primary challenge in these fields often involves distributing limited resources among various activities in the most optimal manner. Real-world scenarios where LP is applicable vary widely, including everything from assigning production facilities to products to allocating national resources for domestic needs, from portfolio selection to determining shipping patterns, and beyond. This unit will cover the mathematical formulation of LPP, the graphical method for solving two-variable LPP, as well as the simplex algorithm, duality, dual simplex, and revised simplex methods for solving LPP with any number of variables.

George Bernard Dantzig (8 November 1914 - 13 May 2005)
https://en.wikipedia.org/wiki/George_Dantzig

1.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of Linear Programming and how it is used to solve real-life problems.
2. To Define and formulate Objective Function and Constraints, and convert a problem into an LPP model.
3. To understand basic principles like Feasible Region, Feasible Solution, Optimal Solution. We will learn to solve two-variable Linear Programming Problem using Graphical Method.
4. To solve real-world situations like profit maximization and cost minimization problems through LPP.
5. To analyse conflict and cooperation among competing players.
6. To develop strategies that yield the best possible results.
7. To study decision-making where the outcome is influenced by others' actions.

1.3 LINEAR PROGRAMMING PROBLEM

Definition Of Linear Programming Problem: Linear Programming (LP) is a mathematical technique used for the optimal utilization of limited resources such as manpower, money, materials, machines, and time. It helps to determine the best possible outcome—maximum profit or minimum cost—under given constraints. This technique is widely used in business, economics, engineering, agriculture, transportation, and military operations.

Meaning Of Game Theory: Game Theory is a branch of applied mathematics that deals with strategic decision-making in competitive situations. It studies the behavior and actions of two or more rational players whose decisions affect each other's outcomes. The central idea is to determine the best possible strategy to win or gain maximum advantage in a competitive scenario.

GENERAL LINEAR PROGRAMMING PROBLEM

Mathematically general linear programming can be represented as follows:
Maximize (Or minimize) Z $= c_1x_1 + c_2x_2 + \cdots + c_nx_n$ Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n (\leq, =, \geq) b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n (\leq, =, \geq) b_2$$

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n (\leq, =, \geq) b_i$
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n (\leq, =, \geq) b_m$
and $x_1, x_2, x_3 \ldots, xn \geq 0$

The above programming problem can be rewritten in compact form as,
Maximize (Or Minimize) Z = $\sum_{j=1}^{n} c_j x_j$

Subject to,

$$\sum_{j=1}^{n} a_{ij}x_j (\leq, =, \geq) b_i; i = 1,2, \ldots, m \qquad (2)$$

$$x_j \geq 0; j = 1,2, \ldots, n \qquad (3)$$

The objective is to determine the values of x_j that optimize (maximize or minimize) the objective function (1). These values must satisfy to the constraints (2) as well as non-negativity restrictions (3). In this context, the coefficients c_j are termed as cost coefficients, while a_{ij} represents technological coefficients; a_{ij} denotes the quantity of the i^{th} resource utilized per unit of variable xj, and bi signifies the overall availability of the i^{th} resource.

Example 1: An oil company possesses two refineries - refinery A and refinery B. Refinery A can produce 20 barrels of petrol and 25 barrels of diesel daily, while refinery B can produce 40 barrels of petrol and 20 barrels of diesel per day. The company has a minimum requirement of 1000 barrels of petrol and 800 barrels of diesel. Operating refinery A costs Rs. 300 per day and refinery B costs Rs. 500 per day. How many days should each refinery be operated to minimize costs? Formulate this scenario as a linear programming model.

Solution: To formulate this problem as a linear programming model, let's define our decision variables:

Let x be the number of days refinery A is operated.
Let y be the number of days refinery B is operated.
Our objective is to minimize costs, so we want to minimize the total operating cost: Minimize: 300 x + 500 y Subject to the constraints:

1. Refinery A produces 20 barrels of petrol per day, and refinery B produces 40 barrels. The total petrol production should be at least 1000 barrels: $20x + 40y \geq 1000$
2. Refinery A produces 25 barrels of diesel per day, and refinery B produces 20 barrels. The total diesel production should be at least 800 barrels: $25x + 20y \geq 800$
3. Non-negativity constraints: $x \geq 0, y \geq 0$

This linear programming model represents the problem of minimizing costs while meeting the production requirements for petrol and diesel.

OR

$$\text{Minimize } Z = 300x + 500y$$
$$\text{Subject to,}$$
$$20x + 40y \geq 1000$$
$$25x + 20y \geq 800$$
$$x, y \geq 0$$

Example 2: In a particular factory, three machines, namely $\mathrm{M}_1, \mathrm{M}_2$, and M_3, are utilized in the manufacturing process of two products, P_1 and P_2. Machine M_1 is occupied for 5 minutes for producing one unit of P_1, while M_2 is used for 3 minutes and M_3 for 4 minutes. For one unit of P_2, the time requirements are 1 minute for M_1, 4 minutes for M_2, and 3 minutes for M_3. The profit earned per unit is Rs. 30 for P_1 and Rs. 20 for P_2, regardless of whether the machines operate at full capacity. How can we determine the production plan that maximizes profit? Frame this problem as a linear programming challenge.

Solution: To formulate this problem as a linear programming problem, let's define our decision variables:

Let x_1 be the number of units of product P_1 produced. Let x_2 be the number of units of product P_2 produced.

Our objective is to maximize profit, so we want to maximize the total profit:
Maximize: $30x_1 + 20x_2$

Subject to the constraints:

1. Time constraint for machine M_1: $15x_1 + x_2 \leq T_1$

Where T_1 is the total available time on machine M_1.
2. Time constraint for machine M_2: $3x_1 + 4x_2 \leq T_2$

Where T_2 is the total available time on machine M_2.
3. Time constraint for machine M_3: $4x_1 + 3x_2 \leq T_3$

Where T_3 is the total available time on machine M_3.
4. Non-negativity constraints: $x_1 \geq 0, x_2 \geq 0$

Note: Here, we can take total available time for all machines is 60 i.e., $T_1 = T_2 = T_3 = 60$

This linear programming model represents the problem of determining the production plan that yields the highest profit while considering the time constraints on each machine.

OR
Maximize $Z = 30x_1 + 20x_2$
Subject to,

$5x_1 + x_2 \leq 60$

$3x_1 + 4x_2 \leq 60$
$4x_1 + 3x_2 \leq 60$

$x_1, x_2 \geq 0$

1.4 CONVEXITY AND BASIC FEASIBLE SOLUTION FORMULATION

So far we have derived geometrical properties from simple graphical examples of two dimensions. Now, shall derive these properties, mathematically, for the general linear programming problem. In this chapter, h_i shall draw the conclusion that all the properties that hold true for simple problem (of two or three variables) also hold true for the general linear programming problem (of n variables), if we think of it as being represented graphically in an n-dimensional space.

First, we shall introduce a few important definitions and give proper names to the concepts that we have been using in our discussion. The main topic of this chapter is convex set theory. Recently, however, the theory has found many important applications in linear programming, games theory, economic and statistical decision theory.

An optimal as well as feasible solution to an LP problem is obtained by choosing among several values of decision variables $x_1, x_2, \dots x_n$ the one set of values that satisfy the given set of constraints simultaneously and also provide the optimal (most suitable) value of the given objective function.

Solution having values of decision variables $x_j (j = 1,2, \dots, n)$ which satisfy the constraints of a general LP model is called the solution to that LP model.

Feasible Solution: Solution values of decision variables $x_j (j = 1,2, \dots, n)$ which satisfy the constraints and non-negativity conditions of a general LP model are said to constitute the feasible solution to that LP model.

Basic Solution: For a set of m equations in n variables $(n > m)$, a solution obtained by setting $(n - m)$ variables equal to zero and solving for remaining m equations in m variables is called a basic solution.

The $(n - m)$ variables whose value did not appear in this solution are called non-basic variables and the remaining m variables are called basic variabels.

While obtaining the optimal solution to the LP problem by the graphical method, the statement of the following theorems of linear programming is used

(i) The collection of all feasible solutions to an LP problem constitutes a convex set whose extreme points correspond to the basic feasible solution.
(ii) There are finite number of basic feasible solutions within the feasible solution space.
(iii) If the convex of the feasible solutions of the system $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$, is a convex polyhedron, then at leas one of the extreme points gives an optimal solution.

1.5 CANONICAL FORMS

The general linear programming problem discussed above can always be put in the following form, called the canonical form:

$$\text{Maximize} Z = \sum_{j=1}^{n} c_j x_j$$

subject to
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1,2, \ldots, m,$$
$$x_j \geq 0, \quad j = 1,2, \ldots, n.$$

The characteristics of this form are
(a) all decision variables are non-negative,
(b) all constraints are of the ($\leq$) type, and
(c) objective function is of maximization type.

Any linear programming problem can be put in the canonical form by the

1. The minimization of a function, $f(x)$, is equivalent to the maximization use of some elementary transformations. of the negative expression of this function, - $f(x)$. For example, the linear objective function

$$\text{minimize} Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

is equivalent to

$$\text{maximize} G = -\text{Z} = -c_1 x_1 - c_2 x_2 - \cdots - c_n x_n,$$

with $\text{Z} = -\text{G}$. Therefore, for all linear programming problems the objective function can be expressed in the maximization form.\

2. An inequality in one direction ($\leq$ or $\geq$) can be changed to an inequality in the opposite direction ($\geq$ or $\leq$) by multiplying both sides of the inequality by -1. For example, the linear constraint

$$a_1 x_1 + a_2 x_2 \geq b$$

is equivalent to

$$-a_1 x_1 - a_2 x_2 \leq -b.$$

Also

$$p_1 x_1 + p_2 x_2 \leq q$$

is equivalent to

$$-p_1 x_1 - p_2 x_2 \geq -q.$$

3. An equation may be replaced by two weak inequalities in opposite directions. For example, $a_1x_1 + a_2x_2 = b$ is equivalent to the two simultaneous constraints

$$a_1x_1 + a_2x_2 \leq b \quad \text{and} \quad a_1x_1 + a_2x_2 \geq b$$
$$\text{or} \quad a_1x_1 + a_2x_2 \leq b \quad \text{and} \quad -a_1x_1 - a_2x_2 \leq -b.$$

4. So far, we have assumed the decision variables $x_1, x_2, \dots, x_n$ to be all non-negative. It is possible, in actual practice, that a variable may be unconstrained in sign, i.e., it may be positive or negative (it may vary from $-$- to +). If a variable is unconstrained, it is expressed as the difference between two nonnegative variables. For example, if x is an unconstrained variable, then it can be expressed as

$$x = x' - x'', \text{ where } x' \geq 0 \text{ and } x'' \geq 0.$$

1.6 STANDARD FORMS

The characteristics of the standard form are

1. All the constraints are expressed in the form of equations, except the non-negativity constraints which remain inequalities (≥ 0).
2. The right-hand side of each constraint equation is non-negative.
3. All the decision variables are non-negative.
4. The objective function is of the maximization or minimization type.

The inequality constraints are changed to equality constraints by adding or subtracting a non-negative variable from the left-hand sides of such constraints. These new variables are called slack variables or simply slacks. They are added if the constraints are ($\leq$) and subtracted if the constraints are ($\geq$). Since in the case of ($\geq$) constraints the subtracted variable represents the surplus of left-hand side over right-hand side, it is commonly known as surplus variable and is, in fact, a negative slack. In our discussion, however, we shall always use the name "slack" variable and its sign will depend on the inequality sign in the constraint. Both decision variables and slack variables are called admissible variables and are treated in the same manner while finding a solution to a problem.

For example, the constraint

$$a_1x_1 + a_2x_2 \leq \dot{o}, \dot{v} \leq \dot{v}$$

is changed in the standard form to $a_1x_1 + a_2x_2 + s_1 = b$, where $s_1 \geq 0$. Also, constraint

$$p_1x_1 + p_2x_2 \geq q, q \geq 0$$

is changed to $p_1x_1 + p_2x_2 - s_2 = q$, where $s_2 \geq 0$. The quantities s_1 and s_2 are variables and their values depend upon the values. assumed by other x 's in a particular equation.

Before trying for the solution of the linear programming problem, it must be expressed in the standard form. The information given by the standard form is then expressed in the "table form" or "matrix form".

Let us consider the general linear programming problem

$$\text{maximize} Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i, (b_i \geq 0), \quad i = 1,2,3, \ldots, m,$$

$$x_i \geq 0, \quad j = 1,2,3, \ldots, n.$$

This is expressed in the standard form as

$$\text{maximize} Z = \sum_{j=1}^{n} c_j x_j$$

$j = 1$

n

subject to

Such an L.P. problem formed after the introduction of slack or surplus variables is called reformulated L.P. problem.

Now, solving the L.P. problem means determining the set of nonnegative values of variables x_j and s_i which will maximize Z while satisfying the constraint equations. The concept is simple but we have a set of m equations with $(m + n)$ unknowns and an infinite number of solutions is possible. Clearly, a hit and trial method for finding the optimal solution is not feasible. There is a definite need for an efficient and systematic procedure which will yield the desired solution in a finite number of trials. An iterative procedure called simplex technique helps us to reach the optimal solution (if it exists) in a finite number of iterations.

Examples

Express the following linear programming problem in the standard form:

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3,$$
$$\text{subject to } 2x_1 - 3x_2 \leq 3,$$
$$x_1 + 2x_2 + 3x_3 \geq 5,$$
$$3x_1 + 2x_3 \leq 2,$$
$$x_1 \geq 0, x_2 \geq 0.$$

Solution. Here x_1 and x_2 are restricted to be non-negative, while x_3 is unrestricted

Introducing slack variables, the standard form is where $x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, s_1 \geq 0, s_2 \geq 0$ and $s_3 \geq 0$.

$$\text{maximize Z} = 3x_1 + 2x_2 + 5x_3' - 5x''{}_3,$$
$$\text{subject to } 2x_1 - 3x_2 + s_1 = 3,$$
$$x_1 + 2x_2 + 3x_3' - 3x_3'' - s_2 = 5,$$
$$3x_1 + 2x_3' - 2x_3'' + s_3 = 2,$$

CHECK YOUE PROGRESS

1. Linear Programming is a technique used for finding the maximum or minimum value of a linear function subject to certain constraints.
2. The objective function in an LPP is always non-linear.
3. In LPP, all constraints must be linear inequalities or equations.
4. The feasible region of an LPP is always a straight line.
5. An infeasible solution satisfies at least one constraint.

MULTIPLE CHOICE QUESTIONS

1. Linear Programming is used to:
A) Solve non-linear equations
B**)** Find the optimal value of a linear function
C) Minimize non-linear constraints
D) Solve differential equations

2. In a Linear Programming Problem, the constraints are always:
A) Non-linear
B) Quadratic
C) Linear equations or inequalities
D) Cubic

3. The optimal solution of an LPP always lies at:
A) Midpoint of feasible region
B) Center of feasible region
C) Corner point (vertex) of feasible region
D) Outside the feasible region

4. The function to be maximized or minimized in LPP is called:
A) Constraint function
B) Objective function
C) Decision variable
D) Feasible function

5. If the feasible region is unbounded and the objective function increases indefinitely, then:
A) The problem has no solution
B) The problem is infeasible
C) The problem has an unbounded solution
D) The problem is degenerate

1.7 SUMMARY

Linear Programming & Operations Research provides a comprehensive overview of two interconnected disciplines essential for optimizing decision-making processes. Linear programming offers a mathematical approach for resource allocation through the formulation and solution of linear optimization problems. Operations research, on the other hand, extends beyond linear programming to encompass a broader range of mathematical techniques aimed at addressing complex operational challenges across various industries. By exploring these fields, individuals gain valuable insights into modelling real-world problems and devising optimal solutions to enhance organizational efficiency and decision-making effectiveness in diverse domains such as manufacturing, logistics, finance, and healthcare. In this unit we have learned about the basic definitions of LPP, Feasible region, optimal solution, convex set, basic feasible solution, optimal feasible solution and more useful definitions used to solve the linear programming problem.

1.8 GLOSSARY

Linear Programming Problem

Feasible Region

Optimal Solution

Convex Set

Basic Feasible Solution

1.9 REFERENCES

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1.10 SUGGESTED READING

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1.11 TERMINAL QUESTIONS

1. Write down the mathematical form of a Linear Programming Problem.
2. Explain the difference between feasible and infeasible solutions.

1.12 ANSWERS

CYQ1. True **CYQ2.** False **CYQ3.** True **CYQ4.** False
CYQ5. False

MCQ1. B **MCQ2.** C **MCQ3**. C **MCQ4.** B
MCQ5. C

UNIT-2: GRAPHICAL METHOD

CONTENTS:

2.1 *INTRODUCTION*

The graphical method is a fundamental technique in operational research used to solve linear programming Problems (LPPs) involving two decision variables. It provides a visual approach to identify the optimal solution by representing the feasible region, formed by a set of linear constraints, on a two-dimensional graph. Each constraint is expressed as a straight line, and the region satisfying all constraints simultaneously is called the feasible solution space. The objective function, usually aimed at maximization or minimization, is then represented as a family of parallel lines to determine the point that yields the best value within this region. The graphical method is particularly useful for developing an intuitive understanding of linear programming concepts, feasibility, boundedness, and optimality, and serves as a foundational step toward more advanced methods like the simplex Method.

George Bernard Dantzig, born on November 8, 1914, and passing away on May 13, 2005, was an American mathematical scientist renowned for his contributions to a range of fields including industrial engineering, operations research, computer science, economics, and statistics.

His most notable achievement is the development of the simplex algorithm, a groundbreaking method for solving linear programming problems. Dantzig's work in linear programming has had a profound impact across numerous industries and disciplines.

In addition to his work in optimization, Dantzig made significant contributions to statistics. Interestingly, he famously solved two open problems in statistical theory, mistaking them for homework after arriving late to a lecture by Jerzy Neyman. At the time of his passing, Dantzig held the prestigious positions of Professor Emeritus of Transportation Sciences and Professor of Operations Research and Computer Science at Stanford University.

George Bernard Dantzig
(8 November 1914 – 13 May 2005)
https://en.wikipedia.org/wiki/George_Dantzig

2.2 *OBJECTIVE*

The main objectives of the *Graphical Method* unit in operational research are as follows:

- To understand the basic concepts and formulation of a linear programming problem (LPP) with two variables.
- To learn how to represent constraints and objective functions graphically on a coordinate plane.
- To identify and construct the feasible region that satisfies all constraints simultaneously.
- To determine the corner points (vertices) of the feasible region and evaluate the objective function at these points.
- To find the optimal solution (maximum or minimum value) of the objective function using the graphical approach.
- To analyze the conditions of feasibility, infeasibility, unboundedness and multiple optimal solutions in LPPs.
- To develop a clear visual understanding of optimization problems and prepare the foundation for advanced analytical methods like the simplex method.

2.3 *GENERAL LINEAR PROGRAMMING PROBLEM*

Mathematically general linear programming can be represented as follows:

Maximize (Or Minimize) Z= $c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to,

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1j}x_j + ... + a_{1n}x_n\ (\le, =, \ge) b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2j}x_j + ... + a_{2n}x_n\ (\le, =, \ge) b_2$

………………………………………………………

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ... + a_{ij}x_j + ... + a_{in}x_n\ (\le, =, \ge) b_i$

………………………………………………………

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mj}x_j + ... + a_{mn}x_n\ (\le, =, \ge) b_m$

And $x_1, x_2, x_3, ..., x_n \ge 0$

The above programming problem can be rewritten in compact form as,

Maximize (Or Minimize) Z= $\sum_{j=1}^{n} c_j x_j$ …………….. (1)

Subject to,

$\sum_{j=1}^{n} a_{ij}x_j\ (\le, =, \ge)\ b_i ; i = 1,2,..., m$ ………………. (2)

$x_j \ge 0;\ j = 1,2,..., n$ ………………. (3)

The objective is to determine the values of x_j that optimize (maximize or minimize) the objective function (1). These values must satisfy to the constraints (2) as well as non-negativity restrictions (3). In this context, the coefficients c_j are termed as cost coefficients, while a_{ij} represents technological coefficients; a_{ij} denotes the quantity of the i^{th} resource utilized per unit of variable *xj*, and *bi* signifies the overall availability of the i^{th} resource.

Example 1: An oil company possesses two refineries - refinery A and refinery B. Refinery A can produce 20 barrels of petrol and 25 barrels of diesel daily, while refinery B can produce 40 barrels of petrol and 20 barrels of diesel per day. The company has a minimum requirement of 1000 barrels of petrol and 800 barrels of diesel. Operating refinery A costs Rs. 300 per day and refinery B costs Rs. 500 per day. How many days should each refinery be operated to minimize costs? Formulate this scenario as a linear programming model.

Solution: To formulate this problem as a linear programming model, let's define our decision variables:

Let *x* be the number of days refinery A is operated.

Let y be the number of days refinery B is operated.

Our objective is to minimize costs, so we want to minimize the total operating cost:

Minimize: $300x+500y$

Subject to the constraints:

1. Refinery A produces 20 barrels of petrol per day, and refinery B produces 40 barrels. The total petrol production should be at least 1000 barrels: $20x+40y\geq1000$

2. Refinery A produces 25 barrels of diesel per day, and refinery B produces 20 barrels. The total diesel production should be at least 800 barrels: $25x+20y\geq800$

3. Non-negativity constraints: $x\geq0$, $y\geq0$

This linear programming model represents the problem of minimizing costs while meeting the production requirements for petrol and diesel.

OR

$$\text{Minimize } Z = 300x + 500y$$

Subject to,

$$20x + 40y \geq 1000$$

$$25x + 20y \geq 800$$

$$x, y \geq 0$$

Example 2: In a particular factory, three machines, namely M_1, M_2, and M_3, are utilized in the manufacturing process of two products, P_1 and P_2. Machine M_1 is occupied for 5 minutes for producing one unit of P_1, while M_2 is used for 3 minutes and M_3 for 4 minutes. For one unit of P_2, the time requirements are 1 minute for M_1, 4 minutes for M_2, and 3 minutes for M_3. The profit earned per unit is Rs. 30 for P_1 and Rs. 20 for P_2, regardless of whether the machines operate at full capacity. How can we determine the production plan that maximizes profit? Frame this problem as a linear programming challenge.

Solution: To formulate this problem as a linear programming problem, let's define our decision variables:

Let x_1 be the number of units of product P_1 produced. Let x_2 be the number of units of product P_2 produced.

Our objective is to maximize profit, so we want to maximize the total profit:

Maximize: $30x_1+20x_2$

Subject to the constraints:

1. Time constraint for machine M_1: $15x_1+x_2 \leq T_1$
Where T_1 is the total available time on machine M_1.

2. Time constraint for machine M_2: $3x_1+4x_2 \leq T_2$
Where T_2 is the total available time on machine M_2.

3. Time constraint for machine M_3: $4x_1+3x_2 \leq T_3$
Where T_3 is the total available time on machine M_3.

4. Non-negativity constraints: $x_1 \geq 0$, $x_2 \geq 0$

Note: Here, we can take total available time for all machines is 60 i.e., $T_1 = T_2 = T_3 = 60$

This linear programming model represents the problem of determining the production plan that yields the highest profit while considering the time constraints on each machine.

OR

Maximize $Z = 30x_1 + 20x_2$

Subject to,

$$5x_1 + x_2 \leq 60$$

$$3x_1 + 4x_2 \leq 60$$

$$4x_1 + 3x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

2.4 *LP SOLUTION*

First, we will learn some terminologies for solution.

Closed half plane: A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane.

In the context of linear inequalities in two variables, a half plane represents the region of the coordinate plane that satisfies the inequality. The boundary of this region is defined by the corresponding equality or line. When considering both the boundary and the region itself, it's termed as a closed half plane. This closed half plane includes the boundary line and all the points on one side of it.

Convex set: A set is convex if, for any two points within the set, the line segment connecting those points remains entirely within the set. This property holds true for all pairs of points in the set, making it a fundamental characteristic of convexity. Mathematically, A set S is said to be convex set if for all $x, y \in S$, $\lambda x + (1-\lambda) y \in S \; \forall \lambda \in [0,1]$.

For example, the set $S = \{(x, y) : 3x + 2y \leq 12\}$ is convex because for two points (x_1, y_1) and $(x_2, y_2) \in S$, it is easy to see that $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in S \; \forall \lambda \in [0,1]$. While the set, $(S = \{(x, y) : x^2 + y^2 \geq 16\}$ is not convex. Note that two points (4,0) and (0,4) $\in S$ but $\lambda(4,0) + (i-\lambda)(0,4) \notin S$ for $\lambda = 1/2$.

Convex polygon: A convex polygon is indeed a convex set formed by the intersection of a finite number of closed half planes. Each side of the polygon corresponds to a boundary line of a half plane, and the polygon itself includes all the points within its boundaries. This property ensures that the polygon is convex, meaning that any line segment connecting two points within the polygon lies entirely within it.

Extreme points: The extreme points of a convex polygon are precisely the points where the lines that bound the feasible region intersect. These points are crucial because any point within the polygon can be expressed as a convex combination of the extreme points. Thus, they play a fundamental role in characterizing the polygon's shape and properties.

Feasible solution (FS): A feasible solution in optimization refers to any solution that meets all the constraints of the problem while maintaining non-negative values for the decision variables. It's essentially a valid solution that adheres to the problem's requirements.

OR

A feasible solution to the problem is any non-negative solution that complies with every restriction.

Basic solution (BS): In linear programming, particularly when dealing with a set of *m* simultaneous equations in *n* variables (where *n* > *m*), a basic solution is obtained by setting (*n* -

m) variables equal to zero and then solving the resulting system of equations for the remaining *m* variables. These *m* variables are referred to as basic variables, while the (*n* - *m*) variables set to zero are called non-basic variables. Basic solutions play a crucial role in optimization algorithms such as the simplex method.

Basic feasible solution (BFS): A basic solution to a linear programming problem is termed a basic feasible solution (BFS) if it satisfies all the non-negativity constraints.

Furthermore, a BFS is classified as degenerate if at least one of the basic variables has a value of zero. Conversely, a BFS is considered non-degenerate if all of the basic variables have non-zero and positive values.

These distinctions are significant in understanding the behavior of optimization algorithms such as the simplex method.

Optimal basic feasible solution: An optimal basic feasible solution in linear programming is a basic feasible solution that optimizes (maximizes or minimizes) the objective function. It represents the best feasible solution among all basic feasible solutions in terms of achieving the highest (or lowest) objective function value.

In linear programming, the optimal value of the objective function occurs at one of the extreme points of the convex polygon formed by the set of feasible solutions of the linear programming problem (LPP). This property is fundamental and is exploited in optimization algorithms such as the simplex method to efficiently find the optimal solution. By examining the extreme points, we can determine the best feasible solution that maximizes or minimizes the objective function.

Unbounded Solution: An LPP is said to have an unbounded solution if its solution can grow infinitely large without violating any of the constraints. This means that there is no finite optimal solution, and the objective function can be increased (in case of maximization) or decreased (in case of minimization) indefinitely while still satisfying all the constraints.

2.5 *GRAPHICAL METHOD*

The graphical method is indeed suitable for solving Linear Programming Problems (LPPs) with only two decision variables because it allows us to visualize the feasible region and the objective function contour lines on a two-dimensional graph. By graphically identifying the corner points of the feasible region and evaluating the objective function at these points, we can determine the optimal solution.

However, when dealing with three or more decision variables, graphical methods become impractical due to the difficulty of visualization. In such cases, the simplex method is commonly used. The simplex method is an iterative algorithm that systematically moves from one basic feasible solution to another along the edges of the feasible region until the optimal solution is reached. It's a powerful algorithm for solving linear programming problems of any size efficiently.

The simplex method will indeed be discussed further in the next section, as it provides a robust and efficient approach for solving LPPs with three or more variables.

Example 3: Solve the following LPP by graphical method.

$$\text{Minimize } Z = 20x_1 + 10x_2$$

$$\text{Subject to, } x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30$$

$$4x_1 + 3x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

Solution: To solve this Linear Programming Problem (LPP) graphically, we'll start by plotting the feasible region defined by the given constraints and then find the optimal solution within this region.

Let's begin by plotting the constraint equations:

1. $x_1+2x_2 \leq 40$
2. $3x_1+x_2 \geq 30$
3. $4x_1+3x_2 \geq 60$

To plot these equations, we'll first find their intercepts on the axes.

For $x_1+2x_2=40$, intercepts are:

- When $x_1=0$, $2x_2=40$ => $x_2=20$
- When $x_2=0$, $x_1=40$

For $3x_1+x_2=30$, intercepts are:

- When $x_1=0$, $x_2=30$

- When $x_2=0$, $3x_1=30 \Rightarrow x_1=10$

For $4x_1+3x_2=60$, intercepts are:

- When $x_1=0$, $3x_2=60 \Rightarrow x_2=20$
- When $x_2=0$, $4x_1=60 \Rightarrow x_1=15$

Now, we'll plot these points and draw the lines connecting them.

Next, we'll shade the region that satisfies all the inequalities. Since we're minimizing $Z=20x_1+10x_2$, we're looking for the region where Z is the smallest.

Let's get to graphing it!

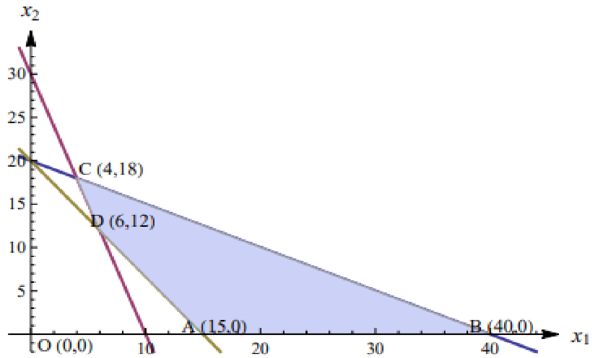

Figure 1: Unique optimal solution of example 3

Hence, the optimal solution of the shaded region is determined by the following table which shows the given LPP has minimum value is $Z_{min} = 240$ at the points $x_1 = 6$, $x_2 = 12$.

Extreme point	Objective function $Z = 20x_1 + 10x_2$
A (15,0)	300
B (40,0)	800
C (4,18)	260
D (6,12)	240

Example 4: Solve the following LPP by graphical method.

$$\text{Minimize } Z = 4x_1 + 3x_2$$

$$\text{Subject to, } x_1 + 2x_2 \leq 6$$

$$2x_1 + x_2 \leq 8$$

$$x_1 \geq 7$$

$$x_1, x_2 \geq 0$$

Solution: As seen in Figure 2, the limitations are plotted on the graph. There is no possible solution to the problem because there is no feasible region in the solution space.

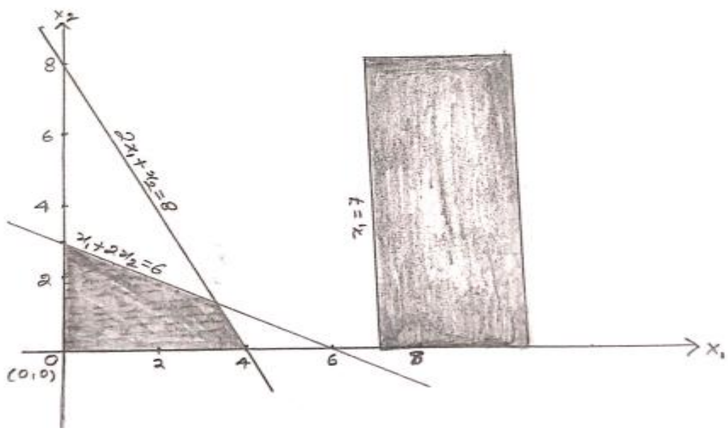

Figure 2: Feasible region of example 4

Example 5: Solve the following LPP by graphical method.

$$\text{Minimize } Z = 3x_1 + 5x_2$$

$$\text{Subject to, } x_1 + 2x_2 \geq 10$$

$$x_1 \geq 5$$

$$x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Solution: It is evident from the graph in Figure 3 that the feasible region is open-ended. As a result, Z's value can be increased indefinitely without going against any of the restrictions. Therefore, the LPP has an infinite solution.

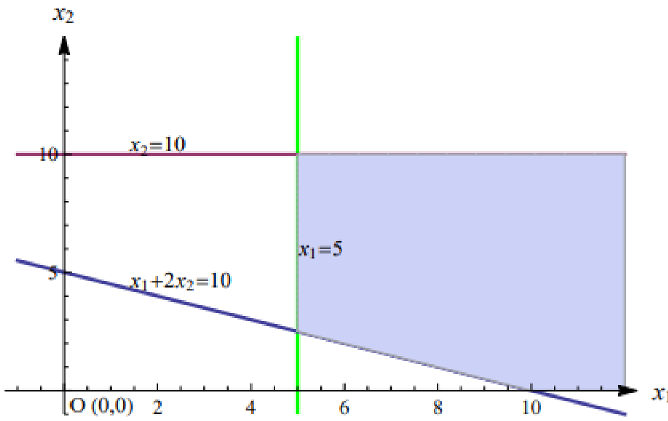

Figure 3: Unbounded solution of example 5

Note: It should be noted that an unbounded feasible zone does not always indicate the absence of a finite optimal solution for an LP problem. Examine the subsequent LPP, which although having an infinite feasible region, has an optimal viable solution.

$$\text{Minimize } Z = 2x_1 - x_2$$

$$\text{Subject to, } x_1 - x_2 \le 1$$

$$x_1 \le 3$$

$$x_1, x_2 \ge 0$$

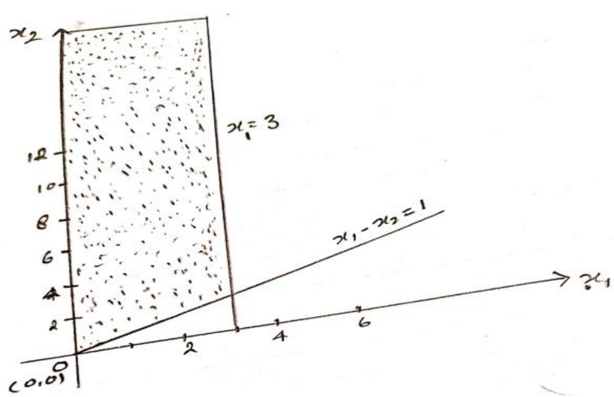

Figure 4: Finite optimal solution

Example 6: Solve the following LPP by graphical method.

$$\text{Minimize } Z = 3x_1 + 2x_2$$

$$\text{Subject to, } 6x_1 + 4x_2 \leq 24$$

$$x_2 \geq 2$$

$$x_1 \leq 3$$

$$x_1, x_2 \geq 0$$

Solution: As seen in Figure 5, the constraints are plotted on a graph by considering them as equations, and the feasible region is then identified using the signs of their inequality.

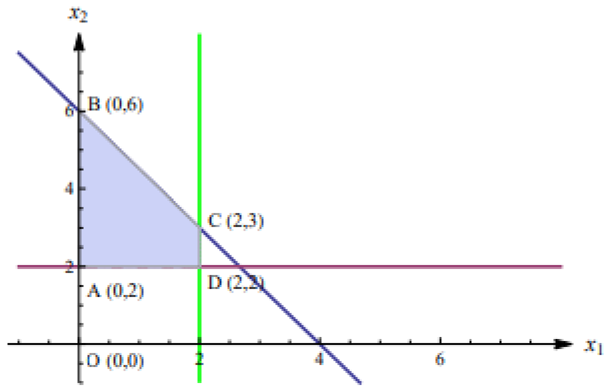

Figure 5: An infinite number of optimal solution of example 6

The extreme points of the region are A(0,2), B(0,6), C(2,3) and D(2,2). As we can easily find that slope of the objective function and one of the constraint $6x_1 + 4x_2 = 24$ coincide at line BC. Also from figure BC is the boundary line of the feasible region. So we can say that the optimal solution of LP problem can be obtained at any point of the line segment BC. From the following table

Corners (x, y)	Objective Function $Z = 3x_1 + 2x_2$
A (0,2)	4
B (0,6)	12
C (2,3)	12
D (2,2)	10

The optimal solution Z=12 is same at two different extreme points B and C. As a result, there exist several combinations of any two locations on the line segment BC that yield identical values for the objective function, thereby serving as optimal solutions for the linear programming problem. As a result, the provided LP issue has an endless number of optimal solutions.

2.6 *OUTCOMES AND LIMITATION OF GRAPHICAL METHOD*

The outcomes of the graphical method in Operational Research are as follows:

1. Ability to formulate and solve linear programming problems with two decision variables using a visual approach.

2. Understanding of how to graphically represent constraints and objective functions on a coordinate plane.

3. Skill to identify the feasible region and determine its corner points (vertices).

4. Capability to find the optimal solution (maximum or minimum) of the objective function by evaluating it at the feasible region's vertices.

5. Understanding of different solution types**-** unique, multiple, unbounded**,** and infeasible solutions- in linear programming problems.

6. Development of a visual and conceptual understanding of optimization, providing a foundation for more advanced methods such as the simplex method.

The limitations of the graphical method in Operational Research are as follows:

1. It is restricted to Linear Programming Problems with only two decision variables, since higher-dimensional problems cannot be easily represented graphically.

2. It becomes impractical and complex when the number of constraints increases, making the feasible region difficult to visualize accurately.

3. The method can only be used for linear relationships; it cannot handle non-linear programming problems.

4. It does not provide sensitivity or post-optimal analysis, which are essential for understanding how changes in parameters affect the solution.

5. The accuracy of the solution depends on precise graphical representation**,** which may lead to approximation errors when plotting or interpreting the graph.

6. It is time-consuming and inefficient for large-scale real-world problems involving multiple variables and constraints.

Check your progress

Problem 1: Using the graphical method solve the following LPP

Minimize, $z = -x + 2y$

Subject to the constraint, $-x + 3y \le 10$; $x + y \le 6$; $x - y \le 2$; $x, y \ge 0$

Answer: $x = 2, y = 0$ and minimum $z = -2$

Problem 2: Using the graphical method solve the following LPP

Minimize, $z = 2x + 3y$

Subject to the constraint, $x + y \le 30$; $x - y \ge 0$; $y \ge 3$; $0 \le x \le 20, 0 \le y \le 12$

Answer: $x = 18, y = 12$ and maximum $z = 72$

2.7 *SUMMARY*

Linear Programming & Operations Research provides a comprehensive overview of two interconnected disciplines essential for optimizing decision-making processes. Linear programming offers a mathematical approach for resource allocation through the formulation and solution of linear optimization problems. Operations research, on the other hand, extends beyond linear programming to encompass a broader range of mathematical techniques aimed at addressing complex operational challenges across various industries. By exploring these fields, individuals gain valuable insights into modeling real-world problems and devising optimal solutions to enhance organizational efficiency and decision-making effectiveness in diverse domains such as manufacturing, logistics, finance, and healthcare. In this unit we have learned about the basic definitions of LPP, Feasible region, optimal solution, convex set, basic feasible solution, optimal feasible solution and more useful definitions used to solve the linear programming problem. The overall summarization of this units are as follows:

- A hyper plane is a convex set.
- Intersection of two convex sets is also a convex set.
- The set of all feasible solutions of an LPP is a convex set.
- The collection of all feasible solutions of an LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.

2.8 *GLOSSARY*

- Linear Programming Problem
- Feasible Region

- Optimal Solution
- Convex Set
- Basic Feasible Solution
- Optimal Basic Feasible Solution
- Graphical Method

2.9 *REFERENCES*

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2.11 *TERMINAL QUESTION*

Long Answer Type Question:

1: Solve graphically the following LPP and find maximum and minimum value of objective function:

Maximize (or minimize) $z = 5x + 3y$

Subject to: $x + y \le 6;\ 2x + 3y \ge 3;\ 0 \le x \ge 3;\ 0 \le y \ge 3$

2: Solve graphically the following LPP and find maximum value of objective function:

Maximize $z = 5x + 3y$

Subject to: $x + y \le 6$; $2x + 3y \ge 6$; $0 \le x \le 4$; $0 \le y \le 3$

3: Solve graphically the following LPP and find maximum value of objective function:

Maximize (or minimize) $z = 3x + 2y$

Subject to: $-2x + y = 1$; $x \le 2$; $x + y \le 3$; $x, y \ge 0$

Short answer type question:

1: Solve graphically the following LPP and find maximum value of objective function:

Maximize $z = 2x + 4y$

Subject to: $x + 2y \le 5$; $x + y \le 4$; $x, y \ge 0$

2: Solve graphically the following LPP and find maximum value of objective function:

Maximize $z = 6x + y$

Subject to: $2x + y \ge 3$; $y - x \ge 0$; $x, y \ge 0$

Objective type question:

1: The graphical method of solving a linear programming problem is applicable when the number of decision variables is:

A) 1

B) 2

C) 3

D) Any number

2: In the graphical method, the feasible region is:

A) The entire plane

B) The area where all constraints overlap

C) The intersection of the objective function and one constraint

D) The area outside the constraints

3: The optimal solution to a linear programming problem using the graphical method is found:

A) At the center of the feasible region

B) At any point within the feasible region

C) At a corner point (vertex) of the feasible region

D) Along the boundary of the feasible region

4: If the feasible region is unbounded, the linear programming problem:

A) Has no solution

B) Always has an optimal solution

C) May have an optimal solution if the objective function is bounded

D) Will have an infinite number of solutions

5: In a maximization problem using the graphical method, the objective function line is shifted:

A) Parallel to itself towards the origin

B) Parallel to itself away from the origin

C) In any random direction

D) To the nearest constraint line

6: If two constraints intersect at a point in the feasible region, this point is called:

A) A feasible solution

B) An infeasible solution

C) A corner point

D) The optimal solution

7: In the graphical method, the area where no constraints overlap is called:

A) The feasible region

B) The infeasible region

C) The optimal region

D) The objective region

8: When solving a linear programming problem graphically, the constraints are represented by:

A) Straight lines

B) Curved lines

C) Dotted lines

D) Points

9: If the objective function is parallel to one of the constraints in the feasible region, then:

A) The problem has a unique solution

B) The problem has no solution

C) The problem has infinitely many solutions

D) The feasible region is empty

10: In a linear programming problem, the feasible region is bounded if:

A) The feasible region extends infinitely in one or more directions

B) The feasible region is a closed polygon

C) The feasible region lies entirely within the first quadrant

D) The objective function has a finite value

Fill in the blanks:

1: The graphical method of solving linear programming problems is only applicable when the number of decision variables is __________.

2: In the graphical method, the __________ region is the area where all constraints overlap.

3: The optimal solution in the graphical method is typically found at a __________ point of the feasible region.

4: The __________ function line is shifted parallel to itself in the graphical method to find the optimal solution.

5: If the feasible region is __________, the problem may have no finite optimal solution**.**

6: In the graphical method, each constraint is represented by a __________ on the graph.

7: If the objective function is __________ to one of the constraints in the feasible region, the problem may have infinitely many optimal solutions.

8: The area on the graph that does not satisfy all the constraints is called the __________ region.

9: The point of intersection of two or more constraints in the graphical method is called a __________ point.

10: In the graphical method, a linear programming problem is said to be __________ if the feasible region is a closed and bounded area.

2.12 *ANSWERS*

Answer of short answer type question

Answer 1: Maximum $z = 10$.

2: Problem has unbounded solution.

Answer of Long answer type question

Answer 1: $x = 3, y = 3$; Optimum $z = 24$

2: $x = 4, y = 2$; Optimum $z = 2$

3: $x = 2, y = 1$; Maximum $z = 8$

Answer of objective type question

Answer 1:	B)	**2:**	B)	**3:**	C)	**4:**	C)
5:	B)	**6:**	C)	**7:**	B)	**8:**	A)
9:	C)	**10:**	B)				

Answer of fill in the question

Answer 1:	**2**	**2:**	Feasible	**3:**	Corner	**4:**	Objective
5:	unbounded	**6:**	straight line	**7:**	Parallel	**8:**	Infeasible
9:	Corner (or vertex)	**10:**	Bounded				

UNIT-3: CONVEX SET AND THEIR PROPERTIES

CONTENTS:

3.1 *INTRODUCTION*

A convex set is a fundamental concept in operational research and optimization, forming the basis for many analytical and computational techniques used to solve decision-making problems. A set is said to be convex if, for any two points within it, the entire line segment joining those points also lies within the set. This simple geometric idea leads to powerful mathematical properties that greatly simplify optimization tasks, particularly in linear and nonlinear programming. Understanding convex sets and their properties such as convex combinations, extreme points, separation theorems, and supporting hyperplanes is essential because many feasible regions in operational research models are convex, ensuring that local optima are global and that efficient solution methods can be applied. This chapter introduces these concepts, develops their key properties, and highlights their importance in constructing and analyzing optimization models.

3.2 *OBJECTIVE*

The main objectives of the chapter "convex set and their properties" presented in point form:

- To introduce the concept of convex sets and explain their importance in operational research.
- To enable students to identify and verify whether a given set is convex.
- To explain convex combinations and their role in defining convexity.
- To study key features of convex sets such as extreme points, convex hulls, and faces.
- To understand geometric properties including separation theorems and supporting hyperplanes.
- To highlight how convexity influences the structure of feasible regions in optimization problems.
- To prepare students to apply convexity concepts in linear programming, nonlinear programming, and other optimization models.

3.3 *HYPERPLANES AND HYPERSPHERE*

A hyperplane is a flat, affine subspace of one dimension less than its ambient space.

Definition: In R^n, a hyperplane is defined as: $H = \{x \in R^n : a^T x = b\}$

where

- $a \in R^n$ is a nonzero normal vector,
- $b \in R$ is a scalar.

Remarks:

- In $a \in R^2$ (plane), a hyperplane is a line.
- In R^3 (3D space), a hyperplane is a plane.
- In R^n, the hyperplane has dimension $n-1$.

Uses in Optimization

- Hyperplanes are used to represent constraints like $a^T x = b$ or to separate convex sets.
- They form the boundaries of feasible regions in linear programming.

2. Hyperspheres

A hypersphere is the generalization of a sphere to higher dimensions.

Definition

In R^n, a hypersphere with center *c* and radius $r > 0$ is:

$S = \{x \in R^n : \|x - c\| = r\}$

Remarks:

- In R^2 a hypersphere is a circle.
- In R^3, it is a sphere.
- In R^n, it is called an (*n*–1)-sphere, having dimension *n*-1.

The corresponding ball (interior region) is:

$$B = \{x : \|x - c\| \leq r\}$$

Simple Visualization (Lower Dimensions)

Dimension	Hyperplane	Hypersphere
R^2	Line	Circle
R^3	Plane	Sphere
R^n	*n*−1-flat	*n*−1-sphere

3.4 *CONVEX SET*

A convex set is a set of points with the property that, for any two points within the set, the entire line segment joining those two points also lies completely inside the set.

Definition: A subset $S \subset R^n$, is said to be convex, if for any two points $a_1, a_2 \in S$, the line segment joining the points a_1 and a_2 is also contained in *S*.

In other word, a subset $S \subset R^n$ is convex if and only if

$a_1, a_2 \in S \Rightarrow \lambda a_1 + (1-\lambda)a_2 \in S;\ 0 < \lambda \leq 1.$

Figure 1 shows the figure of some convex set and non-convex set.

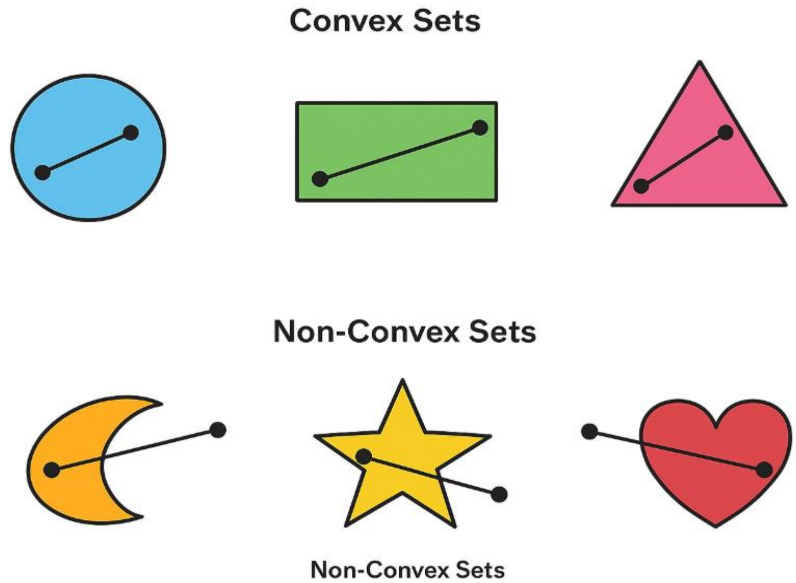

Figure 1: Some figure of convex and non-convex set

Example 1: Show that the set $S = \{(y_1, y_2) : 3{y_1}^2 + 2{y_2}^2 \le 6\}$ is convex set.

Solution: Let $A, B \in S$ where, $X = (a_1, a_2)$ and $Y = (b_1, b_2)$.

The line segment joining $A \,\&\, B$ is the set,

$\{u : u = \lambda A + (1-\lambda)B, 0 \le \lambda \le 1\}$

For some λ, $0 \le \lambda \le 1$, let $u = (u_1, u_2)$ be a point of this set, so that

$u_1 = \lambda a_1 + (1-\lambda)b_1$ and $u_2 = \lambda a_2 + (1-\lambda)b_2$

Now, $3{u_1}^2 + 2{u_2}^2 = 3[\lambda a_1 + (1-\lambda)b_1]^2 + 2[\lambda a_2 + (1-\lambda)b_2]^2$

$= \lambda^2[3{a_1}^2 + 2{b_1}^2] + (1-\lambda)^2[3{a_1}^2 + 2{b_2}^2] + 2\lambda(1-\lambda)(3a_1b_1 + 2a_2b_2)$

$\le 6\lambda^2 + 6(1-\lambda)^2 + 12\lambda(1-\lambda)$,

Since $(3a_1b_1 + 2a_2b_2) \le \sqrt{(x_1\sqrt{3})^2 + (x_1\sqrt{2})^2}\sqrt{(y_1\sqrt{3})^2 + (y_2\sqrt{2})^2}$,

Thus, $3{u_1}^2 + 2{u_2}^2 \le 6$; and hence $u = (u_1, u_2)$ is also a point of *S*.

Hence, *S* is a convex set.

The following results easily follow from the definition of a convex set.

(i) A hyperplane in R^n is a convex set.

(ii) A closed ball in R^n, namely, $\{x : |x - x_0| \le r\}$, where $r > 0$ and $x_0, x \in R^n$, is a convex set.

(*iii*) Hyperspheres are convex sets (their interiors).

(*iv*) They are often used to express distance constraints in optimization and machine learning (e.g., clustering, norm constraints).

3.4.1 *SOME IMPORTANT RESULTS*

Now we will discuss some important theorems and the results which are required to solve LPP.
Theorem 1: A hyperplane is a convex set.
Proof: Consider the hyperplane $S = \{x : cx = z\}$. Let x_1 and x_2 be two points in *S*. Then $cx_1 = z$ and $cx_2 = z$. Now, let a point x_3 be given by the convex combination of x_1 and x_2 as
$x_3 = \lambda x_1 + (1-\lambda)x_2, 0 \le \lambda \le 1$. Then

$$cx_3 = c\{\lambda x_1 + (1-\lambda)x_2\}$$
$$= c\lambda x_1 + (1-\lambda)x_2$$
$$= cz + (1-\lambda)z = z$$

Therefore, x_3 satisfies $cx = z$ and hence $x_3 \in S$. x_3 being the convex combination of x_1 and x_2 in *S*, *S* is a convex set. Thus a hyperplane is a convex set.

(Proof by another way) A hyperplane is typically defined as an affine subspace of dimension $n-1$ in an n-dimensional vector space.

To prove that a hyperplane is a convex set, we need to show that for any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in the hyperplane, the line segment connecting them lies entirely within the hyperplane.

Consider the equation of a hyperplane in *n*-dimensional space:

$\mathbf{a}^T\mathbf{x} = b$

where **a** is a non-zero vector normal to the hyperplane, **x** is a point in the hyperplane, and *b* is a scalar constant.

Now, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two arbitrary points in the hyperplane, satisfying:

$$\mathbf{a}^T\mathbf{x}_1 = b$$

$$\mathbf{a}^T\mathbf{x}_2 = b$$

Consider any point **x** on the line segment connecting **x**1 and **x**2. This point can be expressed as:

$\mathbf{x} = t\,\mathbf{x}_1 + (1-t)\mathbf{x}_2$

where, $0 \le t \le 1$.

Now, let's compute the dot product of **a** with **x**:

$\mathbf{a}^T\mathbf{x} = \mathbf{a}^T(t\mathbf{x}_1+(1-t)\mathbf{x}_2)$

$=t\mathbf{a}^T\mathbf{x}_1+(1-t)\mathbf{a}^T\mathbf{x}_2 = tb+(1-t)b=b$

Thus, $\mathbf{a}^T\mathbf{x}=b$, showing that **x** also lies on the hyperplane. Since this is true for any **x** on the line segment connecting $\mathbf{x}_1$ and $\mathbf{x}_2$, the entire line segment lies within the hyperplane.

Therefore, a hyperplane is indeed a convex set.

Theorem 2: Intersection of two convex sets is also a convex set.

Proof: The intersection of two convex sets is indeed a convex set. This property is a fundamental result in convex geometry and is known as the "intersection theorem" or "convexity preserving property of intersections."

To prove this, let's suppose we have two convex sets A and B in some vector space. We want to show that their intersection, denoted by $A \cap B$, is also convex.

Let x, y be any two points in $A \cap B$, and let λ be a scalar such that $0 \leq \lambda \leq 1$.

Since x and y belong to $A \cap B$, they must belong to both A and B. Because A and B are convex sets, the line segment connecting x and y lies entirely within both A and B.

Since A is convex, $\lambda x+(1-\lambda)y$ lies in A. Similarly, since B is convex, $\lambda x + (1-\lambda)y$ lies in B. Therefore, $\lambda x+(1-\lambda)y$ lies in both A and B, which means it lies in their intersection $A \cap B$.

Thus, $A \cap B$ is convex, as any point on the line segment between any two points in $A \cap B$ also lies within $A \cap B$.

Theorem 3: The set of all feasible solutions of an LPP is a convex set.

Proof: The set of all feasible solutions of a Linear Programming Problem (LPP) forms a convex set.

An LPP typically involves optimizing a linear objective function subject to linear constraints. The feasible region, which is the set of all points that satisfy these constraints, is typically a convex set.

To see why, consider the constraints of an LPP:

$\mathbf{Ax} \leq \mathbf{b}$

where **A** is a matrix of coefficients, **x** is the vector of decision variables, and **b** is a vector of constants.

Each constraint $\mathbf{a}_i^T\mathbf{x} \leq bi$ represents a half-space in n-dimensional space, defined by a hyperplane. The intersection of all these half-spaces forms the feasible region.

Since each constraint defines a convex set (a half-space is convex), the intersection of convex sets (feasible region) is also convex. This means that any convex combination of feasible solutions remains feasible, ensuring the convexity of the feasible set.

Therefore, the set of all feasible solutions of an LPP is indeed a convex set. This property is crucial for the efficient solution of linear programming problems using convex optimization techniques.

Remark: An LPP has an infinite number of feasible solutions if it has two feasible solutions, since a feasible solution might be any convex combination of the two feasible solutions.

Theorem 4: The collection of all feasible solutions of an LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.

Proof: Let's break down the proof into two parts:

1. The Feasible Region is Convex: To prove that the collection of all feasible solutions of an LPP constitutes a convex set, we need to show that any convex combination of two feasible solutions is also a feasible solution.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two feasible solutions, meaning they satisfy all the constraints of the linear programming problem. Now, consider the convex combination:

$\mathbf{x}=\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2$

where $0 \leq\lambda \leq1$.

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the constraints, it follows that:

$\mathbf{Ax}_1 \leq \mathbf{b}$

$\mathbf{Ax}_2 \leq \mathbf{b}$

Multiplying these inequalities by λ and $1-\lambda$ respectively and summing them, we get:

$\lambda(\mathbf{Ax}_1)+(1-\lambda)(\mathbf{Ax}_2) \leq \lambda\mathbf{b}+(1-\lambda)\mathbf{b}$

Simplifying:

$\mathbf{A}(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2) \leq \mathbf{b}$

Thus, **x** also satisfies the constraints, making it a feasible solution. Since this holds for any λ in the range $0\leq \lambda \leq1$, the feasible region is convex.

2. Extreme Points Correspond to Basic Feasible Solutions: To prove that extreme points of the feasible region correspond to basic feasible solutions, we need to show that each extreme point is indeed a basic feasible solution, and conversely, every basic feasible solution is an extreme point.

- **Extreme Points as Basic Feasible Solutions**: Any convex combination of two distinct feasible solutions lies strictly within the line segment connecting those two solutions. Since an extreme point cannot be expressed as such a convex combination of two distinct points, it must satisfy a minimal set of constraints, making it a basic feasible solution.
- **Basic Feasible Solutions as Extreme Points**: Basic feasible solutions are those solutions where a minimal set of constraints is active. If a solution is not a basic feasible solution, it means it can be expressed as a convex combination of two distinct basic feasible solutions. Therefore, it cannot be an extreme point.

Therefore, the extreme points of the feasible region correspond to the basic feasible solutions, completing the proof.

3.5 *EXTREME POINT*

In convex analysis, extreme points help describe the "corners" or boundary-structure of a convex set. Here is a clear and concise explanation.

Definition: An extreme point (vertex) of a convex set is a point of the set which does not lie on any segment joining two other point of the set.

Thus, a point x of a convex set *S* is an extreme point of the set, if there does not exist any pair of points x_1, x_2 in *S*, such that

$x = \lambda x_1 + (1-\lambda)x_2; 0 < \lambda < 1$

Note: We should observe that the inequality on λ is required to be strict. An extreme point always lies on the boundary of the set, but a boundary point of a convex set is not necessarily an extreme point.

Definition (convex combination of vectors): Given a set of vectors $\{x_1, x_2, x_3, ..., x_k\}$, a linear combination

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 +, ... + \lambda_k x_k\}$$

is called the convex linear combination of the given vectors, if

$$\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1$$

Theorem 5: The set of all convex combinations of a finite number of points of $S \subset R^n$ is a convex set.

Proof: Let, $S = \{x : x = \sum_{i=1}^{m} \lambda_i x_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}, x_i \in S$

We have to show that *S* is a convex set.

Let $x', x'' \in S \subset R^n$, so that

$x^{'} = \sum_{i=1}^{m} \lambda_i^{'} x_i$, where $\lambda_i^{'} \geq 0$, $\sum_{i=1}^{m} \lambda_i^{'} = 1$

And $x^{"} = \sum_{i=1}^{m} \lambda_i^{"} x_i$, where $\lambda_i^{"} \geq 0$, $\sum_{i=1}^{m} \lambda_i^{"} = 1$

Consider now the vector,

$x = \lambda x^{'} + (1-\lambda)x^{"}; 0 \leq \lambda \leq 1$

$= \lambda \sum_{i=1}^{m} \lambda_i^{'} x_i + (1-\lambda) \sum_{i=1}^{m} \lambda_i^{"} x_i$

$= \sum_{i=1}^{m} [\lambda \lambda_i^{'} + (1-\lambda)\lambda_i^{"}] x_i = \sum_{i=1}^{m} \mu_i x_i$

Where, $\mu_i = \lambda \lambda_i^{'} + (1-\lambda)\lambda_i^{"}$; $i = 1, 2, 3, ..., m$

Since, $0 \leq \lambda \leq 1$, $\lambda_i^{'} \geq 0$, $\lambda_i^{"} \geq 0$, therefore $\mu_i \geq 0$ for each *i*.

Also, $\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} [\lambda \lambda_i^{'} + (1-\lambda)\lambda_i^{"}] = \lambda \sum_{i=1}^{m} \lambda_i^{'} + (1-\lambda) \sum_{i=1}^{m} \lambda_i^{"} = \lambda + (1-\lambda) = 1$

Hence x is a convex combination of the vectors $x_1, x_2, x_3, ..., x_m$ or $x \in S$.

Thus for each pair of points $x^{'}, x^{"} \in S$, the line segment joining them is contained in the set. Hence S is a convex set.

3.6 *CONVEX POLYHEDRON, CONVEX CONE AND CONVEX HULL*

Definition (Convex polyhedron): A convex polyhedron is the set formed by all convex combinations of a finite collection of linearly independent vectors.

The convex polyhedron generated by the finite set of linearly independent vectors $x_1, x_2, x_3, ..., x_m$ is the set

$$\left\{ x : x = \sum_{i=1}^{m} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

Definition (Convex cone): A non-empty subset $C \subset R^n$, is called a cone if for each $x \in C$, and $\lambda \geq 0$, the vector λx is also in *C*.

A cone is called a convex cone if it is a convex set.

Example 2: If A be an $m \times n$ matrix, then the set of $n-$vectors x satisfying $Ax > 0$ is a convex cone in R^n. It is a cone, because if $Ax > 0$, then $A\lambda x > 0$ for non-negative λ. It is convex because if $Ax^{(1)} \geq 0$ and $Ax^{(2)} \geq 0$, then $A[\lambda x^{(1)} + (1-\lambda)x^{(2)}] \geq 0$.

Definition (Convex hull of a set): Given a set $Y \subset R^n$, the smallest convex set containing Y is called the convex hull of Y, and is denoted by $< Y >$.

It can be easily seen that the convex hull of a set is the intersection of all convex set containing Y.

Example 3: Let $A = \{x_1, x_2\}$. Then the line segment joining x_1, x_2 is a convex set. Also if S is any convex set containing A, then it must contain the line segment joining x_1, x_2.

$< A >= \{x : x = \lambda x_1 + (1-\lambda)x_2 . 0 < \lambda < 1\}$

Remarks: If A is finite subset of vectors in R^n, then the convex hull of A is the set of all convex combinations of vectors in A.

3.7 *SUPPORTING AND SEPRATING HYPERPLANES*

Here is a clear and structured explanation of supporting and separating hyper planes**,** commonly studied in convex analysis and operational research:

Supporting Hyperplanes

A supporting hyperplane to a convex set $C \subset R^n$ is a hyperplane that touches the set but does not cut through it. Formally, a hyperplane

$$H = \{x \in R^n : a^T x = b\}, a \neq 0$$

is said to support C if:

1. $C \subseteq \{x : a^T x \leq b\} (or \geq b)$, and
2. The hyperplane intersects the boundary of C:

   $H \cap C \neq \phi$

Intuition

- A supporting hyperplane just "touches" the convex set from one side (like a tangent line touching a convex curve).
- Every boundary point of a convex set has at least one supporting hyperplane.

Example

For a convex polygon, any line that touches the polygon at an edge or a vertex without cutting through it is a supporting hyperplane.

Separating Hyperplanes

A separating hyperplane is a hyperplane that places two disjoint sets on opposite sides.
Given two sets $C_1, C_2 \subset R^n$, a hyperplane

$$H = \{x : a^T x = b\}$$

Separates them if: $a^T x \leq b \,\forall\, x \in C_1$

And $a^T x \geq b \,\forall\, x \in C_2$

Types of Separation

1. **Weak Separation:**
   The hyperplane may touch one or both sets: $a^T x \leq b \,\forall\, x \in C_1$, $a^T x \geq b \,\forall\, x \in C_2$
2. **Strong Separation:**
   There is a *strict gap* between the sets: $a^T x < b < a^T y \,\forall\, x \in C_1, y \in C_2$.

Remark: If two nonempty, disjoint, convex sets and at least one is open then a separating hyperplane always exists.

Geometric Intuition

- A supporting hyperplane touches a convex set from the outside.
- A separating hyperplane stands between two disjoint sets and divides space so they lie on different sides.

3.8 *CONVEX FUNCTIONS*

A convex function is a fundamental concept in optimization and mathematical analysis, characterized by the property that the line segment between any two points on its graph lies above or on the graph itself. Defined on a convex set, a function *f* is convex if for any two points in its domain, the function value at any weighted average of these points does not exceed the weighted average of their function values. This simple geometric attribute leads to powerful analytical advantages most notably, any local minimum of a convex function is also a global minimum, making convex functions central to optimization theory, economics, engineering, and machine learning. Their structural simplicity and predictable behavior make them essential tools for solving a wide range of real-world optimization problems.

Definition (convex function): A function $f : R^n \to R$ is called convex on a convex set C if, for all $x, y \in C$ and for all $\lambda \in [0,1]$:

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y).$$

Definition (strictly convex function): Let S be a non-empty convex subset of R^n. A function $f(x)$ on S is said to be strictly convex if for two different vectors x_1 and x_2 in S.

$$f[\lambda x+(1-\lambda)y)]<\lambda f(x)+(1-\lambda)f(y), 0<\lambda<1$$

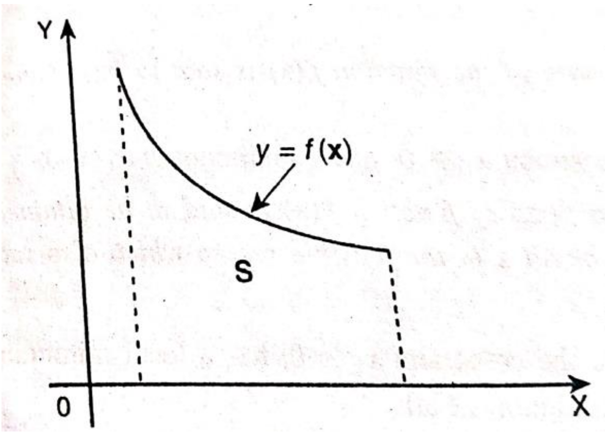

Figure 2: Strictly convex function

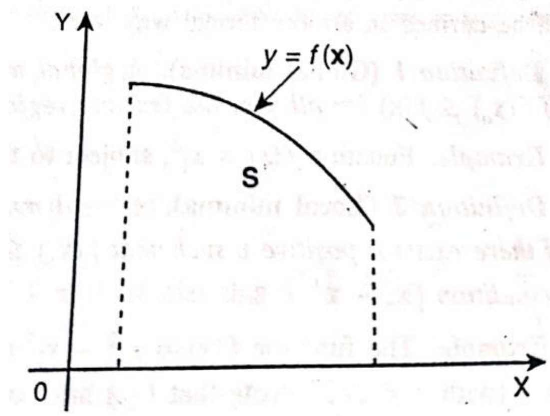

Figure 3: Strictly concave function

From the two definitions given above, it is clear that every strictly convex function is inherently convex. Figure 2 presents the graph of a strictly convex function for illustration.

Definition [Concave (strictly concave) function]: A function $f(x)$ on a non-empty subset S of R^n is said to be concave (strictly concave) if $-f(x)$ is convex (strictly convex).

Clearly, every strictly concave function is also concave, as follows directly from their definitions. Figure 3 illustrates the graph of a strictly concave function for better understanding .

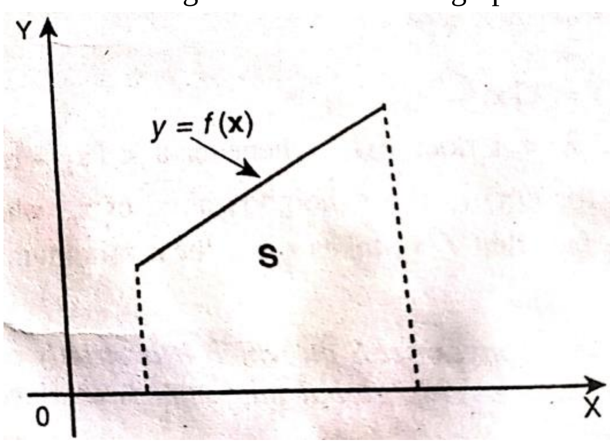

Figure 4: Both convex and concave function

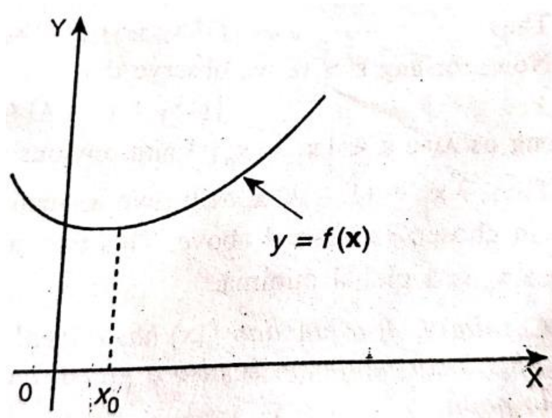

Figure 5

It is possible for a function to be both convex and concave. For example, $f(x) = x$ is such a function (Fig 4). The function in Figure 5 is strictly convex for $x \geq x_0$ but not strictly convex for $x < x_0$.

Check your progress

Problem 1: Prove that the set $x = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 4\}$ is a convex set.

Problem 2: Show that the following functions are convex over R^1

(*i*) $f(x) = 3x^2$ **(*ii*)** $f(x) = e^x$ and **(*iii*)** $f(x) = |x|$

3.9 *SUMMARY*

A convex set is fundamentally defined by the property that the line segment connecting any two points within the set lies entirely within the set, formally expressed as containing all convex combinations θx + (1-θ)y for any x, y in the set and 0≤θ≤1. This unit establishes that key operations preserve convexity, including intersections, affine transformations (scaling and translation), and linear combinations, while unions generally do not. Essential examples are presented, such as hyperplanes, hypersphere, convex polyhedron, convex cone, convex hull, supporting and separating hyperplanes and convex function which form the building blocks for more complex convex structures. The concepts of convex hulls (the smallest convex set containing a given collection) and extreme points (which cannot be expressed as convex combinations of other set members) are introduced to characterize set boundaries. These geometric foundations are crucial for optimization, as they ensure that local minima in convex problems are global minima, and they underpin the theory behind linear programming, quadratic programming, and more general convex optimization frameworks in operational research..

3.10 *GLOSSARY*

- Hyperplanes
- Hypersphere
- Extreme point
- Convex polyhedron
- Convex cone
- Convex hull
- Supporting and seprating hyperplanes

- Convex function

3.11 *REFERENCES*

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3.12 *SUGGESTED READING*

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3.13 *TERMINAL QUESTION*

Long Answer Type Question:

1: Examine whether the following set is convex or not:

(i) $S=\{(x_1,x_2):x_1x_2\geq 1, x_1\geq 0, x_2\geq 0\}$

(ii) $S=\{(x_1,x_2):5x_1+2x_2\geq 10, 2x_1+5x_2\geq 10\}$

(iii) $S=\{(x_1,x_2):x_1^{\,2}+x_2^{\,2}\leq 1, x_1+x_2\geq 1\}$

2: Show that $S=\{(x_1,x_2,x_3):2x_1-x_2+x_3\leq 4\}\subset R^3$ is a convex set.

3: Determine the convex hull of the following sets:

(*i*) $A = \{(x_1, x_2) : x_1^{\ 2} + x_2^{\ 2} = 1\}$

(*ii*) $A = \{(x_1, x_2)\}$

4: Prove that the set of all convex combinations of a finite number of points of $S \subset R^n$ is a convex set.

5: Prove that a hyperplane is a convex set.

Short answer type question:

1: Show that the function $f(x) = 2x_1^{\ 2} + x_2^{\ 2}$ is a convex function over all of R^2

2: Prove that intersection of two convex sets is also a convex set.

Objective type question:

1: A set $C \subset R^n$ is convex if and only if:

A. It contains the origin
B. It contains all its boundary points
C. It contains the line segment between any two points in the set
D. It is closed and bounded

2: Which of the following sets is always convex?

A. The set of all integer points
B. The set of all solutions to a system of linear inequalities
C. The union of two convex sets
D. The boundary of a circle

3: The intersection of any collection of convex sets is:

A. Always convex
B. Always non-convex
C. Convex only when the collection is finite
D. Convex only when the sets are closed

4: A set is called strictly convex if:

A. Every line segment between two points in the set lies completely outside the set
B. Every line segment lies on or outside the boundary
C. Every interior point of a line segment between two distinct points lies inside the set
D. It contains only one extreme point

5: Which of the following is not a convex set?

A. A disk (solid circle)
B. A line in R^2
C. A triangle including its interior
D. A hollow circular ring (annulus)

6: A half-space defined by $a^T x \leq b$ is:

A. Always convex
B. Always non-convex
C. Convex only if $a = 0$
D. Convex only if $b > 0$

7: A supporting hyperplane to a convex set C must:

A. Pass through the origin
B. Intersect the interior of C
C. Touch the set without cutting through it
D. Divide the set into two equal parts

8: If two convex sets are non-empty, closed, and disjoint, then:

A. They can never be separated
B. A separating hyperplane always exists
C. They must intersect at a boundary point
D. They must lie in different dimensions

9: Extreme points of a convex set are points that:

A. Lie outside the set but on the boundary
B. Cannot be expressed as a convex combination of other points in the set
C. Always lie in the interior
D. Form the center of the set

10: The convex hull of a set of points is:

A. The set of all concave combinations of the points
B. The largest convex set containing the points

C. The intersection of all convex sets not containing the points
D. The smallest convex set containing the points

Fill in the blanks:

1: A set C is said to be convex if it contains the ________ between any two of its points.

2: A function f is convex if its ________ is a convex set.

3: The intersection of any number of convex sets is always ________.

4: A set is strictly convex if every interior point of the line segment between two distinct points of the set lies ________ the set.

5: A hyperplane is defined as the set of all points x satisfying the equation a^x = ________.

6: A half-space of the form $a^T x \leq b$ is always a ________ set.

7: A separating hyperplane places two disjoint convex sets on ________ sides of the hyperplane.

8: A supporting hyperplane touches the convex set but does not ________ it.

9: The convex hull of a set of points is the ________ convex set containing those points.

10: A point that cannot be expressed as a convex combination of any other two points in the set is called an ________ point.

3.14 *ANSWERS*

Answer of long answer type question:

Answer 1: (i) convex **(ii)** convex **(iii)** convex

Answer 3: (i) $< A >= \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$

(ii) $< A >= \{x : x = \lambda x_1 + (1-\lambda) x_2;\ 0 \leq \lambda \leq 1\}$

Answer of objective type question:

Answer 1:	C)	**2:**	B)	**3:**	A)	**4:**	C)
5:	D)	**6:**	A)	**7:**	C)	**8:**	B)
9:	B)	**10:**	D)				

Answer of fill in the blanks

Answer 1:	line segment	**2:**	epigraph	**3:**	convex	**4:**	inside
5:	b	**6:**	convex	**7:**	opposite	**8:**	cut through
9:	smallest	**10:**	extreme				

BLOCK-II

SIMPLEX METHOD

UNIT- 4: SIMPLEX METHOD

CONTENTS:

4.1 *INTRODUCTION*

The Simplex method is an iterative algebraic technique for solving Linear Programming Problems (LPPs) with more than two decision variables by systematically moving from one corner point of the feasible region to another until the optimal solution is found. It involves converting inequalities to equations using slack, surplus, or artificial variables, creating a **simplex tableau**, and then performing row operations (pivoting) to find the entering and leaving variables until all variables in the objective function row satisfy the optimality condition (non-negative for maximization or non-positive for minimization).

4.2 *OBJECTIVE*

After reading this unit learners will be able to

- Various types of variables like slack and surplus variable.

- Visualized the canonical and standard forms of an LPP.
- Implementation of simplex method and visualized the algorithm to solve the given LPP by simplex method.

4.3 *SIMPLEX METHOD*

George Dantzig created the simplex approach in 1947 as an effective way to solve LP problems with many of variables. The graphical technique and the simplex method both involve examining the extreme points of the feasible region in order to get the best possible solution. In this case, the ideal solution is located at a multi-dimensional polyhedron's extreme point. The foundation of the simplex technique is the fact that, in the event that an ideal solution exists, it can always be found inside one of the most basic feasible options.

4.3.1 *CANONICAL AND STANDARD FORMS OF AN LPP*

Any linear programming problem is said to be in canonical form if it can be expressed as,

Maximize, $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to,

$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \le b_i, i = 1,2,...,m$

$x_1, x_2,..., x_n \ge 0$

These are some characteristics of this form,

(i) The objective function is of maximization type Or Maximize Z. If we have given minimize Z, we convert it to maximize by taking negative of Z i.e., Maximize (-Z).

(ii) All constraints should be of the type "$\le$", except the non-negative restrictions.

(iii) All variables are non-negative.

An LPP in such form known as **Standard form**:

Maximize (or minimize), $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to,

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n = b_i, i = 1,2,...,m$$

$$x_1, x_2,..., x_n \ge 0$$

OR

Maximize (or minimize), $Z = cx$

Subject to,

$$Ax = b, i = 1,2,...,m$$

$$x \ge 0 \text{ (null vector)}$$

Where $c = (c_1, c_2, ..., c_n)$ and n-component row vector; $x = (x_1, x_2, ..., x_m)$ an m-component column vector; $b = [b_1, b_2, ..., b_m]$ an m-component column vector and the matrix $A = (a_{ij})_{m\times n}$. The characteristic of this form are as follows:

(i) All constraints are expressed in the form of equations, except the non-negative restrictions.

(ii) The RHS of each constraint equation is non-negative.

4.3.2 *SLACK AND SURPLUS VARIABLES*

1. **Slack Variables**:
   Slack variables are introduced to convert inequality constraints into equations.

- For each inequality constraint of the form $\mathbf{a}_i^T\mathbf{x} \le b_i$, a slack variable s_i is added such that $\mathbf{a}_i^T\mathbf{x} + s_i = b_i$.
- Slack variables represent the amount by which the left-hand side of the constraint falls short of the right-hand side to satisfy the constraint.
- In the simplex method, slack variables start with a value of zero in the initial basic feasible solution.

OR

Slack variable: A variable which is added to the LHS of a "≤" type constraint to convert the constraint into an equality is called slack variable.

2. **Surplus Variables**:

- Surplus variables are introduced to convert inequality constraints into equations when the inequalities are of the form $\mathbf{a}_i^T\mathbf{x} \ge b_i$.
- For each inequality constraint of the form $\mathbf{a}_i^T\mathbf{x} \ge b_i$, a surplus variable s_i is added such that $\mathbf{a}_iT\mathbf{x} - s_i = b_i$.
- Surplus variables represent the amount by which the left-hand side of the constraint exceeds the right-hand side.
- In the simplex method, surplus variables start with a value of zero in the initial basic feasible solution.

Surplus variable: A variable which is subtracted from the LHS of a "≥" type constraint to convert the constraint into an equality is called surplus variable.

These additional variables allow the LP problem to be formulated in canonical form, where all constraints are equations. The simplex method operates on LP problems in canonical form, making it easier to identify and move between basic feasible solutions efficiently.

4.3.3 *BASIC SOLUTION*

Consider a set of m linear simultaneous equations of n ($n > m$) variables.

$$Ax = b,$$

where A is an $m \times n$ matrix of rank m. If any $m \times m$ non-singular matrix B is chosen from A and if all the $(n - m)$ variables not associated with the chosen matrix are set equal to zero, then the solution to the resulting system of equations is a basic solution (BS).

Basic solution has not more than m non-zero variables called basic variables. Thus the m vectors associated with m basic variables are linearly independent. The variables which are not basic, are termed as non-basic variables. If the number of non-zero basic variables is less than m, then the solution is called degenerate basic solution. On the other hand, if none of the basic variables vanish, then the solution is called non-degenerate basic solution. The possible number of basic solutions in a system of m equations in unknowns is $^{n}C_{m} = \dfrac{n!}{m!(n-m)!}$.

Theorem 1: The necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of $Ax = b$ is that every set of m columns of the augmented matrix $[A, b]$ is linearly independent.

Proof: To prove the theorem stating that the existence and non-degeneracy of all basic solutions of $Ax = b$ depend on every set of m columns of the augmented matrix $[A, b]$ being linearly independent, we'll need to establish both the necessity and sufficiency of this condition.

Necessary condition: Let's first prove the necessity part. We want to show that if all basic solutions exist and are non-degenerate, then every set of m columns of $[A, b]$ must be linearly independent.

Suppose that there exists a set of m columns of $[A, b]$ that is linearly dependent. This implies that there exists a nontrivial linear combination of these columns that equals the zero vector. Without loss of generality, let's assume that the linear combination involves the last column, corresponding to the vector b.

$c_1A_1+c_2A_2+\ldots+c_mA_m+c_{m+1}b=0$

Where A_i represents the i^{th} column of A and c_i are coefficients not all zero.

Since the last column of $[A, b]$ is linearly dependent on the other columns, it means that the system $Ax = b$ has at least one redundant equation. In other words, the last component of b can be expressed as a linear combination of the other components, rendering the system degenerate.

Hence, if every set of m columns of $[A, b]$ is linearly independent, then the system $Ax = b$ cannot have any redundant equations, ensuring the existence and non-degeneracy of all basic solutions.

Sufficient condition: Now let's prove the sufficiency part, i.e., if every set of m columns of $[A, b]$ is linearly independent, then all basic solutions exist and are non-degenerate.

Suppose all sets of m columns of $[A, b]$ are linearly independent. This implies that each component of b cannot be expressed as a linear combination of the other components. Therefore, each equation in the system $Ax = b$ contributes uniquely to the determination of the solution.

Since there are no redundant equations, every basic solution of the system $Ax = b$ corresponds to a unique set of pivot variables, making the solution non-degenerate. Furthermore, since each equation is necessary for determining the solution, all basic solutions exist.

Therefore, the sufficiency part is proven.

Conclusion: Combining the necessity and sufficiency proofs, we conclude that the existence and non-degeneracy of all basic solutions of $Ax = b$ are guaranteed if and only if every set of m columns of the augmented matrix $[A, b]$ is linearly independent.

4.3.4 *BASIC FEASIBLE SOLUTION (BFS)*

As we know that, if a feasible solution x is also basic, meaning that it corresponds to a set of linearly independent columns of the constraint matrix A, then it is termed a basic feasible solution (BFS). Basic feasible solutions are important in LP because they often correspond to the vertices of the feasible region (in the case of bounded LP problems), and they serve as starting points for various optimization algorithms such as the simplex method.

In summary, a basic feasible solution is a feasible solution that satisfies the additional condition of being basic, implying that it corresponds to a set of linearly independent constraints.

OR

An LPP's feasible solution is one that meets all of its constraints and non-negativity restrictions. A viable solution is referred to as basic feasible solution (BFS) if it is basic once more.

Theorem 2: The necessary and sufficient condition for the existence and non-degeneracy of all possible basic feasible solutions of $Ax = b$, $x \geq 0$ is the linear independent of every set of m columns of the augmented matrix [A, b], where A is the $m \times n$ coefficient matrix.

Proof: To prove the theorem that the existence and non-degeneracy of all possible basic feasible solutions of the linear programming problem $Ax = b$, $x \geq 0$ depend on the linear independence of every set of m columns of the augmented matrix $[A, b]$, we need to establish both the necessity and sufficiency of this condition.

Necessary condition: Let's first prove the necessity part. We want to show that if all possible basic feasible solutions exist and are non-degenerate, then every set of *m* columns of [*A*, *b*] must be linearly independent.

Suppose that there exists a set of *m* columns of [*A*, *b*] that is linearly dependent. This implies that there exists a nontrivial linear combination of these columns that equals the zero vector. Without loss of generality, let's assume that the linear combination involves the last column, corresponding to the vector *b*.

$c_1A_1+c_2A_2+\ldots+c_mA_m+c_{m+1}b=0$

Where *Ai* represents the i^{th} column of *A* and c_i are coefficients not all zero.

Since the last column of [*A*, *b*] is linearly dependent on the other columns, it means that the system $Ax = b$ has at least one redundant equation. In other words, the last component of *b* can be expressed as a linear combination of the other components, violating the non-negativity constraint $x \geq 0$ and rendering the system degenerate.

Hence, if every set of *m* columns of [*A*, *b*] is linearly independent, then the system $Ax = b$ cannot have any redundant equations, ensuring the existence and non-degeneracy of all possible basic feasible solutions.

Sufficient condition: Now let's prove the sufficiency part, i.e., if every set of *m* columns of [*A*, *b*] is linearly independent, then all possible basic feasible solutions exist and are non-degenerate.

Suppose all sets of *m* columns of [*A*, *b*] are linearly independent. This implies that each component of *b* cannot be expressed as a linear combination of the other components. Therefore, each equation in the system $Ax=b$ contributes uniquely to the determination of the solution.

Since there are no redundant equations, every basic feasible solution of the system $Ax = b$ corresponds to a unique set of pivot variables, making the solution non-degenerate. Furthermore, since each equation is necessary for determining the solution, all possible basic feasible solutions exist.

Therefore, the sufficiency part is proven.

Conclusion:

Combining the necessity and sufficiency proofs, we conclude that the existence and non-degeneracy of all possible basic feasible solutions of the linear programming problem $Ax = b$, $x \geq 0$ are guaranteed if and only if every set of m columns of the augmented matrix $[A, b]$ is linearly independent.

Example 1: Find out the basic feasible solution for the system of linear equations

$2x_1 + 3x_2 - x_3 + 4x_4 = 8$

$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

$x_1, x_2, x_3, x_4 \geq 0$

Proof: The given system of equations can be written as

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$$

where $a_1 = [2,1], a_2 = [3,-2], a_3 = [-1,6], a_4 = [4,-7]$ and $b = [8,-3]$. The maximum number of basic solutions that can be obtained is ${}^4C_2 = 6$. The six sets of 2 vectors out of 4 are

$$B_1 = [a_1, a_2],\ B_2 = [a_1, a_3],\ B_3 = [a_1, a_4]$$
$$B_4 = [a_2, a_3],\ B_5 = [a_2, a_4],\ B_6 = [a_3, a_4].$$

Here $|B_1| = -7$, $|B_2| = 18$, $|B_3| = -18$, $|B_4| = 16$, $|B_5| = -13$, and $|B_6| = -17$. Since none of these determinants vanishes, hence every set B_i of two vectors is linearly independent. Therefore, the vectors of the basic variables associated to each set B_i, i = 1,2,3,4,5,6 are given by,

$$x_{B_1} = B_1^{-1}b = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_{B_2} = B_2^{-1}b = \frac{1}{13}\begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/13 \\ -14/13 \end{bmatrix}$$

$$x_{B_3} = B_3^{-1}b = \frac{1}{18}\begin{bmatrix} -7 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 22/9 \\ 7/9 \end{bmatrix}$$

$$x_{B_4} = B_4^{-1}b = \frac{1}{16}\begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/16 \\ 7/16 \end{bmatrix}$$

$$x_{B_5} = B_5^{-1}b = -\frac{1}{13}\begin{bmatrix} -7 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/13 \\ -7/13 \end{bmatrix}$$

And $x_{B_6} = B_6^{-1}b = -\frac{1}{17}\begin{bmatrix} -7 & -4 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/17 \\ 45/17 \end{bmatrix}$

From above, we see that the possible basic feasible solutions are $x_1 = [1, 2, 0, 0]$, $x_2 = [22/9, 0, 0, 7/9]$, $x_3 = [0, 45/16, 7/16, 0]$ and $x_4 = [0, 0, 44/17, 45/17]$ which are also non-degenerate. The other basic solutions are not feasible.

Theorem 3: (Fundamental theorems of LP): If a linear programming problem has an optimal solution, then the optimal solution will coincide with at least one basic feasible solutions of the problem.

Proof: Let us consider that x^* is an optimal solution of the following LPP:

Maximize, $z = cx$

Subject to $Ax = b, x \geq 0$ … (1)

Without loss of generality, we assume that the initial *p* component of optimal solution x^* are non-zero and the remaining $(n-p)$ component of x^* are non-zero. Thus,

$x^* = [x_1, x_2, ..., x_p, 0, 0, ..., 0]$

Then, from (1), $Ax^* = b$ gives $\sum_{j=1}^{p} a_{ij} x_j = b_i, i = 1, 2, ..., m$

Also, $A = [a_1, a_2, ..., a_p, a_{p+1}, ..., a_n]$ gives

$a_1 x_1 + a_2 x_2 + ..., a_p x_p = b$ … (2)

Also, $z^* = z_{\max} = \sum_{j=1}^{p} c_j x_j$ … (3)

Now, if the vectors $a_1, a_2, ..., a_p$ correspond to the non-zero components of x^* are linearly independent, then, by definition of x^* is a basic solution by definition, and the theorem is valid in this instance. The simplest possible solution is non-degenerate if *p* = *m*. Conversely, in the event where *p* is less than *m*, a degenerate basic feasible option will be formed, wherein the basic variables (*m* - *p*) equal zero.

If vectors are not linearly independent, then they must be linearly dependent i.e., there exists scalars $\lambda_j, j = 1, 2, ..., p$ of which at least one of the λ_j's is non-zero such that

$\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_p a_p = 0$ … (4)

Suppose that at least one $\lambda_j > 0$. If the non-zero λ_j is not positive, then we can multiply (4) by (-1) to get a positive λ_j.

Let $\mu = \underset{1 \leq j \leq p}{Max} \left\{ \frac{\lambda_i}{x_j} \right\}$ … (5)

Then μ is positive as $x_j > 0 \; \forall \; j = 1, 2, ..., p$ and at least one λ_j is positive. Dividing (4) by μ and subtracting it from (2), we get

$$\left(x_1 - \frac{\lambda_1}{\mu} \right) a_1 + \left(x_2 - \frac{\lambda_2}{\mu} \right) a_2 + ... + \left(x_p - \frac{\lambda_p}{\mu} \right) a_p = b$$

And hence $x = \left[\left(x_1 - \frac{\lambda_1}{\mu} \right) a_1, \left(x_2 - \frac{\lambda_2}{\mu} \right) a_2, ..., \left(x_p - \frac{\lambda_p}{\mu} \right), 0, 0, ..., 0 \right]$ … (6)

is a solution of the systems of equations $Ax = b$

Again from (5), we have

$$\mu \geq \frac{\lambda_i}{x_j} \text{ for } j = 1, 2, ..., p$$

Or $\mu \geq \frac{\lambda_i}{x_j}$ for $j = 1, 2, ..., p$

This implies that all the components of x_1 are non-negative and hence x_1 is feasible solution of $Ax = b, x \geq 0$. Again, for at least one value of j, we have, from (5), $x_j - \frac{\lambda_i}{\mu} = 0$, for at least one value of j.

As a result, we may observe that the feasible solution x_1 will have one extra zero than what was demonstrated in (6). Therefore, there can be no more than $(p-1)$ non-zero variables in the possible solution x_1. As a result, we have demonstrated that it is possible to decrease the number of positive variables that provide an optimal solution.

$$z' = cx_1 = \sum_{j=1}^{p} c_j \left(x_j - \frac{\lambda_j}{\mu} \right) = \sum_{j=1}^{p} c_j x_j - \sum_{j=1}^{p} c_j \frac{\lambda_j}{\mu} = z^* - \frac{1}{\mu} \sum_{j=1}^{p} c_j \lambda_j \text{, by (3)} \quad \ldots\ldots (7)$$

Now, if we can show that,

$$\sum_{j=1}^{p} c_j \lambda_j = 0 \quad \ldots (8)$$

Then $z' = z^*$ and this will prove that x_1 is an optimal solution.

We assume that equation (8) does not hold and we find a suitable real number γ, such that

$$\gamma(c_1\lambda_1 + c_1\lambda_2 + ... + c_p\lambda_p) > 0$$

i.e., $c_1(\gamma\lambda_1) + c_2(\gamma\lambda_2) + ... + c_p(\gamma\lambda_p) > 0$.

Adding $(c_1x_1 + c_2x_2 + ... + c_px_p)$ to both sides, we get

$$c_1(x_1 + \gamma\lambda_1) + c_2(x_2 + \gamma\lambda_2) + ... + c_p(x_p + \gamma\lambda_p) > c_1x_1 + c_2x_2 + ... + c_px_p = z^* \quad \ldots (9)$$

Again, multiply (4) by γ and adding to (2), we get

$$(x_1 + \gamma\lambda_1)a_1 + (x_2 + \gamma\lambda_2)a_2 + ... + (x_p + \gamma\lambda_p)a_p = b$$

So that

$$[(x_1 + \gamma\lambda_1), (x_2 + \gamma\lambda_2), ..., (x_p + \gamma\lambda_p), 0, 0, 0] \quad \ldots (10)$$

is also a solution of the system $Ax = b$

Now, we choose γ such that,

$$x_j + \gamma\lambda_j \geq 0 \; \forall \; j = 1, 2, ..., p$$

or $\gamma \geq -\frac{x_j}{\lambda_j} \; if \; \lambda_j > 0$

and $\gamma \le -\dfrac{x_j}{\lambda_j}$ if $\lambda_j > 0$

and γ is unrestricted, if $\lambda_j = 0$.

Now equation (10) becomes a feasible solution of $Ax = b, x \ge 0$.

Thus choosing γ is such a way that,

$$\underset{\substack{j \\ \lambda_j > 0}}{Max}\left\{-\frac{x_j}{\lambda_j}\right\} \le \gamma \le \underset{\substack{j \\ \lambda_j < 0}}{Max}\left\{-\frac{x_j}{\lambda_j}\right\}$$

We see from equation (9) that the feasible solution equation (10) gives a greater value of the objective function than z^*. Which is the contradiction our assumption that z^* is optimal value. Thus we can say that

$$\sum_{j=1}^{p} c_j \lambda_j = 0$$

Thus, x_1 is likewise the best option. As a result, we demonstrate that the number of non-zero variables in the given optimal solution is fewer than that of the one that was provided. If the additional non-zero variables' corresponding vectors are linearly independent, then the theorem follows because the new solution will be a fundamentally workable solution. We can further reduce the number of non-zero variables as previously mentioned to obtain a new set of optimal solutions if the new solution is once more not a fundamentally possible option. We can keep going until we arrive at an ideal solution that is also a basically feasible solution.

4.4 *SIMPLEX ALGORITHM*

Any LP problem that may be solved using a simplex algorithm always assumes the existence of a starting BFS. We shall talk about the LP issue of maximizing kind using the simplex approach here. Here's a simplified explanation of how it works:

1. **Initialization**: Start with an initial feasible solution. This can be achieved by solving a set of linear equations or inequalities that satisfy the constraints of the problem.
2. **Iteration**: The algorithm iterates through a series of steps to improve the solution. At each iteration, it selects a variable to enter the solution and a variable to leave the solution, moving towards the optimal solution.
3. **Optimality Test**: At each iteration, the algorithm checks if the current solution is optimal. If it is, the algorithm terminates. Otherwise, it proceeds to the next step.
4. **Pivoting**: If the current solution is not optimal, the algorithm performs a pivoting operation to improve the solution. This involves selecting a pivot element in the current tableau (a table representing the problem), and using it to update the tableau in a way that improves the objective function value.
5. **Repeat**: Steps 3 and 4 are repeated until an optimal solution is found.

The Simplex algorithm is efficient and can handle large-scale linear programming problems with thousands or even millions of variables and constraints. However, it's worth noting that in some cases, the algorithm may take exponential time to find the optimal solution, although this is rare in practice.

The following are the steps involved in computing an optimal solution:

Step 1: If the given LPP is of minimization type, then convert the objective function to maximizing type. Additionally, change all m constraints to non-negative b_i's (i = 1, 2, ..., m). Next, create an equation for each inequality constraint by adding a slack or surplus variable, and give that variable a zero cost coefficient in the objective function.

Step 2: If necessary, introduce artificial variable(s) and take (-M) as the coefficient of each artificial variables in the objective function.

Step 3: Obtain the initial basic feasible solution $x_B = B^{-1}b$, where B is the basis matrix (Which is here an identity matrix).

Step 4: Calculate the net evaluation $z_j - c_j = c_B x_{Bj} - c_j$.

(i) If $z_j - c_j \geq 0 \ \forall \ j$ then x_B is an optimum BFS.

(ii) If at least once $z_j - c_j < 0$. Then to improve the next solution we proceed in next step.

Step 5: If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_k - c_k$ for some $j = k$.

(i) If all $a_{ik} < 0$ (i = 1, 2, …., m), then there exist an unbounded solution to the given problem.

(ii) If at least one $a_{ik} > 0 \, (i = 1, 2, ..., m)$ then the corresponding vector a_k enters the basis B. This column is called the *key* or *pivot column*.

Step 6: Divide each value of x_B (i.e., b_i) by the corresponding (but positive) number in the key column and select a row which has the ratio non-negative and minimum i.e.,

$$\frac{x_{Br}}{a_{rk}} = Min\left\{\frac{x_{Bi}}{a_{ik}}; a_{ik} > 0\right\}$$

We refer to this rule as the minimum ratio rule. This kind of row selection is known as the pivot or key row, and it stands for the variable that will be eliminated from the fundamental solution. The key or pivot element (let's say ark) is the element that is located where the simplex table's key row and key column intersect.

Step 7: Use the relation to convert all other elements in its column to zeros and the leading element to unity by dividing its row by the key element itself:

$$\hat{a}_{rj} = \frac{a_{rj}}{a_{rk}} \text{ and } \hat{x}_{Br} = \frac{x_{Br}}{a_{rk}}, \ i = r; \ j = 1, 2, ...n$$

$$\hat{a}_{ij} = a_{ij} - \frac{a_{rj}}{a_{rk}} a_{ik} \text{ and } \hat{x}_{Br} = x_{Bi} - \frac{x_{Br}}{a_{rk}} a_{ik}, \ \ i = 1, 2, ...m; i \neq r$$

Step 8: Now go to the step 4 and repeat the procedure until all entries in $(z_j - c_j)$ are either positive or zero, or there is an indication of an unbounded solution.

4.4.1 *SIMPLEX TABLE*

The simplex for a standard LPP

Maximize, $z = cx$

Subject to, $Ax = b$

$x \geq 0$

is given below:

Where, $B = (a_{B1}, a_{B2}, ..., a_{Bm})$, basis matrix

$x_B = (x_{B1}, x_{B2}, ..., x_{Bm})$, basic variables

$c_B = [c_{B1}, c_{B2}, ..., c_{Bm}]$

$A = (a_{ij})_{m \times n}$

$b = [b_1, b_2, ..., b_m]$

$c = (c_1, c_2, ..., c_n)$

$x = (x_1, x_2, ..., x_n)$

			$c_j \rightarrow$	c_1	c_2	...	c_n
$\mathbf{c_B}$	**B**	$\mathbf{x_B}$	**b**	a_1	a_2	...	a_n
c_{B1}	a_{B1}	x_{B1}	b_1	a_{11}	a_{12}	...	a_{1n}
c_{B2}	a_{B2}	x_{B2}	b_2	a_{21}	a_{22}	...	a_{2n}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
c_{Bm}	a_{Bm}	x_{Bm}	b_m	a_{m1}	a_{m2}	...	a_{mn}
			$z_j - c_j$	$z_1 - c_1$	$z_2 - c_2$	...	$z_n - c_n$

Table 2.1: Simplex table

Example 2: Using simplex method solve the following LPP.

Maximize, $Z = x_1 - 3x_2 + 2x_3$

Subject to,

$3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Solution: We have given the problem of minimization. So, at first we convert the problem into maximization.

So, we have Max(Z_1) $= Min(-Z) = -x_1 + 3x_2 - 2x_3$. Now, introduced the slack variable x_4, x_5 and x_6, then problem can be put in the standard form as

Max(Z_1) $= -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to,

$3x_1 - x_2 + 2x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Now, we apply the simplex algorithm (Step 2 to Step 8). The outcomes of each iteration are displayed in Table 2.2. Since, $z_j - c_j \geq 0 \; \forall \; j$ in the last iteration Table 2, condition of optimality is satisfied. The optimal solutions are $x_1 = 4, x_2 = 5, x_3 = 0$ and the corresponding objective function is $(Z_1)_{\max} = 11$. Hence the solution of the original problem is $x_1 = 4, x_2 = 5, x_3 = 0$ and $(Z_1)_{\min} = -11$

			$c_j \rightarrow$	-1	3	-2	0	0	0	Mini
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Ratio
0	a_4	x_4	7	3	-1	2	1	0	0	-
0	a_5	x_5	12	-2	4	0	0	1	0	12/4=3
0	a_6	x_6	10	-4	3	8	0	0	1	10/3=3.33
		$z_j - c_j$		1	-3	2	0	0	0	
0	a_4	x_4	10	5/2	0	2	1	1/4	0	
3	a_2	x_2	3	-1/2	1	0	0	1/4	0	
0	a_6	x_6	1	-5/2	0	8	0	-3/4	1	
		$z_j - c_j$		-1/2	0	2	0	3/4	0	
-1	a_1	x_1	4	1	0	4/5	2/5	1/10	0	
3	a_2	x_2	5	0	1	2/5	1/5	3/10	0	
0	a_6	x_6	11	0	0	10	1	-1/2	1	
		$z_j - c_j$		0	0	12/5	1/5	16/20	0	

Table 2.2: Simplex table

Example 3: Niki works at Job I and Job II, two part-time employment. She has a strict limit of 12 hours per week that she would never work. She has calculated that she requires two hours of preparation time for every hour she works at Job I, and one hour of preparation time for every hour she works at Job II. She has also decided that she cannot spend more than sixteen hours preparing. How many hours a week should she work at each job to optimize her income if she makes $40 an hour at Job I and $30 an hour at Job II?

Solution: (*Solution of this example is described in step wise procedure and also described many questions which can be arise on mind during solving by simplex method***)** We will use the above-mentioned algorithm to solve this problem.

Step 1: Define the issue. Write the constraints and the goal function.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3 and so on.

$x_1 =$ The number of hours per week Niki will work at Job I and

$x_2 =$ The number of hours per week Niki will work at Job II.

Traditionally, Z is selected as the variable to be maximized.

The formulation of the problem is the same as it was in the previous chapter.

Maximize, $Z = 40x_1 + 30x_2$

Subject to,

$x_1 + x_2 \le 12$

$2x_1 + x_2 \le 16$

$x_1, x_2 \ge 0$

Step 2: Convert the inequalities into equations: For every inequality, one slack variable is added to achieve this. Convert the equality into inequality $x_1 + x_2 \le 12$. We add a non-negative variable y_1, and we get

$$x_1 + x_2 + y_1 = 12$$

Here the variable y_1 picks up the slack, and it represents the amount by which x_1+x_2 falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z = 40x_1+30x_2 = 401+302$ as $-40x_1-30x_2+Z = 0$

Subject to the constraints: $-40x_1 - 30x_2 + Z = 0$

$x_1 + x_2 + y_1 = 12$

$2x_1 + x_2 + y_2 = 16$

$x_1 \ge 0 : x_2 \ge 0$

Step 3: Construction of initial table of simplex method: Every inequality constraint is displayed in a separate row. (In the simplex tableau, the non-negativity constraints are not represented as rows.) Assign the bottom row to the objective function.

After the inequalities have been transformed into equations, we can use the following augmented matrix representation of the issue to create the initial simplex tableau.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

The left and right sides of the equations are divided in this instance by a vertical line. The goal function and constraints are divided by the horizontal line. Column C is the representation of the right side of the equation.

It is important for the reader to note that the final four columns of this matrix resemble the final matrix obtained by solving a system of equations. If we select $x_1 = 0$ and $x_2 = 0$ at random, we obtain

$$\begin{bmatrix} y_1 & y_2 & Z & | & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Which reads

y_1=0, y_2=16, Z = 0

The basic solution related to the tableau is the result of solving for the remaining variables after randomly allocating values to some of the variables. Thus, the basic solution for the original simplex tableau is the one mentioned above. As indicated in the table below, we can identify the fundamental solution variable to the right of the final column.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z

Step 4: The pivot column is indicated by the lowest row's most negative entry: Since the bottom row's most negative entry is -40, column 1 is recognized.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z
↑						

Question: Why do we select the lowest row entry that is the most negative?

Answer: The coefficient whose entry will raise the value of the objective function the fastest is the greatest coefficient in the objective function, and it is represented by the most negative entry in the bottom row.

The simplex approach starts at a corner where all of the primary variables, variables with symbols like x_1, x_2, x_3, etc. have zero values. Next, it advances from one corner point to the next, always raising the goal function's value. Increasing the value of x_1 will make more sense in the case of the objective function $Z = 40x_1+30x_2$ than x_2 will. The number of hours a week that Niki works at Job I is represented by the variable x_1. The variable x_1 will raise the goal function by $40 for every unit increment in the variable x_1, as Job I pays $40 per hour whereas Job II only pays $30.

Step 5: Do the quotient calculations. The row is identified by the least quotient. The pivot element is the one that is found at the intersection of the row found in this step and the column found in step 4.

We divide the items in the far right column by the entries in column 1, omitting the entry in the bottom row, in accordance with the algorithm to find the quotient.

x_1	x_2	y_1	y_2	Z			
1	1	1	0	0	12	y_1	$12 \div 1 = 12$
[2]	1	0	1	0	16	y_2	$\leftarrow 16 \div 2 = 8$
–40	–30	0	0	1	0	Z	
↑							

Of the two quotients, 12 and 8, 8 is the smallest. Row 2 is thus identified. The highlighted entry number two is located at the junction of row 2 and column 1. This is the key component for us.

Question: Why do we look for quotients, and how does a row become identified by its smallest quotient?

Answer: By adding the variable x_1, we want to raise the value of the objective function when we select the entry in the bottom row that is the most negative. However, we are unable to select a value for x_1. Can we allow for x_1=100? Absolutely not! This is due to

Niki's insistence on never working more than 12 hours at both jobs put together: $x_1+x_2 \leq 12$. Can we allow for $x_1=12$? Once more, the answer is no, as the time required to prepare for task I is twice that of the actual task. Niki can work no more than 16 ÷ 2 = 8 hours since she never wants to spend more than 16 hours preparing.

You now understand why it is necessary to compute the quotients; doing so ensures that we do not go against the limitations when identifying the pivot element.

Question: Why is the pivot element identified?

Answer: The simplex approach, as previously discussed, starts at one corner point and advances to the next, always increasing the value of the objective function. By altering the number of units of the variables, the objective function's value is increased. One variable's number of units may be increased while the units of another are subtracted. We can accomplish just that by pivoting.

The variable that is being added units is referred to as the entering variable, while the variable that is being replaced units are referred to as the departing variable. The highest negative item in the bottom row of the above table indicates that x_1 is the entering variable. The lowest of all quotients was used to identify the departure variable, y_2.

Step 6: Perform pivoting to make all other entries in this column zero

To get the row echelon form of an augmented matrix, we pivot the matrix. Getting a 1 at the pivot element's location and then setting all other values in that column to zeros is the process of pivoting. It is now our task to divide the entire second row by two in order to turn our pivot element into a 1. The outcome is as follows.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

We add row 1 to the second row after multiplying it by -1 to get a zero in the entry above the pivot element. We obtain

x_1	x_2	y_1	y_2	Z	
0	1/2	1	–1/2	0	4
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

We multiply the second row by 40 and add it to the last row in order to get a zero in the element below the pivot.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
[1]	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

We now ascertain the fundamental solution linked to this tableau. Upon selecting $x_2 = 0$ and $y_2 = 0$ at random, we arrive at $x_1 = 8$, $y_1 = 4$, and $z = 320$. The following matrix states the same thing if we write the augmented matrix, whose left side is a matrix with one column having a 1 and all other entries zeros.

$$\begin{bmatrix} x_1 & y_1 & Z & | & C \\ 0 & 1 & 0 & | & 4 \\ 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & | & 320 \end{bmatrix}$$

The answer linked to this matrix can be restated as follows: $z = 320$, $y_1 = 4$, $y_2 = 0$, $x_1 = 8$, and $x_2 = 0$. At this point in the game, Niki's profit Z is $320 if she works 8 hours at Job I and 0 hours at Job II.

Step 7: We are done when the bottom row contains no more negative entries; if not, we go back to step 4 and repeat the process.

We must start over at step 4 because there is still a negative entry, -10, in the bottom row. Instead of going over each step in detail again, this time we will find the column and row that contain the pivot element and highlight it. This is the outcome.

x1	x2	y1	y2	Z			
0	[1/2]	1	–1/2	0	4	y1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x1	8 ÷ 1/2 = 16
0	–10 ↑	0	20	1	320	Z	

By multiplying row 1 by 2, we create the pivot element 1, and we obtain

x1	x2	y1	y2	Z	
0	[1]	2	–1	0	8
1	1/2	0	1/2	0	8
0	–10	0	20	1	320

We are done because there are no more negative entries in the bottom row.

Question: When there are no negative entries in the bottom row, why are we done?

Answer: The bottom row has the solution. The equation and the bottom row match.

$0x_1 + 0x_2 + 20y_1 + 10y_2 + Z = 400$ **or**

$Z = 400 - 20y_1 - 10y_2$

The maximum number Z can ever reach is 400 because all variables are non-negative, and that can only occur when both y_1 and y_2 are 0.

Step 8: Now that we have determined the fundamental solution linked to the final simplex tableau, we read off our solutions. Once more, we examine the columns containing a 1 and all other entries are zeros. We choose $y_1 = 0$ and $y_2 = 0$ at random because the columns with the labels y_1 and y_2 are not such columns, and we obtain

$$\begin{bmatrix} x_1 & x_2 & Z & | & C \\ 0 & 1 & 0 & | & 8 \\ 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & | & 400 \end{bmatrix}$$

The values of the matrix are z = 400, x_1 = 4, and x_2 = 8. According to the final solution, Niki will maximize her income to $400 if she works 4 hours at Job I and 8 hours at Job II. She would have used up all of the working and preparation time because both of the slack variables are 0, meaning that none will be left.

Example 4: Using Simplex method solve the following LP problem

Maximize, $Z = 4x_1 + 10x_2$

Subject to,

$2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \ge 0$

Solution Step 1: Introducing the slack variable.

Maximize, $Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to,

$2x_1 + x_2 + s_1 = 50$

$2x_1 + 5x_2 + s_2 = 100$

$2x_1 + 3x_2 + s_3 = 90$

$x_1, x_2, s_1, s_2 \ge 0$

So, the given L.P.P. converted to the following system of linear equations.

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

Step 2: So, the basic feasible solution is given by $x_B = B^{-1}b$

i.e., $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$

Here, $B=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=B^{-1}$ and $b=\begin{bmatrix}50\\100\\90\end{bmatrix}$

Step 3: Now compute y_j and $(z_j - c_j)$ as follows:

$$y_1 = B^{-1}a_1 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{bmatrix}2\\2\\2\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$y_2 = B^{-1}a_2 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{bmatrix}1\\5\\3\end{bmatrix}=\begin{bmatrix}1\\5\\3\end{bmatrix}$$

$y_3 = B^{-1}e_1 = e_1, y_4 = B^{-1}e_2 = e_2$ and $y_5 = B^{-1}e_3 = e_3$

$$z_1 - c_1 - = c_B y_1 - c_1 = (0,0,0)\begin{bmatrix}2\\2\\2\end{bmatrix} - 4 = -4$$

$$z_2 - c_2 - = c_B y_2 - c_2 = (0,0,0)\begin{bmatrix}1\\5\\3\end{bmatrix} - 10 = -10$$

$$z_3 - c_3 - = c_B y_3 - c_3 = (0,0,0)\begin{bmatrix}1\\0\\0\end{bmatrix} - 0 = 0$$

$$z_4 - c_4 - = c_B y_4 - c_4 = (0,0,0)\begin{bmatrix}0\\1\\0\end{bmatrix} - 0 = 0$$

$$z_5 - c_5 - = c_B y_5 - c_5 = (0,0,0)\begin{bmatrix}0\\0\\1\end{bmatrix} - 0 = 0$$

Step 4: The simplex table below now displays the initial basic feasible answer.

		c_j	4	10	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	50	2	1	1	0	0
0	y_4	100	2	**5**	0	1	0
0	y_5	90	2	3	0	0	1
	z_j		0	0	0	0	0

	z_j- c_j	z (=0)	-4	-10	0	0	0

It is clear from the tableau that two of the zj-cj are negative. We select -10, which is the most negative of these. The corresponding column vector (y_2) enters the basis.

Step 5: Given that y_2's entries are all positive. We compute $Min\left\{\frac{x_{Bi}}{y_{ir}}; y_{jr} > 0\right\}$ i.e.,

$Min\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\} = \frac{100}{5}$. This occur for the element y_{22} = (=5) i.e. (element of second row and second column). As a result, the column element becomes the leading element for the first iteration and the vector y_4 departs from the basis y_B.

Step 6: Utilizing the following transformation, convert all of y_2's elements to zeros and the leading element, y_{22}, to unity:

$$\hat{y}_{ij} = y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2}; i = 1,2,3,4 \,\&\, i \neq 2$$

$$\hat{y}_{2j} = \frac{y_{2j}}{y_{22}};\ j = 0,1,2,3,4,5$$

$$\therefore \hat{y}_{21} = \frac{y_{21}}{y_{22}} = \frac{2}{5}; y_{20} = \frac{y_{20}}{y_{22}} = \frac{100}{5}\ or\ 20\ etc.$$

$$\hat{y}_{10} = y_{10} - \frac{y_{20}}{y_{22}} y_{12} = 50 - \frac{100}{5} \times 1 = 30$$

$$\hat{y}_{30} = y_{30} - \frac{y_{20}}{y_{22}} y_{32} = 90 - \frac{100}{5} \times 3 = 30$$

$$\hat{y}_{31} = y_{31} - \frac{y_{21}}{y_{22}} y_{32} = 2 - \frac{2}{5} \times 3 = \frac{4}{5}$$

$$\hat{y}_{11} = y_{11} - \frac{y_{21}}{y_{22}} y_{12} = 2 - \frac{2}{5} \times 1 = \frac{8}{5}$$

$$\hat{y}_{14} = y_{14} - \frac{y_{24}}{y_{22}} y_{12} = 0 - \frac{1}{5} \times 1 = -\frac{1}{5}$$, and so on.

Step 7: By using above mentioned calculation, the simplex table is given below:

		c_j	4	10	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	30	8/5	0	1	-1/5	0
10	y_2	20	2/5	1	0	1/5	0
0	y_5	30	4/5	0	0	-3/5	1
	z_j		4	10	0	2	0

z_j- c_j	z (=200)	0	0	0	2	0

With an increasing value of z, the following simplex table produces a new fundamental feasible solution. Moreover, since z_j- c_j > 0, there is no chance for z to increase any further. Thus, using only the most basic variables x_2, s_1 and s_3 we have arrived at our ideal answer. So, the optimal/maximal basic feasible solution of given LPP is $x_1 = 0, x_2 = 20$ with maximum $z = 200$.

Example 5: Using Simplex method solve the following LP problem

Minimize, $Z = x_2 - 3x_3 + 2x_5$

Subject to,

$3x_2 - x_3 + 2x_5 \le 7$

$-2x_2 + 4x_3 \le 12$

$-4x_2 + 3x_3 + 8x_5 \le 10$

$x_2, x_3, x_5 \ge 0$

Solution Step 1: Introducing the slack variable.

Maximize, $Z^* = -(x_2 - 3x_3 + 2x_5) + 0s_1 + 0s_2 + 0s_3$

Subject to,

$3x_2 - x_3 + 2x_5 + s_1 = 7$

$-2x_2 + 4x_3 + 0.x_5 + s_2 = 12$

$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$

$x_2, x_3, x_5, s_1, s_2, s_3 \ge 0$

So, the given L.P.P. converted to the following system of linear equations.

$$\begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_5 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} \text{ or } Ax = b$$

So, the obvious initial basic feasible solution is $x_B = B^{-1}b$ where $B = I_3$, and $x_B =$basic variable corresponding to columns of basis matrix $B(= I)$.

Step 2: So, the basic feasible solution is given by $x_B = B^{-1}b$

i.e., $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$

Here, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^{-1}$ and $b = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

Now, the simplex table is:

		c_j	-1	3	-2	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	7	3	-1	2	1	0	0
0	y_5	12	-2	**4**	0	0	1	0
0	y_6	10	-4	3	8	0	0	1
	z	0	1	-3	2	0	0	0

Since there is at least one negative z_j-c_j, or z_2-c_2, the existing basic feasible option is not optimal. We choose the column corresponding to z_2-c_2, *i.e.*, column vector y_2 enters the basis y_B (Since at least one $y_{i\,2}$>0). Further, since minimum $Min\left\{\frac{x_{Bi}}{y_{i2}}; y_{i2} > 0\right\}$ is $\frac{12}{4}(=3)$, current basis vector y_5 leaves the basis. y_{22} (= 4) is thus identified as the leading element. We now change all other elements of the incoming column vector y_2 to zero and the leading element to unity using E-Row operations. As seen in the following simplex table, we obtain the improved basic feasible solution.

		c_j	-1	3	-2	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	10	**5/2**	0	2	1	1/4	0
3	y_2	3	-1/2	1	0	0	1/4	0
0	y_6	1	-5/2	0	8	0	-3/4	1
	z	9	-1/2	0	2	0	3/4	0

Over that z_1-c_1 is negative and thus the current basic feasible solution is not optimum. The column corresponding to z_1-c_1 enters the next basis y_B (since $y_{i\,1}$>0). Further, since only

y_{11}>0 both y_{12}<0 and y_{13}<0); current basis vector y_4 leave the basis. This gives $y_{1\,1}$ (= 5/2) as the leading element. We change the leading element to unity and the other values in its column y_1 to zero using E-row operations. The next simplex table displays the improved basic feasible solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
-1	y_1	4	1	0	4/5	2/5	1/10	0
3	y_2	5	0	1	2/5	1/5	3/10	10
0	y_6	11	0	0	10	2/5	-1/2	1
	z	11	0	0	12/5	1/5	8/10	0

In this table, all z_i-c_i ≥ 0, an optimal BFS has been attained. So, the optimal solution of the given L.P.P. is,

Minimum Z = - Maximum $Z^* = -11$ with $x_2 = 4, x_3 = 5$ and $x_5 = 0$.

Check your progress

Problem 1: Using Simplex method solve the following LP problem

Maximize, $Z = 107x_1 + x_2 + 2x_3$

Subject to,

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2, x_3, x_4 \geq 0$

Answer: Unbounded Solution

Problem 2: Using Simplex method solve the following LP problem

Maximize, $Z = 3x_1 + 2x_2$

Subject to,

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2 \geq 0$

Answer: $x_1 = 3, x_2 = 1, Z = 11$

4.5 *SUMMARY*

The simplex method is a powerful algorithm used to solve linear programming problems by iteratively improving upon a feasible solution until an optimal solution is reached. The overall summarization of this units are as follows:

- The necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of $Ax = b$ is that every set of m columns of the augmented matrix $[A, b]$ is linearly independent.
- The necessary and sufficient condition for the existence and non-degeneracy of all possible basic feasible solutions of $Ax = b$, $x \geq 0$ is the linear independent of every set of m columns of the augmented matrix [A, b], where A is the m × n coefficient matrix.
- If a linear programming problem has an optimal solution, then the optimal solution will coincide with at least one basic feasible solutions of the problem.

After completion of this Unit learners will be able to solve the given LPP by using the simplex method more effectively.

4.6 *GLOSSARY*

- Slack and Surplus variable
- Simplex Method

4.7 *REFERENCES*

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4.9 *TERMINAL QUESTION*

Long Answer Type Question:

1. Using Simplex method solve the following LP problem

Maximize, $Z = 3x_1 + 2x_2 + 5x_3$

Subject to,

$x_1 + 2x_2 + x_3 \leq 430$; $3x_1 + 2x_3 \leq 460$; $x_1 + 4x_3 \leq 420$

$x_1, x_2, x_3 \geq 0$

2. Using Simplex method solve the following LP problem

Maximize, $Z = 2x_1 + 4x_2 + x_3 + x_4$

Subject to,

$x_1 + 3x_2 + x_4 \leq 4$; $2x_1 + x_2 \leq 3$; $x_2 + 4x_3 + x_4 \leq 3$

$x_1, x_2, x_3, x_4 \geq 0$

3. Using Simplex method solve the following LP problem

Maximize, $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$

Subject to,

$x_1 + 2x_2 + 2x_3 + 4x_4 \leq 80$; $2x_1 + 2x_3 + x_4 \leq 60$; $3x_1 + 3x_2 + x_3 + x_4 \leq 80$

$x_1, x_2, x_3, x_4 \geq 0$

4. Using Simplex method solve the following LP problem

Maximize, $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

Subject to,

$x_1 + x_2 + x_3 + x_4 \leq 15$; $7x_1 + 5x_3 + 3x_3 + 2x_4 \leq 120$; $3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$

$x_1, x_2, x_3, x_4 \geq 0$.

5. Using Simplex method solve the following LP problem

Maximize, $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

Subject to,

$2x_1 + x_2 + 5x_3 + 0.6x_4 \leq 10$; $3x_1 + x_2 + 3x_3 + 0.25x_4 \leq 12$; $7x_1 + x_4 \leq 35$

$x_1, x_2, x_3, x_4 \geq 0$.

6. Using Simplex method solve the following LP problem

Maximize, $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

Subject to,

$2x_1 + x_2 + 5x_3 + 0.6x_4 \leq 10$; $3x_1 + x_2 + 3x_3 + 0.25x_4 \leq 12$; $7x_1 + x_4 \leq 35$

$x_1, x_2, x_3, x_4 \geq 0$.

7. Using Simplex method solve the following LP problem

Maximize, $Z = 4x_1 + x_2 + 3x_3 + 5x_4$

Subject to,

$4x_1 - x_2 - 5x_3 - 4x_4 \leq -20$; $3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$; $8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

has an unbounded solution.

Short answer type question:

1. Using Simplex method solve the following LP problem

Maximize, $Z = 3x_1 + 2x_2$

Subject to,

$x_1 + x_2 \le 6$; $2x_1 + x_2 \le 6$;

$x_1, x_2 \ge 0$

2. Using Simplex method solve the following LP problem

Maximize, $Z = 2x_1 + 3x_2$

Subject to,

$x_1 + x_2 \le 4$; $-x_1 + x_2 \le 1$; $x_1 + 2x_2 \le 5$

$x_1, x_2 \ge 0$

3. Using Simplex method solve the following LP problem

Maximize, $Z = 5x_1 + 3x_2$

Subject to,

$x_1 + x_2 \le 2$; $5x_1 + 2x_2 \le 10$; $3x_1 + 8x_2 \le 12$

$x_1, x_2 \ge 0$

4. Using Simplex method solve the following LP problem

Maximize, $Z = 5x_1 + 3x_2$

Subject to,

$x_1 \le 4$; $x_2 \le 3$; $x_1 + 2x_2 \le 18$; $x_1 + x_2 \le 9$

$x_1, x_2 \ge 0$

5. Using Simplex method solve the following LP problem

Maximize, $Z = x_1 + 2x_2 + 3x_3$

Subject to,

$x_1 + 2x_2 + 3x_3 \le 10$; $x_1 + x_2 \le 5$

$x_1, x_2 \ge 0$

Answer:

Fill in the blanks:

1: A variable which is added to the LHS of a "≤" type constraint to convert the constraint into an equality is called …………………….

2: A variable which is subtracted from the LHS of a "≥" type constraint to convert the constraint into an equality is called ………………

3: The necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of $Ax = b$ is that every set of m columns of the augmented matrix $[A, b]$ is ……………..

4.10 *ANSWERS*

Answer of long answer type question

1: $x_1 = 0, x_2 = 100, x_3 = 230; Maximum(Z) = 1350$

2: $x_1 = 1, x_2 = 1, x_3 = 1/2, x_4 = 0; Maximum(Z) = 13/2$

3: $x_1 = 280/13, x_2 = 0, x_3 = 20/13, x_4 = 180/13; Maximum(Z) = 2280/13$

4: $x_1 = 50/7, x_2 = 0, x_3 = 55/7, x_4 = 0; Maximum(Z) = 695/7$

5: Unbounded solution.

Answer of short answer type question

1: $x_1 = 0, x_2 = 6, Maximum(Z) = 12$

2: $x_1 = 4\, and\; x_2 = 0\, or\; x_4 = 1\, and\; x_2 = 2; Maximum(Z) = 8$

3: $x_1 = 2, x_2 = 0, Max\, Z = 10$

4: $x_1 = 4, x_2 = 3, Max\, Z = 29$

5: $x_1 = 1, x_2 = 2, x_3 = 1.67;\; Max\, Z = 10$

$x_1 = 1, x_2 = 0, x_3 = 3;\; Max\, Z = 25$

Answer of fill in the blank question

1: Slack Variable

2: Surplus variable

3: Linearly independent

UNIT-5: ARTIFICIAL VARIABLE, TWO-PHASE AND BIG-M METHOD

CONTENTS:

5.1 *INTRODUCTION*

The Big-M method is a technique used in linear programming to solve problems with constraints that cannot be directly incorporated into the standard form. In linear programming, problems are typically formulated to maximize or minimize a linear objective function subject to linear equality and inequality constraints.

The Big-M method involves introducing a large positive constant (M) into the objective function for each constraint that needs to be converted from inequality to equality form. This constant ensures that the original problem's solution remains feasible even after converting the inequality constraints into equality constraints.

Since, both methods are used for solving linear programming problems, the simplex method focuses on iteratively improving a feasible solution to reach optimality, while the Big M method specifically addresses inequality constraints by introducing artificial variables and a large penalty constant to guide the optimization process.

Generally, solution of linear programming problem having artificial variables are evaluated by these two methods:

1. Big-M Method or Method of Penalties.
2. Two-Phase Method.

5.2 *OBJECTIVE*

After reading this unit learners will be able to

- Understand the basic concept of Big-M method or Method of Penalty
- Implement the Big-M method for the solution of LPP.

5.3 *Big-M METHOD*

An **artificial variable** is a temporary, fictitious variable added to equality (=) or greater-than-or-equal-to (≥) constraints in a Linear Programming problem. Its purpose is to help find an initial basic feasible solution, which is necessary for methods like the Simplex method. These variables are assigned a large penalty in the objective function so they are driven out of the final solution, as they have no physical meaning.

LPP in which constraints may also have >, and = signs after ensuring that all bi ≥ 0 are considered in this section. In such cases basis matrix cannot be obtained as an identify matrix in the stm1ing simplex table, therefore we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The **artificial variable** technique is a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.

(i) The Big M Method or Method of Penalties.

(ii) The Two-phase Simplex Method.

A non-negative variable is added to the left side of each equation in the usual form of an LPP when the choice variables, slack variables, and surplus variables are unable to pay for the original basic variables. We refer to this variable as an artificial variable.

One technique for solving LPP using artificial variables is the Big M approach. This strategy assigns a very big negative price (−M) (where M is positive) to every fake variable in the maximization type objective function. The issue can be resolved using the standard simplex approach once the artificial variable or variables have been introduced. Nevertheless, the following inferences are made from the final table when solving in simplex.

1: Providing the optimality requirement is met and there are no artificial variables left in the basis, the current solution is an optimal BFS.

2: If the optimality requirement is met and at least one artificial variable occurs in the basis at zero level, the present solution is an optimal degenerate BFS.

3: If at least one artificial variable exists in the basis at a positive level and the optimality criterion is met, then the issue has no feasible solution.

5.3.1 *Algorithm for Big-M Method*

The steps involved in applying the Big-M method generally include:

1. Convert inequality constraints to equality constraints by introducing slack variables.
2. Introduce artificial variables for any inequality constraints that have a "> =" or "=" sign, but not for those with "< ,=".
3. Introduce a term in the objective function for each artificial variable multiplied by a large positive constant (M). This term penalizes the objective function for the presence of artificial variables.
4. Solve the modified linear programming problem using standard techniques, such as the simplex method.
5. If any artificial variables remain positive in the optimal solution, it indicates that the original problem is infeasible.
6. If the problem is feasible, eliminate the artificial variables from the solution to obtain the optimal solution to the original problem.

The Big-M method is a widely used approach in linear programming, especially in introductory courses and textbooks, as it provides a systematic way to handle constraints of different types. However, care must be taken in choosing the value of M to avoid numerical instability or other computational issues.

Example 1: Using method of penalty (or Big M) solve the following LP problem

Maximize, $Z = 6x_1 + 4x_2$

Subject to,

$2x_1 + 3x_2 \leq 30$; $3x_1 + 2x_2 \leq 24$; $x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Is the solution unique? If it is not, then find two different solutions.

Solution: Introducing the slack variable.

Maximize, $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to,

$2x_1 + 3x_2 + s_1 = 30$

$3x_1 + 2x_2 + s_2 = 24$

$x_1 + 3x_2 - s_3 = 3$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Here, we can easily see that there is no initial basic feasible solution. So, we introduce an artificial variable $A_1 \geq 0$ in the third constraints. Then the initial basic feasible solution are, $s_1 = 30, s_2 = 24$ and $A_1 = 3$.

Now, corresponding to the basic variables s_1, s_2 and A_1, the matrix $Y = B^{-1}A$ (where $B = I$, the identity matrix) and the net evaluation $z_j - c_j$ $(j = 1,2,3,4,5,6)$ are computed, Where $c_B = (0,0,-M)$.

So, the table will be

		c_j	6	4	0	0	0	-M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	30	2	3	1	0	0	0
0	y_4	24	3	2	0	1	0	0
-M	y_6	3	**1**	1	0	0	-1	1
		z (= -3M)	-M-6	-M-4	0	0	M	0

In the above table we can easily see that $z_1 - c_1$ and $z_2 - c_2$ are negative. Among these two $z_1 - c_1$ has most negative value (Since M is very large), Therefore, y_1 enters the basis. Since, $Min\left\{\frac{x_{Bi}}{y_{i1}}; y_{i1} > 0\right\} = \frac{3}{1}$. This indicates y_6 leaves the basis and y_{31} becomes the leading element. Since corresponding y_6, A_1 is the artificial variable. So, we drop it from the objective function.

		c_j	6	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	24	0	1	1	0	2
0	y_4	15	0	-1	0	1	**3**
6	y_1	3	1	1	0	0	-1
		z (= 18)	0	2	0	0	-6

Since $z_5 - c_5 < 0$, y_5 enters the basis, Further, $Min\left\{\frac{x_{Bi}}{y_{i5}}; y_{i5} > 0\right\} = \frac{15}{3}$.

$\therefore y_4$ leaves the basis and y_{25} becomes the leading element.

		c_j	6	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	14	0	5/3	1	-2/3	0
0	y_5	5	0	-1/3	0	1/3	1
6	y_1	8	1	2/3	0	1/3	1
		z (= 48)	0	0	0	2	0

Since all $z_j - c_j \geq 0$. Thus, the optimal BFS of the given LPP is,

$x_1 = 8, x_2 = 0$ with max. Since $Z = 48$.

Example 2: Using method of penalty (or Big M) solve the following LP problem

Maximize, $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to,

$x_1 + 2x_2 + 3x_3 = 15;\ 2x_1 + x_2 + 5x_3 = 20;\ x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Solution: We can see from the problem's requirements that the starting **B** does not have the necessary identity column to make an identity matrix. So, we introduce artificial variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first and second constraints respectively. An initial basic feasible solution, then, is

$A_1 = 15, A_2 = 20$ and $x_4 = 10$.

Now corresponding to basic variables, A_1, A_2 and x_4, the basis matrix $Y = B^{-1}A$ and the net evaluations $z_j - c_j\ (j = 1, 2, 3, 5, 6)$ are computed, where $c_B = (-M\ \ -M\ \ -1)$. So the simplex is,

		c_j	6	4	0	0	0	-M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
$-M$	y_6	15	1	2	3	0	0	1
$-M$	y_5	20	2	1	**5**	0	1	0
-1	y_4	10	1	2	1	1	0	0
	z	$-35M$-10	$-3M$-2	$-3M$-4	$-8M$-4	0	0	0

Since the most negative $(z_3 - c_3)$ corresponds to y_3, it enters the basis. Further, $Min\left\{\frac{x_{Bi}}{y_{i3}}; y_{i3} > 0\right\} = \frac{20}{5}$, the current basis vector y_5 leaves the basis and y_{23} becomes the leading element. As y_5 corresponds to an artificial variable A_2, we drop y_5 column from subsequent simplex tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_6
$-M$	y_6	3	-1/5	**7/5**	0	0	1
3	y_3	4	2/5	1/5	1	0	0
-1	y_4	6	3/5	9/5	0	1	0

z	-3M+6	$\frac{M}{5}-\frac{2}{5}$	$\frac{-7M}{5}-\frac{16}{5}$	0	0	0

Clearly, $(z_2 - c_2)$ is the only negative and hence y_5 enters the basis. Further $Min\left\{\frac{x_{Bi}}{y_{i2}}; y_{i2} > 0\right\}$ correspond to y_6. So, y_6 leaves the basis and y_{12} becomes the leading element. Again, y_6 corresponds to the artificial variable A_1 and therefore we drop the artificial column y_6 in the subsequent tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4
2	y_2	15/7	-1/7	1	0	0
3	y_3	25/7	3/7	0	1	0
-1	y_4	15/7	**6/7**	0	0	1
	z	90/7	$-6/7$	0	0	0

Clearly, $z_2 - c_2 < 0$ and, therefore, y_1 enters the basis. Further, $Min\left\{\frac{x_{Bi}}{y_{i1}}; y_{i1} > 0\right\}$ corresponds to y_4. So, y_4 leaves the basis and y_{31} becomes the leading element.

c_B	y_B	x_B	y_1	y_2	y_3	y_4
2	y_2	15/6	0	1	0	1/6
3	y_3	15/6	0	0	1	-3/6
1	y_1	15/6	1	0	0	7/6
	z	15	0	0	0	1

Since, all $z_j - c_j$ are positive, therefore, an optimum basic feasible solution has been attained. Hence, for the given LPP the optimal solution is,

Maximize, $z = 15$; $x_1 = x_2 = x_3 = 5/2$ and $x_4 = 0$

Example 3: Using method of penalty (or Big M) solve the following LP problem

Minimize, $Z = 2x_1 + 3x_2$

Subject to,

$x_1 + x_2 \geq 15$; $x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0$

Proof: Introducing artificial and surplus variable, the given problem written in the standard form as,

Maximize $(Z^{'}) = \text{Minimize}\,(-Z) = -2x_1 - 3x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$

Subject to,

$x_1 + x_2 - x_3 + x_5 = 5$; $x_1 + 2x_2 - x_4 + x_6 = 6$; $x_1, x_2, x_3, ..., x_6 \geq 0$

			$c_j \rightarrow$	-2	-3	0	0	-M	-M	Mini Ratio
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	
-M	a_5	x_5	5	1	1	-1	0	1	0	5/1=5
-M	a_6	x_6	6	1	2	0	-1	0	1	6/2=3
$z_j - c_j$				-2M+2	-3M+3	M	M	0	0	
-M	a_5	x_5	2	1/2	0	-1	1/2	1	$\times$	2/(1/2)=4
-3	a_2	x_2	3	1/2	1	0	-1/2	0	$\times$	3/(1/2)=6
$z_j - c_j$				-(M+1)/2	0	M	(-M+3)/2	0	$\times$	

-2	a_1	x_1	4	1	0	-2	1	×	×	
-3	a_2	x_2	1	0	1	1	-1	×	×	
$z_j - c_j$				0	0	1	1	×	×	

From the last iteration in above mentioned table we see that $z_j - c_j \geq 0$ for all j. Hence, the optimality is satisfied. So, the optimal solution of given LPP is $x_1 = 4, x_2 = 1$ and the corresponding $Z_{\min} = 11$

5.4 *TWO PHASE METHOD*

The Two-Phase Method is an algorithm used to solve linear programming problems, particularly those that are not initially in standard form. It's a systematic approach to convert such problems into standard form and then solve them using the simplex method. Here's a detailed explanation of the Two-Phase Method:

1. **Problem Setup**:

- Begin with a linear programming problem that may not be expressed in the standard form, i.e., it may contain inequalities, non-negativity constraints, or objective functions that are not in the form of maximization or minimization.

2. **Phase I**:

- **Objective**: The objective of Phase I is to convert the original problem into an equivalent problem that can be solved using the simplex method. This involves introducing artificial variables to transform the problem into standard form.
- **Artificial Variables**: Artificial variables are introduced for each inequality constraint in the problem. These artificial variables help create an initial basic feasible solution. The objective function in Phase I aims to minimize the sum of these artificial variables.
- **Initial Solution**: The simplex method is then applied to this modified problem to find a basic feasible solution.

- **Feasibility Check**: If the minimum value of the artificial variables is zero, indicating that the original problem is feasible, the method proceeds to Phase II. If the minimum value is positive, it suggests that the original problem is infeasible.

3. **Phase II**:

- **Objective**: In Phase II, the artificial variables introduced in Phase I are eliminated, and the original objective function is reintroduced.

- **Optimization**: The simplex method is applied to optimize the original objective function while maintaining feasibility. The basic feasible solution obtained from Phase I serves as the starting point for Phase II.

- **Optimal Solution**: The optimal solution found in Phase II is the solution to the original linear programming problem.

4. **Conclusion**:

- Once Phase II is completed, the optimal solution provides the values of decision variables that maximize or minimize the objective function, subject to the given constraints.

- The Two-Phase Method ensures that linear programming problems can be solved even if they are not initially presented in standard form, providing a systematic approach to conversion and solution.

5.4.1 *PROBLEM SOLVBING OF TWO PHASE METHOD*

To obtain a basic feasible solution to the original L.P.P., the first part of this method involves minimizing the sum of the artificial variable, subject to the stated constraints (known as the auxiliary L.P.P.). Beginning with the fundamentally feasible solution found at the conclusion of phase 1, the second step optimizes the original objective function.

The algorithm's iterative process can be summed up as follows:

Step 1: Put the provided L.P.P. into standard form and see if there is a feasible, basic solution already in place.

(a) Proceed to phase 2 if a fundamental, feasible solution is available at the present time.

(b) Proceed to the following step if a ready, basic, and feasible solution is not available.

Phase I

Step 2: Add the artificial variable on the left side of every equation in which the initial basic variables are missing. Construct an auxiliary objective function with the goal of minimizing the overall sum of artificial variables.

Thus, the new objective is to

Minimize $z = A_1 + A_2 + ... + A_n$

i.e., Maximize $z^* = -A_1 - A_2 - ... - A_n$

where, $A_i (i = 1,2,..., m)$ are the non-negative artificial variables.

Step 3: Use the specially created L.P.P. and the simplex method. The least possible interaction could result in either of the following three cases:

a. $Max(Z^*) < 0$ and atleast one artificial vector appear in the optimum basis at a positive level. In this instance, there is no feasible solution for the provided problem.

b. $Max(Z^*) = 0$ and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase-II.

c. $Max(Z^*) = 0$ and no one artificial vector appears in the optimum basis. In this case also proceed to phase-II.

Phase II

Step 4: At this point, give each artificial variable that shows up in the basis at the zero level a zero cost, and assign the actual cost to each variable in the objective function. Now, with the specified restrictions, the simplex approach maximizes this new objective function. The modified simplex table that was created at the end of phase I is subjected to the simplex method until an optimal basic feasible solution is reached. At the conclusion of phase, I, the artificial variables that are not basic are eliminated.

Note: It is possible to completely remove artificial variables from the simplex table that do not occur in the fundamental solution.

Example 1: Using two-phase method solve the following LP problem

Maximize, $Z = 5x_1 + 3x_2$

Subject to,

$2x_1 + x_2 \leq 1$; $x_1 + 4x_2 \geq 6$;

$x_1, x_2 \geq 0$

Solution: Introducing the slack variable $s_1 \geq 0$, a surplus variables $s_2 \geq 0$ and an artificial variables $A_1 \geq 0$ in the constraints of the linear programming problem.

So, the initial basic feasible solution is; $s_1 = 1$ and $A_1 = 6$ with l_2 as the basis matrix.

Phase 1: The objective function of the auxiliary L.P.P. is to maximize $Z^* = -A_1$. Using now simplex algorithm to the auxiliary linear programming problem, the simplex table is,

Initial iteration: Dropping of y_3 and introducing of y_2.

		c_j	0	0	0	0	-1
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	1	2	**1**	1	0	0
-1	y_5	6	1	4	0	-1	1
	z_j	-	-1	-4	0	1	-1
	z_j - c_j	z (= -6)	-1	-4	0	1	0

Since $z_1 - c_1$ and $z_2 - c_2$ are negative, we choose the most negative of these, *viz.*, -4. The corresponding column vector y_2 enters the basis, Therefore, y_1 enters the basis. Further, since, $Min\left\{\frac{x_{Bi}}{y_{i2}}; y_{i2} > 0\right\} = 1$, which occurs for element y_{12}, y_3 leaves the basis.

Final iteration: Optimal solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_2	1	2	1	1	0	0
-1	y_5	2	-7	0	-4	-1	1
		z (= -2)	7	0	4	1	0

Since all $(z_j - c_j) \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. is obtained . But max. $z^* < 0$ and an artificial variables is in the basis at a positive level. Thus, there isn't a feasible option in the original L.P.P.

Example 2: Using two-phase method solve the following LP problem

Maximize, $Z = 5x_1 - 4x_2 + 3x_3$

Subject to the constraints,

$2x_1 + x_2 - 6x_3 = 20; 6x_1 + 5x_2 + 10x_3 \leq 76;\ 8x_1 - 3x_2 + 6x_3 \leq 50;$

$x_1, x_2, x_3 \geq 0$

Solution: Introducing the slack variables $s_1 \geq 0$ and $s_2 \geq 0$, the given L.P.P. in the standard form is:

Maximize, $Z = 5x_1 - 4x_2 + 3x_3$, subject to the constraints:

$2x_1 + x_2 - 6x_3 = 20; 6x_1 + 5x_2 + 10x_3 + s_1 = 76;\ 8x_1 - 3x_2 + 6x_3 + s_2 = 50;$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

In the matrix form the set of constraints is,

$$\begin{pmatrix} 2 & 1 & -6 & 0 & 0 \\ 6 & 5 & 10 & 1 & 0 \\ 8 & -3 & 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{bmatrix} 20 \\ 76 \\ 50 \end{bmatrix}$$

Given that there are no identity matrix columns that can be used as the initial basis matrix. To complete the identity basis matrix, we add the necessary identity column or columns. Put in the identity column [1 0 0] as the new column y_6, in other words. Clearly, this amounts to the adding an artificial variable $A_1 \geq 0$ in the 1^{st} constraints.

Now, an initial basic feasible solution is $A_1 = 20$, $s_1 = 76$ and $s_2 = 50$.

Phase 1: The objective function of the auxiliary L.P.P. is $z^* = -A_1$. The iterative simplex tables for the auxiliary L.P.P. are;

Initial iteration: Introduce y_1 and drop y_5.

		c_j	0	0	0	0	0	-1
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
-1	y_6	20	2	1	-6	0	0	1
0	y_5	76	6	5	10	1	0	0
0	y_4	50	**8**	-3	6	0	1	0
		z^* (= -20)	-2	-1	6	0	0	0

Since, $z_1 - c_1$ is negative, the column vector y_1 enters the basis. Further, since $Min\left\{\frac{x_{Bi}}{y_{i1}}; y_{i1} > 0\right\} = \frac{50}{8}$; y_5 leaves the basis. The element y_{31} (=8) becomes the leading element.

First iteration: Introduce y_2 and drop y_6.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
-1	y_6	15/2	0	**7/4**	-15/2	0	-1/4	1
0	y_4	77/2	0	29/4	11/2	1	-3/4	0
0	y_1	25/4	1	-3/8	3/4	0	1/8	0
		z^* (= -15/2)	0	-7/4	15/2	0	1/4	0

Here, $z_2 - c_2$ is the only negative $z_j - c_j$. This indicates that y_2 enters the basis. Also $Min\left\{\frac{x_{B2}}{y_{i2}}; y_{i2} > 0\right\} = \frac{(15/2)}{(7/4)}$ suggested that y_6 must leave the basis, thereby $y_{12}\,(-7/4)$ becomes the leading element.

Final iteration: Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6

0	y_2	30/7	0	1	-30/7	0	-1/7	4/7
0	y_4	52/7	0	0	256/7	1	2/7	-29/7
0	y_1	55/7	1	0	-6/7	0	1/14	3/14
		z^* (= 0)	0	0	0	0	0	1

Since all $z_j - c_j \geq 0$ an optimum solution to the auxiliary L.P.P has been reached. Moreover, the table makes it clear that there are no artificial variables in the base.

Phase 2: Now, we consider the actual costs associated with the original variables. So, the objective function is,

$$Z = 5x_1 - 4x_2 + 3x_3 + 0.s_1 + 0.s_2$$

The iterative simplex table for this phase is:

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
-4	y_2	30/7	0	1	-30/7	0	-1/7
0	y_4	52/7	0	0	256/7	1	2/7
5	y_1	55/7	1	0	-6/7	0	1/14
		z^* (= 155/7)	0	0	69/7	0	13/4

Since all $z_j - c_j \geq 0$ an optimum basic feasible solution has been reached. Hence an optimum basic feasible solution to the given L.P.P. is,

$x_1 = 55/7, x_2 = 30/7, x_3 = 0;$ maximum $z = 155/7,$

Example 3: Maximize, $Z = 3x_1 - x_2$

Subject to the constraints,

$$2x_1 + x_2 \geq 2; x_1 + 3x_2 \leq 2; x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution: Maximize, $Z = 3x_1 - x_2$

Subject to the constraints,

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

So, the auxiliary LPP will become

Maximize, $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$

Subject to,

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-1	2	[2]	1	-1	0	0	1	1→
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
	$Z^* = -2$		↑ -2	-1	1	0	0	0	←Δ_j
x_1	0	1	1	1/2	-1/2	0	0	x	
s_2	0	1	0	5/2	1/2	1	0	x	
s_3	0	4	0	1	0	0	1	x	
	$Z^* = 0$		0	0	0	0	0	x	←Δ_j

Since all $\Delta_j \geq 0, MaxZ^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$	3	-1	0	0	0	

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B/X_k
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	[1/2]	1	0	2→
s_3	0	4	0	1	0	0	1	-
	Z = 3		0	5/2	↑ -3/2	0	0	←Δ_j
x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	←Δ_j

Since all $\Delta_j \geq 0,$ optimal basic feasible solution is obtained.

Hence, the solution is $Max\, Z = 6, x_1 = 2, x_2 = 0$.

Example 4: Maximize, $Z = 5x_1 + 8x_2$

Subject to the constraints,

$3x_1 + 2x_2 \geq 3; x_1 + 4x_2 \geq 4; x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

Solution: Maximize, $Z = 5x_1 + 8x_2$

Subject to the constraints,

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

So, the auxiliary LPP will become

Maximize, $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to,

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B/X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	[4]	0	-1	0	0	1	1→
s_3	0	5	1	1	0	0	1	0	0	5
		$Z^* = -7$	-4	↑ -6	1	1	0	0	0	←Δ_j
a_1	-1	1	[5/2]	0	-1	1/2	0	1	x	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	x	4
s_3	0	4	3/4	0	0	1/4	1	0	x	16/3
		$Z^* = -1$	↑ -5/2	0	1	-1/2	0	0	x	←Δ_j
x_1	0	2/5	1	0	-2/5	1/5	0	x	x	
x_2	0	9/10	0	1	1/10	-3/10	0	x	x	
s_3	0	37/10	0	0	3/10	1/10	1	x	x	
		$Z^* = 0$	0	0	0	0	0	x	x	←Δ_j

Since all $\Delta_j \geq 0$, $Max Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B/X_k
x_1	5	2/5	1	0	-2/5	[1/5]	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
	Z = 46/5		0	0	-6/5	↑ -7/5	0	←Δ_j
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	[1/2]	0	1	7→
	Z = 12		7	0	↑ -4	0	0	←Δ_j
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	Z = 40		3	0	0	0	4	

Since all $\Delta_j \geq 0,$ optimal basic feasible solution is obtained. Therefore, the solution is,

$Max\, Z = 40, x_1 = 0, x_2 = 5$

Check your progress

Problem 1: Using penalty method to solve the following LP problem

Maximize, $Z = 2x_1 + 3x_2$

Subject to,

$x_1 + 2x_2 \leq 4$

$x_1 + x_2 = 3$

$x_1, x_2 \geq 0$

Answer: $x_1 = 2, x_2 = 1$, Maximum $Z = 7$

Problem 2: Using Big-*M* method to solve the following LP problem

Minimize, $Z = 12x_1 + 20x_2$

Subject to,

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1, x_2 \geq 0$

Answer: $x_1 = 15, x_2 = 5/4$, Minimum $Z = 205$

5.5 *SUMMARY*

The Big-M method is a technique used in linear programming to solve problems involving artificial variables, typically in cases where constraints cannot be easily transformed into standard form. It involves adding artificial variables with a very large positive or negative coefficient, represented by "*M*," to the objective function. The purpose of these artificial variables is to facilitate finding an initial feasible solution. The algorithm then proceeds to minimize the impact of these artificial variables by driving their coefficients to zero, effectively removing them from the solution. If any artificial variables remain in the final solution with non-zero values, it indicates that the original problem has no feasible solution. The Big-*M* method is particularly useful in dealing with complex constraints and ensures that the artificial variables do not influence the optimal solution, unless they are necessary to indicate infeasibility.

5.6 *GLOSSARY*

- Big-*M* method or Method of penalty

5.7 *REFERENCES*

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5.9 *TERMINAL QUESTION*

Long Answer Type Question:

1. Using Big-*M* method to solve the following LP problem

Minimize, $Z = 5x_1 - 6x_2 - 7x_3$

Subject to,

$x_1 + 5x_2 - 3x_3 \geq 15$; $5x_1 - 6x_2 + 10x_3 \geq 0$; $x_1 + x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

2. Using Big-*M* method to solve the following LP problem

Maximize, $Z = 2x_1 + x_2 + 3x_3$

Subject to,

$x_1 + x_2 + 2x_3 \leq 5$; $2x_1 + 3x_2 + 4x_3 = 12$;

$x_1, x_2, x_3 \geq 0$

3. Using Big-*M* method (Penalty method) to solve the following LP problem

Maximize, $Z = 8x_2$

Subject to the constraints;

$x_1 - x_2 \geq 0$; $2x_1 + 3x_2 \leq 6$; $3x_1 + 3x_2 + x_3 + x_4 \leq 80$

x_1, x_2 are unrestricted

Short answer type question:

1. Using Big-*M* method (Penalty method) to solve the following LP problem

Maximize, $Z = 2x_1 + x_2$

Subject to,

$3x_1 + x_2 = 3; 4x_1 + 3x_2 \geq 6; x_1 + 2x_2 \leq 3;$

$x_1, x_2 \geq 0$

2. Using Big-*M* method (Penalty method) to solve the following LP problem

Maximize, $Z = 3x_1 + 2x_2 + x_3$

Subject to,

$2x_1 + x_2 + x_3 = 12;\ \ 3x_1 + 4x_3 = 11;$

$x_2, x_3 \geq 0$ and x_1 is unrestricted.

Fill in the blanks:

1: Linear programming is a technique of finding the …………………….

2: Any solution to a linear programming problem which also satisfies the non-negative notification of the problem has ………………

5.10 *ANSWERS*

Answer of long answer type question

1: No feasible solution.

2: $x_1 = 3, x_2 = 2, x_3 = 0; Maximum(Z) = 8$

3: $x_1 = -6/5, x_2 = -6/5; Maximum(Z) = -48/5$

Answer of short answer type question

1: $x_1 = 3/5, x_2 = 6/5, Minimum(Z) = 12/5$

2: $x_1 = 11/3, x_2 = 0, x_3 = 14/3, Maximize(Z) = 47/3$

Answer of fill in the blank question

1: Optimal value

2: Feasible solution

UNIT- 6: RESOLUTION OF DEGENERACY

CONTENTS:

6.1 *INTRODUCTION*

Degeneracy in linear programming occurs when a basic feasible solution has at least one basic variable with a value of zero. This can happen when there is a tie for the minimum positive replacement ratio in the simplex method, leading to a situation where an arbitrary choice is made for the leaving variable. While it doesn't affect the feasibility or optimality of the solution, degeneracy can cause the simplex method to take more iterations, potentially leading to a cycling problem where the same set of basic feasible solutions repeat without improving the objective function.

Recall that the simplex algorithm tries to increase a non-basic variable X_s . If there is no degeneracy, then xs will be positive after the pivot, and the objective value will improve. Recall also that each solution produced by the simplex algorithm is a basic feasible solution with m basic variables, where m is the number of constraints.

There are a finite number of ways of choosing the basic variables. (An upper bound is n! / (n - m)! m!, which is the number of ways of selecting m basic variables out of n.) So, the simplex algorithm moves from bfs to bfs. And it never repeats a bfs because the objective is constantly improving. This shows that the simplex method is finite, so long as there is no degeneracy.

6.2 *OBJECTIVE*

After reading this unit learners will be able to

- Understand the concept of Degeneracy in LPP.

6.3 *DEGENRACY IN LINEAR PROGRAMMING*

In this section of linear programming, degeneracy occurs when a feasible solution has more than one way to be optimal, or more technically, when the number of basic variables is less than the number of constraints at a basic feasible solution. Here's a detailed look at degeneracy in linear programming:

1. Understanding Degeneracy

Definition:

- **Degeneracy at a Vertex:** In a linear programming problem, a vertex of the feasible region is degenerate if there are more constraints (hyperplanes) intersecting at that vertex than the number of dimensions (basic variables).
- **Degeneracy in the Simplex Method:** A basic feasible solution (BFS) is degenerate if one or more of the basic variables are zero.

2. Causes of Degeneracy

- **Redundant Constraints:** Extra constraints that do not change the feasible region but increase the number of intersections.
- **Multiple Optimal Solutions:** When the objective function is parallel to a constraint boundary, leading to multiple solutions along that boundary.

3. Implications of Degeneracy

- **Cycling:** In the simplex method, degeneracy can cause the algorithm to revisit the same BFS repeatedly, potentially leading to an infinite loop (cycling).

- **Stalling:** The simplex method might make a pivot that does not improve the objective function, causing the algorithm to "stall" and take longer to find the optimal solution.

4. Handling Degeneracy

Anti-Cycling Rules:

- **Bland's Rule:** Choose the entering and leaving variables using a fixed order to prevent cycling.
- **Lexicographic Ordering:** Maintain a lexicographic ordering of the variables to ensure progress in each step.

Perturbation Techniques:

- Slightly modify the right-hand side of the constraints to break ties and remove degeneracy artificially.

Interior-Point Methods:

- These methods approach the optimal solution from within the feasible region rather than along the edges, avoiding degeneracy issues inherent in vertex-based methods like the simplex algorithm.

Degeneracy: Generativity is the property of obtaining a degenerate fundamental feasible solution in a linear programming problem.

In an L.P.P., degeneracy can occur (i) at the beginning and (ii) at any point during the subsequence iteration.

In case (i), every basic variable in the first basic feasible solution is zero. In case (ii), however, multiple vectors are allowed to exit the basis at any time during a simplex method iteration. As a result, the subsequent simplex iteration yields a degenerate solution where every basic variable is zero. This implies that the objective function's value might not increase in the ensuing iterations. Therefore, without enhancing the answer, the same simplex iteration subsequence can be repeated indefinitely. We call this idea "cycling."

Generally speaking, degeneracy is not problematic—that is, unless cycling happens. If there is a tie in the replacement ratios, it usually suffices to choose a row at random. However, by following these guidelines, the number of iterations needed to reach the optimal can be reduced.

(***i***) Using the matching positive elements of the entering column vector, go from left to right to divide the coefficients of basic variables (element of the column vector of the basic matrix) in the simplex table where degeneracy is identified.

(***ii***) The corresponding current basis vector departs the basis and the row with the least ratio, measured from left to right column wise, becomes the pivot row.

Example 4: Maximize, $Z = 22x_1 + 30x_2 + 25x_3$

Subject to the constraints,

$2x_1 + 2x_2 \le 100;\ 2x_1 + x_2 + x_3 \le 100,\ x_1 + 2x_2 + 2x_3 \le 100;$

$x_1, x_2, x_3 \ge 0$

Solution: By introducing slack variables $s_1 \ge 0, s_2 \ge 0$ and $s_3 \ge 0$ in the respective inequalities, the set of constraints can be written as $Ax = b$, where

$$A = \begin{bmatrix} 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{bmatrix}$$

An obvious initial (starting) basic feasible solution in $x_B = B^{-1}b$, where $x_B = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$, $B = I_3$ and $b = [100\ 100\ 100]$.

$$x_B = I^{-1}b = Ib \text{ gives } \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} = [100\ 100\ 100]$$

Using now simplex method, the iterative simplex table are:

Initial iteration: Introduce y_2 and drop y_6

		c_j	22	30	25	0	0	-0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	100	2	2	0	1	0	0
0	y_5	100	2	1	1	0	1	0

0	y_6	100	1	**2**	2	0	0	1
	z	0	-22	-30	-25	0	0	0

Since, $z_2 - c_2$ is the most negative $z_j - c_j$, y_2 enters the basis. Further, $Min\left\{\frac{x_{Bi}}{y_{i2}}; y_{i2} > 0\right\} = 50$ occurs for the element y_{12} and y_{32}. Thus, there is tie among the ratios in the first and third rows, *i.e*. among the basis vectors y_4 and y_6. To obtain the unique current basis vector that will leave the basis, we compute the ratios $\left\{\frac{y_{ij}}{y_{i2}}; y_{i2} > 0\right\}$ instead of $\left\{\frac{x_{Bi}}{y_{i2}}; y_{i2} > 0\right\}$ for those column vector which are in the basis. Here, since y_4, y_5 and y_6 are in the basis and there is a tie among y_4 and y_6 for leaving the basis, we write the coefficients (elements) from above table:

	y_4	y_5	y_6
y_4	1	0	0
y_6	0	0	1

Dividing these coefficients by the corresponding element of the entering column, i.e., of y_2, we obtain the following ratios:

	y_4	y_5	y_6
y_4	1/2	0/2	0/2
y_6	0/2	0/2	1/2

On comparing of ratios in the first column $y_6 -$row yields the smallest ratio and hence y_6 leaves the basis.

First iteration: Introduce y_1 and drop y_4

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	0	**1**	0	-2	1	0	-1
0	y_5	50	3/2	0	0	0	1	-1/2
30	y_2	50	1/2	1	1	0	0	1/2
	z	1500	-7	0	5	0	0	15

It is apparent from the above table that, $(z_1 - c_1) < 0$ and therefore y_1 enters the basis. Further, since $Min\left\{\frac{x_{Bi}}{y_{i1}}; y_{i1} > 0\right\} = \frac{0}{1} = 0$, on the current basis vector y_4 leaves the basis and y_{11} becomes the leading element.

Second iteration: Introduce y_3 and drop y_5.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
22	y_1	0	1	0	-2	1	0	-1
0	y_5	50	0	0	**3**	-3/2	1	-1
30	y_2	50	0	1	2	-1/2	0	1
	z	1,500	0	0	-9	7	0	8

Clearly, the solution is still not optimum, since $(z_3 - c_3) < 0$. So, y_3 enters the basis. Further, since $Min\left\{\frac{x_{Bi}}{y_{i3}}; y_{i3} > 0\right\} = \frac{50}{3}$, the current basis vector y_5 leaves the basis and y_{23} becomes the leading element.

Final iteration:

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
22	y_1	100/3	1	0	0	0	2/3	-1/3
25	y_3	50/3	0	0	1	-1/2	1/3	1/3

30	y_2	50/3	0	1	0	1/2	-2/3	1/3
	z	1,650	0	0	0	5/2	3	11

Since, all $(z_j - c_j) \geq 0$, an optimum basic feasible solution is,

$x_1 = 100/3, x_1 = 100/3, x_3 = 50/3$ and maximum $z = 1,650$.

Check your progress

Problem 1: Using two phase method to solve the following LP problem

Maximize, $Z = 10x_1 + 20x_2$

Subject to the constraint,

$2x_1 + x_2 = 1$

$x_1 + 2x_2 = 5$

$x_1, x_2 \geq 0$

Answer: $x_1 = 0, x_2 = 3$, Maximum $z = 60$

Problem 2: Using two phase method to solve the following LP problem

Minimize, $Z = 2x_1 + 4x_2$

Subject to, $2x_1 + x_2 \geq 14$; $x_1 + 3x_2 \geq 18$; $x_1 + x_2 \geq 12$; $x_1, x_2 \geq 0$

Answer: $x_1 = 18, x_2 = 0$, Minimum $z = 36$

6.4 *SUMMARY*

Degeneracy is important because we want the simplex method to be finite, and the generic simplex method is not finite if bases are permitted to be degenerate.

In principle, cycling can occur if there is degeneracy. In practice, cycling does not arise, but no one really knows why not. Perhaps it does occur, but people assume that the simplex algorithm is just taking too long for some other reason, and they never discover the cycling. • Researchers have developed several different approaches to ensure the finiteness of the simplex method, even if the bases can be degenerate. Bob Bland developed a very simple rule that prevents cycling.

Degeneracy in linear programming is a common occurrence, especially in large and complex problems. While it can complicate the solution process by causing cycling or stalling, several strategies like Bland’s Rule, perturbation techniques, and the use of interior-point methods effectively address these issues. Understanding and handling degeneracy is crucial for efficient and accurate linear programming solutions.

6.5 *GLOSSARY*

- Degeneracy in linear programming

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6.8 *TERMINAL QUESTION*

Long Answer Type Question:

1. Solve the L.P.P.

Maximize, $z = 5x_1 - 2x_2 + 3x_3$

Subject to,

$2x_1 + 2x_2 - x_3 \geq 2$; $3x_1 - 4x_2 \leq 3$; $x_2 + 3x_3 \leq 5$;

$x_1, x_2, x_3 \geq 0$

2. Solve the L.P.P.

Maximize, $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$

Subject to,

$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$; $2x_1 + x_2 + x_3 + 5x_4 \leq 4$; $2x_1 + 6x_2 - 8x_3 + 4x_4 \leq 5$;

$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$; $x_1, x_2, x_3 \geq 0$

3. Using Two-phase method

Maximize, $z = x_1 + 2x_2 + 3x_3$

Subject to,

$x_1 - x_2 + x_3 \geq 4$; $x_1 + x_2 + 2x_3 \leq 8$; $x_1 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

4. Using Two-phase method

Maximize, $z = 12x_1 + 15x_2 + 9x_3$

Subject to,

$8x_1 + 16x_2 + 12x_3 \leq 250$; $4x_1 + 8x_2 + 10x_3 \geq 80$; $7x_1 + 9x_2 + 8x_3 = 105$

$x_1, x_2, x_3 \geq 0$

Short answer type question:

1: Solve the L.P.P.

Maximize, $z = 3x_1 + 2x_2 + 3x_3$

Subject to,

$2x_1 + x_2 + x_3 \leq 2$; $3x_1 + 4x_2 + 2x_3 \geq 8$;

$x_1, x_2, x_3 \geq 0$

2: Using two-phase method solve the following L.P.P.

Minimize, $z = 2x_1 + 4x_2$

Subject to,

$2x_1 + x_2 \geq 14$; $x_1 + 3x_2 \geq 18$; $x_1 + x_2 \geq 12$; $x_1, x_2 \geq 0$

3: Using two-phase method solve the following L.P.P.

Minimize, $z = 3x_1 - x_2$

Subject to, $2x_1 + x_2 \geq 2$; $x_1 + 3x_2 \leq 2$; $x_2 \leq 4$; $x_1, x_2 \geq 0$

4: Using two-phase method solve the following L.P.P.

Maximize, $z = 5x_1 + 8x_2$

Subject to,

$3x_1 + 2x_2 \geq 3$; $x_1 + 4x_2 \geq 4$; $x_1 + x_2 \leq 5$; $x_1, x_2 \geq 0$

5. Using two-phase method solve the following L.P.P.

Minimize, $z = x_1 + x_2 + x_3$

Subject to, $x_1 - 3x_2 + 4x_3 = 5$; $x_1 - 2x_2 \leq 3$; $2x_2 + x_3 \geq 4$; $x_1, x_2 \geq 0$

Objective type question:

1: What is the purpose of the Two-Phase Method in linear programming?

a) To find the optimal solution directly

b) To handle problems where the initial basic feasible solution is not readily apparent

c) To maximize the objective function

d) To minimize the objective function

2: In the first phase of the Two-Phase Method, the objective function is:

a) The original objective function

b) An artificial objective function, usually the sum of artificial variables

c) A constant value

d) Unchanged

3: Which of the following is introduced in the first phase of the Two-Phase Method?

a) Slack variables

b) Surplus variables

c) Artificial variables

d) All of the above

4: If the minimum value of the artificial objective function at the end of the first phase is zero, this indicates:

a) The original problem has no feasible solution

b) The original problem is unbounded

c) A feasible solution to the original problem has been found

d) The problem needs to be reformulated

5: What happens if the artificial variables are still in the basis at the end of Phase 1?

a) The original problem has multiple optimal solutions

b) The original problem is infeasible

c) The original problem is unbounded

d) The artificial variables are ignored in Phase 2

6: In Phase 2 of the Two-Phase Method, what is done after removing the artificial variables?

a) The original objective function is optimized using the feasible basis found in Phase 1

b) The process is restarted from Phase 1

c) New artificial variables are introduced

d) The solution is checked for optimality and feasibility

7: Why are artificial variables introduced in the Two-Phase Method?

a) To convert inequalities into equalities

b) To provide an initial basic feasible solution when one is not apparent

c) To increase the complexity of the problem

d) To ensure the problem is bounded

8: Which of the following statements is true regarding the Two-Phase Method?

a) It guarantees an optimal solution in all cases

b) It is used when the primal problem has a readily available basic feasible solution

c) The second phase deals with the original linear programming problem after feasibility is ensured in the first phase

d) It is only applicable to problems with all constraints as equalities

6.9 *ANSWERS*

Answer of long answer type question

Answer 1: $x_1 = 23/3, x_2 = 0, x_3 = 5$; Maximum $z = 85/3$

2: $x_1 = 0; x_2 = 1.2; x_3 = 0.9; x_4 = 0;$; Maximum $z = 3.6$

3: $x_1 = 18/5, x_2 = 6/5, x_3 = 8/5$; Maximum $z = 108$

4: $x_1 = 6, x_2 = 7, x_3 = 0$; Maximum $z = 177$

Answer of short answer type question

Answer 1: $x_1 = 0, x_2 = 2, x_3 = 0$; Maximum $z = 4$

2: $x_1 = 18, x_2 = 0,$; Minimum $z = 36$

3: $x_1 = 3, x_2 = 0,$; Maximum $z = 9$

4: $x_1 = 0, x_2 = 5,$; Maximum $z = 40$

5: $x_1 = 0, x_2 = 5,$; Maximum $z = 40$

Answer of objective type question

Answer 1:	b)	**2:**	b)	**3:**	c)
4:	c)	**5:**	b)	**6:**	a)
7:	b)	**8:**	c)		

UNIT- 7: REVISED SIMPLEX METHOD

CONTENTS:

7.1 *INTRODUCTION*

The revised simplex method is a computational variant of the traditional simplex method for solving linear programming problems that uses matrix operations to improve efficiency and accuracy, especially for large problems. Instead of manipulating a full tableau, it maintains and updates a representation of the inverse of the basis matrix, only computing the necessary data for each iteration. Key steps include calculating the inverse of the basis matrix to find the optimal solution for the current iteration and determining the entering and leaving variables by solving a system of equations.

7.2 *OBJECTIVE*

After reading this unit learners will be able to

- revised simplex method

7.3 REVISED SIMPLEX METHOS

Revised simplex method is a modification of the simplex method in the sense that it is more economical on the computer, as it computes and stores only the relevant information needed currently for testing and/or updating the current solution. Moreover, in the revised method, the crux of the computations is rooted in the basis B and its inverse B^{-1}.

Computational Procedure:

Consider the L.P.P.:

Maximize z = $c^T X$ subject to the constraints: Ax = b, x ≥ 0:

where c^T, x ∈ R^n, b^T, ∈ R^n, and A is an m × n real matrix. In order to solve this L.P.P. by the revised simplex method, we consider the objective function equation z = cx also as one of the constraints and then seek a solution to the new system of (m + 1) simultaneous linear equations in (n + 1) variables z, $x_1, x_2,$ ......, x_n such that z is as large as possible The set of constraints can thus be represented as

$$\mathbf{Ax} = \mathbf{b}, \quad z - \mathbf{c}^T\mathbf{x} = 0, \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{c}^T & 1 \end{pmatrix} \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{x} \geq \mathbf{0}.$$

Let B be an initial basis submatrix of A and $X_B = B^{-1}b$ be an initial basic feasible solution to the original problem. Then, an initial basic feasible solution to the reformulated problem is given by

$$\hat{X}_B = \hat{B}^{-1}\hat{b}, \text{ where } \hat{X}_B = [X_B^T \ z]^*, \ \hat{b} = [b^T \ 0] \text{ and } \hat{B} = \begin{pmatrix} B & 0 \\ -C_B^T & 1 \end{pmatrix}$$

Clearly, since B is invertible, therefore we can write

$$\hat{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ C_B^T B^{-1} & 1 \end{pmatrix}$$

By assumption B^{-1} is known, $C_B{}^T B^{-1}$is known and hence all the elements of $\widehat{B^{-1}}$ are known. The V reader may check that $\hat{B}\,\widehat{B^{-1}} = I_{m+1}$.

Let us define a new $(m+1) \times n$ matrix

$$\hat{\mathbf{Y}} = \hat{\mathbf{B}}^{-1}\hat{\mathbf{A}} \quad \text{where} \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ -\mathbf{c}^T \end{bmatrix}$$

$$\therefore \quad \hat{\mathbf{y}}_j = \begin{pmatrix} \mathbf{B}^{-1} & 0 \\ \mathbf{c}_B{}^T\mathbf{B}^{-1} & 1 \end{pmatrix} \begin{bmatrix} \mathbf{A} \\ -\mathbf{c}^T \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{A} \\ \mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T \end{bmatrix}$$

or

$$\hat{\mathbf{Y}} = \begin{bmatrix} \mathbf{y}_j \\ z_j - c_j \end{bmatrix}, \qquad j = 1, 2, \ldots, n$$

where $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$ and $z_j = \mathbf{c}_B{}^T\mathbf{B}^{-1}\mathbf{a}_j$.

Thus, we arrive at an interesting conclusion that the first m components of $\hat{\mathbf{y}}_j$ constitute the vector $\mathbf{y}_j$ and the $(m+1)^{th}$ component is $z_j - c_j$; the first m components of $\hat{\mathbf{x}}_B$ constitute $\mathbf{x}_B$ and the $(m+1)^{th}$ component is z (being treated as a variable).

The above discussion enables us to give the computational procedure for solving linear programming problems by Revised Simplex Method which is summarized below:

Revised Simplex Algorithm Major steps for the computation of an optimum solution of any L.P.P. by revised simplex method are summarized below:

Step 1. Introduce slack and surplus variables, if needed, and restate the given L.P.P. maximization standard form.

Step 2. Begin with an initial basis $B = I_m$ and form the auxiliary matrix $\hat{B}$ and write down $\widehat{B^{-1}}$.

Step 3. State the objective relation z = cx as an addition constraint and form $\hat{A}$ and $\hat{b}$, where.

$$\hat{A} = \begin{pmatrix} A \\ -c^T \end{pmatrix} \quad \text{and} \quad \hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Step 4. Compute the net evaluations $(z_j - c_j)$, $j = 1, 2, \ldots, n$ by multiplying the successive columns of $\hat{A}$ with the last row of $\hat{B}^{-1}$, that is, by using the relation

$$(z_j - c_j) = (c_B B^{-1} \quad 1) \begin{pmatrix} A \\ -c^T \end{pmatrix}.$$

If all $(z_j - c_j)$ are non-negative, the current basic solution is an optimum one.

If at least one $(z_j - c_j)$ is negative, determine the most negative of them, say $(z_k - c_k)$, the corresponding vector y_k enters the basis. Go to Step 5.

If there is a tie for the most negative $(z_j - c_j)$, resolve the tie by any standard method. Go to Step 5.

Step 5. Compute $\hat{y}_k = \hat{B}^{-1}_{curr} \cdot \hat{a}_k$. If all $y_{ik} \le 0$, there exists an unbounded optimum solution to the given problem.

If at least one $y_{ik} > 0$, consider the current x_B and determine the departing vector. Go to Step 6.

Step 6. Write down the results obtained in Step 2 through Step 5 in a tabular form known as revised simplex table.

Step 7. Convert the leading element to unity and all other elements of the entering column to zero by suitable row operations and update the current basic feasible solution.

Step 8. Go to step 4 and repeat the procedure until an optimum basis feasible solution is obtained or there is an unbounded solution.

Key differences from the standard simplex method

- **Representation:**

The standard method uses a full tableau that directly shows the constraints scaled to the basic variables. The revised method uses a matrix representation of the basis, storing only the inverse of the basis matrix (B^{-1}).

- **Efficiency:**

For very large problems, the full tableau may not fit in computer memory, and even if it does, calculating every value in the table is inefficient. The revised method only computes the data needed for the current iteration, saving computer resources.

- **Calculations:**

It uses matrix operations and solves systems of equations to find the entering and leaving variables, rather than the row operations of the standard method.

- **Accuracy:**

The revised method can reduce the accumulation of rounding errors compared to the traditional method.

Remarks:

1. Benefit of revised simplex method is clearly comprehended in case of large LP problems.
2. In simplex method the entire simplex tableau is updated while a small part of it is used.
3. The revised simplex method uses exactly the same steps as those in simplex method.
4. The only difference occurs in the details of computing the entering variables and departing variable.

Question 1. Use revised simplex method to solve the following L.P.P.:

Maximize $z = 3x_1 + 5x_2$ subject to the constraints:

$x_1 \leq 4$, $x_2 \leq 6$, $3x_1 + 2x_2 \leq 18$; and $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Step 1. Introducing the slack variables $s_1, \geq 0$, $s_1 \geq 0$ and $s_1 \geq 0$,

the given L.P.P. can be restated in the standard form as

Maximize $z = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$,

where $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 18 \end{bmatrix}$ and $\mathbf{c}^T = (3, 5, 0, 0, 0)$.

Step 2. An initial basic feasible solution is : $s_1 = 4$, $s_2 = 6$ and $s_3 = 18$ with $\mathbf{I}_3$ as the initial basis matrix.

Step 3. $\hat{A} = \begin{pmatrix} A \\ -\mathbf{c}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \\ \hline -3 & -5 & 0 & 0 & 0 \end{pmatrix}$, $\hat{\mathbf{B}}^{-1}_{curr} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{c}_B{}^T\mathbf{B}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{pmatrix}$

$\mathbf{c}_B{}^T = (0, 0, 0)$.

Initial Iteration.

Step 4. The net evaluations are given by

$$(z_j - c_j) = (\mathbf{c}_B{}^T \mathbf{B}^{-1} \quad 1)\,\hat{\mathbf{A}} = (\mathbf{c}_B{}^T \mathbf{B}^{-1} \quad 1)\,[\hat{\mathbf{a}}_1 \;\; \hat{\mathbf{a}}_2 \;\; \hat{\mathbf{a}}_3 \;\; \hat{\mathbf{a}}_4 \;\; \hat{\mathbf{a}}_5] = (-3,\; -5,\; 0,\; 0,\; 0).$$

Since $(z_2 - c_2)$ is the most negative, $\hat{\mathbf{y}}_2$ enters the basis.

Step 5. Now,

$$\hat{\mathbf{y}}_2 = \hat{\mathbf{B}}^{-1}{}_{curr}\,\hat{\mathbf{a}}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ \hdashline -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ \hdashline -5 \end{pmatrix}$$

and

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}{}_{curr}\,\hat{\mathbf{b}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 18 \\ \hdashline 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 18 \\ \hdashline 0 \end{pmatrix} \begin{matrix} \mathbf{x}_B \\ \\ z \end{matrix}$$

Basic variable to be removed from the basis is determined by using

$$\min_i \left\{ \frac{x_{Bi}}{y_{i2}},\; y_{i2} > 0 \right\} = \min. \left\{ \frac{6}{1},\; \frac{18}{2} \right\} = 6.$$

Since, this minimum ratio corresponds to basic variable s_2, $\mathbf{y}_4$ leaves the basis. Thus, y_{22} becomes the leading element.

The initial revised simplex table, therefore, is :

$\hat{\mathbf{y}}_B$			$\hat{\mathbf{B}}^{-1}$		$\hat{\mathbf{y}}_2$	$\hat{\mathbf{x}}_B$
$\hat{\mathbf{y}}_3$	1	0	0	0	0	4
$\hat{\mathbf{y}}_4$	0	1	0	0	(1)	6
$\hat{\mathbf{y}}_5$	0	0	1	0	2	18
•	0	0	0	1	–5	0

First Iteration.

Step 3. Converting the leading element to unity and all other elements of the entering coloumn to zero, we get

$$\hat{\mathbf{B}}^{-1}{}_{next} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ \hdashline 0 & 5 & 0 & 1 \end{pmatrix}$$

This directly gives the new current quantities as

$$\mathbf{B}^{-1}{}_{curr} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c}_B{}^T \mathbf{B}^{-1}{}_{curr} = (0 \quad 5 \quad 0)$$

Using these current quantities, an improved solution is

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}_{curr}\,[\mathbf{b}^T \quad 0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ \hdashline 0 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 18 \\ \hdashline 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \\ \hdashline 30 \end{pmatrix}_z \mathbf{x}_B \quad (s_1,\ x_2 \text{ and } s_3 \text{ basic})$$

Step 4. The net evaluations are given by

$$(z_j - c_j) = (\mathbf{c}_B{}^T\mathbf{B}^{-1} \quad 1)\,[\hat{\mathbf{a}}_1 \quad \hat{\mathbf{a}}_2 \quad \hat{\mathbf{a}}_3 \quad \hat{\mathbf{a}}_4 \quad \hat{\mathbf{a}}_5]$$
$$= (-3 \quad 0 \quad 0 \quad 5 \quad 0).$$

Since $(z_1 - c_1)$ is the only negative, $\hat{\mathbf{y}}_1$ enters the basis.

Step 5. Now $$\hat{\mathbf{y}}_1 = \hat{\mathbf{B}}^{-1}_{curr}\,\hat{\mathbf{a}}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ \hdashline 0 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ \hdashline -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ \hdashline -3 \end{pmatrix},$$

and $$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}_{curr}\,\hat{\mathbf{b}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ \hdashline 0 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 18 \\ \hdashline 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \\ \hdashline 30 \end{pmatrix}_z \mathbf{x}_B$$

Basic variable to be removed from the basis is determined by using

$$\min_i \left\{ \frac{x_{Bi}}{y_{ii}},\ y_{ij} > 0 \right\} = \min. \left\{ \frac{4}{1}, \frac{6}{3} \right\} = 2.$$

Since, this minimum ratio corresponds to the basic variable s_3, $\mathbf{y}_5$ leaves the basis. The revised simplex table, therefore, is

$\hat{\mathbf{y}}_B$	$\hat{\mathbf{B}}^{-1}$				$\hat{\mathbf{y}}_1$	$\hat{\mathbf{x}}_B$
$\hat{\mathbf{y}}_3$	1	0	0	0	1	4
$\hat{\mathbf{y}}_2$	0	1	0	0	0	6
$\hat{\mathbf{y}}_5$	0	–2	1	0	(3)	6
z	0	5	0	1	–3	30

Final Iteration.

Step 3. Converting the leading element to unity and all other elements of the entering coloumn to zero, we get

$$\hat{\mathbf{B}}^{-1}_{next} = \begin{pmatrix} 1 & 2/3 & -1/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1/3 & 0 \\ \hdashline 0 & 3 & 1 & 1 \end{pmatrix}$$

$$\therefore \quad \hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}_{next}\,\hat{\mathbf{b}} = \begin{pmatrix} 1 & 2/3 & -1/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1/3 & 0 \\ \hdashline 0 & 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 18 \\ \hdashline 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \\ \hdashline 36 \end{pmatrix}$$

Step 4. The net evaluations are given by

$$(z_j - c_j) = (0,\ 3,\ 1,\ 1)\,\hat{\mathbf{A}} = (0,\ 0,\ 0,\ 3,\ 1).$$

Since all $(z_j - c_j) \geq 0$, an optimum basic feasible solution is obtained which is given by

$$x_1 = 2,\ x_2 = 6,\ x_3 = 2 \text{ and maximum } z = 36.$$

Question 2.

Use revised simplex method to solve the L.P.P.,

Maximize $z = 3x_1 + 2x_2 + 5x_3$ *subject to the constraints :*

$$x_1 + 2x_2 + x_3 \leq 430,\ 3x_1 + 2x_3 \leq 460,\ x_1 + 4x_2 \leq 420,\ x_1, x_2, x_3 \geq 0.$$

Solution:

Solution. *Step 1.* By introducing the slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$, the given L.P.P. can be re-written as

Maximize $z = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$,

where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix}$ and $\mathbf{c}^T = (3\ \ 2\ \ 5\ \ 0\ \ 0\ \ 0)$.

Step 2. An obvious initial basic solution is $s_1 = 430$, $s_2 = 460$ and $s_3 = 420$ with $\mathbf{B} = \mathbf{I}_3$ as the initial basis matrix. Now

$$\hat{\mathbf{A}} = \begin{pmatrix} \mathbf{A} \\ -\mathbf{c}^T \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \\ \hdashline -3 & -2 & -5 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{B}}^{-1}_{curr} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{pmatrix}.$$

with $\mathbf{c}_B = (0\ \ 0\ \ 0)$.

Initial Iteration.

Step 3. The net evaluations for non-basic variables are

$$(z_j - c_j) = (\mathbf{c}_B^T \mathbf{B}^{-1}, \quad 1)\,\hat{\mathbf{a}}_j; \qquad j = 1, 2, 3.$$

$$= (0 \quad 0, \quad 0, \quad 1)\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 4 & 0 \\ -3 & -2 & -5 \end{pmatrix} = (-3, \quad -2, \quad -5)$$

Since $(z_3 - c_3)$ is the most negative, $\hat{\mathbf{y}}_3$ enters the basis.

Step 4. Now, $$\hat{\mathbf{y}}_3 = \hat{\mathbf{B}}^{-1}{}_{curr}\,\hat{\mathbf{a}}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ \cdots \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ \cdots \\ -5 \end{pmatrix}$$

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}{}_{curr}\,\hat{\mathbf{b}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 430 \\ 460 \\ 420 \\ \cdots \\ 0 \end{pmatrix} = \begin{pmatrix} 430 \\ 460 \\ 420 \\ \cdots \\ 0 \end{pmatrix}$$

Basic variable to be removed from the basis is determined by using

$$\min_i \left\{\frac{x_{Bi}}{y_{i3}},\ y_{i3} > 0\right\} = \min\left\{\frac{430}{1}, \frac{460}{2}\right\} = 230.$$

Since this minimum ratio corresponds to basic variables s_2, $\hat{\mathbf{y}}_5$ leaves the basis. Thus y_{35} becomes the leading element.

The initial revised simplex table therefore is :

$\hat{\mathbf{y}}_B$		$\hat{\mathbf{B}}^{-1}$			$\hat{\mathbf{y}}_3$	$\hat{\mathbf{x}}_B$
$\hat{\mathbf{y}}_4$	1	0	0	0	1	430
$\hat{\mathbf{y}}_5$	0	1	0	0	(2)	460
$\hat{\mathbf{y}}_6$	0	0	1	0	0	420
z	0	0	0	1	−5	0

First Iteration.

Step 2. Converting the leading element to unity and all other elements of the entering column to zero, we get

$$\hat{\mathbf{B}}^{-1}{}_{next} = \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 5/2 & 0 & 1 \end{pmatrix}$$

This directly gives the new current quantities as

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{c}_B{}^T\mathbf{B}^{-1} = (0, \ 5/2, \ 0).$$

Making use of these current quantities, an improved solution is

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}{}_{curr}[\mathbf{b}^T \ \ 0] = \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 430 \\ 460 \\ 420 \\ 0 \end{pmatrix}$$

$$= [\underbrace{200, \ 230, \ 420,}_{\mathbf{x}_B} \ \vdots \ \underbrace{1150}_{z}] \qquad (s_1, x_3 \textit{ and } s_3 \textit{ basic})$$

Step 3. The net evaluations for the non-basic variables are given by

$$(z_j - c_j) = (\mathbf{c}_B{}^T\mathbf{B}^{-1}, \ 1)\,[\hat{\mathbf{a}}_1, \ \hat{\mathbf{a}}_2, \ \hat{\mathbf{a}}_3]$$

$$= (0, \ 5/2, \ 0, \ 1)\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix}$$

$$= (9/2, \ -2, \ 5/2).$$

Since $(z_2 - c_2)$ is the only negative, $\hat{\mathbf{y}}_2$ enters the basis.

Step 5. Now $\hat{\mathbf{y}}_2 = \hat{\mathbf{B}}^{-1}{}_{curr}\,\hat{\mathbf{a}}_2 = [2, \ 0, \ 4 \ : \ -2]$ (current = next of *previous step*)

Also $\underset{i}{\text{Min}} = \left\{ \dfrac{x_{Bi}}{y_{i2}}, \ y_{i2} > 0 \right\} = \underset{i}{\text{Min}} \left\{ \dfrac{200}{2}, \dfrac{420}{4} \right\} = 100$

Since the minimum ratio corresponds to basic variable s_1, $\hat{\mathbf{y}}_4$ leaves the basis. The revised simplex table is :

$\hat{\mathbf{y}}_B$	$\hat{\mathbf{B}}^{-1}$				$\hat{\mathbf{y}}_2$	$\hat{\mathbf{x}}_B$
$\hat{\mathbf{y}}_4$	1	−1/2	0	0	(2)	200
$\hat{\mathbf{y}}_3$	0	1/2	0	0	0	230
$\hat{\mathbf{y}}_6$	0	0	1	0	4	420
z	0	5/2	0	1	−2	1150

Final Iteration. Converting the leading element to unity and all other elements of the entering column to zero, we get

$$\hat{\mathbf{B}}^{-1}{}_{next} = \begin{pmatrix} 1/2 & -1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

The net evaluation for non-basic variables are given by

$$(z_j - c_j) = (1,\ 2,\ 0,\ 1)[\ \hat{\mathbf{a}}_1,\ \hat{\mathbf{a}}_4,\ \hat{\mathbf{a}}_5\ ] = (4,\ 1,\ 3)$$

Since all $z_j - c_j \geq 0$, an optimum basic feasible solution is attained :

$$\therefore \quad \hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}{}_{curr}\ \hat{\mathbf{b}} = \begin{pmatrix} 1/2 & -1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 430 \\ 460 \\ 420 \\ \cdots \\ 0 \end{pmatrix}$$

$$= [\underbrace{100,\ 230,\ 665}_{\mathbf{x}_B}\ :\ \underbrace{1350}_{z}]$$

Hence, $x_1 = 0$, $x_2 = 100$, $x_3 = 230$ and maximum $z = 1350$.

Question 2.

Use revised simplex method to solve the following L.P.P. :

Minimize $z = x_1 + x_2$ *subject to the constraints :*

$$x_1 + 2x_2 \geq 7,\ 4x_1 + x_2 \geq 6,\ x_1, x_2 \geq 0.$$

Solution. By introducing surplus variables $s_1 \geq 0$, $s_2 \geq 0$ and artificial variables $A_1 \geq 0$, $A_2 \geq 0$, the given L.P.P. is written in the standard form :

Maximize $z^* = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$

where $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 4 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\mathbf{c}^T = (-1, -1, 0, 0, -M, -M)$.

An initial basic feasible solution is $A_1 = 7$ and $A_2 = 6$ with $\mathbf{I}_2$ as the initial basis.

Now, $$\hat{\mathbf{A}} = \begin{pmatrix} \mathbf{A} \\ \cdots \\ -\mathbf{c} \end{pmatrix} = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 4 & 1 & 0 & -1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & M & M \end{bmatrix}$$

and $$\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ c_B^T B^{-1} & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ -M & -M & 1 \end{bmatrix}$$

with $c_B^T = (-M, -M)$.

Initial Iteration. The net evaluations for non-basic variables are computed as :

$$(z_j - c_j) = (-M, -M, 1)[\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4] = (-5M + 1, -3M + 1, M, M)$$

This shows that $\hat{y}_1$ enters the basis, since $(z_1 - c_1)$ is the most negative.

Now $$\hat{y}_1 = \hat{B}^{-1}_{curr}\,\hat{a}_1 = [\underbrace{1, 4,}_{y_1} : \underbrace{-5M + 1}_{z_1 - c_1}]$$

and $$\hat{x}_B = \hat{B}^{-1}_{curr}[b^T, 0] = [\underbrace{7, 6}_{x_B} : \underbrace{-13M}_{z}]$$

The initial revised simplex table, therefore, is

$\hat{y}_B$		$\hat{B}^{-1}$		$\hat{y}_1$	$\hat{x}_B$
$\hat{y}_5$	1	0	0	1	7
$\hat{y}_6$	0	1	0	(4*)	6
z	$-M$	$-M$	1	$-5M + 1$	$-13M$

First Iteration. The updated basis inverse is

$$\hat{B}^{-1}_{next} = \begin{pmatrix} 1 & -1/4 & 0 \\ 0 & 1/4 & 0 \\ \vdots & \vdots & \vdots \\ -M & (M-1)/4 & 1 \end{pmatrix}, \text{ and therefore}$$

$$(z_j - c_j) = \left(-M, \frac{M-1}{4}, 1\right)[\hat{a}_2, \hat{a}_3, \hat{a}_4] = \left(\frac{-7M+3}{4}, M, \frac{-M+1}{4}\right)$$

Since $(z_2 - c_2)$ is the most negative, $\hat{y}_2$ enters the basis.

Therefore, the next revised simplex table is :

$\hat{y}_B$		$\hat{B}^{-1}$		$\hat{y}_2$	$\hat{x}_B$
$\hat{y}_5$	1	–1/4	0	7/4*	11/2
$\hat{y}_1$	0	1/4	0	1/4	3/2
z*	–M	$\frac{M-1}{4}$	1	$\frac{7M+3}{4}$	$\frac{-11M-3}{2}$

Final Iteration. The updated basis inverse is

$$\hat{B}^{-1}_{next} = \begin{pmatrix} 4/7 & -1/7 & 0 \\ -1/7 & 2/7 & 0 \\ -3/7 & -1/7 & 1 \end{pmatrix}.$$

Therefore, $(z_j - c_j) = (-3/7, -1/7, 1)\ [\hat{a}_3,\ \hat{a}_4] = (3/7, 1/7)$.

Since, all $(z_i - c_j) \geq 0$ an optimum basis feasible solution is attained. Hence, an optimum basic feasible solution is $x_1 = 5/7, x_2 = 22/7$; minimum $z = 27/7$.

- **REMARK 1. SIMPLEX METHOD VERSUS REVISED SIMPLEX METHOD**

Considered the general L.P.P. as of maximizing z = cx, subject to the constraints:
Ax = **b** and **x** ≥ 0 where A is an mn matrix $(m \times n)$ and x as well as **b** are $m \times 1$ matrices. In solving the L.P.P. by simplex method suppose that artificial variables are not needed. Then, we have to carry out the calculations of $(n + 1)$ columns (columns corresponding to columns of **A** and one column corresponding to xa. the basic solution) at each iteration. At each iteration of simplex method, one non- basic variable is introduced into basis and one current basic variable is removed from the basis.
Thus, in total we compute for $m + n - 1$ columns. Furthermore, for each of these columns, we have to transform m + 1 elements (m corresponding to y_j, and one corresponding to $z_j - c_j$). For moving from one iteration to another we also need to calculate minimum ratio X_{iB}/y_{jk} Hence, in all we have to perform multiplication (m + 1)(n - m + 1) times and addition m (n - m + 1) times.
In the revised simplex are m + 1 rows and m + 2 columns. So, for moving from one iteration to another we have to make $(m + n)^2$ multiplication operations to get an improved solution in addition to $m(n - m)$ operations for calculating $z_j - c_j$. The major differences between the two methods of solution are the following:
(i) In the revised simplex method, we need to make (m + 1) (m + 2) entries in each table while in simplex method there are (m + 1) (m + 1) entries in each
(ii) In the simplex method all y, are updated at each iteration, whereas in the revised simplex method only the column of entering variable is updated.
(iii) If the number of variables, is significantly larger than number of constraints m, then the computational efforts of the revised simplex method is smaller than that of the simplex method.

(iv) The inverse of the current basis matrix is obtained automatically in the revised simplex method.

- **Bounded variables:**

Very often a linear programming problem may have, in addition to the given constraints, some (or all) variables with lower and upper limits. In such cases the standard form of L.P.P. will look like :

$$\text{Maximize } z = \mathbf{cx} \text{ subject to the constraints : } \mathbf{Ax} = \mathbf{b}, \quad \mathbf{l} \leq \mathbf{x} \leq \mathbf{u},$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]$, $\mathbf{b}^T = [b_1, b_2, \ldots, b_m]$, $\mathbf{c}^T = (c_1, c_2, \ldots, c_n)$, $\mathbf{l} = [l_1, l_2, \ldots, l_n]$, $\mathbf{u} = [u_1, u_2, \ldots, u_n]$, and $\mathbf{A} = (a_{ij})$; $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

Here, $\mathbf{A}$ is an $m \times n$ real matrix and $\mathbf{l}$ and $\mathbf{u}$ denote the lower and upper bounds of $\mathbf{x}$ respectively. In cases of unbounded variable, these limits are 0 and ∞ respectively.

The inequality constraints $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$ can be converted into equality constraints by introducing slack and/or surplus variables $\mathbf{s}'$ and $\mathbf{s}''$ as shown below :

$$\mathbf{x} + \mathbf{s}' = \mathbf{u} \quad \text{and} \quad \mathbf{x} - \mathbf{s}'' = \mathbf{l},$$

where $\mathbf{x} \geq \mathbf{0}$, $\mathbf{s}' \geq \mathbf{0}$ and $\mathbf{s}'' \geq \mathbf{0}$.

The lower bound constraint can be written as $\mathbf{x} = \mathbf{l} + \mathbf{s}''$ and thus $\mathbf{x}$ can be eliminated from all the remaining constraints.

The upper bound constraint can be written as x = u - s'. This does not serve the purpose, since there is no guarantee that x will be non-negative.

The difficulty is overcome by using a special technique known as bounded variable simplex method.

In bounded variable simplex method, the optimality condition for a solution is the same as the simplex method, discussed earlier. But the inclusion of constraints x + $s' =$ u in the simplex table requires modification in the feasibility condition of the simplex method due to the following reasons:

(i) A basic variable should become a non-basic variable at its upper bound (in usual simplex method all non-basic variables are at zero level).

(ii) When a non-basic variable becomes basic variable, its value should not exceed its upper bound and also should not disturb the non-negativity and upper bound conditions of all existing basic variables.

Check your progress

Question 1. What is the revised simplex method?

The revised simplex method is technically equivalent to the traditional simplex method, but it is implemented differently.

Question 2. Why do we use the revised simplex method?

The revised simplex approach is more efficient and accurate in terms of computing.

Question 3. Which of the following is an advantage of the Revised Simplex Method over the standard Simplex Method?

a) It is more computationally efficient and accurate.

b) It requires more manual calculations and a larger tableau.

c) It is less effective at handling large-scale problems.

d) It cannot be used for maximization problems.

Question 4

In the Revised Simplex Method, what is Standard Form-II typically used for?

a) When an identity matrix is obtained after adding slack variables.

b) When artificial variables are needed to form an identity matrix.

c) When the problem has only one constraint.

d) When the objective function is being minimized.

Question 5

In the Revised Simplex Method tableau, what does the column denoted as '$B^{-1}{}'$ represent?

a) The inverse of the constraint matrix.

b) The inverse of the basis matrix.

c) The objective function coefficients.

d) The non-basic variables.

Question 6

Which of the following statements is NOT a characteristic of the Revised Simplex Method?

a) It avoids the need to calculate an initial basic feasible solution for certain problems.

b) It directly works with the inverse of the basis matrix (B^{-1})

c) It requires solving the dual problem to find the primal solution.

d) It is more computationally efficient, especially for large problems.

7.4 *SUMMARY*

When using the regular simplex approach to solve a linear programming problem on a digital computer, the full simplex table must be stored in the computer table's memory, which may not be possible for particularly big problems. However, each iteration must include the calculation of each table. The revised simplex method, which is a variation of the original approach, uses fewer computer resources since it computes and maintains only the data that is currently needed for testing and/or improving the current solution. To put it another way, it only requires a small amount of effort. i.e.,

- The non-basic variable that reaches the basis is determined using the net evaluation row Δ_j.
- The pivoting column
- To establish the minimal positive ratio, first, identify the present basis variables and their values (X_B column), and then identify the basis variable to exit the basis.

By using the inverse of the current basis matrix at any iteration, the above information can be directly extracted from the original equations.

7.5 *GLOSSARY*

➢ Degeneracy in linear programming

7.6 *REFERENCES*

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7.7 *SUGGESTED READING*

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➢ Frederick S. Hillierand Gerald J. Lieberman: *Introduction to Operations Research* (10th edition). McGraw-Hill Education, 2015.

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7.8 *TERMINAL QUESTION*

Long Answer Type Question: Use revised simplex method to solve the following linear programming problems:

1.

Maximize $Z = 2x_1 + x_2$

such that:

$3x_1 + 4x_2 \le 6$

$6x_1 + x_2 \le 3$

$x_1, x_2 \ge 0$

2. Maximize z = $x_1 + 2x_2$ subject to the constraints: $x_1 + x_2 \le 3$, $x_1 + 2x_2 \le 5$ $3x_1 + x_2 \le 6$ and $x_1, x_2 \ge 0$.

3. Maximize z = $x_1 + x_2$ subject to the constraints: $3x_1 + 2x_2 \le 6$, $x_1 + 4x_2 \le 4$

$x_1, x_2 \ge 0$.

4. Maximize z = $2x_1 + x_2$ subject to the constraints: $3x_1 + 4x_2 \le 6$, $6x_1 + x_2 \le 3$

$x_1, x_2 \ge 0$.

7.9 *ANSWERS*

Answer of long answer type question

Answer 1: $x_1 = 2/7$, $x_2 = 9/7$ and maximum value $Z = 13/7$.

- Check Your Progress
  Q 3 (a)
  Q 4 (b)
  Q 5 (b)
  Q 6 (c)

BLOCK-III
DUALITY

UNIT-8: DUALITY

CONTENTS:

8.1 *INTRODUCTION*

The history of duality in linear programming (LPP) is linked to the work of John von Neumann, who conjectured the duality theorem shortly after George Dantzig introduced LPP. The theorem states that every linear program has an associated "dual" problem, and solving one can provide the solution to the other. The concept of duality was rigorously proven in 1948 and provides valuable insights like shadow prices and a way to solve problems that are computationally easier in their dual form.

Duality in Linear Programming (LPP) is the principle that every LPP (called the primal) has a related LPP, called the dual, which can be systematically constructed from it. The optimal solutions of the primal and dual problems are interconnected: the optimal value of one provides information about the optimal value of the other. For example, if the primal problem is a maximization problem, the dual problem will be a minimization problem, and the objective values will be equal at the optimum.

Key concepts of duality

- **Primal and Dual:** The original LPP is called the "primal," and the derived LPP is the "dual".
- **Objective function:** The type of objective function is reversed. A maximization problem becomes a minimization problem, and vice versa.
- **Variables:** Each constraint in the primal problem corresponds to a variable in the dual problem. The number of constraints in the primal equals the number of variables in the dual.
- **Constraints:** The coefficients of the primal objective function become the right-hand side values of the dual's constraints, and the right-hand side values of the primal constraints become the coefficients of the dual's objective function.

8.2 *OBJECTIVE*

After reading this unit learners will be able to

- Understand the basic concept of Duality.

8.3 *PRIMAL PROBLEM*

Consider the standard form of a linear programming problem:

Primal LP: Minimize $c^T x$ ……………. (1)

Subject to $Ax \geq b$

$x \geq 0$

Where:

- x is the vector of decision variables.
- c is the vector of coefficients for the objective function.
- A is the matrix of coefficients for the constraints.
- b is the vector of constants on the right-hand side of the constraints.

8.4 *DUAL PROBLEM*

The concept of duality in Linear Programming Problems (LPP) is a fundamental aspect of optimization theory. The dual problem provides deep insights into the structure of the original (or primal) problem and can often be used to derive bounds on the optimal value of the objective function.

Here are some key points about the dual problem in LPP:

1. Duality Principle:

- For every linear programming problem, known as the primal problem, there exists a corresponding dual problem.
- The solutions to the dual problem provide valuable information about the primal problem and vice versa.

2. Fomulation:

Given a primal problem in the standard form:

Maximize $c^T x$

Subject to, $Ax \le b,\ x \ge 0$

The corresponding dual problem is:

Minimize $b^T y$

Subject to, $A^T y \ge c,\ y \ge 0$

Here, A is the matrix of coefficients, c and b are vectors, x and y are the variables for the primal and dual problems, respectively.

The dual of the above primal problem (1) is formulated as follows:

Dual LP: Maximize $b^T y$

Subject to $A^T y \le c$

$y \ge 0$

Where:

- y is the vector of decision variables for the dual problem.
- A^T is the transpose of matrix A.
- b is the same vector as in the primal problem.
- c is the same vector as in the primal problem.

Remarks: One can readily detect the following from the definitions above:

(a) There is a dual variable for each primal constraint.

(b) There is a dual constraint for each primal variable.

(c) The primal and dual variable coefficients in the constraints are same except that they are transposed; i.e., the columns in the primal coefficient matrix becomes the rows in the dual coefficient matrix.

(d) While the number of primal variables and the number of dual constraints are exactly equal, whereas the number of dual variables is exactly equal to the number of primal constraints.

(e) The right-hand side constants of the dual constraints become the objective coefficients of the primal problem, whereas the objective coefficients of the primal variables become the right-hand side constants of the dual constraints.

The following table can be used to summarize information about the dual variables' signs, the type of restrictions, and the primal-dual objective:

Standard primal objective	Dual		
	Objective	Constraints	Variables
Maximization	Minimization	≥	Unrestricted
Minimization	Maximization	≤	Unrestricted

8.5 *STEP-WISE PROCEDURE FOR FORMULATING DUAL PROBLEM*

The process of formulating a prime-dual pair involves several steps:

Step 1: In standard form, solve the given linear programming problem. Think of it as the primal problem.

Step 2: Determine the factors that will be applied to the dual problem. These variables have the same number as the constraint equations in the primal.

Step 3: Using the constants on the right side of the primal restrictions, write out the objective function of the dual.

The dual will be a minimization problem if the primal problem is of the maximization type, and vice versa.

Step 4: Write the constraints for the dual problem using the dual variable found in Step 2.

(a) If the primal is a maximization problem, the dual constraints must be all of ≥ type. If the primal is a minimization problem, the dual constraints must be all of ≤ type.

(b) The dual constraints' row coefficients are derived from the primal constraints' column coefficients.

(c) The dual constraints' constants on the right side are the fundamental objective function's coefficients.

(d) It is defined that the dual variables have an unrestricted sign.

Step 5: Using steps 3 and 4, write down the dual of the given L.P.P.

Note: It is never required to take into account the dual constraints related to an artificial variable since, in the standard form of the primal, the dual constraint relating to an artificial variable is always redundant.

Remark 1: Primal-dual pairs are symmetric if the given linear programming problem is in its canonical form.

2: The primal-dual pair is considered unsymmetric if the provided linear programming problem is in its standard form.

Solved Example

Example 2: Find the dual of the following linear programming problem.

Maximize $z = 5x_1 + 3x_2$, subject to the constraints

$3x_1 + 5x_2 \leq 15,\ 5x_1 + 2x_2 \leq 10,\ x_1 \geq 0$ and $x_2 \geq 0$

Solution: Standard primal: Introducing slack variables $s_1, s_2 \geq 0$, the standard linear programming problem is:

Maximize $z = 5x_1 + 3x_2 + 0.s_1 + 0.s_2$, subject to the constraints

$3x_1 + 5x_2 + s_1 + 0.s_2 = 15, 5x_1 + 2x_2 + 0.s_1 + s_2 = 10, x_1, x_2, s_1, s_2 \geq 0$

Dual: Let w_1 and w_2 be the dual variables corresponding to the primal constraints. Then, the dual problem will be:

Minimize $z^* = 5w_1 + 10w_2$, subject to the constraints:

$3w_1 + 5w_2 \geq 5, 5w_1 + 2w_2 \geq 3$

$$\left.\begin{array}{l} w_1 + 0.w_2 \geq 0 \\ 0.w_1 + w_2 \geq 0 \end{array}\right\} \Rightarrow w_1 \geq 0 \ and \ w_2 \geq 0 \text{ unrestricted (redundant)}$$

Here w_1 *and* w_2 unrestricted (redundant).

The dual variables w_1 *and* w_2 unrestricted" are dominated by $w_1 \geq 0$ *and* $w_2 \geq 0$. Eliminating redundancy, the restricted variables are $w_1 \geq 0$ *and* $w_2 \geq 0$.

Example 3: Find the dual of the following linear programming problem.

Minimize $z = 4x_1 + 6x_2 + 18x_3$, subject to the constraints

$x_1 + 3x_2 \geq 3,\ x_2 + 2x_3 \geq 5,\ x_1, x_2, x_3 \geq 0$.

Solution: Standard primal: Introducing slack variables $s_1, s_2 \geq 0$, the standard linear programming problem is:

Minimize $z = 4x_1 + 6x_2 + 18x_3 + 0.s_1 + 0.s_2$, subject to the constraints

Dual: Let w_1 and w_2 be the dual variables corresponding to the primal constraints. Then, the dual problem will be:

Minimize $z^* = 3w_1 + 5w_2$, subject to the constraints:

$w_1 + 0.w_2 \leq 4, 3w_1 + w_2 \leq 6,\ 0.w_1 + 2w_2 \leq 18$

$$\left.\begin{array}{l}-w_1+0.w_2\leq 0\\ 0.w_1-w_2\leq 0\end{array}\right\}\Rightarrow w_1\geq 0\ and\ w_2\geq 0$$ unrestricted (redundant)

Eliminating redundancy, the dual problem is:

Maximize $z*=3w_1+5w_2$ subject to the constraints:

$w_1\leq 4, 3w_1+w_2\leq 6, 2w_2\leq 18;\ w_1\geq 0\ and\ w_2\geq 0.$

Example 4: Find the dual of the following linear programming problem.

Minimize $z=3x_1-2x_2+4x_3$, subject to the constraints

$3x_1+5x_2+4x_3\geq 7,\ \ 6x_1+x_2+3x_3\geq 4,$

$7x_1-2x_2-x_3\leq 10,\ \ x_1-2x_2+5x_3\geq 3,\ \ 4x_1+7x_2-2x_3\geq 2,.$

$x_1,x_2,x_3\geq 0$

Solution: Introducing the slack variable $s_3\geq 0$ surplus variables $s_1\geq 0, s_2\geq 0, s_4\geq 0, s_5\geq 0$.

Minimize: $z=3x_1-2x_2+4x_3+0.s_1+0.s_2+0.s_4+0.s_5$

Subject to constraint, $3x_1-2x_2+4x_3-s_1=7$

$6x_1+x_2+3x_3-s_2=4$

$7x_1-2x_2-x_3+s_3=10$

$x_1-2x_2+5x_3-s_4=3$

$4x_1+7x_2-2x_3-s_5=2$

$x_1,x_2,x_3,s_1,s_2,s_3,s_4,s_5\geq 0$

Dual: If $w_j\,(j=1,2,3,4,5)$ are the dual variables corresponding to mentioned five primal constraints, So, the dual of the given L.P.P. will be;

Maximize $z^*=7w_1+4w_2+10w_3+3w_4+2w_5$, subject to the constraints:

$3w_1+6w_2+7w_3+w_4+4w_5\leq 3$

$5w_1+w_2-2w_3-2w_4+7w_5\leq -2$

$4w_1+3w_2-w_3+5w_4-2w_5\leq 4$

$-w_1\leq 0, -w_2\leq 0, w_3\leq 0, -w_4\leq 0, -w_5\leq 0$

$w_j\ (j=1,2,3,4,5)$ are unrestricted in sign.

Hence, after eliminating the redundancy, the dual variables are: $w_1\geq 0, w_2\geq 0, w_3\leq 0, w_4\geq 0,\ and\ w_5\geq 0$

Example 5: Find the dual of the following linear programming problem.

Minimize $z=x_1-3x_2-2x_3$, subject to the constraints

$3x_1-x_2+2x_3\leq 7,\ \ 2x_1-4x_2\geq 12, -4x_1+3x_2+8x_3=10\ \ x_1,x_2\geq 0$ and x_3 is unrestricted.

Solution: Initially, we introduced the slack and surplus variable $s_1\geq 0$ and $s_2\geq 0$ respectively, the primal problem is restated as, Minimize $z = cx$; subject to the constraints: $Ax = b, x \geq 0$

Where $x = [x_1, x_2, x_3', x_3'', s_1, s_2]$, $c = [1, -3, -2, 2, 0, 0], b = [7, 12, 10]$ and

$$A = \begin{pmatrix} 3 & -1 & 2 & -2 & 1 & 0 \\ 2 & -4 & 0 & 0 & 0 & -1 \\ -4 & 3 & 8 & -8 & 0 & 0 \end{pmatrix}, \text{ when } x_3 = x_3' - x_3''$$

Dual: If $w = (w_1, w_2, w_3)$ are the dual variables , then the dual of the given prima is

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the

$3w_1 + 2w_2 - 4w_3 \leq 1$

$-w_1 - 4w_2 + 3w_3 \leq -3 \Rightarrow w_1 + 4w_2 - 3w_3 \geq 3$

$$\left.\begin{matrix} 2w_1 + 8w_3 \leq -2 \\ -2w_1 - 8w_3 \leq 2 \end{matrix}\right\} \Rightarrow -2w_1 - 8w_2 = 2$$

$$\left.\begin{matrix} w_1 \leq 0 \\ -w_2 \leq 0 \end{matrix}\right\} \Rightarrow w_1 \leq 0 \;\; and \;\; w_2 \geq 0$$

Where w_1, w_2 *and* w_3 unrestricted.

Eliminating redundancy, dual variables are $w_1 \leq 0, w_2 \geq 0$ *and* w_3 unrestricted. So, this is re-written as follows:

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the constraints:

$3w_1 + 2w_2 - 4w_3 \leq 1$; $w_1 + 4w_2 - 3w_3 \geq 3$; $-2w_1 - 8w_2 = 2$;

$w_1 \leq 0$ *and* $w_2 \geq 0$, w_3 unrestricted.

Example 6:

Min $Z_x = 2x_2 + 5x_3$

Subject to

$x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Solution:

Max $Z_x' = -2x_2 - 5x_3$

Subject to

$-x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 \leq 4$

$-x_1 + x_2 - 3x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

Dual

Min $Z_w = -2w_1 + 6w_2 + 4w_3 - 4w_4$

Subject to

$-w_1 + 2w_2 + w_3 - w_4 \geq 0$

$-w_1 + w_2 - w_3 + w_4 \geq -2$

$6w_2 + 3w_3 - 3w_4 \geq -5$

$w_1, w_2, w_3, w_4 \geq 0$

Some important theorems related to duality:

1. The dual of the dual is the primal.

Proof: Let the Primal L.P.P. be to determine $x^T \in \mathbb{R}^n$ so as to

Maximize $f(x) = cx, c \in \mathbb{R}^n$ subject to the constraints:

$Ax = b$ and $x \geq 0,\ b^T \in \mathbb{R}^m$

where A is an m × n real matrix.

The dual of this primal is the L.P.P. of determining $w^T \in \mathbb{R}^m$ so as to

Minimize $f(w) = b^T w, b^T \in \mathbb{R}^m$ subject to constraints:

$A^T w \geq c^T$, w is unrestricted, , $c \in \mathbb{R}^n$.

Now, introduce surplus variables $s \geq 0$ in the constraints of the dual and write $w = w_1 - w_2$, where, $w_1 \geq 0$ and $w_1 \geq 0$.

The standard form of dual then is to

Minimize $g(w) = b^T(w_1 - w_2), b^T \in \mathbb{R}^m$ subject to constraints:

$A^T(w_1 - w_2) - I_n s = c^T$, $c \in \mathbb{R}^n$.

$w_1,\ w_2$ and $s \geq 0$.

Considering this linear programming problem as our standard primal, the associated dual problem will be to

Maximize $h(y) = cy$, $c \in \mathbb{R}^n$ subject to the constraints:

$(A^T)^T y \leq (b^T)^T, -(A^T)^T y \leq -(b^T)^T$

$-y \leq 0$ ($\Longrightarrow y \geq 0$) and y is unrestricted.

Eliminating redundancy, the dual problem may be re-written as:

Maximize $h(y) = cy$, $c \in \mathbb{R}^n$ subject to the constraints:

$Ay \leq b$ and $Ay \geq b$ and $y \geq 0 \Longrightarrow Ay = 0, b^T \in \mathbb{R}^m$

This problem, which is the dual of the dual problem, is just the primal problem we had started with.

This completes the proof.

2. **(Weak- Duality Theorem)** Let x_0 be a feasible solution to the primal problem, Maximize $f(x) = cx$ subject to: $Ax \le b, x \le 0$

Where x^T and $c \in R^n, b^T \in R^m$ and A is $m \times n$ real matrix. If w_0 be a feasible solution to the dual of the primal, namely

Minimize $g(w) = b^T w$, subject to: $A^T w \ge c^T$, $w \ge 0$

Where $w^T \in R^m$, then $cx_0 \in b^T w_0$

3. **(Basic duality theorem)** Let a primal problem be

Maximize $f(x) = cx$ subject to: $Ax \le b, x \ge 0, x^T, c \in R^n$

And the associated dual be

Minimize $g(w) = b^T w$ subject to: $A^T w \ge c^T, w \ge 0, w^T, b^T \in R^m$

Important characteristics of Duality

1. Dual of dual is primal

2. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.

3. If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.

4. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.

5. If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.

6. If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.

2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.

3. In physics, it is used in parallel circuit and series circuit theory.

4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.

5. When a problem does not yield any solution in primal, it can be verified with dual. **6.** Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

Symmetry property

For any primal problem and its dual problem, all relationships between them must be symmetric because dual of dual is primal.

Fundamental duality theorem

- If one problem has feasible solution and a bounded objective function (optimal solution) then the other problem has a finite optimal solution.
- If one problem has feasible solution and an unbounded optimal solution then the other problem has no feasible solution.
- If one problem has no feasible solution then the other problem has either no feasible solution or an unbounded solution.

If k th constraint of primal is equality then the dual variable w_k is unrestricted in sign If p th variable of primal is unrestricted in sign then p th constraint of dual is an equality.

CHECK YOUR PROGRESS

1. **Which statement about duality in LPP is always TRUE?**

(A) The dual always has a solution if the primal does.

(B) The optimal value of the primal equals the optimal value of the dual if both are feasible and optimal.

(C) The dual is always a minimization problem.

(D) The dual variables must be non-negative.

2. **What is the dual of a dual LPP?**

a) The dual problem itself.

b) The primal problem.

c) An entirely new problem with no relation to the primal.

d) A problem with reversed constraints and objective function.

3. In a standard LPP, if the i-th constraint in the primal problem is an equality (=) constraint, the corresponding i-th dual variable will be:
   a) Non-negative (≥ 0)
   b) Non-negative (≤ 0)
   c) Unrestricted in sign
   d) Zero
4. **If the primal linear programming problem is unbounded, what can be said about its dual problem?**
   a) The dual problem will always be unbounded.
   b) The dual problem will have a finite optimal solution.
   c) The dual problem will always be **infeasible** (have no feasible region).
   d) The feasibility of the dual cannot be determined without more information.
5. The duality theory in linear programming relates the primal problem to a corresponding dual problem, and states that:
   a) The optimal solution of the dual is always greater than the primal.
   b) The primal and dual problems can have different optimal values if a solution exists.
   c) Both primal and dual have the **same optimum value**, provided an optimal solution exists for either problem.
   d) The dual problem is only used for minimization problems.

6. Solution which satifies all the constraints of linear programming problem is called
   (a) Feasible solution
   (b) Bounded solution
   (c) Unbounded solution
   (d) None of these

7. Any feasible solution of a canonical maximization (respectively minimization) linear programming problem which maximizes (respectively minimizes) the objective function is called

   (a) Feasible solution

   (b) Optimal solution

(c) Unbounded solution

(d) Bounded solution

8.6 SUMMARY

The summary of this unit are as follows:

- Every linear programming problem (primal) has a corresponding dual problem.
- The primal problem involves maximizing or minimizing an objective function subject to constraints and non-negativity restrictions.
- The dual problem is derived from the primal problem and involves a different set of variables and constraints, effectively reversing the roles of the constraints and the objective function.
- For any feasible solutions to the primal and dual problems, the value of the objective function in the primal problem is less than or equal to the value in the dual problem (for maximization) or greater than or equal to the value in the dual problem (for minimization).
- If the primal problem has an optimal solution, the dual problem also has an optimal solution, and the optimal values of their objective functions are equal.
- Sometimes, solving the dual problem is easier than solving the primal problem. This can be especially true when the dual problem has fewer constraints or variables.
- Duality theory is also used in sensitivity analysis to understand how changes in the coefficients of the primal problem affect the optimal solution.

Understanding duality is essential for grasping the deeper structure of linear programming problems, providing insights that can be leveraged in both theoretical analyses and practical applications.

8.7 GLOSSARY

- Duality

8.8 REFERENCES

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8.9 *SUGGESTED READING*

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8.10 *TERMINAL QUESTION*

Short Answer Type Question:

1. Find the dual of the following problem

Maximize, $z = 2000x_1 + 3000x_2$

Subject to the constraints,

$6x_1 + 9x_2 \leq 100;\ 2x_1 + x_2 \leq 20;\ x_1, x_2 \geq 0$

Answer: $x_1 = 23/3, x_2 = 0, x_3 = 5$; Maximum $z = 85/3$

2. Find the dual of the following problem

Maximize, $z = 10x + 8y$

Subject to,

$x + 2y \geq 5;\ 2x - y \geq 12;\ x + 3y \geq 4$

$x \geq 0$ and y is unrestricted.

Answer: $x_1 = 0; x_2 = 1.2; x_3 = 0.9; x_4 = 0;$; Maximum $z = 3.6$

3. Find the dual of the following problem

Maximize, $z = 2x + 5y + 6z$

Subject to,

$5x+6y-z\le 3;\ -2x+y+4z\le 4;\ x-5y+3z\le 1;\ -3x-3y+7z\le 6;$

$x, y, z \ge 0$

Answer: $x_1 = 18/5, x_2 = 6/5, x_3 = 8/5$; Maximum $z = 108$

4. Find the dual of the following problem

Minimize, $z = x_1 + 2y$

Subject to,

$2x+4y\le 160;\ x-y=30;\ x\ge 10;\ x\ge 0$ and $y\ge 0$

Answer: $x_1 = 6, x_2 = 7, x_3 = 0$; Maximum $z = 177$

5. Find the dual of the following problem

Maximize, $z = 2x+3y+z$

Subject to,

$4x+3y+z=6;\ x+2y+5z=4;$

$x, y, z \ge 0$

6. Find the dual of the following problem

Maximize, $z = 3x+5y+7z$

Subject to,

$x+y+3z\le 10;\ 4x-y+2z\ge 15$

$x, y \ge 0$ and z is unrestricted.

7. Find the dual of the following problem

Minimize, $z = x+y+z$

Subject to,

$x-3y+4z=5;\ x-2y\le 3; 2y-z\ge 4$

$x, y \geq 0$ and z is unrestricted.

8. Find the dual of the following problem

Minimize, $z = 2x + 3y + 4z$

Subject to,

$2x + 3y + 5z \geq 2;\ 3x + y + 7z = 3; x + 4y + 6z \leq 5$

$x, y \geq 0$ and z is unrestricted.

9. Find the dual of the following problem

Maximize, $z = 6x + 6y + z + 7w + 5s$

Subject to,

$3x + 7y + 8z + 5w + s = 2;\ 2x + y + 3z + 2w + 9s = 6$

$x, y, z, w \geq 0$ and t is unrestricted.

Check Your Progress

1. (b)
2. (b)
3. (c)
4. (c)
5. (c)
6. (a)
7. (b)

UNIT- 9: DUAL SIMPLEX ALGORITHM

CONTENTS:

9.1 *INTRODUCTION*

Any LPP for which it is possible to find infeasible but better than optimal initial basic solution can be solved by using dual simplex method. Such a situation can be recognized by first expressing the constraints in '≤' form and the objective function in the maximization form. After adding slack variables, if any right-hand side element is negative and the optimality condition is satisfied then the problem can be solved by dual simplex method. Negative element on the right-hand side suggests that the corresponding slack variable is negative. This means that the problem starts with optimal but infeasible basic solution and we proceed towards its feasibility. The dual simplex method is similar to the standard simplex method except that in the latter the starting initial basic solution is feasible but not optimum while in the former it is infeasible but optimum or better than optimum. The dual simplex method works towards feasibility while simplex method works towards optimality.

9.2 *OBJECTIVE*

After reading this unit learners will be able to

- Dual simplex method

9.3 *DUAL SIMPLEX METHOD*

- **Formulate the Dual Problem**: Given a primal problem, formulate its dual.
- **Solve the Dual Problem Using Simplex Method**: Sometimes, it is easier to solve the dual problem than the primal problem. The solution to the dual provides information about the primal solution.
- **Interpreting the Dual Solution**: The values of the dual variables provide the shadow prices or the marginal values of the resources in the primal problem.
- **Complementary Slackness**: This principle helps in validating the solutions. For each pair of primal and dual variables, at least one in the pair must be zero in the optimal solution.

OR

If the primal problem is a maximization problem, the following set of rules govern the derivation of the optimal solution:

Rule 1: The corresponding net evaluations of the initial primal variables are equal to the difference between the left and right sides of the dual constraints associated with these initial primal variables.

Rule 2: The negative of the corresponding net evaluations of the initial dual variables is equal to the difference between the left and right sides of the primal constraints associated with these initial dual variables.

Rule 3: If the primal (dual) problem is unbounded, then the dual (primal) problem has no feasible solution.

Note: In rule 2, solve the dual problem by changing its objective from minimization to maximization.

Solved Examples

Example 6: Using duality solve the following L.P.P

Maximize $z = 2x_1 + x_2$, subject to the constraints

$x_1 + 2x_2 \leq 10;\ \ x_1 + x_2 \leq 6;\ \ x_1 - x_2 \leq 2;\ \ x_1 - 2x_2 \leq 1;\ \ x_1, x_2 \geq 0$

Solution: The dual problem for the given problem is as follows:

Minimize $z^* = 10w_1 + 6w_2 + 2w_3 + w_4$, subject to the constraints

$w_1 + w_2 + w_3 + w_4 \geq 2;\ \ 2w_1 + w_2 - w_3 - 2w_4 \geq 1;\ \ w_1, w_2, w_3, w_4 \geq 0$

Introducing surplus variables $s_1 \geq 0, s_2 \geq 0$ and artificial variables $A_1 \geq 0, A_2 \geq 0$, an initial basic feasible solution is $A_1 = 2, A_2 = 1$. (The primal constraints associated with s_1, s_2, A_1, A_2 are: $-x_1 \leq 0, -x_2 \leq 0, x_1 \leq M$ *and* $x_2 \leq M$).

The iterative simplex table are:

Initial Iteration: Introduce y_1 and y_8

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-*M*	y_7	2	1	1	1	1	-1	0	1	0
-*M*	y_8	1	**2**	1	-1	-2	0	-1	0	1
	*z**	-3M	-3M+10	-2M+6	2	M+1	M	M	0	0

First Iteration: Introduce y_3 and drop y_7

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-*M*	y_7	3/2	0	1/2	**3/2**	2	-1	1/2	1	-1/2
-10	y_1	1/2	1	1/2	-1/2	-1	0	-1/2	0	1/2
	*z**	-5-3M/2	0	1-M/2	7-3M/2	11-2M	M	5-M/2	0	-5+3M/2

Second Iteration: Introduce y_2 and drop y_1

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1	0	1/3	1	4/3	-2/3	1/3	2/3	-1/3
-10	y_1	1	1	**2/3**	0	-1/3	-1/3	-1/3	1/3	13
	*z**	-12	0	-4/3	0	5/3	14/3	8/3	M-14/3	M-8/3

Final Iteration: Optimal Solution.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1/2	-1/2	0	1	3/2	-1/2	1/2	1/2	-1/2
-6	y_2	3/2	3/2	1	0	-1/2	-1/2	-1/2	1/2	1/2
	z^*	-10	2	0	0	1	4	2	M-4	M-2

Thus, an optimum feasible solution to the dual problem is.

$w_1 = 0, w_2 = 3/2 \ and \ w_3 = 1/2; \ \min(Z^*) = -(-10) = 10$.

Also the primal constraints associated with the dual variables A_1, A_2 are $x_1 \le M$ and $x_2 \le M$. Thus, by applying duality rules, the optimal solution to the primal problem is derived as follows:

Starting dual variables	A_1	A_2
Corresponding $\{-(z_j - c_j)\}$	$-(M-4)$	-(M-2)
The difference between the left and right sides of the primal constraints associated with the initial dual variables	$x_1 - M$	$x_2 - M$

Making use of Rule 2, we get

$x_1 - M = -M + 4$ and $x_2 - M = -M + 2$

$x_1 = 4 \ and \ x_2 = 2$

Hence , Maximum $z = Minimum \ \ z^* = 10$

Example 7: Consider the linear programming

Maximize $z = 3x_1 + 2x_2 + 5x_3$, subject to the constraints

$x_1 + 2x_2 + x_3 \le a_1; \ 3x_1 + 2x_2 \le a_2; \ x_1 + 4x_2 \le a_3;$

Where a_1, a_2, a_3 are constant. For specific values of a_1, a_2, a_3 the optimal solution is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution

Z	4	0	0	c_1	c_2	0	1350
x_2	b_1	1	0	1/2	-1/4	0	100
x_3	b_2	0	1	0	1/2	0	c_3
x_6	b_3	0	0	-2	1	1	20

Where b_i's and c_i's are constant. Determine:

(i) The values of a_1, a_2 and a_3 that yield the given optimal solution.
(ii) The values of b_1, b_2 , b_3 and c_1, c_2 , c_3 in the optimal tableau.
(iii) The optimal dual solution.

Solution: The optimal table indicates that slack variables x_4, x_5, x_6 are introduced in the three primal constraints. They happen to be the starting primal basic variables also. Thus the optimal basis inverse is given by $B^{-1} = [y_4 \;\; y_5 \;\; y_6]$ from the optimal table.

(i) We have $B^{-1}b = x_B$

$$\begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 100 \\ c_3 \\ 20 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}a_1 - \frac{1}{4}a_2 = 100, \frac{1}{2}a_2 = c_3, -2a_1 + a_2 + a_3 = 20$$

Also, $z = c_B x_B \Rightarrow 1350 = 200 + 5c_3 \Rightarrow c_3 = 230$, where $c_B = [2\ 5\ 0]$.

Thus, we get $a_1 = 430, a_2 = 460$ and $a_3 = 480$

(ii) The z-row gives:

$$4 = c_B y_1 - c_1 = 2b_1 + 5b_2 - 3 \Rightarrow 2b_1 + 5b_2 = 7$$

$$c_1 = c_B y_4 - c_4 = 1 - 0 = 1$$

$$c_2 = c_B y_5 - c_5 = -1/2 + 5/2 - 0 = 2$$

To obtain the value of b_1, b_2 and b_3, we perform iteration on the starting primal table:

Initial Iteration: Introduce y_3 and drop y_5

			3	2	5	0	0	0

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	$a_1 = 430$	1	2	1	1	0	0
0	y_5	$a_2 = 460$	3	0	**2**	0	1	0
0	y_6	$a_3 = 480$	1	4	0	0	0	1
		0	-3	-2	-5	0	0	0

First Iteration: Introduce y_2 and drop y_4

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	200	-1/2	2	0	1	-1/2	0
0	y_5	230	3/2	0	1	0	1/2	0
0	y_6	480	1	4	0	0	0	1
		1150	9/2	-2	0	0	5/2	0

Second Iteration: Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_2	100	-1/4	1	0	1/2	-1/4	0
5	y_3	230	3/2	0	1	0	1/2	0
0	y_6	80	-4	0	0	-2	1/2	1
		1350	4	0	0	1	2	0

Comparing it with the given optimal table, we get

$b_1 = -1/4, b_2 = 3/2$ and $b_3 = -4$

(Note that the values of c_1, c_2 are also readily available.)

(iii) The dual problem is,

Minimize $z^* = a_1w_1 + a_2w_1 + a_3w_3$, subject to the constraints:

$w_1 + 3w_2 + w_3 \geq 3,\ 2w_1 + w_2 + 4w_3 \geq 2,\ w_1 + 2w_2 + 0w_3 \geq 5$

$w_1 \geq 0,\ w_2 \geq 0\ and\ w_3 \geq 0$

The dual constraints associated with the starting primal variables x_4, x_5 and x_6 and $w_1 \geq 0, w_2 \geq 0$ and $w_3 \geq 0$

Thus we have the following information:

Starting primal variables	x_4	x_5	x_6
Left minus right sides of the associated dual constraint Net evaluation primal optimal table	$w_1 - 0$ c_1	$w_2 - 0$ c_2	$w_3 - 0$ 0

This using Rule 1 we get

$c_1 = w_1 - 0 \Rightarrow w_1^* = c_1 = 1$

$c_2 = w_2 - 0 \Rightarrow w_2^* = c_2 = 2$

$0 = w_3 - 0 \Rightarrow w_3^* = 0$

The optimal dual objective is Min. $z^* = 1350 = a_1w_1^* + a_2w_2^*$

Check Your Progress

1. In dual simplex, the solution is always primal feasible and dual infeasible.

2. In the dual simplex method, we maintain dual feasibility at every step and work to achieve primal feasibility.

3. in contrast, the primal simplex maintains primal feasibility and works toward dual feasibility (optimality).

4. Which condition necessitates the use of the Dual Simplex Method?

a) All constraints are of '≤' type with positive RHS.

b) The initial basic solution is optimal but infeasible (negative RHS).

c) The problem has multiple optimal solutions.

d) The objective function is to be minimized.

5. In the Dual Simplex Method, the entering variable is chosen based on:

a) The most negative value in the objective row ($Zj - Cj$)

b) The most negative value in the RHS (Basic Variable) column.

c) The ratio of the RHS value to the corresponding negative coefficient in the pivot row.

d) The least positive value in the RHS column.

6. What does a negative value in the RHS column of the final simplex tableau indicate in the Dual Simplex Method?

a) Optimality has been reached.

b) The solution is feasible.

c) The current solution is infeasible, requiring further iterations.

d) An alternative optimal solution exists.

9.4 SUMMARY

The summary of this unit are as follows:

The Dual Simplex Method solves Linear Programming Problems (LPPs) by starting with an optimal but infeasible solution, unlike the standard Simplex Method which begins feasible and moves towards optimality. It works by iteratively removing the most negative basic variable (maintaining optimality) and introducing a new variable (maintaining dual feasibility) until all basic variables are non-negative, achieving both feasibility and optimality. It's ideal for situations where adding constraints (like in Integer Programming) makes the current solution infeasible but keeps the optimality conditions met. By contrast, the **dual simplex method** takes a reversed approach. Rather than starting from a feasible solution and moving towards optimality, it begins from a solution that would be optimal *if* the primal constraints were fully satisfied. In other words, the objective row is already optimal, but some RHS values are negative, indicating that the current solution is *not* primal feasible. The dual simplex method then works to eliminate these infeasibilities. Once feasibility is restored, the solution is guaranteed to be optimal.

9.5 *GLOSSARY*

- Duality

9.6 *REFERENCES*

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9.7 *SUGGESTED READING*

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9.8 *TERMINAL QUESTION*

Long answer type question:

1. Solve the following LPP by using dual of the following problem

Maximize, $z = 8x + 4y$

Subject to,

$4x + 2y \leq 30;\ 2x + 4y \leq 24$

$x, y \geq 0$

2. Solve the following LPP by using dual of the following problem

Minimize, $z = 15x + 10y$

Subject to,

$3x + 5y \geq 5;\ 5x + 2y \geq 3$

$x, y \geq 0$

3. Solve the following LPP by using dual of the following problem

Maximize, $z = 5x + 2y$

Subject to,

$6x + y \geq 6;\ 4x + 3y \geq 12;\ x + 2y \geq 4$ and $x, y \geq 0$

4. Solve the following LPP by using dual of the following problem

Maximize, $z = 2x + 9y + z$

Subject to,

$x + 4y + 2z \geq 5;\ 3x + y + 2z \geq 4;$ and $x, y, z \geq 0$

5. Solve the following LPP by using dual of the following problem

Maximize, $z = x + 5y + 3z$

Subject to,

$x + 2y + z = 3;\ 2x - y = 4;$ and $x, y, z \geq 0$

6. Solve the following LPP by using dual of the following problem

Minimize, $z = 10x + 4y + 5z + w$

Subject to,

$5x - 7y + 3z + 0.5w \geq 150;$ and $x, y, z, w \geq 0$

7. Solve the following LPP by using dual of the following problem

Maximize, $z = x - y + 3z + 2w$

Subject to,

$x + y \geq -1;\ x - 3y - z \leq 7;\ x + z - 3w = -2$ and $x, y, z, w \geq 0$

8. Solve the following LPP by using dual of the following problem

Maximize, $z = 2y - 5z$

Subject to,

$x+z \geq 2;\ 2x+y+6z \leq 6; x-y+3z=0$ and $x, y, z, w \geq 0$

9.9 *ANSWERS*

Answer of long answer type question

1: Min (z^*) $= 30x_1 + 24x_2$

Subject to, $4x_1 + 2x_2 \geq 8;\ 2x_1 + 4x_2 \geq 4; x_1 \geq 0; x_2 \geq 0$

The optimal solution is $w_1 = 6$ and $w_2 = 3$, max $z = 60$

2: The optimal solution is $w_1 = 5/19$ and $w_2 = 16/19$, min $z = 235/19$

3: Unbounded solution

4: $w_1 = 0$, $w_2 = 0$ and $w_3 = 5/2$; min $z = 5/2$

5: Minimize $(z^*) = 3x_1 + 4x_2$

Subject to, $x_1 + 2x_2 \geq 1;\ 2x_1 - x_2 \geq 5; x_1 \geq 3$ and x_2 is unrestricted.

The optimal solution is $x_1 = 3$ and $x_2 = -1$, min $z^* = 5$.

6: The optimal solution is $x_1 = 0$; $x_2 = 0$; $x_3 = 50$; $x_4 = 0$; min $z^* = 250$.

7: Unbounded solution.

8: Minimize $z^* = 2x_1 + 6x_2$

Subject to the constraint, $x_1 + 2x_2 + x_3 \geq 0;\ x_2 - x_3 \geq 2;\ x_1 + 6x_2 + 3x_3 \geq -5$ and $x_1 \leq 0,\ x_2 \geq 0$ and x_3 is unrestricted in sign.

The optimal solution is $x_1 = 0$; $x_2 = 2/3$; $x_3 = -4/3$; min $z^* = 4$.

Check Your Progress

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. (b)

CYQ 5. (c)

CYQ 6. (c)

BLOCK IV- SENSITIVE ANALYSIS, LINEAR AND INTEGER PROGRAMMING

UNIT -10 SENSITIVE ANALYSIS

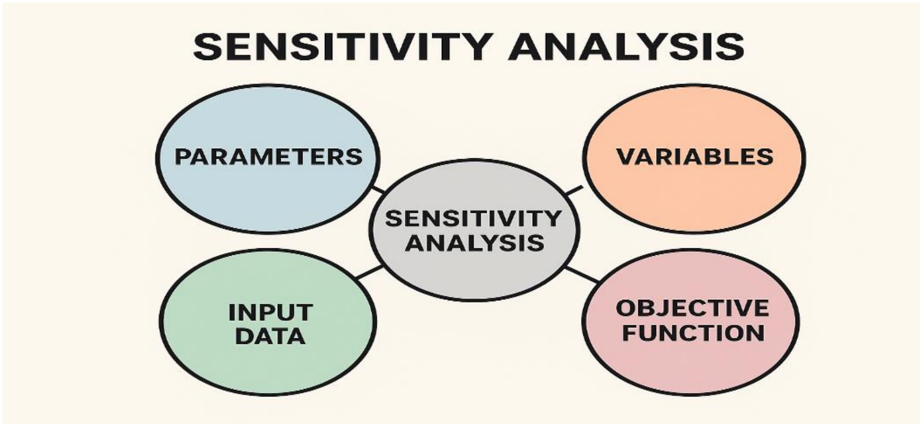
CONTENTS:

10.1 INTRODUCTION

Sensitivity Analysis is a technique used to determine how the change in input parameters (like cost, resources, or profit coefficients) affects the optimal solution of a mathematical or decision-making model — especially in Linear Programming Problems (LPP). It helps to check how sensitive or stable the optimal solution is when there are small changes in the data or assumptions.

Once the optimal solution to a linear programming problem has been attained, it may be desirable to study how the current solution changes when the parameters of the problem are changed. In many practical problems this information is much more important than the single result provided by the optimal solution. Such an analysis converts the static linear programming solution into a dynamic tool to study the effect of changing conditions such as in business and industry.

The change in parameters of the problem may be discrete or continuous. The study of the effect of discrete changes in parameters on the optimal solution is called sensitivity analysis or post optimality analysis, while that of continuous changes in parameters is called parametric programming.



This figure is a concept diagram explaining what *Sensitivity Analysis* studies in a mathematical or optimization model (such as Linear Programming). The central circle is Sensitivity Analysis, and the four surrounding circles show the main components whose changes are analyzed

10.2 OBJECTIVES

After studying this unit learner will be able

1. Understand the concept of Sensitivity Analysis.
2. Explain the importance of Sensitivity Analysis in evaluating the stability.
3. To Determine the range of optimality for objective function coefficients.
4. To Apply Sensitivity Analysis tools in managerial decision-making and resource allocation.

10.3 CHANGES IN THE RIGHT-HAND SIDE OF THE CONSTRAINTS b_i

Suppose that an optimal solution to a linear programming problem has already been found and it is desired to find the effect of increasing or decreasing some resource. Clearly, this will affect not only the objective function but also the solution. Large changes in the limiting resources may even change the variables in the solution.

Examples:10.3.1 (a) Solve the problem

$$\text{maximize} Z = 5x_1 + 12x_2 + 4x_3$$
$$\text{subject to} \quad x_1 + 2x_2 + x_3 \leq 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

(b) Discuss the effect of changing the requirement vector from $\begin{bmatrix}5\\2\end{bmatrix}$ to $\begin{bmatrix}7\\2\end{bmatrix}$ on the optimum solution.
(c) Discuss the effect of changing the requirement vector from $\begin{bmatrix}5\\2\end{bmatrix}$ to $\begin{bmatrix}3\\9\end{bmatrix}$ on the optimum solution.
(d) Which resource should be increased and how much to achieve the best marginal increase in the value of the objective function?
(e) Which resource should be decreased and how much to achieve the best marginal increase in the value of the objective function?

Solution

(a) The standard form of this problem is
maximize $Z = 5x_1 + 12x_2 + 4x_3 + 0s_1 - MA_1$,
subject to $x_1 + 2x_2 + x_3 + s_1 = 5$,

$$2x_1 - x_2 + 3x_3 + A_1 = 2$$
$$x_1, x_2, x_3, s_1, \ A_1 \geq 0$$

Putting $x_1 = x_2 = x_3 = 0$ in the constraint equations, we get $s_1 = 5$ and $A_1 = 2$ as the initial basic solution which can be expressed in the form of a simple matrix or **table10.3.1-1**

C_B		Objective function c_j	5	12	4	0	-M	
0		variables in current solution	x_1	x_2	x_3	s_1	A_1	b
0		S_1	1	2	1	1	0	5
-M		A_1	2	-1	3	0	1	2
			Initial basic feasible solution to the artificial system					

First Iteration: (i) Perform optimality test.

Table10.3.1-2

c_j		5	12	4	0	-M	b	
c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	5	
0	s_1	1	2	1	1	0	5	2β key row
-M	A_1	2	-1	(3)	0	1	2	
$E_j = \Sigma_{c_B} a_{ij}$		-2 M	M	-3 M	0	-M		
$c_j = c_j - E_j$		$5 + 2M$	$12 - M$	$4 + 3M$	0	0		$c_j E_j$

(ii) Make key element unity.

Table 10.3.1-3

c_8	c.s.v.	x_1	x_2	x_3	s_1	A_1	b	
0	s_1	1	2	1	1	0	5	
-M	A_1	$2/3$	$-1/3$	(1)	0	$1/3$	$2/3$	Key element unity

(ii) Replace A_1 by x_3.

Table 10.3.1-4

	c_j	5	12	4	0	-M	b	θ
c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	13/3	13/7 ←Key row
0	s_1	1/3	(7/3)	0	1	-1/3	2/3	-2
4	x_3	2/3	-1/3	1	0	1/3		

$E_j = \Sigma_{c_B} a_{ij}$		8/3	-4/3	4	0	4/3		
$\overline{c_j} = c_j - E_j$		7/3	40/3	0	0	-M-$\frac{4}{3}$		
			↑ K		Second feasible solution			

Second Iteration. (i) Make key element unity. .

Table 10.3.1-5

c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
0	s_1	$\frac{1}{7}$	(1)	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$
4	x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{2}{3}$
							Key element unity

(ii) Replace s_1 by x_2.

Table 10.3.1-6

	c_j	5	12	4	0	-M	b	θ
c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	13/7	13
12	s_1	1/7	1	0	3/7	-1/7	9/7	9/5 ←Key row
4	x_3	5/7	0	1	1/7	2/7		
$E_j = \Sigma_{c_B} a_{ij}$		32/7	12	4	40/7	-4/7		
$\overline{c_j} = c_j - E_j$		3/7	4	0	-40/7	-M-$\frac{4}{7}$		
	↑ K				Third feasible solution			

Third Iteration. (i) Make key element unity.

Table 10.3.1-7

c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
12	x_2	1/7	1	0	3/7	-1/7	13/7
4	x_3	(1)	0	7/5	1/5	2/5	9/5 Key element

(ii) Replace x_3 by x_1.

Table 10.3.1-8

	C_j	5	12	4	0	-M	b
C_B	c.s.v	$\boldsymbol{x_1}$	x_2	x_3	s_1	A_1	
12	x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{9}{5}$
$E_j = \Sigma \boldsymbol{c_B} a_{ij}$		5	12	$\frac{23}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$	
$\bar{c}_j = c_j - E_j$		0	0	$-\frac{3}{5}$	$-\frac{29}{5}$	$-M + \frac{2}{5}$	
Optimal feasible solution							

Thus, the optimal solution is $x_1 = 9/5, x_2 = 8/5, x_3 = 0$,

$$Z_{\max} = 5 \times 9/5 + 12 \times 8/5 + 0 = 141/5.$$

(b) New values of the current basic variables are given by

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} - \frac{2}{5} \\ \frac{7}{5} + \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{11}{5} \end{bmatrix}.$$

Since both x_1 and x_2 are non-negative, the current basic solution consisting of x_1 and x_2 remains feasible and optimal at the new values $x_1 = 11/5, x_2 = 12/5$ and $x_3 = 0$. The new optimum value of Z is $5 \times 11/5 + 12 \times 12/5 + 4 \times 0 = 199/5$.

(c) New values of the current basic variables are

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} - \frac{9}{5} \\ \frac{3}{5} + \frac{18}{5} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ \frac{21}{5} \end{bmatrix}.$$

Since x_2 becomes $-ve$, the current optimal solution becomes infeasible. As discussed dual simplex method may be used to clear infeasibility of the problem. Table 6.44 is modified and written as below.

Table 10.3.1-9

	C_j	5	12	4	0	-M	b
C_B	c.s.v	$\boldsymbol{x_1}$	x_2	x_3	s_1	A_1	
12	x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{-3}{5}$key row
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{21}{5}$
$E_j = \Sigma \boldsymbol{c}_{\mathbf{B}} a_{ij}$		5	12	$\frac{23}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$	
$\bar{c}_j = c_j - E_j$		0	0	$-\frac{3}{5}$	$-\frac{29}{5}$	$-M + \frac{2}{5}$	
↑k							

As $b_1 = -\frac{3}{5}$, the first row is the key row and x_2 is the outgoing variable. Find the ratios of nonbasic elements of c_j row to the elements of key row. Neglect the ratios corresponding to positive or zero elements of key row and choose the lowest ratio. The desired ratio is $\frac{-3/5}{-1/5} = 3$. Hence ' x_3 '-column is the key column, x_3 is the incoming variable and $\left(-\frac{1}{5}\right)$ is the key element. Make the key element unity. This is shown in table **10.3.1-10**

Table 10.3.1-10

c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
12	x_2	0	-5	(1)	-2	1	3
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{21}{5}$
							Key element unity

Replace x_2 by x_3. This is shown in table 6.47.

Table 10.3.1-11

	C_j	5	12	4	0	-M	b
C_B	c.s.v	$\boldsymbol{x_1}$	x_2	x_3	s_1	A_1	
4	x_3	0	-5	1	-2	1	3
5	x_1	1	7	0	$\frac{1}{5}$	-1	0
$E_j = \Sigma \boldsymbol{c_B} a_{ij}$		5	15	4	$\frac{29}{5}$	-1	
$\bar{c}_j = c_j - E_j$		0	-3	0	$-\frac{29}{5}$	$-M+1$	

As all elements in $\bar{c}_j$ row are negative or zero and all b_i are positive, the solution given by table 6.47 is optimal. The optimal solution is

$$x_1 = 0, x_2 = 0, x_3 = 3$$
$$Z_{\max} = 5(0) + 12(0) + 4 \times 3 = 12$$

(d) In order to find the resource that should be increased (or decreased), we shall write the final objective function, which is

$$G = 5y_1 + 2y_2,$$

where $y_1 = 29/5$ and $y_2 = -2/5$ are the optimal dual variables. Thus the first resource should be increased as each additional unit of the first resource increases the objective function by $29/5$. Next we are to find how much the first resource should be increased so that each additional unit continues to increase the objective function by $29/5$. This requirement will be met so long as the primal problem remains feasible. If Δ be the increase in the first resource, it can be determined from the condition

$$\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5+\Delta \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10/5 + 2\Delta/5 - 2/5 \\ 5/5 + \Delta/5 + 4/5 \end{bmatrix} = \begin{bmatrix} \frac{8+2\Delta}{5} \\ \frac{9+\Delta}{5} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As x_1 and x_2 remain feasible (≥ 0) for all values of $\Delta > 0$, the first resource can be increased indefinitely while maintaining the condition that each additional unit will increase the objective function by 29/5.

(e) The second resource should be decreased as each additional unit of the second resource decreases the objective function by $2/5$. Let Δ be the decrease in the second resource. To find its extent, we make use of the condition that the current solution remains feasible so long as

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 \\ 2-\Delta \end{bmatrix}$$

$$= \begin{bmatrix} 10/5 - 2/5 + 2\Delta/5 \\ 5/5 + 4/5 - 2\Delta/5 \end{bmatrix} = \begin{bmatrix} \frac{8+\Delta}{5} \\ \frac{9-2\Delta}{5} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Evidently x_1 remains positive only so long as $\frac{9-2\Delta}{5} \geq 0$ or $\Delta \leq 9/2$. If $\Delta > 9/2, x_1$ becomes negative and must leave the solution.

10.4 CHANGES IN THE COST COEFFICIENTS

Changes in the coefficients of the objective function may take place due to a change in cost or profit of either basic variables or non-basic variables. Each of these two cases will first be considered separately. The discussion, will then, be followed by a combined case. All the three cases will be studied by considering an example.

A company wants to produce three products A, B and C . The unit profits

EXAMPLE 10.4.1

on these products are Rs. 4, Rs. 6 and Rs. 2 respectively. These products require two types of resources-man-power and material. The following L.P. model is formulated for determining the optimal product mix:

$$\begin{aligned} &\textbf{Maximize } Z = && 4x_1 + 6x_2 + 2x_3, \\ &\textbf{subject to} && x_1 + x_2 + x_3 \leq 3, \textbf{ (man-power)} \\ & && x_1 + 4x_2 + 7x_3 \leq 9, \textbf{ (material)} \\ & && x_1, x_2, x_3 \geq 0, \end{aligned}$$

Where x_1, x_2, x_3 are the $x_1, x_2, x_3 \geq 0$,
(a) Find the optimal product mix and the corresponding profit to the
(b) (i) Find the range on the values of non-basic variable coefficient c_3 company. such that the current optimal product mix remains optimal.
(ii) What happens if c_3 is increased to Rs. 12? What is the new optimal product mix in this case?

(c) (i) Find the range on basic variable coefficient c_1 such that the current optimal product mix remains optimal.
(ii) Find the effect when c_1 = Rs. 8 on the optimal product mix.
(d) Find the effect of changing the objective function to $Z = 2x_1 + 8x_2 + 4x_3$ on the current optimal product mix.

Solution. The standard form of the problem is

$$\text{maximize} Z = 4x_1 + 6x_2 + 2x_3 + 0x_4 + 0x_5$$
$$\text{subject to } x_1 + x_2 + x_3 + x_4 = 3$$
$$x_1 + 4x_2 + 7x_3 + x_5 = 9$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Putting $x_1 = x_2 = x_3 = 0$ in the constraint equations, we get $x_4 = 3$ and $x_5 = 9$ as the initial basic feasible solution which can be expressed in the form of a simple matrix or table shown below.

Table 10.4.1-1

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
0	x_4	1	1	1	1	0	3
0	x_5	1	4	7	0	1	9
Initial basic feasible solution							

First Iteration: Perform optimally test

Table 10.4.1-2

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b	θ
0	x_4	1	1	1	1	0	3	3
0	x_5	1	(4)	7	0	1	9	9/4← $Key\ row$
$E_j = \Sigma c_B a_{ij}$		0	0	0	0	0		

$c_j = c_j - E_j$		4	6	2	0	0		
			$\uparrow K$		Initial basic feasible solution			

(ii) Make Key element unity

Table 10.4.1-3

c_B	c_j	x_1	x_2	x_3	x_4	x_5	b
0	x_4	1	1	1	1	0	3
0	x_5	1/4	(1)	7/4	0	1/4	9/4
					Key element		

(iii) Replace x_5 by x_2.

Table 10.4.1-4

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b	θ
0	x_4	(3/4)	0	-3/4	1	-1/4	3/4	1
6	x_2	1/4	1	7/4	0	1/4	9/4	9 $\leftarrow Key\ row$
$E_j = \Sigma c_B a_{ij}$		3/2	6	21/2	0	3/2		
$c_j = c_j - E_j$		5/2	0	-17/2	0	-3/2		
		$\uparrow K$				Second feasible solution		

Second Iteration: (i) Make key element unity.

Table 10.4.1-5

c_B	c_j	x_1	x_2	x_3	x_4	x_5	b

0	x_4	1	0	-1	4/3	-1/3	1
6	x_2	1/4	1	7/4	0	1/4	9/4
Key element unity							

(ii) Replace x_4 by x_1

Table 10.4.1-6

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
4	x_1	1	0	-1	4/3	-1/3	1
6	x_2	0	1	2	-1/3	1/3	2
$E_j = \Sigma c_B a_{ij}$		4	6	8	10/3	2/3	
$c_j = c_j - E_j$		0	0	-6	-10/3	-2/3	
Optimal feasible solution							

∴ Optimal solution is $x_1 = 1, x_2 = 2, x_3 = 0$ and $Z_{\max} =$ Rs. $(4 \times 1 + 6 \times 2 + 2 \times 0)$ = Rs. 16.

Effect of changing the objective function coefficient of a nonbasic variable

(b) (i) The coefficient c_3 corresponds to the non-basic variable x_3 for product C . In the optimal product mix shown in table 10.4.1-6 , product C is not produced because of the low associated profit of Rs. 2 per unit (c_3). Clearly, if c_3 further decreases, it will have no effect on the current optimal product mix. However, if c_3 is increased beyond a certain value, it may become profitable to produce

As a rule, the sensitivity of the current optimal solution is determined by the product C . studying how the current optimal solution given in table 10.4.1-6 changes as a result of changes in the input data. When value of c_3 changes, the value of net evaluation (relative profit coefficient) of the non-basic variable x_3 i.e., $\bar{c}_3$ in table 10.4.1-6 also changes. The table will remain optimal as long as $\bar{c}_3$ remains

nonpositive.
∴ For table 6.53 to remain optimal, $\bar{c}_3 \leq 0$ or

$$c_3 - (4,6)\begin{bmatrix}-1\\2\end{bmatrix} \leq 0$$

This means that as long as the unit profit of product C is less than Rs. 8, it
or $c_3 - (-4 + 12) \leq 0$, or $c_3 \leq 8$. is not profitable to produce it.
(ii) If $c_3 = 12, \bar{c}_3 = c_3 - (4,6)\begin{bmatrix}-1\\2\end{bmatrix}$
or $\bar{c}_3 = 12 - (-4 + 12) = 12 - 8 = +4$.
As $\bar{c}_3$, becomes positive, the current product mix given by table 10.4.1-6 does not remain optimal. The optimum profit can be increased further by producing product C. Non-basic variable x_3 can enter the solution to increase Z. This is shown in table 10.4.1-7.

Table 10.4.1-7

	c_j	4	6	12	0	0		
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b	θ
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	-1
6	x_2	0	1	(2)	$-\frac{1}{3}$	$\frac{1}{3}$	2	1
$E_j = \Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$		
$\bar{c}_j = c_j - E_j$		0	0	4	$-\frac{10}{3}$	$-\frac{2}{3}$		
				↑ K				

First Iteration. (i) Make key element unity.

Table 10.4.1-8

c_3	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1

6	x_2	0	$\frac{1}{2}$	(1)	$-\frac{1}{6}$	$\frac{1}{6}$	1

Key element unity

(ii) Replace x_2 by x_3.

Table 10.4.1-9

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
4	x_1	1	1/2	0	7/6	-1/6	2
12	x_3	0	1/2	1	-1/6	1/6	1
$E_j = \Sigma c_B a_{ij}$		4	8	12	8/3	4/3	
$c_j = c_j - E_j$		0	-2	0	-8/3	-4/3	
Optimal feasible solution							

∴. New optimal product mix is $x_1 = 2, x_2 = 0, x_3 = 1$ and $Z_{\max} =$ Rs. $(4 \times 2 + 6 \times 0 + 12 \times 1)$ = Rs. 20.

Effect of changing the objective function coefficient of a basic variable

(c) (i) Clearly, when c_1 decreases below a certain level, it may no longer remain profitable to produce product A . On the other hand, if c_1 increases beyond a certain value, it may become so profitable that it is most paying to produce only product A . In either case the optimal product mix will change and hence there is lower as well as upper limit on c_1 within which the optimal product mix will not be affected.

Referring again to table 10.4.1-6, it can be seen that any variation in c_1 (and/or in c_2 also) will not change $\bar{c}_1$ and $\bar{c}_2$ (i.e., they remain zero), while $\bar{c}_3, \bar{c}_4, \bar{c}_5$ will change. However, as long as $\bar{c}_j (j = 3,4,5)$ remain non-positive, table 10.4.1-6 will remain optimal. $\bar{c}_3, \bar{c}_4$ and $\bar{c}_5$ can be expressed as functions of c_1 as follows:

$$\bar{c}_3 = 2 - (c_1, 6)\begin{bmatrix}-1\\2\end{bmatrix} = 2 - (-c_1 + 12) = c_1 - 10,$$

$$\bar{c}_4 = 0 - (c_1, 6)\begin{bmatrix}\frac{4}{3}\\-\frac{1}{3}\end{bmatrix} = 0 - \left(\frac{4}{3}c_1 - 2\right) = -\frac{4}{3}c_1 + 2,$$

$$\bar{c}_5 = 0 - (c_1, 6)\begin{bmatrix}-\frac{1}{3}\\\frac{1}{3}\end{bmatrix} = 0 - \left(-\frac{1}{3}c_1 + 2\right) = \frac{1}{3}c_1 - 2.$$

For $\bar{c}_3$ to be $\leq 0, c_1 - 10 \leq 0$ or $c_1 \leq 10$,
for $\bar{c}_4$ to be $\leq 0, -\frac{4}{3}c_1 + 2 \leq 0$ or $c_1 \geq \frac{3}{2}$,
for $\bar{c}_5$ to be $\leq 0, \frac{1}{3}\dot{c}_1 - 2 \leq 0$ or $c_1 \leq 6$.
∴ Range on c_1 for the optimal product mix to remain optimal is $\frac{3}{2} \leq c_1 \leq 6$.
Thus so long as c_1 lies within these limits, the optimal solution in table 6.53 viz ., $x_1 = 1, x_2 = 2, x_3 = 0$ remains optimal. However, within this range, as the value of c_1 is changed, $Z_{\max}$ undergoes a change. For example, when $c_1 = 3$, $Z_{\max}$ = Rs. $(3 \times 1 + 6 \times 2) =$ Rs. 15.
(ii) When $c_1 = 8, \bar{c}_3 = c_1 - 10 = 8 - 10 = -2$,

$$\bar{c}_4 = -4\beta c_1 + 2 = -4/3 \times 8 + 2 = -26/3,$$
$$\bar{c}_5 = 1/3c_1 - 2 = 8/3 - 2 = +2/3,$$
$$\bar{c}_1 = \bar{c}_2 = 0.$$

As $\bar{c}_5$ becomes positive, the solution given in table 10.4.1-6 no longer remains optimal. Slack variable x_5 enters the solution. This is shown in table 10.4.1-7.

Table 10.4.1-10

c_B	c_j C.S.V.	8	6	2	0	0			
		x_1	x_2	x_3	x_4	x_5		θ	
8	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	-3	
6	x_2	0	1	2	$-\frac{1}{3}$	$\left(\frac{1}{3}\right)^3$	2	6	← Key row

$E_j = \Sigma c_B a_{ij}$		8	6	4	$\frac{26}{3}$	$-\frac{2}{3}$			
$\bar{c}_j = c_j - E_j$		0	0	-2	$-\frac{26}{3}$	$\frac{2}{3}$			

First Iteration. (i) Make key element unity.

Table 10.4.1-11

c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b	
8	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	
6	x_2	0	3	6	-1	(1)	6	Key element unity

(ii) Replace x_2 by s_2

Table 10.4.1-12

	c_j	8	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
c_B	x_1	1	1	1	1	0	3
8	x_5	0	3	6	-1	1	6
0		4	8	12	8/3	4/3	
$E_j = \Sigma c_B a_{ij}$		8	8	8	8	0	
$c_j = c_j - E_j$		0	-2	-6	-8	0	
				Optimal feasible solution			

Thus, the optimal product mix changes to $x_1 = 3$ units with $Z_{\max} =$ Rs. 24. Effect of changing the objective function coefficients of basic as well as non-basic variable.

(d) The effect on the optimal product mix can be determined by checking whether the $\bar{c}_j$ row in table 10.4.1-6 remains nonpositive.

$$\bar{c}_1 = 0,$$
$$\bar{c}_2 = 0,$$
$$\bar{c}_3 = 4 - (2,8)\begin{bmatrix}-1\\2\end{bmatrix} = 4 - (-2 + 16) = -10 < 0,$$
$$\bar{c}_4 = 0 - (2,8)\begin{bmatrix}\frac{4}{3}\\-\frac{1}{3}\end{bmatrix} = 0 - \left(\frac{8}{3} - \frac{8}{3}\right) = 0,$$

$$\bar{c}_5 = 0 - (2,8)\begin{bmatrix}-\frac{1}{3}\\\frac{1}{3}\end{bmatrix} = 0 - \left(\frac{2}{3} + \frac{8}{3}\right) = -2 < 0.$$

Hence the optimal solution does not change. The optimal product mix remains $x_1 = 1, x_2 = 2, x_3 = 0$ and $Z_{\max} =$ Rs. $(1 \times 2 + 2 \times 8 + 0 \times 4) =$ Rs. 18. There is indication of an alternate optimal solution since $\bar{c}_4 = 0$.

10.5 ADDITION OF A NEW VARIABLE

let us suppose that Research and Development department of the company has proposed a fourth product D which requires 1 unit of manpower and 1 unit of material and earns a unit profit of Rs. 3 when sold in the market. It is desired to find whether it is profitable to produce product D.

Addition of this product in the already existing product mix is equivalent to addition of a new variable (say x_4) and a column $\begin{bmatrix}1\\1\end{bmatrix}$ in the initial Table 10.4.1-1. Now the present optimal product mix given by the table 10.4.1-6 remains optimal so long as the relative profit coefficient (net evaluation) of this new product, say $\bar{c}_6$ remains non-positive.

Now from the revised simplex method we know that

$$\bar{c}_6 = c_6 - \mathbf{c}_{\mathrm{B}}\overline{\mathbf{P}}_6 = c_6 - \mathbf{c}_{\mathrm{B}} \cdot \mathbf{B}^{-1}\mathbf{P}_6 = c_6 - \pi\mathbf{P}_6,$$

where $c_6 =$ Rs. $3, \mathbf{P}_6 = \begin{bmatrix}1\\1\end{bmatrix}$ and π is the simplex multiplier corresponding to the current optimal solution contained in Table 10.4.1-6 and is given by

$$\pi = \mathbf{c}_{\mathrm{B}}\mathbf{B}^{-1}$$
$$= (4,6)\begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \left(\frac{10}{3}, \frac{2}{3}\right).$$

$$\therefore\ \bar{c}_6 = 3 - \left(\frac{10}{3}, \frac{2}{3}\right)\begin{bmatrix}1\\1\end{bmatrix} = 3 - \left(\frac{10}{3} + \frac{2}{3}\right) = -1.$$

As $\bar{c}_6$ is non-positive, the present optimal solution does not change even after the product D is introduced. As product D cannot improve the present value of the maximum profit, it should not be produced.

If, however, $\bar{c}_6$ turns out to be positive, it follows that product D can increase the value of maximum profit; simplex method can then be applied to find the new optimal solution.

EXAMPLE 10.5.1

Consider the problem

$$\begin{aligned} \text{maximize} \quad & Z = 45x_1 + 100x_2 + 30x_3 + 50x_4, \\ \text{subject to} \quad & 7x_1 + 10x_2 + 4x_3 + 9x_4 \leq 1{,}200, \\ & 3x_1 + 40x_2 + x_3 + x_4 \leq 800, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

The optimal table for this problem is given below.

Table 10.5.1-1

	c_j	45	100	30	50	0	0	
c_{B}	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b
30	x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	$\frac{800}{3}$
		$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	$\frac{40}{3}$
100	x_2	$\frac{25}{3}$	0	0	$-\frac{50}{3}$	$-\frac{22}{3}$	$-\frac{2}{3}$	
$\bar{c}_j = c_j - \mathrm{E}_j$		$-\frac{25}{3}$						

If a new variable x_7 is added to this problem with a column $\begin{bmatrix}10\\10\end{bmatrix}$ and $c_7 = 120$, find the change in the optimal solution.

Solution

$$\bar{c}_7 = c_7 - \mathbf{c_B}\overline{\mathbf{P}}_7 = c_7 - \mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{P}_7 = c_7 - \pi\mathbf{P}_7,$$

where $c_7 = 120, P_7 = \begin{bmatrix}10\\10\end{bmatrix}$ and π, the simplex multiplier corresponding to the original optimal solution in table 10.5.1-1 is given by $\pi = (\pi_1, \pi_2) = \mathbf{c_B}\mathbf{B}^{-1} = (30,100)\begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\ -\frac{1}{150} & \frac{2}{75}\end{bmatrix} = \left(\frac{22}{3}, \frac{2}{3}\right)$.

$$\therefore \quad \bar{c}_7 = c_7 - \pi \mathrm{P}_7 = 120 - \left(\frac{22}{3}, \frac{2}{3}\right)\begin{bmatrix}10\\10\end{bmatrix}$$
$$= 120 - \left(\frac{220}{3} + \frac{20}{3}\right) = +40.$$

Since $\bar{c}_7$ is positive

Since $\bar{c}_7$ is positive, the existing optimal solution can be improved.

Now $\overline{\mathbf{P}}_7 = \mathbf{B}^{-1}\mathbf{P}_7 = \begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\ -\frac{1}{150} & \frac{2}{75}\end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix} = \begin{bmatrix}2\\ \frac{1}{5}\end{bmatrix}$.

Now we start with the original optimal table (table 10.5.1-1) and add entries corresponding to variable x_7 as follows:

Table 10.5.1-2

	c_j	45	100	30	50	0	0	120		
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	$\boldsymbol{\theta}$
30	x_3	5/3	0	1	7/3	4/15	-1/15	2	800/3	400/3
100	x_2	1/30	1	0	-1/30	-1/150	2/75	(1/5)	40/3	200/3← *Key row*

$c_j = c_j - E_j$		-25/3	0	0	-50/3	-22/3	-2/3	+40 $\uparrow K$		

Make key element unity.

Table 10.5.1-3

c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
30	x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	2	$\frac{800}{3}$
100	x_2	$\frac{1}{6}$	5	0	$-\frac{1}{6}$	$-\frac{1}{30}$	$\frac{2}{15}$	(1)	$\frac{200}{3}$

Replace x_2 by x_7.

Table 10.5.1-4

	c_j	45	100	30	50	0	0	120	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
30	x_3	4/3	-10	1	8/3	1/3	-1/3	0	400/3
120	x_7	1/6	5	0	-1/6	-1/30	2/15	1	200/3
$E_j = \Sigma c_B a_{ij}$		60	300	30	60	6	6	120	
$\bar{c}_j = c_j - E_j$		-15	-200	0	-10	-6	-6	0	-15
Optimal feasible solution									

Optimal feasible solution

Since $\bar{c}_j$ is negative, table 10.5.1-4 gives the optimal solution with $x_3 = 400/3, x_7 = 200/3$ (basic variables), $x_1 = x_2 = x_4 = x_5 = x_6 = 0$ (non-basic variables) and $Z_{\max} = 30 \times 400/3 + 120 \times$

$200/3 = 4{,}000 + 8{,}000 = 12{,}000.$

10.6 CHANGES IN THE COEFFICIENTS OF THE CONSTRAINTS A_{IJ}

When changes take place in the constraint coefficients of a non-basic variable in a current optimal solution, feasibility of the solution is not affected. The only effect, if any, may be on the optimality of the solution.

However, if the constraint coefficients of a basic variable get changed, things become more complicated since the feasibility of the current optimal solution may also be affected (lost). The basic matrix is affected, which, in turn, may affect all the quantities given in the current optimal table. Under such circumstances, it may be better to solve the problem over again.

EXAMPLE 10.5.2

Find the effect of the following changes in the original optimal table 10.5.1-1of problem 10.5.1

(a) ' x_1 '-column in the problem changes from $\begin{bmatrix}7\\3\end{bmatrix}$ to $\begin{bmatrix}7\\5\end{bmatrix}$.

. (b) ' x_1 '-column changes from $\begin{bmatrix}7\\3\end{bmatrix}$ to $\begin{bmatrix}5\\8\end{bmatrix}$.

Solution

(a) x_1 is a nonbasic variable in the optimal solution.

$$\bar{c}_1 = c_1 - \mathbf{c_B}\overline{\mathbf{P}}_1 = c_1 - \mathbf{c_B}\mathbf{B}^{-1}\mathbf{P}_1$$
$$= c_1 - \pi\mathbf{P}_1,\ \text{where } c_1 = 45, \mathbf{P}_1 = \begin{bmatrix}7\\5\end{bmatrix},$$

and $\pi = \mathbf{c_B}\mathbf{B}^{-1} = (30,100)\begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\ -\frac{1}{150} & \frac{2}{75}\end{bmatrix} = \left(\frac{22}{3}, \frac{2}{3}\right).$

$\therefore\ \bar{c}_1 = 45 - \left(\frac{22}{3}, \frac{2}{3}\right)\begin{bmatrix}7\\5\end{bmatrix} = 45 - \left(\frac{154}{3} + \frac{10}{3}\right) = 45 - \frac{164}{3} = -\frac{29}{3}.$

Since c_1 remains non-positive, the original optimum solution remains optimum for the new problem also.

(b) $\bar{c}_1 = c_1 - \mathbf{c_B}\overline{\mathbf{P}}_1 = c_1 - \mathbf{c_B}\mathbf{B}^{-1}\mathbf{P}_1$

$$= c_1 - \pi\mathrm{P}_1 = 45 - \left(\frac{22}{3}, \frac{2}{3}\right)\begin{bmatrix}5\\8\end{bmatrix} = 45 - \left(\frac{110}{3} + \frac{16}{3}\right) = +3$$

As $\bar{c}_1$ is positive, the existing optimum solution can be improved.

Now $\overline{\mathbf{P}}_1 = \mathbf{B}^{-1}\mathbf{P}_1 = \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{4}{75} \\ \frac{27}{150} \end{bmatrix}$.

Now we start with the original optimal table (table 10.5.1-1 and incorporate the changes due to variable x_1.

Table 10.5.2.1

	c_j	45	100	30	50	0	0		
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b	θ
30	x_3	4/5	0	1	7/3	4/15	-1/15	800/3	1000/3
120	x_2	27/150	1	0	-1/30	-1/150	2/75	40/3	2000/27← *Key row*
$\bar{c}_j = c_j - E_j$		+3 ↑ K	0	0	-50/3	-22/3	-2/3		

Make key element unity.

Table 10.5.2.2

c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	4/5	0	1	7/3	4/15	-1/15	800/3
x_2	1	150/27	0	-5/27	-1/27	4/27	2000/27
Key Unity Element							

Replace x_2 by x_1.

Table 10.5.2.3

	c_j	45	100	30	50	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b

30	x_3	0	-40/9	1	67/27	8/27	-5/27	5600/27
120	x_1	1	50/9	0	-5/27	-1/27	4/27	2000/27
$E_j = \Sigma c_B a_{ij}$		45	350/3	30	595/9	65/9	10/9	
$\bar{c}_j = c_j - E_j$		0	-50/3	0	-145/9	-65/9	-10/9	
Optimal feasible solution								

Since $\bar{c}_j$ is non-positive, table 10.5.2-3 gives the optimal solution with

$$x_1 = \frac{2{,}000}{27}, x_3 = \frac{5{,}600}{27} \text{ (basic variables)}$$

$$x_2 = x_4 = x_5 = x_6 = 0 \text{ (non-basic variables)}$$

$$Z_{\max} = \frac{2{,}000}{27} \times 45 + \frac{5{,}600}{27} \times 30 = \frac{10{,}000}{3} + \frac{56{,}000}{9} = \frac{86{,}000}{9}$$

CHECK YOUE PROGRESS

Problem 1: Obtain the optimum solution of the LPP

Maximize $Z = 15x_1 + 45x_2$ subject to the constraints:

$x_1 + 6x_2 \le 240;\ 5x_1 + 2x_2 \le 162;\ x_2 \le 50; x_1, x_2 \ge 0$

If maximum $z = \sum c_j x_j,\ j = 1, 2$ and c_2 is kept fixed at 45, determine how much can c_1 be changed without affecting the above optimal solution.

Answer: Optimal solution: $x_1 = 27.1, x_2 = 13.3$; Maximum $z = 1005$; $2.8 \le c_1 \le 112.5$

MULTIPLE CHOICE QUESTIONS

1: Choose the correct correct option for the statement "post-optimal analysis is a technique to"

(a) Analyse how the optimal solution to a Linear Programming Problem (LPP) is affected by changes in the problem inputs.

(b) Distribute resources in the most effective way.

(c) Minimize the operational costs.

(d) Describe the relationship between the dual problem and its primal.

2: Addition of a new constraints in the existing constraints will ensure a

(a) Change in the coefficient a_{ij}.

(b) Change in the objective function coefficient c_j.

(c) Both (a) and (b).

(d) Neither (a) nor (b)

3. To achieve the maximum marginal increase in the objective function value, it is advisable to increase the value of a resource with the highest shadow price

(a) Smaller.

(b) larger.

(c) Both (a) and (b).

(d) Neither (a) nor (b)

Fill in the blanks:

1. Post-optimality analysis study only the continuous changes in the parameter of …………………….
2. Optimum solution to an LPP is not very sensitive to the changes in the RHS values of the ………………

3. The optimality of the current solution may be affected if right hand side of the constraints is ………..

10.7 SUMMARY

Sensitivity Analysis is an important part of Linear Programming and decision-making models. It studies how changes in the input data (such as profit coefficients, resource availability, or constraint values) affect the optimal solution **of** a problem.

It helps decision-makers understand how stable or **sensitive** the current solution is when real-life conditions change. The analysis is performed after obtaining the optimal solution, so it is also called Post-Optimality Analysis**.**

10.8 GLOSSARY

Sensitivity Analysis: A method used to determine how changes in input values (like profit, cost, or resources) affect the optimal solution.
Post-Optimality Analysis: Another name for sensitivity analysis; performed after obtaining the optimal solution to check its stability.

Optimal Solution: The best possible solution that maximizes profit or minimizes cost in a Linear Programming Problem (LPP).

Objective Function Coefficient: The numerical value (like profit or cost per unit) associated with a decision variable in the objective function.

Range of Optimality: The range within which the coefficient of the objective function can change without altering the optimal solution.

Range of Feasibility: The range of values for the right-hand side (RHS) of constraints within which the current solution remains feasible.

Constraint: A condition or limitation (like labour hours, materials, or budget) that restricts the values of decision variables.

Right-Hand Side (RHS): The constant term in a constraint that represents the total available resources.

10.9 REFERENCES

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10.10 SUGGESTED READING

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10.11 TERMINAL QUESTIONS

1. Determine the range within which c_3, c_4 and b_2 can be varied while preserving the optimality of the current solution in the specified Linear Programming Problem (LPP).

Maximize, $z = 3x + 5y + 4z$

Subject to the constraints,

$2x + 3y \le 8;\ 2x + 5y \le 10;\ 3x + 2y + 4z \le 15; x, y, z \ge 0$;

2. In the given LPP

Minimize, $z = 3x + 6y + z$

Subject to the constraints, $x+y+z\geq 6;\ x+5y-z\geq 4; x+5y+z\geq 24; x,y,z\geq 0$;

Solve the Linear Programming Problem (LPP) and analyze the impact of modifying the requirement vector from [6, 4, 24] to [6, 2, 12] on the optimal solution.

3. In the given LPP

Maximize, $z=4x_1+3x_2+4x_3+6x_4$

Subject to, $x_1+2x_2+2x_3+4x_4\leq 8; 2x_1+2x_3+x_4\leq 6$;

$3x_1+3x_2+x_3+x_4\leq 8; x_1,x_2,x_3,x_4\geq 0$

(a) Identify the individual ranges for discrete changes in a_{12}, a_{22} and a_{23} that are consistent with maintaining the optimal solution of the given LPP.

(b) If a_{11} is changed to $a_{11}+\Delta a_{11}$, determine the allowable limit for the discrete change Δa_{11} in order to preserve the optimality of the current solution.

10.12 ANSWERS

MCQ 1:(a) **MCQ 2:** (c) **MCQ 3:**(b)

TQ1. $1.25\leq c_3\leq 11.5$; $c_4\leq 1.10$; $3.75\leq b_2\leq 17.42$

TQ2. (a) $x_1=14; x_2=0; x_3=10;$ Minimum $z=52$

(b) $x_1=7; x_2=0; x_3=5;$ Minimum = 26

TQ3. (a) $(15/16)\leq a_{12}$; $(-17/6)\leq a_{22}$ and $(-1/8)\leq a_{23}$

(b) $-3\leq \Delta a_{11}\leq 17/21$

Fill in the blank question

1: LPP **2:** Constraints **3:** Changed

UNIT -11 PARAMETRIC LINEAR PROGRAMMING AND INTEGER PROGRAMMING

CONTENTS:

11.1 INTRODUCTION

In real-world decision-making problems, the parameters used in a Linear Programming Problem (LPP) such as objective function coefficients, resource availability, or constraint values, may not always remain constant. To study how changes in these parameters affect the optimal solution, we use Parametric Linear Programming (PLP).

Parametric Linear Programming is an extension of the standard LPP that analyses how the optimal solution changes when one or more parameters of the problem vary continuously within a specified range. It helps decision-makers understand the behaviour of the optimal solution under different scenarios without resolving the problem repeatedly. This concept is particularly useful in business, economics, and engineering applications where input data may fluctuate due to uncertain market or resource conditions.

11.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of Parametric Linear Programming and how it extends the traditional LPP.
2. To Explain the need for studying the effect of changing parameters in an optimization model.
3. To Analyse how variations in objective function coefficients or constraint values influence the optimal solution.
4. To Determine the range of parameter values for which the current optimal solution remains valid.

11.3 PARAMETRIC PROGRAMMING

In previous unit on sensitivity analysis discussed the effect of discrete changes in the input coefficients of the linear programming problem on its optimal solution. However, if there is continuous change in the values of these coefficients, none of the results derived in that section are applicable. Parametric linear programming investigates the effect of predetermined continuous variations of these coefficients on the optimal solution. It is simply an extension of sensitivity analysis and aims at finding the various basic solutions that become optimal, one after the other, as the coefficients of the problem change continuously. The coefficients change as a linear function of a single parameter, hence the name parametric linear programming for this computational technique. As in sensitivity analysis, the purpose of this technique is to reduce the additional computations required to obtain the changes in the optimal solution.

Let the linear programming problem before parameterization be minimize $Z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$,

$$X \geq 0,$$

where $\mathbf{C}$ is the given cost vector.
Let this cost vector change to $\mathbf{C} + \lambda\mathbf{C}'$ so that the parametric cost problem becomes

$$\text{minimize} Z = (\mathbf{C} + \lambda \mathbf{C}')\mathbf{X},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b},$$
$$\mathbf{X} \geq \mathbf{0},$$

where C' is the given predetermined cost variation vector and λ is an unknown (positive or negative) parameter. As λ changes, the cost coefficients of all variables also change. We wish to determine the family of optimal solutions as λ changes from $-\infty$ to $+\infty$.

This problem is solved by using the simplex method and sensitivity analysis. When $\lambda = 0$, the parametric cost problem reduces to the original L.P. problem; simplex method is used to find its optimál solution Let $\mathbf{B}$ and $\mathbf{X_B}$ represent the optimal basis matrix and the optimal basic feasible solution respectively for $\lambda = 0$. The net evaluations or relative cost coefficients are all non-negative (minimization problem) and are given by

$$\bar{c}_j = c_j - \mathrm{E}_j = c_j - \Sigma \boldsymbol{c}_{\mathbf{B}} a_{ij} = c_j - \mathbf{c_B} \overline{\mathbf{P}}_j,$$

where $\mathbf{c_B}$ is the cost vector of the basic variables and $\overline{\mathbf{P}}_j$ is the j th column (corresponding to the variable x_j) in the optimal table.

As λ changes from zero to a positive or negative value, the feasible region and values of the basic variables $\mathbf{X_B}$ remain unaltered, but the relative cost coefficients change. For any variable x_j, the relative cost coefficient is given by

$$\bar{c}_j(\lambda) = \left(c_j + \lambda c_j'\right) - (\mathbf{c_B} + \lambda \mathbf{c}_{\mathbf{B}}')\overline{\mathbf{P}}_j$$
$$= \left(c_j - \mathbf{c_B}\overline{\mathbf{P}}_j\right) + \lambda\left(c_j' - \mathbf{c}_{\mathbf{B}}'\overline{\mathbf{P}}_j\right) = \bar{c}_j + \lambda \bar{c}_j'$$

Since vectors $\mathbf{C}$ and $\mathbf{C}'$ are known, $\bar{c}_j$ and $\bar{c}_j'$ can be determined. For the current minimization problem, $\bar{c}_j(\lambda)$ must be non-negative for the solution to be optimal $[\bar{c}_j(\lambda)$ must be non-positive for a maximization problem]. Thus

$$\bar{c}_j(\lambda) \geq 0, \text{ or } \bar{c}_j + \lambda \bar{c}_j' \geq 0.$$

In other words, for a given solution, we can determine the range for λ within which the solution remains optimal.

Example 11.3.1 Consider the linear programming problem

$$\begin{aligned} \text{maximize Z} = \quad & 4x_1 + 6x_2 + 2x_3, \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 3, \\ & x_1 + 4x_2 + 7x_3 \leq 9, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The optimal solution to this problem is given by the following table:

Table 11.3.1-1

	c_j	4	6	2	0	0	
C_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2
$E_j = \Sigma_{c_B} a_{ij}$		4	6	8	10/3	2/3	
$c_j = c_j - E_j$		0	0	-6	-10/3	-2/3	

solve this problem if the variation cost vector $\mathbf{C}' = (2, -2,2,0,0)$. Identify all critical values of the parameter λ.

Solution. The given parametric cost problem is

$$\begin{aligned} \text{maximize Z} = \quad & (4 + 2\lambda)x_1 + (6 - 2\lambda)x_2 + (2 + 2\lambda)x_3 + 0x_4 + 0x_5 \\ \text{subject to} \quad & 1 + x_2 + x_3 + x_4 = 3 \\ & x_1 + 4x_2 + 7x_3 + x_5 = 9 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

When $\lambda = 0$, the problem reduces to the L.P. problem, whose optimal solution is given by table 11.3.1-1. The relative profit coefficients in this optimal table are all non-positive. For values of λ other than zero, the relative profit coefficients become linear functions of λ_e To compute them we, first, add a new relative profit row called $\bar{c}'_j$ row to table11.3.1-1 . This is shown in table Table 11.3.1-2

Table 11.3.1-2

		c'_j	2	-2	2	0	0	
		c_j	4	6	2	0	0	
c'_B		c.s.v.	x_1	x_2	x_3	x_4	x_5	b
	4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
	6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2
-2	$\bar{c}_j$		0	0	-6	$-\frac{10}{3}$	$-\frac{2}{3}$	$\text{Z} = 16$
	$\bar{c}'_j$		0	0	8	$-\frac{10}{3}$	$\frac{4}{3}$	$Z' = -2$

In table 11.3.1-2, $\bar{c}'_j$ is calculated just as $\bar{c}_j$ row except that vector C is replaced by $\mathbf{C}'$. For example,

$$\bar{c}_2 = c_2 - \text{E}_2 = c_2 - \Sigma \mathbf{c}_{\text{B}} a_{i2} = c_2 - \mathbf{c}_{\text{B}} \overline{\mathbf{P}}_2$$
$$= 6 - (4,6)\begin{bmatrix}0\\1\end{bmatrix} = 6 - 6 = 0$$
$$\therefore\ \bar{c}'_1 = c'_1 - \mathbf{c}'_{\text{B}} \overline{\mathbf{P}}_1$$
$$= 2 - (2,-2)\begin{bmatrix}1\\0\end{bmatrix} = 0$$
$$\bar{c}'_2 = -2 - (2,-2)\begin{bmatrix}0\\1\end{bmatrix} = 0$$
$$\bar{c}'_3 = 2 - (2,-2)\begin{bmatrix}-1\\2\end{bmatrix} = 2 - (-2 - 4) = 8$$
$$\bar{c}'_4 = 0 - (2,-2)\begin{bmatrix}\frac{4}{3}\\-\frac{1}{3}\end{bmatrix} = -\left(\frac{8}{3} + \frac{2}{3}\right) = -\frac{10}{3},$$
$$\bar{c}'_5 = 0 - (2,-2)\begin{bmatrix}-\frac{1}{3}\\\frac{1}{3}\end{bmatrix} = -\left(-\frac{2}{3} - \frac{2}{3}\right) = \frac{4}{3},$$
$$Z' = 1 \times 2 - 2 \times 2 = -2.$$

table 11.3.1-2, represents a basic feasible solution for the given parametric cost problem. It is given by

$$x_1 = 1, x_2 = 2, x_3 = x_4 = x_5 = 0.$$

Value of the objective function, $Z(\lambda) = Z + \lambda Z' = 16 - 2\lambda$. The relative profit coefficients, which are linear functions of λ, are given by

$$\bar{c}_j(\lambda) = \bar{c}_j + \lambda \bar{c}'_j, j = 1,2,3,4,5.$$

table 11.3.1-2, will be optimal if $\bar{c}_j(\lambda) \leq 0$ for $j = 3,4,5$. Thus we can determine the range of λ for which table 11.3.1-2, remains optimal as follows:

$$\bar{c}_3(\lambda) = \bar{c}_3 + \lambda \bar{c}'_3 = -6 + 8\lambda \leq 0 \text{ or } \lambda \leq 3/4,$$
$$\bar{c}_4(\lambda) = \bar{c}_4 + \lambda \bar{c}'_4 = -\frac{10}{3} - \frac{10}{3}\lambda \leq 0 \text{ or } \lambda \geq -1,$$
$$\bar{c}_5(\lambda) = \bar{c}_5 + \lambda \bar{c}'_5 = -\frac{2}{3} + \frac{4}{3}\lambda \leq 0 \text{ or } \lambda \leq \frac{1}{2}.$$

Thus $x_1 = 1, x_2 = 2, x_3 = x_4 = x_5 = 0$ is an optimal solution for the given -parametric problem for all values of λ between -1 and $1/2$ and $Z_{\max} = 16 - 2\lambda$.

- For $\lambda > 1/2$, the relative profit coefficient of the non-basic variable x_5, 'namely $\bar{c}_5(\lambda)$ becomes positive and table 11.3.1-2, no longer remains optimal. Regular simplex method is used to iterate towards optimality. x_5 is the entering variable and computation of ' θ '-column indicates x_2 to be the variable that leaves the basis matrix so that the key element is $1/3$. The key element is made unity in table 11.3.1-2,

Table 11.3.1-2,

c'_B	c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
2	4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
-2	6	x_2	j	3	6	-1	(1)	6

Table 11.3.1-3

	c'_j		2	-2	2	0	0	
	c_j		4	6	2	0	0	
c'_B	c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
2	4	x_1	1	1	1	1	0	3
0	0	x_5	0	3	6	-1	1	6

		$\bar{c}_j$	0	2	-2	-4	0	$Z = 12$
		$\bar{c}'_j$	0	-4	0	-2	0	$Z' = 6$

Table 11.3.1-3 will be optimal if $\bar{c}_j(\lambda) \leq 0$, for $j = 2,3,4$. Now $\bar{c}_2(\lambda) = \bar{c}_2 + \lambda \bar{c}'_2 = 2 - 4\lambda \leq 0 \;\therefore \lambda \geq \frac{1}{2}$,

$\bar{c}_3(\lambda) = \bar{c}_3 + \lambda \bar{c}'_3 = -2 \leq 0$, which is true,
$\bar{c}_4(\lambda) = \bar{c}_4 + \lambda \bar{c}'_4 = -4 - 2\lambda \leq 0 \;\therefore \lambda \geq -2.$

$\therefore$ For all $\lambda \geq \frac{1}{2}$, the optimal solution is given by

$$x_1 = 3, x_2 = x_3 = x_4 = 0, x_5 = 6 \text{ and } Z_{\max} = 12 + 6\lambda$$

For $\lambda < -1$, the relative profit coefficient of the non-basic variable x_4 namely $\bar{c}_4(\lambda)$ becomes positive and again table no longer remains optimal. x_4 becomes the entering variable and x_1 the leaving variable. Key element is $4/3$. This element is made unity in table 11.3.1-4

Table 11.3.1-4

c'_B	c_B	c.s.v.	x_1	$\boldsymbol{x_2}$	x_3	x_4	x_5	b
2	4	x_1	$\frac{3}{4}$	0	$-\frac{3}{4}$	(1)	$-\frac{1}{4}$	$\frac{3}{4}$
-2	6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2
								Key element unity

Table 11.3.1-5

	c_j		2	-2	2	0	0	
c'_B	c_B	c.s.v.	4	6	2	0	0	
			x_1	$\boldsymbol{x_2}$	x_3	x_4	x_5	b
	0	x_4	$\frac{3}{4}$	0	$-\frac{3}{4}$	1	$-\frac{1}{4}$	$\frac{3}{4}$
-2	6	x_2	$\frac{1}{4}$	1	$\frac{7}{4}$	0	$\frac{1}{4}$	$\frac{9}{4}$
		$\bar{c}_j$	$\frac{5}{2}$	0	$-\frac{17}{2}$	0	$-\frac{3}{2}$	$Z = \frac{27}{2}$

		$\bar{c}'_j$	$\frac{5}{2}$	0	$\frac{11}{2}$	0		$\frac{1}{2}$ $Z' = -\frac{9}{2}$

Table 11.3.1-5 will be optimal if $\bar{c}_j(\lambda) \leq 0$ for $j = 1,3,5$. Now $\bar{c}_1(\lambda) = \bar{c}_1 + \lambda\bar{c}'_1 = \frac{5}{2} + \frac{5}{2}\lambda \leq 0 \ \therefore \lambda \leq -1$,

$$\bar{c}_3(\lambda) = \bar{c}_3 + \lambda\bar{c}'_3 = -\frac{17}{2} + \frac{11}{2}\lambda \leq 0 \ \therefore \lambda \leq \frac{17}{11},$$
$$\bar{c}_5(\lambda) = \bar{c}_5 + \lambda\bar{c}'_5 = -\frac{3}{2} + \frac{1}{2}\lambda \leq 0 \ \therefore \lambda \leq 3:$$

For all $\lambda \leq -1$, the optimal solution is given by $x_1 = 0, x_2 = \frac{9}{4}, x_3 = 0, x_4 = \frac{3}{4}, x_5 = 0$ and $Z_{\max} = \frac{27}{2} - \frac{9}{2}\lambda$. Thus tables $11.3.1 - 2, 11.3.1 - 4$ and 11.3.1-6 give families of optimal solutions for $-1 \leq \lambda \leq \frac{1}{2}, \lambda \geq \frac{1}{2}$, and $\lambda \leq -1$ respectively.

11.4 PARAMETRIC RIGHT-HAND-SIDE PROBLEM

The right-hand-side constants in a linear programming problem represent the limits in the resources and the outputs. In some practical problems all the resources are not independent of one another. A shortage of one resource may cause shortage of other resources at varying levels. Same is true for outputs also. For example, consider a firm manufacturing electrical appliance. A shortage in electric power will decrease the demand of all the electric items produced in varying degrees depending upon the electric energy consumed by them. In all such problems, we are to consider simultaneous changes in the right-hand-side constants, which are functions of one parameter and study how the optimal solution is affected by these changes.

Let the linear programming problem before parameterization be

$$\text{maximize} \mathbf{Z} = \mathbf{cX},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b},$$
$$\mathbf{X} = 0,$$

where $\mathbf{b}$ is the known requirement (right-hand-side) vector. Let this requirement vector $\mathbf{b}$ change to $\mathbf{b} + \boldsymbol{\lambda}\mathbf{b}'$ so that parametric right-hand-side problem becomes

$$\text{maximize} Z = \mathbf{cX},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b} + \lambda\mathbf{b}',$$
$$\mathbf{X} \geq 0,$$

Where $\mathbf{b}'$ is the given and predetermined variation vector and λ is an unknown parameter. As λ changes, the right-hand-constants also change. We wish to decrmine the family of optimal solutions as λ changes from $-\infty$ to $+\infty$.

When $\lambda = 0$, the parametric problem reduces to the original L. P. problem; simplex method is used to find its optimal solution.

Let $\mathbf{B}$ and $\mathbf{X_B}$ represent the optimal basis matrix and the optimal basic feasible solution respectively for $\lambda = 0$. Then $\mathbf{X}_{\mathrm{B}} = \mathbf{B}^{-1}\mathbf{b}$. As λ changes from ^{260} to a positive or negative value, the values of the basic variables change and the new values are given by

$$\mathbf{X_B} = \mathbf{B}^{-1}(\mathbf{b} + \lambda\mathbf{b}') = \mathbf{B}^{-1}\mathbf{b} + \lambda\mathbf{B}^{-1}\mathbf{b}'$$
$$= \overline{\mathbf{b}} + \lambda\overline{\mathbf{b}}'$$

A change in λ has no effect on the values of relative profit coefficients $\bar{c}_j$ i.e., $\bar{c}_j$ values remain non-positive (maximization problem). For a given basis matrix $\mathbf{B}$, values of $\overline{\mathbf{b}}$ and $\overline{\mathbf{b}}'$ can be calculated. The solution $\mathbf{X_B} = \overline{\mathbf{b}} + \lambda\overline{\mathbf{b}}'$ is feasible and optimal as long as $\overline{\mathbf{b}} + \lambda\overline{\mathbf{b}}'$ is ≥ 0. In other words, for a given solution we can determine the range for λ within which the solution remains optimal.

11.5 INTEGER PROGRAMMING

Integer programming is a specialized branch of mathematical optimization that focuses on problems where some or all of the decision variables are required to be integers. This is particularly important in situations where fractional solutions are not practical or possible, such as in scheduling, allocation, or logistics where quantities must be whole numbers.

11.6 *PURE AND MIXED INTEGER PROGRAMMING PROBLEMS*

Integer programming problems can be categorized into two main types: pure integer programming and mixed integer programming. Both types involve optimization where some or all of the decision variables are required to be integers. Here’s a closer look at each type:

Pure Integer Programming: In pure integer programming, all the decision variables are constrained to be integers. This type of problem is used when all variables in the optimization model must take on whole number values.

Example 1: Consider a factory that produces two types of products (Product 1 and Product 2). The objective is to maximize the profit given the constraints on resources (e.g., labour, materials).

Maximize: $Z = 4x_1 + 7x_2$
Subject to: $3x_1 + 2x_2 \leq 12$

$5x_1 + 3x_2 \leq 15$

$x_1 \geq 0, x_2 \geq 0$; $x_1, x_2 \in Z$

In this example, x_1 and x_2 represent the quantities of Product 1 and Product 2, respectively, and both must be integers.

Mixed Integer Programming (MIP): In mixed integer programming, only some of the decision variables are required to be integers, while others can be continuous. This type of problem is useful when some decisions are inherently discrete (e.g., number of units produced), while others can vary continuously (e.g., amounts of resources used).

Example 2: Consider a company that wants to determine the optimal production quantities for two products, where one of the products can be produced in fractional quantities (e.g., a liquid), and the other must be in whole units.

Maximize: $Z = 4x_1 + 7x_2$
Subject to: $3x_1 + 2x_2 \leq 12$; $5x_1 + 3x_2 \leq 15$; $x_1 \geq 0$; $x_2 \geq 0$; $x_1 \in Z$

Here, x_1 (the integer variable) might represent the number of whole units of a product, while x_2 (the continuous variable) represents a product that can be produced in any amount.

11.7 *GOMORY's ALL I.P.P. METHOD*

Gomory's All-Integer Programming Problem (All-I.P.P.) method is a technique used to solve integer programming problems. Developed by Ralph Gomory in the 1950s, this method is a cutting-plane algorithm specifically designed to handle integer constraints. The general approach involves solving a series of linear programming relaxations and iteratively adding cuts (constraints) to eliminate non-integer solutions, gradually converging to an integer solution.

Steps in Gomory's All-I.P.P. Method

1. **Solve the Linear Programming Relaxation**: Solve the original integer programming problem without the integer constraints, treating it as a standard linear programming problem.

This step provides an optimal solution to the relaxed problem, which may not be an integer solution.

2. **Identify the Fractional Variables**: Examine the optimal solution from the LP relaxation. Identify any decision variables that have non-integer values.
3. **Generate Gomory Cuts:** Write the equation corresponding to the chosen fractional basic variable from the simplex tableau: $x_i + \sum_j a_{ij} x_j = b_i$

Isolate the fractional parts of the coefficients and the right-hand side: $f_i + \sum_j a_{ij} x_j = b_i$

The Gomory cut is derived as: $\sum_j \left(a_{ij} - \lfloor a_{ij} \rfloor\right) x_j \geq b_i - \lfloor b_i \rfloor$

Here, $\lfloor . \rfloor$ denotes the floor function, which returns the greatest integer less than or equal to the given number.

4. **Add the Cut to the LP**: Add the newly generated Gomory cut to the original set of constraints.

This modifies the feasible region of the LP by cutting off the current non-integer solution.

5. **Re-Solve the LP**: Solve the modified linear programming problem with the added cut.

Repeat the process of generating cuts and re-solving until an integer solution is found.

6. **Check for Optimality**: Once an integer solution is obtained, check if it is optimal.

If it is not optimal, further cuts might be necessary, or another branch-and-bound approach may be combined to refine the solution.

Example 3: Consider a simple integer programming problem

Objective Function: Maximize $Z = 3x_1 + 2x_2$

Constraints:

$2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

$x_1, x_2 \in Z$

Step-by-Step Illustration:

1. **Solve LP Relaxation**:

Relax the integer constraints and solve the LP problem.

Suppose the optimal solution to the LP relaxation is $x_1 = 1.5, x_2 = 1.0$

2. **Identify Fractional Variables**:

The current solution is $x_1 = 1.5$, which is fractional.

3. **Write the Equation for the Fractional Basic Variable**:

Suppose the optimal tableau provides the following equation for x_1:
$x_1 + 0.5x_2 = 1.5$

4. **Isolate the Fractional Parts**:

The fractional part of $x_1 = 1.5$ is 0.5

The equation can be written as: $0.5 + 0.5x_2 = 1.5$

5. **Generate the Gomory Cut**:

The Gomory cut is derived as: $0.5x_2 \geq 1.5 - 1$

Simplifying this, we get: $0.5x_2 \geq 0.5 \Rightarrow x_2 \geq 1$

6. **Add the Gomory Cut to the Original Constraints**:

The new constraint $x_2 \geq 1$ is added to the original set of constraints.

This modifies the feasible region of the LP to exclude the current non-integer solution.

7. **Re-Solve the LP**:

Solve the modified LP problem with the added cut.

Repeat the process until an integer solution is found.

Gomory's constraints are a powerful tool in integer programming, helping to iteratively eliminate non-integer solutions and converge to an optimal integer solution. The process involves generating cuts from the fractional parts of the basic variables in the optimal simplex tableau and adding these cuts to the set of constraints. This method is systematic and can be combined with other techniques like branch-and-bound to solve complex integer programming problems efficiently.

11.8 FRACTIONAL CUT METHOD-ALL INTEGER LPP

The Fractional Cut Method is a technique used specifically for solving all-integer linear programming problems. It involves iteratively adding constraints (cuts) to eliminate non-integer solutions while preserving feasible integer solutions. Here's a detailed breakdown of the method:

Steps in the Fractional Cut Method

1. **Solve the Linear Programming Relaxation**: Solve the integer programming problem (IPP) by first relaxing the integer constraints, treating it as a linear programming problem (LPP). This provides an optimal solution to the relaxed problem.
2. **Identify Fractional Solutions**: Examine the optimal solution of the LP relaxation. Identify which variables have fractional

values. If the solution is entirely integer, then it is already optimal.

3. **Generate Fractional Cuts**: For each fractional solution, generate a cutting plane (cut) that eliminates the current fractional solution while keeping all feasible integer solutions. The cut is derived from the simplex tableau.
4. **Add the Cut to the LP**: Incorporate the new cut into the existing constraints of the LP. This effectively narrows the feasible region to exclude the fractional solution.
5. **Re-Solve the LP**: Solve the modified LP problem with the added cut. Repeat the process of identifying fractional solutions and generating cuts until an integer solution is found.
6. **Check for Integer Solutions**: After adding each cut and solving, check if the resulting solution is an integer. If it is, then this is the optimal integer solution. If not, continue with the process of generating and adding cuts.

Example 4: Consider the following integer programming problem:

Objective Function: Maximize $Z = 2x_1 + 3x_2$

Constraints:

$x_1 + 2x_2 \le 5$

$2x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

$x_1, x_2 \in Z$

Step-by-Step Illustration:

1. **Solve the LP Relaxation**:

Relax the integer constraints and solve the LP problem. Suppose the optimal solution is $x_1 = 2.5, x_2 = 1.5$.

2. **Identify Fractional Variables**:

The optimal solution $x_1 = 2.5, x_2 = 1.5$ are both fractional.

3. **Generate Fractional Cuts**:

From the simplex tableau, we might find a cut for a fractional basic variable. Assume the cut derived is: $x_1 + x_2 \leq 3$

4. Add the Cut to the LP:

Add the constraint $x_1 + x_2 \leq 3$ to the LP constraints.

5. Re-Solve the LP:

Solve the modified LP problem. Suppose the new optimal solution is $x_1 = 2, x_2 = 1$, which is an integer solution.

6. Check for Integer Solutions:

The new solution $x_1 = 2, x_2 = 1$ is an integer and satisfies the constraints.

Example 5: In the given LPP evaluate the optimum integer solution

Maximize $Z = x_1 + 4x_2$

Subject to; $2x_1 + 4x_2 \leq 7$; $5x_1 + 3x_2 \leq 15$; $x_1, x_2 \geq 0$ such that $x_1, x_2 \in Z$

Solution: Initially, we add the slack variable $s_1 \geq 0$ and $s_2 \geq 0$, an initial basic feasible solution is $s_1 = 7$ and $s_2 = 15$. Using the simplex method, an optimal non-integer solution is achieved, and it is presented in the following simplex table:

Initial iteration: non-integer optimum solution

c_B	y_B	x_B	y_1	y_2	y_3	y_4
4	y_2	7/4	1/2	1	1/4	0
0	y_4	39/4	7/2	0	-3/4	1
	z	7	1	0	1	0

Step 2: Because of the optimal solution is not an integer, we focus solely on the fractional parts of $x_{B1} = \frac{7}{4}\left(=1+\frac{3}{4}\right)$ and $x_{B2} = \frac{39}{4}\left(=9+\frac{3}{4}\right)$.

Step 3: Max $\{f_1, f_2\} = Max\left\{\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$ and $x_{B2} = \frac{39}{4}\left(=9+\frac{3}{4}\right)$ i.e., both f_1 and f_2 are equal. Therefore, we arbitrarily select one of these fractional parts. For instance, let's choose f_2.

Step 4: In the second row, since $y_{23} = -3/4$, we write
$y_{23} = -1+1/4$

Step 5: Let G_1 denote the first Gomory slack. We can then express it as follows:

$$G_1 = -f_{20} + f_{21} + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 = -\frac{3}{4} + \frac{1}{2}x_1 + 0.x_2 + \frac{1}{4}x_3 + 0.x_4$$

Step 6: By adding this additional constraint to the optimal simplex table, we obtain:

First Iteration: Drop G_1 and introduce y_1

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	7/4	1/2	1	1/4	0	0
0	y_4	39/4	7/2	0	-3/4	1	0
0	G_1	-3/4	**-1/2**	0	-1/4	0	1
	z	7	1	0	1	0	0

Since, the optimum solution is still non-integral, we introduce the second Gomorian constraints.

Now, $x_{B3} = -3/4$ only is negative, this basic variable leaves the basis. Further, since Max.

$$Max\left\{\frac{(z_j - c_j)}{y_{3j}}, y_{3j} < 0\right\} = \max\left\{\frac{1}{-1/2}, \frac{1}{-1/4}\right\} = -2, y_1$$ enters the basis, i.e., x_1 becomes basic variable in place of G_1.

Second iteration: non-integer optimal solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	1	0	1	0	0	1
0	y_4	9/2	0	0	-5/2	1	7
1	y_1	3/2	1	0	1/2	0	-2
	z	11/2	0	0	1/2	0	2

Since the optimal solution is still non-integral, we introduce the second Gomory constraint. Now,

$$X_{B2} = \frac{9}{2}\left(= 4 + \frac{1}{2}\right) \text{ and } X_{B3} = \frac{3}{2}\left(= 1 + \frac{1}{2}\right)$$

Since, Max $\{f_2, f_3\} = Max\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$ i.e., both f_2 and f_3 are equal. So, let us choose $f_2 = 1/2$ and write $y_{23} = -6 + 1/2$.

$$\therefore G_2 = -f_{20} + f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{21}x_4 = -1/2 + 0.x_1 + 0.x_2 + (1/2)x_3 + 0.x_4$$

Adding these additional constraints in the second iterative table, we have

Third iteration: Drop G_2 and introduce y_3.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	9/2	0	0	-5/2	1	7	0

1	y_1	3/2	1	0	1/2	0	-2	0
0	G_2	-1/2	0	0	$-1/2*$	0	0	1
	z	11/2	0	0	1/2	0	2	0

Final iteration: Optimal Solution in the integers.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	7	0	0	0	1	7	-5
1	y_1	1	1	0	0	0	-2	1
0	y_3	1	0	0	1	0	0	-2
	z	5	0	0	0	0	2	1

The table indicates that the optimal basic feasible solution has been achieved. Therefore, the optimal solution is

$x_1 = 1, x_2 = 1$ and Maximum $z = 5$.

The Fractional Cut Method, also known as Gomory's Cut Method, can be applied to Mixed Integer Linear Programming Problems (MILPP). These problems involve both integer and non-integer (continuous) variables. The method iteratively adds constraints to the linear programming relaxation to eliminate non-integer solutions for the integer-constrained variables.

11.9 BRANCH AND BOUND METHOD

The Branch and Bound method is a widely used algorithm for solving integer programming problems, including both pure integer and mixed-integer problems. It systematically explores all potential solutions to find the optimal one while efficiently pruning suboptimal solutions to reduce computational effort.

Steps in the Branch and Bound Method

1. **Initialization**: Solve the linear programming relaxation of the integer programming problem (i.e., ignore the integer constraints) to obtain an initial solution. This gives an upper bound for maximization problems and a lower bound for minimization problems.
2. **Branching**: Identify a variable that has a fractional value in the current solution. Create two new subproblems (branches) by adding constraints to this variable to take its floor and ceiling values.

For example, if $x_i = 3.5$ in the current solution, create two new problems: one with $x_i \le 3$ and the other with $x_i \ge 4$.

3. **Bounding**: Solve the LP relaxation of each new subproblem to obtain new bounds.

If a subproblem yields an integer solution, compare it with the current best solution and update the best solution if this one is better.

If the subproblem's bound is worse than the current best solution or infeasible, discard (prune) that branch.

4. **Pruning**: Eliminate branches that cannot yield a better solution than the current best solution. This is done by comparing the bounds of the subproblems to the current best known integer solution.

Discard infeasible branches or those that lead to worse solutions than the current best solution.

5. **Repeat**: Continue the branching, bounding, and pruning process until all branches have been either explored or pruned. The best solution found during this process is the optimal integer solution.

Example 8: Consider the following integer programming problem:

Objective Function: Maximize $z = 3x_1 + 2x_2$

Constraints:

$x_1 + x_2 \le 4$

$x_1 - x_2 \le 1$

$x_1, x_2 \geq 0$

$x_1, x_2 \in Z$

Step-by-Step Illustration:

1. **Initialization**:

   Solve the LP relaxation:

   Maximization $z = 3x_1 + 2x_2$

   Subject to, $x_1 + x_2 \leq 4$; $x_1 - x_2 \leq 1$; $x_1, x_2 \geq 0$

   Suppose the optimal solution is $x_1 = 2.5, x_2 = 1.5$ with $z = 10.5$.

2. **Branching**:

   Create two new subproblems by branching on x_1:

   Subproblem 1: $x_1 \leq 2$

   Subproblem 2: $x_1 \geq 3$

3. **Bounding**:

   Solve the LP relaxation for Subproblem 1:

   Maximize, $Z = 3x_1 + 2x_2$

   Subject to, $x_1 + x_2 \leq 4$; $x_1 - x_2 \leq 1$; $x_1 \leq 2$; $x_1, x_2 \geq 0$

   Suppose the optimal solution is $x_1 = 2, x_2 = 2,$ with $z = 10$ (an integer solution).

   Solve the LP relaxation for Subproblem 2:

   Maximize, $Z = 3x_1 + 2x_2$

   Subject to, $x_1 + x_2 \leq 4$; $x_1 - x_2 \leq 1$; $x_1 \geq 3$; $x_1, x_2 \geq 0$

   Suppose this subproblem is infeasible.

4. **Pruning**: Subproblem 2 is pruned because it is infeasible. The solution from Subproblem 1 is an integer solution with $z=10$, so we update our best solution.
5. **Repeat**: Continue the process for any remaining branches (if applicable). In this example, Subproblem 1 has provided an integer solution that is feasible and maximizes z.

The Branch and Bound method is an effective algorithm for solving integer programming problems by exploring and eliminating suboptimal branches systematically. It is widely used in operations research, scheduling, logistics, and other fields where optimization of discrete decisions is crucial.

Example 9: Solve the following LPP using branch and bound method.

Maximize, $Z = 7x_1 + 9x_2$

Subject to the constraints, $-x_1 + 3x_2 \le 6; 7x_1 + x_2 \le 35; x_2 \le 7$; $x_1, x_2 \ge 0$ and $x_1, x_2 \in Z$

Solution: Step 1: Disregarding the integer constraints, the optimal solution to the given linear programming problem can be readily obtained as:

$x_1 = 9/2, x_2 = 7/2$ and Maximum $Z = 63$

Step 2: Since the solution is not an integer, let's select x_1 i.e., $x_1^* = 9/2$ as the variable with the largest fractional value.

Step 3: Taking the value of z as the initial upper bound, i.e., $z = 63$; the lower bound is found by rounding off the values of x_1 and x_2 to the nearest integers, i.e., $x_1 = 4$ and $x_2 = 3$. Thus, the lower bound is $z_1 = 55$.

Step 4: Since $[x^*{}_1] = [9/2] = 4;$ where, $[.]$ denote the greatest integer.

Sub-problem 1: Maximize $z = 7x_1 + 9x_2$

Subject to constraints, $-x_1 + 3x_2 \le 6;\ 7x_1 + x_2 \le 35;\ 0 \le x_1 \le 4$ *and* $0 \le x_2 \le 7;$

Sub-problem 2: Maximize $z = 7x_1 + 9x_2$ subject to constraints,

$-x_1 + 3x_2 \le 6;\ 7x_1 + x_2 \le 35;\ 0 \le x_1 \ge 5\ \textit{and}\ 0 \le x_2 \le 7;$

Step 5 Optimal solutions to the sub-problem are determined as follows

Sub-problem 1: Maximize $x_1 = 4, x_2 = 10/3$ and Maximum $z = 58$

Sub-problem 2: Maximize $x_1 = 5, x_2 = 0$ and Maximum $z = 35$

Since the solution to sub-problem 1 is not in integers, we further divide it into the following two sub-problems:
Sub-problem 3: Maximize $z = 7x_1 + 9x_2$
Subject to, $-x_1 + 3x_2 \le 6;\ 7x_1 + x_2 \le 35;\ 0 \le x_1 \le 4\ \textit{and}\ 0 \le x_2 \le 3$
Sub-problem 4: Maximize $z = 7x_1 + 9x_2$
Subject to, $-x_1 + 3x_2 \le 6;\ 7x_1 + x_2 \le 35;\ 0 \le x_1 \le 4\ \textit{and}\ 0 \le x_2 \ge 4$
Step 6: The optimal solutions to the sub-problem 3 and 4 are:
Sub-problem 3: $x_1 = 4, x_2 = 3$ and maximum $z = 55$.
Sub-problem 3: No feasible solution.
Step 7: Among the recorded integer-valued solutions, the highest value of z is 55; therefore, the required optimal solution is:

$x_1 = 4, x_2 = 3$ and maximum $z = 55$.

The entire branch and bound procedure for the given problem is shown below:

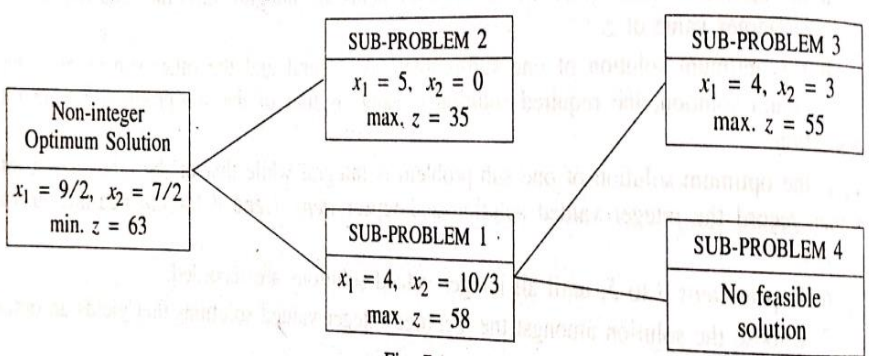

Figure 1

MULTIPLE CHOICE QUESTIONS

1: In integer programming, which type of decision variables are used?

A) Only continuous variables

B) Only binary variables

C) Only integer variables

D) Both integer and continuous variables

2: Which method is commonly used to solve integer programming problems?

A) Gradient Descent

B) Branch and Bound

C) Newton's Method

D) Least Squares

3: What type of optimization problem is an integer programming problem classified as?

A) Linear

B) Non-linear

C) NP-hard

D) Polynomial-time

4: In a mixed integer programming problem, some of the decision variables are:

A) Real numbers

B) Integer numbers

C) Binary numbers

D) Both A and B

5: Which of the following is NOT a method used for integer programming?

A) Simplex method

B) Cutting Plane method

C) Branch and Bound method

D) Genetic Algorithm

6: What is a Gomory cut?

A) A technique for dividing problems into subproblems

B) A type of cutting plane used to eliminate fractional solutions

C) A method for rounding solutions to the nearest integer

D) A type of constraint that ensures non-negativity

11.10 SUMMARY

Integer programming is a powerful tool for solving discrete optimization problems, but it requires sophisticated methods to handle its computational challenges. Its applications are vast and impactful in various fields such as operations research, logistics, finance, and more. Integer programming is used in various fields, including:

Operations Research: Scheduling, resource allocation, production planning.

Logistics: Vehicle routing, supply chain optimization.

Finance: Portfolio optimization, capital budgeting.

Telecommunications: Network design, bandwidth allocation.

11.11 GLOSSARY

Integer Programming

Gomory's method

Fractional cut method

Branch and bound method

11.12 REFERENCES

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11.13 SUGGESTED READING

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11.14 TERMINAL QUESTIONS

1. Solve the following IPP

Maximize, $z = 3y$

Subject to the constraints,

$3x + 2y \le 7;\ -x + y \le 2;\ x, y \ge 0$; and $x, y \in Z$

2: Find the optimal solution of the following IPP

Maximize, $z = x - y$

Subject to the constraints,

$x + 2y \le 4;\ 6x + 2y \le 9;\ x, y \ge 0$; and $x, y \in Z$

3: Find the optimal solution of the following IPP

Maximize, $z = 2x + 3y$

Subject to the constraints,

$-3x + 7y \le 14;\ 7x - 3y \le 14;\ x, y \ge 0$; and $x, y \in Z$

4.. Solve the following integer linear programming problems using the branch and bound method.

Maximize, $z = 2x + 3y$

Subject to the constraints,

$5x + 7y \le 35; 4x + 9y \le 36;\ x, y \ge 0$ and $x, y \in Z$

11.15 ANSWERS

TQ 1: $x_1 = 0, x_2 = 2,$; Maximum $z = 6$ TQ**2:** $x = 1, y = 0$; Maximum $z = 1$ TQ**3:** $x = 3, y = 3$; Maximum $z = 15$ TQ4. $x_1 = 4, x_2 = 2,$ and Maximum $z = 14$

MCQ 1: D MCQ **2:** B MCQ **3:** C

MCQ 4: D MCQ **5:** A MCQ **6:** B

BLOCK V: APPLICATION

UNIT 12: -Assignment Problem

CONTENTS:

12.1 *INTRODUCTION:-*

The Assignment Problem is a fundamental topic in Operations Research that deals with the optimal allocation of limited resources to various tasks. The objective is to assign a set of agents (such as workers, machines, or employees) to an equal number of tasks (such as jobs, projects, or machines) in such a way that the total cost, time, or distance is minimized (or the total profit is maximized).

Typically, the Assignment Problem is represented using a cost matrix, where each cell indicates the cost of assigning a particular agent to a specific task. The challenge is to make one-to-one assignments—each agent gets exactly one task, and each task is assigned to exactly one agent—while minimizing the total cost or maximizing efficiency.

This problem has wide applications in fields such as production planning, scheduling, transportation, and human resource management. It is often solved using methods like the Hungarian Algorithm, which provides an efficient solution for the optimal assignment.

12.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- Define Assignment Problem.
- Understand the Hungarian Method for solution of the Assignment Problem

12.3 *DEFINITION OF ASSIGNMENT PROBLEM: -*

The cost of assigning the $i-th$ facility (person) to the $j-th$ work is represented by c_{ij}, and the assignment problem can be expressed as a $n \times n$ matrix $[c_{ij}]$, also known as the cost or effectiveness matrix.

$$\begin{array}{cc} & \text{Jobs} \\ & \begin{array}{ccccccc} 1 & 2 & 3 & \dots & j & \dots & n \end{array} \\ \text{Persons} \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \boxed{\begin{array}{ccccccc} c_{11} & c_{12} & c_{13} & \dots & c_{1j} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2j} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{i1} & c_{i2} & c_{i3} & \dots & c_{ij} & \dots & c_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nj} & \dots & c_{nn} \end{array}} \end{array}$$

Effectiveness matrix

n persons can be assigned to n jobs in $n!$ possible ways. One method may be to find all possible $n!$ assignments and evaluate total costs in all cases. Then the assignment with minimum cost (as required) will give the optimal assignment. But this method is extremely laborious. For example if $n = 8$ then the number of such possible assignments is $8! = 40320$. The evaluation of costs for all these allocations will take a large time. Thus, there is a need to develop an easy computational technique for the solution of assignment problems.

12.4 *MATHEMATICAL REPRESENTATIONOF ASSIGNMENT PROBLEM: -*

Mathematically, the assignment problem can be expressed as follows:

$$x_{ij} = \begin{cases} 0, \text{if the } i\text{th person is not assigned to } j\text{th job.} \\ 1, \text{if the } i\text{th person assigned to } j\text{th job.} \end{cases}$$

Then the problem is given by

$$minimize\ Z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} \left\{ = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \right\}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1; i = 1,2, \dots, n \ (one\ job\ is\ assigened\ to\ the\ ith\ person)$$

$$\sum_{i=1}^{n} x_{ij} = 1; j = 1,2, \dots, n \ (one\ person\ is\ assigened\ to\ the\ jth\ job)$$

$$and\ x_{ij} = 0\ or\ 1\ \left(or\ x_{ij} = {x_{ij}}^2\right)$$

Theorem1. (Reduction Theorem): If, in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix $[c_{ij}]$, then an assignment which minimize the total cost for one matrix, also minimizes the total cost for the other matrix.

Or

Mathematically the theorem may state as follows:

If $x_{ij} = X_{ij}$,

$$minimize\ Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}\ \forall x_{ij}\ s.t.$$

$$\sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}\ \&\ x_{ij} \geq 0$$

Then $x_{ij} = X_{ij}$ also

$$minimize\ Z' = \sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij}$$

Where $c'_{ij} = c_{ij} \pm a_i \pm b_j$, a_i, b_j are constants, $i = 1,2, \dots, n; j = 1,2, \dots, n$

Proof: we have

$$Z' = \sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} \pm a_i \pm b_j)x_{ij}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} \pm \sum_{j=1}^{n}\sum_{i=1}^{n} a_i x_{ij} \pm \sum_{j=1}^{n}\sum_{i=1}^{n} b_j x_{ij}$$

$$= Z \pm \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} x_{ij}\right) \pm \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} x_{ij}\right)$$

$$= Z \pm \sum_{i=1}^{n} a_i \,.\, 1 \pm \sum_{j=1}^{n} b_j \,.\, 1$$

$$= Z \pm \sum_{i=1}^{n} a_i \pm \sum_{j=1}^{n} b_j$$

Since $\sum_{i=1}^{n} a_i \pm \sum_{j=1}^{n} b_j$ are independent of x_{ij}.

It follows that Z' is minimized when Z is minimized.

Hence $x_{ij} = X_{ij}$ which minimizes Z also minimizes Z'.

Theorem2. If all $c_{ij} \geq 0$ and there exists a solution

$$x_{ij} = X_{ij} \; s.t. \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} = 0$$

Then this solution is an optimal solution (i.e. this solution minimizes Z).

Proof: Since all $c_{ij} \geq 0$

$\therefore \quad Z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$ can not be negative.

Thus its minimum value is zero, when $x_{ij} = X_{ij}$.

Hence the solution $x_{ij} = X_{ij}$ for which $\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} = 0$

is an optimal solution.

12.5 *THE HUNGARIAN METHOD FOR SOLUTION OF THE ASSIGNMENT PROBLEM: -*

The "assignment algorithm" is a powerful approach for resolving assignment problems that is derived from the two previously discussed theorems. The following is the process for the solution:

Step 1: Subtract from each element of the corresponding row the minimal element of each row in the cost matrix $[c_{ij}]$.

Step2: Subtract from each element of the corresponding column the minimal element of each column in the reduced matrix that was produced in step 1.

Step3: (a) Examine each row of the matrix obtained in step 2 one after the other until a row containing exactly one zero element is discovered. Since an assignment will be made there, mark (□) at this zero. To indicate that they cannot be used to create new assignments, mark (×) at each additional zero in the column that we mark. Continue doing this until the final row has been examined.

(b) Once every row has been thoroughly examined, continue by looking at the columns. Start by looking at column 1 and continue until you find a column with exactly one unmarked zero. At this zero, mark (□), and at every other zero in the marked(□) row, mark ×. Continue doing this until the final column has been examined.

(c) Repeat steps (a) and (b) one after the other until we arrive at one of the two situations.

(i) Every zero has been crossed or marked(□).

(ii) There are at least two unmarked zeros left in each row and column.

We have a maximal assignment (as much as we can) in case (i), and we still have some zeros to deal in case (ii). To avoid using a very complex algorithm, we use the trial-and-error method.

There are now two options:

(i) The full optimal assignment is obtained if it has an assignment in each row and each column (i.e., the total number of marked (□) zeros is exactly n). (Refer to example 1)

(ii) The cost (effectiveness) matrix must be modified by adding or subtracting to add more zeros if it does not contain assignment in every row and column (that is, if the total number of marked (□) zeros is fewer than n). To do this, move on to step 4.

Step 4: When the matrix obtained in step 3 does not contain assignment in every row and every column then we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. For this the following procedure is adopted.

(i) Mark(√) all rows for which assignment have not been made.

(ii) Mark (√) column which have zeros is marked rows.

(iii) Mark (√) rows (not already marked) which have assignment in marked columns.

(iv) Repeat step (ii) and (iii) until the chain of marking ends.
(v) Draw minimum number of lines through unmarked rows and through marked columns to cover all the zeros.
This procedure will yield the minimum number of lines (equal to the number of assignments in the maximal assignment obtained in step 3) that will pass through all zeros.

Step 5: Select the smallest of the elements that do not have a line through them, subtract it from all these elements that do not have a line through them, add it to every element that lies at the intersection of two lines and leave the remaining elements of the matrix unchanged.

Step 6: At the end of step 5, number of zeros are increased (never decreased) in the matrix than that in step 3.

Now re-apply the step 3 to the modified matrix obtained in step 5, to obtain the desired solution.

Example1: Solve the following minimal assignment problem:

Man→ / Job↓	1	2	3	4
I	12	30	21	15
II	18	33	9	31
III	44	25	24	21
IV	23	30	28	14

Solution: For the clear understanding, this example is solved step by step systematically.
Step1: Substracting the smallest element of each row from every element of the corresponding roiw, we get the following matrix:

	1	2	3	4
I	0	18	9	3
II	9	24	0	22
III	23	4	3	0
IV	9	16	14	0

Step2: subtracting smallest element of each column from every element of the corresponding column, we get the followi9ng matrix:

	1	2	3	4
I	0	14	9	3
II	9	20	0	22
III	23	0	3	0
IV	9	12	14	0

Step3: now we test whether it is possible to make an assignment using the zeros by the method described in step 3 in $ 12.5

Starting with row I, we mark □ in the row containing only one zero and cross(×) the zeros in the corresponding column in which □ lies. Thus, we get the following table.

	1	2	3	4
I	[0]	14	9	3
II	9	20	[0]	22
III	23	0	3	~~0~~
IV	9	12	14	[0]

Again starting with column 1, we mark □ in the column containing only one unmarked zero in the above table and cross out the zeros in the corresponding row in which this assignment is marked. Thus, we get the following table.

	1	2	3	4
I	[0]	14	9	3
II	9	20	[0]	22
III	23	[0]	3	~~0~~
IV	9	12	14	[0]

Since in the last table, every row and every column have one assignment, so we have the complete optimal zero assignment.

Job:	I	II	III	IV
Man:	1	3	2	4

$$i.e. I \rightarrow 1, II \rightarrow 3, III \rightarrow 2, IV \rightarrow 4$$

Which is the optimal assignment.

Example2. A department head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the task be allocated, one to a man, so as to minimize the total man hour?

subordinates

		I	II	III	IV
	A	8	26	17	11
Tasks	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution: Step 1: Subtracting the smallest element in each row from every element of the corresponding row, we get the following matrix.

	I	II	III	IV
A	0	18	9	13
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

Step 2: Subtracting the smallest element in each column of the above matrix from every element of the corresponding column, we get the following matrix.

	I	II	III	IV
A	0	14	9	13
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

The above matrix is the same as obtained in step 3 in example 1, therefore for minimum man hours the allotment should be as follows:

Tasks	A	B	C	D
Subordinates	I	III	II	IV
Man hours	8	4	19	10

$$i.e. A \rightarrow I, B \rightarrow III, C \rightarrow II, D \rightarrow IV$$

The total man hours are $8 + 4 + 19 + 10 = 41$.

12.6 *VARIATIONS OF THE ASSIGNMENT PROBLEM: -*

1. Non-square matrix (Unbalanced assignment problem): Such a problem is found to exist when the number of facilities is not equal to the number of jobs. Since the Hungarian method of solution requires a square matrix, fictitious facilities or jobs may be added and zero costs be assigned to the corresponding cells of the matrix. These cells are then treated the same way as the real cost cells during the solution procedure.

2. Maximization problem: Sometimes the assignment problem may deal with maximization of the objective function. The maximization problem has to be changed to minimization before the Hungarian method may be applied. This transformation may be done in either of the following two ways:

(a) By subtracting all the elements from the largest element of the matrix,

(b) By multiplying the matrix elements by 1.

The Hungarian method can then be applied to this equivalent minimization problem to obtain the optimal solution.

3. Restrictions on assignments: Sometimes technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problems can be solved by assigning a very heavy cost (infinite cost) i.e, ∞ $or\ M$ to the corresponding cell. Such a job will then be automatically excluded from further consideration (making assignments).

4. Alternate optimal solutions: Sometimes, it is possible to have two or more ways to strike off all zero elements in the reduced matrix for a given problem. In such cases, there will be alternate optimal solutions with the same cost. Alternate optimal solutions offer a great flexibility to the management since it can select the one which is most suitable to its requirement.

Example3. (Unbalanced Assignment Problem): A department head has four tasks to be performed and three subordinates. The subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, are given below in the matrix. How should he allocate the tasks, one to each man, so as to minimize the total man hours?

subordinates

1 2 3

Tasks	I	9	26	15
	II	13	27	6
	III	35	20	15
	IV	18	30	20

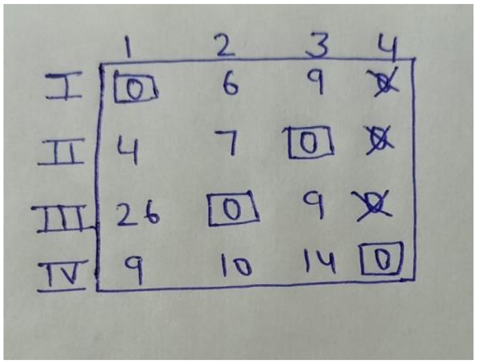
Solution: Since the matrix is not square, it an unbalanced assignment problem we introduce one fictitious subordinate (4th column with zero costs) to get a square matrix. Thus the resulting matrix is shown in the following table. Now the problem can be solved by usual method.

	1	2	3	4
I	9	26	15	0
II	13	27	6	0
III	35	20	15	0
IV	18	30	20	0

Step1: Subtracting the minimum element of each row from every element of the corresponding row and then subtracting the minimum element of each column from every element of the corresponding column, the matrix reduces to

	1	2	3	4
I	0	6	9	0
II	4	7	0	0
III	26	0	9	0
IV	9	10	14	0

Step2: Giving zero assignments in the usual manner, we observe that, each row and each column have zero assignments.



Hence the optimal assignment is as follows.

Tasks $\rightarrow$ subordinates, $I \rightarrow 1, II \rightarrow 3. III \rightarrow 2.$

Task IV remains unassigned.

From the original matrix, the total time (man hours) $= 9 + 6 + 20 = 35$ hours.

Example4. (Maximization problem): Alpha Corporation has four plants each of which can manufacture any of the four products. Production costs differ from plant to plant as do sales revenue. From the following data, obtain which product each plant should produce to maximize profit?

Sales revenue (₹1000)

Plant ↓	Product 1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

Production cost (₹1000)

Plant ↓	Product 1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution: Since, $\text{Profit} = \text{Sales revenue} - \text{Product cost}$, so the profit matrix is as follows.

	1	2	3	4
A	1	8	4	1
B	5	7	6	5
C	3	5	4	2
D	3	1	6	3

This is a maximization problem. We shall solve this problem by converting it to minimization problem by both methods discussed in article 12.6.

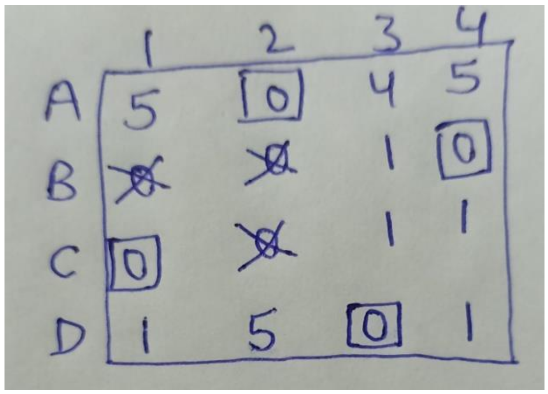
1. **By Method 1:** Subtracting each element of the above matrix from the greatest element 8 of the matrix, the equivalent loss matrix is

	1	2	3	4
A	7	0	4	7
B	3	1	2	3
C	5	3	4	6
D	5	7	2	5

Step 1 and 2: Subtracting the minimum element of cach row from all the elements of the corresponding row and then subtracting minimum element of each column from all the elements of the corresponding column, we get the following matrix.

	1	2	3	4
A	5	0	4	5
B	0	0	1	0
C	0	0	1	1
D	1	5	0	1

Step 3: Giving zero assignments in the usual manner, we get the following matrix.



In the above table there is an assignment in each row and each column. Hence the optimal assignment for maximum profit is

$$A \to 2, B \to 4, C \to 1, D \to 3$$

$$and\ Max. Profit = ₹(8 + 5 + 3 + 6) \times 1000 = ₹22000$$

2. **By Method 2:** Placing negative sign before each element of the profit matrix, the equivalent loss matrix is

	1	2	3	4
A	−1	− 8	− 4	− 1
B	−5	− 7	− 6	− 5
C	−3	− 5	− 4	− 2
D	−3	− 1	− 6	− 3

Now subtracting the minimum element of each row from every elements of the corresponding row and then subtracting the minimum element of each column from every element of the corresponding column, we get the following matrix.

1	2	3	4

A	5	0	4	5
B	0	0	1	0
C	0	0	1	1
D	1	5	0	1

Which the same matrix is as obtained in step 1 and 2 in methods 1. Hence giving zero assignments we get the same optimal solution as in method 1.

Example5: (restrictions on assignment): Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed to cach engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

		Projects			
		A	B	C	D
	1	12	10	17	11
Engineers	2	14	not suitable	4	26
	3	6	10	16	4
	4	8	10	9	7

Solution: To avoid the assignment $2 \rightarrow B$. we take its time to be very large (say). Then the cost matrix of the resulting assignment problem is shown in the following

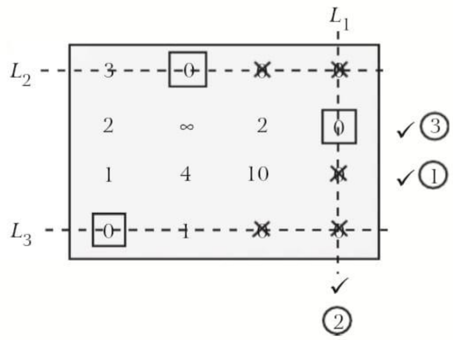
	A	B	C	D
1	12	10	17	11
2	14	∞	4	26
3	6	10	16	4
4	8	10	9	7

Now we apply the assignment technique in the usual manner.

Step1: Subtracting the minimum element of each row from every element of the corresponding row and then subtracting minimum element of each column from every clement of the corresponding column, the reduced matrix is

	A	B	C	D
1	3	0	0	0
2	2	∞	2	0
3	1	4	10	0
4	0	1	0	0

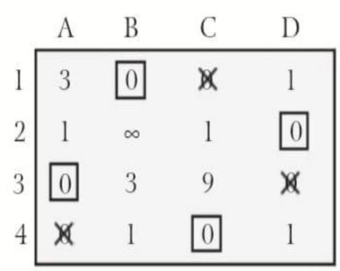
Step 2: Giving zero assignments in the usual manner, we observe that row 3 and column 3 have no zero assignments. So we draw minimum number of lines to cover all zeros at least once. Number of such zeros is 3.



Step 3: In the above table, the smallest of the uncovered elements is 1. Subtracting this clement 1 from all uncovered elements, adding to each element that lies at the intersection of two lines and leaving remaining elements unchanged we get the following matrix.

	A	B	C	D
1	3	0	0	1
2	1	∞	1	0
3	0	3	9	0
4	0	1	0	1

Step 4: Giving zero assignments in the usual manner, we observe that each row and each column have a zero assignment.



Hence the optimal assignment is

$$\text{Engineer} \rightarrow \text{ Project: } 1 \rightarrow \text{ B}, 2 \rightarrow \text{D}, 3 \rightarrow \text{A}, 4 \rightarrow \text{C}$$

From the given matrix total minimum time$= 10 + 11 + 6 + 9 = 36.$

SELF CHECK QUESTIONS

1. what is an Assignment Problem?
2. How is the Assignment Problem related to the Transportation Problem?
3. State the main objective of the Assignment Problem.
4. What are the assumptions made in an Assignment Problem?
5. Define the term “feasible solution” in the context of the Assignment Problem.

6. Explain the difference between a balanced and an unbalanced Assignment Problem.
7. What is a cost matrix?
8. What is meant by an optimal assignment?

12.7 *SUMMARY*: -

In this unit, we have studied that the Assignment Problem is a special case of the Transportation Problem in which the objective is to assign a number of resources or tasks to an equal number of activities or agents in such a way that the total cost or time is minimized (or profit is maximized). In this problem, each resource can be assigned to only one task, and each task must be assigned to only one resource. It is commonly applied in situations such as assigning workers to jobs, machines to tasks, or salesmen to territories. The cost or effectiveness of each possible assignment is represented in a matrix form, and the goal is to find the optimal one-to-one assignment that results in the minimum total cost. The Hungarian Method is the most widely used algorithm to solve the Assignment Problem efficiently. This problem plays an important role in operations research and decision-making processes to ensure the optimal utilization of available resources.

12.8 *GLOSSARY*: -

- **Assignment Problem:** A special type of Transportation Problem that deals with assigning a number of resources (like workers or machines) to an equal number of tasks or jobs in a way that minimizes total cost or time (or maximizes profit).
- **Agent (Resource):** The person, machine, or resource that needs to be assigned to a particular task.
- **Task (Job or Activity):** The specific work or operation that must be completed by an agent or resource.
- **Cost Matrix:** A square matrix that shows the cost, time, or effectiveness of assigning each agent to each task.
- **Feasible Assignment:** An arrangement in which each agent is assigned to exactly one task, and each task is assigned to exactly one agent.
- **Optimal Assignment:** The assignment that results in the minimum total cost or maximum total profit.

- **Hungarian Method:** A systematic algorithm used to find the optimal solution of the Assignment Problem efficiently.
- **Balanced Assignment Problem:** An Assignment Problem in which the number of agents is equal to the number of tasks.
- **Unbalanced Assignment Problem:** A problem in which the number of agents and tasks are not equal. It can be made balanced by adding dummy rows or columns with zero cost.
- **Dummy Row/Column:** An artificial row or column added to balance an unbalanced assignment problem, usually containing zero costs.
- **Minimization Problem:** A type of assignment problem in which the goal is to minimize the total cost or time of performing all tasks.
- **Maximization Problem:** A type of assignment problem in which the goal is to maximize total profit or efficiency.
- **Opportunity Cost:** The difference between the cost of a selected assignment and the minimum cost in the same row or column, used during optimization steps in the Hungarian Method.
- **Operations Research:** A branch of applied mathematics that uses analytical methods to make better decisions, under which the Assignment Problem is studied.
- **Optimal Solution:** The best possible assignment that satisfies all constraints and achieves the objective of minimum cost or maximum profit.

12.9 *REFERENCES*: -

- Wagner, H. M. (2010). Principles of Operations Research: With Applications to Managerial Decisions. Prentice Hall of India.
- Hamdy, A. Taha. (2003). Linear Programming and Network Flows. Pearson Education.
- Ravindran, A. (2008). Operations Research and Management Science Handbook. CRC Press.
- Winston, W. L. (2004). Operations Research: Applications and Algorithms (4th Edition). Thomson Brooks/Cole.

12.10 SUGGESTED *READING*: -

- Dr. R.K.Gupta (2nd Eddition, 2012), Krishna Publication, Operation Research
- Er. Prem Kumar Gupta and Dr. D.S. Hira (7th Edition,2014), S.Chand & Company PVT. LTD., Operations Research

12.11 *TERMINAL QUESTIONS: -*

(TQ-1) Solve minimal assignment problem whose effectiveness matrix is

	I	II	III	IV
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

(TQ-2) Suggest optimum solution to the following assignment problem and also the maximum sales:

Salesman	Market (sales in Lakhs₹)			
	I	II	III	IV
A	44	80	52	60
B	60	56	40	72
C	36	60	48	48
D	52	76	36	40

(TQ-3) A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the natrue of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below.

		District			
		1	2	3	4
	A	16	10	14	11
Salesman	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

(TQ-4) Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed to each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

Projects

		A	B	C	D
Engineers	1	16	14	14	12
	2	16	--	17	13
	3	11	15	21	9
	4	8		10	9
			7		

(TQ-5) Find the optimal assignment for the problem with the following matrix:

	I	*II*	*III*	*IV*
1	5	3	1	8
2	7	9	2	6
3	6	4	5	7
4	3	7	7	6

(TQ-6) Find the optimal assignment for the problem having the following cost matrix:

	I	*II*	*III*	*IV*
1	30	25	26	28
2	26	32	24	20
3	20	22	18	27
4	23	20	21	19

(TQ-7) Solve the following assignment problem:

	P	*Q*	*R*	*S*
A	85	50	30	40
B	90	40	70	45
C	70	60	60	50
D	75	45	35	55

(TQ-8) Solve the following minimal assignment problem:

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

(TQ-9) A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

		machine			
		W	X	Y	Z
jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost?

(TQ-10) Solve the following minimal assignment problem:

	1	2	3	4	5
A	9	11	15	10	11
B	12	9	$-$	10	9
C	$-$	11	14	11	7
D	14	8	12	7	8

12.12 ANSWERS: -

(TQ-1) $A \to II, B \to III, C \to IV, D \to I$

Minimum cost= ₹20

(TQ-2) $A \to II, B \to IV, C \to III, D \to I$

Maximum sales = ₹252

(TQ-3) $salesman \to district, A \to 2, B \to 4, C \to 1, D \to 3$

Maximum profit = ₹61

(TQ-4) Engineer → Project: 1 → B, 2 → D, 3 → A, 4 → C

Minimum total time= 47 *hours*

(TQ-5) $Min. cost =$ ₹16

(TQ-6) $Min. cost =$ ₹86

(TQ-7) $Min. cost =$ ₹9

(TQ-8) $Min. cost =$ ₹38

(TQ-9) $A \to W, B \to X, C \to Y$, No job is assigned to machine Z.

(TQ-10) $A \to I, B \to II, C \to V, D \to IV$, job III remains undone.

$Min. cost = 32\ units$.

UNIT 13: -Transportation Problem

CONTENTS:

13.1 *INTRODUCTION: -*

The Transportation Problem is a classic optimization problem in Operations Research (OR) that deals with the efficient distribution of goods from multiple sources (such as factories or warehouses) to multiple destinations (such as markets or retail outlets). The main objective is to minimize the total transportation cost while satisfying the supply and demand constraints at each source and destination.

In practical terms, it helps organizations decide how much of a product should be shipped from each origin to each destination so that the overall shipping cost is minimized, and all supply and demand requirements are met

13.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- Define Transportation Problem.
- Understand Mathematical Formulation of Transportation Problem.
- Explain Methods of Finding Initial Feasible Solution.
- Explain The Stepping Stone Method and MODI Method or u-v Method.
- Define Unbalanced Transportation Problem.

13.3 *TRANSPORTATION PROBLEM/ MODEL: -*

When a single homogeneous commodity needs to be delivered in different quantities to many locations known as "destinations," there are multiple centers, often referred to as "origins" or "sources," in the transportation problem or model. In this case, each origin (or source) has a capacity (i.e. availability), and each destination has a requirement, so that the total of the capacities (i.e. available) at all origins (sources) equals the total of the requirements at all destinations. There are known and distinct transportation costs from each origin to each destination. Transporting the entire quantity available from all sources to all destinations in order to satisfy their needs while keeping the overall cost of transportation as low as possible is the goal.

Therefore, the transportation problem (or model) focuses on reducing transportation costs by satisfying the needs of every destination using the complete amount of resources available at all sources.

13.4 *NOTATIONS: -*

The following notations will be applied to the transportation problem.

$m =$ the numberof sources

$n =$ the number of destinations

$a_i =$ the availability (or supply)at the $i-th$ source

$b_j =$ the requirement (or demand) at the $j-th$ destination

$c_{ij} =$ the cost of transportation of one unit from ith source to j − th destination

$x_{ij} =$ the number of units to the transported from the $i-th$ source to $j-th$ destination.

13.5 *TABULAR REPRESENTATION OF TRANSPORTATION PROBLEM*: -

The above-described transportation problem can be shown in the following tabular form, called a transportation table:

$Destinations \rightarrow$ $sources \downarrow$	W_1	W_2	…	W_j	…	W_n	Capacity of the sources
F_1	c_{11}	c_{12}	…	c_{1j}	…	c_{1n}	a_1
F_2	c_{21}	c_{22}		c_{2j}		c_{2n}	a_2
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_i	c_{i1}	c_{i2}		c_{ij}		c_{in}	a_i
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_m	c_{m1}	c_{m2}	…	c_{mj}	…	c_{mn}	a_m
$Requirements \rightarrow$	b_1	b_2	…	b_j	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$Destinations \rightarrow$ $sources \downarrow$	W_1	W_2	…	W_j	…	W_n	Capacity of the sources
F_1	x_{11}	x_{12}	…	x_{1j}	…	x_{1n}	a_1
F_2	x_{21}	x_{22}		x_{2j}		x_{2n}	a_2
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_i	x_{i1}	x_{i2}		x_{ij}		x_{in}	a_i
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_m	x_{m1}	x_{m2}	…	x_{mj}	…	x_{mn}	a_m
$Requirements \rightarrow$	b_1	b_2	…	b_j	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The above two tables can be combined together by writing the costs c_{ij} within the bracket (), as follows:

$Destinations \rightarrow$ $sources \downarrow$	W_1	W_2	…	W_j	…	W_n	Capacity of the sources
F_1	$x_{11}(c_{11})$	$x_{12}(c_{12})$	…	$x_{1j}(c_{1j})$	…	$x_{1n}(c_{1n})$	a_1
F_2	$x_{21}(c_{21})$	$x_{22}(c_{22})$		$x_{2j}(c_{2j})$		$x_{2n}(c_{2n})$	a_2
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_i	$x_{i1}(c_{i1})$	$x_{i2}(c_{i2})$		$x_{ij}(c_{ij})$		$x_{in}(c_{in})$	a_i
⋮	…	…	…	…	…	…	⋮
⋮	…	…	…	…	…	…	⋮
F_m	$x_{m1}(c_{m1})$	$x_{m2}(c_{m2})$	…	$x_{mj}(c_{mj})$	…	$x_{mn}(c_{mn})$	a_m

$Requirements \rightarrow$	b_1 b_2 ... b_j ... b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

13.6 *MATHEMATICAL FORMULATION OF TRANSPORTATION PROBLEM*: -

Mathematically, a transportation problem can be stated as a linear programming problem as follows:
Find $x_{ij}(i = 1,2,\dots,m, j = 1,2,\dots,n)$ to minimize the total transportation cost, i.e.

$$Min. Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad \dots(1)$$

Subject to constraints

$$\left.\begin{aligned} x_{11} + x_{12} + \cdots + x_{1n} &= a_1 \\ x_{21} + x_{22} + \cdots + x_{2n} &= a_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{m1} + x_{m2} + \cdots + x_{mn} &= a_m \end{aligned}\right\} i.e. \sum_{i=1}^{n} x_{ij} = a_i \; for \; i = 1,2,\dots,m \qquad \dots(2)$$

Demand constraints

$$\left.\begin{aligned} x_{11} + x_{21} + \cdots + x_{m1} &= b_1 \\ x_{12} + x_{22} + \cdots + x_{m2} &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{1n} + x_{2n} + \cdots + x_{mn} &= b_n \end{aligned}\right\} i.e. \sum_{j=1}^{m} x_{ij} = b_j \; for \; j = 1,2,\dots,n \qquad \dots(3)$$

$$a_1 + a_2 + \cdots + a_m = b_1 + b_2 + \cdots + b_n \; i.e. \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \qquad \dots(4)$$

And $x_{ij} \geq 0 \; \forall i = 1,2,\dots,m, j = 1,2,\dots,n$
Thus the transportation problem is a L.P.P. of special type, where we are required to find the values of m, n variable that minimizes the objective function Z given by (1), satisfying $(m + n)$ constraints given in (2) and (3), constraint (4) and the non-negative restriction of variables.

13.7 *TERMINOLOGY*: -

We will define a few terminologies used in transportation-related problems here.

1. **A Feasible Solution (F.S.)**

   A set of non-negative individual allocations ($x_{ij} \geq 0$) that satisfy the row and column sum restrictions, or the constraints of the problems or models, represents a feasible solution to a transportation problem. In order for a feasible solution to be found, it must

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

2. **Basic Feasible Solution(B.F.S.)**

   If the total number of positive allocations is less than or equal to $m + n - 1$ (one less than the sum of the number of rows and columns), then a transportation problem's feasible solution of m is considered basic. In other words, a B.F.S. of a T.P. does not contain more than $m + n - 1$ positive allocation.

3. **Optimal Solution**

   A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

4. **Non-degenerate Basic Feasible Solution**

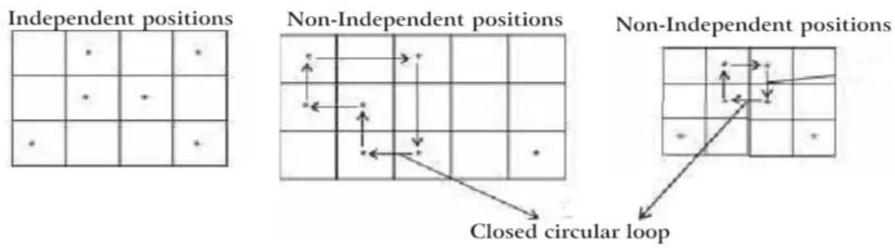
   The term "non-degenerate basic feasible solution" refers to a feasible solution to the transportation problem if

   (i) $(m + n - 1)$ is precisely the total number of positive allocations.

   (ii) These allocations are situated independently.

   To put it another way, a B.F.S. is considered non-degenerate if it involves exactly $(m + n - 1)$ individual positive allocations and these allocations are used in independent positions; if not, it is considered degenerate.

   By "independent positions of the allocation," we mean that combining any or all of these allocations with horizontal and vertical lines would never result in a closed circuit (loop). Only for the allocations in independent places may one return to itself by making a sequence of jumps from one occupied cell to another occupied cell, both vertically and horizontally, without actually reversing the path. View the following tables, where allocation positions are shown by



5. **Balanced Transportation Problem**
   If the total supply (availability) at all origins (sources) equals the total demand at all destinations, the transportation problem is considered balanced. i.e.
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

6. **Unbalanced Transportation Problem**
   If the total supply (availability) at all origins (sources) is not equal to the total demand at all destinations, the transportation problem is considered unbalanced. i.e.
$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

7. **Rim Requirements**
   The quantity required for distribution, or $\sum_{j=1}^{n} b_j$, in a transportation problem is referred to as the "rim requirement".

Theorem 1: Existence of feasible solution
A necessary and sufficient condition for the existence of feasible solution of a men transportation problem is
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \ (Rim\ condition)$$
Theorem 2: Out of $(m+n)$ equations (constraints) in a $m \times n$ transportation problem, one (any) is redundant and remaining $(m+n-1)$ equations form a linearly independent set.
Theorem 3: Existence of an optimal solution
There always exists an optimal solution to balanced transportation problem.

13.8 *SOLUTION OF A TRANSPORTATION PROBLEM*: -

A transportation problem is solved, in general, by the following step by step procedure.

Step 1: To make a transportation table
Set up the problem in the form of a transportation if not given so,

Step 2: To check the balance in supply and requirements (demands)
Check whether the given T.P. is balanced or not. For a balanced $m \times n$ T.P.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

i.e., the total supply (availability) at the sources must be equal to the total requirements (demands) at all destinations whenever it is not so, a dummy origin (sources) or destination (as the case may be) is created to balance the supply and demands.

Step 3: To find an initial basic feasible solution (B.F.S.)
First we find an initial B.F.S. of the given T.P. by any of the methods given in article 13.9. It is in general better to find initial B.F.S. by VAM which will save their valuable time to reach the optimal solution of the problem. Then

(1) Check whether the B.F.S. has allocations in exactly $(m + n - 1)$ cells or not. If the number of allocations is less than $(m + n - 1)$, then it is case of degeneracy i.e., the B.F.S. is degenerate.
(ii) Check whether the B.F.S. has $(m + n - 1)$ allocations in independent positions or not. If not then either shift source allocation from an occupied cell to an occupied (empty) cell or find B.F.S. by other method to get exactly $(m + n - 1)$ allocations in independent positions.

Step 4: To check the solution for optimality
Make optimality test to check the above non-degenerate solution obtained in step 3 for optimality. If this solution is not optimal then proceed to the next step.

Step 5: To find the modified (revised) solution
If the solution in step 3 is not optimal, then modify the solution by shifting an allocation from an occupied all (cell having allocation) to an unoccupied cell (cell having no allocation) so that total transportation cost is not increased and the allocations remain in independent positions.

Step 6: Repeat steps 4 and 5 until an optimal solution is obtained.

13.9 *METHODS OF FINDING INITIAL FEASIBLE SOLUTION*: -

Finding the first basic feasible solution to the given transportation problem can be done in a number of ways. Here, we outline the following three simple methods.

1. Method 1: North-West Corner Method (NWCM)

2. Method 2: Lowest Cost Entry Method (LCEM) or Matrix Minima Method (MMM)

3. Method 3: Vogel's Approximation Method (VAM) or Unit Cost Penalty Method (UCPM)

13.9.1 North-West Corner Method (NWCM): -

In this method, we have the following steps:

Step 1: Start with the $(1, 1)$ at the north-west corner ie., the upper left corner of the transportation table and allocate it the maximum possible amount x_{11} which is equal to the minimum of available supply a_1 in row 1 and the demand required b_1 in the column 1 i.e..

$$x_{11} = min.(a_1, b_1)$$

i.e., if $a_1 < b_1$, then allocate $x_{11} = a_1$ and move to cell (2,1) vertically below cell $(1, 1)$ as the supply in row 1 is exhausted.

if $a_1 > b_1$ then allocate $x_{11} = b_1$ and move to cell (1,2) horizontally right to cell $(1, 1)$ as the requirement in column one is exhausted.

if $a_1 = b_1$ then allocate $x_{11} = a_1 = b_1$ and move to cell (2,2), diagonally to cell $(1, 1)$ as the supply in row 1 and requirement in column 1 are exhausted simultaneously.

Step 2: Adjust the supply and demand units in the respective rows and columns i.e., in row 1 and column 1 through the cell $(1, 1)$.

Step 3: Repeat the steps 1 and 2 with the new cell moved.

Step 4: Continue in this manner, step by step until the total available supply is fully allocated to the cells destinations as required.

Note: that the procedure will end at the cell in the south-east corner (i.e., lowest right corner).

13.9.2 Lowest Cost Entry Method (LCEM) or Matrix Minima Method (MMM): -

In this method, we have the following steps:

Step 1: Write all the costs within the brackets () in the transportation table.

Step 2: Examine the cost matrix carefully and find the lowest cost. Let it be c_{ij}. Then allocate x_{ij} as much as possible in the cell $(i, j), i.e. x_{ij} = min.(a_i, b_j)$

(i). If $a_i < b_j$, $x_{ij} = a_i$, then the capacity of the ith origin is completely exhausted. In this case cross out the ith row of the transportation table and decrease the requirement b_j by a_i. Now go to step 3.

(ii) If $a_i > b_j$, $x_{ij} = b_j$, then the requirement of j-th destination is completely satisfied. In this case cross out the j-th column of the transportation table and decrease a_i by b_j. Now go to step 3.

(iii) If $x_{ij} = a_i = b_j$ then either cross-out the ith row or j-th column but not both. Now go to step 3.
If such cell of lowest cost is not unique, select the cell where we can allocate more amounts.

Step 3: Adjust the supply and demand in the row and column through the cell in which allocation is made.
Step 4: Repeat the steps 2 and 3 leaving the cost of the cells in the row or column already crossed, until all the supply is exhausted or all the requirements are satisfied.

13.9.3 Method 3: Vogel's Approximation Method (VAM): -

In this method, each allocation is made on the basis of the penalty (or opportunity) cost that would have incurred if allocation in certain cells with minimum cost were missed. In this method, we have the following steps:

Step 1: Write the differences of the smallest and the second smallest costs (i.e. penalties) in each row to the right of the corresponding row and write the similar differences (penalties) of each column below the corresponding column.

Step 2: Select the row or column for which the penalty is the largest and allocate the maximum possible amount to the cell with lowest cost in that particular row or column.

If the largest penalty among rows and columns is not unique, select that row or column in which we can allocate more amount in the lowest cost cell of that row or column.

Step 3: Adjust the supply and demand units in the respective row and column through the cell in which allocation is made and cross (or leave) out the row (or column) in which the supply (or demand) is exhausted.

Step 4: Repeat the steps 1, 2 and 3 with the costs in the remaining rows and columns left after crossing the exhausted row or column in previous step till all the supply and demands are exhausted.

13.10 *OPTIMALITY TEST*: -

After obtaining the initial F.S. of a transportation problem, we evaluate this solution for optimality, meaning we determine whether or not the feasible solution found minimizes the overall cost of transportation. There, we begin the optimality test to a non-degenerated B.F.S., or an F.S. with $(m + n - 1)$ allocations in independent positions.

For the test of the solution's optimality, the two approaches listed below are generally used:

1. The stepping-stone method
2. The modified distribution (MODI) or u-v method.

13.11 *THE STEPPING STONE METHOD: -*

Examine the matrix that provides the transportation problem's basic feasible solution (a feasible solution with exactly $m + n - 1$ allocations), where m is the number of rows and n is the number of columns.

Follow these steps to test the B.F.S. for optimality using the stepping-stone method:

Step 1: Choose an unoccupied cell, or one that has no allocation. Starting from this cell, move horizontally and vertically along a closed path that passes through at least three occupied cells utilized in the solution before returning to this cell. Only the cells at the turning points in this closed path can be taken into account, ignoring the other occupied and unoccupied cells that cross the path. These cells at the path's turning points are referred to as stepping stones.

Step 2: Allocate $+1$ unit to this chosen unoccupied cell, then alternately assign -1 and $+1$ units to each of the above closed path's corner cells so that the total of the row and column allocations remain unchanged.

Step 3: Add the unit costs in the cells with $+1$ allocations and subtract the unit costs in the cells with -1 allocations to determine the net change in the cost along this closed path. The cell evaluation of the unoccupied cell chosen in step 1 refers to this net change in cost, which can be either positive or negative. The positive cell evaluation implies that the above adjustment would increase the total cost while the negative cell evaluation implies that this adjustment would decrease the total cost.

Step 4: For every unoccupied cell in the solution, repeat steps 1 to 4.

Step 5: Examine each cell evaluation's signs. The solution under evaluation is optimal and cannot be further improved if all are larger than or equal to zero. The present solution is not ideal and can be further improved if at least one of these cell evaluations is negative. In this case, move on to step 6.

Step 6: Choose the unoccupied cell with the highest negative cell evaluation, allocate a maximum number of units to it, and modify the units in other occupied cells so that the total of the allocations in the rows and columns remains the same.

Step 7: For every unoccupied cell in this new solution, repeat steps 1 to 6. Continue this process until an optimal solution is found. Because there are $mn - (m + n - 1) = (m - 1)(n - 1)$ unoccupied cells, we must compute $(m - 1)(n - 1)$ such cell evaluations for each solution, which is extremely difficult.

We provide the following theorem (without proof) for the simultaneous computation of cell evaluations for every unoccupied cell in order to avoid this difficulty.

Theorem 4: If we have a B.F.S. consisting of $(m + n - 1)$ independent positive allocations, and a set of arbitrary numbers $u_i \; and \; v_j, i = 1, 2, \ldots, m; j = 1, 2 \ldots \ldots n$; such that

$$c_{rs} = u_r + v_s$$

for all occupied cells (r, s) then the cell evaluations d_{ij} corresponding to each empty cell (i, j) is given by

$$d_{ij} = c_{ij} - (u_i + v_j)$$

Example1. Solve the following transportation problem.

		To			Supply
		1	2	3	
From	1	2	7	4	5
	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
Demand		7	9	18	34

Solution:

Step 1: The initial B.F.S. of the above problem (by VAM) is given in the following table.

Total transportation cost

$$= ₹(5 \times 2 + 2 \times 1 + 7 \times 4 + 2 \times 6 + 8 \times 1 + 10 \times 2) = ₹80$$

			a_i
(2) 5	(7)	(4)	5
(3)	(3)	(1) 8	8
(5)	(4) 7	(7)	7
(1) 2	(6) 2	(2) 10	14
b_j 7	9	18	

Step 2: Now we determine a set of u_i *and* v_j s.t. for each occupied cell $(r,s), c_{rs} = u_r + v_s$. For this we choose $u_4 = 0$ (since row 4 contains maximum number of allocations).
Since

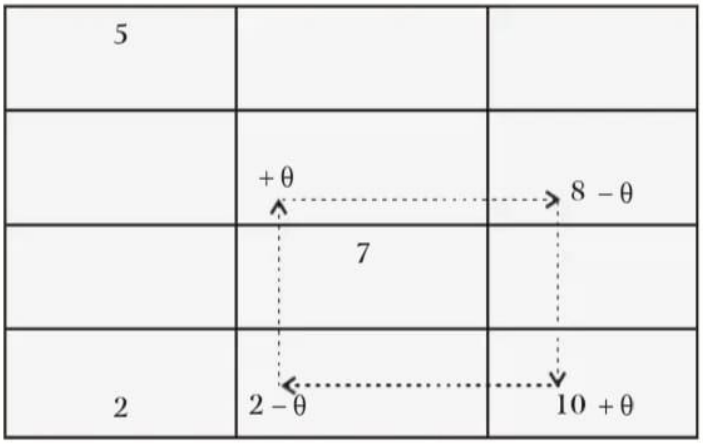
$$c_{41} = 1 = u_4 + v_1, c_{42} = 6 = u_4 + v_2, c_{43} = 2 = u_4 + v_3$$
$$\therefore \quad v_1 = 1 - u_4 = 1, v_2 = 6 - u_4 = 6, v_3 = 2 - u_4 = 2$$
$$also \quad c_{11} = 2 = u_1 + v_1, c_{23} = 1 = u_2 + v_3, c_{32} = 4 = u_3 + v_2$$
$$\therefore \quad u_1 = 2 - v_1 = 1, u_2 = 1 - v_3 = -1, u_3 = 4 - v_2 = -2$$

Step 3: Then we find the cell evaluations $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding unoccupied cell.
Step 4: Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (i.e., the difference of the upper right corner entry from the upper left corner entry) for each unoccupied cell (i,j) and enterat the lower right corner of the corresponding unoccupied cell.
Thus, we get the following table:

			u_i
(2) 5	(7) (7) (0)	(4) (3) (1)	1 (u_1)
(3) (0) (3)	(3) (5) (− 2*)	(1) 8	−1 (u_2)
(5) (−1) (6)	(4) 7	(7) (0) (7)	−2 (u_3)
(1) 2	(6) 2	(2) 10	0 (u_4)
v_j 1 (v_1)	6 (v_2)	2 (v_3)	

Step 5: Since cell evaluation $d_{22} = -2 < 0$, so the solution under test is not optimal.

Step 6: Since minimum d_{ij} is $d_{22} = -2 < 0$ (negative), so we give maximum allocation θ to this cell from an occupied cell and make the necessary changes in other allocations as shown in the following table.



Since minimum allocation containing $-\theta$ is $2-\theta$

∴ taking $2-\theta=0$, we get $\theta=2$

Step 7: The new B.F.S. (allocations in independent positions) thus obtained is shown in the following table. For this B.F.S. total transportation cost

$= ₹\,(5\times 2 + 2\times 1 + 2\times 3 + 7\times 4 + 6\times 1 + 12\times 2) = ₹76$

This is less than that for the initial B.F.S.

				a_i
	(2) 5	(7)	(4)	5
	(5)	(3) (2)	(1) (6)	8
	(5)	(4) 7	(7)	7
	(1) (2)	(6)	(2) (12)	14
b_j	7	9	18	

Step 8: Proceeding as in step 2, 3 and 4 (to test the optimality of the above B.F.S.) we get the following table:

				u_i
	(2) 5	(7) (5) (2)	(4) (3) (1)	$1(u_1)$
	(3) (0) (3)	(3) 2	(1) 6	$-1(u_2)$
	(5) (1) (4)	(4) 7	(7) (2) (5)	$0\ (u_3)$
	(1) 2	(6) (4) (2)	(2) 12	$0\ (u_4)$
v_j	1 (v_1)	4 (v_2)	2 (v_3)	

Since all $d_{ij} > 0$. Hence, the B.F.S. shown by table in step 8 is an optimal solution which is also unique.

Thus, the solution of the given transportation problem is
from source 1 transport 5 units to destination 1.
from source 2 transport 2 and 6 units to destinations 2 and 3 respectively.
from source 3 transport 7 units to destination 2.
And from source 4 transport 2 and 12 units to destinations 1 and 3 respectively which can also be written as

$$x_{11} = 5, x_{22} = 2, x_{23} = 6, x_{32} = 7, x_{41} = 2, x_{43} = 12$$

where x_{ij} is the number of units to be transported from i-th source to j-th destination.

And the total transportation cost (optimal) = ₹76.

13.12 *MODI METHOD OR u-v METHOD: -*

The MODI Method's iterative process for identifying the most optimal solution to a minimization transportation problem is as follows:

Step 1: Create a transportation table by entering the requirements $b_1, b_2, \dots, b_n$ and the sources' capacity $a_1, a_2, \dots, a_m$. In each cell's upper left corner, enter the different costs c_{ij}. Use any of the techniques listed in article 13.9 to find the problem's initial B.F. solution (allocation in independent positions). Put the allocations in the cell centers.

Step 2: Choose a set of $(m+n)$ numbers u_i *and* $v_j, i = 1, 2, \dots, m; j = 1, 2 \dots \dots n$ such that for each occupied cell (r, s)

$$c_{rs} = u_r + v_s$$

In order to do this, we give one of the $u_i's\ or\ v_j's$ an arbitrary value. The remaining ones $(m + n - 1)$ may then be solved algebraically using the relation $c_{rs} = u_r + v_s$ for occupied cells. generally, we select the row or column with the greatest number of individual allocations, $u_i\ or\ v_j = 0$.

Step 3: For every unoccupied cell (i, j), find the cell evaluation $u_i + v_j$ and insert it in the upper right corner of the matching cell (i, j).

Step 4: For each unoccupied cell (i, j), get the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ and enter them in the associated cells' lower right corners.

Step 5: Examine for unoccupied cells in the cell evaluations d_{ij} and determine that

(i) The solution under test is optimum and unique if all $d_{ij} > 0$.

(ii) The solution under test is optimal and there is an alternate optimal solution if all $d_{ij} \geq 0$ and at least one $d_{ij} = 0$.

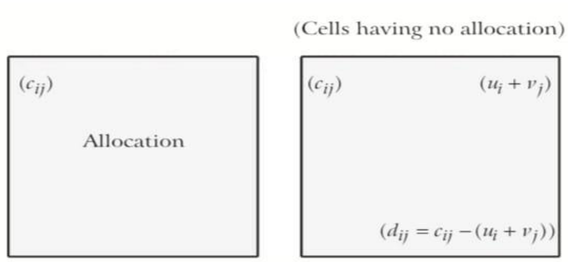
(iii) The solution is not optimal if there is at least one $d_{ij} < 0$. In the last case, move on to step 6.

Step 6: Create a new B.F.S. by making an occupied cell empty, providing the maximum allocation to the cell for which d overline eta is the greatest negative, and modifying the units in the remaining occupied cells so that the total of the allocations in the rows and columns remain unchanged.

Step 7: To determine whether this new B.F. solution is optimal, repeat steps (2) to (5).

Until the optimal solution is found, continue improving the B.F.S. interactively using steps 2 to (6).

The occupied (cells with allocations) and unoccupied (cells without allocations) cells in the table will therefore look like this after all entries have been made:



13.13 UNBALANCED *TRANSPORTATION PROBLEM*: -

An unbalanced transportation problem is defined as one in which the total of all available quantities does not equal the total of all requirements. That is, if

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

A transportation problem that is unbalance gets transformed into one that is balanced by creating a fictitious source or destination that will supply the surplus supply or demand. Transporting a unit from the fictitious source (or to the fictitious destination) is assumed to have no cost. By introducing a fictional source or destination, the unbalanced transportation problem is transformed into a balanced transportation problem, which is then resolved using the previous techniques.

Example2. Determine the optimal transportation plan from the following table given the plant to market shipping costs and quantities at each market and available at each plant:

Plant	W_1	W_2	W_3	W_4	Availability
F_1	11	20	7	8	50
F_2	21	16	10	12	40
F_3	8	12	18	9	70
Requirements	30	25	35	40	

Solution: Here total requirement of the market $= 30 + 25 + 35 + 40 = 130$ and total availability at the plants $= 50 + 40 + 70 = 160$

Since the total availability at three plants is 30 more than the total requirements in four markets W_1, W_2, W_3, W_4 . Therefore, this transportation problem is unbalanced and so we convert this problem to a balanced one by introducing a fictitious, market W_5 with requirement 30 such that the cost of transportation from plants to this market W_5 are zero.
Thus, the balanced transportation problem is given by the following table:

Plant	W_1	W_2	W_3	W_4	W_5	Availability
F_1	11	20	7	8	0	50
F_2	21	16	10	12	0	40
F_3	8	12	18	9	0	70

Requirements	30	25	35	40	30	Total 160

By 'VAM', we get the following B.F. solution of the problem:

Market

Plant	W_1	W_2	W_3	W_4	W_5	a_i	u_i
F_1	(11) (7) (4)	(20) (11) (9)	(7) **25**	(8) **25**	(0) (– 3) (3)	**50**	– 1
F_2	(21) (10) (11)	(16) (14) (2)	(10) **10**	(12) (11) (1)	(0) **30**	40	2
F_3	(8) **30**	(12) **25**	(18) (8) (10)	(9) **15**	(0) (–2) (2)	70	0
b_j	30	25	35	40	30	160	
v_j	8	12	8	9	– 2		

The solution given in the above table is an optimal solution as all $d_{ij} \geq 0$.
Thus the optimal solution is
transport from plant F_1 to market W_3, 25 units.
transport from plant F_1 to market W_4, 25 units.
transport from plant F_2 to market W_3, 10 units.
transport from plant F_3 to market W_1, 30 units.
transport from plant F_3 to market W_2, 25 units.
transport from plant F_3 to market W_4, 15 units..
$i.e.$ $x_{13} = 25, x_{14} = 25, x_{23} = 10, x_{32} = 25, x_{31} = 30, x_{34} = 15$
Total transportation cost

$$= ₹(25 \times 7 + 25 \times 8 + 10 \times 10 + 30 \times 8 + 25 \times 12 + 15 \times 9)$$
$$= ₹1150$$

It is important to note that 30 units are dispatched from plant F_2 to market (Fictitious) W_5, In other words, we can say that 30 units are left undispatched at the plant F_2.

SELF CHECK QUESTIONS

1. What is a Transportation Problem?
2. How is the Transportation Problem related to Linear Programming?
3. What is the main objective of the Transportation Problem?
4. Define the term **initial feasible solution**.
5. What is the purpose of the **Stepping Stone Method**?
6. Explain the **Modified Distribution Method (MODI)**.
7. What are the conditions for the **optimal solution** in a Transportation Problem?
8. What is the difference between a **Transportation Problem** and an **Assignment Problem**?
9. Give one real-life example where a Transportation Problem can be applied.

13.14 *SUMMARY*: -

In this unit, we learned that the Transportation Problem is a type of Linear Programming Problem (LPP) that deals with the efficient allocation of resources in order to reduce the cost of transporting goods from multiple sources (such as factories or warehouses) to multiple destinations. The primary goal of this challenge is to discover the best cost-effective method for distributing a commodity while meeting supply and demand limitations at each source and destination. Each route between a source and a destination has a transportation cost per unit, and the solution seeks to reduce the overall transportation cost. The problem can be solved using methods such as the North-West Corner Rule, Least Cost Method, and Vogel's Approximation Method (VAM) to obtain an initial feasible solution, followed by optimization techniques such as the Stepping Stone Method or Modified Distribution Method (MODI) to achieve the optimal solution.

13.15 *GLOSSARY*: -

- **Transportation Problem:** A special type of Linear Programming Problem (LPP) that focuses on minimizing the cost of transporting goods from multiple sources to multiple destinations while meeting supply and demand constraints.
- **Source (Origin):** The point or location (such as a factory or warehouse) from where goods are supplied or transported.
- **Destination:** The point or location (such as a market or store) where goods are required or demanded.
- **Supply:** The quantity of goods available at each source for transportation.
- **Demand:** The quantity of goods required at each destination.
- Transportation Cost: The cost incurred in transporting one unit of a product from a source to a destination.
- **Transportation Table (Matrix):** A tabular representation showing sources, destinations, supply, demand, and unit transportation costs.
- **Feasible Solution:** A solution that satisfies all supply and demand constraints without violating the non-negativity condition.
- **Initial Basic Feasible Solution (IBFS):** The starting solution that satisfies all constraints before applying optimization methods.

- **North-West Corner Rule:** A method used to find an initial feasible solution by starting from the top-left (north-west) cell of the transportation table.
- **Least Cost Method:** A method of finding an initial feasible solution by selecting cells with the lowest transportation cost first.
- **Vogel's Approximation Method (VAM):** A method used to obtain an initial feasible solution by considering penalties (difference between the two lowest costs in each row and column).
- **Stepping Stone Method**: An optimization method used to test whether the current feasible solution is optimal and to improve it if not.
- **Modified Distribution Method (MODI Method):** An efficient method for testing the optimality of a transportation problem and improving the current solution.
- **Balanced Transportation Problem:** A transportation problem in which the total supply equals the total demand.
- **Unbalanced Transportation Problem:** A transportation problem in which total supply does not equal total demand. It can be balanced by adding a dummy source or destination.
- **Degeneracy:** A condition that occurs when the number of occupied cells in the transportation table is less than $(m + n - 1)$, where m is the number of sources and n is the number of destinations.
- **Optimal Solution:** The feasible solution that results in the minimum total transportation cost.

13.16 *REFERENCES*: -

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- Sharma, J. K. (2018). Operations Research: Theory and Applications (6th Edition). Macmillan Publishers India.

13.17 *SUGGESTED READING: -*

- Dr. R.K.Gupta (2nd Eddition, 2012), Krishna Publication, Operation Research
- Er. Prem Kumar Gupta and Dr. D.S. Hira (7th Edition,2014), S.Chand & Company PVT. LTD., Operations Research

13.18 *TERMINAL QUESTIONS: -*

(TQ-1) Solve the following transportation problems for minimum cost.

To

		I	II	III	IV	
	A	15	10	17	18	2
From	B	16	13	12	13	6
	C	12	17	20	11	7
		3	3	4	5	

(TQ-2) Solve the following transportation problems for minimum cost.

Sources	Destination			Supply
	X	Y	Z	
A	2	7	4	50
B	3	3	7	70
C	5	4	1	80
D	1	6	2	140
Demand	70	90	180	340

(TQ-3) Obtain an optimal B.F.S. to the following T.P.

To

		I	II	III	Available
From	A	7	3	4	2
	B	2	1	3	3
	C	3	4	6	5
	Demand	4	1	5	10

(TQ-4) A company has three plants A, B, C and three were houses X, Y and Z. Number of units, available at the plants are 60, 70 and 80 respectively. Demands at X, Y and Z are 50, 80 and 80 respectively. Unit's costs of transportation are as follows:

	X	Y	Z
A	8	7	3
B	3	8	9

C	11	3	5

(TQ-5) Solve the following transportation problem by North West corner method.

		To			Supply
		I	*II*	*III*	
From	1	19	16	12	14
	2	22	13	19	16
	3	14	28	8	12
Demand		10	15	17	

(TQ-6) Solve the following transportation problem by North West corner method.

		To				Supply
		I	*II*	*III*	*IV*	
From	1	14	25	45	5	6
	2	65	25	35	55	8
	3	35	3	65	15	16
Demand		7	7	6	13	

(TQ-7) Solve the following transportation problem by Lowest cost entry method.

		Warehouse				Supply
		I	*II*	*III*	*IV*	
Factory	1	19	39	50	10	7
	2	70	30	40	60	9
	3	40	8	70	20	18
Demand		5	8	7	14	

(TQ-8) Solve the following transportation problem by VAM method.

		Destination				Supply
		I	*II*	*III*	*IV*	
Sources	1	5	8	3	6	30
	2	4	5	7	4	50
	3	6	2	4	6	20
Demand		30	40	20	10	

(TQ-9) Solve the following transportation problem.

		To				Supply
		I	*II*	*III*	*IV*	
	1	21	16	25	13	11

From	2	17	18	14	23	13
	3	32	27	28	41	19
Demand		6	10	12	15	

(TQ-10) Solve the following transportation problem.

		Market				Supply
		I	*II*	*III*	*IV*	
Plant	1	14	9	18	6	11
	2	10	11	7	16	13
	3	25	20	11	34	19
Demand		6	10	12	15	

13.19 *ANSWERS: -*

(TQ-1) $min. cost = ₹174$

(TQ-2) $min. cost = ₹710$

(TQ-3) $min. cost = ₹33$

(TQ-4) $min. cost = ₹750$

(TQ-5) $x_{11} = 10, x_{22} = 11, x_{12} = 4, x_{23} = 5, x_{33} = 12, T.C. = ₹588$

(TQ-6) $x_{11} = 4, x_{22} = 5, x_{12} = 2, x_{23} = 3, x_{33} = 3, x_{34} = 13, T.C. = ₹726$

(TQ-7) $x_{14} = 7, x_{21} = 2, x_{31} = 3, x_{23} = 7, x_{32} = 8, x_{34} = 7, T.C. = ₹814$

(TQ-8) $x_{11} = 10, x_{21} = 20, x_{13} = 20, x_{22} = 20, x_{24} = 10, x_{32} = 20, T.C. = ₹370$

(TQ-9) : $T.C. = ₹796$

(TQ-10) $OP.T.C. = ₹495$

UNIT 14: -Game Theory

CONTENTS:

14.1 *INTRODUCTION: -*

Game Theory is a branch of applied mathematics and economics that studies how people (or players) make decisions in situations where the outcome depends on the choices of all participants. It helps to understand competition, cooperation, and strategic behavior.

For example, in business, politics, or even everyday life, people often have to make choices considering what others might do. Game theory provides tools to analyze these situations and find the best strategies.

14.2 *OBJECTIVES: -*

After studying this unit, the learner’s will be able to

- Explain game theory.
- Understand rules for game theory.
- Define n persons Zero-Sum games.

14.3 *DEFINITION: -*

The general characteristics of competitive situations are considered by the mathematical theory of games, sometimes known as game theory or competitive strategies. When two or more people or organizations with competing goals attempt to make decisions, this theory can be useful. In these circumstances, a decision made by one decision-maker influences the decisions made by one or more of the other decision-makers, and the decision made by all parties determines the final result. In the field of business, industry, economics, sociology, and military training, such situations often arise. This theory can be applied to a wide range of scenarios, including two players trying to win a game of chess, candidates competing in an election, two adversaries preparing a war strategy, businesses trying to hold onto market share, the start of advertising campaigns by businesses promoting rival products, talks between organizations and unions, etc. These situations are not the same as the ones we have discussed, where nature was seen as a harmless opponent.

J. Von Neumann's minimax principle, which states that each competitor will try to minimize his miximum loss (or maximize his minimal gain) or attain best of the worst, is the foundation of game theory. This mathematical theory has only been used to analyze basic competitive problems so far. How a game should be played is not explained by the theory. It merely outlines the process and guidelines for choosing plays.

Von Neumann, the "father of game theory," invented the theory of games in 1928, but it wasn't until he and Morgenstern published "Theory of Games and Economic Behaviour" in 1944 that the theory got the attention it deserved. There has been a significant gap between what the theory can manage and the most real-world business and industrial scenarios because it has only been able to analyze quite basic situations thus far. Therefore, rather than its formal application to the resolution of actual issues, game theory's main contribution has been its notions.

14.3.1 GAME MODELS

There are various types of game models. They are based on the factors like the number of players participating, the sum of gains or losses and the number of strategies available, etc.

1. **Number of persons:** If a game involves only two players, it is called two-person game, if there are more than two players, it is named n-person game. An n-person game does not imply that exactly n players

are involved in it. Rather it means that the participants can be classified into mutually exclusive groups, with all members in a group having identical interests.

2. **Sum of payoffs:** If the sum of payoffs (gains and losses) to the players is zero, the game is called zero-sum or constant-sum game, otherwise non zero-sum game.

3. **Number of strategies:** If the number of strategies (moves or choices) is finite, the game is called a finite game, if not, it is called infinite game.

14.4 *CHARACTERISTICS OF GAMES: -*

A competitive game has the following characteristics:

(1) There are finite numbers of participants or competitors. If the number of participants is 2, the game is called two-person game, for number greater than two; it is called n-person game.

(2) Each participant has available to him a list of finite number of possible courses of action. The list may not be same for each participant.

(3) Each participant knows all the possible choices available to others but does not know which of them is going to be chosen by them.

(4) A play is said to occur when each of the participants chooses one of the courses of action available to him. The choices are assumed to be made simultaneously so that no participant knows the choices made by others until he has decided his own.

(5) Every combination of courses of action determines an outcome which results in gains to the participants. The gain may be positive, negative or zero. Negative gain is called a loss.

(6) The gain of a participant depends not only on his own actions but also those of others.

(7) The gains (payoffs) for each and every play are fixed and specified in advance and are known to each player. Thus each player knows fully the information contained in the payoff matrix.

(8) The players make individual decisions without direct communication.

14.5 *TERMINOLOGY*: -

1. **Game:** It is an activity between two or more people in which each of them does acts according to a set of rules, resulting in some gain

$(+ve, -ve, or\ zero)$ for each. If the actions in a game are determined by abilities, it is considered a game of strategy; if they are determined by chance, it is called a game of chance. Furthermore, a game can be finite or infinite. A finite game has a limited number of moves and options, whereas an infinite game has an infinite number of them.

2. **Player:** Every participant or competitor in a game is referred to as a player. Each player takes an equally intelligent and sensible approach.
3. **Play:** A game play is defined as when each player selects one of his options.
4. **Strategy:** It is the predetermined rule by which a player selects a course of action from his list of options during the game. The player does not need to know the opponent's strategy in order to decide a particular strategy.
5. **Pure Strategy:** It is the choice rule to always take a certain course of action. It is commonly represented by a numerical value that corresponds to the course of activity.
6. **Mixed Strategy:** It is a decision made ahead of time to select a course of action for each play based on a probability distribution. Thus, a mixed strategy is a choice between pure strategies with certain fixed probabilities (proportions). The advantage of a mixed approach, once the game pattern is established, is that opponents are kept wondering as to which line of action a player would take.
7. **Optimal Strategy:** An optimal strategy is one that puts the player in the best position possible, regardless of his opponents' strategies. Any deviation from this strategy would reduce his payoff.
8. Z**ero-Sum game:** It is a game in which the total amount paid to all players at the end of the game is zero. In such a game, the gain of players that win is exactly equal to the loss of players that lose. For example, two candidates fighting in elections, where the gain of votes by one is the loss of votes by the other.
9. **Two-Person zero-sum game:** It is a game with only two participants in which one's gain equals the other's loss. It is also known as a rectangle game or matrix game because the payment matrix is rectangular in shape. If there are participants and the sum of the game is zero, it is called a "person zero-sum game." Iwo-person zero-sum games have the following characteristics:

   (a) There are only two players participating.
   (b) Each player has a limited amount of strategies to use.

(c) Each given method has a payoff,

(d) The total payoff for the two players at the end of each play is zero.

10. **Nonzero-sum game:** Here a third party receives or makes some payment.
11. **Payoff:** It is the result of the game. The payoff (gain or game) matrix is a table that shows the sums received by the player labeled on the left after all conceivable game plays. The payment is made by the person listed at the top of the table.

14.6 *RULES FOR GAME THEORY*: -

The basic rules employed in solving games are described below:

14.6.1 RULE 1. LOOK FOR A PURE STRATEGY (SADDLE POINT)

Example1. (Two-Person Zero-Sum Game with Saddle Point): In a certain game, player A has three possible choices $L,\ M\ and\ N$, while player B has two possible choices $P\ and\ Q$. Payments are to be made according to the choices made.

Table: 1

Choices	Payments
L, P	$A\ pays\ B$ ₹3
L, Q	$B\ pays\ A$ ₹3
M, P	$A\ pays\ B$ ₹2
M, Q	$B\ pays\ A$ ₹4
N, P	$B\ pays\ A$ ₹2
N, Q	$B\ pays\ A$ ₹3

What are the best strategies for players $A\ and\ B$ in this game? What is the value for the game $A\ and\ B$?

Solution: The above payments can be easily arranged in the form of a matrix. Let positive number represent a payment from $B\ to\ A$ and negative number a payment from $A\ to\ B$. We, then, have the payoff matrix shown in table 2.

Minimax and maximum values are also shown on the matrix. When player A plays his first strategy (namely L), he may gain$-3\ or\ 3$ depending upon player $B's$ selected strategy. He can guarantee, however, a gain of at least min. $\{-3,3\} = -3$ regardless of B's selected strategy. Similarly, if A plays his second strategy (namely M), he guarantees an income of at least min. $\{-2,4\} = -2$, if he plays his third strategy (namely N) he guarantees an income of at least min. $\{2, 3\} = 2$. Thus the minimum value in each row represents the minimum gain guaranteed to A if he plays his pure (grand) strategies.

These values are indicated in the matrix under 'Minimum of row Now, player A, by selecting his third strategy (N), is maximizing his minimum gain. This gain is given by $\max\{-3,-2,2\} = 2$. This selection of player A is called the maximin strategy and his corresponding gain is called the maximin or lower value of the game.

Table: 2

		Player B Plans (choices)		
		P	Q	*Minimum of row*
Player A (plans choices)	L	– 3	3	– 3
	M	– 2	4	– 2
	N	2	3	(2) *maximin*
Miximum of column		(2) *minimax*	4	

Player B, on the other hand, wants to minimize his losses. He realizes that if he plays his first pure strategy (namely P), he can lose no more than max. $\{-3,-2,2\} = 2$, regardless of $A's$ selections. Similarly, if he plays his second pure strategy (Q), the maximum he loses is max $\{3,4,3\} = 4$. These values are indicated in the above matrix by "Maximum of column. Player B will select the strategy that minimizes his maximum loss. This is given by strategy P and his corresponding loss is given by min. $\{2,4\} = 2$. Player $B's$ selection is called the minimax strategy and his corresponding loss is called the minimax (or upper) value of the game.

It is seen from the conditions governing the minimax criterion that the minimax (upper) value is greater than or equal to the maximin (lower) value. When the two are equal (minimax value maximin value), the corresponding pure strategies are called optimal strategies and the game is anid to have a saddle point or equilibrium point and is called a stable game. The value of the game is given by the saddle point and is equal to the maximin and minimax values. Thus the saddle point is the point of intersection of the two courses of action and the gain at this point is the value of the game. The game is said to be fair if $maximin\ value = minimax\ value = 0$, and is said to be strictly determinable inable if $maximin\ value = minimax\ value \neq 0$. Note that neither player can improve his position by selecting any other strategy. Saddle point is the number which is lowest in its row and highest in its column.

In the above example, $minimax\ value = maximin\ value = 2$. The value of the game is thus equal to 2. The game has a saddle point given by the entry (N, P) of the matrix. As the game value is 2, (and not zero), the game is not fait, though it is strictly determinable.

The saddle point solution guarantees that neither player is tempted to select a better strategy If B moves to the other strategy Q. player A may move to strategy M, which means that B will lose ₹4, rather than ₹2 at present. Likewise, A does not want to use a different strategy because if A moves to strategy, say L, player B will adopt strategy P so that A will lose₹3, rather than winning ₹2 presently.

We summaries below the steps required to detect a saddle point

(1) At the right of each row, write the row minimum and ring the largest of them.

(2) At the bottom of each column, write the column maximum and ring the smallest of them.

(3) If these two elements are same, the cell where the corresponding row and column meet is a saddle point and the element in that cell is the value of the game

(4) If the two ringed elements are unequal, there is no saddle point, and the value of the game has between these two values.

(5) If there are more than one saddle points then there will be more than one solution, each solution corresponding to each saddle point.

We give below a few more examples of games. Saddle points, if they exist, have been ringed Optimum strategies and game values are also indicated.

$$\begin{array}{c} \quad B \\ A\begin{bmatrix} -4 & 3 \\ -3 & -7 \end{bmatrix} \end{array}$$

No saddle point exists since there is no element which is both the lowest in its row and highest in its column.

$$\begin{array}{c} \quad B \\ A\begin{bmatrix} 3 & 2 \\ -2 & -3 \\ -4 & -5 \end{bmatrix}\begin{matrix} (2) \\ -3 \\ -5 \end{matrix} \\ \quad 3 \quad (2) \end{array}$$

Strategies: A, row 1 and B, column 2.
Saddle point : (1, 2)
Game value: + 2.

$$\begin{array}{c} \quad B \\ A\begin{bmatrix} 1 & 13 & 11 \\ -9 & 5 & -11 \\ 0 & -3 & 13 \end{bmatrix}\begin{matrix} (1) \\ -11 \\ -3 \end{matrix} \\ \quad (1) \quad 13 \quad 13 \end{array}$$

Saddle point: (1, 1)
Strategies: A, row 1; B, column 1.
Game value: + 1.

$$\begin{array}{c} \quad B \\ A\begin{bmatrix} 16 & 4 & 0 & 14 & -2 \\ 10 & 8 & 6 & 10 & 12 \\ 2 & 6 & 4 & 8 & 14 \\ 8 & 10 & 2 & 2 & 0 \end{bmatrix}\begin{matrix} -2 \\ (6) \\ 2 \\ 0 \end{matrix} \\ \quad 16 \quad 10 \quad (6) \quad 14 \quad 14 \end{array}$$

Saddle point: (2, 3)
Strategies: A, row 2; B, column 3.
Game value: + 6.

If there is no saddle point, neither player can optimize his chances by using a pure strategy, they must mix some or all of their courses of sction, resulting in mixed strategies.

Example2: Consider the game G with the following payoff.

		Player B	
		B_1	B_2
Player A	A_1	2	6
	A_2	-2	λ

(a) Show that G is strictly determinable, whatever λ may be.
(b) Determine the value of G.

Solution: a) Ignoring whatever the value of λ. may be, the given payoff matrix represents

	B_1	B_2	*Row minima*
A_1	2	6	2
A_2	-2	λ	-2
Column maxima	2	6	

$\therefore$ Maximin value$=2$ and minimax value$=2$.
$\therefore$ The game G is strictly determinable, whatever λ may be.
b) Value of the game$=2$
Strategies: A, row 1; B, column1.

14.6.2 RULE 2. REDUCE GAME BY DOMINANCE

If there are no pure strategies, the next step is to eliminate certain strategies (rows and/or columns) through dominance. Rows and/or columns of the payoff matrix that are less than at least one of the remaining rows and/or columns are removed from further consideration. The resulting game can be solved using a mixed strategy.

Example 3: (3 x 3 Game, Matrix Reduction by Dominance) Two players P and Q play a game. Each of them has to choose one of the three colours, white (W), black (B) and red (R) independently of the other. Thereafter the colours are compared. If both P and have chosen white (W. W), neither wins anything. If player P selects white and player Q black (W, B), player P loses 2 or player wins the same amount and so on. The complete payoff table is shown below (Table: 3). Find the optimum strategies for P and Q and the value of the game.

Table: 3

		Colour chosen by Q		
		W	R	B
Colour chosen by P	W	0	-2	7
	R	2	5	6
	B	3	-3	8

Solution: This matrix has no saddle point. Evidently, player Q will not play strategy R since this will result in heaviest losses to him and highest gains to player P. He can do better by playing columns W or B. Thus column R is to be deleted and strategy R is called dominated strategy.

The dominance rule for columns is: Every value in the dominating column(s) must be less than or equal to the corresponding value of the dominated column. The resulting matrix is

Table: 4

Player Q

Player P		W	B
	W	0	−2
	B	2	5
	R	3	−3

From table 4, it is clear that player P will not play row W since it will give him returns lower than given by row B. Hence row W is dominated by row B and can be deleted.

The dominance rule for rows is: Every value in the dominating row(s) must be greater than or equal to the corresponding value of the dominated row. The resulting matrix is

Table:5

Player Q

Player P		W	B
	B	2	5
	R	3	−3

This 2×2 matrix can be easily solved as discussed later.

Dominance need not be based on the superiority of pure strategies only.

A given strategy is also said to be dominated if it is inferior to some convex linear combination (e.g., average) of two or more pure strategies. To illustrate this let us consider the following game:

Table:6

B

A		1	2	3
	1	6	1	3
	2	0	9	7
	3	2	3	4

This game has no saddle point. Further, none of the pure strategies of A is inferior to any of his other pure strategies. However, average of A's first and second pure strategies gives us

$$\left(\frac{6+0}{2}, \frac{1+9}{2}, \frac{3+7}{2}\right) = (3,5,5)$$

This is obviously superior to A's third pure strategy. Therefore, the third strategy may be deleted from the matrix. The resulting matrix becomes

Table: 7

B

1	2	3

	1	6	1	3
A	2	0	9	7

14.6.3 RULE 3. SOLVE FOR A MIXED STRATEGY

When there is no saddle point and dominance has been exploited to shrink the game matrix, players will choose varied strategies. A few distinct techniques will be given to optimize each player's winning chances and solve the game. One of the players must decide how much time to spend on each row, while the other must figure out how much time to spend on the each column. The payoffs earned will be the expected payoffs, and the game's worth will be its expected value. These games are called unstable games.

14.7 $MIXED\ STRATEGIES(2 \times 2\ GAMES)$:-

Arithmetic and algebraic methods are utilized to determine optimal strategies and game values for a 2×2 game. Each of these strategies will be discussed in further depth now.

14.7.1 Arithmetic Method (Odds Method or Short Cut Method) Optimum Strategies and Game Value for Finding

It provides a simple approach for determining the best strategy for each player in a 2×2 game with no saddle points. It includes the following steps.

(a) Subtract the two digits from column 1 and write the difference in column 2, ignoring the sign.

(b) Subtract the two digits from column 2 and write the difference in column 1, ignoring the sign.

(c) Repeat for the two rows.

These values are referred to as oddments. They are the frequency at which players must implement their optimal strategy.

Example 4: (Two-person zero-sum game without saddle point) In a game of matching coins, player A wins ₹2 if there are two heads, wins nothing if there are two tails and loses ₹1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A.

Solution: The payoff matrix for A will be

Player B

		H	T
Player A	H	2	-1
	T	-1	0

Since there is no saddle point, the optimal strategies will be mixed strategies. Using the steps described above we get

Player B

		H	T		
Player A	H	2	−1	1	$1/3+1=0.25$
	T	−1	0	3	$3/3+1=0.75$
		1	3		
		0.25	0.75		

Thus for optimum gains, player A should use strategy H for 25% of the time and strategy T for 75% of the time, while player B should use strategy H 25% of the time and strategy T 75% of the time.

To obtain the value of the game any of the following expressions may be used:

Using A's oddments

$$B\ plays\ H;\ value\ of\ the\ game, V\ = ₹\left(\frac{1\times 2-3\times 1}{3+1}\right) = ₹\left(-\frac{1}{4}\right)$$

$$B\ plays\ T;\ value\ of\ the\ game, V\ = ₹\left(\frac{1\times -1+3\times 0}{3+1}\right) = ₹\left(-\frac{1}{4}\right)$$

Using B's oddments

$$A\ plays\ H;\ value\ of\ the\ game, V\ = ₹\left(\frac{1\times 2-1\times 3}{3+1}\right) = ₹\left(-\frac{1}{4}\right)$$

$$A\ plays\ T;\ value\ of\ the\ game, V\ = ₹\left(\frac{-1\times 1+0\times 3}{3+1}\right) = ₹\left(-\frac{1}{4}\right)$$

The above values of V are equal only if sum of the oddments vertically and horizontally are equal. Cases in which it is not so are treated later.

Thus the full solution of the game is

$$A\ (1,3);\ B\ (1,3);\ V = ₹(-1/4)$$

This is the value of the game to A i.e., A gains $₹(-1/4)$ ie, he loses $₹1/4$ which B, in turn, gets. Arithmetic method is easier than algebraic method but it cannot be applied to larger games.

14.7.2 Algebraic Method for Finding Optimum Strategies and Game Value

When using this method, it is expected that x represents the fraction of time (frequency) in which player A employs strategy 1 and $(1-x)$ represents the fraction of time (frequency) in which he uses strategy 2. Similarly, y and $(1-y)$ denote the fraction of time that player B uses methods 1 and 2, respectively.

Example 5: (Two-person zero-sum game without saddle point): The two armies are at war. Army A has two airbases, one of which is thrice as valuable as the other. Army B can destroy an undefended airbase, but it can destroy only one of them. Army A can also defend only one of them. Find the best strategy for A to minimize its losses.

Solution: Since both armies have only two possible courses of action, the gain matrix for army A is

			Army B	
			1	2
			Attack the smaller airbase	Attack the larger airbase
Army A	Defend smaller airbase	1	0 Both survive	-3 The larger one destroyed
	Defend larger airbase	2	-1 The smaller one destroyed	0 Both survived

There is no saddle point. Under this method, army A wants to divide its plays between the two rows so that the expected winnings by playing the first row are exactly equal to the expected winnings by playing the second row irrespective of what army B does. In order to arrive at the optimum strategies for Army A, it is necessary to equate its expected winnings when army B plays column 1 to its expected winnings when army B plays column 2.

$i.e., when\ 0x + (-1)(1-x) = -3x + 0(\ (1\ -\ x)$

$or\ when\ -\ 1\ +\ x\ = -\ 3x\ i.e., 4x\ =\ 1 \therefore\ x\ =\ ¼$

Thus army A should play first row 1/4th of the time and second row 3/4th $(= 1 -\ x\ )$ of the time.

Similarly, army B wants to divide its time between columns 1 and 2 so that the expected winnings are same by playing each column, no matter what army A does. Optimum strategies for army B will be found by equating its expected winnings when army A plays row 1 to its expected winnings when army A plays row 2.

$i.e., when\ 0.y\ -\ 3(1\ -\ y)\ =\ -1.y\ +\ 0(1\ -\ y)$

$or\ when -\ 3\ +\ 3y\ =\ -\ y$

$or\ when\ 4y\ =\ 3\ or\ when\ y\ =\ 3/4$

Thus army B should play first column 3/4th of the time and second column 1/4th $(-1 -\ y)$ of the time. These optimum strategies can be shown on the gain-matrix, which becomes

		Army B		
		1	2	
Army A	1	0	-3	$1/4$
	2	-1	0	$3/4$
		$3/4$	$1/4$	

The game value can be found either for army A or for army B.

Game value for army A: While army B plays column 1, $3/4$ of time, army A wins zero for $1/4$ time and -1 for $3/4$ time, also while army B plays column 2 for $1/4$ of time, army A wins -3 for $1/4$ time and zero for $3/4$ time.

∴Total expected winnings for army A are

$$game\ value\ = \frac{3}{4}\left(0\times\frac{1}{4}-1\times\frac{3}{4}\right)+\frac{1}{4}\left(-3\times\frac{1}{4}+0\times\frac{3}{4}\right)=-\frac{9}{16}-\frac{3}{16}$$
$$=-\frac{3}{4}$$

Game value for army B: While army A plays row 1, $1/4$ of time, army B wins zero for $3/4$ of time and 3 for $1/4$ of time, also while army A plays row 2, $3/4$ of time, army B wins -1 for $3/4$ of time and zero for $1/4$ of time.

∴Game value for army B

$$=\frac{1}{4}\left(0\times\frac{3}{4}-3\times\frac{1}{4}\right)+\frac{3}{4}\left(-1\times\frac{3}{4}+0\times\frac{1}{4}\right)=-\frac{3}{16}-\frac{9}{16}=-\frac{3}{4}$$

Thus the full solution of the game is

$$army\ A{:}\ army\left(\frac{1}{4},\frac{3}{4}\right), army\ B{:}\ \left(\frac{3}{4},\frac{1}{4}\right)\ game\ =\ -\ \frac{3}{4},$$

14.8 *n PERSONS ZERO-SUM GAMES:-*

These games are typically considered as two coalitions established by the n-persons involved. The characteristics of such a game are the values of various games played by every possible coalition pair. For example, players A, B, C, and D can create the following coalitions.

$A\ against\ B, C, D$;
$B\ against\ A, C, D$;
$C\ against\ A, B, D$;
$D\ against\ A, B, C$;
$A, B\ against\ C, D$;
$A, C\ against\ B. D$;
$A. D\ against\ B, C$.

If the value of the game for B, C, D coalition is V, then the value of the game for A is -V, since it is zero-sum game. Thus in a four-person zero-sum game there will be seven values or characteristics for the game, which are obtained from the seven different coalitions

Example 6: Find the values of the three-person zero-sum game in which player A has two choices X_1, X_2; player B has two choices Y_1, Y_2 and player C also has two choices Z_1, Z_2. The payoff matrix is shown in table below.

Choice	Payoff

A	B	C	A	B	C
X_1	Y_1	Z_1	3	2	−2
X_1	Y_1	Z_2	0	2	1
X_1	Y_2	Z_1	0	−1	4
X_1	Y_2	Z_2	1	3	−1
X_2	Y_1	Z_1	4	−1	0
X_2	Y_1	Z_2	1	1	3
X_2	Y_2	Z_1	1	0	2
X_2	Y_2	Z_2	0	2	1

Solution: There are three possible coalitions:
1. $A\ against\ B, C;$
2. $B\ against\ A, C,$
3. $C\ against\ A, B.$

We shall solve each of the resulting game.

1. $\boldsymbol{A\ against\ B, C.}$ The payoff matrix in $A's$ terms is shown in table below.

		B, C				
		Y_1, Z_1	Y_1, Z_2	Y_2, Z_1	Y_2, Z_2	
A	X_1	3	0	0	1	(0)
	X_2	4	−1	1	0	−1
		4	(0)	1	1	

The first step is to look for a saddle point. The game has a saddle point. Thus, we have the following solution for $A\ against\ B, C.$
$A's$ best strategy is tilde Y_1 ,
$B's\ and\ C's$ best combination of strategies is Y_1, Z_2 value of the game for A is zero, value of the game for B, C is zero.

2. $\boldsymbol{B\ against\ A, C}$**:** The payoff matrix in $B's$ terms is shown in table below.

		A, C			
		X_1, Z_1	X_1, Z_2	X_2, Z_1	X_2, Z_2
B	Y_1	2	2	−1	1
	Y_2	−1	3	0	2

The first step is to look for saddle point. In this game there is none. The next step is to reduce the game by the rules of dominance. Columns X_1, Z_2 and X_2, Z_2 are dominated and should, therefore, be deleted. The resulting reduced matrix is shown in this table below.

		A, C			
		X_1, Z_1	X_2, Z_1		
B	Y_1	2	−1	1	1/4
	Y_2	−1	0	3	3/4

1	3
1/4	3/4

Solving this 2×2 game by arithmetic method we get the following result: $B's$ best strategy is to play choice Y_1 with a frequency of $1/4$ and choice Y_2 with a frequency of $3/4$. $A's\ and\ C's$ best strategy is for C to play Z_1 and for A to play X_1 with frequency of $1/4$ and X_2 with a frequency of $3/4$

$$Value\ of\ the\ game\ for\ B\ = \frac{2/4 - 3/4}{1/4 + 3/4} = -\frac{1}{4}$$

$$value\ of\ the\ game\ for\ A, C\ =\ 1/4$$

3. **C against A, B:** The payoff matrix in $C's$ terms is shown in table below.

C		A,B: X_1, Y_1	X_1, Y_2	X_2, Y_1	X_2, Y_2
	Z_1	-2	4	0	2
	Z_2	1	-1	3	1

The first step is to look for saddle point. In this case there is none. The next step is to reduce the game by the rules of dominance. Columns X_2, Y_1 andX_2, Y_2 are dominated by column X_1 and the resulting reduced matrix is shown in table below.

C		A,B: X_1, Y_1	X_1, Y_2		
	Z_1	-2	4	2	2/8
	Z_2	1	-1	6	6/8
		5	3		
		5/8	3/8		

Solving it by the arithmetic method we get the following results:
$C's$ best strategy is to play choice Z_1 with a frequency of $2/8$ and choice Z_2 with a frequency of $6/8$. $A's\ and\ B's$ best strategy is for A to play X_1 and for B to play Y_1 with a frequency of $5/8$ and Y_2 with a frequency of $3/8$.

$$Value\ of\ the\ game\ for\ C\ is = \frac{-10/8 + 12/8}{5/8 + 3/8} = \frac{2/8}{1} = \frac{1}{4}$$

$$value\ of\ the\ game\ for\ A, B\ =\ -1/4$$

Therefore, the characteristics of the game are

$$V(A) = 0, V(B, C) = 0,$$
$$V(B) = -\frac{1}{4}, V(A, C) = \frac{1}{4},$$

$$V(C) = \frac{1}{4}, V(A,B) = -\frac{1}{4}$$

14.9 *LIMITATIONS OF GAME THEORY: -*

Our game theory discussion has only covered two- and n-person zero-sum games. There are relatively few applications of game theory to real-world situations. This is due to the following factors:

1. Management choices are rarely made in a two-person environment; the government or society is frequently an outside party participating in decision-making.
2. Nonzero-sum games occur when the sum of the opponents' gains and losses is not zero.
3. In actual life, it is rare for both parties to have equal information and intelligence.
4. It is difficult to precisely calculate the payout matrix's values. Inaccurate values in the matrix will produce misleading results. It is not difficult to prove that one outcome is superior to the other, but it is much more difficult to determine how much superior.
5. The game's solution is based on maximin or minimax principles, which require players to select strategies that maximize the minimal gains or minimize the highest losses. In actual life, managers may not take such a cautious attitude and instead choose to take risks. Furthermore, data about available strategies and payoffs may be incomplete and uncertain.
6. In the real world, the chosen strategy is usually continued for a sufficiently long period of time, which equates to long-term planning, and for short durations, this method may be incorrect.

Game theory has not yet realized its full potential. It may become increasingly popular for solving O.R. marketing difficulties as more companies use computers to model their operations. The combination of game theory and simulation for the solution of management marketing problems is likely to provide game theory the boost it needs to become a significant instrument for quantitative decision-making.

SELF CHECK QUESTIONS

1. What is Game Theory?
2. Define a **two-person zero-sum game**.
3. What are the **characteristics of a game**?
4. What do you mean by a **payoff matrix**?
5. What is a **pure strategy**?
6. Define a **mixed strategy**.
7. What is a **saddle point** in a game?
8. How is the **value of a game** determined?

9. What is meant by **maximin** and **minimax** principles?
10. What is a **dominant strategy**?
11. Explain the difference between **pure** and **mixed** strategies.
12. What are **n-person games**, and how do they differ from two-person games?
13. What are the **rules** for solving a game without a saddle point?

14.10 *SUMMARY*:-

In this unit, we have studied the following topics: game characteristics, terminology, game theory rules, mixed strategies, n-person zero-sum games, and game theory limitations. Game theory is a mathematical framework for analyzing situations involving conflict and cooperation among rational decision-makers known as players. It aids in establishing optimal strategies in situations where the outcome of one player is dependent on the strategies used by others. We investigated game characteristics such as the number of participants, strategy types, and payout structure; game theory terminology; and strategic decision-making procedures.The notion of mixed strategies was created to deal with scenarios in which participants randomly select their options in order to attain the best potential results. We also investigated n-person zero-sum games, which extend two-player competitive scenarios to numerous players and have the total wins and losses of all participants sum to zero. Finally, the constraints of game theory were explored, emphasizing how real-world scenarios frequently contain imperfect knowledge, irrational conduct, and shifting preferences, making practical application difficult.

14.11 *GLOSSARY*: -

- **Game Theory:** A branch of applied mathematics and operations research that studies strategic interactions between rational decision-makers (players).
- **Player:** An individual or decision-maker involved in the game who selects strategies to achieve the best possible outcome.
- **Strategy:** A complete plan of action a player follows during the game to achieve the desired payoff.
- **Pure Strategy:** A strategy in which a player consistently chooses the same action every time the game is played.
- **Mixed Strategy:** A strategy in which a player randomly selects among two or more pure strategies based on specific probabilities.

- **Payoff:** The reward or outcome received by a player as a result of the combination of strategies chosen by all players.
- **Payoff Matrix:** A table that shows the payoffs for each player for all possible combinations of strategies.
- **Zero-Sum Game:** A game in which one player's gain is exactly equal to the other player's loss, so the total payoff remains constant (sum equals zero).
- **Non-Zero-Sum Game:** A game in which the total payoff to all players is not necessarily zero; both players can gain or lose simultaneously.
- **Two-Person Game:** A game involving exactly two players competing against each other.
- **n-Person Game:** A game involving more than two players, where strategies and payoffs depend on the collective actions of all participants.
- **Saddle Point:** A position in the payoff matrix that represents the equilibrium point where both players' strategies are optimal, giving a stable solution to the game.
- **Value of the Game:** The expected payoff to a player when both players follow their optimal strategies in a zero-sum game.
- **Dominance:** A rule used to simplify a game by eliminating strategies that are less effective compared to others for a player.
- **Equilibrium Point (Nash Equilibrium):** A situation where no player can improve their payoff by unilaterally changing their strategy, assuming the other players keep theirs unchanged.
- **Cooperative Game:** A game where players can form coalitions and make binding agreements to maximize their collective payoff.
- **Non-Cooperative Game:** A game in which players make independent decisions without forming alliances or agreements.
- **Limitations of Game Theory:** Refers to the practical difficulties in applying game theory due to assumptions of perfect rationality, complete information, and static preferences, which may not hold true in real-life situations.

14.12 *REFERENCES*: -

- Taha, H. A. (2017). Operations Research: An Introduction (10th Edition). Pearson Education.

- Kanti Swarup, Gupta, P. K., & Man Mohan. (2014). Operations Research. Sultan Chand & Sons.
- Hillier, F. S., & Lieberman, G. J. (2021). Introduction to Operations Research (11th Edition). McGraw-Hill Education.

14.13 *SUGGESTED READING*: -

- Dr. R.K.Gupta (2nd Eddition, 2012), Krishna Publication, Operation Research.
- Er. Prem Kumar Gupta and Dr. D.S. Hira (7th Edition,2014), S.Chand & Company PVT. LTD., Operations Research.

14.14 *TERMINAL QUESTIONS: -*

(TQ-1) For what value of λ, the game with following payoff matrix is strictly determinable?

		Player B		
		B_1	B_2	B_3
Player A	A_1	λ	6	2
	A_2	-1	λ	-7
	A_3	-2	4	λ

(TQ-2) Reduce the following dominance and find the game value:

		Player B			
		I	II	III	IV
Player A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

(TQ-3) For any 2×2 two-person zero-sum game without any saddle point, having payoff matrix for player A as

		player B	
		B_1	B_2
playerA	A_1	a_{11}	a_{12}
	A_2	a_{21}	a_{22}

Find the optimal mixed strategies and value of the game.

(TQ-4) Define game theory and explain rules for game theory.

(TQ-5) Find the ranges of values of p and q which will render the entry (2,2) a saddle point for the game

		Player B		
		B_1	B_2	B_3
Player A	A_1	2	4	5
	A_2	10	7	q
	A_3	4	p	6

(TQ-6) Reduce the following game by dominance property and solve it:

		Player B				
		I	II	III	IV	V
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

(TQ-7) Solve the following game by using the principle of dominance:

		Player B					
		I	II	III	IV	V	VI
Player A	1	4	2	0	2	1	1
	2	4	3	1	3	2	2
	3	4	3	7	-5	1	2
	4	4	3	4	-1	2	2
	5	4	3	3	-2	2	2

(TQ-8) Solve the following game:

		Player B	
		B_1	B_2
Player A	A_1	30	2
	A_2	4	14
	A_3	6	9

(TQ-9) Define n person zero sum game.

(TQ-10) A and B play a game in which each has three coins $a\ 5p, a\ 10p\ and\ a\ 20p$. Each player selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B' s coin; if the sum is even, B wins A's coin. Find the best strategy for each player and the value of the game.

14.15 *ANSWERS: -*

(TQ-1) $-1 \le \lambda \le 2$

(TQ-2) $game\ value\ for\ A = 8/3$

(TQ-3)

$$value = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

(TQ-5) $p \leq 7, q \geq 7$

(TQ-6) optimal strategies for $A: III$, optimal strategy for $B: II$, game value for$A = 5$

(TQ-7) game value$13/7$.

(TQ-8) game value$206/19$.

(TQ-10) optimal strategies for $A: (1/2,1/2,0)$, optimal strategy for $B: (2/3,1/3,0)$, game value for$A = 0$

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University,
Haldwani, Nainital-263139
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,

E-mail:info@uou.ac.in
Website: https://www.uou.ac.in/