NUMERICAL ANALYSIS MT(N)-221

BECHELOR OF SCIENCE
(FOURTH SEMESTER)

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: NUMERICAL ANALYSIS

COURSE CODE: MT(N) 221

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India,
263139

Editor

Dr. Arvind Bhatt
Professor
Department of Mathematics
Uttarakhand Open University
Haldwani, Uttarakhand

Unit Writer	Block	Unit
Dr. Deepa Bisht Department of Mathematics Uttarakhand Open University Haldwani	**I- IV**	**1-14**

Course Title and Code : Numerical Analysis (MT(N) 221)
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Edition : 2025
Published By : Uttarakhand Open University, Haldwani, Nainital- 263139

CONTENTS

NUMERICAL ANALYSIS
MT(N) - 221

BLOCK I: INTERPOLATION

UNIT 1:
NUMERICAL METHOD FOR SOLVING ALZEBRAIC AND TRANSCENDENTAL EQUATIONS

CONTENTS:

1.1 INTRODUCTION

Numerical analysis is a branch of mathematics that solve continuous problem using numerical approximation. Numerical methods for solving algebraic and transcendental equations are computational techniques which is used to find approximate solutions that cannot be solved analytically. These methods are essential in various fields, such as physics, engineering, economics, and computer science, where equations are used to model real-world problems. These equations are essential for solving complex problems in various fields.

In this unit we defined algebraic and transcendental equations, local and global transaction error. We also explain about methods for finding the initial value of the roots like, graphical methods, bisection method, iteration method etc. and examples based on these methods.

1.2 OBJECTIVES

After studying this unit, the learner will be able to

1. Defined algebraic and transcendental equation.
2. Describe approximate solutions to algebraic and transcendental equations, which may not have exact analytical solutions.
3. Explain and solve real world problem in various fields, such as physics, engineering, economics and biology.

1.3 ALZEBRAIC OR TANSCENDENTAL EQUATIONS

An equation f (x) = 0 is called polynomial or algebraic or transcendental equation of degree n if the function is purely a polynomial in x or contains some other functions such as logarithmic, exponential and trigonometric function etc., e.g.
1+ cos x- 5x, x tan x- cos h x, e^{-x} -sinx etc.

A polynomial in x of degree n is an expression of the form

f (x)= a_0x^n +a_1x^{n-1} +....................a_{n-1} x + a_n , where a's are constant and n is positive integer. The zeros or the roots of the polynomial f(x) are those values of x for which f(x) is zero. Geometrically if the graph of f(x) crosses the x-axis at the point x = a then x=a is the root of the equation f(x) =0. We conclude that a is a root of the equation f(x) =0 if and only if f (a) =0.

By finding the solution of an equation f(x)=0, we mean to find zeros of f (x).

If f(x) is an algebraic polynomial of degree less than or equal to 4, direct methods for finding the roots of such equation are available. But if f(x) of higher degree or it involves transcendental functions, direct methods do not exist and we need to apply numerical methods to find the roots of the equation f(x) =0

In this chapter we shall discuss some numerical methods for the solution of equation of the form f(x) =0. Here f(x) may be or algebraic or transcendental or a combination of both. Before finding the solution of algebraic and transcendental equation we give some properties of algebraic equation and transcendental equations we give some properties of algebraic equation which help for locating the roots.

(i) An algebraic equation of degree n where n is a positive integer, has n and only n roots.

(ii) a + i b is a root of f(x)=0, then a- i b is also a root of the equation i.e., complex roots occur in conjugate pairs.

(iii) **Descartes's rule of signs.** In an algebraic equation f(x) = 0 with real coefficients the number of positive roots cannot exceed the number of changes of signs from positive to negative and from negative to positive in f(x). Also, the number of negative roots in f(x) =0 cannot exceed the number of variations in f(-x).

(iv) If the value of f(a) and f(b) are of opposite signs, when we substitute two real quantities a and b for x in any polynomial f(x) then at least one or an odd number of real roots of the equation f(x)=0 lie between a and b. If f(a)and f(b) are of the same sign, then either there is no real root or an even number of real roots of f(x)=0 lie between a and b.

(v) Every equation of odd degree has at at least one real root. Every equation of an even degree with last term negative has at least two real roots, one positive and other negative.

(vi) The largest root of the polynomial equation $a_0x^n + a_1x^{n-1} +$....................a_{n-1} x + a_n = 0, $a_n \neq 0$ may be approximated by the root of the equation a_0 x +a_1 x +a_2=0 which is larger in absolute value. Similarly, the smallest root can be obtained approximately by the root of the equation a_{n-1} x + a_n =0 or by that root of $a_{n-2}\,x^2$+a_{n-1} x +a_n =0, which is the smallest value.

(vii) If x=a is a root of the equation f(x) =0 then f(x) is exactly divisible by (x-a) and conversely, if f(x) is exactly divisible by (x-a), a is a root of the equation f(x) = 0.

13.1 Synthetic Division: To find the quotient and the remainder when a polynomial is divided by a binomial.

Let f(x) = $a_0x^n + a_1x^{n-1} +$....................a_{n-1} x + a_n be the polynomial o fn[th] degree

and let it be divided by the binomial x-a. if Q=$b_0x^{n-1} + b_1\,x^{n-2}$+……….$b_{n-2}$x + b_{n-1} be

the quotient and R the remainder, then the coefficient of Q and R can be

Exhibited in the following manner:

a	a_0 a_n	a_1	a_2	a_3	…………		A_{n-1}
		ab_0	ab_1	ab_2	....	ab_{n-2}	ab_{n-1}
R	b_0	b_1	b_2	b_3	…………		b_{n-1}

1.3 (2) Rule for synthetic division: First we make the polynomial f(x) complete (if it is not so) by supplying the missing term with zero coefficients.

In the first row we write the successive coefficients a_0, a_1, a_2 ………a_n of the polynomial f(x).

If we are to divide the polynomial by x-a we write a in the corner as shown above.

In the third row write b_0 below a_0 (note $b_0=a_0$). The first term in the second row is obtained by multiplying b_0 (a_0) by a. the product ab_0 is written under a_1. Adding ab_0 to a_1, we get b_1, which is the second term in the third row. Multiplying b_1 by a product ab_1 is written under a_2. Adding ab_1 to a_2, we get b_2, which is the third term in the third row. We continue this process. The last term the third row is the value of the remainder R while the last but one term in the third row is the value of b_{n-1}.

Note: In case R=0 we can say that a is a root of the equation f(x) =0 and the equation f(x) =0 can be depressed by one dimension. In case R is not equal to zero, we get R= f (a) by reminder theorem.

1.4 RATE OF CONVERGENCE OF NEWTON'S METHOD

Let a be the exact value of the root. Suppose x_n differs from a by the small quantity ε_n.

Then $x_n = a + \varepsilon_n$ => $x_{n+1} = a + \varepsilon_{n+1}$

By Newton- Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e., $$a + \varepsilon_{n+1} = a + \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$$

or $$\varepsilon_{n+1} = \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)} \quad \text{... (1)}$$

by taylor's theorem, we have

$$f(a+\varepsilon_n) = f(a) + \varepsilon_n f'(a) + \frac{1}{2!}\varepsilon_n^2 f''(a) + \ldots\ldots$$

$$= \varepsilon_n f'(a) + \frac{1}{2!}\varepsilon_n^2 f''(a) + \ldots\ldots \qquad [\text{since } f(a)=0]$$

and $f'(a+\varepsilon_n) = f'(a) + \varepsilon_n f''(a) + \ldots\ldots$

Substituting this value in (1), we get

$$\varepsilon_{n+1} = \varepsilon_n - \varepsilon_n\left[\frac{f'(a) + \frac{1}{2!}\varepsilon_n f''(a) + \cdots\ldots}{f'(a) + \varepsilon_n f''(a) + \cdots\ldots}\right]$$

$$= \frac{\varepsilon_n^2 f''(a)}{2[f'(a)+\varepsilon_n f''(a)]} = \frac{\varepsilon_n^2 f''(a)}{2 f'(a)\left[1+\varepsilon_n \frac{f''(a)}{f'(a)}\right]}$$

$$= \frac{\varepsilon_n^2 f''(a)}{2 f'(a)}\left[1+\varepsilon_n \frac{f''(a)}{f'(a)}\right]^{-1} = \frac{\varepsilon_n^2 f''(a)}{2 f'(a)} \qquad \ldots (2)$$

From (2) we observe that the subsequent error is proportional to the square of the previous error so that the Newton – Raphson method has a second – order or quadratic convergence.

Rate of Convergence of Newton's Method When There Exist Double Roots

Proceeding as earlier, we get

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$$

$$= \varepsilon_n - \frac{f(a)+\varepsilon_n f'(a) + \frac{1}{2!}\varepsilon_n^2 f''(a) + \frac{1}{3!}\varepsilon_n^3 f'''(a) + \cdots}{f'(a)+\varepsilon_n f''(a) + \frac{1}{2!}\varepsilon_n^2 f'''(a) + \frac{1}{3!}\varepsilon_n^3 f^{iv}(a) + \cdots}$$

$$= \varepsilon_n - \frac{\frac{1}{2!}\varepsilon_n^2 f''(a) + \frac{1}{3!}\varepsilon_n^3 f'''(a) + \cdots}{\varepsilon_n f''(a) + \frac{1}{2!}\varepsilon_n^2 f'''(a) + \cdots} \quad \text{[Since } f(a)=0=f'(a)\text{]}$$

$$= \varepsilon_n - \varepsilon_n \frac{\frac{1}{2} f''(a) + \frac{1}{6}\varepsilon_n f'''(a)}{f''(a) + \frac{1}{2}\varepsilon_n f'''(a)}, \quad \text{neglecting higher powers}$$

$$= \varepsilon_n \frac{\frac{1}{2} f''(a) + \frac{1}{3}\varepsilon_n f'''(a)}{f''(a) + \frac{1}{2}\varepsilon_n f'''(a)} = \varepsilon_n.\frac{1}{2}\frac{f''(a)}{f''(a)}$$

$$= \frac{1}{2}\varepsilon_n = \frac{2-1}{2}\varepsilon_n.$$

This implies that the convergence is linear. In general, we can prove that if x= a is a root of multiplicity m, where m> 1 then the speed of convergence is given by

$\varepsilon_{n+1} \cong \frac{m-1}{m}\varepsilon_n$, which is also linear.

1.5 Method For Finding The Initial Approximate Value Of The Root

To find the real root of a numerical equation by any method except that of Graeffe, it is necessary first to find an approximate value of the root by any method. The general technique is that we start with an approximate value say x_0 and then find the better approximations $x_1, x_2, \ldots\ldots x_n$ successive approximations approach the root and more closely, then we say that the method converges.

(i) **Graphical Method:** Suppose we are to find the roots of the equation f(x) =0. Taking a set of rectangular coordinates axes we plot the graph of y=f(x). Then the real root of the given equation is the abscissa of the points where the graph crosses the x- axis. Obviously, at these points y is zero and so the equation f(x) =0 is satisfied. Hence from the graph of the given equation, approximate values for the real roots of the equation can be found. Sometime when f(x) involves difference of two functions, the approximate values of the real roots of f(x) =0 can be found by writing the equation in the form $f_1(x) = f_2(x)$ where $f_1(x)$ and $f_2(x)$ are both function of x. then we plot the two equation $y_1= f_1(x)$ and $y_2= f_2(x)$ on the same axes. The real roots of the given equation are the abscissa of the points of intersection of these two curves because at these points $y_1= y_2$ and so $f_1(x) = f_2(x)$.

(ii) **Bisection method:** we know that if a function f(x) is continuous between a and b and f(a) and f(b) are of opposite signs, then there exists at least one root between a and b. let f(a) be negative and f(b) be positive. Also let the approximate value of the root be given by $x_0 = (a+b)/2$. Now if $f(x_0) = 0$ then it ensures that x_0 is a root of the equation f(x) =0. If $f(x) \neq 0$ then the root either lies between x_0 and b or between x_0 and a. it depends on whether $f(x_0)$ is negative of opposite. Then again, we bisect the interval and repeat the process until the root is obtained to the desired accuracy.

(iii) **The method of false position (regular- falsi method):**

It is the oldest method for computing the real root of a numerical equation f(x) =0. It is closely similar to the bisection method.

In this method we find sufficient small interval (x_1, x_2) in which the root lies. Since the root lies between x_1 and x_2, the graph of y=f(x) must cross the x axes between $x= x_1$ and $x=x_2$ and hence y_1 and y_2 must be of opposite signs.

This method is based upon the principal that any portion of a smooth curve is practically straight for a short distance. Hence, we assume that the graph of y = f (x) is a straight line between the point (x_1, y_1) and (x_2, y_2). The points are on opposite site on x –axes.

The x- coordinate of the point of intersection of the straight line joining (x_1, y_1) and (x_2, y_2) and the axis of x gives an approximate value of the desirable root. The figure 1 represents a magnified view of the part of the graph which lies between (x_1, y_1) and (x_2,y_2).

Now from the similar triangle PAR and PSQ, *we have*

$$\frac{AR}{AP} = \frac{SQ}{SP} \quad or \quad \frac{h}{|y1|} = \frac{x2-x1}{|y1|+|y2|} \quad => \quad h = \frac{(x2-x1)\,|y1|}{|y1|+|y2|}$$

Hence the approximate value of the desired root.

$$= x_1 + AR = x_1 + h = x_1 + \frac{(x2-x1)\,|y1|}{|y1|+|y2|}$$

Let this value be denoted by x^1. Then to find the better approximation,

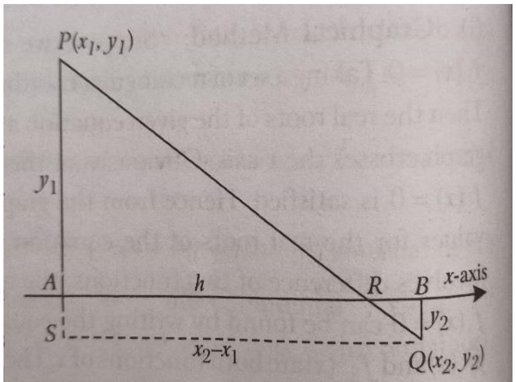


Fig. 1

we find $y^{(1)}$ by $y^{(1)} = f(x^{(1)})$. Now either $y^{(1)}$ and y_1 and $y^{(1)}$ and y_2 will be of opposite signs. If $y^{(1)}$ and y_1 are of opposite signs then one root lies in the interval $(x_1, x^{(1)})$. We again apply the method of false position to this interval and get the second approximation.

If $y^{(1)}$ and y_2 are of opposite signs, then second approximation can be obtained by using the method of false position to the interval $(x^{(1)}, x_2)$. Continuing this process, we can obtain the root to the desired degree of accuracy.

(iv) The secent method: This method is similar to that of regula-falsi method except the condition that $f(x_1).\ f(x_2) < 0$. In this the graph of the function y=f(x) in the neighborhood of the root is approximated by a chord line. Here it is not necessary that the interval at each iteration should contain the root. let x_1 and x_2 be the limit of interval initially then the first approximation is given by

$$X_3 = \frac{x2+\ (x2-x1)f(x2)}{f(x1)-f(x2)}$$

The formula for successive approximation in general form is given by

$$X_{n+1} = x_n + \frac{(xn-xn-1)f(xn)}{\{f(xn-1)-f(xn)\}} \quad , n \geq 2$$

If at any iteration we have $f(x_n) = f(x_{n-1})$ then the secant method fails. Hence the method does not converge always while the Regula-falsi method converges surely. But if the secent method converges then it converges more rapidly than the Regula -falsi method.

(v) Iteration method: In this method for finding the root of the equation f(x)=0 it is expressed in the form $x = \emptyset(x)$ let x_0 be an initial approximation to the solution of $x = \emptyset(x)$

Let x_0 be an initial approximation to the solution of $x = \emptyset(x)$. Substituting it in $\emptyset(x)$ the next approximation x_1 is given by $x_1 = \emptyset(x_0)$. Again, substituting $x=x_1$ in $\emptyset(x)$, we get the next approximation as $x_2 = \emptyset(x_1)$.

Continuing in this way, we get

$X_n = \emptyset(x_{n-1})$ or $X_{n+1} = \emptyset(xn)$

Thus, we get a sequence of successive approximations which may converge to the desired root. For using this method, the |equation f(x)=0 can be put as $x = \emptyset(x)$ in many different ways.

Let $f(x)= x^2-x-1 =0$. It can be written as

(i) $x= x^2-1$,
(ii) $x^2= x+1$ or $x=\sqrt{x+1}$
(iii) $x^2= x+1$ or $x=1+\left(\frac{1}{x}\right)$,
(iv) $x=x-(x^2-x-1)$ or $x = 2x - x^2 + 1$

in order to find the root of f (x)=0, i.e, $x = \emptyset(x)$, we are to find the abscissa of the point of intersection of the line y=x and the curve $y= \emptyset(x)$. These two curves do not intersect, then the equation of f (x)=0 has no real root.

Note: The iteration method is convergent conditionally and the condition is that $|\emptyset'(x)| < 1$ in the neighborhood of the real root x=a

(v) Newton Raphson Method

Let X_0 denote an approximate value of the desired root of the equation f(x)=0 and let h be the correction which must be applied to x_0 to get the exact value of the root. Then $x_0 + h$ is a root of the equation f(x)=0, so that $f(x_0+h) =0$ ...(1)

Expanding $f(x_0+h)$ by Taylor's theorem, we get

$$F(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots\ldots\ldots = 0.$$

Now if h is sufficiently small, we may neglect the terms containing second and higher power of h and get simple relation $f(x_0) + h f'(x_0) = 0$.

This gives $h = - f(x_0)/f'(x_0)$, provided f'(x) provided $f'(x) \neq 0$. The improved value of the roots is

$$X_1 = x_0 + h = x_0 - f(x_0)/f'(x_0) \qquad ...(2)$$

Successive approximation are given by $x_2, x_3, \ldots\ldots\ldots\ldots, x_{n+1}$, where

Successive approximation are given by $x_2, x_3, \ldots\ldots\ldots\ldots x_{n+1}$, where

$X_{n+1} = x_n - f(x_n)/f'(x_n)$ …(3)

Formula (3) known as Newton – Raphson formula.

In this method we have assumed that h is a small quantity which is so if the derivative f'(x) is large. In other words, the correct value the root can be obtained more rapidly and with very little labour when the graph is nearly vertical where it crosses the axis. If f'(x) is small in the neighborhood of the root then the value of h is large and by this method the computation of the root will be a slow process or might even fail together. Hence this method is not suitable in cases when the graph of f(x) is nearly horizontal where it crosses the x-axes. In such case the regula – falsi method should be used.

Geometric Significance of the Newton Raphson Method

A magnified view of the graph of y=f(x), where it crosses the axis is represent in Figure. 2.

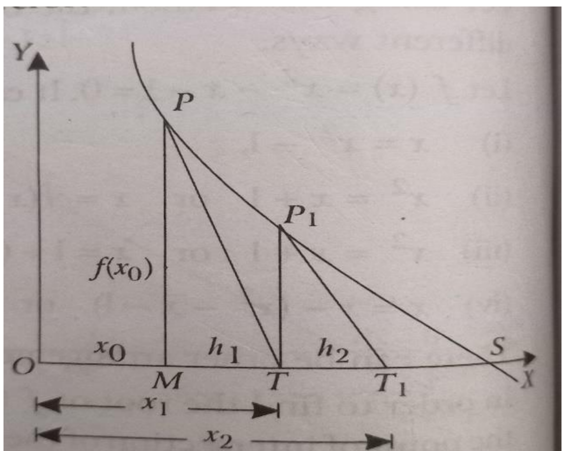


Fig. 2

Let us draw a tangent at a point P whose -coordinate is. It intersects the -axis at some point T. Then we draw a tangent at the point P_1 whose abscissa is OT. Suppose it meets the -axis at some point T_1which lies between T and S. Further we draw a tangent at the point P_2 whose abscissa is OT_1. This tangent intersects the -axis at a point T_2 which lies between T_1 and S. We continue this process. Let the curvature of the graph do not change sign between P and S. Then the points T, T_1, T_2, …. will approach the point S as a limit or in other words the intercepts OT, OT_1, OT_2 …. will tend to the intercept OS as a limit.

But OS denotes the real root of the equation f(x) =0 So OT, OT_1, OT_2 …. denote successive approximations to the desired root.

From this figure we derive the fundamental formula. Let MT = h_1 and TT_1 =h_2 etc.

We have PM = $f(x_0)$, slope at P = tan XTP = - f $(x_0)/h_1$

Also, the slop of the graph at P is $f'(x_0)$.

Thus, we get $f'(x_0)$ = - f $(x_0)/h_1$ => -f $(x_0)/f'(x_0)$

Similarly, we find from the P_1TT_1 that

h_2 = -f $(x_1)/f'(x_1)$

From this discussion we conclude that in this method we replace the graph of the given function by a tangent at each successive step in the approximation process. It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.

Newton's iterative formula for obtaining inverse, square root, cube root etc.

1**. Inverse**: The quantity a^{-1} can be considered as a root of the equation

Here $\left(\frac{1}{x}\right)$-a =0

So, f'(x) =- $\frac{1}{x^2}$

Hence by Newton's formula, we get the simple recursion formula

$X_{n+1} = x_n + \frac{\left(\frac{1}{x}\right)-a}{\frac{1}{{x_n}^2}}$

Or $X_{n+1} = x_n\,(2-ax_n)$.

2. Square root: The quantity can be considered as a root of the equation from this we get the recursion formula

$X_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

3. Inverse Square root: The inverse square root of a is the root of the equation

$\frac{1}{x^2}$-a =0. From this we get the iterative formula

$x_{n+1} = \frac{1}{2}x_n\,(3- a{x_n}^2)$.

ILLUSTRATIVE EXAMPLES

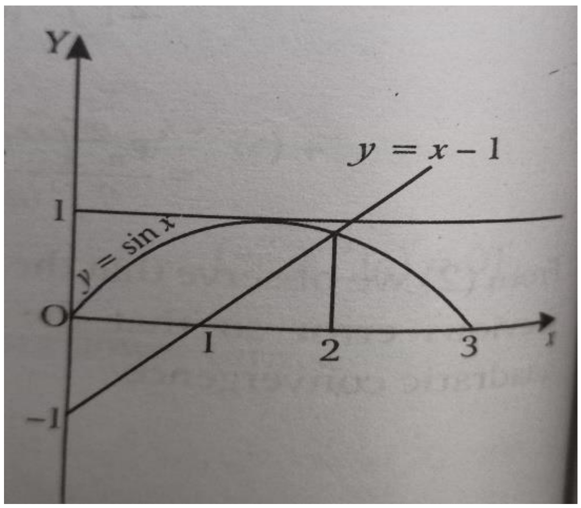
Example 1: Find the approximate value of the root of x - sinx -1=0

Solution: Since the L.H.S of the equation is the difference of two functions, we can put it in the form

X-1 = sin x.

Then we plot separately on the Same set of coordinate axes the two equations Y_1 = X-1 and y_2= sin x.

The abscissa of the point of intersection of the graphs of this equation is seen to be 1.9.



Example 2: Find the approximate value of the root of the equation

$3x - \sqrt{1+\sin x} = 0$

Solution: let f(x) = 3x- $\sqrt{1+\sin x}$. Then the given equation is f(x)=0

We have f (0) = 3 × 0- $\sqrt{1+\sin 0}$ = -1

And f (1) = 3 × 1- $\sqrt{1+\sin 1}$ = 3 – $(1+0.8147)^{\frac{1}{2}}$ = 3 - 1.3570 = 1.6430.

Since f (0) and f (1) are of opposite signs so at least one root of f(x) =0 lies in the interval (0.1).

Now f (.3) = 3 ×.3 - $\sqrt{1+\sin.3}$ = .9 - $\sqrt{1+.29552}$

And f (.4) = 3 × .4 - $\sqrt{1+\sin.4}$ =1.2 – $\sqrt{1.38912}$

= 1.2 – 1.1787 = 0.0213.

Since f (.3) and f (-4) are of opposite signs to at least one real root lies between x=.3 and x=.4

Further f (.38) = 3 × .38 - $\sqrt{\{1+\sin(.38)\}}$

= 1.14 – $\sqrt{\{1+.37092}$}

= 1.14 – 1. 1708 = - 0.0808

And f (.40) =0.0213

Hence at least one root lies between x= .38 and x=.40. so, x=.38 can be taken as an approximate value of the root which lies in the interval (.38, .40).

Example 3: The equation $x^6 - x^4 - x^3$ - 1 =0 has one root between 1.4 and 1.5. find the root to four decimals by false position method.

Solution: The successive approximations are given below:

	X	y	
First approx.	1.4	-.056064	$h_1 = \frac{.1 \times .056064}{2.009189}$ = .003
	1.5	1.953125	$x^{(1)}$ = 1.4 +.003 = 1.403
Diff.	.1	2.009189	
Second approx.	1.403	-0.167	$h_2 = \frac{.001 \times .0167}{0.223}$ = 0.0007
	1.404	0.0056	$x^{(2)}$ = 1.403 + 0.007 =1. 4037
	.001	0.223	
Third approx.	1.4036	-.0001	$h_3 = \frac{.0001 \times .0001}{0.0005}$ =0.22222
	1.4037	0.0004	$x^{(3)}$ =1.4036 +0.00002 = 1.4037
	0.0001	0.0005	

Hence the value of the root is 1.4036.

Example 4: Compute the real root of x$\log_{10} x$ -1.2 =0 Correct to five decimal places.

Solution: Let f(x) = x $\log_{10} x$ -1.2

We have f (2) = -0.60 and f (3) = 0.23.

It shows that the correct root lies between 2 and 3 and it is nearer to 3. The following table shows successive approximations where corrections are computed by

$$h = \frac{(x_2 - x_2)|y_1|}{|y_1| + |y_2|}$$

	X	y	
First approx.	2	-.056064	$h_1 = \frac{1 \times .06}{0.83} = 0.72$
	3	1.953125	$x^{(1)} = 2 + 0.72 = 2.72$
Diff.	1	2.009189	
Second approx.	2.7	-0.04	$h_2 = \frac{0.1 \times .0.04}{0.09} = 0.044$
	2.8	0.05	$x^{(2)} = 2.7 + 0.044 = 0.044$
	0.1	0.09	
Third approx.	2.74	-0.0006	$h_3 = \frac{.01 \times 0.0006}{0.0087} = 0.0007$
	2.75	+ 0.0081	$x^{(3)} = 2.74 + 0.0007 = 2.7407$
	0.01	0.0087	
Fourth approx.	2.7406	-0.000039	$h_4 = \frac{.0001 \times 0.000046}{0.000046}$
	2.7407	+0.000045	$= 0.000046$
	0.0001	0.000084	

$X^{(4)} = 2.7406 + 0.000046$

$= 2.74065.$

Hence the value of the root is 2.74065.

Example 5 : Find $\sqrt{12}$ to five places of decimal by Newton- Rapshson method.

Soluton: let x= $\sqrt{12}$ =$(12)^{\frac{1}{2}}$

Then x^2 = 12 or x^2 - 12 =0

Therefore, the square root of 12 is nothing but the solution of the equation

f(x) = $x^2 - 12 = 0$ ......(1)

we have f(x) = $x^2 - 12$ and f'(x) =2x.

also f (3) = $3^2 - 12$ = 9-12 = -3 which is –ve

and f (4) = $4^2 - 12$ = 16 -12 = 4 which is +ve.

So, a root of (1) lies between 3 and 4.

To shorten this interval we observe the signs of the values of f (3.1), f (3.2), f (3.3) etc.

We have f (3.4) = $(3.4)^2$ -12 =11.56 -12, which is –ve.

And f (3.5) = $(3.5)^2$ = 12.25 -12, which is +ve.

So, $\sqrt{12}$ i.e the root of (1) lies between 3.4 and 3.5.

We take x_0 = 3.4 and obtain successive approximations x_1, x_2, x_3 etc., using Newton - Raphson Formula

$$x_{n+1} = x_n - \frac{f(x)_n}{f'(x)_n}$$

$$x_n - \frac{x_n^2 - 12}{2x_n} = \frac{2x_n^2 - x_n^2 + 12}{2x_n}$$

$$\frac{x_n^2 + 12}{2x_n} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right).$$

Thus $x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right)$.

Taking n=0 we have from (2), the first approximation

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = \frac{1}{2}\left(3.4 + \frac{12}{3.4}\right)$$

$= \frac{1}{2}$ (3.4 +3.5294118) = 3.4647059.

Taking n=1 we have from (2), the second approximation

$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = \frac{1}{2}\left(3.4647059 + \frac{12}{3.4647059}\right)$

$= \frac{1}{2}$ (3.4647059 +3.4634974) = 3.4641017.

Taking n=2 we have from (3), the third approximation

$x_3 = \frac{1}{2}\left(x_2 + \frac{12}{x_2}\right) = \frac{1}{2}\left(3.4641017 + \frac{12}{3.4641017}\right)$

$= \frac{1}{2}$ (3.4641017 +3.4641015) = 3.4641016.

We see that up to five places of decimal, $x_2 = x_3$. So, we stop the calculation work here.

Hence, up to five places of decimals, $\sqrt{12}$ = 3.46410.

Example 6 : Find the approximate value for the real root of $x\log_{10}x$ -1.2 =0 correct to five decimal places by Newton Raphson method.

Solution: Here f(x) = $x\log_{10}x$ -12.

f'(x) = $\log_{10}x$ +x. (1/x). $\log_{10}e$ =$\log_{10}x$ + .43429.

we have $x_{n+1} = x_n - \frac{f(x)_n}{f'(x)_n}$,n=0,1,2……….

$$x_{n+1} = x_n - \frac{x_n log_{10}(x_n) - 1.2}{log_{10}(x_n) + .43429} = \frac{43429\ \text{xn} + 1.2}{log_{10}(x_n) + .43429}$$

Taking initial value x_0 = we get

$x_1 = \frac{.43429\ (2) + 1.2}{log_{10}2 + .43429}$ =2.81

$x_2 = \frac{.43429\ (2.81) + 1.2}{log_{10}2.81 + .43429}$ =2.741

$x_3 = \frac{.43429\ (2.741) + 1.2}{log_{10}2.741 + .43429}$ =2.7406

$x_4 = \frac{.43429\ (2.7406) + 1.2}{log_{10}2.7406 + .43429}$ =2.74065

$x_5 = \frac{.43429\ (2.74065) + 1.2}{log_{10}2.74065 + .43429}$ = 2.74065.

Hence the required value of the root is 2.74065.

Example 7: Find the root of the equation 2x= cos x +3 correct to three decimal places by using iteration method.

Solution: The given equation can be put in the form

X= $\frac{1}{2}$ (cos x+3)

Here $\phi(x) = \frac{1}{2}$ (cos x+3)

$\phi'(x) = \frac{1}{2}$ (- sinx).

We have | $\phi'(x)$ | = $\left|\frac{\sin x}{2}\right| <$

Hence the iterative method is applicable and starting with $x_0 = \frac{\pi}{2}$, we obtained successive approximations as follows:

$X_1 = 1.5$, $x_2 = 1.535$, $x_3 = 1.518$, $x_4 = 1.526$,

$X_{5=}$ 1.522, x_6= 1.524, x_7 1.523, x_8= 1.524.

Hence, we take the value of the root as 1.524 correct to three decimal places.

Example 8: Use synthetic division to solve f(x) =x^3 - x^2- 1.0001x +0.999 =0, in the neighborhood of x=1.

Solution: First we shall find f (1) and f' (1) by synthetic division.

1	1	-1	-1.0001	0.9999
		1	0	-1.0001
1	1	0	-1.0001	-0.0001 = f (1)
		1	1	
	1	1	-0.0001 =f' (1)	
1		1		
	1		2=$\frac{1}{2}$ f'' (1)	

We observed that f (1) and f' (1) are very small. Hence two nearly equal roots exist. Taking x_0 =1, we shall use the formula $x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$ to modify the root.

So, first approximation

$X_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{f'(1)}{f''(1)}$

$1-\frac{(-0.0001)}{4}$ = 1.000025 =c, say.

Again, we have $x = c \pm \sqrt{\left\{\frac{-2f(c)}{f''(c)}\right\}} = 1.000025 \pm \sqrt{\left\{\frac{-2f(1.000025)}{f''(1.000025)}\right\}}$ … (1)

Now we calculate f (1.000025) and f'' (1.000025) by synthetic division.

1.000025	1	-1	-1.0001	0.9999
		1.000025	0.000025	-1.0001
1.000025	1	0.00 0025	-1.000075	-0.0002 = f (1.000025)
		1.000025	1.000075	
1.000025	1	1.000050	0 =f' (1.000025)	
		1.000025		
	1	2.000075 = f''(1.000025)		

Putting the values f (1.000025) =-0.0002 and f'' (1.000025) =2× 2.000075 in (1), we get

$X = 1.000025 \pm \sqrt{\left(\frac{.000200}{2.000075}\right)} = 1.000025 \pm 0.009$

= 1.009025, 0.991025.

Example 9: Solve x^3 -8x^2 +17x -10= 0 by Graeff's method, squaring three times.

Solution: The given equation is

x^3 - $8x^2$ +17x -10= 0 ….(1)

Let the roots of (1) be p_1, $p_{2,}$ p_3

The equation (1) can be written as

x^3 +17x = $8x^2 + 10$ or x (x^2 +17) = $8x^2$ +10.

Squaring, we get $x^2(x^2 +17)^2 = (8x^2 +10)^2$.

Putting x^2 = y, we have

$y (y +17)^2 = (8y +10)^2$

$y^3 – 30y^2 +129y -100 =0$ ... (2)

The root of (2) are the square of the roots of the given equation ((1).

Again from (2) bringing terms contain odd powers of y on one side and even power of y on the other side, we get

y (y^2 +129) = 30 y^2 +100.

Squaring, we get

$Y^2(y^2 +129)^2 = (30 y^2 +100)^2$.

Putting y^2 =z, we have

$Z (z +129)^2 = (30 z +100)^2$

Or z^3 -642 z^2+10641z -10^4=0 … (3)

From (3), z (z^2 + 10641) = 642 z^2+10000

Or $z^2 (z^2 +10641)^2 = (642 z^2+10000)^2$

Or $t (t+10641)^2 = (642t +10000)^2$ [replacing z^2 by t]

t^3- 390882t^2 +100390881t -10^8 =0

The root of (4) and 8^{th} power of the roots of the given equation (1).

Let the roots of (4) be $q_{1,}$ q_2, q_3.

Then $q_1 = p_1^8 = |p_1|^8$, $q_2 = |p_2|^8$, $q_3 = |p_3|^8$.

From equation (4), we have

$q_1 \approx 390882$; $q_1q_2 \approx 100390881$; $q_1q_2q_3 \approx 10^8$.

So $|p_1|^8 = q_1 \approx 390882$

$|p_2|^8 = q_2 \approx \frac{100390881}{390882}$

$|p_3|^8 = q_2 \approx \frac{10^8}{100390881}$

Taking square root times, we get

$|p_1| = (390882)^{\frac{1}{8}} = 5.00041$

$|p_2| = (100390881/390882)^{\frac{1}{8}} = 2.00081$

$|p_3| = (10^8/100390881)^{1/8} = 0.999512.$

Now $|p_1| = 5.00041 \Rightarrow p_1 = \pm 5.00041.$

By actually substituting in (1), we see that $p_1 = 5.00041$.

Satisfies (1) while $p_1 = -5.00041$. does not satisfy (1). So, $p_1 = 5.00041$.

Similarly, we see that the admissible values of p_2, p_3 are

$P_2 = 2.00081$, $p_3 = 0.999512$.

Hence, the required roots of the given equation (1) are

5.00041, 2.00081, 0.999512.

We see that the exact values of the roots of (1) are 5,2,1. The approximate values of the roots given by Graaff's root square method are sufficiently close to them.

CHECK YOUR PROGRESS

True or false questions

1. If f(x) is exactly divisible by (x-a), a is root of the equation f(x) = 0.
2. An equation of odd degree has no root.
3. Newton- Raphson method cannot be used when the roots are complex.
4. In an algebraic equation f(x) =0 with real coefficients the number of positive roots cannot exceed the number of changes of signs from positive to negative and from negative to positive in f(x).

Multiple choice question

1. **a is a root of the equation f(x)=0 if and only if**

(a) f(a)=0
(b) f(a) $\neq$ 0
(c) f'(a)=0
(d) f'(a)$\neq$0

2. **An algebraic equation of degree n, where n is a positive integer, has**

(a) n roots
(b) n^2 roots
(c) n^3 roots
(d) none of these

3. **One root of the equation x^3 – x-1 =0 lies between**

(a) 1 and 2
(b) 0and1
(c) 2and 3
(d) None of these

4. **Newton Raphson method is suitable for**

(a) f'(x) is small
(b) f'(x) is large
(c) f'(x) is negative
(d) f'(x) is positive

5. **Which of the following is also known as the Newton Raphson method?**

(a) Chord method
(b) Tangent method
(c) Diameter method
(d) Secant method

1.6 SUMMARY

In this unit we explained Algebraic equations and Transcendental equations. We also discussed method for finding the initial approximate value of the root **i**.e Graphical method, secant method, newton's Raphson method, regular- false method etc.

1.7 GLOSSARY

Convergence: The property of a numerical method to approach the exact solution as the number of iterations increases.

Divergence: The property of a numerical method to move away from the exact solution as the number of iterations increases.

Stability: The property of a numerical method to resist the growth of errors.

Error: The difference between the approximate solution and the exact solution.

1.8 REFERENCES

1. K. Atkinson: An Introduction to Numerical Analysis, Wiley, (2nd ed.), 1989.
2. P.G. Ciarlet and J. L. Lions (eds), Handbook of Numerical Analysis, North Holland, 1990.
3. E. W. Cheney and D. R. Kincaid: Numerical Mathematics and Computing, Brooks Cole, 6 editions, 2007.

1.9 SUGGESTED READING

1. Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India (1989).

2. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

1.10 TERMINAL AND MODEL QUESTIONS

1. Find the root of log x = cos x, by Newton- Raphson method up to five decimal places.

2. Find a root of $x=\frac{1}{2}$ + sin x near =1.5

3. Find $\sqrt{30}$ by using iterative process.

4. Use Graeffe's method to solve the equation given below,

$X^3 - 5x^2 - 17x + 20$

1.11 ANSWERS

CYQ1 . TRUE	**MCQ 1.** (a)	**TQ1.** 1.30295
CYQ2. FALSE	**MCQ 2.** (b)	**TQ2.** 1.4973
CYQ3. FALSE	**MCQ 3.** (a)	**TQ3.** 5.477225
CYQ 4. TRUE	**MCQ 4.** (b)	**TQ4.** 7.01751, -2.97443, 0.95817
	MCQ 5. (b)	

UNIT 2: ERROR ANALYSIS AND NUMERICAL COMPUTATIONS

CONTENTS:

2.1 *INTRODUCTION*

Error analysis and numerical computations are essential components of scientific computing, engineering, and data analysis. The primary goal of error analysis is to quantify and understand the errors that arise during numerical computations, while numerical computations involve the use of mathematical algorithms to solve complex problems. It is a critical aspect of numerical computations, as it helps to ensure the accuracy, reliability and optimality of numerical results. By understanding and quantifying errors, we can improve the quality of numerical computations and make informed decisions in a wide range of fields. Numerical computations involve the use of numerical methods and algorithms to solve mathematical problems. Numerical computations are essential in various fields, including Scientific simulations, Engineering design and Data analysis. In applied mathematics, numerical results are crucial due to the numerous instances where practical applications require precise numerical solutions.

2.2 OBJECTIVES

After studying this unit, the learner will be able to

1. Identify and quantify errors in mathematical models, algorithms, and numerical

computations.
2. Develop error analysis techniques and mathematical theories.
3. Evaluate the accuracy and reliability of numerical results.
4. Determine the sources and causes of errors.

2.3 ROUND OF ERROR

If we consider a number 270/131 = 2.061068702, which never terminates, then in order to use this number in a practical computation, must cut in down to a manageable number of digits such as 2.06 or 2.061 or 2.06107, etc.

The process of cutting off unwanted digits and retaining as many as desired is called rounding off.

To round of a number to n significant digits, discard all digit to the right of n th digit according to the following rule:

(i) If the discarded number is less than 5 at the (n+1) th place, leave the nth digit as such for example:

The following numbers are rounded off correctly to three significant difit:

1.963 becomes 1.96

2.354comes 2.35

(ii) If the discarded number is greater then 5 at (n + 1) th place, and 1 to the digit 4.457 comes 4.46

(iii) the discarded number is exactly 5 at (n+1) th place, leave the nth digit unchanged if it is even. 48.365 becomes 48.36

(iv) If the discard number is exactly 5 at (n+1) th place, add 1 to the nth digit if it is odd. 48.365 becomes 48.36

A list of the numbers rounded – off to three significant digit is given as:

7.894 becomes 7.89

12.865 becomes 12.9

6.4356 becomes 6.44

3.4567 becomes 34600

3.8254 becomes 3.82

2.4 LOCAL AND GLOBAL TRUNCATION ERROR

The error caused by one iteration is called Local truncation error and the cumulative error caused by many iterations is called global truncation error.

Truncation errors are the difference between the actual value of the given function. The transacted value of the functions is the approximated value up to a given number of digits. Truncation errors mainly arise when we approximated functions represented by an infinite series instead of using the actual value. We shall use Taylor’s and Maclaurin’s series to calculate the truncation error.

Inherent errors: Error which are already present in the statement of a problem before its solution are called inherent errors.
Such errors arise either due to the given data which are being approximated or due to the limitation of the computing aids: such as mathematical tables, desk calculators, or the digital computer.

Rounding errors: Error which arises in the process of rounding -off the numbers during computations, are called rounding errors.

Absolute error: Absolute error is defined as the numerical difference between the true value of a quantity and its approximate value:
If X be the true value of a quantity and X' be its approximates value, then $|X - X'|$ is the absolute error. it is denoted by E_a. that is,
$E_a = |X - X'|$.

Relative error: the relative error E_r is defined by
$E_r = \left|\frac{|X-X'|}{X}\right|$.
Percentages error: Percentage error $E_p = 100\ E_r = 100 \left|\frac{|X-X'|}{X}\right|$.
Absolute accuracy: If there is a number $\overline{X}$ such that $E_a \leq \overline{X}$ then $\overline{X}$ is an upper limit on the magnitude of absolute error and measures the absolute accuracy.

2.5 TAYLOR'S SERIES TO APPROXIMATED TRUNCATION ERRORS

If f is a continuously differentiable function on an interval containing the points a and x, then the value of the function f at x is given by

$$f(x)= f(a) +f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}(x-a)^3 + \dots\dots\dots + \frac{fn(a)}{n!}(x-a)^n + R_n$$

Where R_n is the remainder, defined as

$$R_n = \frac{f^{(n+1)}(c)}{(n+1!)}(x-a)^{n+1}$$

C is any quantity between a and x. It is Lagrange's form of remainder.
Taylor's series is very useful for representing any continuously differentiable function as a polynomial function of infinite order.

To understand the approximation of truncation error by Taylor's series, let us take an example. let f (x)=e^x, clearly which is continuously differentiable function. Taking a=0, we have f (0) =1 and f'(0)=1 = $f^{(k)}(0)$ such that

$$e^x = f(0) +f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots\dots\dots + \frac{fn(0)}{n!}(x-0)^n$$

finally, we have

$e^x = 1+x+\frac{x}{2!} + ... + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}+............$

If we approximate the function up to (n+1) terms, the truncation error is the remaining terms from (n+1) onwards.

Approximated function	Truncation error
$e^x = 1+x+\frac{x}{2!} + ... + \frac{x^n}{n!}$	$+ \frac{x^{n+1}}{(n+1)!}+............$

The remainder of Taylor's series calculates the value of the truncation's error. It is to be noted more the number of terms in the approximation smaller the truncation error.

Solved Example on Errors

Example 1: Calculate the truncation error for approximation of the up to 5 terms of the infinite series $1+x^2+x^4+x^6+x^8+..........$

Solution:

Given infinite geometric series,

$1+x^2+x^4+x^6+x^8+..........$

Now, $|R_5| = t_5+t_6+t_7+.......... = x^8+x^{10}+x^{12}+..........$

Which is again a infinite geometric series with term x^8 and common ratio x^2, then

$|R_5| = \frac{X^8}{(1-X^2)}$.

Example 2: Round off the numbers 865250 and 37.46235 to four significant digit and compute E_a, E_r, E_p in each case.

Solution (i) : 865250 is rounded off to four significant digits = 865200

Here X = 865250 and X' = 865200

$\therefore \quad E_a = |X - X'|$.

$= |865250 - 865200| = 50$

$E_r = \left|\frac{|X-X'|}{X}\right| = \frac{50}{865250} = 6.71 \times 10^{-5}$

And $E_p = 100\, E_r = 100 \times 6.71 \times 10^{-5} = 6.71 \times 10^{-3}$

(i) 37.46235 is rounded – off to four significant digits = 37.46.

In this case = 37.46253 and X' = 37.46

$\therefore E_a = |X - X'|$

$= |37.46235 - 37.46| = 0.00\,235$

$E_r = \left|\frac{|X-X'|}{X}\right| = \frac{0.00235}{37.46235} = 6.71 \times 10^{-5}$

And $E_p = 100E_r = 100 \times 6.71 \times 10^{-5} = 6.71 \times 10^{-3}$

Example 3: Find the absolute, relative and percentage error if x is rounded off to three decimal digits. Given x = 0.005998.

Solution: Rounded – off to three decimal digits of given number = 0.006.
Thus X = 0.005998, X' = 0.006.
∴ Absolut error $E_a = |X - X'|$
$= |0.005998 - 0.006| = 0.0000002$
Relative error $E_r = \left|\frac{|X - X'|}{X}\right| = \frac{0.000002}{0.005998} = 0.0003344$
And Percentage error $E_p = 100E_r = 100\ (0.0003344) = 0.03344$.

Example 4: Suppose 1.414 is used as an approximation to $\sqrt{2}$. Find the absolute and relative errors.
Solution: Here X (True value) = $\sqrt{2}$ 1.414213562 and X' (Approximate value) = 1.414= $|1.414213562 - 1.414|$
= 0.0002135637 = 0.21356237 × 10^{-3}
Relative error $E_r = \frac{E_a}{X} = \frac{0.21356237 \times 10^{-3}}{1.414213562} = 0.151011 \times 10^{-3}$.

Example 5: An approximate value of π is given by 3.1428571 and its true value is 3.1415926. Find absolute and relative errors.

Solution: Here X = 3.1415926
And X' = 3.1415926
∴ Absolute error $E_a = |X - X'|$
$= |3.1415926 - 3.1428571|$
$= |-0.0012645| = 0.0012645$.
Relative error $E_r = \frac{E_a}{X} = \frac{0.0012645}{3.1415926} = 0.000402502$.

2.6 COMPUTATION ARITHMETIC

In computer there are two types of arithmetic operation available in a computer.

(i) **Integer arithmetic.** This arithmetic only deals with integer operands which are used in counting and are used as sub**scripts.**

(ii) **Real or floating-point arithmetic:** This arithmetic uses the numbers with fractional parts as operands and its used in most computations. Due to economic consideration the digital computer is designed such that its memory has separately cells which are called **'words'.** Each word contains the same number of binary digits, called **'bits'** and having only a finite number of digits. These number of digits which can be stored in a computer is known as its **word length.**

The number are stored in a computer in two forms:

(i) Fixed point.

(ii) Floating point.

The fixed-point mode is used to represent integers while the floating-point mode is used to represent real number.

An n- digit floating point number in base β has the form

$X = \pm(d_1 d_2 \ldots d_n)_\beta \ \beta^e$ ... (1)

Where $d_1 d_2 \ldots d_n$ are all digits in the base in the base β and d's lie between 0 and β, and e is an integer called the exponent and it is such $m \leq e \leq M$, where m and M very with computer.

$(d_1 d_2 \ldots d_n)_\beta = d_1 \times \beta^{-1} + d_2 \times \beta^{-2} + \ldots + d_n \beta^{-n}$

Is called the mantissa which lies between -1 and 1 and the size of number.

For example: The number 24.35×10^6 may be rewritten as 0.2435×10^8 or 0.2435E8 (E8 is used to). Hence 0.2435 is the mantissa and E8 the exponent.

Translation of a number to floating point made: let fl (x) denote a floating-point number corresponding to a real number x.

A real number x can be translated into an nβ - digit floating point number fl (x) in two ways:

(i) **Rounding:** In this case fl(x) is chosen as the normalized floating-point number nearest x, where symmetric rounding digit is used in case of a tie.

(ii) **Chopping:** fl (x) is chosen as the nearest normalized floating-point number between x and 0.

Above definition of f (x) is modified when $|X| \geq \beta^m$ or $0|x| \leq \beta^{m-n}$,

Where m and M are the bounds on the exponents.

For example: If x = 2/3, then in rounding its floating fl (x) is given by\

fl (x) =$\left(\frac{2}{3}\right) = (0.67)10^0$

And in chopping it is given by

fl(x) =$\left(\frac{2}{3}\right) = (0.66)10^0$

For example: If x = -838, then in rounding

fl (-838) = - (0.84) 10^3

and in chopping, it is

fl(-838) = - (0.84) 10^3.

Arithmetic Operation:

(i) Addition or normalized floating points:

To add two normalized floating points we make their exponents equal by shifting the mantissa appropriately.

(ii) Subtraction of normalized floating points:

The operation of subtraction is performed by adding of a negative normalized and the exponent is suitably adjusted.

For example: Add 0.6756E4 and 0.7644E6.

Solution: The exponent of a number with the smallest exponent is increased by 2 so that 0.6756E4 becomes 0.0067E6.

Then 0.6756E4 +0.7644E6 = 0.0067E6 + 0.7644E6

= **0.**7711E6.

(iii) Multiplication of normalized floating points:

In order to multiply two floating points, we multiply their mantissa and add their exponents. In doing so, the resulting mantissa is normalized and the exponent is suitably adjusted. the mantissa is only four digits of the resulting mantissa which are retained by dropping the rest of the digits.

For example: Evaluate 0.1234E12 × 0.1111E9.

Solution: 0.1234E12 × 0.1111E9 = 0.0137097E21

= **0.**137097E20 [∵ Leading digit is mantissa should be non-zero]

= 0.1370E20 [∵ Resulting mantissa should be in 4 digits]

(iv) Division of two normalized floating points:

In order to divide a floating point be another, the mantissa of numerators is divided by the mantissa of the denominator and the exponent of denominator is subtracted from the exponent of the numerator. In doing so the quotient mantissa is then normalized retaining 4 digit and the exponent is suitable adjusted.

For example: Divide 0.1000E5 by 0.8889E3.

Solution: Here $\frac{0.10}{0.8889}$ = 0.11245

and $\frac{E5}{E3}$ = E2

∴ $\frac{0.1000E5}{0.8889E3} = 0.1124E2$

ILUSSTARTIVE EXAMPLE

Example 1: Convert $(\mathbf{11100111.101})_{\mathbf{2}}$, binary numbers into decimal form:

Solution: $(\mathbf{11100111.101})_{\mathbf{2}} = 1\times 2^7 \times 1\times 2^6 + 1\times 2^5 +$
$0\times 2^4 + 0\times 2^3 + 1\times 2^2 + 1\times 2^1 + 1\times 2^0 + 1\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3}$
$= 128 + 64 + 32 + 0 + 0 + 4 + 2 + 1 + 0.5 + 0.125$
$= (\mathbf{231.625})_{\mathbf{10}}$.

Example 2: Convert the number 17.375 from decimal to binary.

Solution: To convert integral part

2	17	Remainder
2	8	1
2	4	0
2	2	0
2	1	0
	0	1

$\therefore\ (17)_{10} = (10001)_2$

Integral part	0.375 ×2
1	0.75 ×2
0	0.50 ×2
1	0.00

$\therefore\ (0.375)_{10} = (0.11)_2$
Hence $=(17.375)_{10} == (10001.011)_2$.

Example 3: Multiply the following floating-point numbers.
0.1111E10 and 0.1234E15

Solution: $0.1111 \times 0.1234 = 0.01370974$ and
$E10\times E15 = E25$
0.1111E10×0.1234E15 = 0.01370974E25
= 0.01370974E24
= 0.01370E24.

CHECK YOUR PROGRESS

True or false questions

1. Rounding errors are unavailable in most of the calculations due to the limitation of computing aids.
2. 0.0050 is the maximum relative error if the number p is correct to 3 significant digits.
3. Errors are an important part of numerical computations.
4. Error analysis can help identify sources of error and how to reduce them.
5. There are several types of errors, including absolute, relative, and percentage errors.
6. Sources of error include rounding, truncation, and initial data inaccuracies.

Multiple choice question

1. The binary form of the decimal number 187.625 is
   (a) $(10111011)_2$
   (b) $(101110)_2$
   (c) $(1011.1011)_2$
   (d) None of these
2. The decimal form of the binary number 1101101 is
   (a) 21
   (b) 24
   (c) 26
   (d) 30
3. The error due to the discretization of the partial differential equation is called as
   (a) round-off error
   (b) discretization error
   (c) truncation error
   (d) iteration error
4. Which of these conditions is unstable?
   (a) Error is amplified in increasing iterations
   (b) Error is decreasing in increasing iterations
   (c) Error is amplified in decreasing iterations
   (d) Error is maintained in increasing iterations

2.7 SUMMARY

In this unit we discuss about the Local and global truncation errors, Truncation error, Rounding error and Absolute error
There are two types of Floating-point arithmetic one is integer arithmetic and other is real or floating-point arithmetic. We also explained Taylor's series to approximated truncation error and computer Arithmetic.

2.8 GLOSSARY

Algorithm: A step-by-step procedure for solving a problem numerically.
Absolute error : The absolute difference between the exact and approximate values of a quantity.
Relative error : The ratio of the absolute error to the exact value of a quantity
Truncation error : The error introduced by truncating an infinite series or sequence.
Accuracy: The closeness of a numerical solution to the exact solution.

2.9 REFERENCES

1. K. Atkinson: An Introduction to Numerical Analysis, Wiley, (2nd ed.), 1989.
2. "Error Analysis" by R. L. Burden and J. D. Faires: This book provides a comprehensive introduction to error analysis, including types of errors, error propagation, and error bounds.
3. E. W. Cheney and D. R. Kincaid: Numerical Mathematics and Computing, Brooks Cole, 6 editions, 2007.
4. "Error Analysis and Computational Complexity" by James L. Ortega: This paper discusses the relationship between error analysis and computational complexity.
5. MIT Open Course Ware: Numerical Analysis: This online course provides lectures, notes, and assignments on numerical analysis, including error analysis.

2.10 SUGGESTED READING

1.Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India (1989).

2. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

2.11 TERMINAL AND MODAL QUESTION

1. Find the percentage error if 625.483 is approximated to three significant figures.
2. Find the relative error of the number 8.6 if both of its digits are correct.
3. Convert the following number from decimal to binary
(i) 634.640625 **(ii)** 17.375
4. Convert the following numbers from binary to decimal:
(i) 0.1010101 **(ii)** 1.0110101
5. Show that $(176)_8 = (126)_{10}$
6. Subtract the following two floating point numbers:
0.36143448E3 - 0.36132346E7

2.12 ANSWERS

CYQ1. True
CYQ2. True
CYQ3. True
CYQ4. True
CYQ5. True
CYQ6. True
MCQ1. (a)
MCQ.2 (a)
MCQ.(c)
MCQ.(4)
TQ1. 0.0772
TQ2. 0.00581
TQ3. (i) $(1001111010.101001)_2$ (ii) $(10001.011)_2$
TQ4. (1) 0.6640625 (ii) 1.4140625
TQ6. 0.1110E4

UNIT 3: FINITE DIFFERENCE

CONTENTS:

3.1 INTRODUCTION

The Finite Difference Method (FDM) is a numerical technique used to solve partial differential equations (PDEs) and ordinary differential equations (ODEs) by discretizing the spatial and temporal domains. It is a powerful tool for solving differential equations and partial differential equations. With the help of finite difference method, we can solve wide range of problem. There are three types of finite difference formulas, namely the forward, the backward, and central difference. The method of finite difference converts ordinary differential equation or partial differential equation, which may be nonlinear, into a system of linear equations that can be solved by matrix algebra techniques. Finite difference was introduced by Brook Taylor in 1715.

3.2 OBJECTIVES

After studying this unit, the learner will be able to

1. Solve differential equation and partial differential equations.
2. Find the operator E, Δ, ∇ and the difference between the operators. They will calculate one or more missing terms in the given problem and they will be representing any given polynomial in factorial notation.
3. Analyzing the stability of numerical solutions and estimating the error in numerical solutions.

3.3 THE OPERATOR E Δ AND ∇

Shift operator E: Suppose y = f (x) be any function of x. By operating E on f (x). By operating E on f (x) we mean to simply give an increment to the value of x in the function f (x). If this increment be denoted by h, then the operator of E on f (x) means that put x + h in the function f (x) wherever there is x i .e,

$E f(x) = f(x + h)$.

Here we should note that E f (x) does not simply the multiplication of E and f (x) but it implies that E is operator on f (x). The operator is known as the shift operator. By $E^2 f(x)$ we mean that operator E is applied twice on the function f (x) i.e,

$E^2 f(x) = E\{E f(x)\}$

$= E\{f(x + h)\}$, by def. of E

$= f(x + h + h)$, by def. of E

$= f(x + 2h)$.

Similarly, $E^n f(x)$ means that the operator E is applied n times on the function f (x) i.e.,

$E^n f(x) = EE\ldots E$ (n times) f (x)

$=E^{n-1}\{E f(x)\} = E^{n-1}\{f(x + h)\}$

$= E^{n-2} E f(x + h)$

$=E^{n-3} f(x + 3h) = \ldots\ldots = f(x + \overline{n-1}h) = f(x + nh)$.

Remark: The operator E^{-1} is the inverse operator of the operator E and is defined as

$E^{-1} f(x) = f\{x + (-1)h\} = f(x - h)$.

The operator Δ : Suppose y = f (x) be a function of x. let the values of consecutive values of x be a, a + h , a + 2 h ,…… a + n h differing by h. then the corresponding values of y are f (a), f (a + h), f (a + 2 h),…… f (a + n h).

The independent variable x is known as argument and the dependent variable y is known as entry. Thus, we are given a set of values of argument and entry. The number a is called the interval of differencing.

The difference f (a + h) – f (a) is called the first forward difference of the function f (x) at the point x=a and we denote it by $\Delta f(a)$i.e.,
$\Delta f(a)$ = f (a + h) – f (a).

Again, the difference f (a + 2 h) – f (a + h) is called the first forward difference of the function f (x) at the point x = a + h and is denoted by $\Delta f(a+h) i.e.,$
$\Delta f(a+h)$ = f (a + 2 h) – f (a + h).
Continuing in a similar manner, we ultimately have

Δf (a + $\overline{n-1}$h) = f (a + n h) – f (a + $\overline{n-1}$ h).

The operator Δ is called the forward or descending difference operator.The difference $\Delta f(a)$, $\Delta f(a+h)$ etc. are called first forward differences. Thus, the first forward difference of f (x) is defined as

$\Delta f(x)$ = f (x + h) – f (x).

The difference $\Delta f(x)$ = f (x + h) – f (x). The difference $\Delta f(a$ + h) – f (a) is known as the second forward differences of f (x) at the point x = a and is denoted by $\Delta^2 f(a) i.e.,$

$\Delta^2 f(a)$ = $\Delta f(a$ + h) – f (a)

= {f (a + 2h) – f (a + h)} – {f (a + h) – f (a)}

= f (a + 2 h) – 2 f (a + h) + f (a).

The difference $\Delta f(a$ +2 h) – Δf (a + h) is known as the second forward differences of f (x) at the point x = a + h and is denoted by $\Delta^2 f(a+h)$.

Thus $\Delta^2 f(a+h)$ = Δf (a + 2h) – Δf (a + h)

= {f (a + 3 h) – f (a + 2 h)} – {f (a + 2 h) – f (a + h)}

= f (a+3 h) – 2 f (a + 2 h) +f (a + h).

In general, the second forward difference of f (x) is given by

$\Delta^2 f(x)$ = Δ [$\Delta f(x)$] = $\Delta [f(x+h) - f(x)]$ = Δf (x + h) - $\Delta f(x)$

= {f (x + 2 h) – f (x + h)} – {f (x + h) – f (x)}

= f (x + 2 h) – 2f (x + h) +f (x).

The differences of the second forward differences are called third forward differences and are denoted by $\Delta^3 f(a)$, $\Delta^3 f(a+h)$ etc.

Thus $\Delta^2 f(a) = \Delta^2 f(a+h) - \Delta^2 f(a)$

$= \{f(a+3h) - 2f(a+2h) + f(a+h)\} - f(a+2h) - 2f(a+h) + f(a)\}$

$= f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$.

In general, the n^{th} forward difference of f (x) is given by

$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$.

Note 1: If the function f(x) is a constant, say (x) = c, then

$$\Delta f(x) = f(x+h) - f(x) = c - c = 0$$

Thus, the first forward difference of a constant function is zero.

Note 1: It should be noted that by $\Delta^2 f(x)$, we mean that the operator Δ is to be applied twice on the function f (x).

The operator ∇: The difference $f(a+h) - f(a)$ is called the first backward difference of f (x) at x = a + h and is denoted by $\nabla f(a+h)$ i.e.,

$\nabla f(a+h) = f(a+h) - f(a)$.

Similarly, $\nabla f(a+2h) = f(a+2h) - f(a+h)$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$\nabla f(a+nh) = f(a+nh) - f(a+\overline{n-1}\,h)$.

The operator ∇ is called the backward or ascending difference operator. The differences of first backward difference are called second backward difference and are denoted by

$\Delta^2 f(a+2h)$, $\Delta^2 f(a+3h)$ etc.

Thus $\Delta^2 f(a+2h) = \nabla f(a+2h) - \nabla f(a+h)$

$= \{f(a+2h) - f(a+h)\} - \{f(a+h) - f(a)\}$

$= f(a+2h) - 2f(a+h) + f(a)$.

In general, the first backward difference of $f(x)$ is defined as

$\nabla^2 f(x) = \nabla\{\nabla f(x)\} = \nabla\{f(x) - f(x-h)\}$

$= \nabla f(x) - \nabla f(x-h)$

$= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\}$

$= f(x) - 2f(x-h) + f(x-2h).$

Similarly, the n^{th} backward difference of $f(x)$ is defined as

$\nabla^n f(x) = \nabla\{\nabla^{n-1} f(x)\} = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h).$

The identity operator I: The operator I, defined by I $f(x) = f(x)$, is called the identity operator. The operator I is also often denoted by the symbol 1. The operator I, defined by if I $(x) = f(x)$, is called the identity operator. It is denoted by I.

3.4 ALGEBRAIC PROPERTIES OF THE OPERATORS E AND Δ

There are some algebraic properties of the operators E and **Δ.**

(i) Operators E and Δ are distributive. Let u (x) be any function which is the sum of the functions $f(x)$, $g(x)$, $p(x)$,….so that

$u(x) = f(x) + g(x) + p(x) + \ldots\ldots\ldots.$

Thus $E\,u(x) = E\,f(x) + E\,g(x) + E\,p(x) + \ldots\ldots.$

Similarly, $\Delta u(x) = \Delta f(x) + \Delta g(x) + \Delta p(x) + \ldots\ldots$

(ii) E and Δ are commutative with regard to a constant, i.e.,

$E\{c f(x)\} = c\,E f(x)$ and $\Delta\{c f(x)\} = c\Delta f(x).$

(iii) E and Δ obey the law of indices i. e.,

$E^n E^n f(x) = E^{m+n} f(x)$ and $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x).$

(iv) E and Δ are not commutative w.r.t. variables, i.e., if

$u(x) = f(x)\,g(x),$

then $E\,u(x) \neq f(x)\,E\,g(x)$ and $u(x) \neq f(x)\,\Delta\,g(x).$

(v) Operators E and Δ are linear i.e.,

$E\{a f(x) + b g(x) = a E f(x) + b E g(x)$

and $\Delta \{a f(x) + b g(x) = a \Delta f(x) + b \Delta g(x)$

(vi) $E^{-n} f(x) = f(x - nh)$

3.5 DIFFERENCE OF PRODUCT AND QUOTIENT OF TWO FUNCTIONS

(i) The operator E and Δ are linear operators:

As $\Delta [f_1(x) + f_2(x)] = \{f_1(x+h) + f_2(x+h)\} - \{f_1(x) + f_2(x)\}$

$= \{f_1(x+h) - f_1(x)\} + \{f_2(x+h) - f_2(x)\}$

$= \Delta f_1(x) + f_2(x)$

And $\Delta[a f(x)] = a f(x+h) - a f(x)$

$= a [f(x+h) - f(x)] = a \Delta f(x)$,

Therefore Δ is a linear operator.

Similarly, we can show that E is a linear operator.

(ii) **The operator Δ and E are commutative in operation with respect to constants.**

If c is a constant, then

$\Delta \{c f(x)\} = c f(x+h) - c f(x) = c \Delta f(x)$

And $E \{c f(x)\} = c f(x+h) = c E f(x)$.

Note: The operator Δ and E not are commutative w.r.t. the function of x i.e., (variables)

i.e., if $u_x = f(x). g(x)$,

then $\Delta u_x \neq f(x) . \Delta g(x)$ and $E u_x \neq f(x) . E g(x)$.

(iii) The operator Δ and E are distributive:

We have $\Delta [f_1(x) + f_2(x) + \ldots\ldots\ldots] = \Delta f_1(x) + f_2(x) + \ldots\ldots$

And $E [f_1(x) + f_2(x) + \ldots\ldots\ldots] = E f_1(x) + E f_2(x) + \ldots.$

(iv) The operator Δ and E are commutative:

$(\Delta E) f(x) = [E f(x)] = \Delta f(x+h)$

$= [f(x+2h) - f(x+h) = E f(x+h) - E f(x)$

$= E [f(x+h) - f(x)] = (\Delta E) f(x)$

Thus $\Delta E = E \Delta.$

(v) The operator Δ and E are associative:

$(\Delta E) \Delta f(x) = \Delta (E \Delta) f(x).$

(vi) The operator Δ and E obey the law of indices:

$\Delta^m \Delta^n f(x) = (\Delta\Delta\ldots\ldots m \text{ times}) f(x)$

$= [\Delta\Delta\ldots\ldots (m+n) \text{ times}] f(x)$

$= \Delta^{m+n} f(x).$

Similarly $E^m E^n f(x) = E^{m+n} f(x).$

(vii) Difference of the product of two function:

$\Delta [f(x) g(x)] = [E f(x)]. \Delta g(x) + g(x). \Delta f(x).$

We have $\Delta [f(x). g(x)] = f(x+h). g(x+h) - f(x). g(x)$, by definition of Δ

$= f(x+h). g(x+h) - f(x+h) g(x) + f(x+h). g(x) - f(x). g(x)$

$= f(x+h) [g(x+h) - g(x)] + g(x) [f(x+h) - f(x)]$

$= f(x+h). \Delta g(x) + g(x). \Delta f(x)$

$= E f(x). \Delta g(x) + g(x). \Delta f(x).$

(viii) Difference of the quotient of two functions:

$$\Delta \left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\, \Delta f(x) - f(x). \Delta g(x)}{g(x). E\, g(x)}$$

We have $\Delta \left[\frac{f(x)}{g(x)}\right] = \left[\frac{f(x+h)}{g(x+h)}\right] - \left[\frac{f(x)}{g(x)}\right]$

$$= \frac{f(x+h)\, g(x) - f(x).\ g(x+h)}{g(x). g(x+h)}$$

$$= \frac{[f(x+h) - f(x)] g(x) - [g(x+h) - g(x)] f(x)}{g(x). g(x+h)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x).\Delta g(x)}{g(x).E\,g(x)}$$

3.6 RELATION BETWEEN THE OPERATORS

$E \equiv 1+\Delta$ or $\Delta \equiv E - 1$.

We have $\Delta f(x) = f(x + h) - f(x) = E f(x) - f(x) = (E-1) f(x)$.

Thus $\Delta f(x) = (E - 1) f(x)$, for any function $f(x)$.

So $\Delta \equiv E - 1$ or $E \equiv 1+\Delta$.

(a) $\Delta \equiv 1- E^{-1}$ or $E^{-1} \equiv 1- \nabla$.

We have $\nabla f(x) = f(x) - f(x - h)$

$= f(x) - E^{-1} f(x) = (1-E^{-1}) f(x)$.

Thus $\nabla f(x) = (1-E^{-1}) f(x)$, for any function f(x).

So $\nabla \equiv 1- E^{-1}$ or $E^{-1} \equiv 1- \nabla$.

(b) $E\nabla \equiv \Delta E \equiv \Delta$.

We have $(E\nabla) f(x) = E\{\nabla f(x)\}$

$= E\{f(x) - f(x-h)$

$= E f(x) - E f(x - h)$

$= f(x + h) - f(x) = \Delta f(x)$ … (1)

Also $(\nabla E) f(x) = \nabla \{E f(x)\} = \nabla f(x + h)$

$= f(x + h) - f(x) = \Delta f(x)$ ... (2)

From (1) and (2), we have

$E\nabla \equiv \Delta$ and $\Delta E \equiv \Delta$.
Thus $E\nabla \equiv \Delta E \equiv \Delta$.

(c) $\Delta - \nabla \equiv \Delta\nabla$.

We have $\Delta f(x) = f(x + h) - f(x)$

And $\Delta f(x) = f(x) - f(x - h)$, where h is the interval of differencing.

Now $(\Delta\nabla) f(x) = \Delta\{\nabla f(x)\} = \Delta\{f(x) - f(x-h)\}$

$= \Delta f(x) - \Delta f(x-h)$

$= \{f(x+h) - f(x)\} - \{f(x) - f(x-h)\}$

$= \Delta f(x) - \nabla f(x).$

$= (\Delta - \nabla) f(x).$

Thus $(\Delta\nabla) f(x) = (\Delta - \nabla) f(x)$, for any function $f(x)$.

So $\quad \Delta\nabla \equiv (\Delta - \nabla).$

(c) $\quad (1+\Delta)(1-\nabla) \equiv 1.$

We have $(1+\Delta)(1-\nabla) f(x) = (1+\Delta)\{(1-\nabla) f(x)\}$

$= (1+\Delta)\{f(x) - \nabla f(x)\}$

$= (1+\Delta)[f(x) - \{f(x) - f(x-h)$

$= (1+\Delta) f(x-h)$

$= E f(x-h) \qquad$ [since $E \equiv (1+\Delta)$]

$= f(x) = 1. f(x).$

Thus $(1+\Delta)(1-\nabla) f(x) = 1. f(x)$ for any function $f(x)$.

So $(1+\Delta)(1-\nabla) \equiv 1.$

(d) $E \equiv e^{hd} \equiv 1+\Delta$, where D is the differential operator of differential calculus.

We have $E f(x) = f(x+h)$

$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots\dots\dots$

[By Taylor's theorem of differential calculus].

$= 1. f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \dots\dots\dots\dots$

$= \left\{1 + hD + \frac{h^2}{2!} D^2 f(x) + \cdots\right\} f(x)$

$= e^{hd} f(x).$

Thus, E f (x) = e^{hd} f (x), for any function f(x).

So E = ehd .

Again, we Know that E ≡ 1+ Δ. Therefore

E ≡ e^{hd} ≡ 1+ Δ.

3.7 THE DIFFERENCE TABLE

Suppose y = f (x) is a function of x; x being given at an equal interval. Let the values of x be a, a+ h, a + 2 h and so on and the corresponding values of y be f (a), f (a + h), f (a+2h) and so on. Then the forward and backward differences can be calculated from the forward difference table and backward difference table respectively.

Table I

Forward Difference Table

Arugment x	Entry y=f (x)	First differences Δ f (x	second differences $\Delta^2 f$ (x)
a	f (a)		
		f(a+h) – f (a) = Δ f (a)	
a+h	f (a+h)		Δ f (a+h) - Δ f (a) = $\Delta^2 f$ (a)
		f(a+2h) – f (a+h) = Δ f (a+h)	
a+2h	f (a+2h)		
			Δ f (a+2h) - Δ f (a+h) = $\Delta^2 f$ (a +h)
a+3h	f (a+3h)		
		f (a+3h) – f (a+2h) = Δ f (a+2h)	
a+4h	f (a+4h)		
		f (a+4h) – f (a+3h) = Δ f (a+3h)	Δ f (a+3h) - Δ f (a+2h) =$\Delta^2 f$ (a+2h)

Similarly, we can calculate third and higher order difference from the table. Here we have taken forward differences therefore this table is known as forward difference table.

Backward Difference Table

Arugment x	Entry $y=f(x)$	First differences $\Delta f(x$	second differences $\Delta^2 f(x)$
a	$f(a)$		
		$f(a+h) - f(a) = \nabla f(a)$	
a + h	$f(a+h)$		
			$\nabla f(a+h) - \nabla f(a) = \nabla^2 f(a)$
		$f(a+2h) - f(a+h) = \nabla f(a+h)$	
a + 2 h	f (a+2 h)		
			$\nabla f(a+2h) - \nabla f(a+h) = \nabla^2 f(a+h)$
a + 3 h	f (a+3 h)		
		$f(a+3h) - f(a+2h) = \nabla f(a+2h)$	
a + 4 h	f (a+4 h)	$f(a+4h) - f(a+3h) = \nabla f(a+3h)$	$\nabla f(a+3h) - \nabla f(a+2h) = \nabla^2 f(a+2h)$

Similarly, we can calculate the differences of higher order from this table.

3.8 FUNDAMENTAL THEOREM OF THE DIFFERENCE CALCULUS

The n^{th} difference of a polynomial of degree n is constant and higher order differences are zero i.e., if f (x) is a polynomial of degree n in x, then the n^{th} difference of f (x) is constant and $(n+1)^{th}$ is zero.

Proof: let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$,

Where n is a positive integer and a_0 , $a_1, \ldots\ldots\ldots a_n$ are constant.
We have $\Delta f(x) = f(x+h) - f(x)$ by def. of Δ

$= [a_0 + a_1 (x+h) + a_2 (x+h)^2 + \ldots\ldots + a_n (x+h)^n - [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n],$

$= = a_1 h + a_2[(x+h)^2 - x^2] + a_3 [(x+h)^3 - x^3] + \ldots\ldots + a_n [(x+h)^n - x^n]$

$= a_1 h + a_2 [{}^2c_1 \; xh + h^2] + a_3 [{}^3c_1 \; x^2 h + {}^3c_2 \; xh^2 + h^3] + \ldots.+ a_n [\,{}^nc_1 \; x^{n-1} h + {}^nc_2$

$x^{n-2} h^2 + \ldots\ldots + {}^nc_n \; h^n]$

$= b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_{n-2} x^{n-2} + n a_n h x^{n-1}$, …….(1)

Where b_0 ,b_1, $b_2, \ldots\ldots\ldots b_{n-2}$ are constant coefficient. From (1) we see that $\Delta f(x)$ is a polynomial of degree n-1 in x. Thus, the first difference of a polynomial $f(x)$ of degree n is again a polynomial of degree n-1 in which the coefficient of x^{n-1}

= n. h. the coefficient of x^n in f (x).

Now let $\Delta f(x) = \phi(x)$, where $\phi(x)$ is a polynomial of degree n-1.

Then $\Delta^2 f(x) = \Delta [\Delta f(x)] = \Delta \phi(x)$.

But by (1), $\Delta \phi(x)$ is a polynomial of degree n-2 in which the coefficient of x^{n-2}

= (n-1). h. the coefficient of x^{n-1} in $\phi(x)$

= (n-1). h. n. h. $a_n = n(n-1) h^2 a_n$.

Thus $\Delta^2 f(x)$ is a polynomial of degree n-2 in which the coefficient of x^{n-2}

$= n(n-1) h^2 a_n$.

Continuing the above process, we see that the n^{th} difference of $f(x)$ is a polynomial of degree zero

i.e., $\Delta^n f(x) = n(n-1)(n-2) \ldots.1.\, h^n a_n$.

$= n!\, h\, a_n x^0 = n!\, h\, a_n$.

Thus, the n^{th} difference of $f(x)$ is constant. So, all higher order differences are zero, i.e., $(n+1)^{th}$ and higher differences of a polynomial of degree n are zero.

Note 1: The converse of the above result is not true i.e., if the n^{th} differences of a tabulated function are constant when values of the independent variable are taken at equal interval, the function is a polynomial of degree n.

Note 2: Basic assumption of calculus of finite differences.

If we are given n +1 pairs

(a, f(a), (a + h), f(a + h), (a+2 h), f(a+2 h)), ……, (a + n h), f(a + n h)) of the values of the argument x and the entry f (x), then f (x) can be represented by a polynomial

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

of degree n. so n^{th} differences of f (x) are all constant and differences of f (x) of order

higher than n are zero.

To express any value of the function in terms of the leading term and the leading differences of the difference table

Show that $f(a + nh) = f(a) + {}^n c_1 \Delta f(a) + {}^n c_2 \Delta^2 f(a) + \ldots + {}^n c_n \Delta^n f(a)$.

We have $f(a + nh) = E^n f(a) = (1+\Delta)^n f(a)$

$$= \{1 + {}^n c_1 \Delta + {}^n c_2 \Delta^2 + {}^n c_3 \Delta^3 + \ldots + {}^n c_n \Delta^n\} f(a)$$

$$= f(a) + {}^n c_1 \Delta f(a) + {}^n c_2 \Delta^2 f(a) + \ldots + {}^n c_n \Delta^n f(a).$$

3.9 ONE OR MORE MISSING TERMS (EQUAL INTERVAL)

Sometime we may be given a set of equidistant terms with some (one or two or more) missing. The problem of estimating such terms can be easily tackled by the use of the operators E and Δ.

Let us suppose that we are given (n+1) equidistant arguments, (x= 0, 1, 2, n, say), but the entry y_r corresponding to any one of them, say $(r+1)^{th}$ argument, is not given and we want to estimates it . Since we are given n entries, the data can be represented by a polynomial of $(n-1)^{th}$ degree.

So $\Delta^{n-1} y_x$ = constant and

$$\Delta^n y_x = 0, \; x = 0,1,2,\ldots.n. \qquad \ldots(1)$$

In particular

$\Delta^n y_0 = 0$ i.e., $(E-1)\, y_0 = 0$

⇨ $[E^n - {}^n c_1 E^{n-1} + {}^n c_2 E^{n-2} - \ldots + (-1)^n]\, y_0 = 0$
⇨ $E^n y_0 - {}^n c_1 E^{n-1} y_0 + {}^n c_2 E^{n-2} y_0 - \ldots + (-1)^n]\, y_0 = 0$
⇨ $Y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2}$ ----- $+ (-1)\, y_0 = 0.$
⇨
From this equation, the missing entry can be easily calculated.
If in a set of (n+1) equidistant arguments, two entries y_r and y_s are missing, then the data can be represented by a polynomial of degree n-2. So, proceeding as above we have

$$\Delta^{n-1} y_0 = 0$$

And $\Delta^{n-1}y_1 = 0$

⇨ $(E - 1)^{n-1} y_0 = 0$
and $(E - 1)^{n-1} y_1 = 0$.
Expanding and simplifying as above the two missing terms can be estimated by solving the equations (2). Similarly in a set of n+1 equidistant arguments the three missing terms can be estimated by solving the equations
$\Delta^{n-2}y_0 = 0, \Delta^{n-2}y_1 = 0, \Delta^{n-2}y_2 = 0,$

3.10 FACTORIAL NOTATION

The product of n consecutive factors each at a constant difference, say h, the first factor being x

is called a factorial function or a factorial polynomial of degree n and is denoted by $x^{(n)}$. Thus

$x^{(n)} = x (x-h) (x-2h) \ldots x- n-1 h).$

in particular, if h = 1, then

$x^{(n)} = x (x-1) (x-2) \ldots (x- n+1).$

These functions, because of their properties, play an important role in the theory of finite differences.

The factorial function help in finding the various order differences of a polynomial directly by simple rule of differentiation and similarly given any difference of a function in factorial we can find the corresponding function by simple integration.

Result 1. $\Delta^n x^{(n)} = n!\, h^n$ and $\Delta^{n+1} x^{(n)} = 0$.

Proof: By definition of Δ, we have

$\Delta x^{(n)} = (x + h)^{(n)} - x^{(n)}$

$= (x + h) (x + h - h) (x + h - 2 h) \ldots\ldots (x + h - n - 1 h) - [x (x-h) \ldots (x-\overline{n-1}\, h)]$

$= (x + h) x (x - h) \ldots\ldots (x + h - n - 2 h) - x (x - h) \ldots(x-n-2 h)] (x-\overline{n-1}\, h)$

$= x (x-h) \ldots.. (x-\overline{n-2}\, h) [(x+h) - (x-\overline{n-1}\, h)]$

$= x(x-h) \ldots.. (x-\overline{n-2}\, h) [(x+h) - (x-\overline{n-1}\, h)]$

$= x^{(n-1)} [x+h - x + n h - h] = n h x^{(n-1)} \ldots\ldots\ldots (1)$

=or equivalently $\frac{\Delta x(n)}{\Delta x} = n x^{(n-1)}$ because $\Delta x = x + h - x = h$

Again $\Delta^2 x^{(n)} = \Delta\Delta x^{(n)}$

$= \Delta \{n\, h\, x^{(n-1)}\}$ [from (1)]

$= n\, h\, \Delta\, x^{(n-1)}$, because n h is a constant.

Now replacing n by n-1 in the relation (1), we have

$\Delta\, x^{(n-1)} = (n-1)\, h\, x^{(n-2)}$.

So $\Delta^2 x^{(n)} = n\, h. (n-1)\, h\, x\, (n-2) = n\, (n-1)\, h^2\, x^{(n-2)}$.

Proceeding in the same manner, we get

$\Delta^{n-1} x^{(n)} = n\, (n-1) \ldots\ldots 2. h^{n-1}\, x$.

So $\Delta^n x^{(n)} = n\, (n-1) \ldots\ldots 2. h^{n-1}\, \Delta\, x$.

$= n\, (n-1) \ldots\ldots 2. h^{n-1}\, (x + h - x)$

$= n\, (n-1) \ldots\ldots 2.\, 1 h^{n-1}. h$

$= n!\, h^{(n)}$. …. (2)

So $\Delta^{n-1} x^{(n)} = \Delta\, (n!\, h) = n!\, h^n - n! h^n = 0$

In particular when h=1 we get from (1) and (2)

$\Delta\, x^{(n)} = n\, x^{(n-1)}$ and $\Delta^n x^{(n)} = n!$. …. (3)

In case of factorial notation, the operator Δ is equivalent to the operator ion of differentiation if the interval of differencing is unity.

Result 2. $\Delta^{-1} x^{(n)} = \frac{x\,(n+1)}{(n+1)\, h}$ +t (x), where t (x) is a periodic function of period h.

Proof: we have $\Delta^{(2)} = 2.h\, x^{(1)}$ [since $\Delta^{(n)} = n\, h\, (x^{(n-1)}$]

Or $\Delta\left[\frac{x^2}{2h}\right] = x^{(1)}$

Or $\Delta^{-1} x^{(1)} = \frac{x^{(2)}}{2h} + t\,(x)$,

Where t (x) is a periodic function of period h.

Similarly, $\Delta^{-1} x^{(2)} = \frac{x^{(3)}}{5h} + t\,(x)$

In general, $\Delta^{-1} x^{(n)} = \frac{x^{(n+1)}}{(n+1)h} + t\,(x)$.

Note: From above we observe that the operator Δ^{-1} behaves on polynomial of factorial function in a similar way as the integration behaves on ordinary polynomials.

3.11 METHOD OF REPRESENTING ANY GIVEN POLYNOMIAL IN FACTORIAL NOTATION

First method (direct method)

Express $2\ x^3\ -3\ x^2 + 3\ x\ -10 \equiv A\ x^{(3)} + B\ x^{(2)} + C\ x^{(1)} + D$

$$= A\ x\ (x-1)\ (x-2) + B\ x\ (x-1) + C\ x + D. \quad \ldots.\ (1)$$

Where A, B, C and D are constant to be determined.

Putting $x = 0$ in (1), we get

$D = -10.$

Again, putting $x=1$ in (1), we get

$2-3+3-10 = C+D$

⇨ $C=2.$

Putting $x=2$ in (1), we get $16 -12+6-10 = 2B+2C+D$

⇨ $0 = 2B + 4 -10 \Rightarrow B=3$

Equating the coefficients of x^3 on both side of (1), we get $A=2$.

Putting the values of A, B, C and D in (1), we get

$f\ (x) = 2x^3 -3x^2 +3x -10$

$\quad = 2\ x^{(3)} +3\ x^{(2)} + 2\ x^{(1)} -10.$

By the rule of simple differentiation, we have

$\Delta\ f\ (x) = 6\ x^{(2)} +6\ x^{(1)} +2$

$\Delta^2\ f\ (x) =12\ x^{(1)} + 6$

$\Delta^3\ f\ (x) = 12.$

Step in First Method

(i) The given function is expressed terms by term in functions with certain unknown coefficient as shown in (1).

(ii) To get the values of the unknown coefficient we put x=0, 1, 2,……successively in the L.H.S and R.H.S of (1)

And the resulting equation are solved to find the value of A, B, C, etc.

(iii) The value of A, B, C, etc. thus found are substituted in the R.H.S of (1) to get the given polynomial in factorial notation.

Second method (method of detached coefficient or synthetic division)

We have $2x^3 - 3x^2 + 3x - 10 \equiv A x^{(3)} + B x^{(2)} + C x^{(1)} + D$ ….(1)

$= A x (x-1)(x-2) + B x(x-1) + C x + D$

$= x [A (x-1)(x-2) + B(x-1) + C] + D$

If we divide the given polynomial by x then the remainder will be -10 and the quotient s $2x^2 - 3x + 3$. The value of D in (1) is taken as -10.

Again, divide the quotient $2x^2 - 3x + 3$ as done below:

$$\begin{array}{r|l} & 2x-1 \\ \hline x-1) & 2x^2 - 3x + 3 \\ & 2x^2 - 2x \\ \hline & -x + 3 \\ & -x + 1 \\ \hline & 2 \end{array}$$

So, the quotient is 2x-1 and the remainder is value of C i.e., C =2.

Again, divide 2x-1 by x-2 as done below:

$$\begin{array}{r|l} & 2 \\ \hline x-2) & 2x - 1 \\ & 2x - 4 \\ \hline & 3 \end{array}$$

The quotient 2 is the value of A and the remainder 3 is B. Thus, the given polynomial when expresses in factorial notation is

$$2x^3 - 3x^2 + 3x - 10 = 2x^{(3)} + 3x^{(2)} + 2x - 10.$$

The above method can be simplified by the procedure of detached coefficients in the following way:

Taking the coefficient of the various powers of x in the given polynomial, we have

	2	-3	3	-10	=D	… (a)
1						
	0	2	-1			
	2	-1	2	= C		…. (b)
2						
	0	4				
	2	3 = B				…. (c)
3						
	0					
	2 =A					

Steps in the second method of detached coefficients

(i) First make the given polynomial complete (if it is not say) by supplying the missing terms zero coefficients. Then write the coefficients of different powers of x in order beginning with the coefficient of highest power of x. The constant term -10 is the value of D.

(ii) Put 1 in the left-hand side column of (a) and write zero below the coefficient of highest power of x. In this case we have written 0 below 2 which is the coefficient of x^3. The sum of 2 and 0 is 2 which we write below 0 in the third row. Now we multiply 2 by 1 of the left-hand column of (a) and write their product 2 in the second row below -3 of the first row. Adding -3 and -2 we get -1 which we put in the third below 2 of the second row.

(iii) Now we get -1 which we put in the third row below 2 of the second row. Now we multiply -1 by -1 of the left-hand column of (a) and then write their product -1 n the second row below 3 of the first row. Adding 3 and -1 we get 2 which we put below -1. This 2 is the value of C.

(iv) Now we write 2 in the left-hand column of (b). Below 2 of the third row we write 0 and adding 2 and 0 we get 2 which we write in the fifth row below 0 of the fourth row. Now we multiply 2 of the fifth row by 2 of the left-hand columns of (b) and write their product 4 in the fourth row below -1 of the third row. Adding -1 and 4 we get 3 and we write it in the fifth row below 4of the fourth row.

This 3 is the value of B.

(v) Now we write 3 in the left-hand column of (c). Below 2 of the fifth row we write 0 and adding 2 and 0 we get 2 which we write in the seventh row below 0 of the sixth row. This 2 is the value of A.

3.12 DIFFERENCE OF ZERO

If n and m are positive integers, we have in usual notation of finite difference calculus

$\Delta^n x^m = (E - 1)^n x^m$

$= [E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - \ldots + (-1)^{n-1}\, {}^nC_{n-1}E + (-1)^n]\, x^m$

$= E^n x^m - C_1 E^{n-1} x^m + {}^nC_2 E^{n-2} x^m - \ldots + {}^nC_{n-1} E (-1)^{n-1} E x^m + (-1)^n x^m$

$= (x+n)^m - {}^nC_1 (x+n-1)^m + {}^nC_2 (x+n-2)^m - \ldots + {}^nC_{n-1} (-1)^{n-1} (x+1)^m + (-1)^n x^{m.}$

= putting x = 0 and writing $[\Delta^n, x^m]_{x=0}$ as $\Delta^n 0^{m,}$ we have

$\Delta^n 0^m = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots + {}^nC_{n-1} - (-1)^{n-1} \ldots$ (1)

The quantities $\Delta^n 0^m$ are known as differences of zero because the leading term is always zero.

We can calculate the values of $\Delta^n 0^m$ for various integral values of n and m. For example,

If n=1, m =3, $\Delta\, 03 = 1^3 = 1$

n =2, m=3, $\Delta^2 0^3 = 2^3 - 2.\,1^3 = 6$

n=3, m=3, $\Delta^3\, 03 = 3^3 - 3.\,2^3 + 3.1^3 = 6$, etc.

Recurrence elation between $\Delta^n 0^{m,}$ $\Delta^{n-1} 0^{m-1}$ and $\Delta^n 0^{m-1}$. From (1), we have

$\Delta^n 0^m = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-1)^m + {}^nC_2 (n-2)^m - \ldots + {}^nC_{n-1} - (-1)^{n-1}$

$= n^m - n (n-1)^m + \frac{n(n-1)}{2!} (n-2)^m - \ldots + n (-1)^{n-1}$

$= \left[n^{m-1} - (n-1)^m + \frac{n(n-1)}{2!} (n-2)^m - \cdots . + n (-1)^{n-1}\right]$

$= n \left[n^{m-1} - (n-1)^m + \frac{n(n-1)}{2!} (n-2)^m - \cdots + (-1)^{n-1}\right]$

$= n\, [n^{m-1} - {}^{n-1}C_1 (n-1)^{m-1} + {}^{n-1}C_2 (n-2)^{m-1} - \ldots + (-1)^{n-1}]$

$= n\, [(1+n-1)^{m-1} - {}^{n-1}C_1 (1+n-2)^{m-1} + {}^{n-1}C_2 (1+n-3)^{m-1} - \ldots + (-1)^{n-1}]$

$= n\, [E^{n-1} (1)^{m-1} - {}^{n-1}C_1 E^{n-2} (1)^{m-1} + {}^{n-1}C_2 E^{n-3} (1)^{m-1} - \ldots + (-1)^{n-1} (1)^{m-1}]$

$[\because (1)^{m-1} = 1]$

$= n\,[E-1]^{n-1}\,(1)^{m-1} = n\,\Delta^{n-1}\,(1)^{m-1}$

$= n\,\Delta^{n-1}\,E\,(0)^{m-1}$ [∵ E (0)=0]

$= n\,\Delta^{n-1}\,(1+\Delta)\,0^{m-1}$

$= n\,\Delta^{n-1}\,0^{m-1} + n\,\Delta^{n}\,0^{m-1}$

3.13 LEIBNIT'S RULE

To find the n^{th} difference of product of two functions, in differential calculus, we use Leibnitz's theorem

$D^n(u.\,v) = (D^n u).\,v + {}^nC_1\,(D^{n-1}u).\,D\,v + \ldots\ldots + u.\,D^n\,(v).$

To find the n^{th} difference (Δ^n) of product of two functions an analogous formula is given by

$\Delta^n\,(u.\,v) = (\Delta^{n\,u}).\,v + {}^nC_1\,(\Delta^{n-1}\,E\,u).\,(\Delta\,v) + \ldots\ldots + {}^nC_2\,(\Delta^{n-2}\,E^2).\,(\Delta^2\,v) + \ldots.$

$+ (E^n u).\,(\Delta^n v).$

This is called Leibnitz's rule for differences.

Note: For convenience a polynomial in x considered as the second function i.e., *v*

ILLUSTRATIVE EXAMPLES

Example 1: Find the value of E^2x^2 when the values of x vary by a constant increment of 2.

Solution: We have

$E^2x^2 = EE\,x^2 = E\,(x+2)^2$

[∵ interval of differencing is 2]

$(x+2+2)^2 = (x+4)^2$

$= x^2 + 8x + 16$

Example 2: Evaluate $E^n e^x$ when interval of differencing is h.

Solution: we have $E(e^x) = e^{x+h}$

$E^2(e^x) = EE\,e^x = E\,e^{x+h}\;\; e^{x+2h}$

$E^3(e^x) = EE^2\,e^x = E\,e^{x+2h}\;\; e^{x+3h}$

$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots.$
$E^n e^x = e^{x+h}$.

Example 3: Evalute the following

(i) Δ^3 (1- x) (1-2x) (1-3x)
(ii) Δ^n (e^{ax+b})

The interval of differencing being unity.

Solution: (1) Here f (x) = (1-x) (1-2 x) (1-3x)

$= - 6x^2 + 11\, x^2 - 6x + 1$

The polynomial is of degree 3.

We know that for a polynomial of n^{th} degree, the n^{th} difference is constant being equal to a_n h n! where a_n is the coefficient of x^n in the polynomial, h is the interval of differencing.

Here a_n = - 6, h=1, n=3

$\therefore \quad \Delta^3 f(x) = - (6). (1)^3 .3! = -36.$

(ii) Here f (x) = $e^{\,ax+b}$

Now Δ f (x) = f (x+1) –f (x).

$\therefore \qquad \Delta\, e^{ax+b} = e^{a(x+1)+b} - e^{ax+b} = e^{ax+b}\,(e^{a-1})$

$\Delta^2 (e^{ax+b}) = \Delta\,(\Delta\, e^{ax+b}) = \Delta\,\{e^{ax+b}\,(e^{a-1})\}$

$= (e^{a-1})\,(\Delta\, e^{ax+b})$

$= (e^{a-1})\, e^{ax+b}\,(e^{a-1})$

$= (e^{a-1})^2\, e^{ax+b}.$

Proceeding in this way we get

$\Delta^n\,(e^{ax+b}) = (e^{a-1})^{n\, e\, ax+b}.$

Example 4: Evalute

(i) Δ^2 (Cos 2x)
(ii) $\Delta\, tan^{-1}$x the interval of differencing being h.

Solution: (i) we have Δ^2 (cos 2x) = $(E-1)^2$ cos 2x

$= (E^2 - 2E + 1)$ cos 2x $\qquad [\because \Delta \equiv E-1]$

$= E^2$ cos 2x – 2E cos 2x +cos 2x

$= \cos\{2(x+2h)\} - 2\cos 2(x+h) + \cos 2x$

$= \cos(2x+4h) - 2\cos(2x+2h) + \cos 2x$

$= \cos(2x+4h) - 2\cos(2x+2h) - \cos(2x+2h) + \cos 2x$

$= 2\sin(2x+3h)\sin(-h) + 2\sin(2x+h)\sin h$

$= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)]$

$= -2\sin h\,[2\sin(2x+2h) - \sin h]$

$= -4\sin^2 h\cos(2x+2h).$

(ii) $\Delta\, tan^{-1} x = tan^{-1}(x+h) - tan^{-1} x$

$$= tan^{-1}\frac{(x+h)-x}{1+(x+h)\,x} = tan^{-1}\left[\frac{h}{1+hx+x^2}\right].$$

Example 5: Construct a forward difference table for the following values:

x:	**0**	**5**	**10**	**15**	**20**	**25**
f(x):	**7**	**11**	**14**	**18**	**24**	**32**

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
5	11		-1			
		3		2		
10	14		1		-1	
		4		1		0
15	18		2		-1	
		6		0		
20	24		2			
		8				
25	32					

Example 6: Show that

$$\sum_{K=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0.$$

Solution: We have $\sum_{K=0}^{n-1} \Delta^2 f_k = \sum_{K=0}^{n-1} \Delta (\Delta f_k)$

$= \sum_{K=0}^{n-1} (\Delta f_{k+1} - \Delta f_k)$

[∴ interval of differencing is 1]

$= (\Delta f_1 - (\Delta f_0) + (\Delta f_2 - (\Delta f_1) + (\Delta f_3 - (\Delta f_2) + ... + (\Delta f_{n-1} - (\Delta f_{n-2}) +$
$(\Delta f_n - \Delta f_{n-1})$

$= (\Delta f_n - \Delta f_0)$.

Example 7: Obtain the missing terms in the following table:

x:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
f(x):	**1**	**8**	**?**	**64**	**?**	**216**	**343**	**512**

Solution: Here we are given six values, so a polynomial of degree 5 may be ditted which will have its 6th difference as zero,

i.e., $\Delta^6 f(x) = 0$ for all values of x

i.e., $(E-1)^6 f(x) = 0 \; \forall \; x$ [$\because \Delta \equiv E-1$]

i.e., $(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0 \; \forall \; x$

i.e., $E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6E f(x) +$
$f(x) = 0 \; \forall \; x$

i.e., $f(x+6) - 6 f(x+5) + 15 f(x+4) - 20 f(x+3) + 15 f(x+2) - 6 f(x+1) +$
$f(x) = 0 \; \forall \; x$ ……. (1) here interval of differencing is unity.

Putting x = 1 and 2 in (1), we get

$f(7) - 6 f(6) + 15 f(5) - 20 f(4) + 15 f(3) - 6 f(2) + f(1) = 0$ ....(2) and

$f(8) - 6 f(7) + 15 f(6) - 20 f(5) + 15 f(4) - 6 f(3) + f(2) = 0$ .... (3)

putting the value of $f(8), f(7), f(6), f(4), f(2), f(1)$, we get

$343 - 6\times 216 + 15 f(5) - 20\times 64 + 15 f(3) - 6 \times 8 + 1 = 0$ and

$512 - 6\times 343 + 15 \times 216 - 20 f(5) + 15\times 64 - 6 f(3) + 8 = 0$

i.e., $15 f(5) + 15 f(3) = 2280$ and

$20 f(5) + 6 f(3) = 2662$

i.e., $f(5) + f(3) = 152$ .... (4)

and $10 f(5) + 3f(3) = 1331$ .... (5)

solving (4) and (5); we get

$f(3) = 27$, $f(5) = 125$

Example 8: Find the first term of the series whose second and subsequent term are 8,3,0, -1,0.

Solution: Given

x:	0	1	2	3	4	5
$f(x)$:	?	8	3	0	-1	0

As five values of [x, f(x)] are given, we have

$$\Delta^5 f(x) = 0 \quad \text{or } (E-1)^5 f(x) = 0$$

or $[E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - {}^5C_5] f(x) = 0$

or $E^5 f(x) - 5 E^4 f(x) + 10 E^3 f(x) - 10 E^2 f(x) + 5E f(x) - f(x) = 0$

$$f(x+5) - 5 f(x+4) + 10 f(x+3) - 10 f(x+2) + 5 f(x+1) - f(x) = 0$$

for x=0 we have

$$(5) - 5 f(4) + 10 f(3) - 10 f(2) + 5 f(1) - f(0) = 0$$

or $0 - 5 \times (-1) + 10 \times 0 - 10 \times 3 + 5 \times 8 - f(0) = 0$

or $15 - f(0) = 0$ given $f(0) = 15$

Hence first term of the series is 15.

Example 9 : Represent the function $f(x) = x^4 - 12x^3 + 12x^2 - 30x + 9$ and its successive difference in factorial notation.

Solution:

let $f(x) = x^4 - 12x^3 + 12x^2 - 30x + 9 \equiv Ax^{(4)} + Bx^3 + C(x)^{(2)} + (Dx)^1 + E$

using this method of synthetic division (method of detached coefficients,) as shown in the table, we get

$$x^4 - 12x^3 + 12x^2 - 30x + 9 \equiv x^{(4)} + 6x^3 + 13x^{(2)} + x^{(1)} + 9.$$

Since in the factorial notation, the operator Δ is equivalent to differentiating we get

$\Delta f(x) = 4x^{(3)} - 18x^{(2)} + 26x + 1$

$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} + 26$

Δ^3 f (x) = 24x -36

Δ^4 f (x)= 24.

Table for finding A, B, C

1	1	-12	42	-30	9=E
	0	1	-11	31	
2	1	-11	31	1= D	
	0	2	-18		
3	1	-9	13 = C		
	0	3			
4	1	-6 = B			
	0				
	= 1A				

NOTE 1: While using the method of synthetic division, we should first of all see whether the given expression is complete or not. If any power of x missing, we should first make the expression complete by taking its coefficient zero and then employ the method of synthetic division.

Example 10 : **Find the function whose first difference is 9 x^2 +11 x +5.**

Solution: let f (x) be the required function.

Then $\Delta f(x) = 9x^2 + 11x + 5$.

Let $9x^2 + 11x + 5 \equiv 9(x)^{(2)} + Ax^{(1)} + B$

$= 9x(x-1) + Ax + B$.

Putting x = 0, we get B = 5.

Putting x = 1, we get A= 20.

$\therefore$ $\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5$.

Integrating it, we get

$$f(x) = 9.\frac{x^{(3)}}{3} + 20.\frac{x^{(2)}}{2} + 5..\frac{x^{(1)}}{1} + C, \text{ where C is a constant}$$

$\therefore$ $f(x) = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + C$

$= 3x^3 + x^2 + x + C$.

Example 11: **Find the function whose first difference is e^x.**

Solution: We know that

$$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1),$$

Where h is the interval of differencing.

$\therefore \qquad e^x = \frac{1}{e^h - 1} \; \Delta e^x = \Delta\left(\frac{e^x}{e^h - 1}\right).$

Hence the required function f (x) is $\left(\frac{e^x}{e^h - 1}\right)$.

Example 12: Find $\Delta^3(x^2.a^x)$ by Leibnitz's rule.

Solution: we have $\Delta^3(x^2.a^x) = \Delta^3(a^x.x^2)$

$= (\Delta^3 a^x).x^2 + {}^3C_1 (\Delta^2 E a^x)(\Delta x^2) + {}^3C_2 (\Delta E a^x).(\Delta^2 x^2) + {}^3C_3 (E^3 a^x).(\Delta^3 x^2)$

$= a^x (a^h - 1)^3. x^2 + 3 (\Delta^2 a^{x+h}).\{(x+h)^2 - x^2\} + 3.(\Delta a^{x+2h}).(2h^2) + 0$

$= a^x (a^h - 1)^3. x^2 + 3 a^{x+h} (a^h - 1)^2 (2xh + h^2) + 3a^{x+2h} (a^h - 1).(2h^2).$

Example 13: Use the method of separation of symbols prove that

$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \ldots\ldots.. e^x [u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \ldots..].$

Solution: L.H.S $= u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \ldots\ldots..$

$= u_0 + \frac{x}{1!} E u_0 + \frac{x^2}{2!} E^2 u_0 + \frac{x^3}{3!} E^2 u_0 + \ldots\ldots..$

$= \left[1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \cdots\ldots\right] u_0$

$= e^{xe} u_0 = \mathbf{e^{x(1+\Delta)}} = e^x e^{x\Delta} u_0$

$= e^x \left[1 + x\Delta + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!}\right] u_0$

$= e^x \left[u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \frac{x^3}{3!} \Delta^3 u_0 + \cdots\ldots\right]$

CHECK YOUR PROGRESS

TRUE OR FALSE

(1) . Operators E and Δ are linear.

(2) . If f(x) and g (x) are any function of x. then Δ [f (x) g (x)] = f (x+1) Δ g (x) + g (x) Δ f (x).

(3) . If Δ f (x) = e^x then f (x) = e^x

(4) . E ≡ e^{hd} ≡ 1+Δ

Multiple choice questions

(1) If $f(x) = 2x^3 - 3x^2 + 3x - 10$, then $\Delta^3 f(x) =$

(a) 3 **(c)** 6

(b) 2 (d)12

(2) If $\Delta u_x = x(x-1)$ then u_x is

(a) $x(x-1)(x-2) + C$ **(c)** $\frac{1}{3}x(x-1)(x-2) + C$

(b) $(x-1)(x-2) + C$ **(d)** $x(x-2) + C$

(3) If $f(0) = -3$, $f(1) = 6$, $f(2) = 8$ and $f(3) = 12$ then $\Delta^3 f(0)$ is

(a) 9 **(c)** 6

(b) 3 **(d)** 5

(4) If if f(x) is a polynomial of degree n, then $\Delta^{n+1} f(x)$ is equal to

(a) Zero **(c)** Constant

(b) $f(x+nh)$ **(d)** None of these

(5) The value of $\Delta^2 \cos 2x$ is

(a) $4\sin^2 h \cos(2x+2h)$ **(c)** $4\cos^2 h \cos(2x+2h)$

(b) $-4\sin^2 h \cos(2x+2h)$ **(d)** $-4\cos^2 h \cos(2x+2h)$

3.14 SUMMARY

In this chapter we defined the derivative of a function *f* at a point x, finite difference for a function *f* (x), Difference table. We also defined fundamental theorem of the difference calculus, Algebraic Properties of the Operators E And Δ, Relation between the operators and Leibnit's Rule **.**

3.15 GLOSSARY

Finite Difference Method (FDM): A numerical technique used to solve partial differential equations (PDEs) and ordinary differential equations (ODEs).:
Forward difference operator: A operator used to approximate derivatives using forward differences.
Backward difference operator: A operator used to approximate derivatives using backward differences.
Central difference operator: A operator used to approximate derivatives using central differences.

3.15 REFERENCES

1. S.D.Conteand Carl de boor, Elementary numerical analysis – an algorithmic approach (3rd edition, McGraw-hill, 1981.
2. K. Atkinson: An Introduction to Numerical Analysis, Wiley, (2nd ed.), 1989.
3. "Finite Difference Methods for Partial Differential Equations" by G. D. Smith.

3.17 SUGGESTED READING

1. Numerical methods for scientists and engineers, R.W. Hamming
2. Analysis of numerical methods, Isaacson and keller.
3. "Finite Difference Methods for Ordinary and Partial Differential Equations" by Randall J. LeVeque
4. Finite Difference Methods: A Tutorial by Prof. J. M. McDonough

3.18 TERMINAL AND MODAL QUESTIONS

1. Define operator E and Δ and show that $E\nabla \equiv \nabla E \equiv \Delta$.
2. Evalute the following

(i) $\frac{\Delta^2 x^3}{Ex^3}$

(ii) Δ tan ax

3. Prove that $\Delta \log x = \log\left(1+\frac{h}{x}\right)$
4. Express $x^2 -3x +1$ in factorials. Hence or otherwise find its third difference.
5. Obtain the function whose first difference is $x^3 +4x^2 + 9x +12$
6. Obtain the missing terms in the following table

x :	0	.1	.2	.3	.4	.5	.6
f (x) :	.135	?	.111	.100	.?	.082	.074

7. Interpolate f (2) from the following data:

x:	1	2	3	4	5
f(x):	7	?	13	21	37

and explain why the value obtained is different from that obtained by putting x =2 in the expression

8. If

x:	1	2	3	4	5
y:	2	5	10	20	30

Find by forward difference table Δ^4 y(1).

9. Use the method of separation of symbols to prove the that

$\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + \ldots\ldots (-1)^n u_x.$

FILL IN THE BLANKS

1. $(1 + \Delta)(1 + \nabla)$ = ………..
2. $E^n f(x)$ = …….
3. $\Delta^2 f(a+h) = f(a+2h) - 2f(a+2h)+$……..
4. If $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$, then $\Delta^4 f(x)$ = ……
5. $E^{-1} f(x)$ = ………

3.19 ANSWERS

CHECK YOUR PROGRESS

CYQ1. True
CYQ2. True
CYQ3. False
CYQ4 True
MCQ1. (d)
MCQ2. (c)
MCQ3. (a)
MCQ4. (a)
MCQ5. (c)
TQ2. (i). 6x +2 (ii) $\frac{\sin a}{\cos ax \cos a(x+1)}$
TQ5. $F(x) = x^{(2)} - 2x^{(1)} + 1$, $\Delta f(x) = 2x^{(1)} - 2$, $\Delta^2 f(x) = 2$, $\Delta^3 f(x) = 0$

TQ6. $\frac{x^4}{4} + \frac{5x^3}{6} - \frac{7x^2}{4} + \frac{49x}{6} + k$
Tq7. F (2.1) = 0.123, f (2.4) = 0.090
TQ8. 9.5
TQ9. -8
FQ1. 0
FQ2. $f(x+nh)$
FQ3. $f(a+h)$
FQ4. 24
FQ5. $f(x-h)$

UNIT 4: INTERPOLATION WITH EQUAL INTERVAL

CONTENT

4.1 INTRODUCTION

Interpolation with equal intervals is a numerical technique used to estimate missing values between known data points that are equally spaced. This method is commonly used in various fields such as engineering, physics, and computer science. When data points are equally spaced, it's called interpolation with equal intervals.

According to theile, interpolation is the art of reading between the lines of the table. This is useful for data analysis, modeling and prediction in various disciplines like science engineering and statistics etc. In this unit we will read about methods for interpolation with equal interval.

4.2 OBJECTIVES

After studying this unit, the learner will be able to

1. Understand interpolation and their methods.
2. Describe interpolation with equal intervals with the help of Newton- Gregory formula for forward and backward interpolation.
3. Improve the accuracy of data by estimating values between known points.

Definition: Interpolation

The value of 'x' for which the value of 'y' is to be eliminated is an intermediate value in the given set of values of 'x' then the method of determining 'y' is called interpolation. In other word it is a technique of obtaining the most likely estimate of a certain quantity under certain assumptions.

If the form of the function f (x) is known, we can find f (x) for any value of x by simple substitution. But in case, the form of the function is unknown or if it is quite complicated, then the problem of determining the nature of the function or replacing the function by a comparatively simpler one is also the problem or parabolic or polynomial interpolation.

If we estimate the value of f (x) for any value of the argument outside the given range of the arguments, the technique s known as extrapolation.

There are some points to be kept n mind for interpolation:

(i) The value of the function should be either in increasing or in decreasing order, i.e., in y = f (x), should either increase with increase in value of x or y should decrease with decrease in the value of x.

(ii) The rice or fall in the values should be uniform e.g.

4.3 METHODS OF INTERPOLATION

There are three methods used for interpolation:

(i) Graphical method

(ii) Method for curve fitting

(iii) Application of the calculus of finite differences formulae.

(i) **Graphical method:** If the given function is y = f (x) then we can easily plot a graph between the values of x and the corresponding values of y. from the graph we can find out the value of y for given value of x.

For example, consider the following data:

Year (x):	1891	1901	1911	1921	1931
Population (y): (In thousand)	46	66	81	93	101

Suppose we have to interpolate population for the year 1925.

Steps in graphical method

(i) Take a suitable scale for the values of x and y and plot the various points on the graph paper, for given values of x and y.

(ii) Draw a free hand passing through the plotted points.

(iii) Find the point on the curve corresponding to x = 1925 and find the corresponding value of y.

There are some Drawbacks in the graphical method

1. It's very approximate method of estimating the value of y.

2. In most of the case it's not reliable:

(i) **Method of curve fitting:** This method can be used only in the case in which the form of the function is known. Then by method of least squares we can fit the curve of known form to the given set of observation and with the help of the fitted curve we can calculate the unknown value.

There are some Drawbacks in the curve fitting method

1. This method is not exact.
2. When the number of observations is sufficiently large the method becomes complicated.
3. The form of the function for the given set of observations is assumed to be known.
4. When some additional observations are included in the data then the calculation for finding the unknown constants are to be done afresh.

The only merit of this method lies in the fact that it gives closer approximation than the graphical method.
Let us consider the above example, in which we have to find the population for the year 1925. Let us assume that the function y =f (x) is a first-degree polynomial of the form

$$y = a + bx$$

now our problem is to find the value of a and b from the given data to get the fitted curve. The calculations can be done in tabular form.

Table 1

X	X - 1911	z- (x-1911)/10	y	Y z	z^2
1891	-20	-2	46	-92	4
1901	-10	-1	66	-66	1
1911	0	0	81	0	0
1921	10	1	93	93	1
1931	20	2	101	202	4
Total		0	387	137	10

Then by the method of least squares, we have to minimize

S = $\sum(y - a - bz)^2$.

This gives $\frac{\partial S}{\partial a} = 0 \quad => \quad \sum(y - a - bz) = 0$ …. (1)

and $\frac{\partial S}{\partial b} = 0 \quad => \quad \sum(y - a - bz) = 0$ …. (2)

i.e., $\sum y = n a + b \sum z$ ….. (3)

and $\sum yz = a \sum z + b \sum z^2$. …… (4)

Putting the value of $\sum y$, $\sum z$, $\sum y z$, $\sum z^2$ from the table I and putting n =5 in equation (3) and (4), we get

387 = 5 a +0 or a = 77.4

And 137= 0 + 10b or b =13.7

∴ the required fitted polynomial is

$y = 77.4 + 13.7z$ .... (5)

now we have to find the population for

x = 1925 i.e., for z = $\frac{1925-1911}{10}$ = 1.4.

putting z = 1.4 in (5), we get

y = 77.4 + (13.7) (1.4) = 96.58.

Hence the population for the year 1925 is estimated to be 96.58 thousand.

(ii) Application of the Calculus of Finite Difference Formulae:

The use of finite difference calculus for the purpose of interpolation can be divided into three cases which are as follows:

1. The techniques of interpolation with equal interval.
2. The techniques of interpolation with unequal intervals.
3. The technique of central differences.

Merits: **(i)** These methods do not assume the form of the function to be known.
(ii) These methods are less approximate than the method of graph.
(iii) The calculation remains simple even if some additional observations are include in the given data.

There are some drawbacks of this method:

There is no definite rule to verify whether the assumption for the application of finite difference calculus is verify for the given set of observations.

4.4 INTERPOLATION WITH EQUAL INTERVALS

(i) Newton- Gregory Formula for Forward Interpolation.

If a, a+h, a+2h,….. a +n h are (n+1) equidistant values of the argument x so that a function y =f(x) assumes the values f(a), f(a +h), f(a+2h) ,….. f(a+ nh), then

$$f(a+uh) = f(a) + \frac{u^{(1)}}{1!}\Delta f(a) + \frac{u^{(2)}}{2!}\Delta^2 f(a) + \ldots\ldots + \frac{u^{(n)}}{n!}\Delta^n f(a)$$

$$= \sum_{r=0}^{n} \frac{u^{(r)}}{r!}\Delta^r f(a),$$

Where $u^{(r)}$= u (u-1) (u-2)……. $\{u-(r-1)\}$.

Proof. Since we are known (n+1) pairs of values of the argument x and the entry f (x), therefore f (x) can be represented as a polynomial in x of degree n.

Let $f(x) = A_0 + A_1(x - a) + A_2(x - a)(x - a - h) + A_3(x - a)(x - a - h)(x - a - 2h) + \dots + A_n(x - a)(x - a - h) \dots (x - a - \overline{n-1}\,h)$, …….(1)

Where $A_0, A_1, A_2, \dots A_n$ are constants.

To find the constants $A_0, A_1, A_2, \dots A_n$ we put successively the values In (1) for x. thus, we get

$f(a) = A_0 \Rightarrow A_0 = f(a)$,

$f(a + h) = A_0 + A_1 h \Rightarrow A_1 = \frac{f(a+h) - f(a)}{h} = \frac{\Delta f(a)}{h}$,

$f(a + 2h) = A_0 + 2h A_1 + 2h.h A_2$.

$\Rightarrow A_2 = \frac{f(a+2h) - 2[f(a+h - f(a)] - f(a)}{2h^2} = \frac{1}{2!h^2}\Delta^2 f(a)$.

Similarly, $A_3 = \frac{1}{3!}\Delta^3 f(a)$,

…. …. ….

$A_n = \frac{1}{n!h^n}\Delta^n f(a)$.

Putting the value of $A_0, A_1, \dots A_n$ found above in (1), we get

$f(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!h^2}(x - a)(x - a - h) + \frac{\Delta^3 f(a)}{3!h^3}(x - a)(x - a - h)(x - a - 2h) + \dots + \frac{\Delta^n f(a)}{n!h^n}(x - a)(x - a - h)(x - a - 2h) \dots (x - a - \overline{n-1\,h})$.

This is known as Newton- Gregory formula for forward interpolation.

Putting $\frac{x-a}{h} = u$ or $x = a + hu$, the formula takes the form

$f(a+uh) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + u(u-1)(u-2)\Delta^3 f(a) + \dots + u(u-1)(u-2)\dots(u - n + 1)\Delta^n f(a)$ …….. (2)

the result (2) can be written as

$f(a + uh) = f(a) + u^{(1)}\Delta f(a) + \frac{1}{2!}u^{(2)}\Delta^2 f(a) + \frac{1}{n!}u^{(n)}\Delta^n f(a)$

where $u^{(n)} = u(u - 1)(u - 2) \dots (u - n + 1)$.

This is the form in which Newton – Gregory formula for forward interpolation is often written.

(ii) Newton- Gregory Formula for backward Interpolation:

Suppose a, a+h, a+2h,…..a +nh are (n+1) equidistant values of the argument x so that a function $y = f(x)$ assumes the values $f(a)$, $f(a+h)$, $f(a+2h)$,….. $f(a+ nh)$, then

$$f(a + nh + uh) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!}\nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!}\Delta^3 f(a + nh) + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!}\Delta^n f(a + nh).$$

Proof: Since we are known (n+1) pairs of values of the argument x and the entry f (x), therefore f (x) can be represented as a polynomial in x of degree n.

Let f (x) = A_0 +A_1 (x – a – nh) + A_2 (x – a- nh) (x – a – nh+h) + A_3 (x – a - nh) (x – a – nh + h) (x – a – nh+2h) +………+ A_n (x – a- nh) (x – a – nh+h) (x – a – h), ………… (1)

Where Where A_0 , A_1, A_2, …….A_n are constants.

To find the constants A_0 , A_1, A_2, …….A_n we put successively the values

a+ nh, a + (n – 1) h, a + (n – 2) h,…….a + h, a

In (1) for x. thus, we get

f (a + nh) = A_0 or A_0 =f (a + nh),

f (a + nh - h) = A_0 +A_1(- h)

or $A_1 = \frac{f(a+nh) - f\,(a+nh-h)}{h} = \frac{\Delta f\ (\text{a+nh})}{h}$,

f (a +n h – 2h) = A_0 +A_1(-2h) + A_2 (-2h).(-h).

or $A_2 = \frac{-\,A_0 + 2A_1 h + f\,(a+\overline{n-2h})}{2h^2}$

$= \frac{-f\,(a+nh) + 2\,[(a+nh) - f(a+nh-h)] + f(a+\overline{n-2h})}{2h^2}$

$= \frac{f\,(a+nh) - 2f(a+\overline{n-1h}) + 2\,f(a+\overline{n-2h})}{2h^2}$

$= \frac{1}{2!h^2}\,\nabla^2$f (a + nh).

Similarly, $A_3 = \frac{1}{3!}\nabla^3 f$ (a +nh),…….. $A_n = \frac{1}{n!h^n}\,\nabla^n f$ (a +nh).

Sunstituting the values of A_0, A_1,…..A_n in (1), we get

f (x) = f (a + nh) +$\frac{\nabla f(a+nh)}{h}$ (x- $\overline{a+nh}$) + $\frac{\nabla^2 f\ (a+nh)}{2!h^2}$(x- $\overline{a+nh}$) (x- $\overline{a+nh-h}$)+….+ $\frac{\nabla^n f\ (a+nh)}{n!h^n}$(x- $\overline{a+nh}$) (x- $\overline{a+nh-h}$ …. (x – a – h).

this is Newton- Gregory formula for backword interpolation.

Putting u = $\frac{x-(a\ +n\ h)}{h}$ or x = a +n h +uh, we get

f (a+ n h +u h) = f (a + n h) + u∇ f (a + n h) +$\frac{u\ (u+1)}{2!}$ ∇^2f (a + n h) +$\frac{u\ (u+1)(u+2)}{3!}$ ∇^3f (a +n h) +$\frac{u\ (u+1)\ldots\ldots(u+n-1)}{n!}$ ∇^nf (a+ n h)…..(2)

this is the form in which Newton- Gregory formula for backward interpolation is often written.

ILLUSTRATIVE EXAMPLES

Example1: From the following table, estimate the number of students who obtained marks between 40 to 45.

Marks	No. of Students
30-40	31
40-50	42
50-60	51
60-70	35
70-80	31

Solution : We prepare the cumulative frequency table first, as given below:

Marks above 30 but less than X	No. of Students f (x)
40	31
50	73
60	124
70	159
80	190

Now we prepare the difference table for the data.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Now we shall find:
f(45) = no. of students with marks less then 45.
Taking a+u h = 45, we get
40 +u ×10 = 45 => u = $\frac{1}{2}$.
By using neuton forward interpretation formula, we get

$$f(45) = f(40) + \frac{1}{2}\Delta f(40) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{\Delta^2 f(40)}{2!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{\Delta^4 f(40)}{4!}$$

$$= 31 + \frac{1}{2} \times 42 + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!} \times 9 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}(-25) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{4!} \times 37$$

= 47.868 (on simplification).
Hence the number of students with marks less than 45 is 47.868 i.e., 48.
But the number of students with marks less than 40 is 31.
Hence the required number of students getting marks between 40 and 45 is
= 48-31 = 17.

Example 2: Given $\sin 45^0$ = 0.7071, $\sin 50^0$ = 0.7660
$\sin 55^0$ = 0.8192, $\sin 60^0$ = 0.8660.
Find $\sin 52^0$ by using any method of interpolation.

Solution: Here, we have

X:	45^0	50^0	55^0	60^0
f(x):	0.7071	0.7660	0.8192	0.8660

The difference table for the given data is as as follows:

X	f(x)	Δf (x)	Δ^2f (x)	Δ^3f (x)
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

We want f (52^0) = f (a +uh), say.
$\therefore\ 52^0 = a + uh \Rightarrow 52^0 = 45^0 + u \times 5^0 \Rightarrow u = \frac{7}{5} = 1.4,$
By newton's forward interpolation formula, we get

$$f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a).$$

$$\therefore f(52^0) = f(45^0) + 1.4\,\Delta f(45^0) + \frac{(1.4)(0.4)}{2!}\Delta^2 f(45^0) + \frac{(1.4)(0.4)(-0.6)}{3!}\Delta^3 f(45^0).$$

$$= 0.7071 + 1.4 \times 0.0589 + \frac{(1.4)(0.4)}{2!}(-0.0057) + \frac{(1.4)(0.4)(-0.6)}{3!}(0.0007)$$

= 0. 7071 +0.08246 – 0.001596 +0.0000392 = 0.7880032.

Thus sin 52^0 = 0.7880032 = 0.7880 approx.

Example 3: The population of a country in the decennial cansus were as under Estimate the population for the year 1925.

Year x :	**1891**	**1901**	**1911**	**1921**	**1931**
Population y :	**46**	**66**	**81**	**93**	**101**
(in thousand)					

Solution: let us introduce a new variable u given by u = $\frac{x-1891}{10}$.

∴ u takes the value 0,1,2,3,4.

The difference table is as follows:

u	Y	Δy	Δ^2y	Δ^3y	Δ^4y
0	46				
		20			
1	66		-5		
		15		2	
2	81		-3		-3
		12		-1	
3	93		-4		
		8			
4	101				

Since five value are given, we must assume that fourth differences are contant.

We want the entry for x = 125, i.e., for u=

We have

$Y_{3.4} = E^{3.4}\, y_0 = (1+\Delta)^{3.4}\, y_0$

$= y_0 + 3.4\, y_0 + \frac{3.4 \times 2.4}{1\times 2}\Delta^2 y_0 + \frac{3.4 \times 2.4 \times 1.4}{1\times 2\times 3}\Delta^3 y_0 + \frac{3.4 \times 2.4 \times 1.4 \times .4}{1\times 2\times 3\times 4}\Delta^4 y_0$

$= 46\ 3.4 \times 20 + \frac{3.4 \times 2.4}{2}(-5) + \frac{3.4 \times 2.4 \times 1.4}{6} \times 2 + \frac{3.4 \times 2.4 \times 1.4 \times .4}{24}(-3)$

= 46 + 68-20.4 +3.808 – 0.5712 = 96.8368 thousand.

Hence the population for 1925 is estimated to be 96.8368 thousand.

Example 4: Given

x :	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
f **(x):**	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**

Find f (7.5).

Solution: The value to be interpolated les at the end of the given observations i.e., near x= 8. So, in this case Newton's backward formula will be more suitable.

Here u = $\frac{x-(a+nh)}{h} = \frac{7.5-8}{1}$ = - 0.5.

To calculate backward differences $\nabla f(a+h), \nabla^2$ f (a+nh), …… we prepare the following difference table.

x	f (x)	∇f (x)	∇^2f (x)	∇^3f (x)
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

Since ∇^3 f (x) is constant, so that we can leave higher order differences.

By Newton's backward interpolation formula,

f (a + n h + u h) = f (a + n h) + u ∇ f (a + n h) + $\frac{u(u+1)}{2!}$ $\nabla^2 f$ (a + n h) + $\frac{u(u+1)(u+2)}{3!}$ ∇^3 f (a + n h)

$\therefore$ f (7.5) = f (8) + (-0.5) ∇ f (8) + $\frac{(-0.5)\ (-0.5+1)}{2!}$ $\nabla^2 f$ (8) + $\frac{(-0.5)(-0.5+1)(0.5+2)}{6}$ $\nabla^3 f$ (8)

= 512 + (- 0.5) × 169 + $\frac{(-0.5)\ (0.5)}{2}$ × 42 + $\frac{(-0.5)(0.5)(1.5)}{6}$ × 6

= 512 – 84.5 – 5.25 - .375 = 421.875.

Example : Show that Newton- Gregory interpolation formula can be put in the form $u_x = u_0 + x\,\Delta u_0 - a\,x\,\Delta^2 u_0 + a\,b\,x\,\Delta^3 u_0 - a\,b\,c\ x\ \Delta^4 u_0 + \ldots\ldots$
Where a= 1- $\frac{1}{2}$ (x+1), b = 1- $\frac{1}{3}$ (x+1), c = 1- $\frac{1}{4}$ (x+1) etc.

Solution: Here a= 1- $\frac{1}{2}$ (x+1).

-a = $\frac{1}{2}$ (x+1) -1 = $\frac{1}{2}$ (x-1).

Also, b = 1- $\frac{1}{3}$ (x+1).

-b = $\frac{1}{3}$ (x+1) -1 = $\frac{1}{3}$ (x-2).

Similarly –c = $\frac{1}{4}$ (x-3), -d = $\frac{1}{5}$ (x-4), etc.

Newton- Gregory interpolation formula **gives**

$u_x = u_0 + \frac{x}{1!}\Delta u_0 + \frac{x\,(x-1)}{2!}\Delta^2 u_0\ \frac{x\,(x-1)(x-2)}{3!}\Delta^3 u_0 + \frac{x\,(x-1)(x-2)(x-3)}{4!}\Delta^4 u_0 + \ldots\ldots$

$= u_0 + x\,\Delta\, u_0 + \frac{x\,(x-1)}{2!}\Delta^2 u_0 + \frac{x\,(x-1)}{2}\left(\frac{x-2}{3}\right)\Delta^3 u_0$

$+\frac{x\,(x-1)}{2}\left(\frac{x-2}{3}\right)\left(\frac{x-3}{4}\right)\left(\frac{x-2}{3}\right)\Delta^4 u_0 + \ldots\ldots..$

$= u_0 + x\Delta\, u_0 + x\,(-a)\,\Delta^2 u_0 + x\,(-a)\,(-b)\,\Delta^3 u_0 + x\,(-a)\,(-b)(-c)\,\Delta^4 u_0 + \ldots\ldots$

$= u_0 + x\,\Delta\, u_0 - x\,a\,\Delta^2 u_0 + a\,b\,x\,\Delta^3 u_0 - a\,b\,c\,x\,\Delta^4 u_0 + \ldots\ldots$

CHECK YOUR PROGRESS

TRUE OR FALSE

1. The determination of value f (y) at any point y inside the interval $[x_1, x_n]$ is called extrapolation.
2. The n^{th} difference of a polynomial of degree n is constant.
3. Newton- Gregory formula forward interpolation is used for equal interval.

Multiple choice questions

1. For the values f (0) = 3, f (1) = 6, f (2) = 11, f (3) = 27, the form of the function f (x) is

(a) $x^2 + 2x + 3$ **(c)** $x^2 + 2x + 5$

(b) $x^2 + 2x + 1$ **(d)** $x^2 + 2x + 4$

2. The nth term of the Newton's Gregory formula is:

(***a***) $\frac{\Delta^{n+1} f(a)}{(n+1)!h^{n+1}}(x-a)\,(x-(a+h))\ldots\ldots..(x-(a+nh))$

(***b***) $\frac{\Delta^{n+1} f(a)}{(n+1)!h^{n}}(x-a)\,(x-(a+h))\ldots\ldots..(x-(a+nh))$

(***c***) $\frac{\Delta^{n} f(a)}{n!h(n)}(x-a)\,(x-(a+h))\ldots\ldots..(x-(a+nh))$

(***d***) $\frac{\Delta^{n} f(a)}{n!h^{n}}(x-a)\,(x-(a+h))\ldots\ldots..(x-(a+(n-1)h))$

4.5 SUMMARY

1. Interpolation with equal interval: Estimating values between known data points with equal spacing.
2. Graphical method
3. Newton- Gregory Formula for Forward Interpolation.
4. Newton- Gregory Formula for backward interpolation.

4.6 GLOSSARY

1. Interpolation: The process of estimating missing values between known data points.
2. Equal Intervals: The data points are equally spaced, meaning the distance between consecutive points is constant.
3. Polynomial Interpolation: A method of interpolation that uses a polynomial function to estimate missing values.

4.7 REFERENCES

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3. K. Atkinson: An Introduction to Numerical Analysis, Wiley, (2nd ed.), 1989.
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5. E. W. Cheney and D. R. Kincaid: Numerical Mathematics and Computing, Brooks Cole, 6 editions, 2007.

4.8 SUGGESTED READING

1. Interpolation and Approximation" by Philip J. Davis .
2. "Interpolation with equal intervals" by J.M. Hammersley (Numerische Mathematik, 1956).
3. "On the interpolation with equal intervals" by S.D. Conte (Journal of Numerical Analysis, 1965)
4. "Interpolation with equal intervals: A review" by R.W. Hamming (Journal of Computational and Applied Mathematics, 1973)

4.9 TERMINAL AND MODAL QUESTIONS

1. Derive an interpolation formula for equal interval.
2. Find y when x = 8 for

X :	0	5	10	15	20	25
Y:	7	11	14	18	24	32

What do you mean by interpolation?

3. From the following table, for what value of x, y is minimum? Also find this value of y.

X:	3	4	5	6	7	8
Y:	0.205	0.240	0.259	0.260	0.250	0.224

FILL IN THE BLANKS

1. The technique of obtaining the most likely estimate of a certain quantity under certain assumptions is called ………..
2. The technique of obtaining the value of f (x) for any value of the argument outside the given range of the arguments is known as ……….

4.10 ANSWERS

CYQ1. False
CYQ2. True
CYQ3. True
MCQ1. (a)
MCQ2. (d)
TQ2. 13 approx.
TQ4. 5.6875, 0.2628
FQ1. Interpolation
FQ2. Extrapolation

UNIT 5 : INTERPOLATION WITH UNEQUAL INTERVALS

CONTENT:

5.1 *INTRODUCTION*

In the preceding chapter the interpolation formulae are applicable only when the value of the function is given at equidistant intervals of the independent variable. sometime it is inconvenient, or even impossible, to obtain value of a function at equidistant values of its argument, and in such case, it is desirable to have interpolation formulae which are applicable when the functional value are given at unequal interval of the argument. Two such formulae are newton's formula for unequal interval and Lagrange's formula. In this unit we discussed about divide differences and the method of interpolation with unequal interval and their examples.

5.2 OBJECTIVES

After studying this unit, the learner will be able to

1. Explain interpolation with unequal intervals
2. Understand Newton's and lagrenges's formula for unequal intervals and solve problem related them.

5.3 DIVIDE DIFFERENCES

Suppose f (x_0), f (x_1), f (x_2),….. f (x_n) be the entries corresponding to the arguments x_0, x_1, x_2,….. x_n where the intervals x_1- x_0, x_2 –x_1,… x_n – x_{n-1} may not be equal , i.e., nt necessarily equally spaced . Then the first divided difference of f (x) for arugments x_0, x_1 is defined as

$\frac{f(x_1)-f(x_0)}{x_1-x_0}$ or $\frac{f(x_0)-f(x_1)}{x_0-x_1}$ and is denoted by f (x_0, x_1) or by Δ_{x1} f (x_0)

Similarly the other first divided difference of f (x) for the arguments x_1, x_2, x_3,…., xn-1 are

$$f(x_1, x_2) = \frac{f(x_2) - f(1)}{x_2 - x_1} = \Delta_{x2} f(x_1)$$

$$f(x_2, x_3) = \frac{f(x_3) - f(2)}{x_3 - x_2} = \Delta_{x3} f(x_2)$$

…. …….. ……

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(n-1)}{x_n - x_{n-1}} = \Delta_{xn} f(x_{n-1}).$$

The second divide difference of *f* (x) for three arguments x_0, x_1 and x_2 s defined as

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_1} = \Delta^2_{x1, x2} f(x_{n-1}).$$

The nth divide difference is given by

$$f(x_0, x_1, x_2 \ldots\ldots, x_n) = \frac{f(x_1, x_2, \ldots. xn) - f(x_0, x_1, \ldots.. x_{n-1})}{x_n - x_0}$$

$$= \frac{f(x_0 \, x_1, \ldots. xn-1) - f(x_1, x_2, \ldots.. x_n)}{x_0 - x_n} = \Delta^n_{x1, x2, \ldots..xn} f(x_0).$$

Note : if two of the argument concide, the divide difference can be given a meaning assigned by taking the limit. Thus

$$F(x_0, x_0) = \lim_{e \to 0} f(x_0, x_0 + \varepsilon)$$

$$= \lim_{e \to 0} f(x_0 + \varepsilon x_0 + \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon} = f'(x_0),$$

If f (x) is differentiable.

Similarly, f (x_0, x_0, x_0) = $\frac{1}{2!}$ f'' (x_0),…., f (x_0, x_0, x_0) = $\frac{1}{r!}$ $f^{(r)}$ (x_0).

(r+1) arguments

Divide Difference table:

A useful table called the divided difference table showing the value of the argument x, the entries and the divided difference is given below.

Divided difference table

X	**f (x)**	**Δ f(x)**	**Δ^2 f(x)**	**Δ^3 f(x)**
x_0	$f(x_0)$	$f(x_0,x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$		
			f $(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$	
x_1	$f(x_1)$	$f(x_1,x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	f $(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$	f $(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
x_2	$f(x_2)$	$f(x_2,x_3) = \frac{f(x_3) - f(2)}{x_3 - x_2}$		f $(x_1, x_2, x_3, x_4) = \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$
		$f(x_3,x_4) = \frac{f(x_4) - f(x_3)}{x_4 - x_3}$	f $(x_1, x_2, x_3) = \frac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2}$	
x_3	$f(x_3)$			

Theorum 1: The divided difference are symmetric functions of their arguments, that is the value of any divided difference is independent of the order of the arguments.

Proof: We have

$$f(x_{0,}x_0) = \frac{f(x_1-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1, x_0)$$

$$= \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0} = \sum \frac{f(x_0)}{x_0-x_1}.$$

Showing that $f(x_0, x_1)$ is symmetrical in $x_{0,}\ x_{1.}$

Again $f(x_0, x_{1,} x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

$$\frac{1}{(x_2-x_0)} = \left[\left\{\frac{f(x_1)}{x_1-x_2} + \frac{f(x_2)}{x_2-x_1}\right\}\right] - \left[\left\{\frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}\right\}\right]$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)[(x_1-x_0)-(x_1-x_2)]}{(x_2-x_0)(x_1-x_2)(x_1-x_0)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

Or $f(x_0, x_{1,} x_2) = \sum \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$,

Showing that $f(x_0, x_{1,} x_2)$ is symmetrical in $x_0, x_{1,} x_2$.

Let us assume similar symmetrical expression for the (n-1) th divided differences

i.e., let us assumes that

$$f(x_0, x_{1,\ldots\ldots} x_{n-1}) = \frac{f(x_0)}{(x_0-x_1)(x_0-x_{n-1})} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_{n-1})}$$

$$+\ldots+ \frac{f(x_{n-1})}{(x_{n-1}-x_0)(x_{n-1}-1)\ldots\ldots(x_{n-1}-x_{n-2})}$$

$= \sum \frac{f(x_0)}{(x_0-x_1)(x_0-x_{n-1})}$ and similar expression for the other (n-1)th divided differences. Then

f $(x_0, x_{1,\ldots\ldots} x_n) = \frac{f(x_{0,\ldots\ldots,}x_{n-1}) - f(x_{1,\ldots\ldots,}x_n)}{x_0-x_n}$

$$= \frac{1}{(x_0-x_n)} = \left[\left\{\frac{f(x_0)}{(x_0-x_1)(x_0-x_{n-1})} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_{n-1})} + \ldots + \frac{f(x_{n-1})}{(x_{n-1}-x_0)\ldots\ldots(x_{n-1}-x_{n-2})}\right\} - \left\{\frac{f(x_1)}{(x_1-x_2)(x_1-x_n)} + \frac{f(x_2)}{(x_2-x_1)\ldots\ldots(x_2-x_n)} + \cdots + \frac{f(x_n)}{(x_n-x_1)(x_n-x_{n-1})}\right\}\right]$$

$$=\frac{f(x_0)}{(x_0-x_1)(x_0-x_n)}+\frac{f(x_1)}{(x_1-x_0)......(x_1-x_n)}+\cdots..+\frac{f(x_n)}{(x_n-x_0)..........(x_n-x_{n-1})}$$

$$=\sum\frac{f(x_0)}{(x_0-x_1)........(x_0-x_n)}$$

Showing that the nth divide difference $f(x_0, x_1,...... x_n)$ is also symmetrical in $x_0, x_1,...... x_n$. this is the proof of the theorum by mathematical induction.

5.4 NEWTON'S FORMULA FOR UNEQUAL INTERVAL

Suppose f (x_0), f (x_1),….. f (x_n) be the value of f (x) corresponding to the arguments $x_0, x_1,...... x_n$, not necessarily equally speed . From the definition of
Divided difference,

f (x, x_0) = $\frac{f(x)-f(x_0)}{x-x_0}$

Or f (x) $=f(x_0) + (x-x_0) f(x, x_0)$. …(1)

Also $f(x, x_0, x_1) = \frac{f(x, x_0)-f(x_0, x_1)}{x-x_1}$

Or $f(x, x_0) = f(x_0, x_1) + (x - x_1)\ f(x, x_0, x_1)$. …(2)

Similarly $f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2)\ f(x, x_0, x_1,$ …(3)

…… ….. ….. ….. ….

……. …… …… …… …..

$f(x, x_0, x_1, ...x_{n-1}) = f(x_0, x_1, ...x_n) + (x - x_n)\ f(x, x_0, x_1, ...x_n)$ ….(4)

multiplying the equation (2) by $(x-x_0)$, (3) by $(x-x_0)$ $(x-x_1)$ and so on and

finally the equation (1), we have

f (x) = $f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) +....+$

$(x-x_0)(x-x_1).....(x-x_{n-1}) f\left(x_0, x_1, ...x_n\right) + R_n,$

Where the remainder R_n is given by

$R_n = (x-x_0)(x-x_1).....(x-x_{n-1}) f\left(x_0, x_1, ...x_n\right)$.

Assuming that f (x) is a polynomial of degree n, $f\left(x_0, x_1, ...x_n\right)$ vanishes so that

f (x) = $f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) +....+$

$(x-x_0)(x-x_1).....(x-x_{n-1}) f\left(x_0, x_1, ...x_n\right)$. ...(5)

This formula is called Newton's divided difference interpolation formula.

5.5 LAGRANGE'S INTERPOLATION FORMULA OR UNEQUAL INTERVALS

Let f (x_0), f (x_1), f (x_2),……. f (x_n), be the (n+1) values of the function y = f (x)

corresponding to the arugments $x_0, x_1, x_2, x_3, \ldots\ldots x_n$ not necessarily equaly spaced. It

is assume that the function f (x) is a polynomial in x since (n+1) , values of f (x) are given so (n+1)th difference are zero. Thus f (x) is supposed to be a polynomial in x of degree n. Writing

f (x) = A_0 $(x-x_0)(x-x_2)\ldots\ldots(x-x_n)+$ …. + A_n $(x-x_0)(x-x_2)\ldots\ldots(x-x_{n-1})$, …..(1)

where A's are constant. The relation (1) is true for all values of x. to determine A_0, put x = x_0.

∴ f (x_0) $=(x_0-x_1)\ (x_0-x_2)\ldots\ldots(x_0-x_n)$

Or $A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}$

Similarly $A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}$

….. ….. …..

$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_0)\ldots\ldots(x_n-x_{n-1})}$

Putting these values of A's in (1)

f (x) = $\frac{(x_0-x_1)\ (x_0-x_2)\ldots\ldots(x_0-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}$ f (x_0) $+\frac{(x-x_0)\ (x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}$ f (x_1) +….+$\frac{(x-x_0)\ (x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}$ f (x_n) ……….(2)

which is lagrange's interpolation formula.

5.6 ITERATIVE METHOD

Newton's forward difference formula is given by

$y_n = y_0 + u\,\Delta\, y_0 + \frac{u\,(u-1)}{2!}\,\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \ldots\ldots.$

⇨ $u = \frac{1}{2\Delta y_0}\left[y_n - y_0 - \frac{u\,(u-1)}{2!}\ \Delta^2 y_0 - \frac{u\,(u-1)(u-2)}{3!}\ \Delta^3 y_0 - \cdots\right]$ … (1)

If we neglet the second and higher differences, obtain the first approximaton u as

$u_1 = \frac{(y_n-y_0)}{\Delta y_0}$ … (2)

To find the second approximation we retain the terms up to second differences

$$\Rightarrow \quad u_2 = \frac{1}{\Delta y_0}\left[y_n - y_0 - \frac{u\,(u-1)}{2!}\ \Delta^2 y_0\right] \qquad ...(3)$$

Further, to find the third approximation, we again retain the terms with third differences in (1) and get

$$u_3 = \frac{1}{\Delta y_0}\left[y_n - y_0 - \frac{u_2\,(u_2-1)}{2!}\ \Delta^2 y_0 - \frac{u_2\,(u_2-1)(u_2-2}{3!} 3y_0\right] \text{ etc.}$$

The process is continued till two successive approximations agree with each other.

5.7 HERMITE'S INTERPOLATION FORMULA

As earlier we discussed various interpolation formulae which make use only of a certain number of function values. Now we derive an interpolation formula in which both of function and its first derivatives are to be assigned at each point of interpolation.

The problem of interpolation is then: given the set of data points ($x_i y_i y_i{}'$) , i= 0,1,…n , it is required to find a polynomial of the least degree , say

And
$$\left.\begin{aligned} \emptyset_{2n+1\,(x_i)} &= y_{i,} \\ \emptyset'_{2n+1\,(x_i)} &= y_i{}',\ \ i = 0,1,\ldots..n \end{aligned}\right\} \qquad ...(1)$$

Here we have (2n+2) conditions and we note that a polynomial of degree (2n+1) has (2n+2) coefficient to be determined.

We write $\emptyset_{2n+1}$ (x) = $\sum_{i=0}^{n} u_i\,(x) y_i + \sum_{i=0}^{n} v_i\,(x)\, y_i{}'$, ...(2)

Where $u_i\,(x)\ \ v_i\,(x)$ are polynomials in x of degree (2n+1). Using the conditions (1), we get

$$\left.\begin{aligned} u_i(x_j) = 1 \ \ if\ i = j,\ \ v_i(x_j) &= 0; \\ = 0\ if\ i \neq j;\ v_i{}'(x_j) &= 1\ if\ i = j \\ u_i{}'(x_j) = 0 \qquad\qquad &= 0\ if\ i \neq j \end{aligned}\right\} \qquad \ldots (3)$$

We therefore choose

$$\left.\begin{aligned} u_i\,(x) &= a_i\ (x)\,[l_i(x)]^2 \\ and \quad v_i\,(x) &= b_i\ (x)\,[l_i(x)]^2 \end{aligned}\right\} \qquad \ldots(4)$$

Where $l_i\,(x)$ is defined as

$$l_i\,(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots\ldots(x-x_{i-1})(x-x_{i+1})\ldots\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots\ldots\ldots(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_n)}\ .$$

Since $l_i\,(x)$ *is a polynomal* of degree n , it follows that $a_i\,(x)$ and $b_i\ (x)$ are both linear functions.

By using the conditions (3), we get

$$\left.\begin{aligned} u_i\,x_i = 1 \qquad &; \qquad b_i(x_i) = 0\,; \\ a_i{}'(x_i) = -2\,l_i{}'(x_j)\ &; \qquad b_i(x_i) = 1 \end{aligned}\right\} \qquad \ldots(5)$$

From which it follow that

$a_i(x)$ = 1-2$l_i'(x_i)$ (x- x_i)

And $b_i(x)$ = x- x_i .

Hence (2) becomes
$\emptyset_{2n+1}(x) = \sum_0^1 \left[\{1-2l_i'(x_i)\,(x-x_i)\}\{l_i(x)\}^2 y_i\right] + \sum_{i=0}^{n}[(x-x_i)\{l_i(x)\}^2 y_i']$ ….(6)
Which is Hermite's interpolation formula

5.8 SHEPPARD'S RULE

We see in Newton's divides difference formula the function f (x) is expressed in term of leading term and leading differences of a divided difference table . But now we shall give a rule known as Sheppard's rule which can be used to write a divide difference formula with any value of f (x) as the initial term.

Newton divide difference formula for interpolation is
f(x) = p(x)
= f (x_0) + (x-x_0) Δ_{x1} f (x_0) + (x-x_0) (x-x_1) $\Delta^2_{x1,x2}$ f (x_0)+…..+ (x-x_0) (x-x_1) …… (x-x_{n-1}) Δ^n x1….,xn f (x_0) ……(1)

In (1), put
x- x_0 = X_0 , x- x_1 = X_1,….. x - x_{n-1}= X_{n-1}.
Then we have
f(x) = f (x_0) + X_0 Δ_{x1} f (x_0) +Δ^2 x1, x2 X_0 X_1 f (x_0) +……+ X_0...... X_{n-1} Δ^n x1….,xn f (x_0) ……(2)

In this R.H.S of (2) we observed that the coefficient of various terms are written n capital letters X_0, X_1,….. etc. and the suffixes of the operator and operand are given in small letters x_0, x_1 etc. In the first term of R.H.S of (2), the coefficient of f (x_0) is one and the suffix of the operand is x_0. In the second term, the coefficient is X_0 and there are two suffixes x_0 *and* x_1 in the operator and operand. . In the third term the coefficient consist of two letters X_0, X_1 there are three suffixes x_0, x_1 and x_2 in the operator and operand so on . We conclude from this that small letter in each term of R.H.S of (2) are one more in number then the number of capital letters. Now we give the small and capital letters term by term.
Small letters: x_0 x_0x_1 $x_0x_1x_2$ ……….$x_0x_1 \dots \dots x_{n-1}x_n$
Capital letters: 1 X_0 X_0 X_1 .... X_0 X_1.......... X_{n-1}.

This characteristic of Newton's divided difference formula has given rise to a rule which is called Sheppard's rule or Zigzag rule, and is as follows:

(i) Start with any initial term.
(ii) In order to get the second term, take any first order difference, either moving upward

and downward in the divided difference table, which contain the suffix of the initial term and multiply this first order difference by the term obtained by subtracting from x and the value of the suffix of the initial term.

(iii) In the same way find the third term, keeping in mind the suffixes of second term. In this way complete the formula.

ILLUSTRATIVE EXAMPLES

Example 1: By means of Newton's divided difference formula, find the value of f (8) and f (15) from the following table:

X :	**4**	**5**	**7**	**10**	**11**	**13**
f (x) :	**48**	**100**	**294**	**900**	**1210**	**2028**

Solution: The divide difference table is:

x	f (x)	Δf (x)	$\Delta^2 f$ (x)	$\Delta^3 f$ (x)	$\Delta^4 f$ (x)
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100				
			$\frac{97-52}{7-4} = 15$		
7	294				
		$\frac{294-100}{7-5} = 97$			
				$\frac{21-15}{10-4} = 1$	
10	900				
			$\frac{202-97}{10-5} = 21$		0
11	1210	$\frac{900-294}{10-7} = 202$		$\frac{100-48}{5-4} = 1$	
13	2028				
			$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
			$\frac{409-310}{13-10} = 33$		
		$\frac{2023-1210}{13-11} = 409$			

Using Newton's interpolation formula for unequal intervals i.e., Newton's divided difference formula , we get

$f(8) = 48 + (8 - 4) \times 52 + (8-4)(8-5) \times 15 + (8 - 4)(8 - 5)(8 - 7) \times 1$
$= 448,$

And $f(15) = 48 + (15 - 4) \times 52 + (15 - 4)(15 - 5) \times 15 + (15-4)(15 - 5)(15-7) \times 1 = 3150.$

Example 2: If f (x) = $\frac{1}{x^2}$, find the divded differences of f (a,b) , f (a,b,c) and f (a,b,c,d).

Solution: We have $f(a,b) = \frac{f(b) - f(a)}{b-a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a}$

$$= -\frac{(b^2-a^2)}{(b-a)b^2a^2} = -\frac{b+a}{a^2b^2} \qquad \ldots(1)$$

Again $f(a, b, c) = \frac{f(b,c) - f(a,b)}{c-a}$

$$= \frac{1}{c-a}\left[-\frac{b+c}{b^2c^2} + \frac{a+b}{a^2b^2}\right], \qquad \text{using (1)}$$

$$= -\frac{1}{(c-a)}\left[\frac{b+c}{b^2c^2} - \frac{a+b}{a^2b^2}\right]$$

$$= -\frac{1}{(c-a)}\left[\frac{a^2(b+c) - c^2(a+b)}{a^2b^2c^2}\right]$$

$$= -\left[\frac{b(a^2-c^2) + ac(a-c)}{(c-a)a^2b^2c^2}\right] = -\frac{(a-c)\{b(a+c)+ac\}}{(c-a)a^2b^2c^2}$$

$$= \frac{ab+bc+ca}{a^2b^2c^2}. \qquad \ldots.(2)$$

Now $f(a, b, c, d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$

$$= \frac{1}{d-a}\left[\frac{bc+cd+db}{b^2c^2d^2} + \frac{ab+bc+ca}{a^2b^2c^2}\right]$$

$$= \frac{1}{d-a}\left[\frac{a^2(bc+cd+db) - d^2(ab+bc+ca)}{a^2b^2c^2d^2}\right] \qquad \text{using (2)}$$

$$= \frac{1}{d-a}\left[\frac{bc(a^2-d^2) + acd(a-d) + abd(a-d)}{a^2b^2c^2d^2}\right]$$

$$= \frac{a-d}{d-a}\left[\frac{bc(a+d) + acd + abd)}{a^2b^2c^2d^2}\right]$$

$$= -\frac{abc+bcd+acd+abd}{a^2b^2c^2d^2} =$$

Example 3: Construct a divided difference table for the following:

X: 1 2 4 7

F (x): 22 30 82 106

Solution : The divide difference table for the given data is as follows:

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
1	22	$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-08}{4-1}=$ 6		
4	82	$\frac{82-30}{4-2}=26$		$\frac{(-3.6)-6}{7-1}$ = -1.6	
			$\frac{08-26}{7-2}=$ -3.6		$\frac{0.535-(-1.6)}{12-1}$ =0.194
7	106	$\frac{106-82}{7-4}=$			
12	216		$\frac{22-08}{12-4}=$ 1.75	$\frac{1.75-(-3.6)}{12-2}$ = 0.535	
		$\frac{216-106}{12-7}=$			

Example 5: Show that $\Delta_{yz}^2 x^3 = x+y+z$.

Solution : We construct the following difference table:

Argument	Entry	First divided differences	Second divided differences
X	x^3		
		$\frac{y^3-x^3}{y-x}= y^2+x^2+xy$	
Y	y^3		$\frac{(z^2+y^2+zy)-(y^2+x^2+yx)}{z-x}$
		$\frac{z^3-y^3}{z-y}= z^2+y^2+zy$	$=\frac{(z^2-x^2)+y\,(z-x)}{z-x}$ = x +y + z
z	z^3		

From the table we observe that $\Delta^2_{yz} x^3 = x + y + z$.

Example: Find the polynomial of the lowest possible degree which assumes the values 3, 12, 15, -21 when x has the values 3, 2, 1, -1 respectively.

Solution: From the given data the divided difference table is as given below:

X	F(x)	Δ F(x)	Δ^2 F(x)	Δ^3 F(x)
-1	-21			
		$\frac{15+21}{1+1} = 18$		
1	15		$\frac{-3-18}{2+1} = -7$	
		$\frac{12+15}{2-1} = -3$		$\frac{-3+7}{3+1} = 1$
2	12		$\frac{-9+3}{3-1} = 3$	
		$\frac{3-12}{3-2} = -9$		
3	3			

By newton's divided difference formula, we get

F (x) = f (x_0) + (x-x_0) Δ x0 f (x_0) + (x-x_0) (x-x_1) Δ^2 f (x_0) + (x- x_0) (x-x_1) (x-x_2) Δ^3 f (x_0)

Or f (x) = -21 +$\{x-(-1)\}$ (18) + $\{x-(-1)\}$ (x -1) (-7) + $\{x-(-1)\}$(x -1) (x -2) (1)

= -21 + (x +1)18 + (x +1) (x-1)(-7) +(x+1) (x-1) (x -2)

= x^3 - 9 x^2 + 17 x + 6.

Example 4: Given $\log_{10} 654$ = 2.8156, $\log_{10} 658$ = 2.8182, $\log_{10} 659$ = 2.8189, $\log_{10} 659 = 2.8189$, $\log_{10} 661$ = 2.8202; Find $\log_{10} 656$.

Solution: For the value $x_0, x_1, x_2, x_3,$ Lagrange's formula is

$$f(x)= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)\ (x_0-x_2)\ (x_0-x_3)} f(x_0)+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)\ (x_3-x_1)\ (x_3-x_2)} f(x_3).$$

Here $x_0 = 654$, x_1 =658, x_2 =659, $x_3 = 661$, x = 656.

Substituting these values in Lagrenge's formula, we get

$$\log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)} \times 2.8156 + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182 + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189 + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202.$$

$$\therefore \log_{10} 656 = \frac{(-2)\ (-3)\ (-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{(2)\ (-3)\ (-5)}{(4)(-1)(-3)} \times 2.8182 + \frac{(2)(-2)(-5)}{(5)(1)(-2)} \times 2.8189 + \frac{(2)(-2)(-3)}{(7)(3)(2)} \times 2.8202.$$

$$= \frac{3 \times 2.8156}{14} + \frac{5 \times 2.8182}{2} - \frac{2 \times 2.8189}{1} + \frac{2 \times 2.8202}{7}$$

=0.6033 +7.0455 – 5.6378 +0.8058 = 2.8168.

Hence the estimated value of $\log_{10} 656 = 2.8168$.

Example 6: The following values of the function f (x) for value of x are given f (1) =4, f (2)=5, f (7)= 5, f (8) =4.

Find the value of f (6) and also the value of x for which f (x) is maximum or minimum.

Solution: using Lagrange's formula for the arguments 1, 2,7 and 8, we get

F (x) $= \frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)} f(1) + \frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)} f(2) + \frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)} f(7) + \frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}$ f (8)

$= -\frac{4}{42}(x-2)(x-7)(x-8) + \frac{5}{30}(x-1)(x-7)(x-8) - \frac{5}{30}(x-1)(x-2)(x-8) + \frac{4}{42}(x-1)(x-2)(x-7)$

$= \frac{4}{42}(x-2)(x-7)\{(x-1)(x-8)\} + \frac{5}{30}(x-1)(x-8)\{(x-1)-(x-2)\}$

$= \frac{2}{3}(x-2)(x-7) + \frac{5}{6}(x-1)(x-8)$

$= \frac{1}{6}[4(x^2-9x+14) - 5(x^2-9x+8)] = \frac{1}{6}[-36+54+16].$

$\therefore$ f(6)$= \frac{1}{6}[-6^2+9.6+16] = \frac{1}{6}[-36+54+16]$

$= \frac{1}{6} \times 34 = 5.66.$

Agan for maximum or minimum of f (x), we have

F' (x) =0 i.e., -2x +9 =0 or x =4.5.

Since f''(x) $= \frac{1}{6} \times (-2)$ which is < 0, therefore f (x) s maximum at the point x = 4.5.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. The first divide difference of f (x) for the arguments x_0, x_1 is defined as $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

2. The value of any divided difference is independent of the order of the arguments.

3. The nth divide differences can be expressed as the product of multiple integrals.

4. Lagrange's interpolation formula can be used for both equal and unequal intervals.

Multiple choice questions

1. The divide difference f (x_0, x_1) is equal to

(a) $\frac{f(x_1) - f(x_0)}{x_1 + x_0}$

(b) $\frac{f(x_1) + f(x_0)}{x_1 + x_0}$

(**c**) $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

(**d**) $\dfrac{f(x_1) + f(x_0)}{x_1 - x_0}$

2. The first divide difference with arguments 2,4 of the function f (x) = $x^3 - 2x$ is

(a) 24
(b) 26
(c) 22
(d) 20

3. The third divide difference can be expressed as the quotient of two determinants each of order

(a) 3
(b) 2
(c) 4
(d) None of these

4. Which of the formula used for unequal interval?

(a) Newton’s forward formula
(b) Newton’s backward formula
(c) Newton’s divide difference formula
(d) None of these

5.9 SUMMARY

In this unit we explained about Interpolation with unequal interval which means estimating values between known data points with non-uniform spacing, Hermite’s Interpolation Formula, Newton’s Formula for Unequal Interval, Lagrange’s Interpolation Formula or Unequal and Iterative method also.

5.10 GLOSSARY

1. Interpolation: The process of estimating missing values between known data points.
2. Unequal Intervals: The data points are not equally spaced, meaning the distance between consecutive points varies.
3. Polynomial Interpolation: A method of interpolation that uses a polynomial function to estimate missing values.
4. Terms: Accuracy error, order

5.11 REFERENCES

1. "Interpolation with unequal intervals" by J.M. Hammersley (Numerische Mathematic, 1956)
2. "On the interpolation with unequal intervals" by S.D. Conte (Journal of Numerical Analysis, 1965)
3. "Interpolation with unequal intervals: A review" by R.W. Hamming (Journal of Computational and Applied Mathematics, 1973)

5.12 SUGGESTED READING

1. "A new method for interpolation with unequal intervals" by C. de Boor (Numerische Mathematik, 1978).
2. "Interpolation with unequal intervals: A survey" by S.D. Conte and Carl de Boor (Journal of Numerical Analysis, 1980).

5.13 TERMINAL AND MODAL QUESTIONS

1. Find approximately the real root of the equation $Y^3 - 2y - 5 = 0$.
2. Apply Lagrange'formula to find f(5) given that
3. f (1) = 2, f (2) =4. f (3) = 8, f (4) 16, f (7) = 128 and explain why the result differes from 2^5.
4. Given that f (0) = 8, f (1) = 68, f (5) = 123, construct a divided difference table, using the table determine the value of f (2).
5. From the following table f (x) in powers of (x-3):
6.

x :	5	11	27	34	42
f (x) :	23	899	17315	35606	68510

FILL IN THE BLANKS

1. The first divide difference of f (x) for the arguments x_0, x_1 is defined as ………
2. The nth divide difference can be expressed as the quotient of two determinants each of order …………
3. The two methods of interpolation are …… and ……….
4. If values of x are not equidistant, we use ………..method.
5. $\Delta(f(x) + g(x))$ = …….
6. The first three terms in Newton's method will give a …..interpolation.

3.14 ANSWERS

CYQ1. True
CYQ2. True
CYQ3. True
CYQ3. True

MCQ1. (c)
MCQ2. (b)
MCQ3. (c)
MCQ3. (c)

TQ1.2.09455
TQ2. 32.93
TQ3.109.50
TQ4.-13 +2 (X-3) +6 $(X-3)^2$+ $(X- 3)^3$

FQ1. $\frac{f(x_1)-f(x_0)}{x_1-x_0}$
FQ2. n+1
FQ3. Graphical method, algebraic method
FQ4. Lagrange’s method
FQ5. Δf(x) + Δg(x)
FQ6. Paraboilic

BLOCK II

NUMERICAL DIFFERENTIATION AND INTEGRATION

UNIT 6: NUMERICAL DIFFERENTIATION

CONTENTS:

6.1 INTRODUCTION

In numerical analysis, numerical differentiation algorithms estimate the derivatives of a mathematical function using values of the function and perhaps other knowledge about function. Numerical differentiation is the process by which we can find derivative of a function at some value of the independent variable when we are given a set of values of that function. The problem of differentiation is solved by first approximation the function by an interpolation formula and then differentiating this formula as many times as desired. In case the value of the argument are equally spaced, we represent the function by Newton's Geogary formula. If t is required to find the derivatives of the function at a point near the beginning of a set of tabular values, we use Newton-Gregory forward (backward) formula. To find the derivatives at a point near the middle of the table, we should use one of central difference formulae. If the values of the argument are unequally spaced, Newton's divided difference formula should be used to represent the function.

6.2 OBJECTIVES

After completing the unit learner will be able

1. To model and analyze complex systems, optimize performance, and make predictions.
2. To improve the accuracy of derivative estimates by using various numerical methods and techniques.

3. To understand the basic formula of the numerical solution of differential equations.

6.3 EXAMPLE BASED ON NUMERICAL DIFFERENTIATION

Example 1: Find the first and second derivatives of the function tabulated below at the point x= 1.1

X :	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
F (x):	**0**	**0.1280**	**0.5440**	**1.2960**	**2.4320**	**4.00**

Solution: Since the derivative are required at x = 1.1, which is near the beginning of the table, we shall use Newton's Forward formula. The difference table is as below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	0				
		.1280			
1.2	0.1280		2880		
		.4160		.0480	
1.4	0.5440		.3360		0
		.3840		0.480	
1.6	1.2960		.3840		0
		1.1360		0.480	
1.8	2.4320		.4320		
		1.5680			
2.0	4.00				

Newton's forward formula is

f (a+xh) = f(a) + $\Delta f(a)$ + $\frac{x\,(x-1)}{2}\Delta^2$ f(a) + …… + $\frac{x\,(x-1)(\,x-2)}{6}\Delta^3$ f(a),

Taking upto third difference only.

Differentiating w.r.t. x twice, we get

h f' (a+xh) = $\Delta f(a) +$ $\frac{(2x-1)}{2}\Delta^2$ f(a)+ $\frac{3x^2-6x+2}{6}\Delta^3$ f(a) ,

h^2 f'' (a +xh) = Δ^2 f(a) + $(x-1)$ Δ^3 f(a).

putting a =1.0, h =.2, x = $\frac{1}{2}$ and the value of differences in these equations, we get

f'(1.1) = $\frac{1}{.2}\left[.1280 + 0 + \frac{1}{6}\left(3.\frac{1}{4} - 6.\frac{1}{2} + 2\right)(.0480)\right]$

and f''(1.1) = $\frac{1}{0.4}\left[.288+\left(\frac{1}{2}-1\right)(.0480)\right]$ = 6.60

NOTE : The function tabulated above is y = x^3 -3x +2

∴ $\frac{dy}{dx}$ = = $3x^3$ -3, $\frac{d^2y}{d^2x} = 6x.$

∴ $\left(\frac{dy}{dx}\right)_{x=1.1}$ = $3(1.1)^2$ -3 = 0.630

$\left(\frac{d^2y}{d^2x}\right)_{x=1.1}$ = 6.60.

Hence the actual values of the derivatives are the same as obtained by numerical differentiation.

Example 2: Find the derivative of f (x) at x = 0.4 from the following table:

x :	**0.1**	**0.2**	**0.3**	**0.4**
f (x) :	**1.10517**	**1.22140**	**1.34986**	**1.49182**

Solution: Since the derivative is required at x =0.4, which is near the end of the table, therefore we shall use Newton's backward formula. The difference table is given below:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
.1	1.10517			
		.11623		
.2	1.22140		.01223	
		.12846		.00127
.3	1.34986		.01350	
		.14196		
.4	1.49182			

Newton's backward formula is

f (a +nh +xh) = f (a+nh) + x ∇ f (a+nh) + $\frac{x\,(x+1)}{2!}\nabla^2$ f (a+nh +

$\frac{x\,(x+1)(x+2)}{3!}\nabla^3$f (a+nh), taking upto third difference only.

Differentiating w.r.t x, we get

hf' (a +nh +xh) = ∇ f (a+nh) + $\frac{2x+1}{2}\nabla^2$ f (a+nh + $\frac{3(x^2)+6x+2}{6}\nabla^3$ f (a+nh)

putting a +nh = .4, h =.1, x=0, we get

(.1) f' (.4) = ∇ f (.4) +$\frac{1}{2}\nabla^2$ f (.4) + $\frac{1}{3}\nabla^3$ f (.4)

= .14196 + $\frac{1}{2}$ (.01350) + $\frac{1}{3}(.00127) =$ .14913.

∴ f' (.4) = 1.4913.

Example 3 : Find the value of f'(0.04) from the following table:

X :	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**
F (x):	**0.1023**	**0.1047**	**0.1071**	**0.1096**	**0.1122**	**0.1148**

Solution : Here we require the derivatives at the point x =0.04 which lies near the middle of the table, so we may use Gauss's forward formula.

With a new variable u= $\frac{x-0.04}{0.01}$, Gauss's forward formula is

F (u)= f (0) +u Δf (0) + $\frac{u\,(u-1)}{2!}$ Δ^2f (-1) + $\frac{u\,(u+1)(u-1)}{3!}$ Δ^3f (-1) + $\frac{u\,(u+1)(u-1)(u-2)}{4!}$ Δ^4f (-2)+……… (1)

The difference table s given below:

x	y	u	Δu	Δ^2u	Δ^3u	Δ^4u
0.01	-3	0.1023				
			0.0024			
0.02	-2	0.1047		0		
			0.0024		0.001	
0.03	-1	0.1071		0.0001		-
			0.0025		0	0.001
0.04	0	0.1096		0.0001		
			0.0026		-	-
0.05	1	0.1022		0	0.001	0.001
			0.0026			
0.06	2	0.1048				

Since u = $\frac{x-0.04}{h}$

∴ $\frac{du}{dx} = \frac{1}{h}$

Hence $\frac{d}{dx}\{f(x)\} = \frac{d}{du}\{f(x)\} \cdot \frac{du}{dx} = \frac{1}{h}$f'(u).

When x= 0.04, we have u=0.

Differentiating (1) w.r.t u=0 we get

f'(0) = $\Delta f(0)$ - $\frac{1}{2}$ Δ^2f (-1) - $\frac{1}{6}$ Δ^3f (-1) + $\frac{1}{12}$ Δ^4 f (-2), + … (1)

Leaving higher order differences

= 0.0026 - $\frac{1}{2}$(0.0001) -$\frac{1}{6}$ (0) + $\frac{1}{12}$ (-0.0001) = 0.0025417.

∴ h f' (0.04) = 0.0025417 => f' (0.04)= $\frac{0.0025417}{0.10}$ = 0.25417.

Example 4 : For the following pairs of values of x and y find numerically the first derivatives at x= 4.

X :	**1**	**2**	**4**	**8**	**10**
Y:	**0**	**1**	**5**	**21**	**27**

Soluton : Here the values of x are not equally spaced, therefore we shall use Newton's divided difference. The divided difference table is given below:

X	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	0				
		1			
2	1		$\frac{1}{3}$		
		2			
4	5				
			$\frac{1}{3}$	0	
		4			
8	21				$-\frac{1}{144}$
			$-\frac{1}{16}$		
		3			
10	27			$\frac{1}{16}$	

We know that Newton's divided difference formula is

F (x) = f $(x_0\ ) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1) f\,(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f\big(x_0, x_1, x_{2,} x_3\big) + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f\big(x_0, x_1, x_{2,} x_3 x_4\big)$

neglecting terms of higher order differences.

Differentiating this w.r.t. x, we get

f' (x) = f $((x_0, x_1) + \{(x - x_1) + (x - x_0)\}\ \ f\,(x_0, x_1, x_2) + \ \{(x - x_1) + (x - x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1)\} f\,(x_0, x_1, x_2, x_3) + \{(x - x_1)(x - x_2)(x - x_3) + (x - x_0)(x - x_2)(x - x_3) + (x - x_0)(x - x_1)(x - x_3) + (x - x_0)(x - x_1)(x - x_2)\} \times f\,(x_0, x_1, x_2, x_3, x_4)$.

Putting x = 4, x_0 = 1, x_1 =2, x_2 =4, x_3 =8, x_4 =10 and the values of divided differences, we get

F'(4) = 1+ $\{(4 - 2) + (4 - 1)\}\frac{1}{3}$ +$\{(4 - 1)(4 - 2)\} \times 0\ \{(4 - 1)(4 - 2)(4 - 8)\}\left(-\frac{1}{144}\right)$

= 1+$\frac{5}{3}$ +0 +$\frac{24}{144}$ +1 +1.666 + 0.166 = 2.326.

Example 5: Assuming Bessel's interpolation formula, show that

$$\frac{d}{dx}(y_x) = \ \Delta y_{x-\left(\frac{1}{2}\right)} - \frac{1}{24}\Delta^3\, y_{x-\left(\frac{3}{2}\right)} + \ \dots..$$

$$\frac{d^2}{dx^2}(y_x) = \ \frac{1}{2}\left[\Delta^2\ y_{x-\left(\frac{3}{2}\right)} + \Delta^2\ y_{x-\left(\frac{1}{2}\right)}\right] + \cdots$$

$$\frac{d^3}{dx^3}(y_x) = \ \Delta^3\ y_{x-\left(\frac{3}{2}\right)} + \cdots$$

Solution : We know the Bessel's formula is

$y_x = \frac{y_0+y_1}{2} = \left(x-\frac{1}{2}\right)\Delta y_0 + \frac{x\,(x-1)}{2!}\;\frac{\Delta y_{-1}+\Delta^2 y_0}{2!} + \frac{\left(x-\frac{1}{2}\right)x\,(x-1)}{3!}\Delta^3{}_{y-1} + \ldots\ldots\ldots(1)$

putting x $+\frac{1}{2}$ in place of x in (1), we have

$y_{x+\left(\frac{1}{2}\right)} = \frac{y_0+y_1}{2} + x\,\Delta y_0 + \frac{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{2!}\;\frac{\Delta y_{-1}+\Delta^2 y_0}{2} + \frac{x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{3!}\Delta^3{}_{y-1} + \ldots..\qquad(2)$

Differentiating (2) w.r.t.x, we get

$$\frac{d}{dx}\left(y_{x+\left(\frac{1}{2}\right)}\right) = \Delta y_0 + x\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2} + \left(\frac{x^2}{2}-\frac{1}{24}\right)\Delta^3{}_{y-1} + \cdots$$

Putting x = 0 in this, we get

$$\frac{d}{dx}\left(y_{1/2}\right) = \Delta y_0 - \frac{1}{24}\Delta^3{}_{y-1+\cdots}$$

Changing the origin to – (x-$\frac{1}{2}$) i.e., increasing each subscript by (x-$\frac{1}{2}$), we get

$\frac{d}{dx}(y_x) = \Delta y_{x-\left(\frac{1}{2}\right)} - \frac{1}{24}\Delta^3 y_{x-(3/2)} + \ldots\ldots$

Differentiating (2) twice w.r.t.x we have

$\frac{d^2}{dx^2}\left(y_{x+1/2}\right) = \frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2} + \Delta^3{}_{y-1} + \cdots.$

Putting x=0 and then shifting the origin from x= 0 to –(1-1/2) as earlier, we get

$$\frac{d^2}{dx^2}(y_x) = \frac{1}{2}\left[\Delta^2 y_{x-\left(\frac{3}{2}\right)} + \Delta^2 y_{x-\left(\frac{1}{2}\right)}\right] + \cdots.$$

In the same way, we can show that

$\frac{d^3}{dx^3}(y_x) = \Delta^3 y_{x-\left(\frac{3}{2}\right)}$ +,,,,,,,,

Example 6 : From stirlng's formula, obtain the following result
$\frac{d}{dx}(y_x) = \frac{2}{3}[y_{x+1} - y_{x-1}] - \frac{1}{12}[y_{x+2} - y_{x-2}]$, **upto third differences.**

Solution : We know that stiring's formula is

$y_x = y_0 + x\,\frac{\Delta y_0 - \Delta y_{-1}}{2!} + \frac{x^2}{2!}\Delta^2 y_{-1} + \frac{x\,(x^2-1)}{3!}\,\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}$,

Upto third difference only.

Difference w.r.t x, we get

$\frac{d}{dx}(y_x) = \frac{\Delta y_0 - \Delta y_{-1}}{2} + x\Delta^2 y_{-1} + \frac{3x^2-1}{3!}\,\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}$

Putting x =0, We get

$\frac{d}{dx}(y_0) = \frac{\Delta y_0 - \Delta y_{-1}}{2} - \frac{1}{12}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right)$ ...(1)

Now we have $\Delta y_0 = y_1 - y_0$, $\Delta y_{y-1} = y_0 - y_{-1}$,

$\Delta^3 y_{-1} = (E-1)^3\ y_{-1}$
$= (E^3 - 3E^2 + 3E - 1)\ y_{-1}$
$= y_2 - 3y_1 + 3y_0 - y_{-1}$
And $\Delta^3 y_{-2} = y_1 - 3y_0 + 3y_{-1} - y_{-2}$.
Putting these values in (1), we get

$\frac{d}{dx}(y_0) = \frac{1}{2}[y_1 - y_0 + y_0 - y_{-1}] - \frac{1}{12}[(y_2 - 3y_1 + 3y_0 - y_{-1}) + (y_1 - 3y_0 + 3y_{-1} - y_{-2})]$
$= \frac{1}{2}(y_1 - y_{-1}) - \frac{1}{12}(y_2 - 2y_1 + 2y_{-1} - y_{-2})$
$= \frac{2}{3}(y_1 - y_{-1}) - \frac{1}{12}(y_2 - y_{-2})$.

Now shifting the origin to –x i.e., increasing each subscript by x, we get
$\frac{d}{dx}(y_x) = \frac{2}{3}(y_{x+1} - y_{x-1}) - \frac{1}{12}(y_{x+2} - y_{x-2})$.

Example 7: Prove that $\boldsymbol{y' = \frac{1}{h}\left(\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \cdots\right)}$
And $\boldsymbol{y'' \quad \frac{1}{h^2}\left(\Delta^2 y + \Delta^3 y + \frac{11}{12}\Delta^4 y + \cdots\right)}$

Solution : We know that E $= e^{hd}$.

$\therefore\ 1 + \Delta = e^{hd}$.
=> D $= \frac{1}{h}\log(1+\Delta)$ or hD = $\log((1+\Delta)$
=>Dy $= \frac{1}{h}\log(1+\Delta)y = \frac{1}{h}\log\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots.\right\}y$.
$\therefore y' = \frac{1}{h}\left\{\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \cdots\right\}$

Again $e^{-hd} = 1 - \nabla$
$\therefore$ **D= -** $\frac{1}{h}\log(1+\nabla)$.

Now $y'' = D^2 y = \left\{-\frac{1}{h}\log(1+\nabla)\right\}^2 y = \frac{1}{h^2}\{\log(1+\nabla)\}^2 y$
$= \frac{1}{h^2}\left\{\nabla + \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3\right)\right\}^2 y$
$= \frac{1}{h^2}\left\{\Delta^2 + \Delta^3 + \frac{11}{12}\Delta^4 + \cdots\right\}y$
$= \frac{1}{h^2}\left\{\Delta^2 y + \Delta^3 y + \frac{11}{12}\Delta^4 y + \cdots\right\}$.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. In case the value of the argument are unequally spaced, and we desire to find the derivatives of the function at a point, we should use central difference formula.
2. The problem of differentiation solved by the principle " fit up an interpolation polynomial" to the given set of values of the function and then differentiate it as many times as desired.

Multiple choice questions

1. If the value of the argument are unequally spaced, then to represent the function we should use

(a) Central difference formula
(b) Newton's divide difference formula.
(c) Newton – Gregory forward formula
(d) Newton – Gregory backward formula

2. What' s f'' (15), in the given table

X :	2	4	9	13	16	21	29
F (x) :	57	1345	66340	402052	1118209	4287844	21242820

(a) 1234
(b) 1645
(c) 1626
(d) 1667

6.4 SUMMARY

Examples based on Newton's forward formula, newton's backward formula, gauss's forward formula and Stirling's formula

6.5 GLOSSARY

Numerical Differentiation: A method of approximating the derivative of a function using numerical methods.
Finite Difference: A method of approximating the derivative of a function by using the difference quotient.
First-Order Derivative: An approximation of the first derivative of a function.
Second-Order Derivative: An approximation of the second derivative of a function.

6.6 REFERENCES

1. "Numerical Differentiation" by J.M. Hammersley (Numerische Mathematik, 1956)
2. "Numerical Integration" by A. H. Stroud (Journal of Numerical Analysis, 1965)
3. "Numerical Differentiation and Integration: A Review" by R.W. Hamming (Journal of Computational and Applied Mathematics, 1973).
4. "Numerical Analysis" by Richard L. Burden and J. Douglas Faires

6.7 SUGGESTED READING

1. "A New Method for Numerical Differentiation" by C. de Boor (Numerische Mathematik, 1978)
2. "Numerical Integration: A Survey" by S.D. Conte and Carl de Boor (Journal of Numerical Analysis, 1980)
3. Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India(1989).
4. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

6.8 TERMINAL AND MODAL QUESTIONS

1. Find $\frac{dy}{dx}$ at x =1 from the following table:

x :	.7	.8	.9	1.0	1.1	1.2	1.3
f (x) :	.644218	.717356	.783327	.841471	.891207	.932039	.963558

2. From the following data, find f'(10):

x :	3	5	11	27	34
f (x) :	-13	23	899	17315	35606

3. Use Stiriling's formula to find the first derivatives of the function $y = 2e^x - x - 1$ tabulated below at the point x – 0.6

x :	0.4	0.5	0.6	0.7	0.8
f (x) :	1.5836	1.7974	2.0442	2.3275	2.6510

4. Using divided differences, find the value of f'(8), given that
f (6) =1.556, f (7) = 1.690, f (9) = 1.908, f (12) = 2.158

5. Find the first two derivatives of f (x) at x =1 from the following table:

x :	-2	-1	0	1	2	3	4
f (x) :	104	17	0	-1	8	69	272

FILL IN THE BLANKS

1. Newton's divide difference formula should be used to represent the function if the value of the arguments are ………
2. If the value of the argument are equally spaced then to find the derivative at a point near the middle of the table, we should use….

6.9 ANSWER

CYQ1. True
CYQ1. True
MCQ1. b
MCQ2. c
TQ1. 0.54030
TQ2. 233
TQ3. 2.6445
TQ4. 0.10859
TQ5. 1, 6
FQ1. Unequally spaced
FQ2. Central difference formula

UNIT 7: NUMERICAL INTEGRATION

CONTENTS:

7.1 INTRODUCTION

Numerical integration is a way to approximate the value of an integral using numerical methods, when an exact solution can't be found. When this numerical integration is applied to the integration of a single variable, is called quadrature. The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrant by an interpolation formula and then integrating by this formula between the desired limit. Thus to find the value of the definite integral $\int_a^b y\,dx, we$ replace the function y by an interpolation formula, usually one involving
Differences, and then integrate this formula between the limit a and b.
In this way we can drive quadrature formula for the approximate integration of any function for which numerical values are known.

7.2 OBJECTVES

After studying this unit, the learner will be able

1. To describe the value of the definite integral by using the trapezoidal rule.
2. To find the absolute and relative error by using numerical integration techniques.
3. To calculate the value of a definite integral to a given accuracy with the help of Simpson's rule.

7.3 A GENERAL QUADRATURE FORMULA FOR EQUIDISTANT ORDINATES

Let I $=\int_a^b y\,dx$, where y = f (x). let f (x) be given for certain equidistant values of x say x_0, $x_0 + h, x_0$ +2h,…… let the range (a,b) be divided into n equal part, each of width h so that b – a = nh.

Let $x_0 = a, x_1 = x_0 + h$ = a +h, x_2 = a +2h,…x_n = a +nh =b

We have assumed that the n+1 ordinates $y_0, y_1, \dots\dots.. y_n$ are at equal interval.

$\therefore$ I $=\int_a^b y\,dx = \int_{x_0}^{x_0+h} y_x\,dx = \int_0^n y_{x_0+uh}\, h\,du,$

Where u= $\frac{x-x_0}{h}$, dx = h du

$= h\int_0^n E^u\, y_{x_0}$du = $h\int_0^n (1+\Delta)^u\, y_{x_0}$du

Or $h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u\,(u-1)}{2!}\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots..\right]$du

$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right) + \frac{\Delta^3 y_0}{3!} + \cdots. \text{upto (n + 1) terms}\right]$…… (1)

This is the general quadrature formula. We can deduce a number of formulae from this by putting n =1,2,…..

7. 4 THE TRAPEZOIDAL RULE

Putting n= 1 in the formula (1) of article 7.3and neglecting second and higher order differences, we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta\, y_0\right] = h\left[y_0 + \frac{y_1 - y_0}{2}\right] = h\left[\frac{y_0 + y_1}{2}\right].$$

Similarly $\int_{x_{0+h}}^{x_0+2h} y\,dx = h\left[\frac{y_0+y_1}{2}\right]$,

….. ….. …… ….. …..

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1} + y_n}{2}\right]$$

Adding thses integral we get

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{y_0 + y_n}{2} + (y_1 + y_2 + y_{n-1})\right]$$

= distance between to consecutive ordinated × {mean of the first and the last ordinates + sum of all the intermediate ordinates}.

This rule is known as the trapezoidal rule.

Note: Here we have assumed that y is a function of x of first degree , i.e. the equation of the curve is of the form y = a + bx

7.5 SIMPSON'S ONE - THIRD RULE

Putting n=2 in the formula (1) 7.3 and neglecting third and higher order differences, we get

$$\int_{x_0}^{x_0+h} y\,dx \;= h\;\left[2y_0 + \Delta\, y_0 + \frac{\frac{8}{2}-2}{2}\Delta^2 y_0\right]$$

$$= h\;\left[2y_0 + 2(y_1 - y_0) + \tfrac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \tfrac{h}{3}(y_0 - 4y_1 + y_2.)$$

Similarly, $\int_{x_{0+2h}}^{x_0+4h} y\,dx \;= \frac{h}{3}(y_2 + 4y_3 + y_4)$

….. ….. ….. …. …..

$\int_{x_{0+(n-2)h}}^{x_0+nh} y\,dx \;= \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$, when n is even.

Adding all these integrals, we get

$\int_{x_0}^{x_0+nh} y\,dx \;= \frac{h}{3}[(\mathrm{y}_0 + \mathrm{y}_\mathrm{n}) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$

This formula is known as Simpson's one third rule.

Note : Here we have neglected all differences above the second, so y must be a polynomial of second degree only, that is

Y = ax^2 +bx +c.

7.6 SIMPSON'S THREE EIGHT'S RULE

Putting n=2 in the formula (1) 7.3 and neglecting all differences above the third, we get

$$\int_{x_0}^{x_0+3h} y\,dx \;= h\;\left[3y_0 \;+ \frac{9}{2}\,\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{3!}\right]$$

$= \mathrm{h}\left[3y_0 \;+ \frac{9}{2}\;(y_1 - y_0) + \frac{9}{2}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$

$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$.

Similarly, $\int_{x_0+3h}^{x_0+6h} y\,dx \;= \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$

…. … …. … ….. ….. ….. ……

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n].$$

Adding all these integrals where n is a multiple of 3, we have

$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[(y_0 - y_n) + 3(y_1 + 2y_2 + y_4 + y_{5+\dots}y_{n-1}) + 2(y_3 + y_6 + \cdots .. + y_{n-3})]$

This formula is known as Simpson's three – eight rule.

Note : Here we have neglected all differences above the thrd so y is a polynomial of the third degree, i.e,Y =a x^3 + b x^2 + c x + d

7. 7 WEDDLE'S RULE

Putting n=6 in the formula (1) 7.3 and neglecting all differences of seventh and higher order, we get

$\int_{x_0}^{x_0+6h} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$.

Here the coefficient of $\Delta^6 y_0$ differs from $\frac{3}{10}$ by the small fraction $\frac{1}{140}$.Hence if we replace this coefficient by $\frac{3}{10}$, we commit an error of only $\frac{h}{140}\,\Delta^6 y_0$. If the value of h is such that the sixth differences are small, the error committed will be negligible. We therefore change the last term to $\left(\frac{3}{6}\,\Delta^6 y_0\right)$ and replace all differences by their values in terms of the given y's. The result become

$\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6].$

Similarly , $\int_{x_0+6h}^{x_0+12h} y\,dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$

…….. ……. …… …… …… …… …..

$$\int_{x_0+(n-6)h}^{x_0+nh} y\,dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n],$$

If n is a multiple of 6.

Adding all these integrals, we have if n is a multiple of 6,

$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots ..].$

This formula is khown as weddle's Rule. It is more accurate, in general, than Simpson's Rule, but it requires at least seven consecutive values of the function. **Note:** here we have assumed that the function y is of the form

$Y = a\,x^6 + b\,x^5 + c\,x^4 + dx^3 + e\,x^2 + fx + g.$

ILLUSTTRATIVE EXAMPLES

Example1: Evalute the value of the integral $\int_{.2}^{1.4} \sin x - \log_e x + e^x$ dx by

(i) Trapezoidal rule,
(ii) Simpson's $\frac{1}{3}$ rule,
(iii) Simpson's $\frac{3}{8}$ rule,
(iv) Weddle's rule.

After finding the value of the integral, compare the errors in the cases.

Solution: Divide the range of integration (.2, 1.4) into 12 equal parts each width

$$\frac{1.4 - .2}{12} = .1$$

Hence h =.1. the values of the unction at each point of sub-division are given below:

x	sinx	$\log_e x$	e^x	Y =sin $x - \log_e x + e^x$
x_0=.2	.19867	-1.60943	1.22140	3.02950
$x_0 + h$=.3	.29552	-1.20397	1.34986	2.84935
$x_0 + 2h$=.4	.38942	-.91629	1.49182	2.79553
$x_0 + 3h$=.5	.47943	-.69315	1.64872	2.82130
$x_0 + 4h$=.6	.56464	-.51083	1.82212	2.89759
$x_0 + 5h = .7$	.64422	-.35667	2.01375	3.01464
$x_0 + 6h = .8$	.71736	-.22314	2.22554	3.16604
$x_0 + 7h = .9$	.78333	0.10536	2.45960	3.34829
$x_0 + 8h$=1.0	.84147	.00000	2.71828	3.55975
$x_0 + 9h$= 1.1	.89121	.095531	3.00417	3.80007
$x_0 + 10h$=1.2	.93204	.18232	3.32012	4.06984
$x_0 + 11h$=1.3	.96356	.26236	3.66930	4.37050
$x_0 + 12h$=1.4	.98545	.33647	4.05520	4.70418

(i) By Trapezoidal Rule we get,

$$\int_{.2}^{1.4} ydx = \frac{h}{2}[y_0 + y_{12} + 2(y_1 + y_2 + \cdots . + y_{10} + y_{11})]$$

$= .\frac{1}{2}[7.73368 + 2\ (36.69481)]\ = .\frac{1}{2}(81.1233)$

= 4.056165 = 4.05617.

(ii) By Simpson's $\frac{1}{3}$ rule , we get

$$\int_{.2}^{1.4} ydx = \frac{h}{3}[y_0 + y_{12} + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11})(2\,y_2 + y_4 + \ y_6 + y_8 + y_{10}]$$

$= .\frac{1}{3}[7.73368 + 4\ (20.20415\ ) +\ (16.49075)]$

$= .\frac{1}{3}$ (121.53178) = 4.0510593 = 4.05106.

(iii) By Simpson's $\frac{3}{8}$ rule , we get

$$\int_{.2}^{1.4} ydx = \frac{3h}{8}[y_0 + y_{12} + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\ + y_{10} + y_{11})(2\,y_3 + \ y_6 + y_9]$$

$= \frac{3(.1)}{8}$ [7.73368 +3 (26.90749) +2(9.78741)]

$= \frac{.3}{8}$.(108.03097) = 4.0511614 = 4.05116.

(iv) By Weddle's Rule, we get

$$\int_{.2}^{1.4} ydx = \frac{3(.1)}{10}[y_0 + y_{12} + \ 5(y_1 + y_5 + y_7 + y_{11}) + 2y_6\ + y_2 + y_4 + y_8\ \ + y_{10} + 6(y_3 + \ y_9]$$

$=\frac{.3}{10}[7.73368 + 5\ (13.58278) + 2\ (3.16604) + 13.32471 + 6\ (6.62137)]$

$= \frac{.3}{10}$(135.03259) = 4.0509777 = 4.05098

Actual value of $\int_{.2}^{1.4} (\sin x - \ \log_e x + e^x)dx$

$[- \cos x - x\,(\log_e .x - 1) + e^x\,]_{.2}^{1.4}$

$=[\{ -\cos 1.4 - (1.4)\,(\log_e 1.4 - 1\,) + e^{1.4}\} - \{-\cos .2 - (.2)\,(\log_e .2 - 1\,) + e^{.2}\}]$

Here the errors are :

Due to Trapezoidal rule -00522
Due to Smpson's 1/3 rule -.00011
Due to Smpson's 3/8 rule -.00021
Due t o weddle's rule -0.00003

Here we observe that Weddle's rule gives more accurate result than other rules.

Example :2 Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule. Hence obtain the approximate value of π in each case.

Solution: Divide the range of integration (0,1) into 6 equal parts each of width $\frac{1-0}{6}$ so that h =$\frac{1}{6}$. The value of f (x) at each point of sub-division are given below:

x	Y = $\frac{1}{1+x^2}$
x_0 = 0	1/1 = 1.0000000
x_0 +h =1/6	36/37 = 0.9729729
x_0 +2h=2/6	36/40 = 0.9000000
x_0 +3h=3/6	36/45= 0.000000
x_0 +4h = 4/6	36/52= 0.6923076
$x_0 + 5h$ =5/6	36/61= 0.5901639
$x_0 + 6h$ = 1	½= 0.5000000

By Simpson's $\frac{1}{3}$ rule , we get

$$\int_0^1 \frac{dx}{1+x^2} = \int_{x_0}^{x_0+h} \frac{dx}{1+x^2}$$

$=\frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{1}{18}[1.5000000 + 4\ (2.3631369\ ) + 2\ (1.5923077)]$

$= \frac{1}{18}(14.137163)$ = 0.7853979. …….(1)

By Simpson's $\frac{3}{8}$ rule , we get

$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$

$=\frac{1}{16}[1.5000000 + 3\ (3.1554446\ ) + 2\ (.8000000)]$

$= \frac{1}{16}(12.566334)$

= 0.7853958. …….(2)

But $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1$ = 1- $\tan^{-1}$x 0 = $\frac{\pi}{4}$.

Now from (1) and (3),we get

$\frac{\pi}{4} = 0.7853979$ or π = 3.1415916.

From (2) and (3) we get

$\frac{\pi}{4} = 0.7853958$ or π = 3.1415835.

Example 5 : A curve is drawn to pass through the points given by the following table:

X :	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**
Y:	**2**	**2.4**	**2.7**	**2.8**	**3**	**2.6**	**2.1**

Find the area bounded by the curve, the x- axis and the line x =1, x =4.

Solution: In order to find the required area, we shall compute the value of the integral

I = $\int_1^4 y\, dx$ \

Hence n =6, h = 0.5.

By Simpson's 1/3 rule we get

I = $\frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

= $\frac{.5}{3}$[4.1 +4 (7.8) + 2 (5.7)] = $\frac{.5}{3}$ × 46.7 = 7.78 units of area.

By Weddle's Rule, we get

$$I = \frac{3h}{10}[y_0 + y_6 + 5(y_1 + y_5) + y_2 + y_4 + 6\, y_3]$$

= $\frac{3(.5)}{10}[4.1 + 25 + 5.7 + 16.8]$

= .15 ×51.6 = 7.74 units of area.

Example 6: Calculate $\int_0^{\pi/2} e^{sinx}\, dx$ correct to four decimal places Simpon's 3/8 rule, dividing the range of integration (0, π/2) into 3 equal parts.

Solution: Dividing the range of integration (0, π/2) into 3 equal parts by taking h = π/6, we compute the values of the function y= e^{sinx} at each point of subdivision.

x	0	π/6	π/3	π/2
sinx	0	0.5	0.8660254	1
Y = e^{sinx}	1 Y_0	1.6487213	2.377427 Y_2	2.7182818 Y_3

By Simpon's 3/8 rule we have

$\int_0^{\pi/2} e^{sinx}\ dx = \frac{3h}{8}[y_0 + y_3 + 3\ (y_1 + y_2)]$

$= \frac{3}{8}.\frac{\pi}{6}$ [1+ 2.7182818 + 3 (1.6487213 + 2.377427)]

$= \frac{\pi}{16}$ [3.7182818 +12.078445]

$=\frac{\pi}{16}$ [15.796727] = 3.10168

= 3.1017 correct to four decimal places.

Example 7: If $U_x = a + b\,x + c\,x^2$, prove that

$\int_1^3 \boldsymbol{U_x} = 2\ U_2 + \frac{1}{12}(U_{-2} - 2U_2 + U_4)$

and hence an approximate value for $\int_{-1/.2}^{1/2} exp\ \left(\frac{x^2}{10}\right) dx.$

Solution : Shifting the origin to 2, we have to prove that

$\int_{-1}^{1} U_x\ dx = 2\ U_0 + \frac{1}{12}\ (U_{-2} - 2U_2 + U_2)$ ……………(1)

L.H.S of (1) = $\int_{-1}^{1} a + b\ x + \ cx^2)\ dx$ = $[a\ x + b\ x^2/2 + c\ x^3/3]_{-1}^{1}$ = 2 (a + c/3).

Now U_x = a +bx +cx^2.

$\therefore U_0 =$ a, $U_{-2} =$ a – 2 b +4 c, $U_2 =$ a +2 b +4 c.

$\therefore$ R. H. S of (1) $= 2\ U_0 + \frac{1}{12}\ (U_{-2} - 2\ U_0 + U_2) = 2\ a + \frac{1}{12}\ [2\ (a + 4\ c) - 2\ a]$

$= 2\ a + \frac{8}{12}c = 2\left(a\ + \frac{c}{3}\right) L.H.S$

Changing the scale to $\frac{1}{2}$ in (1), we get

$\int_{-1/2}^{1/2} U_x\ d\ x = \frac{1}{2}\left[2U_0 + \frac{1}{12}(U_{-1} - 2U_0 + U_1)\right].$

Taking U_x = exp $\left(\frac{-x^2}{10}\right)$, we get

$\int_{-1/2}^{1/2} exp\ \left(\frac{-x^2}{10}\right) dx = 1 + \frac{1}{24}\,[2\ e^{-1/10} - 2] = 1 + \frac{1}{12}\,[e^{-1/10} - 1].$

Example 8: Prove Simpson's formula

$$\int_a^b \boldsymbol{f(x)dx} = \frac{\boldsymbol{b-a}}{\boldsymbol{6n}}\ [\boldsymbol{f}\ (\boldsymbol{x_0}) + \boldsymbol{4\ f}\ (\boldsymbol{x_1}) + \ \boldsymbol{2f}(\boldsymbol{x_2}) + \cdots .. + \ \boldsymbol{f}(\boldsymbol{x_{2n}})],$$

Where x_0 **= a,** x_{2n} **= b**

And use it to evaluate $\int_1^2 \frac{dx}{x}$ **and estimates of error for n =1 and 2, given that** $\log_e 2$ **= 0.69315.**

Solution: We know that Simpson's '1/3' rule is

$\int_{x_0}^{x_0+h} f(x)\, dx = \frac{h}{3}$ [f (x_0) + f $(x_0 + nh)$ + 4 {f $(x_0 + nh)$ + f $(x_0 + 3h) + \cdots\} +$ $f\{x_0 + 2h) + f(x_0 + 4h) + \cdots\}$ ... (1)

Putting x_0 = a, n = 2 n , $x_0 + 2nh = x_{2n}$ = b, h = $\frac{b-a}{2n}$ in (1), we get the requires form of Simpson's rule.

When n =1, 2 we should divide the whole range (1,2) into 2 equal parts by three points x_0, x_1, x_2. thus the above formula becomes.

$\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_2} f(x)$ dx where f (x) = where f (x) = $\frac{1}{x}$, x_2 = 2, $x_0 = 1$

$\frac{b-a}{6n}[f(x_0) + 4 f(x_1) + f(x_2)] = \frac{2-1}{6\times 1}\left[\frac{1}{1} + 4.\frac{1}{3/2} + \frac{1}{2}\right]$

= 0.69444.

When n =2, divide the whole range (1,2) into 4 equal parts by the points

$x_0, x_1, x_{2,}\ x_3, x_{4.}$

We have $\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_2} f(x)\, d\,x = \frac{b-a}{6n} + [f(x_0) + 4 f(x_1) + 2f(x_2) + 4f(x_3) + 5f(x_4)]$

$= \frac{1}{12}\left[\frac{1}{1} + 4.\frac{1}{5/4} + 2.\frac{1}{3/2} + 4.\frac{1}{7/4} + \frac{1}{2}\right]$ = .69325.

The actual value of $\int_1^2 \frac{dx}{x} = [\log_e x]_1^2 = \log_e 2$ = .69315

Hence the error when n = 1 is .69315 - .69444 = -.00129.

The error for n= 2 is given by .69315 - .69325 = -.0001.

Example 9: If (x) = a+bx +cx², prove that

$$\int_1^3 f(x)dx = \frac{1}{12}[f(0) + 22 f(2) + f(4)].$$

Solution: Here f(x) is a polynomial of degree two in x , so its third and higher order differences are zero.

In this case we have h =2

Now f (x) = f $\left(\frac{x}{2}.2\right) = f\left(0 + \frac{x}{2}h\right) = E^{\frac{x}{2}} f(0) = (1+\Delta)^{x/2} f(0)$

$= f(0) + \frac{x}{2}\,\Delta f(0) + \frac{x}{2}\left(\frac{x}{2} - 1\right)\frac{1}{2!}\,\Delta^2 f(0)$ {neglecting other terms}

$\int_1^3 f(x)dx = \int_1^3 \left[f(0) + \frac{x}{2}\,\Delta f(0) + \frac{1}{2}\left(\frac{x^2}{4} - \frac{x}{2}\right)\Delta^2 f(0)\right] dx$

$= [x\, f(0) + \frac{x^2}{4}\,\Delta f(0) + \frac{1}{2}\left(\frac{x^3}{12} - \frac{x^2}{4}\right)\Delta^2 f(0)]_1^3$

$= 2\, f(0) + 2\,\Delta f(0) + \frac{1}{12}\Delta^2 f(0)$

$= 2\, f(0) + 2[\, f(2) - f(0)] + \frac{1}{12}[\, f(4) - 2\, f(2) + f(0)]$

$= \frac{1}{12}[\, f(0) - 22 f(2) + f(4)]$.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. In general quadrature formula we integrate Newton's formula for forward interpolation.
2. Simpson's one third rule require the division of the whole range into an odd number of subintervals of width h.
3. The process of computing the value of a definite integral from a set of numerical values of the integrand is called numerical integration.

Multiple choice questions

1. Which one is not related to numerical integration
(a) Newton's Raphson method
(b) Weddle's rule
(c) Trapezoidal rule
(d) Gauss's Quadrature formula

2. A parabolic formula is also known as
(a) Cote's formula
(b) Weddle's formula
(c) Simpson's one third rule
(d) None of these

3. In trapezoidal rule we have assumed that y is a polynomial is x of degree
(a) One
(b) Two
(c) Three
(d) Four

4. In Simpson's three eight rule we have assumed that y is a polynomial of degree

(a) one
(b) two
(c) three
(d) four

5. In Simpson's one third rule we have assumed that y is a polynomial of degree
(e) one
(f) two
(g) three
(h) four

7.8 SUMMARY

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{y_0+y_n}{2} + (y_1 + y_2 + y_{n-1})\right]$$

= distance between to consecutive ordinated × {mean of the first and the last ordinates + sum of all the intermediate ordinates}.

Simpson's one third rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(\mathrm{y}_0 + \mathrm{y}_\mathrm{n}) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

Simpson's three eight rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[(y_0 - y_n) + 3(y_1 + 2y_2 + y_4 + y_{5+\cdots}y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

Weddell rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots].$$

7.9 GLOSSARY

Numerical Integration: A method of approximating the value of a definite integral using numerical methods.
Numerical Instability: The tendency of a numerical method to produce inaccurate or unstable results.
Convergence
Stability

7.10 REFERENCES

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7.11 SUGGESTED READING

1. Numerical Integration" by A. H. Stroud (Journal of Numerical Analysis, 1965)
2. "Numerical Integration: A Review" by S.D. Conte and Carl de Boor (Journal of Numerical Analysis, 1980)
3. "A New Method for Numerical Integration" by C. de Boor (Numerische Mathematik, 1978)
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7.12 TERMINAL AND MODAL QUESTIONS

1. Using Simpson's one third rule, find $\int_0^6 \frac{dx}{(1+x)^2}$.

2. Show that $\int_0^1 \frac{dx}{1+x}$ = log 2 = 0.69315.

3. Calculate the approximate value of $\int_{-3}^{3} x^4$ dx by using Trapezoidal rule.

4. Find by Weddle's rule the value of the integral I = $\int_{0.4}^{1.6} \frac{x}{sinhx} dx$ by taking 12 sub-interval.

5. Drive the following quadrature formula:

$$\int_{-a}^{b} f(x)dx = \frac{a+b}{6ab}[b(2a-b)f(-a) + (a+b)^2 f(0) + a(2b-a)f(b)]].$$

6. Evaluate $\int_5^{12} \frac{dx}{x}$ by numerical methods.

FILL IN THE BLANKS

1. The formula obtained by putting n =2 in the general quadrature formula is Simpson's …………rule.
2. A formula which is applicable to any number of subintervals whether even or odd, is the………
3. Weddle's rule is obtained by putting n = ….in the general quadrature formula.
4. In cote's method we integrate ……….. interpolation formula.

7.13 ANSWERS

CYQ1. True
CYQ2. False
CYQ3. True
MCQ1. (a)
MCQ2. (c)
MCQ3. (a)
MCQ4. (c)
MCQ5. (b)
TQ1. 0.8946
TQ3. 115
TQ4. 1.0101996
TQ6. 0.87821
FQ1. One third
FQ2. Trapezoidal rule
FQ3. 6
FQ4. Lagrange's

UNIT 8: NUMERICAL SOLUTION OF ORDINARY DIFFERENCE EQUATION

CONTENTS:

8.1 INTRODUCTION

Numerical solution of Ordinary Differential Equation is a mathematical technique Which is used to numerical calculation for solving the ODEs. It is an approximate solution which is convert ODE to discrete.

Let y =f (x) be a function of x. An ordinary differential equation is an equation of the form:

F (x, y, $y', y'', \ldots\ldots\ldots y^n = 0$ where x is the independent variable, y is the dependent variable and $y', y'', \ldots\ldots\ldots y^n$ are derivatives of y with respect to x. there are many analytical methods exist to solve such equation. but these methods can be applied to solve only a selected class of differential equations. Some time a differential equation gives so difficult solution or cannot be solved. In that cases numerical method are used to solve such type of differential equations.

8.2 OBJECTVES

After studying this unit, the learner will be able

1. To find numerical solutions to ODEs
2. To develop numerical methods and analyze numerical stability and convergence.
3. To improve numerical accuracy and efficiency and to develop new numerical techniques.

8.3 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

We consider the differential equation

$\frac{dy}{dx}$ = f (x, y), y $(x_0) = y_0$ … (1)

Integrating this between the corresponding limit for x and y, we get

$\int_{x_0}^{y} dy = \int_{x_0}^{x} f\ (x, y)dx$ or y - y_0 = $\int_{x_0}^{x} f(x, y)dx$

Or y = y_0 + $\int_0^x f(x, y)dx.$ … (2)

In the equation (2) the unknown function y is present under the integral sign. An equation of this type is called an integral equation. Such as equation can be solved by the process of successive approximations or iteration.

To solve by Picard's method of successive approximations, the first approximation toy is obtained by putting y_0 for y on the R.H.S of (2).

We have

$y^{(1)} = y_0 + \int_{x_0}^{x} f\ (x, y_0)dx.$

Now the integrand is a function of x only and the indicated integration can be performed. Hence, we get $y^{(1)}$. Substituting it for y in the integrand of (2) and integrating again, we get the second approximation

$y^{(2)} = y_0 + \int_0^x f\left(x, y^{(1)}\right)dx.$

Proceeding in this way, we get $y^{(3)}$, $y^{(4)}$, ...

The nth approximation is given by

$y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right)dx.$ …. (3)

The process is repeated as many times as may be necessary or desirable. In fact the process is stopped when the two values of y, viz. $y^{(n-1)}$ and $y^{(n)}$ are same to the desired degree of accuracy. Practically it is unsatisfactory because of the difficulties which arises in performing the necessary integrations. Each step gives a better approximation of the desired solution than the preceding one.

Illustrative Examples

Example 1: Use Picard's method to approximate y when x= 0 .2 given that y = 1 when x =0 and $\frac{dy}{dx}$ = x – y.

Solution: here f (x, y) = x – y, $x_0 = 0$, $y_0 = 1$.

We have first approximation,

$y^{(1)} = y_0 + \int_0^x f(x, y_0)dx = 1 + \int_0^x (x-1)dx = 1 - x + \frac{x^2}{2}.$

Second approximation

$y^{(2)} = y_0 + \int_0^x f\left(x, y^{(1)}\right)dx = 1 + \int_0^x f\left(x - y^{(1)}\right)dx$

$1 + \int_0^x (x - 1 + x - \frac{x^2}{2})\, dx = 1 - x + x^2 - \frac{x^3}{6}),$

Third approximation,

$y^{(3)} = y_0 + \int_0^x f\left(x, y^{(2)}\right)dx = 1 + \int_0^x \left(x - y^{(2)}\right)dx$

$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24})\, dx$

$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$.

Fifth approximation,

$y^{(5)} = y_0 + \int_0^x f\left(x, y^{(4)}\right)dx = 1 + \int_0^x \left(x - y^{(4)}\right)dx$

$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{120})\, dx$

$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}.$

When x = 0.2, we get

$y^{(1)}$ = 0.82, $y^{(2)}$= 0.83867, $y^{(3)}$= 0.83740,

$y^{(4)}$= 0.83746 and $y^{(5)}$= 0.83746.

Thus y = 0.837 when x = 0.2.

8.4 PICARD'S METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

Let $\frac{dy}{dx}$= f (x, y, z) and $\frac{dz}{dx}$= g (x, y, z)

Be the simultaneous differential equations with the initial conditions

Y $(x_0) = y_0$ and z $(y_0) = z_0$.

By Picard's method

$y^{(1)} = y_0 + \int_{x_0}^{x} f\,(x, y_0, z_0)dx;$ $z^{(1)} = z_0 + \int_{x_0}^{x} g\,(x, y_0, z_0)dx$

$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}, z^{(1)}\right)dx;$

$z^{(2)} = z_0 + \int_{x_0}^{x} \left(g, y^{(1)}, z^{(1)}\right), dx$

And so on successive approximations.

ILLUSTRATIVE EXAMPLES

Example: Approximate y and z by using Picard's method for the particular solution of

$\frac{dy}{dx}$ **= x + z,** $\frac{dz}{dx}$ **= x -** y^2 **,**

Given that y = 2, z =1 when x = 0.

Solution: let $\emptyset\,(x, y, z)$= x + z and f (x, y, z) = x - y^2.

Here x_0= 0, y_0 = 2, z_0 =1.

We have $\frac{dy}{dx}$= $\emptyset\,(x, y, z)$ => y = y_0 +$\int_{x_0}^{x} \emptyset\,(x, y, z)dx;$ …(1)

$Also\ \frac{dz}{dx} = f\,(x,$ y, z) => z = z_0 +$\int_{x_0}^{x} f\,(x, y, z)dx;$ …(2)

We have first approximation of y,

$y^{(1)} = y_0 + \int_{x_0}^{x} \emptyset\,(x, y_0, z_0)dx =$ 2 + $\int_{x_0}^{x} (x + z_0)dx$

= 2 +$\int_{0}^{x}(x + 1)dx =$ 2 +x +$\frac{1}{2}\,x^2$.

First approximation of z is,

$z^{(1)} = z_0 + \int_{x_0}^{x} f\,(x, y_0, z_0)dx = \int_{0}^{x}(x - {y_0}^2)dx$

= 1+ $\int_{0}^{x}(x - 4\,)dx$ = 1 – 4 x + $\frac{1}{2}\,x^2$.

Second approximation of z,

$z^{(2)} = z_0 + \int_{x_0}^{x} f\left(x, y^{(1)}, z^{(1)}\right), dx$ = 1 + $\int_{0}^{x}[x - \{\,{y_1}^{(1)}\}^2 dx$

$= 1 + \int_0^x \left[x - \left(2 + x + \frac{1}{2}x^2\right) 2\right] dx$

$= 1 + \int_0^x \left[x - \left(4 + x^2 + \frac{1}{4}x^4 + 4\,x + x^3 + 2\,x^2\right)\right] dx$

$= 1 - 4\,x - \frac{3}{2}\,x^2 - x^3 - \left(\frac{x^4}{4}\right) - \left(\frac{x^5}{20}\right).$

8.5 MILNE'S METHOD

Milne's Method is a simple and reasonably accurate method of solving differential equations numerically. this method is introduced by **W.E. Milne.**

This formula is derived from Newton's forward difference formula in the form:

$y' = y_0' + u\,\Delta y_0' + \frac{u\,(u-1)}{2}\,\Delta^2\,y_0' + \frac{u\,(u-1)\,(u-2)}{6}\,\Delta^3\,y_0' + \frac{u\,(u-1)\,(u-2)(u-3)}{24}\,\Delta^4\,y_0' \;\ldots\ldots(1)$

Where x = x_0 + hu, y' = f (x, y) and y_0' = f (x_0, y_0).

Integrating both side of (1) over the interval x_0 to x_0 + 4 h or u =0 to u = 4, we have

$y_4 - y_0 = h \int_0^4 \left[y_0' + u\,\Delta y_0' + \frac{u(u-1)}{2}\,\Delta^2\,y_0' + \frac{u(u-1)(u-2)}{6}\,\Delta^3\,y_0' + \frac{u(u-1)(u-2)(u-3)}{24}\,\Delta^4\,y_0' + \ldots\right] du$

$= h\left[4\,y_0' + 8\,\Delta y_0' + \frac{20}{3}\,\Delta^2\,y_0' + \frac{8}{3}\,\Delta^3\,y_0' + \frac{28}{90}\,\Delta^4\,y_0' + \ldots\right]$

$y_4 - y_0 = \frac{4\,h}{3}\,[2\,y_1' - y_2' + 2\,y_3']$

$y_4 = y_0 + \frac{4\,h}{3}\,[2\,y_1' - y_2' + 2\,y_3'].$ … (2)

This is Milne's Predictor formula which predict the value of y_4 when those of y_0, y_1, y_2 and y_3 are known.

This formula can also be written as

$y_4 = y_0 + \frac{4\,h}{3}\,[2f_1 - f_2 + 2\,f_3]$ … (3)

Where f_1 = f (x_1, y_1), f_2 = f (x_2, y_2), f_3 = f (x_3, y_3).

Now integrating both sides of (1) over the interval x_0 to x_0 +2 h or u = 0 to u = 2, we have

$y_2 - y_0 = h \int_0^2 \left[y_0' + u\,\Delta y_0' + \frac{u(u-1)}{2}\,\Delta^2\,y_0' + \ldots\right] du$

$= h\left[2y_0' + 2\Delta y_0' + \frac{1}{3}\,\Delta^2\, y_0' \;\; + \ldots\right]$

Neglecting third or higher order differences, we obtain

$y_2 \; - y_0 = \frac{h}{3}\,(y_0' + 4y_1' + \; y_2' \;\;)$

$y_2 \; = y_0 + \frac{h}{3}\,(y_0' + 4y_1' + \; y_2' \;\;)$ … (4)

This formula is **Milne's corrector formula.** It can also be written as

$y_2 = y_0 + \frac{h}{3}\,(f_0 + 4f_1 + f_2).$ … (5)

Hence the value of y_4 obtained from (2) can therefore be checked by using (4).

Working rule of applying Milne's formula: from (2) we obtained y_4

with the help of y_0 , $f_{1,}$ f_2 and f_3 and then to improve this value $y_{4,}$ we use the formula (4). In this formula, we take y_4 as y_2 and x_4 as $x_{2,}$ then find f_2 = f (x_2 , y_2) and $y_{3,}$ as y_1 and x_3 asx_1, then find f_1 = f (x_1 , y_1) and finally y_2 asy_0 and x_2 as x_0, then find f_0 = f (x_0 , y_0). Now putting f_0, f_1 and f_2 in the corrector formula (5) which gives improved values of y_4.

Example 1: Find y (2) if y (x) is the solution of $\frac{dy}{dx} = \frac{x+y}{2}$, assuming y (0) = 2, y (0.5) = 2.636, y (1.0) = 3.595 and y (1.5) = 4.968, using Milne's method.

Solution: Here f (x, y) $=\frac{x+y}{2}$, x_0 = 0, x_1 = 0.5, x_2 =1.0, x_3 = 1.5 so that y_0 =2,

y_1 =2.636, y_2 =3.595, y_3 =4.968.

Also, h = 0.5.

$\therefore\; f_0 = f\,(x_0, y_0)$

$= f\,(0, 2) = \frac{0+2}{2} = 1$

$f_1 = f\,(x_1, y_1)$

$= f\,(0.5, 2.636)\; \frac{0.5+2.636}{2} = 1.568$

$f_2 = f\,(x_2, y_2) = f\,(1.0, 3.595)$

And $f_3 = f\,(x_3, y_3) = f\,(1.5, 4.968)$

$= \frac{1.5+4.468}{2} = \; \frac{6.468}{2} = 3.234$

$= \frac{6.468}{2} = 3.234$

By Milne's predictor formula, we have $y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 3f_3]$

$= 2 + \frac{4}{3}(0.5)\,[2(1.568) - 2.975 + 2\,(3.234)]$

$= 2 + 4.871 = 6.871.$

Also, $x_4 = 2.0$, so $f_4 = f\,(x_4, y_4)$

$= f\,(2, 6.871)$

$= \frac{2.6.871}{2} = 4.4355$

Now to improve the value of y_4 we use Milne's corrector formula

$y_4 = y_0 + \frac{h}{3}[f_0 - 4f_1 + f_2]$

Here y_2 is the improve value of y_4 and taking y_0 $(= y_2) = 3.595$

$f_0(= f_2) = 2.2975$, $f_1(= f_3) = 3.234$, $f_2(= f_4) = 4.4355$

$\therefore\ y_2(= y_4) = 3.595 + \frac{0.5}{3}[2.2975 + 4\,(3.234) + 4.4355]$

$= 3.595 + 3.2782$

$= 6.8732$

Hence, y (2) = 6.8732.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. The general formula of the Picard's method of successive approximation is $y^{(n)} = y_0 + \int_{x_0}^{x} f\,(x^n, y^n)$ dx.
2. The method of Picard and Taylor belong to the first form of the solution.

MULTIPLE CHOICE QUESTIONS

1. What is the error term in Milne's corrector formula?
(a) The error term is - h / 90 Δ^4 y_0'
(b) The error term is -h /90 Δ^3 y_0'
(c) The error term is -h/90 Δ^2 y_0'
(d) None of these

2. Given $y' = 1 + y^2$, y (0) = 0, y (0.2) = 0.2027, y (0.4) = 0.4228, y (0.6) = 0.6841, estimate y (0.8), y (1) using Milne's method upto three decimals.

(a) 1.029, 1.555
(b) 1.028, 1.566
(c) 1.027, 1.567
(d) 1.026, 1.455

3. Solve, $y = x - y^2$, y (0) 1 to obtain y (0.4) by Milne's method. Obtain the data you require by any method of your liking.

(a) 0.7798
(b) 0.7797
(c) 0.6796
(d) 0.7796

4. Given $dy/dx + y + y - x^2 = 0$, y (0.1) = 0.9052, y (0.2) = 0.8213 find correct to four places of decimals y (0.4) and y (0.5), using Milne's Predictor corrector method.

(a) y (0.4)p = 0.6897, y (0.4) c = 0.6897, y(0.5)p = 0.6435, y(0.5), = 0.6435
(b) y (0.4)p = 0.7897, y (0.4) c = 0.6897, y(0.5)p = 0.6435, y(0.5), = 0.6435
(c) y (0.4)p = 0.6897, y (0.4) c = 0.6897, y(0.5)p = 0.8435, y(0.5), = 0.6435
(d) None of these

8.6 SUMMARY

In this chapter we explained about Picard method for succussive Approximations and Milne's method

8.7 GLOSSARY

Accuracy
Algorithm
Initial Conditions
Numerical Instability

8.8 REFERENCES

1. "Numerical Analysis" by Richard L. Burden and J. Douglas Faires.
2. C. F. Gerald & P. O. Wheatley, Applied Numerical Analysis (7 th edition), Pearson Education, India, 2008.
3. F. B. Hildebrand, Introduction to Numerical Analysis: (2 nd edition). Dover Publications, 2013.

8.9 SUGGESTED READING

1.Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India (1989).
2. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).
3. "Numerical Methods for Differential Equations" by William F. Ames.

8.10 TERMINAL AND MODAL QUESTIONS

1. Use Picard's method to solve $\frac{dy}{dx}$ = 1 + x y, with $x_0 = 2, y_0 = 0$.
2. Find the value of y for x= 0.1 by picard's method, given that: $\frac{dy}{dx} = \frac{y-x}{y+x}$, y (0) =1.
3. Using Picard's method of successive approximation obtain a solution upto third approximation of the differential equation $\frac{dy}{dx}$ = x + y^2 with y (0) = 0.
4. Solve initial value problem $\frac{dy}{dx}$ = 1 + x y^2, y (0) = 1 for x =0.4, 0.5 by using Milne's predictor corrector method when it is given

X = 0.1	0.2	0.3
Y = 1.105	1.223	1.355

FILL IN THE BLANKS

1. In general , Milne's predictor and corrector formulae are ……
2. Euler's Runge- Kutta Milne etc., belong to the …………form of the soliution .

8.11 ANSWERS

CYQ1. False
CYQ2. True
MCQ1. (a)
MCQ 2. (a)
MCQ 3.(d)
MCQ 4. (a)
TQ1. $y^{(3)} = \frac{x^5}{15} - \frac{x^4}{4} + \frac{x^3}{3} + x - \frac{22}{15}$.
TQ2. y (0.1) = 0.9828
TQ3. $y^{(3)} = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$
TQ4. Y(0.4) = 1.5379
FQ1. $y_{n+1} = y_n + h f(x_n, y_n)$
FQ2. SECOND

UNIT 9. MATRIX INVERSION

CONTENTS:

9.1 INTRODUCTION

Matrix inversion is the process of finding the inverse of a matrix, which is a fundamental concept in linear algebra. A square matrix A (n x n) with det(A) ≠ 0 has an inverse B (n x n) if and only if AB = BA = I, where I is the identity matrix. In another way Given a non-singular square matrix A (n x n) with a non-zero determinant, a square matrix B (n x n) that satisfies the conditions AB = BA = I is called the inverse of A. There are several methods for finding the inverse of a matrix. In this unit we shall describe some methods for finding the inverse of s given matrix.

9.2 OBJECTIVES

After studying this unit learner will be able to

1. Solve systems of linear equations.
2. Find the inverse of a matrix.
3. Determine the existence and uniqueness of solutions.

9.3 GAUSS ELIMINATION METHOD

In this method first we Write the augmented matrix (matrix with constants). Then Perform elementary row operations to transform the matrix into upper triangular form (all entries below the diagonal are zero) and then to unit matrix. In the whole process the matrix is reduced to unit matrix and the adjacent matrix gives the inverse of A. to increase the accuracy the largest element at the first position of the row is taken as the pivot element by using only row transformation.

SOLVED EXAMPLE

Example:1 Find the inverse of the matrix A = $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$ **by Gauss – elimination method.**

Solution: let us put

A= A I

$$\therefore \ A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Interchanging R_1 *and* R_3 we get

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Performing $R_1 \rightarrow R_2 - \frac{1}{2} R_1, R_3 - \frac{1}{2} R_1$, we get

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Performing $R_3 \rightarrow R_3 + R_2$, we get

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & 1 & -1 \end{bmatrix}.$$

Now A has been converted into an upper triangular matrix. By row transformation we convert this upper triangular matrix into Identity matrix.

Multiplying R_2 by 2, we get

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Performing $R_1 \rightarrow R_1 - 3R_2$, we get

$$\sim \begin{bmatrix} 2 & 0 & -14 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & -6 & 4 \\ 0 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Dividing R_1 by 2 and R_3 by 4, we ge

$$\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & -3 & 2 \\ 0 & 2 & -1 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

Performing $R_1 \rightarrow R_1 + 7R_3, \ \ R_2 \rightarrow R_2 - 5R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \sim \mathrm{I}A^{-1}.$$

Hence $A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$

9.4 GAUSS JORDEN METHOD

Gauss Jorden Method is similar to Gauss elimination method. In Gauss elimination we convert the matrix A into an upper triangular matrix and then to unit matrix. But here the matrix A is directly converted into identity matrix by elementary row transformation.

Example 1: Find the inverse of A = $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$ **by Gauss Jordan Method**.

Solution: we have

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad R_1 - 3R_3$$

Hence $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

9.5 TRIANGULARIZATION METHOD (DOOLITTLE METHOD)

In triangular method, we decompose the given matrix as

A = L U …(1)

where

L = = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

Here L is lower triangular matrix having diagonal elements unity and U is an upper triangular matrix.

Now from (1), we get

$A^{-1} = (LU)^{-1} = U^{-1} L^{-1}$

Example 1: Apply Triangularisation method to find the inverse of

A= = $\begin{bmatrix} 2 & -2 & -4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.

Solution: let A = LU

$$\therefore \begin{bmatrix} 2 & -2 & -4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

$\therefore u_{11} = 2, u_{12} = -2, u_{13} = 4$

$l_{21}u_{11} = 2 \Rightarrow l_{21} = 2/u_{11} = 2/2 = 1$

$l_{21}u_{12} + u_{22}$ =3 => u_{22} = 3 - $l_{21}u_{12}$

= 3- 1 (-2) =5

$l_{21}u_{13}$ +u_{23} = 2 =>$u_{23} = 2 - l_{21}u_{13}$

= 2 – 1 (4) = -2

$l_{31}u_{11}$= -1 => l_{31} = -1/u_{11} = -1/2

$l_{31}u_{12}$ +$l_{32}u_{22}$ = 1 => l_{32}= $\frac{1}{u_{22}}(1 - l_{31}u_{12})$

= $\frac{1}{5}\left(1 + \frac{1}{2}(-2)\right) = 0$

$l_{31}u_{13}$ +$l_{32}u_{23}$ + u_{33} = -1

⇨ u_{33} = -1 - $l_{31}u_{13}$ +$l_{32}u_{23}$

= -1 + $\frac{1}{2}$ (4) – 0 = 1

Thus, L = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$, U= $\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Determination of L^{-1} *and* U^{-1} .

Let L^{-1}= X

∴ LX = I

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

x_{11} =1, $x_{11} + x_{21}$ =0 => x_{21} = -x_{11} =-1

$x_{22} = 1$, - $\frac{1}{2}x_{11} + x_{31}$ =0 => $x_{31} = \frac{1}{2}x_{11} = \frac{1}{2}$

x_{32} =0, x_{33} =1

Thus $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Now let U^{-1} = Y

∴ UY = I

$$\begin{bmatrix}2 & -2 & 4\\0 & 5 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_{11} & y_{12} & y_{13}\\0 & x_{22} & y_{23}\\0 & 0 & y_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

$2y_{11}$ =1 => $y_{11} = ½$, y y_{22} = 1 = > y_{22} = 1/5

$2\,y_{12} - 2y_{22}$ =0 => y_{12} = -y_{22} =1/5

$y_{33} = 1,$

$5y_{23}$ - 2 y_{33}= 0 => $y_{23} = 2/5$

And $2y_{13}$- $2y_{23} + 4y_{33}$=0

⇨ $y_{13}=\frac{1}{2}\,(2y_{23} - 4y_{23})$
$=\frac{1}{2}\left(\frac{4}{5}- 4\right)$
$=$ -8/5.

$thus\ U^{-1} = \text{Y}=\begin{bmatrix}1/2 & 1/5 & -8/5\\0 & 1/5 & 2/5\\0 & 0 & 1\end{bmatrix}.$

Hence, $A^{-1} = U^{-1}L^{-1}$

$$=\begin{bmatrix}1/2 & 1/5 & -8/5\\0 & 1/5 & 2/5\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\1/2 & 0 & 1\end{bmatrix}.$$

$$=\begin{bmatrix}-1/2 & 1/5 & -8/5\\0 & 1/5 & 2/5\\1/2 & 0 & 1\end{bmatrix}.$$

9.6 ITERATIVE METHOD

Suppose we want to compute A^{-1} and we know that B is an approximate inverse of A.

Let AB – I = E, so that we get

AB = I +E → $(AB)^{-1} = (I + E)^{-1}$

⇨ $B^{-1}A^{-1}$ = $((I + E)^{-1}$
⇨ $(A)^{-1}$ = B $((I + E)^{-1}$ = B (I – E + E^2- …)
If the series converges.
Thus, we can find further approximation of A^{-1}, i.e., of B by using the relation
A^{-1}= B (I – E + E^2- …).
$if\ we\ take$ I – AB = R

⇨ $(AB)^{-1} = (I - R)^{-1}$
⇨ $B^{-1}A^{-1} = (I - R)^{-1}$
⇨ $BB^{-1}A^{-1}$ = B $(I - R)^{-1}$
⇨ $B\,A^{-1}$ = B $(I - R)^{-1}$
⇨ A^{-1} = B (I + R + R^2 + R^3+ …)
⇨ A^{-1} = B +B R + BR^2 + BR^3+ …)

Which gives successive approximations for A^{-1} .

9.7 CROUT'S METHOD

In crout's method the given matrix is written as

A = L U … (1)

Where is lower triangular matrix and U is an upper triangular matrix with diagonal elements unity.

From (1), we have

$A^{-1} = U^{-1}L^{-1}$.

Now we find L^{-1} and U^{-1} by using the process in 10.5.

Example: Find the inverse of A = $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by Crout's method.

Solution: let A = LU. Then $A^{-1} = U^{-1}L^{-1}$, where L is a lower triangular matrix and U is an upper triangular matrix with diagonal elements as unity.

Let L = = $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

Now A = L U gives

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the matrices on the R. H.S and then equating the corresponding elements, we get

$$l_{11} = 2\,, l_{21} = 2, l_{31} = -1, l_{22} = 5, l_{32} = 0, l_{33} = 1$$

u_{12}= -1, $u_{13} = 2, u_{23} = -2/5$.

Thus, we have L = = $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}$.

To find L^{-1}. Let L^{-1}= X, where X is also a lower triangular matrix. Then we have L X = I

i.e.

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By equality principle of matrices, we get

x_{11} = ½ , x_{21}= -1/5, x_{22} = 1/5, x_{31} = ½, x_{32} =0 , x_{33} =1.

Hence L^{-1}= X = $\begin{bmatrix} 1/2 & 0 & 0 \\ -1/5 & 1/5 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$.

To find U^{-1}. Let U^{-1}= Y, where Y is also a upper triangular matrix with diagonal elements as unity. Then U Y = I

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the matrices on the L.H.S and then equating the corresponding elements, we get

y_{12} =1, y_{13} = - 8/5, y_{23} = 2/5.

Hence U^{-1}= Y= $\begin{bmatrix} 1 & 1 & -8/5 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}$.

Now A^{-1} = $U^{-1}L^{-1}$

$$\begin{bmatrix} 1 & 1 & -8/5 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ -1/5 & 1/5 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/5 & -8/5 \\ 0 & 1/5 & 2/5 \\ 1/2 & 0 & 1 \end{bmatrix}.$$

9.8 CHOLESKI'S METHOD

In this method the matrix A is taken as a symmetric matrix. That is $A' = A$.

Suppose that A = LL' … (1)

Where L is a lower triangular matrix and L' is its transpose. Then the inverse is obtained

A^{-1} = L L') $^{-1}$ = $(L')^{-1}$ L^{-1} … (2)

Now let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & x_{33} \end{bmatrix}$ and L = $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

Then from (1), we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

Multiplying the two matrices on R.H.S and equating both sides the corresponding elements, we obtain

$l_{11}{}^2 = a_{11}$ => $l_{11} = \sqrt{a_{11}}$

$l_{11}l_{21} = a_{12}$ => $l_{21} = \frac{a_{12}}{l_{11}}$

$l_{11}l_{31} = a_{13}$=> $l_{31} = \frac{a_{13}}{l_{11}}$

$l_{21}{}^2 + l_{22}{}^2 = a_{22}$ => $l_{22} = \sqrt{a_{22} - l_{21}{}^2}$

$l_{21}l_{31} + l_{22}l_{32} = a_{23}$=> l_{32}= (a_{23}- $l_{21}l_{31}$)/ l_{22}

$l_{31}{}^2 + l_{32}{}^2 + l_{33}{}^2 = l_{33} = \sqrt{a_{33} - l_{31}{}^2 - l_{32}{}^2}$.

Thus, L is obtained.

As pointed out above some elements of L may be imaginary. this will cause no extra trouble.

Example: By Cholesky's method, find the inverse of A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{1} \end{bmatrix}$**.**

Solution: let us write A = LL', where L is a lower triangular matrix.

$$\therefore \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

From above, we get

$l_{11}{}^2$ =1 => l_{11} = 1

$l_{11}l_{21}$ =2 => l_{21}=2

$l_{11}l_{31}$= 2 => l_{31}=2

${l_{21}}^2 + {l_{22}}^2$ = 1=> l_{22}= $\sqrt{1-4}$= $\sqrt{-3}$ = i$\sqrt{3}$ [$\because$ $i^2 = -1$]

$l_{21}l_{31}$ $+l_{22}l_{32}$ = 2 => l_{32} = -2/$i\sqrt{3}$ = 2i/$\sqrt{3}$

${l_{31}}^2 + {l_{32}}^2$ $+{l_{33}}^2$ =1 = l_{33} =$\sqrt{1-4+4/3}$= $\sqrt{5/3}$ = i/$\sqrt{5/3}$.

Thus, we obtain

L = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & i\sqrt{3} & 0 \\ 2 & 2\,i/\sqrt{3} & i\sqrt{5/3} \end{bmatrix}$

Determination of L^{-1}: let $L^{-1} = X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

Then L X =I

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & i\sqrt{3} & 0 \\ 2 & 2i\sqrt{3} & i\sqrt{5/3} \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

x_{11} =1, 2x_{11} + i$\sqrt{3}$ x_{21} = 0=> x_{21} = -2/ i$\sqrt{3}$ =2 i$\sqrt{3}$

i$\sqrt{3}$ x_{22}= 1 => $x_{22} = -$ i$\sqrt{3}$

2x_{11}+ (2i/$\sqrt{3}$) x_{21}+ (i$\sqrt{5/3}$) $x_{31} = 0$ => x_{31} = 2 i/$\sqrt{15}$

(2 i$\sqrt{3}$) x_{22}+ (i$\sqrt{5/3}$)x_{32}=0 => x_{32} = 2 i/$\sqrt{15}$

(i$5/3$) x_{33}= 1 => x_{33} = - i$\sqrt{3/5}$.

Thus, we obtain

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2\,i/\sqrt{3} & -\,i\sqrt{3} & 0 \\ 2\,i/\sqrt{15} & 2\,i/\sqrt{15} & -\,i\sqrt{3/5} \end{bmatrix}.$$

Hence A^{-1} = (LL') $^{-1}$ = $(L')^{-1}$ L^{-1} = $(L^{-1})'$ L^{-1}

$$= \begin{bmatrix} 1 & 2\,i/\sqrt{3} & 2\,i/\sqrt{15} \\ 0 & -\,i\sqrt{3} & 2\,i/\sqrt{15} \\ 0 & 0 & -\,i\sqrt{3/5} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2\,i/\sqrt{3} & -\,i\sqrt{3} & 0 \\ 2\,i/\sqrt{15} & 2\,i/\sqrt{15} & -\,i\sqrt{3/5} \end{bmatrix}.$$

$$= \begin{bmatrix} 3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}.$$

CHECK YOUR PROGRESS

TRUE OR FALSE

1. In triangular method we write $A^{-1} = (LU)^{-1} = U^{-1} L^{-1}$ where T is upper triangular matrix.
2. In Gauss elimination method it is necessary to fix the largest element as the pivot element while performing row transformation only.

Multiple choice questions

1. The method in which we factorize a symmetric matrix A as A = $T\,T'$
Where T is a lower triangular matrix is
(a) Crout's method
(b) Choleski's method
(c) Escalator method
(d) Doolittle method

2. In iterative method the further approximations of A^{-1}= B (1-E+E +E^2-…) where E =
(a) AB -I
(b) BA-I
(c) AB +I
(**d**) BA +I

9.9 SUMMARY

In this unit we discussed about Matrix inversion , Gauss Elimination Method, Gauss-Jordan Method, Triangularization method , Cholesky's method ,the Escalator method and Iterative method

9.10 GLOSSARY

Matrix Inversion: The process of finding the inverse of a matrix.
Determinant: A scalar value that can be used to determine the invertibility of a matrix..
Non-Singular Matrix: Determinant is non-zero.

Gaussian Elimination: A method for finding the inverse of a matrix by transforming it into upper triangular form.

9.11 REFERENCES

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9.12 SUGGESTED READING

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2. "Matrix Inversion: A Survey of Recent Developments" by G. H. Golub and C. F. Van Loan (SIAM Journal on Matrix Analysis and Applications, 2013)
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9.13 TERMINAL AND MODAL QUESTIONS

1. 1 Find the inverse of the following matrices using Gauss elimination method:

$(\boldsymbol{i})$ $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

2. Find the inverse of the matrix by Jordan's method.

3. $\begin{bmatrix} 1 & 1/3 & 1/5 \\ 1/3 & 1/5 & 1/7 \\ 1/5 & 1/7 & 1/9 \end{bmatrix}$

4. Find the inverse of the following matrix by Cholesky's method:

5. A= $\begin{bmatrix} 2 & 1 & 6 \\ 1 & 3 & 5 \\ 6 & 5 & 4 \end{bmatrix}$

6. Find the inverse of the following matrix by Crout's method:

7. A= $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

8. Apply Gauss – Jordan method to find the inverse of the matrix

(i) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

FILL IN THE BLANKS

1. Cholesky's method is applicable when the given matrix is real ………. matrix.
2. In Doolittle method we decompose the given matrix A as A = L U where L is lower triangular matrix and U is an upper triangular matrix with diagonal elements….
3. In Gauss Jordan method we put an identity matrix by the side of A and convert the matrix A into an …Metrix.
4. In gauss elimination we take an …. matrix of the same order as that of A and place it with A.

9.14 ANSWERS

CYQ1. True
CYQ1. True
MCQ1. (b)
MCQ2. (a)

TQ1 . (i) $\begin{bmatrix} 7/2 & -3/2 & -3/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$ (ii). $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$

TQ2 $\frac{15}{64}\begin{bmatrix} 15 & -70 & 63 \\ -70 & 588 & -630 \\ 63 & -630 & 735 \end{bmatrix}$

TQ3. $\frac{1}{78}\begin{bmatrix} 13 & -26 & 13 \\ -26 & 28 & 4 \\ 13 & 4 & -5 \end{bmatrix}$

TQ4. $\begin{bmatrix} -1/2 & 1/5 & -8/5 \\ 0 & 1/5 & 2/5 \\ 1/2 & 0 & 1 \end{bmatrix}$

TQ5. $\begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & 2- \\ 1 & -1 & 2 \end{bmatrix}$

FQ1. symmetric
FQ2. L.
FQ3. Identity
FQ4.Identity

BLOCK III
APPROXIMATION AND SOLUTIONS THROUGH DIFFERENTIAL EQUATION

UNIT 10 : APPROXIMATION

CONTENTS:

10.1 INTRODUCTION

Approximation is a fundamental concept in numerical analysis, enabling the solution of complex problem in various fields. When approximating a function, we choose the best approximation that minimize the largest error. The approximations of functions which are commonly used, are the polynomial, trigonometric function, exponential functions and rational functions.

10.2 OBJECTIVES

After studying in this unit learner will be able

1. To developed approximations resilient to data noise and variability
2. To reduce complexity of a function or problem.
3. To improve computational speed and efficiency.

10.3 CHEBYSHEV POLYNOMIALS

(i) The Chebyshev polynomial of first kind of degree n over the interval [-1,1] is defined by

$$T_n(x) = cos\,(cos^{-1} x). \qquad \ldots (1)$$

From (1) it is clear that

$$T_n(x) = T_{-n}(x)$$

$if\ (cos^{-1}x)$ = θ then from (1), we get … (2)

$T_n(x)$= cos n θ ... (3)
$Hence\ \ T_0(x)$ = 1, $T_1(x)$ = cos θ = x.

(ii) The Chebyshev polynomial of second kind of degree n over the interval [-1,1] is defined by
$U_n(x) = sin\ (n\, sin^{-1} x).$ … (4)
Clearly
$U_{-n}(x) = -U_n(x)$ … (5)
$Also\ \ U_n(x)$ = sin nθ … (6)
$where\ \ cos^{-1} x$ = θ i.e., x = cos θ.

(iii) Other form of Chebyshev polynomial of first kind:
From (1), we have
$T_n(x) = cos\,(n\, cos^{-1} x).$
= cos nθ where x= cos θ.
$=\frac{1}{2}\left[e^{in\theta} + e^{-in\theta}\right]$
$=\frac{1}{2}[(cos\,\theta + i\, sin\theta)^{\,n} + (cos\,\theta - i\, sin\theta)^{\,n}]$
$= T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right].$ … (7)

(iv) Other form of Chebyshev polynomial of second kind:
From (ii), we have

$$U_n(x) = sin\ (n\, cos^{-1} x)$$

= sin nθ, where x = cos θ
$= \frac{1}{2i}\left[e^{in\theta} - e^{-in\theta}\right]$
$=\frac{1}{2i}[(cos\,\theta + i\, sin\theta)^{\,n} - (cos\,\theta - i\, sin\theta)^{\,n}]$
$=\frac{1}{2i}\left[\left(x + i\sqrt{1-x^2}\right)^n - \left(x - i\sqrt{1-x^2}\right)^n\right].$ … (8)

10.4 CHEBYSHEV DIFFERENTIAL EQUATION

Let y = $T_n(x) = cos\,(n\, cos^{-1} x).$
So $\frac{dy}{dx} = -sin\ (n\, cos^{-1} x)\left[-\frac{n}{\sqrt{1-x^2}}\right]$
$=\frac{n\, sin\ (n\, cos^{-1} x)}{\sqrt{1-x^2}}$
Or $\sqrt{1-x^2}\frac{dy}{dx}$ = n $sin\ (n\, cos^{-1} x)$
Squaring both side we get,

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = n^2 sin^2\, n\, cos^{-1} x$
$= n^2\left(1 - cos\,(n\, cos^{-1} x)\right)$
$= n^2(1-y^2)$
Or $(1-x^2)\left(\frac{dy}{dx}\right)^2 = n^2$ - = $n^2\, y^2$.

Again differentiating, we get

$$2\ (1-\ x^2)\frac{dy}{dx}\frac{d^2\,y}{dx^2}-2\,x\left(\frac{dy}{dx}\right)^2\ =\ -2\,n^2y\,\frac{dy}{dx}$$

$$(1-\ x^2)\,\frac{d^2\,y}{dx^2}-x\,\frac{dy}{dx}+n^2y=0. \qquad \ldots (1)$$

This is the **Chebyshev differential equation.**
Similarly, the equation (1) is also satisfied by $U_n(x)$
Also $T_n(1)$= cos ($n\,cos^{-1}$ 1) = cos 0 = 1
And $U_n(1)$ = sin ($n\,cos^{-1}$ 1) = sin 0.
This shows that $T_n(x)$ can not be expressed in terms of $U_n(x)$. Hence, $T_n(x)$ and $U_n(x)$ are the independent solution of the differential equation (1).

10.5 RECURRENCE RELATION OF $T_n(x)$ and $U_n(x)$

(i) Recurrence relation of $T_n(x)$:

(***a***) $T_{n+1}(x) = 2\,x\,T_n(x)\ -\ T_{n-1}\,(x)$.

By the definition of Chebyshev polynomial of first kind, we have
$T_n(x)$= cons n θ, where x = cos θ … (1)
since we know that
Cos (n +1) θ + cos (n -1) θ = 2 cos n θ cos θ. … (2)
Using (1), equation (2) becomes

$T_{n+1}(x)$ - $T_{n-1}(x)\ -\ T_{n-1}\,(x)$.

(b) $(1-\ x^2)T_n{}'(x)$ = -n x $T_n(x)$ + n T_{n-1}(x).

Differentiating both sides of (1) w. r. t. x we get

$T_n{}'(x)$ = - n sinθ $\frac{d\theta}{dx}$

$=-$ n sin nθ $\frac{1}{-\,sin\theta}$

= $\frac{\text{n sin n}\theta}{\text{sin n}\theta}$

So, $(1-\ x^2)T_n{}'(x)\ =\ (1-\ x^2)\,\frac{\text{n sin n}\theta}{\text{sin n}\theta}$ [∵ x = cos θ]

= $(1-\ cos^2\theta)\,\frac{\text{n sin n}\theta}{\text{sin n}\theta}$

= n sin θ sin n θ

= n sin θ sin n θ + n cos θ cos n θ - n cos θ cos n θ

= n cos (n-1) θ - n cos θ cos n θ

= $n\,T_{n-1}(x)$ – n x $T_n(x)$.

∴ $(1-\ x^2)T_n{}'(x)$ =- n x $T_n(x)$ - $n\,T_{n-1}\,(x)$.

(ii) Recurrence relation of $U_n(x)$:

(***a***) $U_{n+1}(x)$ = 2 x $U_n(x)$ - $U_{n-1}\,(x)$.

By the definition of Chebyshev polynomial of second kind, we have
$U_n(x)$ = sin ($n\,cos^{-1}\,x$) = sin n θ … (3)
Where x = cos θ since we know that
Sin (n +1) θ + sin (n-1) θ = 2 sin n θ cos θ. … (4)
Using (3), equation (4) becomes

$U_{n+1}(x) + U_{n-1}\,(x)$ - = $2\,x\,U_n\,(x)$

$\therefore U_{n+1}(x)$ = 2 x $U_n(x)$ - $U_{n-1}(x)$.

(b) $(1 - x^2)U_n{}'(x)$ =- n x $U_n(x)$ - $n\, U_{n-1}(x)$.

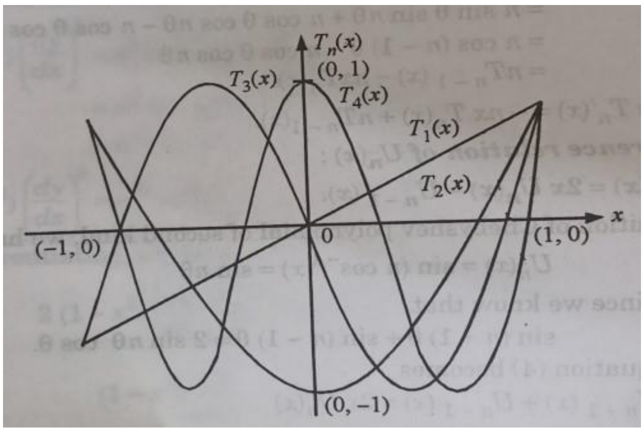
Since $U_n(x)$ = sin ($n\, cos^{-1} x$)
So that $\boldsymbol{U_n{}'(x)}$ = cos ($n\, cos^{-1} x$) $\left(\frac{-n}{\sqrt{1-x^2}}\right)$
Or $(1 - x^2)U_n{}'(x)$ = -n $\sqrt{1 - x^2}$ cos ($n\, cos^{-1} x$)
= = -n $\sqrt{1 - cos^2\theta}$ cos n θ [∴ x = cos θ]
= n sin θ cos nθ
= - n sin θ cos n θ + n cos θ sin n θ - n cos θ sin n θ
= - n cos θ sin n θ + n sin (n – 1) θ
=- n x $U_n(x)$ - $n\, U_{n-1}(x)$
$(1 - x^2)U_n{}'(x)$ =- n x $U_n(x)$ - $n\, U_{n-1}(x)$.

Some Chebyshev Polynomials

Since we have $T_n(x) = cos\ (n\, cos^{-1} x)$
$\therefore T_0(x) = 1,\ \ T_1(x)$ =x …(1)
By Recurrence relation,
$T_{n+1}(x)$ - $2n\, T_n(x) - T_{n-1}(x)$.
Putting n = 1, 2, 3, 4, 5 successively and using (1) , we get the first six Chebyshev polynomials

$T_0(x) = 1$
$T_1(x)$ =x
$T_2(x)$ =2 x^2 -1
$T_3(x)$ =4 x^3 -3 x
$T_4(x)$ =8 x^4 - 8 x^2 +1
$T_5(x)$ =16 x^5 - 20 x^3+5 x
$T_6(x)$ =32 x^6 - 48 x^4+18 x^2 - 1.

The graph of first four Chebyshev polynomials are as shown in the fig. below.



10.6 POWER OF X IN TERMS OF CHEBYSHEV POLYNOMIALS

From above six Chebyshev polynomials, we find

$1 = T_0(x)$
$x = T_1(x)$
$x^2 = \frac{1}{2}[T_0(x) + T_2(x)]$
$x^3 = \frac{1}{4}[3T_1(x) + T_3(x)]$
$x^4 = \frac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)]$
$x^5 = \frac{1}{16}[10T_1(x) + 5T_3(x) + T_5(x)]$
$x^6 = \frac{1}{32}[10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x)]$.
Similarly , we can express higher degree of x in terms of Chebyshev polynomials.

10.7 PROPERTIES OF CHEBYSHEV POLYNOMIALS

(i) We have seen that the first six Chebyshev polynomial, $T_n(x)$ is a polynomials of degree n and the coefficient of x^n in $T_n(x)$ is 2^{n-1}.
(ii) $T_n(x)$ are orthogonal with the function $\frac{1}{\sqrt{1-x^2}}$.
(iii) $Since\ T_n(-x) = -(1)^x\ T_n(x)$, so that $T_n(x)$ is even function if n is even and it is an odd function if n is odd.
(iv) $T_n(x)$ has n simple zeros (roots), which is given by
$x_r = \cos\frac{(2r-1)\pi}{2n}$, for r = 1, 2, 3,..n.
(v) $|T_n(x)| \leq 1 \forall x \in [-1,1]$.

10.8 CHEBYSHEV POLYNOMIAL APPROXIMATION

In approximation theory, one uses monic polynomial, that is Chebyshev polynomials in which the coefficient of x^n is unity.

Since $|T_n(x)| \leq 1$, then $|T_n(x)| \leq 2^{1-n}$ x $\in [-1,1]$.

Thus, in Chebyshev approximation, the maximum error is kept down to a minimal. This is often referred to as minimum principle and the polynomial
$T_n(x)$ is called the minimax polynomial. by this process we can find lower – order approximation, which is called minimax approximation to a given polynomial.
$let\ a_0 + a_1 x + a_2x^2 + \ldots . a_nx^n$ be the required minimax polynomial approximation for a continuous function f (x) defined on the interval [-1,1].
$Suppose\ that$
$$f(x) = \frac{c_0}{2} + \sum_{i-1}^{\infty} c_iT_i(x) \quad \ldots (1)$$
Is the Chebyshev series expansion for f (x). Then the partial sum $S_n(x)$ of (1) given by
$$S_n(x) = \frac{c_0}{2} + \sum_{i=1}^{\infty} c_iT_i(x) \quad \ldots (2)$$
is very close to the solution of the problem.
$\therefore \max_{x \in [-1,1]}\left|f(x) - \sum_{i=0}^{n} a_i\ x^i\right| = \min_{x \in [-1,1]}. |f(x) - \sum_{i=0}^{n} a_i\ x_i|$.
Hence the partial sum (2) is the best uniform approximation to f (x).

10.9 APPLICATION OF CHEBYSHEV SERIES IN THE ECONOMIZATION OF POWER SERIES

If a power series expansion of f (x) converts into an expansion in Chebyshev polynomial then the process is called economization of power series.
By simplification this process, which is given by Lanczos, we consider a power series expansion of f (x) in the form
F (x) = $a_0 + a_1 x + a_2 x^2 + \ldots . a_n x^n + \ldots$ … (1)
For all x ∈ [-1,1].
Now change each power of, x in (1) in terms of Chebyshev polynomial, therefore we obtain,
F (x) = $c_0 T_0(x) + c_1 T_1 x + \ldots \ldots c_n T_n(x) + \ldots$ … (2)
It has been found that for a large number of function f (x), the expansion (2) converges more rapidly than the power series expansion (1).
Let $S_n(x)$ be the partial sum of (2), then
$S_n(x) = c_0 T_0(x) + c_1 T_1(x) + \ldots \ldots c_n T_n(x)$. … (3)
This expansion $S_n(x)$ is good uniform approximation to f (x), because
$|f(x) - S_n(x)| = |c_{n+1} T_{n+1}(x) + c_{n+2} T_{n+2}(x) + \ldots|$
$\therefore \max_{x \in [-1,1]} |f(x) - S_n(x)| \leq |c_{n+1}| + |c_{n+2}| + \ldots \leq \in$ (say)
$[\therefore |T_n(x)| \leq \in$.
$\max_{x \in [-1,1]} |f(x) - S_n(x)| \leq \in$.
So, we can find the number of terms that should be retained in (3). This process is known as Lanczos economization. On replacing $T_i(x)$ in (3) by its polynomial, we get the required economized polynomial approximation for f (x).

ILLUSTRATIVE EXAMPLES

Example 1: Prove that $\sqrt{1 - x^2}\, T_n(x) = U_{n+1}(x) - x\, U_n(x)$.

Solution: Since $T_n(x) = \cos n\theta$, $U_n(x) = \sin n\theta$
Where x = cos θ
R.H.S $U_{n+1}(x) - x\, U_n(x)$
$= \sin(n+1)\theta - \cos\theta \sin n\theta$
$= \sin\theta \cos n\theta$
$= \sqrt{1 - \cos^2\theta} \cos n\theta$
$= \sqrt{1 - x^2}\, T_n(x)$ = R.H.S.

Example 2: Express $T_0(x) + 2T_1(x) + T_2(x)$ as polynomials in x .

Solution: Since $T_0(x) = 1$, $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$.
$\therefore T_0(x) + 2T_1(x) + T_2(x) = 1 + 2x + 2x^2 - 1 = 2x^2 + 2x$.

Example: 1- x^2 +2 2 x^4 as sum of Chebyshev polynomials.
Solution: $x^2 = \frac{1}{2}[T_0(x) + T_2(x)]$

and $x^4 = \frac{1}{8}[3\,T_0(x) + 4\,T_2(x) + T_4(x)]$,
$T_0(x) = 1$.
$\therefore$ 1 - x^2+2 $x^4 = T_0(x) - \frac{1}{2}[T_0(x) + T_2(x)] + \frac{2}{8}\left[3T_0(x) + 4\,T_2(x) + \frac{1}{3}T_4(x)\right]$
$= T_0(x) - \frac{1}{2}T_0(x) + \frac{3}{4}T_0(x) - \frac{1}{2}T_2(x) + T_2(x) + \frac{1}{4}T_4(x)$
$= \frac{5}{4}T_0(x) + \frac{1}{2}T_2(x) + \frac{1}{4}T_4(x)$.

Example 3: Using the Chebyshev polynomial, obtain the least squares approximation of second degree for f (x) = x^4 on [-1,1].

Solution: let p(x) be a second-degree polynomial so it can taken as
P (x) = $c_0T_0(x) + c_1T_1(x) + c_2T_2(x)$.
Let E be the error, then E = f (x) – p(x)
Let f (x) and each $T_0(x)$, $T_1(x)$ and $T_2(x)$ are continuous in [-1,1], then determine c_0, c_1, c_2, such that I(c_0, c_1, c_2) should be minimum, where

I(c_0, c_1, c_2) = $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,[E^2]$ dx
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,[x^4 - c_0T_0(x) - c_1T_1(x) - c_2T_2(x)]^2$ dx.

The normal equations are $\frac{\partial I}{\partial c_0} = 0$, $\frac{\partial I}{\partial c_1} = 0$, $\frac{\partial I}{\partial c_2} = 0$,
i.e., $\frac{\partial I}{\partial c_0} = 0 \Rightarrow \int_{-1}^{1}(x^4 - c_0T_0 - c_1T_1 - c_2T_2)\,\frac{T_0}{\sqrt{1-x^2}}$ dx = …(1)

$\frac{\partial I}{\partial c_1} = 0 \Rightarrow \int_{-1}^{1}(x^4 - c_0T_0 - c_1T_1 - c_2T_2)\,\frac{T_1}{\sqrt{1-x^2}}$ dx =0 … (2)
$\frac{\partial I}{\partial c_2} = 0 \Rightarrow \int_{-1}^{1}(x^4 - c_0T_0 - c_1T_1 - c_2T_2)\,\frac{T_2}{\sqrt{1-x^2}}$ dx =0 … (3)
Now by orthogonal properties, we have

$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}$ dx = 0 when m $\neq n$
$\frac{\pi}{2}$ when m = n $\neq 0$
π when m = n = 0

Therefore, from (1), (2) and (3) successively, we obtain,
$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4T_0}{\sqrt{1-x^2}}$ dx = $\frac{1}{\pi}\int_{-1}^{1} \frac{x^4\,1}{\sqrt{1-x^2}}$ dx
$= \frac{1}{\pi}\int_{-\pi}^{0} \frac{\cos^4\theta}{\sin\theta}$ (-sin θ) dθ [∵ x cos θ]
$= \frac{1}{\pi}\int_{0}^{\pi} \cos^4\theta$ dθ = $\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos^4\theta$ dθ
$= \frac{2}{\pi}\left[\frac{(4-1)(4-3)}{4(4-2)}\frac{\pi}{2}\right]$ $\left[\because \int_0^{\pi} \cos^n\theta\, d\theta = \frac{(n-1)(n-3)\ldots}{n(n-2)\ldots.}\;k,\; k = \frac{\pi}{2}, if\ n\ is\ even, k = 1\ if\ n\ is\ odd\right]$
$= \frac{3}{8}$.
Similarly $c_1 = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4\,T_1}{\sqrt{1-x^2}}$ dx =0

And $c_2 = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4\ 1T_2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}$

Hence the required approximation is

f (x) = p(x) = $\frac{3}{8}T_0 + \frac{1}{2}T_2$.

Example 4: Find a uniform polynomial approximation of degree four or less to e^x on [-1,1] using Lanczos economization with a tolerance of $\in = 0.02$.

Solution: Since we know that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$

Since the tolerance of $\in = 0.02$, which means that e^x is economized to two significant digit accuracy.

But $\frac{1}{20}$ = 0.008 … will produce a change in the decimal place only, therefore the truncated series is

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots +$ …(1)

Now changing each power of x in (2) in terms of Chebyshev polynomials, we obtain

$e^x = T_0(x) + T_1(x) + \frac{1}{4}[T_0(x) + T_2(x)] + \frac{1}{24}[3T_1(x) + T_3(x)] + \frac{1}{192}[3T_0(x) + 4T_2(x) + T_4(x)]$

$e^x = \frac{81}{64}T_0 + \frac{9}{8}T_1 + \frac{13}{48}T_2 + \frac{1}{24}T_3 + \frac{1}{192}T_4.$

again $\frac{1}{192}$ = 0.005… will produce a change in the third decimal place only i.e., it is less than 0.02

Hence the required economized polynomial approximation for e^x is given by

$e^x = \frac{81}{64}T_0 + \frac{9}{8}T_1 + \frac{13}{48}T_2 + \frac{1}{24}T_3$

$e^x = \frac{81}{64} + \frac{9}{8}x + \frac{13}{48}(2x^2 - 1) + \frac{1}{24}(4x^3 - 3x)$

$e^x = \frac{x^3}{6} + \frac{13}{24}x^2 + x + \frac{191}{192} \approx \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{x^3}{6}.$

10.10 SUMMARY

Chebyshev differential equation
Recurrence relation of T_n (x) and U_n (x)
Chebyshev polynomial approximation
Application of Chebyshev series in the economisation of power series

10.11 GLOSSARY

Approximation
Algorithm
Number of polynomial terms

10.12 REFERENCES

1. C. F. Gerald & P. O. Wheatley, Applied Numerical Analysis (7 th edition), Pearson Education, India, 2008.

2. F. B. Hildebrand, Introduction to Numerical Analysis: (2 nd edition). Dover Publications, 2013.

10.13 SUGGESTED READING

1.Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India (1989).
2. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

10.14 TERMINAL AND MODAL QUESTIONS

1. Express 1 +x - x^2+x^3 as the sum of Chebyshev polynomials.
2. Prove that $x^3 = \frac{1}{4}[3T_1(x) + T_3(x)]$
3. Economize the power series cos x = 1-$\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
4. Obtain the best lower degree approximation to the cube x^3+2 x^2.
5. Approximate: f (x) = $\frac{1}{x}\int_0^x \frac{e^t - t}{t}$ dt
6. Obtain the Chebyshev polynomial approximation of second degree to the function f (x) = x^3 on [0,1].

9.15 ANSWER

1. $\frac{1}{2}T_0(x) + \frac{7}{4}T_1(x) - \frac{1}{2}T_2(x) + \frac{1}{4}T_3(x)$

4. $2x^2 - \frac{3}{4}x$

5. P (x) = 0.01071 x^3 +0.05722 x^2 + 0.02499x + 0.99979

6. P (x) = $\frac{1}{32}$ (48x^2 – 18 x +1)

UNIT 11 : SOLUTION OF SYSTEM OF SIMULTANEOUS LINEAR ALGEBRIC EQUATION

CONTENTS:

11.1 INTRODUCTION

If two more algebraic equation share variable such as x and y are called Simultaneous equations. In other way those equations which are solved at the same time are called simultaneous equations. The number of variables in simultaneous equations must match the number of equations for it to be solved. There are two common methods for solving simultaneous linear equations first is substitution and second is elimination method. In this unit we study about different methods for solving simultaneous equations.

11.2 OBJECTVES

After studying this unit, the learner will be able

1. To solve the simultaneous equation by different method.
2. To find the value of unknown variables which is satisfy all the equation at the same time.

11.3 DIFFERENT METHOD OF OBTAINING THE SOLUTION

Direct methods

(a) Matrix Inversion Method:

let A be non – singular i. e., det A $\neq 0$. A^{-1} exists. Now premultiplying both sides of AX = B by A^{-1}, we get

A^{-1} A X A^{-1} B i.e., X = $A^{-1}\,B$. [$\because A^{-1}A =$ I and I X = X]

Here $A^{-1} = \frac{1}{\det A}$ Adj A.

Thus if A^{-1} is known, then the solution vector X can be found from the above matrix relation.

(b) Cramer Rule:

Suppose that det A = D $\neq$ 0 and B $\neq$0. According to Cramer''s rule, the system A X = B has the solution $x_r = \frac{D_r}{D}$, r = 1, 2, 3, …m

Where D_r is the determinant obtained by replacing the rth column in D by B.

Giving different values to r, we can find x_1, x_2,….. x_m and hence the solution is obtained. this method is true from a purely theoretical point of view but from a calculation point of view it is not good. It is feasible when m = 3 or 4.

(c)The elimination method by Gauss:

This is an elementary elimination method which reduces the system of equations to an equivalent upper triangular system which can be solved by back substitution.

Suppose we have a system of n equations in n unknowns as given below:

$$\left.\begin{array}{l} a_{11}\,x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}\,x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{11}\,x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}\,x_n = b_n \end{array}\right\} \quad \dots(1)$$

This system can be put in the form AX =B. let det A $\neq$ 0 and B $\neq$ 0.

We begin by dividing the first equation of (1) by a_{11} (if a_{11} =0, the equations are arranged in a suitable way) and then we subtract this equation multiplied by $a_{21}, a_{31}, \dots\dots\dots a_{n1}$ from the second, third…., nth equation. again, the second equation is divided by the new coefficient a_{22}, of t he variable x_2 (this element is known as the pivot element), and then in a similar way x_2 is eliminated from third, fourth…, nth equations. We continue this procedure as far as possible, and finally we obtained x_n, x_{n-1}, …., x_1 by back substitution.

Suppose the system takes the following form, when the elimination is completed:

$$\left.\begin{array}{r} c_{11}\,x_1 + c_{12}\,x_2 + \cdots . + c_1\,x_n = \ d_1 \\ c_{22}\,x_2 + \cdots . + c_2\,x_n = \ d_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ c_{nn}\,x_n \ = \ d_n \end{array}\right\} \qquad \ldots.(2)$$

The new coefficient matrix is a triangular matrix; the diagonal elements c_{ii} are usually equal to 1.

To find a better result by this method it is desirable to select the largest element of the row as the pivotal element. for this purpose, we recorder the columns and the equations, it necessary.

(d) Jordan's method: It is a modification of the method due to Gauss. In this method the elimination is performed not only in the equation below but also in the equation above. Thus, finally we get a diagonal coefficient matrix and without further computation we have solution.

(e) Crout's method: This method is superior to the gauss elimination method because it requires less calculation. We shall describe this method by considering a system of 3 equation. let the system be

$$\left.\begin{array}{l} a_{11}\,x_1 \ + a_{12}x_2 \ + \ a_{13}\,x_3 \ = \ b_1 \\ a_{21}\,x_1 \ + a_{22}x_2 \ + \ a_{22}\,x_2 \ = \ b_2 \\ a_{31}\,x_1 \ + a_{32}x_2 \ + a_{33}\,x_n \ = \ b_2 \end{array}\right\} \qquad \ldots(1)$$

The system (1) can be written as A X = B ...(2)

Where A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and B= $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The augmented matrix of (2) is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{22} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \qquad \ldots..(3)$$

Now we consider a derived matrix

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{22} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}.$$

Which is to be calculated as follows:

(i) To determine the first column: a'_{il} for all, I the first column is identical with the first column of the coefficient matrix A.

(ii) To determine the first row to the right of the first column:

$a'_{1j} = \frac{a_{1j}}{a_{11}}$, j = 2, 3 i.e., $a'_{12} = \frac{a_{12}}{a_{11}}$, $a'_{13} = \frac{a_{13}}{a_{11}}$; $b'_1 = \frac{b_1}{a_{11}}$

i. e., the first row except the first element is obtained by dividing the corresponding elements of the first row of the matrix (3) by the first element of that row.

(iii) To determine the remaining elements of the second column:

$a'_{i2} = a_{i2} - a'_{12}.\, a'_{i1}$, i=2,3.

Thus $a'_{22} = a_{22} - a'_{12}\, a'_{21}$; $a'_{32} = a_{32} - a'_{12}\, a'_{31}$.

(iv) To determine the remaining second row:

$a'_{2j} = \frac{a_{2j} - a'_{1j}\, a'_{21}}{a'_{22}}$, j = 3 i.e., $\frac{a'_{23} - a'_{13}\, a'_{21}}{a'_{22}}$

(v) To determine the remaining third column:

$a'_{33} = a_{33} - a'_{23} = a'_{13} - a'_{31}$.

(**vi**) to determine the remaining third row:

$b'_3 = \frac{b_3 - b'_2\, a'_{32} - b'_1 - a'_{31}}{a'_{33}}$,

Now the solution is given by

$x_3 = b'_3$, $x_2 = b'_2 - a'_{23}$, $x_1 = b'_1 - a'_{13}\, x_2 - a'_{12}\, x_2$.

We observed that various element of the derived matrix is determined in the following order.

First of all, column; then element of the first row to the right of the first column; then element of the second column below the first row; then element of the second row to the right of the second column, etc.

(f) Method of Factorization or Triangular method (L U decomposition):

In this method we use the fact that a square matrix A can be factorized into the form LU where L is the lower triangular matrix and U is upper triangular, if all the principal minors of A are non – singular, i.e., if

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Let us consider a system of linear equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}. \qquad \ldots (1)$$

This can be put in the form

A X = B

Let A = LU … (2)

Where L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U= $\begin{bmatrix} u_{11} & l_{12} & l_{13} \\ 0 & u_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$.

Writing A = LU in)2), we get LUX = B. … (3)

Setting UX = Y, the equation (3) becomes LU = B … (4)

The equation (4) is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\}. \qquad \ldots (5)$$

By the forward substitution we get the values of y_1, y_2, y_3 .

When we know Y, the system UX =Y gives:

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\}. \qquad \ldots (6)$$

by the backward substitution we get the values of x_1, x_2, x_3.

Now we shall discuss the procedure of computing the matrices L and U. from the relation A = Lu we get

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & l_{12} & l_{13} \\ 0 & u_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Multiplying the matrices on the left hand side and then equation the corresponding elements on both sides, we have

$u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$,

$l_{21}u_{11} = a_{21}$ => $l_{21} = a_{21}/a_{11,}$

$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} - a_{22} - (a_{21}/a_{11})\, a_{12},$

$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - (a_{21}/a_{11})\, a_{13},$

$l_{31}u_{11} = a_{31} \Rightarrow l_{31} = (a_{31}/a_{11}),$

$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{a_{32}-\left(\frac{a_{31}}{a_{11}}\right)a_{12,}}{u_{22}}$

And $l_{31}u_{13} + l_{32}u_{23} + u_{23} = a_{33}$ which gives u_{33}.

Hence in a systematic way the elements of L and U can be evaluated.

(g) Iterative Methods (Indirect Methods): We have discussed some direct methods for the solution of simultaneous linear equations. Now we shall discuss the iterative or indirect methods. In these methods we start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations. We repeat the cycle of computations till the required accuracy is obtained. Thus, in an iterative method the amount of computation depends on the accuracy required, while in a direct method the amount of computation is fixed.

(h) Jacobi Iterative Method: let us system of simultaneous linear equations be

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots (1)$$

Suppose the diagonal coefficient a_{ii} in (1) do not vanish. If it is not so, then we can rearrange the equations so that this condition is satisfied.

Now the system (1) can be written as:

$$\left.\begin{array}{l} x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\ x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \ldots - \frac{a_{2n}}{a_{22}}x_n \\ x_3 = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \ldots - \frac{a_{3n}}{a_{33}}x_n \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots. \\ x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \ldots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1} \end{array}\right\} \quad \ldots (2)$$

Let $x_1^{(1)}, x_2^{(2)}, \ldots x_n^{(1)}$ be any first approximations to the unknowns $x_1, x_2, \ldots\ldots x_{n.}$putting these in the R.H.S of (2), we get a system of second approximations:

$$\left.\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(2)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
x_3^{(2)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} a_1^{(2)} - \dots - \frac{a_{3n}}{a_{33}} x_n^{(1)} \\
\dots \quad \dots & \quad \dots \quad \dots \quad \dots \quad \dots \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \dots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}\right\} \quad \dots (3)$$

In the same way if $x_1^{(n)}, x_2^{(n)}, x_n^{(n)}$ are nth approximations, then the system of next approximations is given by:

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
\dots & \quad \dots \quad \dots \quad \dots \quad \dots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \dots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n)}
\end{aligned}\right\} \quad \dots (4)$$

Writing to in the matrix

X = B X + C,

We can write the iteration formula (4) as

$x^{(n+1)}$ = B $X^{(n)}$ + C.

3. Gauss – Seidel iterative method:

It is a simple modification of Jacobi's method. Substitute the first approximation $(x_1^{(1)}, x_2^{(1)}, \dots x_n^{(1)})$ into the R.H.S of the first equation of (2) and denote the result by $x_1^{(2)}$. now we substitute $(x_1^{(1)}, x_2^{(1)}, \dots x_n^{(1)})$ in the second equation and the result is denoted by $x_2^{(2)}$. We put $(x_1^{(2)}, x_2^{(2)}, \dots x_n^{(1)})$ in the third equation and denote the result by $x_3^{(2)}$. Continuing this process we put $(x_1^{(2)}, x_2^{(2)}, \dots x_n^{(2)}, \dots\dots x_{n-1}^{(2)}, x_n^{(1)})$. In the last equation and denoted the result by $x_n^{(2)}$. Thus first stage of iteration is completed. we repeat the entire process till the values of x_1, x_2, ….. x_n are obtained to the desired accuracy. In this method we use an improved component as soon as it is available.

Gauss Seidel method is called the method of *successive displacement* and Jacobi method is called the method of simultaneous displacements.

For any choice of the first approximation $x_j^{(1)}$(j = 1, 2,.,n) the Jacobi and Gauss Sedal method converge , if the following condition is satisfied by every equation of the system (2).

The sum of the absolute values of the coefficient a_{ij}/a_{ii} is atmost equal to, or in at least one equation less then unity, i.e., if $\sum_{\substack{j=1 \\ j\neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leq (i = 1, 2, \ldots n)$ ...(7)

Where the sign < is valid in the case of at least one equation.

4. Relaxation method:

In this iterative method initially assumed values of the unknown are improved by reducing the so-called residuals, denoted by $R_i^{(m)}$ to zero.

The residue of ith equation at m^{th} iteration is given by

$R_i^{(m)} = b_i - a_{il}\, x_1^{(m)} - a_{i2}\, x_2^{(m)}$- …- $a_{in}\, x_n^{(m)}$,

Where $x_j^{(m)}$ denoted the value of x_j at mth iteration. Thus, we get $R_i^{(m)}$ putting $x_j^{(m)}$ in the L.H.S of ith equation and then subtracting it from the R.H.S. We assume the initial value of unknown at each iteration and then the residuals corresponding to all the equation are calculated. Then we reduce the largest residuals to zero at that iteration. we continuing the process of reducing the residuals till all the residuals becomes zero or negligible.

For the rapid convergences of this procedure, we reformulate the system of equations. First of all, we transpose all the terms to R.H.S. and then recorder the equations so that the largest negative coefficients in the equations appears on the diagonal. Now if at any iteration, the largest residual is $R_{k,}$ then we shall give an increment $d_{xk} = -\frac{R_k}{a_{kk}}$ to x a_{kk} being the coefficient of x_k i.e., x_k change to $x_k + d_{xk}$.

This method converges more rapidly than Gauss – Seidel iterative method because due to known values of residuals we always residuals we always reduce the largest residual to zero i.e., at each iteration we do maximum improvement in the unknown.

ILLUSTRATIVE EXAMPLE

Example 1: Solve the given system by Cramer's rule:

3x + 2y – z + t =1

X – y – 2 z + 4 t =3

2 x – 3 y + z – 2 t = -2

5 x – 2 y + 3 z + 2 t = 0.

Solution: Here **we** have

$$D= \begin{vmatrix} 3 & 2 & -1 & 1 \\ 1 & -1 & 2 & 4 \\ 2 & -3 & 1 & -2 \\ 5 & -2 & 3 & 2 \end{vmatrix} = 50$$

Which is the denominator of the fractions for all the unknowns.

$$D_1 = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & -1 & -2 & 4 \\ -2 & -3 & 1 & -2 \\ 0 & -2 & 3 & 2 \end{vmatrix} = 19, \quad D_2 = \begin{vmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & -2 & 4 \\ 2 & -2 & 1 & -2 \\ 5 & 0 & 3 & 2 \end{vmatrix} = -29$$

$$D_3 = = \begin{vmatrix} 3 & 2 & 1 & 1 \\ 1 & -1 & 3 & 4 \\ 2 & -3 & -2 & -2 \\ 5 & -2 & 0 & 2 \end{vmatrix} = -51, \quad D_4 = \begin{vmatrix} 3 & 2 & -1 & 1 \\ 1 & -1 & -2 & 3 \\ 2 & -3 & 1 & -2 \\ 5 & -2 & 3 & 0 \end{vmatrix}$$

Now $x = \frac{D_1}{D} = \frac{19}{50}$, $y = \frac{D_2}{D} = -\frac{29}{50}$, $z = \frac{D_3}{D} = -\frac{51}{50}$, $t = \frac{D_4}{D} = 0$.

Example 2: **Solve**

$$\left.\begin{aligned} \mathbf{10\,x - 7\,y + 3\,z + 5\,u = 6}, \\ \boldsymbol{-6\,x + 8\,y - z - 4\,u = 5}, \\ \boldsymbol{3\,x + y + 4\,z + 11\,u = 2}, \\ \boldsymbol{5\,x - 9\,y - 2\,z + 4\,u = 7} \end{aligned}\right\}$$

By Jordan Method .

Solution : First we eliminate x, using the first equation:

$$\left.\begin{aligned} X - 0.7\,y + 0.3\,z + 0.5\,u = 0.6 \\ 3.8\,y + 0.8\,z - u = 8.6 \\ 3.1\,y + 3.1\,z + 9.5\,u = 0.2 \\ -5.5\,y - 3.5\,z + 1.5\,u = 4 \end{aligned}\right\} \quad \ldots\ldots.(1)$$

Now we permute the second and fourth equations and then y is eliminated from the first, third and fourth equations by using the second equation. We get the new system as:

$$\left.\begin{aligned} x + 0.74545\,z + 0.30909\,u = 0.09091 \\ y + 0.63636\,z - 0.27273\,u = -0.72727 \\ -1.61818z + 0.03636\,u = 11.36364 \\ 1.12727\,z + 10.34545\,u = 2.45455 \end{aligned}\right\} \quad \ldots(2)$$

Further elimination of z from the first, second and fourth equations, by using the third equation gives:

$$\left.\begin{aligned} x + o.32584\,u = 5.32582 \\ y - 0.25843\,u = 3.74156 \\ z - 0.02447\,u = -7.02248 \\ 10.37078\,u = 10.37078 \end{aligned}\right\} \quad \ldots(3)$$

In the last elimination of u from the first three equations gives:

X = 5 , y = 4, z = -7 , u =1.

Example 3: Solve the system by Crout;s method:

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + x_2 - 3x_3 &= 5 \\ x_1 - 2x_2 - 5x_3 &= 10 \end{aligned}\right\}$$

Solution: The augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 5 \\ 1 & -2 & -5 & 10 \end{bmatrix}$$

Let the derived matrix of the augmented matrix be

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

Then $a'_{11} = 1$, $a'_{21} = 3$, $a'_{31} = 1$ (first column)

$a'_{12} = \frac{1}{1} = 1$, $a'_{13} = \frac{1}{1} = 1$, $b'_1 = \frac{1}{1} = 1$ (remaining first row)

a'_{22} = 1-1.3 = - 2 , $a'_{32} = -2 - 1.1 = -3$ (remaining second colmun)

$a'_{23} = \frac{-3-1.3}{-2} = 3$, $b'_2 = \frac{5-1.3}{-2} = -1$ (remaining second row)

a'_{33} = -5 – (3). (-3) – (1) (1) = 3 (remaining third row)

$b'_3 = \frac{10-(-1).(-3)-(1)(1)}{3} = 2$ (remaining third row)

∴ the derived matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & -2 & 3 & -1 \\ 1 & -3 & -3 & 2 \end{bmatrix}$$

Thus $x_3 = b'_3 = 2$, $x_2 = b'_2 \; a'_{23} \; x_3 = -7$ and $x_1 = b'_1$ - $a'_{13} \, x_3 - a'_{12} \, x_2$ = 6.

Example 4: Solve

$$\left.\begin{aligned} 10\,x - 7\,y + 3\,z + 5\,u &= 6, \\ -6\,x + 8\,y - z - 4\,u &= 5, \\ 3\,x + y + 4\,z + 11\,u &= 2, \\ 5\,x - 9\,y - 2\,z + 4\,u &= 7 \end{aligned}\right\}$$

By Gauss's Elimination Method.

Solution : We can rewrite the given system as

$$\left.\begin{array}{r} x - 0.7\,y + 0.3\,z + 0.5\,u = 0.6, \\ -6\,x + 8\,y - z - 4\,u = 5, \\ 3\,x + y + 4\,z + 11\,u = 2, \\ 5\,x - 9\,y - 2\,z + 4\,u = 7 \end{array}\right\}. \qquad \ldots (1)$$

First, we eliminate x from the second, third and fourth equations, using the first equation. Subtracting (-6) times the first equation from the second equation, 3 times the first equation from the third equation and 5 times the first equation from the fourth equation, we get the new system as

$$\left.\begin{array}{r} x - 0.7\,y + 0.3\,z + 0.5\,u = 0.6, \\ 3.8\,y - 0.8\,z - u = 8.6 \\ 3.1\,y + 3.1\,z + 9.5\,u = 0.2 \\ -5.5y - 3.5\,z + 1.5\,u = 4 \end{array}\right\}. \qquad \ldots (2)$$

Since the numerically largest y- coefficient is in the fourth equation of system (2) so we permute the second and fourth equation. After that, y is eliminated from the third and fourth equations by using the second equation. The new system becomes

$$\left.\begin{array}{r} x - 0.7\,y + 0.3\,z + 0.5\,u = 0.6, \\ y - 0.63636\,z - 0.27273u = -0.72727 \\ -1.61818\,z + 0.03636\,u = 11.36364 \\ 1.12727\,z + 10.34545\,u = 2.45455 \end{array}\right\}. \qquad \ldots (3)$$

Now eliminate of z gives

$$\left.\begin{array}{r} x - 0.7\,y + 0.3\,z + 0.5\,u = 0.6, \\ y - 0.63636\,z - 0.27273u = -0.72727 \\ z - 0.02247\,u = -7.02247 \\ 10.37079\,u = 10\,.37079 \end{array}\right\}. \qquad \ldots (4)$$

Thus, the final solution is given by

U = 1, z = - 7, y = 4 and x = 5.

Gauss – Seidel iteration method: The first approximation $x^{(1)}$ for x is to be obtained from the equation (2) by choosing some values of y and z to start with. For convenience let us choose y= 0 , z = 0.

Note : We can also proceed with some other values.

Starting with y = 0 , z = 0, from (2), we get x = $x^{(1)} = \frac{85}{27}$ = 3.15 = first approximation.

Putting x =3.15, z = 0 in (3), we get

$y^{(1)} = \frac{1}{15}$ (72 – 18.90) = 3.54 = first approx.

Now putting x = 3.15, y = 3.54 in (4) we get

$z^{(1)} = \frac{1}{54}$ (110 – 3.15 – 3.54) = 1.91 = first approx..

Now we obtained the second approximations

$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)}) = \frac{1}{27}$ (85 – 21.24 +1.91) = 2.43,

$y^{(2)} = \frac{1}{15}(72 - 6x^{(2)} - 2\,z^{(1)}) = \frac{1}{15}$(72 – 14.58 – 3.82) = 3.57

$z^{(2)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)}) = \frac{1}{54}$ (110 – 2.43– 3.57)= 1.92

Similarly we get

$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)}) = \frac{1}{27}$ (85 – 21.42 +1.91) = 2.42,

$y^{(3)} = \frac{1}{15}(72 - 6x^{(3)} - 2\,z^{(2)})$= 3.572,

$z^{(3)} = \frac{1}{54}(110 - x^{(3)} - y^{(3)})$= 1.926.

Since $x^{(3)}, y^{(3)}, z^{(3)}$ are sufficiently close to $x^{(2)}y^{(2)}z^{(2)}$ respectively, so the values 2.426, 3.572, 1.926 can be taken as the solution of the given system.

Hence, the solution is

X = 2.426, y = 3.572, z= 1.926.

Jacobi Iterative methos. Starting with x = 0, y = 0, z = 0, we get

$x^{(1)} = \frac{85}{27}$ = 3.15,

$y^{(1)} = \frac{72}{15}$ = 4.8

$z^{(1)} = \frac{110}{54}$ = 2.04.

These are the first approximations. We proceed to obtained the second approximations as follow:

$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)} = \frac{1}{27}$ (85 – 28.8 +2.05) = 2.16,

$y^{(2)} = \frac{1}{15}(72 - 6x^{(1)} - 2\,z^{(1)} = \frac{1}{15}$(72 – 18.9 – 4.08) = 3.27

$z^{(2)} = \frac{1}{54}(110 - x^{(1)} - y^{(1)} = \frac{1}{54}$ (110 – 3.15– 4.8) = 1.89.

Similarly, the next approximations are given by

$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)})$,

$y^{(3)} = \frac{1}{15}(72 - 6x^{(2)} - 2z^{(2)})$,

$z^{(3)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)})$.

Continuing in this way further iterations can be obtained.

Relaxation method. Transposing all the terms to one side, we get

0 = 85 – 27 x – 6 y +z

0 = 72 – 6 x – 15 y – 2 z

0 = 110 – x – y – 54 z.

Here we need not recorder the equations as the largest negative coefficient i.e., -27 , -15 , -54 are already on the diagonal.

Hence the residuals are

R_1 = 85 – 27 x – 6 y +z

R_2 = 72 – 6 x – 15 y – 2 z

R_3 = 110 – x – y +54 z.

Assuming the initial value of the unknowns as x = 0, y = 0, z =0, the results of the calculations are as follows:

Value of unkown			Residuals			Largest	Increment
X	Y	Z	R_1	R_2	R_3	R_k	d_{xk}
0	0	0	85	72	110	110	2.037
0	0	2.037	87.037	67.926	0.002	87.037	3.224
3.224	0	2.037	-0.011	48.582	-3.222	48.582	3.239
3.224	3.239	2.037	-19.445	-0.003	-6.461	-19.445	-0.720
2.505	3.239	1.931	-0.005	4.317	-5.741	-5.741	-0.106
2.505	3.239	1.931	-0.111	4.529	-0.017	4.529	0.302
2.505	3.541	1.931	-1.923	-0.001	-0.319	-1.923	-0.071
2.433	3.541	1.931	-0.0006	0.425	-0.248	0.425	0.028
2.433	3.569	1.931	-0.174	0.005	-0.276	-0.276	-0.005
2.433	3.569	1.926	-0.179	0.015	-0.006	-0.179	-0.007
2.426	3.569	1.926	0.01	0.0057	0.001	0.057	0.004
2.426	3.573	1.926	0.014	-0.003	-0.003		

R_1 , R_2 , R_3 , at this iteration are small enough we may take the values of the unkowns at this iteration as the solution to the given system.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. In Gauss elimination method we reduce the system of equation to a triangular form by eliminating the variable successively.
2. For the solution of simultaneous linear equation by Relaxation iterative method we calculate the residuals corresponding to all equations at each iteration.

Multiple choice questions

1. If we factorize the matrix

$$\begin{bmatrix} 5 & -2 & 2 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Into the form LU where L is lower triangular matrix and U is upper triangular matrix then the elements of the first row of U are
(a) 1, -2, 5
(b) -2 , 1, 5
(c) 5, -2 ,1
(d) 1, 2, 3

2. For the solution of the system of equations
3x +y +2z =3
2x -3y – z = -3
X +2y + z = 4
By crout's method the elements of the first column of the derived matrix are
(a) 3, 2,1
(b) 2, 1,1
(c) 3, 1, 2
(d) 1, 2, 3

11.4 SUMMARY

Methods of obtaining the solution is Substitution method, Elimination method, Matrix method, Cramer's rule, Gauss Seidel iteration method, Relaxation method

11.5 GLOSSARY

System of Linear Equations: A collection of linear equations involving variables and constant.

Coefficient Matrix: A matrix containing the coefficients of the variables in the system of linear equations.

Constant Matrix: A matrix containing the constant terms in the system of linear equations.

Variable Matrix: A matrix containing the variables in the system of linear equations.

Unique Solution: A solution that is unique and satisfies all the equations in the system.

No Solution: A situation where there is no solution that satisfies all the equations in the system.

Infinite Solutions: A situation where there are infinite solutions that satisfy all the equations in the system.

11.6 REFERENCES

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11.7 SUGGESTED READING

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2. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

3. "Gaussian Elimination" by G. H. Golub and C. F. Van Loan (Journal of Linear Algebra and Its Applications, 1979)

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11.8TERMINAL AND MODAL QUESTIONS

1. Solve the following equations by Gauss- Jordan method:
$2x_1$+$4x_2$ +x_3 = 3, $3x_1 + 2x_2$ - $2x_3$ = -2 , x_1- x_2 +x_3 =6.

2. Solve the system
$2x_1$+$4x_2$ +x_3 = 3, $3x_1 + 2x_2$ - $2x_3$ = -2 , x_1- x_2 +x_3 =6.by Gauss's elimination method.

3. Solve the system of Crout's method
2 x – 6 y +8 z = 24
5 x +4 y – 3 z =2
3 x +y + 2 z =16.

4. Solve the system of equations
10 x + 2 y +z = 9, 2 x + 20 y – 2 z = -44, - 2 x +3 y + 10 z =22
By Gauss – Seidel method.

5. Apply Jacobi iteration method to solve
10 x + y + z = 12, 2 x +10 y + x = 13, 2 x + 2 y + 10 z =14.

FILL IN THE BLANKS

1. Jordan method is a modification of the method due to …….
2. In gauss's elimination method, the system of simultaneous linear equations reduces to an equivalent …. triangular system which can be solved by back substitution.
3. For the solution of simultaneous linear equations, the iterative methods are known as …. Methods.

11.9 ANSWERS

CYQ1. True
CYQ2. True
MCQ1. (C)
MCQ2. (a)
TQ1. $x_1 = 2,\ x_2 = -1, x_3 = 3$
TQ2. $x_1 = 2,\ x_2 = -1, x_3 = 3$
TQ3. x =1, y= 3, z =5
TQ4. x =1.0, y = -2 .0, z =3.0
TQ5. x=1, y= 1 z =1
FQ1. Gauss
FQ2. Upper
FQ3. Indirect

UNIT 12: DIFFERENCE EQUATIONS

CONTENTS:

12.1 INTRODUCTION

Difference equations have incredibly diverse applications across various fields, including Economics, Social sciences and computer sciences. In Engineering it is used to optimization and signal processing. Difference equations are essential for anyone interested in modelling and analysing dynamic systems."
In this unit we discussed about difference equation and its solution.

12.2 OBJECTIVES

After studying this unit learner will be able to

1. Solve difference equation and find the order of difference equation.
2. Solve the values of an unknown function y (x) for difference discrete values of x.

12.3 DIFFERENCE EQUATION

An equation involving the values of a function y and its differences Δ_y, $\Delta^2 y$, ….for each value of some set of number is called a difference equation over that set. A difference equation is a relation connecting differences. If y is a function of x, which is defined for all real number x, then the following equations are examples of difference equations over the set of real number:

$\Delta_y + 2y = 0$,

$\Delta^2 y + 2\Delta y + y = 0$,

$\Delta^2 y - xy = 3x + 8$.

If we transfer the origin to y_n, then the above equations can be written as

$\Delta y_h + 2 y_h = 0$ …(1)

$\Delta^2 y_h + 2\Delta y_h + y_h = 0$ …(2)

$\Delta^2 y_h - h y_h = 3h + 8$. …(3)

11.4 TRANSFORMATION OF THE DIFFERENCE EQUATION IN THE FORM OF THE OPERATOR E

$\Delta y_h = E y_h - y_h$, …(4)

$\Delta^2 y_h = E^2 y_h - 2E y_h + y_h$, …(5)

$\Delta^3 y_h = E^3 y_h - 3E^2 y_h + 3E y_h - y_h$, etc. …(6)

Also we know that $E^n y_h = y_{h+nk}$, where k is the interval of differencing .

If k=1, then $E^n y_h = y_{h+n}$ …(7)

Using the result (4) , (5) and (7), the equations (1) , (2) and (3) of article 11.3 reduce to

$Y_{h+1} + y_h = 0$, …(8)

$Y_{h+2} = 0$, …(9)

$Y_{h+2} - 2 Y_{h+1} + (1-h) y_h = 3h + 8$. …(10)

12.5 LINEAR DIFFERENCE EQUATION

Definition: A difference equation defined over a set A is linear over A if it can be written in the form

$F_0 (h) y_{h+n} + f_1 (h) y_{h+n-1} + \ldots\ldots\ldots\ldots + f_n (h) y_h = g(h)$, … (11)

Where each of f_0, f_1, …..f_{n-1}, f_n and g is a function of h (but not of y_h) defined for all values of h in the set of A. The equation (8) ,(9) and (10) are examples of such equations. Also, we observed that coefficient in equations (8) and (9) are constant i.e., they do not depend upon h. the equations of this type come under the category of linear difference equation with constant coefficients. The equation (10) is a linear difference equation with constant coefficients. The equation (10) is a linear difference equation with variable coefficients.

If g(h) =0 in the equation (11), then we have

$F_0 (h) y_{h+n} + f_1 (h) y_{h+n-1} + \ldots\ldots + f_n (h) y_n = 0$

Which is called the homogeneous linear difference equation corresponding to the equation (11).

If g (h) ≠0 , then the equation (11) is called a non-homogeneous linear difference equation.

12.6 ORDER OF A LINEAR DIFFERENCE EQUATION

The order of a linear difference equation defined over the set A and written in the form of equation (11) is n if both f_0 and fn are different from zero at each point of the set A.

In other words, the order of a linear difference equation is the difference between the largest and smallest arguments for the function y involved in the equation. The order of the difference equation y_{h+2}, + $4y_{h+1}$ =3h is of order 1.

12.7 SOLUTION OF A DIFFERENCE EQUATION

A solution of a difference equation is a relation between the independent variable and the dependent variable which satisfies the equation. substituting such a relation in the equation the left hand and right-hand members become identically equal.

A solution which involved n arbitrary periodic constants is called a general solution of a difference equation of order n.

A solution obtained from the general solution by assigning particular periodic constants is called a particular solution.

Let us consider the difference equation

$y_{h+1} - 2y_n$ = 0, h =0 ,1, 2, …. …. (1)

Let $y_h = 2^h$ = 0, 1, 2, ……… …. (2)

The function defined by (2) satisfies the difference equation (1). So, it is said to be a solution of (1). in general (1) satisfied by

$y_h = c.2^h$ … (3)

For any constant c.

The function y given by (2) is the particular solution (1) while the function y in (3) containing the arbitrary constant c , is the general solution of (1).

A difference equation may have no solution just as in the case of algebraic equation, e.g.,

$(y_{h+1} - y_n)^2 + y_n{}^2$ = -1

Is satisfied for no real function y.

12.8 EXISTENCE AND UNIQUENESS THEORUM

Some difference equations have infinitely many solutions whereas others have no solution a+ all. in the case of linear difference equation, we can always find at least one solution, and under certain conditions, only one solution.

Theorem: The linear difference equation of order n

$$f_0(h)y_{k+n} + f_1(h)y_{h+n-1} + \cdots + f_{n-1}(h)y_{h+1} + f_n(h)y_h = g(h), \quad \ldots(1)$$

Over a set A consecutive integral value of h has one, and only one solution y for which values at n consecutive h – values are arbitrarily prescribed.

Proof: To prove this theorem we shall use the method of mathematical induction.

First, we shall prove the theorem for linear difference equation of order two.

Let the equation be

$$f_0(h)y_{h+2} + f_1(h)y_{h+1} + f_2(h)y_h = g(h), \qquad h = 0, 1, 2.. \qquad \ldots(2)$$

$$where\ f_0(h) \neq 0, f_2 \neq 0 .$$

Suppose y_0 and y_1 be given. Putting h =0 in (2) , we get

$$f_0(0)\, y_2 + f_1(0)y_1\, 0 + f_2(0)y_0 = g(0)$$

⇨ $y_2 = \frac{g(0)}{f_0(0)} - \frac{f_1(0)}{f_0(0)} y_1 - \frac{f_2(0)}{f_0(0)} y_{0.}$ $[\because y_0(0) \neq 0]$

Thus being given y_0 and y_1 , we have found y_2 .
To find $y_{3\,,}$ we can use the pairs of known values y_1 and y_2 .
For that putting h =1 in (2), we get
$f_0(1)y_3 + f_1(1)y_2 + f_2(1)y_1 = g(1)$
⇨ $y_3 = \frac{g(1)}{f_0(1)} - \frac{f_1(1)}{f_0(1)} y_1 - \frac{f_2(1)}{f_0(1)} y_{1.}$ $[\because f_0(1) \neq 0]$

Thus y_3 is uniquely determined. We can continue this way the unique solution of the second order equation (2) initiated by the two values y_0 and y_1 .

Let the theorem be true for j th order i.e., the difference equation,

$$f_0(h)y_{h+j} + f_1(h)y_{h+j-1} + \cdots \ldots + f_{i-1}(h)y_{h+1} + f_j(h)y_h = g(h)$$

Of j^{th} order has a unique solution y_{h+j} for the j prescribed values, say

y_h , $y_{h+1,}$ $y_{h+2,}$ ….. $y_{h+j-1,}$

Putting n = j in (1), we get

$$f_0(h)y_{h+j} + f_1(h)y_{h+j-1} + \cdots \ldots + f_{i-1}(h)y_{h+1} + f_j(h)y_h = g(h)$$

⇨ $y_{h+j} = \frac{g(h)}{f_0(h)} - \frac{f_1(h)}{f_0(h)} y_{h+j-1} - \ldots - \frac{f_{j-1}(h)}{f_0(h)} y_{h+1.} - \frac{f_j(h)}{f_0(h)} y_h$
$[\because f_0(h) \neq 0]$

i.e., a unique solution of y_{h+j} is determined.
Now we shall show that the theorem is also true for (j +1) th order.
Putting n = j +1 in (1), we get

$$f_0(h)y_{h+j+1} + f_1(h)y_{h+j} + f_2(h)y_{h+j-1} + \cdots + f_{j+1}(h)y_h = g(h)$$

⇨ $y_{h+j+1} = \frac{g\,(h)}{f_0\,(h)} - \frac{f_1\,(h)}{f_0\,(h)}\, y_{h+j}$ - … - $\frac{f_{\,i+1}\,(h)}{f_0\,(h)}\, y_{n.}$ $[\because f_0\,(h\,) \neq 0]$

The values of y occurring on the R.H.S of this equation are known. Thus y_{h+j+1} is uniquely determined.

But we have already shown that the theorem is true for second order. Hence by mathematical induction it true for any order n, where n is a positive integer. In the case when the theorem is true for order n , we must have

$$y_{h+n} = \frac{g\,(h)}{f_0\,(h)} - \frac{f_1\,(h)}{f_0\,(h)}\, y_{h+n-1} - \dots - \frac{f_{n-1}\,(h)}{f_0\,(h)}\, y_{h+1.} - \frac{f_n\,(h)}{f_0\,(h)}\, y_h$$

Where y_h, y_{h+1}, ….y_{h+n-1}, are known.

Noe suppose y_h, y_{h+1}, …..y_{h+n-1}, are known but instead y_m, y_{m+1}, …..y_{m+n-1}, $(m > n)$ are given . in this case we first successively determine unique values for y_{m-1}, y_{m-2}, …..y_{h+1}, y_h and then proceed as before now our problem is how to obtain y_{m-1}, y_{m-2}, …..y_{h+1}, y_h ?

Putting h =m-1 in (1), we get

$f_0\ (m-1)\ y_{m+n-1}\ + f_1\ (m-1)\ y_{m+n-2}\ + f_2\ (m-1)\ y_{m+n-3}\ + \dots + f_{n-1}\ (m-1)\ y_m\ + f_n\ (m-1)\ y_{m-1}$ = g (m-1)

⇨ $f_n\ (m-1)\ y_{m-1}$ = g (m-1) -$f_0\ (m-1)\ y_{m-1+n}$ - $f_1\ (m-1)\ y_{m-2+n}$ - …- f_{n-1} (m-1) y_m ....(3)

Since the values y_m, y_{m+1}, …..y_{m-1+n}, are given, therefore the R.H.S of (3) is known. Also f_n (m-1) is never zero so we can divide by f_n (m-1) to find $y_{m-1.}$

Similarly we can determine the remaining values y_{m-2},….. y_h .

12.9 SOLUTION OF THE EQUATION $y_{h+1} = Ay_n$ + C

The linear difference equation of first order is

$f_0\ (h)y_{h+1}\ + f_1\ (h)y_h$ = g (h), h = 0, 1,.. … (1)

Over the indicated set of h – values. The functions $f_0\ (h)$ and $f_1\ (h)$ are never zero.

The equation (1) can be written as

$$y_{h+1} = -\frac{f_1\,(h)}{f_0\,(h)}\, y_h + \frac{g\,(h)}{f_0(h)}.$$

Let us suppose that f_0 and f_1 as well as g are constant function. Then the equation can be written in the form

y_{h+1} = A y_n + C , h = 0 , 1 , 2 , ….. … (2)

Where A and C are constants and A $\neq 0$.

We shall find the solution of (2) with y_0 given.

Putting h = in (2), we get

y_1 = A y_0 + C.

Putting h =1 in (2) , we get

y_2 = Ay_1 +C = A (A y_0 + C) + C

$= A^2 y_0$ + C (1 +A).

Putting h =2 in (2), we get

y_3 = A y_2 + C = A {$A^2 y_0$ + C (1+A)}+C = $A^3 y_0$ +C (1 + A +A^2).

In general, we have

y_2 = $A^h y_0$ +C (1 +A +A +…+A^{h-1}).

We know that

1 + A + A^2 + …. +A^{h-1} = $\frac{1- A^h}{1-A}$, if A$\neq$ 1

=h, if A =1

And hence, we have

$$y_n \begin{cases} A^h\ y_0 + \frac{C(1-A^n)}{1-A}, & if\ A\ \neq 1 \\ y_{0+Ch}, & if\ A = 1 \end{cases} \quad \text{…. (3)}$$

ILLUSTRATIVE EXAMPLES

Example 1: Solve the difference equation.

y_{h+1} = 2y_h -1, h = 0, 1, 2, with initial condition y_0 =5.

Solution: Comparing the given equation with y_{n+1} = Ay_h + C.

We get A= 2 C= -1.

Hence the solution is

y_h = A^h y_0 + C $\frac{1- A^h}{1-A}$ = 2^h. 5 – 1. $\frac{1- 2^h}{1-2}$ [$\because y_0 = 5$]

$=$ 5 . 2^h + 1 -2^h = 4. 2^h+1, h = 0 , 1, 2 ,……

Thus we obtain the following sequence of values 5 ,9,17, 33, 65, 129, ..

Example 2: Solve that the function y given by .

y_h = 1 - $\frac{2}{h}$, h = 1,2, 3,……

is a solution of the first order difference equation

(h + 1) y_{h+1} + h y_h = 2h -3, h = 1, 2, 3, …..

Solution: Here we have

$y_h = 1 - \frac{2}{h}$. $\therefore\ y_{h+1} = 1 - \frac{2}{h+1}$

Substituting the values of y_h and y_{h+1} in the given difference equation, the L.H.S.

= (h +1) $\left(1 - \frac{2}{h+1}\right)$+ h $\left(1 - \frac{2}{h}\right)$

= h +1 -2 +h -2 = 2h -3 = the R.H.S

= Thus $y_h = 2 - \frac{2}{h}$ satisfies the given difference equation. Hence it is a solution of the given difference equation.

Example 3: Reduce the following difference equation to the linear form. Also find their order.

(i) $\Delta^2\ y_h + 3\ \Delta\ y_h - 3\ y_h$ = h,
(ii) $\Delta^3\ y_n + \Delta^2\ y_h + \Delta\ y_h + 3\ y_h$ =0.

Solution: By the definition of Δ, we have

$\Delta\ y_h = y_{h+1} - y_h,\ \Delta^2\ y_h = (E-1)^2\ \ y_h = y_{h+2} - 2\ y_{h+1} + y_h$

And $\Delta^3\ y_h = (E-1)^3\ \ y_h = y_{h+3} - 3\ y_{h+2} + 3\ y_{h+1} - y_h$.

(i) Putting the value of $\Delta\ y_h$ and $\Delta^2\ y_h$, the given equation reduce to the form
$y_{h+2} - 2\ y_{h+1} + y_h + 3\ (y_{h+1} - y_h) - 3\ y_h$ = h
Or $y_{h+2} + y_{h+1} - 5\ y_h$ = h.
Order of the differential equation = the difference between the largest and the smallest arguments for the function y.
= (h + 2) – h =2.
(ii) Putting the values of $\Delta\ y_h, \Delta^2\ y_h, \Delta^3\ y_h$ the equation takes the form
$(y_{h+3} - 3\ y_{h+2} + 3\ y_{h+1} - y_h) + y_{h+2} - 2\ y_{h+1} + y_h + (y_{h+1} - y_h) -\ y_h = 0$
Or $y_{h+3} - 2\ y_{h+2} + 2y_{h+1} - y_h)$ =0.
The order of the difference equation = (h +3) – (h +1) =2.

Example: 4 Solve (E -a **)** $y_h =$ h = 0, 1, 2, ….. a is a constant.

Solution: The given difference equation is (E – a) y_h=0

Or y_{h+1} = a y_h.

Comparing it with the difference equation y_{h+1} = A y_h + C, we have A = a , C = 0.

Hence the solution is given by

$y_h = A^h\ y_0 + C\,\frac{1-A^h}{1-A} = a^y\ y_0$.

Hence we get the following sequence of values y_0, a y_0, $a^2 y_0$, …..

Example 5: Solve the differential equation $y_h\, y_{h+2} = y^2{}_{h+1}$

Or $\frac{y_{h+2}}{y_{h+1}} - \frac{y_{h+1}}{y_h} = 0$ or $\Delta\left(\frac{y_{h+1}}{y_h}\right) = 0$

Or $\frac{y_{h+1}}{y_h}$ = constant = a (say) or y_{h+1} = a y_h.

After solving with the help of example 4, we get $y_h = A.a^h$

Hence the solution is $y_h = A.a^h$.

CHECK YOUR PROGRESS

TRUE OR FALSE

1. The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.
2. Finding the order of a difference equation, it must always be expressed in a form free of $\Delta's$.

A linear difference equation is that in which y_{h+1}, y_{h+2} etc. occur to the first degree only and are multiplied together.

Multiple choice questions

1. The order of the difference equation y_{h+2} + 7 y_h = 5
(a) 2
(b) 7
(c) 1
(d) 5

2. The solution of the difference equation y_{x+1} - y_x =1 , x = 0, 1 ,2, …..with y_0 , y_x is
(a) x +1
(b) x+3
(c) x +5
(d) x+7

3. The general solution of the difference equation y_{h+1} - y_h = 4 is
($\boldsymbol{a}$) $y_h = 3^{-h}$
($\boldsymbol{b}$) $y_h = y_0 3^{-h}$
($\boldsymbol{c}$) $y_h = y_0 3^h$
(d) $y_h = 3^h$

4. The order of the linear difference equation y_{h+2} + 6 y_{h+1} – 8 y_h =2h is
(a) 1
(b) 2
(c) 0
(d) 3

12.10 SUMMARY

In this unit we study about linear difference equation and order of linear difference equation, solution of difference equation, existence and uniqueness theorem.

12.11 GLOSSARY

Difference equation : A mathematical equation that describes the relationship between a sequence of numbers and the differences between successive terms.
Initial condition : The starting value of a sequence
Sequence : A list of number in a specific order.

12.12 REFERENCES

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3. "Difference Equations: Theory and Applications" by M. E. Fisher (Advances in Difference Equations, 2005)
4. C. F. Gerald & P. O. Wheatley, Applied Numerical Analysis (7 th edition), Pearson Education, India, 2008.

12.13 SUGGESTED READING

1. "Difference Equations: An Introduction with Applications" by Walter G. Kelley and Allan C. Peterson
2. "Difference Equations and Their Applications" by Kenneth S. Miller
3. "An Introduction to Difference Equations" by Saber Elaydi
4. "Difference Equations: Theory and Applications" by Michael E. Fisher
5. Atkinson K E, An Introduction to Numerical Analysis, John Wiley & Sons, India(1989).
6. Kincaid D and Cheney W, Numerical Analysis: Mathematics of Scientific Computing, Brookes/Cole Publishing Company (1999).

12.14 TERMINAL AND MODAL QUESTIONS

1. Find the order of the following:
(i) $y_{k+2} - 7\, y_k = 5$
(ii) $y_{k+4} - 5\, y_{k+2} + 6\, y_k = 0$
2. solve the difference equation $y_{h+1} = -y_h + 2$, h = 0, 1, 2, …
3. solve (E-a) (E-b) $y_h = 0$, a $\neq b$.
4. prove that the difference equation $(E-a)^2\, y_h = 0$ has solution
$y_h = (A + B\, h)\, a^h$.

FILL IN THE BLANKS

1. The order of the difference equation $\Delta^3 y_h + 2\,\Delta y_h + y_h = h + 3$ is ….
2. The linear form of the difference equation $\Delta^2 y_h + 3\Delta y_h - 3\,y_h = h$ is …..
3. The order of the difference equation $y_{h+3} - 7\Delta y_{h+1} + 2\,y_h = 4$ is ……..

12.15 ANSWERS

CYQ1. True
CYQ2. True
CYQ3. False
MCQ1. (a)
MCQ2. (c)
MCQ3. (c)
MCQ4. (b)
TQ1. (i) 2 (ii) 4
TQ2. Sequence y_0, $-y_0 + 2$, y_0, $-y_0 + 2$, ….
TQ3. $y_n = C_1 a^h + C_2 b^h$
FQ1. 3
FQ2. $y_{h+2} + y_{h+1} - 5\,y_h = h$
FQ3. 3

BLOCK IV

INITIAL AND BOUNDARY VALUE PROBLEM

UNIT 13. INITIAL AND BOUNDARY VALUE PROBLEM

CONTENTS:

13.1 INTRODUCTION

Initial and Boundary Value Problems (IBVPs) are mathematical problems that involve finding solutions to differential equations or systems of equations, subject to certain conditions. It consists of an ordinary differential equation paired with an initial condition, defining the solution's starting value at a particular point.

IVP is a mathematical model that combines a differential equation with a specific initial condition, aiming to determine the system's trajectory. In this unit we discussed about the various numerical methods are used to solve IBVPs, including Finite difference methods, Finite element methods, Shooting methods etc.

13.2 OBJECTIVES

After studying this unit learner will be able to

1. Find the solution that satisfies the differential equations and initial conditions.
   Example dy/dt = f(t,y), y(0) = y0
2. Analyze Stability of solution and develop Mathematical Theory.

13.3 INITIAL VALUE PROBLEM

An Initial Value Problem is a mathematical problem where a differential equation or system of equations is given and initial conditions are specified at a single point (typically t=0). In other words, a IVP is a mathematical problem that combines a differential equation with a specific starting condition, seeking a solution that satisfies both.

13.4 Boundary Value Problem

A Boundary Value Problem is a mathematical problem where a differential equation or system of equations is given.

Or

It is a mathematical problem requiring solution values or derivatives at two or more points, typically the boundaries of the domain.

Examples:

1. Heat equation with temperature conditions at both ends of a rod.
2. Wave equation with displacement conditions at boundaries.
3. Laplace equation with voltage conditions on a surface.

13.5 HIGHER ORDER ONE STEP METHOD

A numerical technique to solve initial value problems (differential equations) with high accuracy. In other words, it is a numerical technique where the solution is computed directly from the previous iteration, without requiring multiple intermediate steps.

Example

Order 2

$y_{j+1} = y_j + \text{h f}\left(y_j + h\frac{2}{f\,(y_j,t_j)}, t_k + \text{h}\ \frac{2}{f}\right)$

Order 3

$y_{j+1} = y_j + \text{h}\frac{4}{k_1 + 3k_3}$ where $k_1 = \text{f}\left(y_j, t_j\right)$, $k_2 = \text{f}\left(y_j + h\ \frac{k_1}{3}, t_j + h\ \frac{3}{f}\right)$, and

$k_3 = \text{f}\left(y_j + 2h\ \frac{k_2}{3}, t_j + h\ \frac{3}{f}\right)$

Order 4

$y_{j+1} = y_j + h \frac{6}{k_1 + 2k_2 + 2k_3 + k_4}$ where $k_1 = f\left(y_j, t_j\right), k_1 = f\left(y_j + h\frac{k_1}{2}, t_j + h\frac{2}{f}\right),$
$k_3 = f\left(y_j + h\frac{k_2}{2}, t_j + h\frac{2}{f}\right)$, and $k_4 = f\left(y_j + h\frac{k_3}{f}, t_j + h\right)$

13.6 EULER'S METHOD

It is one of the oldest and simplest methods but also the crudest.

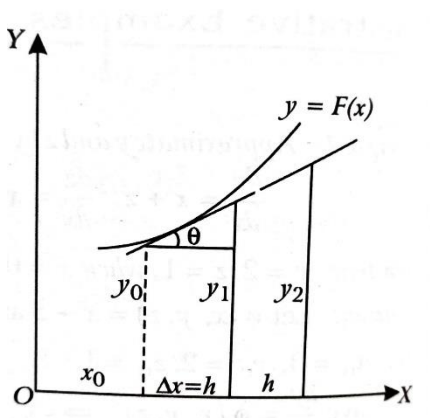
Let the differential equation be

$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0$ …(1)

Integrating (1), we get a relation between y and x which can be written in the form

Y = F(x). …(2)

In the xy-plane the equation (2) represents for a smooth curve is straight for a short distance from any point on it . Hence, we have the approximation relation.



$\Delta y = \Delta x \tan\theta$

$= \Delta x \, (dy/dx)_0 = \Delta x \, f\,(x_0, y_0)$.

$\therefore = y_1 \approx y_0 + \Delta x \,.\, f\,(x_0, y_0)$.

$= y_1 \approx y_0 + h\, f\,(x_0, y_0)$.

This y_1 is the approximate value of y for x = x_1.

Similarly the value of y corresponding to

$x_2 = x_1 + h\,, x_3 = x_2 + h$, etc.

Are given by

$y_3 = y_1 + h + f\left(x_{1,}\ y_1\right)$.

$y_3 = y_2 + h + f\left(x_{2,}\ y_2\right)$.

In general, we obtain

$y_{n+1} = y_n$ + h f (x_n, y_n), n =0, 1, 2,… …(3)

Taking h small enough and continuing in this way we could get the integral of (1) as a set of corresponding values of x and y.

This process is very slow. For practical use, the method is unsuitable because to get reasonable accuracy with this method to give a comparatively smaller value to h. if h is not small then the method is too inaccurate .in this method the actual solution curve is approximated by the sequence of short straight lines which sometimes deviates from the solution curve significant . all these considerations have led to a modification of Euler's method.

Modified Euler's Method:

Starting with the initial value y_0 , an approximate value for y_1 is calculated from the relation

$y_1^{(1)} \approx y_0$ + h (dy/dx)$_0$ $= y_0 + h\, f\ (x_0, y_0)$.

$y_1^{(1)}$ is the first approximation of y_1 at x = $x_{1.}$

Substituting this value of y_1 into the given differential equation (1), we get an approximate value of $\frac{dy}{dx}$

i.e., (dy/dx)$_1^{(1)}$ = f($x_{1\,,}\ y_1^{(1)}$).

Now an improved value of Δy is obtained as

Δy = h .[average of the value of $\frac{dy}{dx}$ at the ends of the interval $x_0\ to\ x_1$]

i.e., $\Delta y \approx$ h. $\frac{\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)}{2}$.

the second approximation for y_1 is

$y_1^{(2)} =\ y_0$ h + $\frac{\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)}{2}$.

Substituting this improve value $y_1^{(2)}$ in the given equation, we get a second approximation for

(dy /dx)$_1$ viz (dy/ dx)$_1^{(2)}$ = f ($x_1, y_1^{(2)}$).

The third approximation for y_1 is given by

$$y_1^{(3)} = y_0 + h + \frac{\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)}{2}.$$

The process is applied until no change is produced in the value of y_1 to the desired degree of accuracy.

In the same manner we make computations for the next interval x_1 to x_2 (= x_1 +h)

i.e., first by finding an approximate value of Δy and then using the averaging process until no improvement is made in the value of y_2.

First approximations to y_2, y_3,…..etc. could be obtained by using the formula

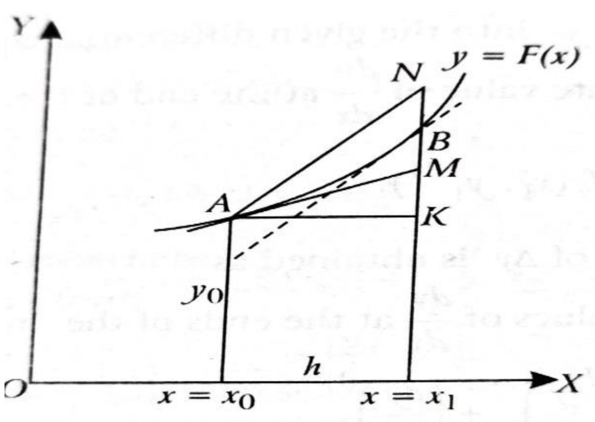
$$y_{n+1} \approx y_n + h\,(dy/dx)_n$$

But two consecutive values of y are known, the first approximation to succeeding y ʿs can be computed more accurately from the relation

$$y_{n+1} = y_{n-1} + 2\,h\,y_n'.$$

This relation can be derived by using Taylor's series in the neighborhood of x_n.

The modifies Euler's method gives a great improvement in accuracy over the original method.



In this figure KM represents the Δy computed by the Euler method. if AN is drawn parallel to the tangent at B, then KN represents

The Δy computed by using the slope the at B. if we take the average of the slopes.

We have

Δy = h (dy/dx)$_0$ + (dy/dx)$_1$/2 = $\frac{1}{2}$[h (dy/dx)$_0$ + h (dy/dx)$_1$]

= $\frac{1}{2}$ [KM +KN] = $\frac{1}{2}$ [KM + KM +MN] = KM + $\frac{1}{2}$ MN,

Which is must close to its true value KB.

Note: in general Euler's modified formula is

$y_1{}^{(n+1)}$=y_0 +$\frac{h}{2}$ [f (x_0, y_0) + f ($x_1, y_1{}^{(n)}$)].

Where $y_1{}^{(n)}$ is the nth approximation of y_1. the initial value of

$y_1{}^{(1)}$ =y_0 +h f (x_0, y_0) is taken from Euler's method.

ILLUSTRATIVE EXAMPLES

Example : **Given** $\frac{dy}{dx}$ = $\frac{y-x}{y+x}$ **, with y =1 for x =0 . find y approximated for x = 0.1 by Euler's Method. (five steps).**

Solution: Here we want the value at x =0.1 from x =0 in five steps. So we break up the interval 0 to 0.1 into five sub- interval by introducing the points x_1, x_2, x_3, x_4, x_5. Let h = 0.02. we shall find the values of y at x = 0.02, 0.04. 0.06, 0.08 and 0.1 successively.

Thus we have

x_0 = 0, y_0 = 1, h =0.02, f (x,y) = $\frac{y-x}{y+x}$.

Using y_{n+1} = y_n + h f (x_n, y_n) , we get

$y_1 = y_0 + h\,f(x_0, y_0)$= 1 +(0.2) $\left[\frac{1-0}{1+0}\right]$ = 1.02

$y_2 = y_1 + h\,f(x_1, y_1)$= 1.02 +(0.2) $\left[\frac{1.02-.02}{1.02+.02}\right]$ = 1.0392

$y_3 = y_2 + h\,f(x_2, y_2)$= 1.0392 + (0.2) $\left[\frac{1.0392-.04}{1.0392+.04}\right]$ = 1.0577

$y_4 = y_3 + h\,f(x_3, y_3)$= 1.0577 + (0.2) $\left[\frac{1.0577-.06}{1.0577+.06}\right]$ = 1.0756

$y_5 = y_4 + h\,f(x_4, y_4)$= 1.0756 +(0.2) $\left[\frac{1.0756-.08}{1.0756+.08}\right]$ = 1.0928.

Hence y = 1.0928 when x = 0.

13.7 RANGE KUTTA METHODS

(i) Range – kutta second order (or quadratic) formula.

Consider the differential equation

$y' = f(x, y)$

With the initial condition y $(x_0) = y_0$.

Let h be the interval between equidistant values of x. Then in the second order Runge - Kutta formula the first increment in y is computed from the formulae

$k_1 = h\, f(x_0, y_0)$,

$k_2 = h\, f(x_0 + h, y_0 + k_1)$,

$\Delta y = \frac{1}{2}(k_1 + k_2)$

Taken in the given order.

Then $x_1 = x_0 + h$ and $y = y_0 + \Delta y = y_0 + \frac{1}{2}(k_1 + k_2)$

In a similar manner the increment in y for the second interval is computed by means of the formulae

$k_1 = h\, f(x_1, y_1)$,

$k_2 = h\, f(x_1 + h, y_1 + k_1)$,

$\Delta y = \frac{1}{2}(k_1 + k_2)$,

And similarly for the next intervals.

The inherent error in the second order Runge -Kutta formula is of order h^3.

(ii) Runge Kutta third order formula.

We consider the differential equation

$y' = f(x, y)$

With the initial condition y $(x_0) = y_0$.

Let h be the interval between equidistance values of x. then the first increment in y is computed from the formulae

k_1 = h f(x_0, y_0),

k_2 = h f $(x_0 + \frac{h}{2}, y_0 + \frac{k_1)}{2}$,

$k_3 =$ f $(x_0 + h, y_0 + y')$; k' = h f $(x_0 + h, y_0 + k_1)$

$\Delta y = \frac{h}{6}(k_1 + 4k_2 + k_3)$

Taken in the given order.

Then $x_1 = x_0 + h$ and $y = y_0 + \Delta y$

We can generalize this for successive approximations.

(iii) Runge kutta Fourth order formula.

It is one of the most widely used method and it is particularly suitable in cases when the computation of higher derivatives is complicated.

We consider the differential equation

y' = f (x, y)

With the initial condition y $(x_0) = y_0$.

Let h be the interval between equidistant values of x. then the first increment in y is computed from the formulae

k_1 = h f(x_0, y_0),

k_2 = h f $(x_0 + \frac{h}{2}, y_0 + \frac{k_1)}{2}$,

$k_3 =$ f $(x_0 + \frac{h}{2}, y_0 + \frac{k_2)}{2})$ …(1)

k_4= h f $(x_0 + h, y_0 + k_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

Taken in the given order.

Then $x_1 = x_0 + h\ and\ y_1 + \Delta\ y$.

In a similar manner the increment in y for the second interval is computed by means of the formulae

k_1 = h f(x_1, y_1),

k_2 = h f $(x_1 + \frac{h}{2}, y_1 + \frac{k_1)}{2}$,

k_3 = f $(x_1 + \frac{h}{2}, y_1 + \frac{k_2)}{2})$ …(1)

k_4= h f $(x_1 + h, y_1 + k_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

And similarly for the next interval.

It is to be noted that the calculations for the first increment are exactly the same as for any other increment. The change in the formulae for the different intervals is only in the values of x and y to be substituted.

Hence to obtain Δy *for the nth interval we* substitute x_{n-1}, y_{n-1}, in the expressions for k_1, k_2 etc.

ILLUSTRATIVE EXAMPLES

Example : Use Range Kutta method to solve the equation $\frac{dy}{dx}$ = 1 +y^2 for x = 0.2 to x = 0.6 with h = 0.2 . given the initially at x =0, y = 0.

Solution: Here $\frac{dy}{dx}$ = 1 +y^2, $x_0 = 0, y_0 = 0$ *and* $h = 0.2$

Now k_1 = h f(x_0, y_0) = (0.2) f (0,0) = (0.2) [1 +0^2]= 0.2.

k_2 = h f $(x_0 + \frac{h}{2}, y_0 + \frac{k_1)}{2} = (0.2) f\left(0 + \frac{0.2}{2}, 0 + \frac{0.2}{2}\right)$

= (0.2) f (0.1,0.1) = (0.2) [1 +$(0.1)^2$] = (0.2) (1.01)= 0.202.

k_3 = f $(x_0 + \frac{h}{2}, y_0 + \frac{k_2)}{2})$ = (0.2) f (0.1 , 0.101)

= (0.2) [1 +$(0.10\ 1)^2$] = f (1 + 0.010201)

= (0.2) (1.0102) = 0.20204.

k_4= h f $(x_0 + h, y_0 + k_3)$ = (0.2) f (0.2, 0.20204)

(0.2) (1+ 0.04082) = 0.208164.

$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

= $= \frac{1}{6}[0.2 + 0.404 + 0.40408 + 0.20816]$ = 0.20271.

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$and\ y_{0.2} = y_0 + \Delta y$ = 0 +0.20271 = 0.20271.

now to determine $y_{0.4}$, we note that

x_1 = 0.2, $y_{0.2}$ = 0.20271.

$\therefore$ for the second interval, we have

k_1 = h f $(x_1, y_1) = (0.2)\ [1 + (0.20271)^2] = 0.20822$

k_2 = h f $(x_1 + \frac{h}{2}, y_1 + \frac{k_1)}{2}$ = (0.2) f (0.3, 0.30682) = (0.2)) $[1 + (0.30682)^2] = 0.21883$.

k_3 = f $(x_1 + \frac{h}{2}, y_1 + \frac{k_2)}{2})$ = (0.2) f (0.3, 0.31213) = (0.2) $[1 + (0.31213)^2] = 0.21948$.

k_4= h f $(x_1 + h, y_1 + k_3)$= (0.2) $[1 + (0.42219)^2]$ =0.23565.

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

$=\frac{1}{6}$ = [0.20822 +2 (0.21883) +2 (0.21948) +0. 23565] = 0.220082.

$x_2 = x_1$ +h = 0.2 + 0.2 = 0.4, $y_{04} = y_1 + \Delta y$ = 0.42279.

Now to determine $y_{0.4}$, we note that

x_2 = 0.4, $y_2 = y_{0.4}$ = 0.42279.

$\therefore$ for the third interval, we have

k_1=h f (x_2, x_2)= (0.2) f (0.4, 0.42279) = (0.2) [1 +$(0.42279)^2$] = 0.23575

k_2 = h f $(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2})$ = (0.2) [1 +$(0.54067)^2$] = 0.25846.

k_3 = h f $(x_2 + \frac{h}{2}, y_2 + \frac{k_2)}{2})$ = 0.260954.

k_4= h f $(x_2 + h, y_2 + k_3)$ =0.29350.

$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

= $\frac{1}{6}$ (0.23575 + 0.51692 + 0.52190 + 0.29350)

= 0.26134.

$\therefore x_3 = x_2$ + h = 0.4 + 0.2 = 0.6

$y_3 = y_{0.6} = y_2$ + Δy = 0.68413.

Hence y (0.2) = 0.20271, y (0.4) = 0.42279 , y (0.6) 0.68413.

13.8 RANGE – KUTTA METHOD FOR SIMULTANEOUS FIRST ORDER EQUATIONS

Consider the simultaneous equations

$\frac{dy}{dx}$ = f (x, y, z) …(1)

$\frac{dz}{dx}$ = g (x, y, z) ...(2)

With the initial condition y $(x_0) = (y_0)$ and z $(x_0) = z_0$. Now, starting from (x_0, y_0, z_0) , the increments k and l in y and z are given by the following formulae:

k_1=h f (x_0, y_0, z_0) ; l_1=h g (x_0, y_0, z_0)

k_2 = h f $(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$; l_2 = h g $(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

k_3 = h f $(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$; l_3 = h g $(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

k_4 = h f $(x_0, h, y_0 + k_3, z_0 + l_3)$; k_4 = h g $(x_0 + h, y_0 + k_3, z_0 + l_3)$

K = $\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, l= $\frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$, Hence $y_1 = y_0 + k_{1,}$ $z_1 = z_0$ +l

To compute y_2, z_2 we simplify replace x_0, y_0 , z_0 by x_1, y_1 , z_1 in the above formulae.

ILLUSTRATIVE EXAMPLES

Example 11: Solve $\frac{dy}{dx}$ = y z + x, $\frac{dy}{dx}$ = x z + y;

Given that y (0) =1, z (0) =-1 for y (0.1), z (0.1)

Solution: f_1 (x, y, z) = y z + x, f_2 (x, y, z) = x z + y

H = 0.1, x_0 = 0, y_0 = 0, z_0 = -1

k_1=h f_1 (x_0,y_0, z_0) = h (($y_0 z_0$ + x_0)= -0.1 l_1=h f_2 (x ,y_0, z_0)= h (($x_0 z_0$+ y_0) = 0.1

k_2 = h f_1 $(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$; l_2 = h f_2 $(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

= h f_1 (0.05, 0.95, -0.95) = -0.08525 = h f_2 (0.05, 0.95, -0.95) = 0.09025

k_3 = h f_1 $(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$ l_3 = h f_2 $(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

h f_1 (0.05, 0.957375) = -0.0864173. = h f_2 90.05, 0.957375, - 0.954875

= 0.0864173

k_4 = h f_1 $(x_0 + h, y_0 + k_3, z_0 + l_3)$; l_4 = h f_2 $(x_0 + h, y_0 + k_3, z_0 + l_3)$

= -0.073048 = 0.822679

K = $\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ l = $\frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$,

-0.0860637 = 0.0907823

Hence $y_1 = y\,(0.1)$ = y_1 + k = 1 – 0.0860637 = 0.9139363

$z_1 = z\,(0.1)$ = z_0 + l = -1+ 0.09078823 = -0.9092176

CHECK YOUR PROGRESS

TRUE OR FALSE

1. Runge's Kutta method is a self-starting method.
2. Euler's method is a predictor Corrector method.
3. The Runge-Kutta method is used to solve differential equations.
4. The Runge-Kutta method is accurate up to a certain number of decimal places.
5. In Euler's method, if h is small, the method is too slow and if h is large, it gives inaccurate value.
6. The modified Euler method is based on the average of points.
7. The initial value problem y′=3y;y(0)=0 has more than one solution.

Multiple choice questions

1. Which method is not applicable for solving differential equation?
(a) Euler's method
(b) Picard's method
(c)Runge- Kutta method
(d)Gauss – Seidel method

2. Ranga kutta fourth order method is
(a) Multi step method
(b) A single step method
(c) Predictor---corrector method
(d) None of these

3. The approximate value of y when x = 0.2, given that y= 1 when x =0 and $\frac{dy}{dx}$ = x- y
(a) 0.83867
(b) 0.73867
(c) 8.02971
(d) None of these

4. Runge – Kutta third order rule is
(a) $\Delta y = \frac{h}{3}(k_1 + 4k_2 + k_3$
(**b**) $\Delta y = \frac{h}{6}(k_1 + 4k_2 + k_3$
(**c**) $\Delta y = \frac{h}{8}(k_1 + 4k_2 + k_3$
(d) **None of these**

13.9 SUMMARY

In this chapter we explained the following topic.
An IVP is a problem that involves finding a function that satisfies a differential equation and an initial condition.
Initial condition: A condition that specifies the value of the unknown function at a specific point.
A BVP is a problem that involves finding a function that satisfies a differential equation and boundary conditions.
Boundary conditions: Conditions that specify the value of the unknown function at specific points.
Euler’s Method and Range- kutta method,

13.10 GLOSSARY

Differential equation
Initial condition
Boundary condition
Solution

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13.13 TERMINAL AND MODAL QUESTIONS

1. Apply Euler's method to solve $\frac{dy}{dx}$ = y, y = 1 when x = 0,
For values x = 0 to x = 0.5, taking Δ x = 0.1,
2. Given $\frac{dy}{dx}$ = y – x, y (0), Find, using Runge-
Kutta fourth order formula y (0.1) and y (0.2) correct to four decimal places.
3. For the differential equation
$\frac{dy}{dx} = - x\, y^2$,
Find by Runge- Kutta method of fourth order y(0.6), given that y = 1.7231 at x = 0.4. take h =0.2.

13.14 ANSWERS

CYQ1. True
CYQ2. True
CYQ3. True
CYQ4. True
CYQ5. True
CYQ6. True
CYQ7. True
MCQ1. (d)
MCQ 2. (b)
MCQ 3. (a)
MCQ 4. (b)
TQ1. 1.1, 1.21, 1.331, 1.4641, 1.61051
TQ2. 2.2052, 2.4213
TQ3. 1.4804

UNIT 14 : APPLICATION OF NUMERICAL ANALYSIS

CONTENTS:

14.1 INTRODUCTION

Numerical analysis is a branch of mathematics that develops effective methods to find numerical solutions to complex problems. Since many scientific and engineering problems can't be solved directly, numerical analysis breaks them down into manageable parts. Numerical analysis is used in mathematics, computer science, and various fields. It solves continuous mathematical problems from algebra, geometry, and calculus.

Numerical Analysis is a multidisciplinary field that combines Mathematics and Computer Science. Numerical Analysis is an interdisciplinary field that leverages mathematical and computational techniques to create, test, and optimize algorithms for solving complex numerical problems in continuous mathematics.

This section focuses on numerical methods, providing an in-depth exploration of their development, analysis, and implementation.

14. 2 OBJECTIVES

After studying this unit learner will be able

1. To solve real world problem and solve complex problem in various fields.
2. To predict and forecast future events such as weather patterns or population growth.
3. To reconstruct and analyze medical images.

14.3 REAL LIFE APPLICATION OF NUMERICAL ANALYSIS

Numerical analysis helps solve complex problems that can't be solved by traditional methods. “Numerical analysis is a multidisciplinary field that integrates mathematical modelling, computational techniques, and algorithmic design to tackle complex problems across various disciplines. Its influence extends to diverse areas, including climate modelling, financial forecasting, and engineering optimization." It is a powerful tool for solving complex problems in various fields, including science, engineering, and finance. By combining mathematical modelling, computational methods, and algorithm design, numerical analysis enables researchers and practitioners to simulate real-world phenomena, make predictions, and optimize systems." Here are some real-life applications:

<u>Weather Forecasting</u>

"Weather forecasting relies heavily on numerical analysis to break down intricate calculations into more manageable components, enabling meteorologists to make accurate predictions." Accurate weather forecasting relies on solving differential equations that describe atmospheric dynamics. “Numerical models are a crucial tool for meteorologists, as they simulate weather patterns by calculating temperature, wind, and humidity across various locations and timeframes. “Numerical methods play a vital role in simulating extreme weather events like hurricanes and tornadoes, allowing for more effective disaster preparedness and response strategies."

Engineering Design

Engineers use numerical analysis to design safe and efficient buildings, bridges, and machines. Numerical methods help engineers analyze the strength of structures, like bridges, to make sure they can handle heavy loads and stresses. In aerospace, numerical analysis simulates airflow around aircraft. This is crucial for designing safer and more efficient aircraft. Car engineers use numerical analysis to make vehicles more fuel-efficient and safer. These methods help reduce the need for physical prototypes, saving time and money in the design process.

Financial Modelling

Sophisticated algorithms utilize numerical analysis to forecast the future value of stocks and bonds, empowering investors to make well-informed decisions. By analyzing historical data, numerical analysis also facilitates the prediction of economic trends, providing valuable insights for investors and policymakers alike. By facilitating the creation of automated trading systems, numerical analysis promotes market efficiency, allowing for faster and more accurate transaction processing.

Image Processing

Image processing uses numerical methods to improve digital images. Medical imaging, like MRI and CT scans, relies on these techniques to provide clear images. This is vital for accurate diagnosis. In astronomy, it enhances images of celestial bodies, helping scientists' study distant planets and stars. Numerical analysis also powers facial recognition technology used in security systems. It is essential in the entertainment industry for creating high-resolution graphics and special effects.

Drug Development

Pharmaceutical companies use numerical analysis to speed up drug development. It makes the process more efficient. By simulating drug interactions at the molecular level, researchers can predict the effectiveness of a drug. Numerical models help understand the behaviour of new drugs in the human body. This reduces the need for extensive clinical trials. It also helps in designing controlled release medications that improve patient compliance and treatment effectiveness. These methods enable researchers to explore more potential treatments in less time.

Environmental Science

Numerical analysis helps in solving environmental issues. It models pollution dispersion in air, water, and soil. This is crucial for environmental protection. Climate models use numerical methods to predict changes in climate. It also assists in managing natural resources, like water and forests, more sustainably. These models help in assessing the impact of human activities on ecosystems, guiding conservation efforts.

Cryptography

Cryptography ensures secure communication. It is used to create algorithms that protect data from unauthorized access. Numerical methods help in the analysis of cryptographic algorithms to ensure they are secure. They are essential in developing new encryption techniques that are harder to break.

Seismic Data Analysis

Numerical analysis helps understand seismic activities to mitigate disaster risks. Geophysicists use numerical models to simulate earthquake scenarios. By analyzing seismic data, scientists can better predict the likelihood of future earthquakes. This is crucial for disaster preparedness. Numerical methods aid in the design of earthquake-resistant structures, enhancing safety and minimizing damage. They also contribute to the exploration of oil and gas by interpreting seismic data to locate reserves.

Power Systems Engineering

Engineers use numerical techniques to model and simulate the behavior of electrical grids. This ensures stability and efficient power distribution. Numerical methods help in optimizing the operation of renewable energy sources like wind turbines and solar panels. They are crucial for designing systems that integrate various types of energy sources, maintaining a stable energy supply. These methods also support the development of smart grids, which automatically respond to changes in energy demand and supply.

Robotics

Robotics integrates numerical analysis to enhance functionality and autonomy.

In robotics, numerical methods are used for motion planning and control.

They are essential in developing algorithms that enable robots to learn from their environment and adapt to new tasks.

14.4 Mathematical Application of Numerical analysis

In algebra Numerical method us for solving equations with the help of Newton Raphson Method. It is also used in linear algebra and Matrix computations.

In Analysis

1. Numerical integration (e.g., Simpson's rule)

2. Numerical differentiation (e.g., finite differences)

3. Optimization techniques (e.g., gradient descent)

In Differential Equations

1. Numerical methods for solving ODEs (e.g., Euler's method)

2. Numerical methods for solving PDEs (e.g., finite element method)

3. Dynamical systems

In Statistics

1. Numerical methods for statistical inference (e.g., maximum likelihood estimation)

2. Time series analysis

3. Data mining

In Computational Mathematics

1. Computational number theory

2. Computational algebra

3. Computational geometry

In Applied Mathematics

1. Mathematical modelling

2. Simulation

3. Optimization

In Numerical Methods for Differential Equations

1. Euler's Method

2. Runge-Kutta Methods

3. Finite Difference Methods

4. Finite Element Methods

5. Boundary Element Methods

In Numerical Methods in Linear Algebra

1. Gaussian Elimination

2. LU Decomposition

3. Cholesky Decomposition

4. QR Decomposition

5. Eigenvalue Decomposition

14.5 SUMMARY

In this unit we discussed about the the application of numerical analysis in various field .

1. Physics and Engineering: Numerical analysis is used to simulate complex physical systems, such as fluid dynamics, heat transfer, and structural mechanics.

2. Computer-Aided Design (CAD): Numerical analysis is used to optimize designs and simulate the behavior of complex systems.

3. Weather Forecasting: Numerical analysis is used to simulate atmospheric conditions and predict weather patterns.

4. Cryptography: Numerical analysis is used to develop and break cryptographic algorithms.

14.6 GLOSSARY

Simulation

Modeling

Optimization

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14.9 TERMINAL AND MODEL QUESTIONS

1. What role does Numerical analysis play in healthcare?

2. How does Numerical analysis benefit weather forecasting?

3. Can Numerical analysis improve engineering designs?

4. What is the importance of Numerical analysis in cryptography?

5. What is seismic data analysis?