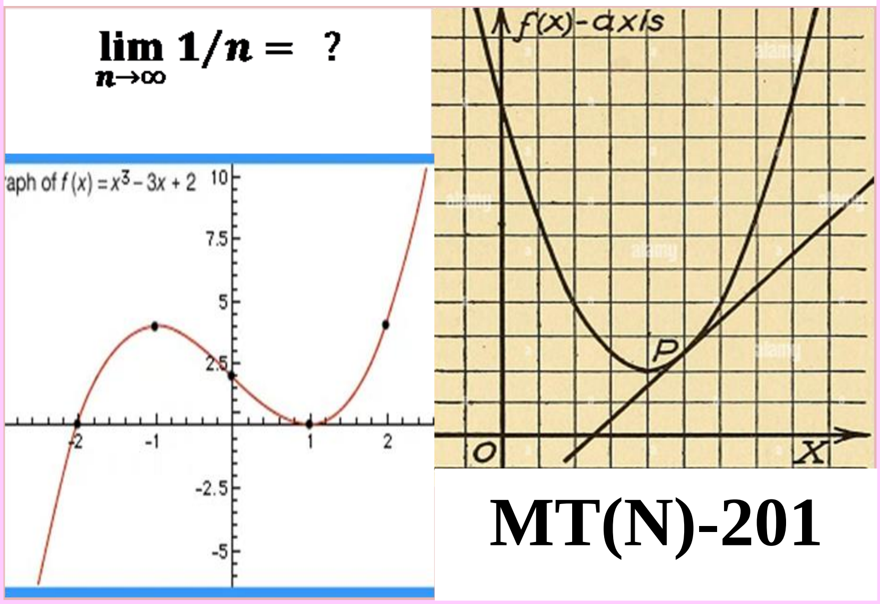
BACHELOR OF SCIENCE/ BACHELOR OF ARTS

(New Education Policy-2020)

REAL ANALYSIS



MT(N)-201

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DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: REAL ANALYSIS

COURSE CODE: MT(N) 201

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BOARD OF STUDIES - 2023

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COURSE INFORMATION

The present self-learning material **“Real Analysis”** has been designed for **B.Sc. (Third Semester)** learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **Real Numbers, Sequence and Series,** in this block Sets and functions, Real Numbers, Limit points, open and closed sets, Real sequences, Infinite Series defined Clearly.

Second block is **Functions Single Variable,** in this block Limits of function, continuous function, Properties of Continuous function, Uniform Continuity, Monotone and Inverse function. Derivative, Mean Value theorem, L Hospital rule defined clearly.

Third block is **Riemann Integration, Uniform convergence and Improper integral,** in this block Riemann integral, Integrability of continuous and monotonic functions, Fundamental theorem of integral calculus, First mean value theorem, Pointwise and uniform convergence of sequence and series of functions, Weierstrass’s M-test, Dirichlet test and Abel’s test for uniform convergence, Uniform convergence and continuity, Uniform convergence and differentiability, Improper integrals, Dirichlet and Abel’s tests for improper integrals are defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

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BLOCK-I

UNIT 1: *SETS AND FUNCTIONS*

CONTENTS:

1.1 *INTRODUCTION*

Set theory, branch of mathematics that deals with the properties of well-defined collections of objects, which may or may not be of a mathematical nature, such as numbers or functions. The theory is less valuable in direct application to ordinary experience than as a basis for precise and adaptable terminology for the definition of complex and sophisticated mathematical concepts.

Between the years 1874 and 1897, the German mathematician and logician Georg Cantor created a theory of abstract sets of entities and made it into a mathematical discipline. This theory grew out of his investigations of some concrete problems regarding certain types of infinite sets of real numbers. A set, wrote Cantor, is a collection of definite, distinguishable objects of perception or thought conceived as a whole. The objects are called elements or members of the set.

1.2 *OBJECTIVES*

After studying this unit, learner will be able to

i. To analyze and predict the behavior of these systems over time.

ii. To provide solutions to problems that cannot be solved using other mathematical techniques.

iii. To understand the definition of differential equation.

1.3 *SETS*

A set is a well - defined collection of distinct objects.

By a ‘well – defined’ collection of objects we mean that there is a rule by means of which it is possible to say, without ambiguity, whether a particular object belongs to the collection or not. The objects in a set are **‘distinct’** means we do not repeat an object over and over again in a set.

Each object belonging to a set is called an element of the set. Sets are usually denoted by capital letters A, B, N, Q, S etc. and the elements by lower case letters a, b, c, x etc.

The symbol ∈ is used to indicate 'belongs to'. Thus $x \in A \Longrightarrow x$ is an element of the set A.

The symbol ∉ is used to indicate 'does not belong to'. Thus $x \notin A \Longrightarrow x$ is not an element of the set A.

Example: Let A = {1, 2, 3, 4, 5} be a set then we say $1 \in A$, $2 \in A$, $3 \in A$, $4 \in A$, $5 \in A$ but $6 \notin A$, $7 \notin A$, $8 \notin A$.

1.4 *METHODS OF DESCRIBING A SET*

There are two methods of describing a set.

(1) Roster Method.

In this method, a set is described by listing all its element, separating by commas and enclosing within curly brackets.

For Example. (i) If A is the set of odd natural numbers less than 10, then in roster form.

$$A = \{1, 3, 5, 7, 9\}$$

(ii) if B is the set of letters of the world FOLLOW, then in roster form.

$$B = \{F, O, L, W\}$$

(2) Set Builder Method.

Listing the element of a set is sometimes difficult and sometimes impossible. We do not have a roster form of the set or rational number or the set of real numbers. In set builder method, a set is described by means of some property which is shared by all the element of the set.

For Example. (i) If P is the set of all prime numbers, then

$$P = \{x : x \text{ is a prime number}\}$$

(ii) if A is the set of all natural numbers between 5 and 50, then

$$A = \{x : x \in N \text{ and } 5 < x < 50\}$$

1.5 *TYPES OF SETS*

(i) Finite set. A set is said to be finite if the number of its elements is Finite i.e. its elements can be counted, by one by one, with counting coming to end.

For Example. (a) the set of letters in the English alphabet is finite set since it has 26 elements.

(b) Set of all multiples of 10 less than 10000 is a finite set.

(ii) Infinite set. A set is said to be infinite if the number of its elements is infinite i.e. we count its elements, one by one, the counting never comes to an end.

For Example. (a) the set of all points in a straight line is an infinite set.

(b) the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$ all are infinite sets.

(iii) Null Set. A set having no element is known as a null set or void set or an empty set and is denoted by $\emptyset$ or {}.

For Example. (a) {x : x is an integer and $x^2 = 3$} = $\emptyset$, because there is no integer whose square is 3.

(iv) Singleton Set. A set having only one element is called a singleton set.

For Example. (a) {a} is a singleton set.

(b) {x: $x^3 + 1 = 0$ and x$\in \mathbb{R}$} = { -1 } is a singleton set.

1.6 *SUBSET, SUPERSET AND POWER SET*

Set A is said to be a subset of Set B if all the elements of Set A are also present in Set B. In other words, set A is contained inside Set B. Example: If set A has {X, Y} and set B has {X, Y, Z}, then A is the subset of B because elements of A are also present in set B.

Subset Symbol

In set theory, a subset is denoted by the symbol $\subseteq$ and read as 'is a subset of'.

Using this symbol we can express subsets as follows:

A ⊆ B; which means Set A is a subset of Set B.

Note: A subset can be equal to the set. That is, a subset can contain all the elements that are present in the set.

All Subsets of a Set

The subsets of any set consists of all possible sets including its elements and the null set. Let us understand with the help of an example.

Example: Find all the subsets of set A = {1,2,3,4}

Solution: Given, A = {1,2,3,4}

Subsets are {},{1}, {2}, {3}, {4}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4},{1,2,3}, {2,3,4}, {1,3,4}, {1,2,4},{1,2,3,4}.

Superset Definition

In set theory, set A is considered as the superset of B, if all the elements of set B are the elements of set A. For example, if set A = {1, 2, 3, 4} and set B = {1, 3, 4}, we can say that set A is the superset of B. As the elements of B [(i.e.,)1, 3, 4] are in set A. We can also say that B is not a superset of A.

Superset Symbol

The superset relationship is represented using the symbol "⊃". For instance, the set A is the superset of set B, and it is symbolically represented by A ⊃ B.

Consider another example,

X = {set of polygons}, Y = {set of irregular polygons}

Then X is the superset of Y (X⊃Y). In other words, we can say that Y is a **subset** of X (Y⊂X).

Proper Superset

The proper superset is also known as a strict superset. The set B is the proper superset of set A, then all the elements of set A are in B, but set B must contain at least one element which is not present in set A.

For example, let us take four sets.

A = {a, b, c}, B = {a, b, c, d}, C = (a, b, c}, D = {a, b, e}

From the sets given above,

B is the proper superset of A, as B is not equal to A

C is a superset of set A, but the set C is not a proper superset of set A, as C = A

D is not a superset of A, as the set D does not contain the element "c" which is present in set A.

Power Set

The set of all subsets of a set A is called the power set of A and denoted by P(A).

i.e. $P(A) = \{S: S \subset A\}$.

For Example. (i) if A = {a}, then P(A) = {∅, A}

(ii) If B = {1, 2} then P{B} = {∅,{1}, {2}, B}

Theorem 1. Every set a subset of itself.

Proof. Let A is any set. Since $x \in A \Longrightarrow x \in A$, therefore $A \subset A$.

Theorem 2. Empty set is a subset of every set.

Proof. Given two sets A and B, let A=∅.
By definition, A is a subset of B if and only if every element in A is also in B.
This means that A would not be a subset of B if there exists an element in A that is not in B.

However, there are no elements in A. This means there cannot exist an element in A that is not in B. Thus, A is a subset of B.
Since A = ∅ and B is an arbitrary set, the ∅ must be a subset of all sets.

Theorem 3. The empty set is unique.

Proof. Let $\emptyset_1 and\ \emptyset_2$ be two empty sets.

Since empty set is a subset of every set .

Therefore $\emptyset_1 \subset \emptyset_2$ and $\emptyset_2 \subset \emptyset_1$

$\Rightarrow \emptyset_1 = \emptyset_2$ that proves the uniqueness of ∅.

Note: if a set has n elements, then the number of subsets is 2^n.

1.7 *OPERATIONS ON A SETS*

1. Union of Sets. The union of two sets X and Y is equal to the set of elements that are present in set X, in set Y, or in both the sets X and Y. This operation can be represented as;

X ∪ Y = {a: a ∈ X or a ∈ Y}

Let us consider an example, say; set A = {1, 3, 5} and set B = {1, 2, 4}

Then A ∪ B = {1, 2, 3, 4, 5}

Properties of Union of Sets

(i) For any two Sets A and B, $A \subset A \cup B$ or $B \subset A \cup B$

Proof. Let x be any element of A. then

$x \in A \Rightarrow x \in A \cup B$

therefore $A \subset A \cup B$

similarly, we can prove $B \subset A \cup B$

(ii) For any set A, $A \cup \emptyset = A$.

Proof. $A \cup \emptyset = \{x: x \in A \text{ or } x \in \emptyset\}$

$= \{x: x \in A\}$ [∵ ∅ has no element]

$= A$

(iii) Union of sets is idempotent i.e. foe any set A, $A \cup A = A$.

Proof. $A \cup A = \{x: x \in A \text{ or } x \in A\}$

$= \{x: x \in A\}$

$= A$

(iv) Union of sets is commutative.

Proof. $A \cup B = \{x: x \in A \text{ or } x \in B\}$

$= \{x: x \in B \text{ or } x \in A\}$
$= B \cup A$

Note: Union of sets is Associative.

2. Intersection of Sets. The intersection of two sets X and Y is the set of all elements which belong to both X and Y. This operation can be represented as;

X ∩ Y = {a: a ∈ X and a ∈ Y}

Let us consider an example, say; set A = {1, 3, 5} and set B = {1, 2, 4}

Then A ∩ B = {1}

Properties of Intersection of Sets

(i) For any two sets A and B, $A \cap B \subset A$ and $A \cap B \subset B$.

Proof. Let x be any element of A ∩ B. then

$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$
$\Rightarrow x \in A$ (in particular)

Therefore $A \cap B \subset A$

Similarly, we can prove $A \cap B \subset B$.

(ii) Intersection of sets is idempotent i.e. foe any set A, $A \cap A = A$.

Proof. $A \cap A = \{x: x \in A \text{ and } x \in A\}$
$= \{x: x \in A\}$
$= A$

(iii) Intersection of sets is commutative.

Proof. $A \cap B = \{x: x \in A \text{ and } x \in B\}$
$= \{x: x \in B \text{ and } x \in A\}$
$= B \cap A$

Note: Intersection of sets is Associative.

3. Difference of Sets. The difference of two sets A and B is the set of all elements which are in A but not in B.

The difference of sets A and B is denoted by A – B.

i.e. $A - B = \{x: x \in A \text{ and } x \notin B\}$

For example. (i) if A = {1, 2, 3, 4, 5} and B {2, 4, 6, 8}, then A – B = {1, 3, 5}, B – A = {6, 8}.

Clearly, $A - B \neq B - A$

Note. The difference of sets is not commutative.

4. Complement of a Set. Let U be the universal set and A ⊂ U. then complement of A is the set of those elements of U which are not in A. the complement of A is denoted by A^c.

Symbolically, A^c = U – A = {x: x ∈ U and x ∉ A} = {x: x ∉ A}

For example. If U is the set of all natural numbers and A is the set of even natural numbers, then

A^c = U – A

= the set of those natural numbers which are not even

= the set of odd natural numbers.

5. Symmetric Difference of Sets. If A and B are any two sets, then the sets (A – B) ∪ (B − A) is called the symmetric difference of A and B. The symmetric difference of A and B is denoted by A Δ B and read as 'A symmetric difference B'.

For Example. If A = {a, b, c, d, e} and B = {c, d, e, f, g}, then

A – B = {a, b}, B – A = {f, g}

Therefore A Δ B = (A – B) ∪ (B − A)

= {a, b} ∪ {f, g} = {a, b, f, g}.

1.8 *DE MORGAN'S LAWS*

For any two sets A and B, prove that

(a) $(\mathbf{A} \cup \mathbf{B})^{\mathbf{c}} = \mathbf{A}^{\mathbf{c}} \cap \mathbf{B}^{\mathbf{c}}$ **(b)** $(\mathbf{A} \cap \mathbf{B})^{\mathbf{c}} = \mathbf{A}^{\mathbf{c}} \cup \mathbf{B}^{\mathbf{c}}$

Proof. (a) We need to prove, $(A \cup B)^c = A^c \cap B^c$

Let X = $(A \cup B)^c$ and Y = $A^c \cap B^c$

Let p be any element of X, then p ∈ X ⇒ p ∈ $(A \cup B)^c$

⇒ p ∉ (A ∪ B)

⇒ p ∉ A or p ∉ B

⇒ p ∈ A' and p ∈ B'

⇒ p ∈ A' ∩ B'

⇒ p ∈ Y

∴ X ⊂ Y . . . (i)

Again, let q be any element of Y, then q ∈ Y ⇒ q ∈ A' ∩ B'

⇒ q ∈ A^c and q ∈ B^c

$\Rightarrow q \notin A$ or $q \notin B$

$\Rightarrow q \notin (A \cup B)$

$\Rightarrow q \in (A \cup B)^c$

$\Rightarrow q \in X$

$\therefore Y \subset X$ . . . (ii)

From (i) and (ii) X = Y

$(A \cup B)^c = A^c \cap B^c$

(b) We need to prove, $(A \cap B)^c = A^c \cup B^c$

Let $X = (A \cap B)^c$ and $Y = A^c \cup B^c$

Let p be any element of X, then $p \in X \Rightarrow p \in (A \cap B)^c$

$\Rightarrow p \notin (A \cap B)$

$\Rightarrow p \notin A$ and $p \notin B$

$\Rightarrow p \in A^c$ or $p \in B^c$

$\Rightarrow p \in A^c \cup B^c \Rightarrow p \in Y$

$\therefore X \subset Y$ —————(i)

Again, let q be any element of Y, then $q \in Y \Rightarrow q \in A^c \cup B^c$

$\Rightarrow q \in A^c$ or $q \in B^c$

$\Rightarrow q \notin A$ and $q \notin B$

$\Rightarrow q \notin (A \cap B)$

$\Rightarrow q \in (A \cap B)^c$

$\Rightarrow q \in X$

$\therefore Y \subset X$ —————(ii)

From (i) and (ii) X = Y

$(A \cap B)^c = A^c \cup B^c$

1.9 *CARTESIAN PRODUCT OF TWO SETS*

Given two non-empty sets A and B. The Cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B,

i.e., $A \times B = \{(p, q): p \in A, q \in B\}$

If either P or Q is the null set, then $A \times B$ will also be an empty set,

i.e., $A \times B = \varphi$

For Example: if A = {1, 2} and B = {3, 4, 5}, then the Cartesian Product of A and B is A × B = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)}.

Cardinality of Cartesian Product?

The cardinality of Cartesian products of sets A and B will be the total number of ordered pairs in the A × B.

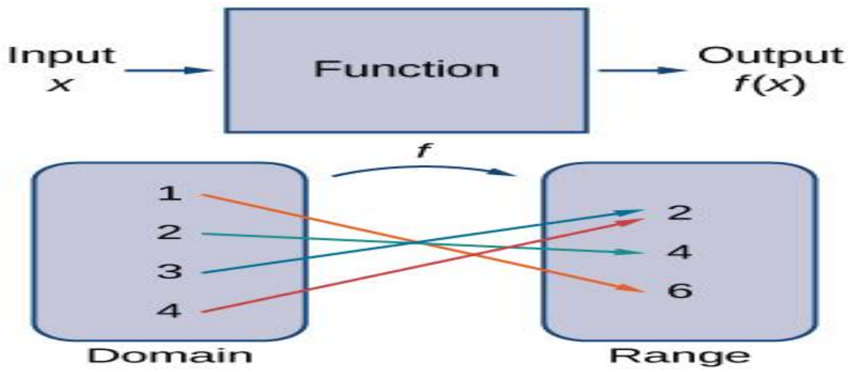
Let p be the number of elements of A and q be the number of elements in B.

So, the number of elements in the Cartesian product of A and B is pq.

i.e. if n(A) = p, n(B) = q, then n(A × B) = pq.

1.10 *FUNCTIONS OR MAPPINGS*

A function can be visualized as an input/output device.



Let A & B be any two non-empty sets. If there exists a rule 'f ' which associates to every element x ∈ A, a unique element y ∈ B, then such rule 'f ' is called a function or mapping from the A to the set A to the set B.

We write f: A ⟶ B read 'f ' is a function from X to Y.

The set **A** is called the domain of f and the set **B** is called the Co-domain of f.

Range of f = f(A) = {f(x): x ∈ A} , clearly f(A) ⊂ B.

1.11 KINDS OF *FUNCTIONS*

(1) Equal Functions. Let A and B be sets and f: A ⟶ B and f: B ⟶ A be functions. We say that f and g are equal and write f = g if f(a) = g(b) for all a ∈ A. If f and g are not equal, we write f ≠ g.

(2) One – One Function (Injective Function). A function f is one-to-one if every element of the range of g corresponds to exactly one element of the domain of f. One-to-one is also written as 1-1. Formally, it is stated as, if f(x) = f(y) implies x=y, then f is one-to-one mapped, or f is 1-1.

Example. Show that f: R→ R defined as $f(a) = 3a^3 – 4$ is one to one function?

Solution: Let $f(a_1) = f(a_2)$ for all $a_1, a_2 \in R$

so $3a_1^3 – 4 = 3a_2^3 – 4$

$a_1^3 = a_2^3$

$a_1^3 – a_2^3 = 0$

$(a_1 – a_2)(a_1 + a_1a_2 + a_2^2) = 0$

$a_1 = a_2$ and $(a_1^2 + a_1a_2 + a_2^2) = 0$

$(a_1^2 + a_1a_2 + a_2^2) = 0$ is not considered because there are no real values of a_1 and a_2.

Therefore, the given function f is one-one.

(3) Onto Function (Surjective Function). Onto function could be explained by considering two sets, Set A and Set B, which consist of elements. If for every element of B, there is at least one or more than one element matching with A, then the function is said to be **onto function** or surjective function.

Note: To show that a function f is an onto function, put y = f(x), and show that we can express x in terms of y for any y ∈ B.

Example 1. Let A = {1, 5, 8, 9) and B {2, 4} And f ={(1, 2), (5, 4), (8, 2), (9, 4)}. Then prove f is a onto function.

Solution: From the question itself we get,

A={1, 5, 8, 9), B = {2, 4} & f={(1, 2), (5, 4), (8, 2), (9, 4)}

So, all the element on B has a domain element on A or we can say element 1 and 8 & 5 and 9 has same range 2 & 4 respectively.

Therefore, f: A → B is a surjective function.

Example 2. How to tell if this function is an onto function? g: R → R defined by $g(x) = 1 + x^2$

Solution: Given the function $g(x) = 1 + x^2$.

For real numbers, we know that $x^2 > 0$. So $1 + x^2 > 1$. $g(x) > 1$ and hence the range of the function is $(1, \infty)$. Whereas, the second set is R (Real Numbers). So the range is not equal to codomain and hence the function is not onto.

Example 3. If f: R → R defined as f(x) = 2x.

Solution. Let y = 2x then $x = \frac{y}{2}$

Thus, for every y R, we have $x = \frac{y}{2} \in R$ such that f(x) = y.

Thus, f is onto.

Example 4. Consider the function f: R → R defined as $f(x)= x^2$.

Solution. Let $y = x^2$ therefore $x = \pm \sqrt{y}$

The square of any real number is non-negative.

It means that $y \geq 0$.

Thus, for $y \leq 0$, we cannot find an element x such that f(x) = y.

Thus, the range of f(x) is the set of non-negative real numbers and the negative real numbers are not in the image of f(x).

As a result, f(x) is not onto.

Note: If you restrict the co-domain to $\mathbb{R}^+ \cup \{0\}$, which is the set of non-negative real numbers, the function becomes onto.

1.12 *INVERSE FUNCTION*

Let f: A → B be a one – one and onto function. Then the function g: B → A which associates to each element b ∈ B the unique element a ∈ A such that f(a) = b is called the inverse function of f. the inverse function of f is denoted by f^{-1}.
Note: every function does not have an inverse. A function f: A → B has inverse iff f is one – one and onto. If f has inverse, then f is said to be invertible and f^{-1}: B → A. also if a ∈ A, then f(a) = b where b ∈ B ⟹ a = $f^{-1}(b)$.

1.13 *COMPOSITE OF FUNCTION*

Let f : A → B and g : B → C be two functions. Then the composition of f and g, denoted by g ∘ f, is defined as the function g ∘ f : A → C given by g ∘ f (x) = g(f (x)), ∀ x ∈ A.

Domain: f(g(x)) is read as f of g of x. In the composition of (f o g) (x) the domain of function f becomes g(x). The domain is a set of all values which go into the function.

Example: If f(x) = 3x+1 and g(x) = x^2, then f of g of x,

$$f(g(x)) = f(x^2) = 3x^2+1.$$

If we reverse the function operation, such as f of f of x,

$g(f(x)) = g(3x+1) = (3x+1)^2$

<u>CHECK YOUR PROGRESS</u>

True or false Questions

Problem 1. function f: R→ R, then f(x) = 2x is injective.

Problem 2. function f: R→ R, then f(x) = 2x+1 is not injective.

Problem 3. The onto function is also called the surjective function.

Problem 4. function f: R→ R, f(x) = |x| is an onto function.

Problem 5. In the surjective function, the range of the function “f” is equal to the codomain.

1.14 *SUMMARY*

1. A set is a well - defined collection of distinct objects.

2. $A \subseteq B$; which means Set A is a subset of Set B.

3. For any two sets A and B, $A \cap B \subset A$ and $A \cap B \subset B$.

4. (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$

5. Let f: A → B and g: B → C be two functions. Then the composition of f and g, denoted by g ∘ f, is defined as the function g ∘ f: A → C given by g ∘ f (x) = g(f (x)), ∀ x ∈ A.

1.15 GLOSSARY

Numbers

letters

Collections of objects

1.16 *REFERENCES*

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1.17 *SUGGESTED READING*

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1.18 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that the function f:ℕ ⟶ ℕ is given by f(x) = x^2 is one – one function.

Q 2. Prove that the function f:ℕ ⟶ ℕ is given by f(x) = x^2 is not onto function.

Q 3. Let A = [– 1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective.

(a) f(x) = $\frac{x}{2}$. (b) f(x) = x^2

Q 4. If f(x) = $3x^2$, then find (f ∘ f)(x).

Q 5. If f (x) = 2x and g(x) = x+1, then find (f ∘ g)(x) if x = 1.

1.19 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. True

CYQ 4. False

CYQ 5. True

TERMINAL QUESTIONS

TQ 3. (a) One - One but not Onto.

(b) Not One - One and not Onto.

TQ 4. 27 X^2

TQ 5. 4

UNIT 2: *REAL NUMBERS*

CONTENTS:

2.1 *INTRODUCTION*

The modern study of set theory was initiated by the German mathematicians Richard Dedekind and Georg Cantor in the 1870s. In particular, Georg Cantor is commonly considered the founder of set theory. The non-formalized systems investigated during this early stage go

2.2 *OBJECTIVES*

After studying this unit, learner will be able to

(i) Real numbers

(ii) Algebraic properties of ℝ

(iii) Order properties of ℝ

(iv) Completeness property of ℝ

(v) Supremum and Infimum property of ℝ

(vi) Archimedean property of ℝ

2.3 *REAL NUMBERS*

A set containing all rational as well as irrational numbers is called the set of all real numbers. The set of real number is denoted by ℝ.

We now describe some fundamental properties of the set ℝ.

1. Algebraic properties of ℝ.

2. Order properties of ℝ.

3. Completeness property of ℝ.

4. Archimedean property of ℝ.

2.4 *ALGEBRAIC PROPERTIES OF* $\mathbb{R}$

Addition and multiplication are defined on the set $\mathbb{R}$ satisfying the following properties:

A1. $a + b \in \mathbb{R}$ for all a, b in $\mathbb{R}$.

A2. $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$.

A3. There exists an element 0 in $\mathbb{R}$ (called zero element) such that

$a + 0 = a$ for all a in $\mathbb{R}$.

A4. For each a in $\mathbb{R}$ there exists an element – a in $\mathbb{R}$ such that

$a + (-a) = 0$.

A5. $a + b = b + a$ for all a, b in $\mathbb{R}$.

M1. $a . b \in \mathbb{R}$ for all a, b in $\mathbb{R}$.

M2. $(a . b) . c = a . (b . c)$ for all a, b, c in $\mathbb{R}$.

M3. There exists an element 1 in $\mathbb{R}$ (called unity element) such that

$a .1 = a$ for all a in $\mathbb{R}$.

M4. For each element $a \neq 0$ in $\mathbb{R}$ there exists an element $\frac{1}{a}$ in $\mathbb{R}$ such that $a . (\frac{1}{a}) = 1$.

M5. $a . b = b . a$ for all a, b in $\mathbb{R}$.

D. $a . (b + c) = a . b + a . c$ for all a, b in $\mathbb{R}$.

Theorem 2.4.1 Let $a, b, c \in \mathbb{R}$. Then

(i) $a + b = a + c$ implies $b = c$ (cancellation law for addition).

(ii) $a \neq 0$ and $a . b = a . c$ implies $b = c$ (cancellation law for multiplication).

Proof. (i) since $a + b = a + c$

$-a \in \mathbb{R}$, since $a \in \mathbb{R}$. Therefore $-a + (a + b) = -a + (a + c)$

Or $(-a + a) + b = (-a + a) + c$, by A2

Or $0 + b = 0 + c$, by A4

Or $b = c$.

(ii) since a.b = a.c

$\frac{1}{a} \in \mathbb{R}$, since $a \neq 0$. Therefore $\left(\frac{1}{a}\right)(a.b) = \left(\frac{1}{a}\right)(a.c)$

Or, $\left(\frac{1}{a}.a\right).b = \left(\frac{1}{a}.a\right).c$, by M2

Or, $1.b = 1.c$, by M4

Or, $b = c$.

Theorem 2.4.2 Let $a, b, c \in \mathbb{R}$. Then $a.b = 0$ implies $a = 0$ or $b = 0$.

Proof. Let $a \neq 0$ then $\frac{1}{a} \in \mathbb{R}$ and $\frac{1}{a}.a = 1$.

$a.b = 0 \Rightarrow \frac{1}{a}.(a.b) = \frac{1}{a}.0 \Rightarrow \left(\frac{1}{a}.a\right).b = 0 \Rightarrow b = 0.$

Therefore $a \neq 0 \Rightarrow b = 0$. Contrapositively, $b \neq 0 \Rightarrow a = 0$.

Therefore either $a = 0$ or $b = 0$.

2.5 *ORDER PROPERTIES OF* $\mathbb{R}$

On the set of $\mathbb{R}$, a linear order relation < is defined by "$a < b$ if $a \in \mathbb{R}$, $b \in \mathbb{R}$ and a is less than b" and it satisfies the following conditions:

O1. If $a, b \in \mathbb{R}$, then exactly one of the following statements holds –

$a < b$, or $a = b$, or $b < a$ (law of **trichotomy**);

O2. $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbb{R}$ (transitivity);

O3. $a < b$ and $a + c < b + c$ for $a, b, c \in \mathbb{R}$;

O4. $a < b$ and $0 < c \Rightarrow ac < bc$ for $a, b, c \in \mathbb{R}$.

Note: 1. The field $\mathbb{R}$ together with the order relation defined on $\mathbb{R}$ satisfying O1 – O4 becomes an ordered field.

Note: 2. $n > 0$ for all $n \in \mathbb{N}$

Note: 3. For all $n \in \mathbb{N}, \frac{1}{n} > 0$.

Theorem 2.5.1 Let $a, b \in \mathbb{R}$. Then $a < b \Rightarrow a < \frac{a+b}{2} < b$.

Proof. $a < b \Rightarrow a + a < a + b$

$\Rightarrow 2a < a + b$

$\Rightarrow \frac{1}{2}.2a < \frac{1}{2}(a + b)$, since $\frac{1}{2} \in \mathbb{R}$ and $\frac{1}{2} > 0$

$$\Rightarrow a < \frac{a+b}{2}.$$

Also $a < b \Rightarrow a + b < b + b$

$$\Rightarrow a + b < 2b$$

$$\Rightarrow \frac{1}{2}(a+b) < \frac{1}{2}.2b, \text{ since } \frac{1}{2} \in \mathbb{R} \text{ and } \frac{1}{2} > 0$$

$$\Rightarrow \frac{a+b}{2} < b.$$

Therefore $a < \frac{a+b}{2} < b$.

Corollary. There is no least positive real number.

If possible, let a be the least positive real number. Then $a > 0$.

$0 < a \Rightarrow 0 < \frac{1}{2}a < a$ by theorem.

This shows that $\frac{1}{2}a$ is a positive real number and $\frac{1}{2}a < a$ indicates that a is not the least positive real number.

It follows that there is no least positive real number.

2.6 *ABSOLUTE VALUE*

Let $a \in \mathbb{R}$. The absolute value of a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a, & if\ a > 0 \\ 0, & if\ a = 0 \\ -a, & if\ a < 0 \end{cases}$$

For example, $|4| = 4, |-10| = 10, |0| = 0$.

It follows from definition that $|a|$ is a non-negative real number. $|a| = 0$ if and only if $a = 0$.

Theorem 2.6.1 Prove that

(i) $|-a| = |a|$ for all $a \in \mathbb{R}$.

(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

(iii) if $a, c \in \mathbb{R}$ $c > 0$, then $|a| < c \Leftrightarrow -c < a < c$.

Proof. (i) Let $a > 0$ then $-a < 0$ and $|-a| = -(-a) = a = |a|$.

Now let $a < 0$ then $-a > 0$ and $|-a| = -a = |a|$.

Let $a = 0$ then $-a = 0$ and $|-a| = 0 = |a|$.

Combining the cases, we have $|-a| = |a|$ for all $a \in \mathbb{R}$.

(ii) Let one or both of a, b be 0. Then $ab = 0$.

In this case $|ab| = 0$ and $|a||b| = 0$. Therefore $|ab| = |a||b|$.

Now let $a > 0$, $b > 0$ then $ab > 0$ and $|ab| = ab, |a| = a, |b| = b$

Therefore $|ab| = |a||b|$.

Now let $a < 0$, $b > 0$ then $ab < 0$ and $|ab| = -ab, |a| = -a, |b| = b$

Therefore $|ab| = |a||b|$.

Now let $a < 0$, $b > 0$ then proof is similar.

Now let $a < 0$, $b < 0$ then $ab > 0$ and $|ab| = -ab, |a| = -a, |b| = -b$

Therefore $|ab| = |a||b|$.

Combining the cases, we have $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

(iii) let $|a| < c$ then if $a \geq 0$, $a < c$ and if $a < 0$, $-a < c$

this implies that

$-c < a$. Therefore $|a| < c \Rightarrow -c < a < c$.

Conversely, let $c > 0$ and $-c < a < c$

Then we have $a < c$, $0 < c$ and $-a < c$.

Combining, we have $|a| < c$.

2.7 *TRIANGLE INEQUALITY*

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Then $-(|a| + |b|) \leq a + b \leq |a| + |b|$.

This implies $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Corollary 1. $|a-b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Proof. Replacing b by $-b$ in triangle inequality we get the inequality.

Corollary 2. $\big||a| - |b|\big| \leq |a-b|$.

Proof. $|a| = |a - b + b| \leq |a-b| + \leq |b|$

Or $|a| - |b| \leq |a-b|$.

Again $|b| = |b - a + a| \leq |b-a| + |a|$

Or $|b| - |a| \leq |b-a| = |a-b|$

So we have $-|a-b| \leq |a| - |b| \leq |a-b|$.

This implies $\big||a| - |b|\big| \leq |a-b|$, since $-c \leq a \leq c \Longrightarrow |a| \leq c$.

Corollary 3. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ then

$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$.

Example. Solve the equation $\left|\frac{x+2}{2x-1}\right| = 3$.

Solution. Given $\left|\frac{x+2}{2x-1}\right| = 3 \Longrightarrow \frac{x+2}{2x-1} = \pm 3$.

If $\frac{x+2}{2x-1} = 3 \Longrightarrow x + 2 = 6x - 3 \Longrightarrow x = 1$

If $\frac{x+2}{2x-1} = -3 \Longrightarrow x + 2 = -6x - 3 \Longrightarrow x = \frac{1}{7}$

Therefore $x = \frac{1}{7}, 1$.

2.8 *COMPLETENESS PROPERTY OF* $\mathbb{R}$

Let P be a Subset of $\mathbb{R}$. A real number u is said to be an upper bound of P if $x \in P \Longrightarrow x \leq u$. A real number l is said to be a lower bound of P of $x \in P \Longrightarrow x \geq l$.

Let P be a subset of ℝ. P is said to be bounded above if P has an upper bound. P is said to be bounded below if P has lower bound.

P is said to be bounded set if P be bounded above as well as bounded below.

Example 1. Let P = $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. P is bounded above, 1 being an upper bound. P is bounded below, 0 being a lower bound.

Example 2. Let P = $\{x \in P: 1 < \text{x} < 2\}$. P is bounded above, 2 being an upper bound. P is bounded below, 1 being a lower bound.

Example 3. Let P = $\emptyset$. Every real number x is an upper bound of the set P. Every real number is a lower bound of the set P. therefore P is bounded set.

2.9 *SUPREMUM AND INFIMUM PROPERTY OF* ℝ

Let P be a subset of ℝ**.** If P is bounded above, then an upper bound of P is said to be the **supremum of P** (or least upper bound of P) if it is less than every other upper bound of P.

If P is bounded below then a lower bound of P is said to be the **infimum** of P (or the greatest lower bound of P) if it is greater than every other lower bound of S.

Note: 1. Every non-empty subset of ℝ that is bounded above has a least upper bound or supremum.

2. Every non-empty subset of ℝ that is bounded below has a greatest lower bound or infimum.

Properties of the supremum and the infimum.

Let P be a non-empty subset of ℝ, which is bounded above. Then supremum of P exists, Let M = sup P. then $M \in \mathbb{R}$ and M satisfies the following conditions:

(i) $x \in \text{P} \implies x \leq \text{M}$, and

(ii) for each $\epsilon > 0$, there exist an element y (depends on ϵ) in P such that $\text{M} - \epsilon < y \leq \text{M}$.

Let P be a non-empty subset of ℝ, which is bounded below. Then infimum of P exists, Let m = inf P. then $m \in \mathbb{R}$ and m satisfies the following conditions:

(i) $x \in$ P $\implies x \geq m$, and

(ii) for each $\epsilon > 0$, there exist an element y (depends on ϵ) in P such that $m < y \leq m + \epsilon$.

Question 1. Prove that the set of natural number (ℕ) is not bounded above.

Solution: The set ℕ is a non – empty subset of ℝ, since 1 ∈ ℕ. Let ℕ be bounded above. Then ℕ being a non – empty subset of ℝ, sup ℕ exist by supremum property of ℝ. Let u = sup ℕ. Then

(i) $x \in \mathbb{N} \implies x \leq$ u, and

(ii) for each $\epsilon > 0$, there exist an element y (depends on ϵ) in ℕ such that $u - \epsilon < y \leq$ u.

Let us choose $\epsilon = 1$. Then there exists an element k in ℕ such that

$u - 1 < k \leq$ u. then $u - 1 < k \implies u < k + 1$.

Since k is a natural number, $k + 1$ is also a natural number. $k + 1 > u$ implies that u is not an upper bound of the set ℕ. Thus we arrive a contradiction. So aur assumption was wrong. Hence the set of natural number (ℕ) is not bounded above.

Question 2. Let P be a non – empty subset of ℝ which is bounded above and T = $\{-x : x \in P\}$. Prove that the set T is bounded below and $infT = -supP$.

Solution: since P is bounded above therefore $supP$ exists. Let u = $sup\ P$. Then $x \in P \implies x \leq u$. let $y \in T$ then $-y \in P$ and therefore $-y \leq u,$ i.e. $y \geq -u$. This implies that $-u$ is a lower bound of T. therefore the set T is bounded below.

Let us choose $\epsilon > 0$. Since $u = supP$, there exists an element p in P such that $u - \epsilon < p \leq u$. Therefore $-u \leq -p < -u + \epsilon$.   **……(i)**

Let $q = -p$. Then $q \in T$.

(i) shows that for a pre-assigned positive ϵ there exists an element q in T such that $-u \leq q < -u + \epsilon$.

This proves that $-u = infT$. Therefore $infT = -\ supP$.

Question 3. Let P be a non – empty bounded subset of ℝ with $supP = M$ and $infP = m$. Prove that the set T = $\{|x - y|: x \in P, y \in P\}$ is bounded above and $supT = M - m$.

Solution: $x \in P \Rightarrow m \leq x \leq M, y \in P \Rightarrow m \leq y \leq M$.

Therefore $m - M \leq x - y \leq M - m$ i.e. $|x - y| \leq M - m$.

This shows that T is bounded above, $M - m$ being an upper bound.

Let $a \in P$. Then $|a - a| \in T$ showing that T is non-empty. By the supremum property of ℝ, $supT$ exists.

We prove that no real number less than $M - m$ is an upper bound of T.

if possible, let $p < M - m$ be an upper bound of T.

Let $(M - m) - p = 2\epsilon$. Then $\epsilon > 0$ and $p + \epsilon = M - m - \epsilon$.

Since $supP = M$, there exist an element $x \in P$ such that

$$M - \frac{\epsilon}{2} < x \leq M$$

Since $inmP = M$, there exist an element $y \in P$ such that

$$m < x \leq m + \frac{\epsilon}{2}.$$

Now $x - y > M - m - \epsilon$ i.e. $x - y < p + \epsilon$.

This shows that p is not an upper bound of T.

Therefore, no real number less than $M - m$ is an upper bound of T.

Hence $supT = M - m$.

Note: Let A, B be bounded subset of ℝ such that $x \in A, y \in B \Rightarrow x \leq y$. Then $supA \leq infB$.

2.10 *ARCHIMEDEAN PROPERTY OF* ℝ

If $x, y \in \mathbb{R}$ and $x > 0, y > 0$, then there exists a natural number n such that $ny > x$.

Proof: If possible, let there exist no natural number n for which $ny > x$.

Then for every natural number $k, ky \leq x$.

Thus, the set S = $\{ky: k \in \mathbb{N}\}$ is bounded above, x being an upper bound.

S is non-empty because $y \in S$.

By the supremum property of $\mathbb{R}$, $supS$ exists. Let $supS$ = b. then $ky \leq b$ for all $k \in \mathbb{N}$.

$b - y < b$ since $y > 0$. This shows that $b - y$ is not an upper bound of S and therefore there exists a natural number p such that

$b - y < py \leq b$. This implies $(p + 1)y > b$ …….. (i)

But $p \in \mathbb{N} \Longrightarrow p + 1 \in \mathbb{N}$ and therefore $(p + 1)y \in S$.

(i) shows that b is not the supremum of S, a contradiction.

Therefore, our assumption is wrong and the existence of a natural number n satisfying $ny > x$ is proved.

Note: (1) If $x \in \mathbb{R}$, then there exists a natural number n such that $n > x$.

Case (i). If $x > 0$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$ and hence existence is proved.

Case (ii). If $x \leq 0$. Then $n = 1$.

(2). If $x \in \mathbb{R}$ and $x > 0$, then there exists a natural number n such that $0 < \frac{1}{n} < x$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $nx > 1$.

Since n is a natural number, $n > 0$ and therefore $\frac{1}{n} > 0$ and also $\frac{1}{n} < x$.

Therefore, we have $0 < \frac{1}{n} < x$.

(3). If $x \in \mathbb{R}$ and $x > 0$, then there exists a natural number m such that $m - 1 \leq x < m$.

Taking $y = 1$ and $x > 0$, by Archimedean property of $\mathbb{R}$ there exist a natural number n such that $n.1 > x$, i.e. $n > x$.

Let $S = \{k \in \mathbb{N}: k > x\}$. Then S is non-empty subset of $\mathbb{N}$, since $n \in S$. By well ordering property of the set $\mathbb{N}$, S has a least element, say m. Since $m \in S, m > x$.

As m is least element of S. $m - 1 \ngtr x$. i.e. $m - 1 \leq x$.

Hence $m - 1 \leq x < m$.

2.11 *EXTENDED SET OF REAL NUMBER*

It is often convenient to extend the set $\mathbb{R}$ by the addition of two elements ∞ and $-\infty$. This enlarged set is called the extended set of real numbers and is often denoted by $\mathbb{R}^*$.

In the extended set $\mathbb{R}^*$ we define –

For all $x \in \mathbb{R}$, $x + \infty = \infty + x = \infty$

$$x + (-\infty) = (-\infty) + x = -\infty.$$

For all $x > 0$, $x.\infty = \infty.x = \infty$ and

$$x.(-\infty) = (-\infty).x = -\infty.$$

For all $x < 0$, $x.\infty = \infty.x = -\infty$ and

$$x.(-\infty) = -\infty.x = \infty.$$

Now $\infty + \infty = \infty$, $(-\infty) + (-\infty) = -\infty$

$$\infty.\infty = \infty, (-\infty).\infty = -\infty, (-\infty).(-\infty) = \infty.$$

And $\infty + (-\infty)$, $(-\infty) + \infty$, $0.\infty$, $\infty.0$, $0 - \infty$, $-\infty.0$ are not defined.

Now, if S be a non-empty subset of $\mathbb{R}$ having no upper bound, we define $supS = \infty$. If S be a non-empty subset of $\mathbb{R}$ having no longer bound, we define $infS = -\infty$.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The set of real number is not bounded above.

Problem 2. The set of natural number is bounded below.

Problem 3. Let $a, b \in \mathbb{R}$. Then $a < b \Rightarrow a < \frac{a+b}{2} < b$.

Problem 4. $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

Problem 5. The supremum of the interval (1, 3) is 4.

2.12 *SUMMARY*

1. A set containing all rational as well as irrational numbers is called the set of all real numbers. The set of real number is denoted by $\mathbb{R}$.

2. There is no least positive real number.

3. Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound or supremum.

4. If $x, y \in \mathbb{R}$ and $x > 0, y > 0$, then there exists a natural number n such that $ny > x$ is called Archimedean property of $\mathbb{R}$.

5. The set of natural number ($\mathbb{N}$) is not bounded above.

2.13 GLOSSARY

Numbers

Sets

Intervals

Modulus

2.14 *REFERENCES*

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2.15 *SUGGESTED READING*

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2.16 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that the set of natural number ℕ is not bounded above.

Q 2. Prove that the set {1, 2, 3, …, 10} is not bounded above.

Q 3. Find the Supremum and infimum of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Q 4. Prove that the set T = $\{|x - y| : x \in P, y \in P\}$ is bounded above.

Q 5. Solve the equation $\left|\frac{x+3}{2x-6}\right| \leq 1$.

2.17 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. True

CYQ 5. False

TERMINAL QUESTIONS

TQ 3. 1, 0

TQ 5. Solution set is $\{x \in \mathbb{R} : x \geq 9\} \cup \{x \in \mathbb{R} : x \leq 1\}$.

UNIT 3: LIMIT POINT, OPEN SET AND CLOSED SETS

CONTENTS:

3.1 *INTRODUCTION*

A point is called a limit point of a set in the Euclidean plane if there is no minimum distance from that point to members of the set; for example, the set of all numbers less than 1 has 1 as a limit point. A set is not connected if it can be divided into two parts such that a point of one part is never a limit point of the other part. The set is connected if it cannot be so divided. For example, if a point is removed from an arc, any remaining points on either side of the break will not be limit points of the other side, so the resulting set is disconnected. If a single point is removed from a simple closed curve such as a circle or polygon, on the other hand, it remains connected; if any two points are removed, it becomes disconnected. A figure-eight curve does not have this property because one point can be removed from each loop and the figure will remain connected.

3.2 *OBJECTIVES*

After studying this unit, learner will be able to

(i) Neighbourhood

(ii) Interior point

(iii) Open set

(iv) Limit point

3.3 *NEIGHBOURHOOD*

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be neighbourhood of c if there exist an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly an open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.

A closed bounded interval containing the point c may not be a neighbourhood of c.

For example: $1 \in [1,5]$ is not a neighbourhood of 1.

Theorem 3.3.1. Let $c \in \mathbb{R}$. The union of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}$, $S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exists open interval (a_1, b_1), (a_2, b_2) such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1$, $a_2 < b_1$ and $a_1 < b_2$, $a_2 < b_2$. Let a_3 = min$\{a_1, a_2\}$, b_3 = max$\{b_1, b_2\}$. Then $(a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$ and $c \in (a_2, b_2)$.

$(a_1, b_1) \subset S_1 \cup S_2$ and $(a_2, b_2) \subset S_1 \cup S_2$

$\Longrightarrow (a_3, b_3) = (a_1, b_1) \cup (a_2, b_2) \subset S_1 \cup S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cup S_2$.

This prove that $S_1 \cup S_2$ is a neighbourhood of c.

Note: The union of finite number of neighbourhoods of c is a neighbourhood of c.

Theorem 3.3.2. Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}$, $S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exists open interval (a_1, b_1), (a_2, b_2) such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1$, $a_2 < b_1$ and $a_1 < b_2$, $a_2 < b_2$. Let a_3 = max$\{a_1, a_2\}$, b_3 = min$\{b_1, b_2\}$. Then $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$ and $c \in (a_2, b_2)$.

$(a_3, b_3) = (a_1, b_1) \cap (a_2, b_2) \subset (a_1, b_1) \subset S_1$ and $(a_3, b_3) = (a_1, b_1) \cap (a_2, b_2) \subset (a_2, b_2) \subset S_2$

$\Longrightarrow (a_3, b_3) \subset S_1 \cap S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cap S_2$.

This prove that $S_1 \cap S_2$ is a neighbourhood of c.

Note: The intersection of finite number of neighbourhoods of c is a neighbourhood of c.

Note: The intersection of infinite number of neighbourhoods of c may or may not be neighbourhood of c.

Example: for every $n \in \mathbb{N}, \left(-\frac{1}{n}, \frac{1}{n}\right)$ is neighbourhood of 0.

$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$. This is not a neighbourhood of 0.

3.4 *INTERIOR POINT*

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an interior point of S if there exists a neighbourhood $N(x)$ of x such that

$N(x) \subset S$.

The set of all interior points of S is said to be interior of S and denoted by **int S** or by S^0.

From definition it follows that $S^0 \subset S$ for any set $S \subset \mathbb{R}$.

Examples 1. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

Let $x \in S$, every neighbourhood of x contain some points not in S. So x can not be an interior point of S. Therefore, int S = $\emptyset$.

Examples 2. Let $S = \mathbb{N}$.

Let $x \in S$, every neighbourhood of x contain some points not in S. So x can not be an interior point of S. Therefore, int S = $\emptyset$.

Examples 3. Let $S = \mathbb{Q}$.

Let $x \in S$, every neighbourhood of x contain some points not in S. So x can not be an interior point of S. Therefore, $S^0 = \emptyset$.

Examples 4. Let $S = \{x \in \mathbb{R}: 1 < x < 3\}$. Each point of S is an interior point of S. so interior of $S = S$.

Examples 5. Let $S = \mathbb{R}$. Each point of S is an interior point of S. so interior of $S^0 = S$.

3.5 *OPEN SET*

Let S be a subset of $\mathbb{R}$. S is said to be open set if each point of S is an interior point of S.

Examples 1. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. no point of S is an interior point of S. therefore, S is not an open set.

Examples 2. Let $S = \mathbb{Z}$.

no point of S is an interior point of S. therefore, S is not an open set.

Examples 3. Let $S = \mathbb{Q}$.

no point of S is an interior point of S. therefore, S is not an open set.

Examples 4. Let $S = \{ x \in \mathbb{R}: 1 < x < 3\}$. Each point of S is an interior point of S. therefore, S is an open set.

Examples 5. Let $S = \mathbb{R}$. Each point of S is an interior point of S. therefore, S is an open set.

Theorem 3.5.1. Let $S \subset \mathbb{R}$. then S is an open set if and only if $S = int\ S$.

Proof. Observe in general that $int\ S. \subseteq S$. Now suppose that S is open. Then for every $x \in S$, there is $\epsilon > 0$ so that $N(x, \epsilon) \subseteq S$ and this just says $x \in int\ S$ and since x was arbitrary we have shown that $S \subseteq int\ S$ or equivalently $S = int\ S$.

Conversely suppose the $int\ S = S$. The if $x \in$ S then x is an interior point and there is an $\epsilon > 0$ so that $N(x, \epsilon) \subseteq S$. But this says S is open.

Theorem 3.5.2. The union of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_1 be two open Sets in $\mathbb{R}$.

Let $x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1$. Since G_1 is an open set and $x \in G_1$, x is an interior point of G_1. Therefore, there exists a neighbourhood $N(x)$ of x such that

$N(x) \subset G_1 \Rightarrow N(x) \subset G_1 \cup G_2$.

This shows that x is an interior point of $G_1 \cup G_2$.

Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is an open set.

If however, $x \in G_2$, we can prove in similar manner that $G_1 \cup G_2$ is an open set. This completes the proof.

Theorem 3.5.3. The intersection of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_1 be two open Sets in $\mathbb{R}$.

Case 1. $G_1 \cap G_2 = \emptyset$. Since $\emptyset$ is an open set, $G_1 \cap G_2$ is an open set.

Case 2. $G_1 \cap G_2 \neq \emptyset$. Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$.

Since G_1 is an open set and $x \in G_1$, x is an interior point of G_1.

Hence there exists a positive δ_1 such that the neighbourhood

$N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2$, x is an interior point of G_2.

Hence there exists a positive δ_2 such that the neighbourhood

$N(x, \delta_2) \subset G_1$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then δ.

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$ and $N(x, \delta) \subset N(x, \delta_2) \subset G_2$.

Consequently, $N(x, \delta) \subset G_1 \cap G_2$.

This shows that x is an interior point of $G_1 \cap G_2$. Since x is arbitrary, $G_1 \cap G_2$ is an open set and this completes the proof.

Theorem 3.5.4. The union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set.

Proof. Let $\{G_\alpha : \alpha \in \wedge\}$, $\wedge$ being the index set, be an arbitrary collection of open sets in $\mathbb{R}$. Let G = $\bigcup_{\alpha \in \wedge} G_\alpha$.

$x \in$ G. then x belongs to at least one open set of the collection, say $G_\lambda, (\lambda \in \wedge)$.

Since G_λ is an open set and $x \in G_\lambda$, x is an interior point of G_λ.

Therefore, there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_\lambda$. $N(x) \subset G_\lambda \Longrightarrow N(x) \subset$ G.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set and this completes the proof.

Note: The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Let us consider the sets G_i Where

$G_1 = \{x \in \mathbb{R}: -1 < x < 1\}$

$G_2 = \left\{x \in \mathbb{R}: -\frac{1}{2} < x < \frac{1}{2}\right\}$

... … … … …

$G_n = \left\{x \in \mathbb{R}: -\frac{1}{n} < x < \frac{1}{n}\right\}$

… … … … …

Each G_i is an open set. $\bigcap_{i=1}^{\infty} G_i = \{0\}$. This is not an open set.

Let us consider the sets G_i Where

$G_1 = \{x \in \mathbb{R}: -1 < x < 1\}$

$G_2 = \{x \in \mathbb{R}: -2 < x < 2\}$

... … … … …

$G_n = \{x \in \mathbb{R}: -n < x < n\}$

… … … … …

Each G_i is an open set. $\bigcap_{i=1}^{\infty} G_i = G_1$. This is an open set.

From these two examples we conclude that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Note: Every open interval is open set.

Theorem 3.5.5. Let S is a subset of $\mathbb{R}$ then $int\ S$ is an open set.

Proof.

Case 1. $int\ S = \emptyset$. Since $\emptyset$ is an open set, $int\ S$ is an open set.

Case 2. $int\ S \neq \emptyset$. let $x \in\ int\ S$. then x is an interior point of S.

therefore, there exist a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

let $x \in N(x)$. Then $N(x)$ is neighbourhood of y also and since

$N(x) \subset S$, y is an interior point of S.

Thus $y \in N(x)$. Then $N(x) \Longrightarrow y \in int\ S$. therefore $N(x)\ ) \subset int\ S$.

This shows that x is an interior point of $int\ S$.

Thus $x \in int\ S \Longrightarrow x$ is an interior point of $int\ S$.

Therefore $int\ S$ is an open set. This completes the proof.

3.6 *LIMIT POINT*

A point $x \in\ \mathbb{R}$ is said to be limit point of a subset of S of $\mathbb{R}$ if every neighbourhood of x has a point of S other that x.

In symbols, a point $x \in\ \mathbb{R}$ is said to be limit point of a subset S of $\mathbb{R}$ if for each neighbourhood N of x,

$$(N \cap S) - \{p\} \neq \emptyset.$$

Note: 1. Limit point of set is also called a limiting point or a cluster point or a condensation point or an accumulation point of the set.

Note: 2. A finite set has no limit point.

Note: 3. A limit point of S may or may not belongs to S.

Note: 4. ISOLATED POINT.

A point $x \in S$ is called an isolated point of S if x is not a limit point of S.

Note:5. Set of all limit points of S is called the **derived set** of S and is denoted by S'. thus

$$S' = \{x: x\ is\ a\ limit\ point\ of\ S\}.$$

Example: 1. Prove that 0 is the limit point of the set

$$S = \left\{\frac{1}{n} : n \in\ \mathbb{N}\right\}.$$

Solution: for each $\epsilon > 0$, $(-\epsilon, \epsilon)$ is neighbourhood of 0.

By Archimedean property of reals, for each $\epsilon > 0$, $\exists\, n \in \mathbb{N}$

such that $n > \frac{1}{\epsilon}$

$\Rightarrow \quad \frac{1}{n} < \epsilon \quad \Rightarrow \quad -\epsilon < 0 < \frac{1}{n} < \epsilon \quad \Rightarrow \quad \frac{1}{n} \in (-\epsilon, \epsilon)$

Thus, every neighbourhood of 0 contains a point of S, namely $\frac{1}{n}$.

$\Rightarrow$ 0 is limit point of S.

Example: 2. Find the derived set of each of the following:

(i) $(1, \infty)$ (ii) $(-\infty, -1)$

Solution:1. Let x be any real number.

If $x < 1$, then for $0 < \epsilon < 1 - x$, $(x - \epsilon, x + \epsilon) \cap (1, \infty) = \emptyset$.

$\Rightarrow$ Any real number < 1 is not a limit point of $(1, \infty)$.

If $x \in [1, \infty)$, then for every $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ contains infinitely many points of $(1, \infty)$ to the right of 1.

$\Rightarrow$ Every element of $[1, \infty)$ is a limit point of $(1, \infty)$.

$\therefore \quad (1, \infty)' = [1, \infty)$.

(ii) Please try yourself. Answer: $(-\infty, -1)$

Example: 3. Find the derived set of each of the following:

$$S = \left\{\frac{1 + (-1)^n}{n} : n \in \mathbb{N}\right\}.$$

Solution: Let $S = \left\{\frac{1 + (-1)^n}{n} : n \in \mathbb{N}\right\}$

When n is odd, $\frac{1 + (-1)^n}{n} = \frac{1 - 1}{n} = 0$

When n is even, $\frac{1 + (-1)^n}{n} = \frac{1 + 1}{n} = \frac{2}{n}$

Therefore, $S = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

$\Rightarrow \quad S' = \{0\}$.

Example: 4. Give one example of each of the following:

(i) an infinite set having no limit point.

(ii) an infinite set having one limit point.

(iii) a set having two limit point.

(iv) a set having an infinite number of limit points.

(v) a set every point of which is a limit point.

(vi) a set with only $\sqrt{2}$ is a limit point.

Solution:

(i) The set of all natural numbers is an infinite set having no limit point set.

(ii) The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is an infinite set having only one limit point, namely 0.

(iii) The set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ has two limit points, namely 0 and 1.

(iv) The sets $\mathbb{Q}$, $\mathbb{R}$ and $(1, 2)$ have an infinite number of limit points.

(v) Every point of a closed interval [1, 2] is a limit point.

(vi) The set $\left\{\sqrt{2} + \frac{1}{n} : n \in \mathbb{N}\right\}$ has only $\sqrt{2}$ as a limit point.

Example: 5. Give one example of each of the following:

(i) an unbounded set having limit points.

(ii) a bounded set having limit points.

(iii) an unbounded set having no limit point.

Solution:

(i) The set of all rational numbers is an unbounded set and derived set of $\mathbb{Q}$ is $\mathbb{R}$.

(ii) The set [1, 2] is bounded and $[1,2]' = [1,2]$.

(iii) The set $\mathbb{Z}$ is unbounded and $\mathbb{Z}' = \emptyset$.

Example: 6. Prove that a finite set has no limit point.

Solution: Let $S = \{p_1, p_2, \dots \dots, p_n\}$ be a finite subset of $\mathbb{R}$. Let p be any real number.

If we choose $\epsilon = \min.\{|p - p_1|, |p - p_2|, \dots \dots, |p - p_n|\}$, then $(p - \epsilon, p + \epsilon)$ is a neighbourhood of p which contains no element of S, i.e.

$(p - \epsilon, p + \epsilon) \cap S = \emptyset$

Therefore, p is not a limit point of S.

Since p is arbitrary, therefore S has no limit point.

Some important theorem:

Theorem 1. (Bolzano – Weierstrass Theorem)

Every infinite and bounded subset of $\mathbb{R}$ has a limit point.

Proof: Let S be an infinite bounded subset of $\mathbb{R}$.

(i) if S is bounded $\Longrightarrow \exists$ real number k and K such that $k \leq s \leq K \; \forall$ $s \in S$.

(ii) Let a set T be defined as follows

$$\text{T} = \{t : t > \text{finitely many elements of } S\}$$

(iii) To prove that $\text{T} \neq \emptyset$.

$k \leq s \; \forall \; s \in S \Longrightarrow$ k is greater than no elements of S

$\Longrightarrow k \in \text{T} \quad \Longrightarrow \text{T} \neq \emptyset$.

(iv) To prove that T is bounded above.

For any $\epsilon > 0$, $K + \epsilon > K \geq s \quad \forall \; s \in S \quad \Longrightarrow K + \epsilon \notin \text{T}, K \notin \text{T}$

$\Longrightarrow \quad \forall \; t \in T, \quad t < K \quad \Longrightarrow$ T is bounded above.

$\because$ T is a non-empty bounded subset of $\mathbb{R}$

$\therefore$ T has a least upper bound say u.

(v) To prove that u is a limit point of S.

Let $(u - \epsilon, u + \epsilon)$ be any neighbourhood of u.

u is least upper bound of T $\Longrightarrow \exists$ some $t \in$ T such that $t > u - \epsilon, \epsilon > 0$.

Now $t \in \text{T} \Longrightarrow t >$ finitely many elements of S

$\Longrightarrow u - \epsilon >$ finitely many elements of S

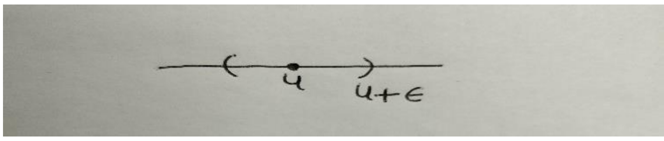
$\Longrightarrow$ finitely many elements of S lie to the left of $u - \epsilon$

$\Longrightarrow$ infinitely many elements of S lie to right of $u - \epsilon$ ……. (1)

Also $u = l.u.b \; of \; T \quad \Longrightarrow u + \epsilon \notin \text{T}$

$\Longrightarrow u + \epsilon >$ infinitely many elements of S

$\Longrightarrow$ infinitely many elements of S lie to left of $u + \epsilon$ ……. (2)



Combining (1) and (2), $(u-\epsilon, u+\epsilon)$ has infinitely many elements of S. but $(u-\epsilon, u+\epsilon)$ is any neighbourhood of u.

Therefore, every neighbourhood of u has infinitely many elements of S.

Hence u is a limit point of S.

Theorem 2. If S is an infinite bounded above subset of ℝ and $u = l.u.b\ of$ S such that $u \notin$ S, then $u \in S'$, i.e. u is limit point of S.

Proof: For each $\epsilon > 0$, $(u-\epsilon, u+\epsilon)$ is neighbourhood of u.

Since $u = l.u.b.$ of S, $u-\epsilon$ is not an upper bound od S.

Therefore ∃ some $x \in$ S such that $x > u-\epsilon$

Also $x < u < u+\epsilon$

Therefore $u-\epsilon < x < u+\epsilon$

$\Rightarrow (u-\epsilon, u+\epsilon) \cap$ S $-\{u\}$ contains at least one point x of S.

$\Rightarrow u$ is a limit point of S.

Note: The derived set of an infinite bounded subset of ℝ is bounded.

3.7 *CLOSED SET*

Let A be a subset of ℝ then A is said to be closed set if its complement A^c = ℝ $- A$ is an open set.

i.e. a set is closed if its complements is open.

For example. (i) $\mathbb{R}^c$ = ℝ − ℝ = ∅ which is open ⟹ ℝ is closed.

(ii) $\emptyset^c$ = ℝ − ∅ = ℝ which is open ⟹ ∅ is closed.

Note: ℝ **and** ∅ are only two sets which are both open and closed.

(iii) $[a,b]^c = (-\infty, a) \cup (b, \infty)$ being the union of two open sets is itself open ⟹ every closed interval [a, b] is a closed set.

Question. Prove that the set ℤ of all integer is a closed set.

Solution. ℤ will be the closed set if $\mathbb{Z}^c$ is open an open set.

Let $x \in \mathbb{Z}^c$ then $x \notin \mathbb{Z}$ i.e. x is not an integer.

If n is an integer nearest to x, then $\exists\ \epsilon = \frac{|x-n|}{2} > 0$ such that

$(x-\epsilon, x+\epsilon)$ does not contain any integer

Therefore, $(x-\epsilon, x+\epsilon) \subset \mathbb{Z}^c \implies \mathbb{Z}^c$ is neighbourhood of x.

But x is any point of $\mathbb{Z}^c$.

Therefore, $\mathbb{Z}^c$ is neighbourhood of each of its points.

$\implies \mathbb{Z}^c$ is open $\implies \mathbb{Z}$ is closed.

Similarly, the set $\mathbb{N}$ of all natural numbers is a closed set.

Question. The set of all rational number is not a closed set.

Solution. Let x be any element of $\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$, the set of irrational numbers.

For every $\epsilon > 0$, $(x-\epsilon, x+\epsilon)$ contains infinitely many rational numbers.

Therefore, $\exists$ no open interval such that $x \in I \subset \mathbb{Q}^c$

$\implies \mathbb{Q}^c$ is not a neighbourhood of x

$\implies \mathbb{Q}^c$ is not an open set.

$\implies \mathbb{Q}$ is not a closed set.

Theorem 3.7.1. The union of two closed sets is a closed set.

Proof: Let A and B be two set closed seats.

$\implies A^c$ and B^c are open sets.

$\implies A^c \cap B^c$ is an open set.

$\implies (A \cup B)^c$ is an open set.

[since $A^c \cap B^c = (A \cup B)^c$ De Morgan's Law]

$\implies A \cup B$ is an open set.

Theorem 3.7.2. The union of a finite number of closed sets is a closed set.

Proof: Let $A_1, A_2, \ldots, A_n$ be n closed sets and S = $\bigcup_{i=1}^{n} A_i$

$\implies {A^c}_1, {A^c}_2, \ldots, {A^c}_n$ are n open sets.

$\implies \bigcup_{i=1}^{n} {A^c}_i$ is an open set.

[since the intersection of a finite collection of open sets is an open set]

$\implies (\bigcup_{i=1}^{n} A_i)^c$ is an open set.

$\implies \bigcup_{i=1}^{n} A_i$ is a closed set.

Theorem 3.7.3. The union of an infinite family of closed sets need not be a closed set.

Proof: Let $A_n = \left[\frac{1}{n}, 1\right]$ $\forall\, n \in \mathbb{N}$, then $\{A_n\}_{n \in \mathbb{N}}$ is an infinite family of closed sets.

$A_1 = \{1\},\ A_2 = \left[\frac{1}{2}, 1\right], A_3 = \left[\frac{1}{3}, 1\right], \ldots$

Therefore, $\bigcup_{n=1}^{\infty} A_n = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \ldots = (0, 1]$ which is not closed.

Theorem 3.7.4. The intersection of two closed sets is a closed set.

Proof: Let A and B be two set closed seats.

$\Rightarrow A^c$ and B^c are open sets.

$\Rightarrow A^c \cup B^c$ is an open set.

$\Rightarrow (A \cap B)^c$ is an open set.

[since $A^c \cup B^c = (A \cap B)^c$ De Morgan's Law]

$\Rightarrow A \cup B$ is closed set.

Note: The intersection of an arbitrary family of closed set is closed.

Note: A set S is said to be **perfect** if $S = S'$.

Example: The set of all real number is perfect but the set of all rational number is not a perfect set.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The set $\{x: 0 \leq x \leq 1\}$ $is\ a\ closed\ set.$

Problem 2. The set of natural number is open set.

Problem 3. The set of rational number is closed set.

Problem 4. The set $\{x: 0 \leq x \leq 1\}$ is neighbourhood of each of its point.

Problem 5. The set $\{x: 0 \leq x \leq 1\}$ has limit point 0.

3.8 *SUMMARY*

1. Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be neighbourhood of c if there exist an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly an open bounded interval containing the point c is a neighbourhood of c.

2. Let S be a subset of $\mathbb{R}$. S is said to be open set if each point of S is an interior point of S.

3. A set is closed if its complements is open.

4. Bolzano – Weierstrass Theorem:

Every infinite and bounded subset of $\mathbb{R}$ has a limit point.

3.9 *GLOSSARY*

Numbers

Intervals

Sets

Functions

3.10 *REFERENCES*

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3.11 *SUGGESTED READING*

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3.12 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that the set of natural number $\mathbb{N}$ is closed set.

Q 2. Prove that the set of rational number $\mathbb{Q}$ is not closed set.

Q 3. The set $\{x: 4 \leq x < 8\}$ is neither open nor closed.

Q 4. Prove that intersection of an arbitrary family of closed sets is a closed set.

Q 5. With the help of examples, prove that intersection of an infinite family of open sets may or may not be an open set.

3.13 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. False

CYQ 4. True

CYQ 5. True

UNIT 4: REAL SEQUENCES

CONTENTS:

4.1 *INTRODUCTION*

So far, we have introduced sets as well as the number systems that we will use in this text. Next, we will study sequences of numbers. Sequences are, basically, countably many numbers arranged in an order that may or may not exhibit certain patterns. Here is the formal definition of a sequence:

4.2 *OBJECTIVES*

After studying this unit, learner will be able to

(i) Neighborhoods

(ii) Interior point

(iii) Open set

(iv) Limit point

4.3 *SEQUENCE*

A real sequence is a function whose domain is the set $\mathbb{N}$ of all natural numbers and range is a subset of the set $\mathbb{R}$ of real numbers.
Symbolically $f: \mathbb{N} \longrightarrow \mathbb{R}$, $\{x_n\}, \{a_n\}, \langle b_n \rangle$ etc.
Example: $\langle \frac{1}{n} \rangle, \langle 2^n \rangle, (-1)^n$ etc.

Range of sequence: The set of all distinct terms of a sequence is called its range.

Note: In a sequence $\{x_n\}$, since $n \in \mathbb{N}$ and $\mathbb{N}$ is an infinite set, the number of terms of a sequence is always infinite. The range of the sequence may be a finite set.
Example: if $x_n = (-1)^n$, then $\{x_n\} = \{-1, \ 1, -1, \ 1, \dots\}$, the range of this sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

Type of Sequence:

1. Constant sequence: A sequence $\{x_n\}$ defined by $x_n = c \in \mathbb{R}\ \forall$ $n \in \mathbb{N}$ is called a constant sequence.
Thus $\{x_n\} = \{c, c, c \dots \dots ..\}$ is a constant sequence with range $= \{c\}$, a singletion.

2. Bounded above sequence: A sequence $\{x_n\}$ is said to be bounded above if $\exists$ a real number K such that

$$x_n \leq K \ \ \forall\, n \in \mathbb{N}$$

i.e. if range of the sequence is bounded above.

3. Bounded Below sequence: A sequence $\{x_n\}$ is said to be bounded below if $\exists$ a real number k such that

$$x_n \geq k \ \ \forall\, n \in \mathbb{N}$$

i.e. if range of the sequence is bounded below.

4. Bounded sequence: A sequence $\{x_n\}$ is said to be bounded if it is bounded above as well as bounded below.
Thus, a sequence $\{x_n\}$ is said to be bounded if $\exists$ a real numbers k and K, such that

$$k \leq x_n \leq K \ \ \forall\, n \in \mathbb{N}$$

i.e. if range of the sequence is bounded.

5. Unbounded sequence: A sequence $\{x_n\}$ is said to be unbonded if it is not bounded.

6. Unbounded above sequence: A sequence $\{x_n\}$ is said to be unbounded above if it not bounded above.
i.e. for every real number K, $\exists\ m \in \mathbb{N}$ such that $x_m > K$.

7. Unbounded below sequence: A sequence $\{x_n\}$ is said to be unbounded below if it not bounded below.
i.e. for every real number k, $\exists\ m \in \mathbb{N}$ such that $x_m < k$.

Example (i) The sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ is bounded,
since $0 < x_n \leq 1$.
(ii) The sequence $\{x_n\}$ defined by $x_n = n$ is bounded below,
because $x_n \geq 1\ \forall\, n \in \mathbb{N}$.
(iii) The sequence $\{x_n\}$ defined by $x_n = (-1)^n$ is bounded,
since $1 \leq x_n \leq 1$.
(iv) every constant sequence is bounded.
(v) The sequence $\{x_n\}$ defined by $x_n = (-1)^n.n$ is neither bounded below nor bounded above.

Theorem 4.3.1. A sequence $\{x_n\}$ is bounded iff $\exists$ a positive real number M such that $|x_n| \leq M \ \forall \ n \in \mathbb{N}$.

Proof: Necessary part

Let $\{x_n\}$ be bounded. Then $\exists$ two real number h and k such that

$$h \leq x_n \leq k \ \forall \ n \in \mathbb{N} \qquad \ldots\ldots. (1)$$

Let M =Max.$\{|h|, |k|\}$, then $|h| \leq M$ and $|k| \leq M$

$\Rightarrow \quad -M \leq h \leq M$ and $-M \leq k \leq M \qquad \ldots\ldots (2)$

From (1) and (2), we have $\quad -M \leq h \leq x_n \leq k \leq M \ \ \forall \ n \in \mathbb{N}$

$\Rightarrow \quad -M \leq x_n \leq M \ \ \forall \ n \in \mathbb{N}$

$\Rightarrow \quad |x_n| \leq M \ \ \forall \ n \in \mathbb{N}$

Sufficient part

Let M be a positive real number such that

$$|x_n| \leq M \ \ \forall \ n \in \mathbb{N}$$

Then $\quad -M \leq x_n \leq M \ \ \forall \ n \in \mathbb{N}$

$\Rightarrow \ \{x_n\}$ is bounded.

Note: The above theorem is used as a definition of a bounded sequence and should be committed to memory.

4.4 *LIMIT OF A SEQUENCE*

Let $\{x_n\}$ be a sequence and $l \in \mathbb{R}$. The real number l is said to be limit of the sequence $\{x_n\}$ it to each $\epsilon > 0$, $\exists \ m \in \mathbb{N}$ (m depending on ϵ) such that $|x_n - l| < \epsilon \ \ \forall \ n \geq m$.

If l is the limit of the sequence $\{x_n\}$, then we write $x_n \longrightarrow l$ as $n \longrightarrow \infty$ or $\lim_{n \to \infty} x_n = l$.

Note: if $|x_n - l| < \epsilon \quad \forall \ n \geq m$

Then $\quad l - \epsilon < x_n < l + \epsilon \quad \forall \ n \geq m$

Then $\quad x_n \in (l - \epsilon, l + \epsilon) \quad \forall \ n \geq m$.

4.5 *CONVERGENT SEQUENCE*

If $\lim_{n \to \infty} x_n = l$, then we say that the sequence $\{x_n\}$ converges to l.

Theorem 4.5.1. Every convergent sequence has a unique limit.

Or

A sequence cannot converge to more than one limit.

Proof: if possible, let a sequence $\{x_n\}$ converse to two distinct real numbers l and l'.

Let $\epsilon = |l - l'|$. Since $l \neq l'$, $|l - l'| > 0$ so that $\epsilon > 0$.

Now the sequence $\{x_n\}$ converges to l

$\Rightarrow$ given $\epsilon > 0$, $\exists$ a positive integer m_1 such that

$$|x_n - l| < \frac{\epsilon}{2} \ \forall \ n \geq m_1$$

Also, the sequence $\{x_n\}$ converges to l'

$\Rightarrow$ given $\epsilon > 0$, $\exists$ a positive integer m_2 such that

$$|x_n - l'| < \frac{\epsilon}{2} \ \forall \ n \geq m_2$$

Let $m = max.\{m_1, m_2\}$

Then $|x_n - l| < \frac{\epsilon}{2}$ and $|x_n - l'| < \frac{\epsilon}{2} \quad \forall\, n \geq m$ ……. (1)

Now $|l - l'| = |(l - x_n) + (x_n - l')| \leq |l - x_n| + |x_n - l'|$

$= |x_n - l| + |x_n - l'| \qquad (\because\ |-x| = |x|\,)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall\, n \geq m$

$\therefore \quad |l - l'| < \epsilon \ \forall\, n \geq m$

Which contradicts the assumption that $\epsilon = \frac{1}{2}|l - l'|$

$\Rightarrow$ Our assumption was wrong. Hence $l = l'$.

4.6 *DIVERGENT SEQUENCE*

(i) A sequence $\{x_n\}$ is said to be **diverge to** $+\infty$ if given any positive real number K, however large there exists a positive integer m (depending on K) such that $x_n > K \ \forall\, n \geq m$

And we write $\lim_{n \to \infty} x_n = \infty$ or $x_n \to \infty$ as $n \to \infty$

(ii) A sequence $\{x_n\}$ is said to be **diverge to** $-\infty$ if given any positive real number K, however large there exists a positive integer m (depending on K) such that $x_n < -K \ \forall\, n \geq m$

And we write $\lim_{n \to \infty} x_n = -\infty$ or $x_n \to \infty$ as $n \to \infty$

(iii) A sequence $\{x_n\}$ is said to be **divergent sequence** if it diverse to $+\infty$ or $-\infty$.

i.e. if $x_n \to \infty$ or $x_n \to -\infty$ as $n \to \infty$.

Example. The sequence $\{n\}$ and $\{n^2\}$ diverge to $+\infty$. Whereas the sequences $\{-n\}$ and $\{-n^2\}$ diverge to $-\infty$.

4.7 *OSCILLATORY SEQUENCE*

If a sequence $\{x_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types:

(i) A bounded sequence which do not converse is said to be oscillate finitely.

For example: Consider a bounded sequence $\{(-1)^n\}$.

Here $x_n = (-1)^n$

Then $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} (-1)^{2n} = 1$ and

$\lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty} (-1)^{2n+1} = -1.$

Thus $\lim\limits_{n\to\infty} x_n$ does not exist

$\Longrightarrow$ the sequence does not converse.

Hence this sequence oscillates finitely.

(ii) An unbounded sequence which do not diverge is said to be oscillate infinitely.

For example: Consider a bounded sequence $\{(-1)^n.n\}$.

Here $x_n = (-1)^n.n$

Then $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} (-1)^{2n}.2n = +\infty$ and

$\lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty} (-1)^{2n+1}.(2n+1) = -\infty.$

Thus, the sequence does not diverge.

$\Longrightarrow$ this sequence oscillates infinitely.

Note: When we say $\lim\limits_{n\to\infty} x_n = l$, it means $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_{2n+1} = l$

Similarly, $\lim\limits_{n\to\infty} x_n = +\infty$, it means $\lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_{2n+1} = +\infty$

4.8 *NULL SEQUENCE*

A sequence $\{x_n\}$ is said to be null sequence if it converges to zero i.e. if $\lim_{n\to\infty} x_n = 0$.

For example: The sequence $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{n^2}\right\}, \left\{\frac{1}{2^n}\right\}$ and $\left\{\frac{(-1)^{n-1}}{n}\right\}$ are null sequences.

4.9 *MONOTONIC SEQUENCE*

(i) A sequence $\{x_n\}$ is said to be **monotonically increasing.**

If $\quad x_{n+1} \geq x_n \quad \forall\, n \in \mathbb{N}$

i.e. if $\quad x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$

(ii) A sequence $\{x_n\}$ is said to be **monotonically decreasing.**

If $\quad x_{n+1} \leq x_n \quad \forall\, n \in \mathbb{N}$

i.e. if $\quad x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$

(iii) A sequence $\{x_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(iv) A sequence $\{x_n\}$ is said to be **strictly monotonically increasing.**

If $\quad x_{n+1} > x_n \quad \forall\, n \in \mathbb{N}$

(v) A sequence $\{x_n\}$ is said to be **strictly monotonically decreasing.**

If $\quad x_{n+1} < x_n \quad \forall\, n \in \mathbb{N}$

(vi) A sequence $\{x_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

Theorem 4.9.1. Every convergent sequence is bounded.

Proof: Let $\{x_n\}$ be a convergent sequence, converging to l.

For $\epsilon = 1$, there exists a positive integer m such that

$$|x_n - l| < 1 \qquad \forall\, n \geq m$$

$$\Longrightarrow \qquad l - 1 < x_n < l + 1 \qquad \forall\, n \geq m$$

Let $k = \text{min.}\{x_1, x_2, \dots, x_{m-1}, l - 1\}$ and

$K = \text{mix.}\{x_1, x_2, \dots, x_{m-1}, l + 1\}$

Then $\quad k \leq x_n \leq K \qquad \forall\, n \geq m$

$\Rightarrow$ The sequence $\{x_n\}$ is a bounded sequence.

Monotone convergence theorem.

Theorem 4.9.2. Every monotonically increasing sequence which is bounded above converge to its least upper bound.

Proof: Let $\{x_n\}$ be a monotonically increasing sequence which is bounded above. Let u be the $l.u.b.$ of the sequence $\{x_n\}$.

We shall show that $\{x_n\}$ converges to u.

Let $\epsilon > 0$ be given.

Since $u - \epsilon < u$, therefore, $u - \epsilon$ is not an upper bound of $\{x_n\}$

$\Rightarrow \exists$ a positive integer m such that $x_n > \ u - \epsilon$

Since $\{x_n\}$ is monotonically increasing.

Therefore $\quad x_n \geq x_m > u - \epsilon \quad \forall\, n \geq m \quad$ ……. (i)

Then $\quad x_n > u - \epsilon \quad \forall\, n \geq m$

Also, u is the $l.u.b.$ of the sequence $\{x_n\}$

$\Rightarrow \quad x_n \leq u + \epsilon \quad \forall\, n \in \mathbb{N}$

$\Rightarrow \quad x_n < u + \epsilon \quad \forall\, n \in \mathbb{N} \quad$ ……. (ii)

From (1) and (2), $u - \epsilon < x_n < u + \epsilon \quad \forall\, n \geq m$

$\Rightarrow \quad |x_n - u| < \ \epsilon \quad \forall\, n \geq m$

$\Rightarrow \quad \lim_{n\to\infty} x_n = u$

$\Rightarrow \{x_n\}$ converges to u.

Theorem 4.9.3. Every monotonically decreasing sequence which is bounded below converges to its greatest lower bound.

Proof: Similar as Theorem 4.9.2.

Note: Every monotonically increasing sequence which is not bounded above diverges to ∞.

Note: Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Note: Every monotonic sequence either converges or diverges.

Examples 4.9.1. By definition show that

(i) The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

(ii) The sequence $\left\{\frac{1}{n^2}\right\}$ converges to 0.

(iii) The sequence $\left\{\frac{1}{3^n}\right\}$ converges to 0.

(iv) The sequence $\left\{\sqrt{n+1}-\sqrt{n}\right\}$ is a null sequence.

Sol. (i) let $a_n = \frac{1}{n}$. let $\epsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{\text{n}} < \epsilon$ if $n > \frac{1}{\epsilon}$

If m is a positive integer $> \frac{1}{\epsilon}$, then $|a_n - 0| < \epsilon \;\; \forall\, n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow$ i.e. sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

(ii) let $a_n = \frac{1}{n^2}$. let $\epsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \epsilon$ if $n^2 > \frac{1}{\epsilon}$

If m is a positive integer $> \frac{1}{\sqrt{\epsilon}}$, then $|a_n - 0| < \epsilon \;\; \forall\, n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow$ i.e. sequence $\left\{\frac{1}{n^2}\right\}$ converges to 0.

(iii) let $a_n = \frac{1}{3^n}$. let $\epsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{3^n} - 0\right| = \left|\frac{1}{3^n}\right| = \frac{1}{3^n} < \epsilon$ if $3^n > \frac{1}{\epsilon}$

i.e. if $n\, log n > log\left(\frac{1}{\epsilon}\right)$

i.e. if $n > \frac{log\left(\frac{1}{\epsilon}\right)}{log(3)}$ [since $log(3) > 0$]

If m is a positive integer $> \frac{log\left(\frac{1}{\epsilon}\right)}{log(3)}$, then $|a_n - 0| < \epsilon \;\; \forall\, n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow$ i.e. sequence $\left\{\frac{1}{3^n}\right\}$ converges to 0.

(iv) let $a_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n-1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{(\text{n}+1)-\text{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

let $\epsilon > 0$ be given.

Now $|a_n - 0| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}} - 0\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} < \epsilon$

if $\sqrt{n} > \frac{1}{\epsilon}$ i.e. if $n > \frac{1}{\epsilon^2}$

If m is a positive integer $> \frac{1}{\epsilon^2}$, then $|a_n - 0| < \epsilon \ \forall\, n \geq m$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

$\Rightarrow$ The given sequence $\{\sqrt{n+1} - \sqrt{n}\}$ is a null sequence.

4.10 *LIMIT POINT OF A SEQUENCE*

A real number l is said to be a limit point of a sequence $\{a_n\}$ if every neighbourhood of l contains infinitely many terms of the sequence.

Note: If $a_n = l$ for infinitely many values of n then l is a limit point of $\{a_n\}$.

Note: Limit of a sequence is a limit point but limit point need not be a limit of sequence.

Note: Limit point of a sequence need not be a term of the sequence.

For example: Sequence $\left\{\frac{1}{n}\right\}$ has 0 as a limit point but no term of $\left\{\frac{1}{n}\right\}$ is 0.

Example 4.10.1. Prove that 0 is a limit point of the Sequence $\left\{\frac{1}{n}\right\}$.

Sol. For $\epsilon > 0, \exists\, m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$

Therefore, for $n \geq m$, $0 < \frac{1}{n} \leq \frac{1}{m} < \epsilon$

$\Rightarrow$ $-\epsilon < 0 < \frac{1}{n} < \epsilon \ \forall\, n \geq m$

$\Rightarrow$ $\frac{1}{n} \in (-\epsilon, \epsilon) \quad \forall\, n \geq m$

$\Rightarrow$ Every neighbourhood of 0 contains infinitely many terms of the sequence $\left\{\frac{1}{n}\right\}$.

$\Rightarrow$ 0 is a limit point of the sequence $\left\{\frac{1}{n}\right\}$.

Example 4.10.2. The sequence $\{(-1)^n\}$ has two limit points.

Sol. Let $a_n = (-1)^n$, then $a_n = -1$ when n is odd and $a_n = 1$ when n is even.

Thus, every neighbourhood of -1 contains all the odd terms of the sequence.

Therefore, -1 is a limit point.

Also, every neighbourhood of 1 contains all the even terms of the sequence.

Therefore, 1 is a limit point.

Therefore, the given sequence has only two limit points.

Example 4.10.3. The sequence $\{n\}$ has no limit points.

Sol. Let l be any real number, then the neighbourhood $\left(l-\frac{1}{4},\ l+\frac{1}{4}\right)$ of l contains at most one term of the sequence $\{n\}$.

$\Rightarrow l$ is not a limit point of the sequence $\{n\}$.

Theorem 4.10.1. If l is a limit point of the range of a sequence $\{a_n\}$, then l is a limit point of the sequence $\{a_n\}$.

Proof: Let S = range of the sequence $\{a_n\}$.

Since l is a limit point of S, every neighbourhood of l contains infinitely many elements of S.

But each element of S is a term of the sequence $\{a_n\}$.

$\therefore$ Every neighbourhood of l contains infinitely many terms of the sequence $\{a_n\}$.

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$.

Note: The converse of above theorem may not be true.

Consider $a_n = 1 + (-1)^n = \begin{cases} 0, & when\ n\ is\ odd \\ 2, & when\ n\ is\ even \end{cases}$

Therefore, $0, 2$ are the limit points of the sequence $\{a_n\}$.

But the range of the sequence $= \{0, 2\}$ is a finite set.

Since the finite set has no limit point, the range of $\{a_n\}$ has no limit point.

Note: If the terms of the sequence are different the limit points of the sequence are the limit points of the range set.

4.11 *BOLZANO – WEIERSTRASS THEOREM FOR SEQUENCES*

Every Bounded sequence has at least one limit point.

Proof: Let $\{a_n\}$ be a bounded sequence and S be its range,

i.e. S = $\{a_n : n \in \mathbb{N}\}$

since $\{a_n\}$ is bounded, then S is bounded.

Case 1. Let S be a finite set.

Then ∃ a real number l such that $a_n = l$ for an infinite number of values of n ∈ ℕ.

⟹ Given $\epsilon > 0$, $a_n \in (l - \epsilon, l + \epsilon)$ for an infinite number of values of n ∈ ℕ.

⟹ Every neighbourhood of l contains infinitely many terms of the sequence $\{a_n\}$.

Therefore l is limit point of the sequence $\{a_n\}$.

Case 2. Let S be an infinite set.

Since S is an infinite bounded set, by Bolzano-Weierstrass theorem, S has at least one limit point, say l.

Now, l is limit point of S.

⟹ Every neighbourhood of l contains infinitely many numbers of the elements of S.

But each element of S is a term of the sequence $\{a_n\}$.

∴ every neighbourhood of l contains an infinite number of terms of the sequence $\{a_n\}$.

Therefore l is a limit point of the sequence $\{a_n\}$.

Corollary: If S is a closed and bounded set, then every sequence in S has a limit point in S.

Corollary: The set of the limit point of the bounded sequence is bounded.

Corollary: The set of the limit points of a sequence is a closed set.

Corollary: The set of limit points of an unbounded sequence may or may not be unbounded.

For example. (i) The sequence $\left\{1,\ \frac{1}{2},\ 2,\ \frac{1}{3},\ 3, \ldots\right\}$ is unbounded but the set of its limit points is $\{0\}$, which is bounded.

(i) The sequence $\left\{2,\ 1+\frac{1}{2},\ 2+\frac{1}{2},\ 1+\frac{1}{3},\ 2+\frac{1}{3},\ 3+\frac{1}{3},\dots\dots\right\}$ is unbounded but the set of its limit points is $\mathbb{N}$, which is unbounded.

4.12 *LIMIT SUPERIOR AND LIMIT INFERIOR OF A SEQUENCES*

Let $\{a_n\}$ be a bounded sequence, then the sequence has the least and greatest limit points.

The least limit point of $\{a_n\}$ is called the limit inferior of $\{a_n\}$ and is denoted by $\lim_{n\to\infty} inf. a_n$.

The greatest limit point of $\{a_n\}$ is called the limit superior of $\{a_n\}$ and is denoted by $\lim_{n\to\infty} sup. a_n$.

Note 1. If $\{a_n\}$ is unbounded above then $\lim_{n\to\infty} sup. a_n = \infty$.

And If $\{a_n\}$ is unbounded below then $\lim_{n\to\infty} inf. a_n = -\infty$.

Note 2. Since the greatest limit point of the sequence $\{a_n\} \geq$ least limit point.

Therefore, $\lim_{n\to\infty} sup. a_n \geq \lim_{n\to\infty} inf. a_n$.

Example 4.12.1. Find $\lim_{n\to\infty} sup. a_n$ and $\lim_{n\to\infty} inf. a_n$ for the sequence $\{a_n\} = \{(-1)^n\}$.

Sol. Since the sequence $\{(-1)^n\}$ has only two limit points -1 and 1.

Therefore, the set of limit points = $\{-1, 1\}$ which is bounded.

Therefore, $\lim_{n\to\infty} inf. a_n = -1$ and $\lim_{n\to\infty} sup. a_n = 1$.

Example 4.12.2. Find $\lim_{n\to\infty} sup. a_n$ and $\lim_{n\to\infty} inf. a_n$ for the sequence $\{a_n\} = \{(-1)^n. n\}$.

Sol. Since the sequence $\{(-1)^n. n\}$ is unbounded above and unbounded below both.

Therefore, $\lim_{n\to\infty} inf. a_n = -\infty$ and $\lim_{n\to\infty} sup. a_n = \infty$.

4.13 *SUBSEQUENCE*

Let $\{a_n\}$ be a sequence of real numbers, and

let $n_1 < n_2 < \cdots < n_k < \cdots$

be a strictly increasing sequence of natural numbers.

Then $\{a_{n_1}, a_{n_2}, \cdots, a_{n_k}, \cdots\}$ is called a subsequence of $\{a_n\}$ and is denoted by $\{a_{n_k}\}$.

Note: A subsequence is formed from a sequence by selecting certain terms from the sequence in order.

For Example. (i) $\{a_{2n}\}$, $\{a_{2n-1}\}$, $\{a_{n^2}\}$ are all subsequence of $\{a_n\}$.

(ii) {2, 4, 6, ….}, {1, 3, 5, …}, {1, 4, 9, 16, …} are all subsequences of the sequence {n}.

Note: Every sequence is a subsequence of itself.

Theorem 4.13.1. If a sequence $\{a_n\}$ converges to l, then every subsequence of $\{a_n\}$ also converges to l.

Proof: Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

Since $\{a_n\}$converges to l.

Therefore, given $\epsilon > 0$, $\exists$ a positive integer m

such that $|a_n - l| < \epsilon \; \forall \; n \geq m$ ……. (i)

We can find a natural number $n_{k_0} \geq m$

If $n_k \geq n_{k_0}$, then $n_k \geq m$.

Therefore, from (i), we have $|a_{n_k} - l| < \epsilon \; \forall \; n_k \geq m$

Hence $\{a_{n_k}\}$converges to l.

4.14 *CAUCHY SEQUENCES*

A sequence $\{a_n\}$ is said to be a Cauchy sequence if given $\epsilon > 0$, however small, there exists a positive integer m (depending on ϵ) such that $|a_n - a_m| < \epsilon \; \forall \; n \geq m$.

Theorem 4.14.1. Every Cauchy sequence is bounded.

Proof: Let $\{a_n\}$ be a Cauchy sequence.

Taking $\epsilon = 1$, by definition, there exists a positive integer m such that

$$|a_n - a_m| < 1 \qquad \forall\, n \geq m$$

$$\Rightarrow \qquad a_m - 1 < a_n < a_m + 1 \quad \forall\, n \geq m$$

Let $\quad k = min.\{ a_1, a_2, \dots, a_{m-1}, a_m - 1\}$

And $\quad K = max.\{ a_1, a_2, \dots, a_{m-1}, a_m + 1\}$

Then $\quad k \leq a_n \leq K \quad \forall\, n$

Hence the sequence $\{a_n\}$ is bounded.

Note: The converse of above theorem is not true, i.e. every bounded sequence need not be a Cauchy sequence.

For Example. The sequence $\{(-1)^n\}$ is bounded but it is not a Cauchy sequence.

Theorem 4.14.2. (Cauchy's convergence criterion)

A sequence is convergent if and only if it is a Cauchy sequence.

Proof: First, let $\{a_n\}$ be a convergent sequence, converging to l.

We shall show that it is a Cauchy sequence.

Let $\epsilon > 0$ be given. Then there exists a positive integer m such that

$$|a_n - l| < \frac{\epsilon}{2} \qquad \forall\, n \geq m \quad \text{..... (i)}$$

In particular, for $n = m$, we have $|a_n - l| < \frac{\epsilon}{2}$ …… (ii)

Now, $|a_n - a_m| = |(a_n - l) - (a_m - l)| \leq |(a_n - l)| + |(a_m - l)|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \forall\, n \geq m \text{ [using (i) and (ii)]}$$

Thus $\quad |a_n - a_m| < \epsilon \quad \forall\, n \geq m$

$\Rightarrow \{a_n\}$ is a Cauchy sequence.

Conversely: Let $\{a_n\}$ is a Cauchy sequence.

Since Every Cauchy sequence is bounded, therefore $\{a_n\}$ is bounded.

Since every bounded sequence has a limit point, $\{a_n\}$ has limit point l (say).

We shall show that $\{a_n\}$ converges to l.

Let $\epsilon > 0$ be given. Since $\{a_n\}$ is Cauchy sequence, $\exists$ a positive integer m such that $\quad |a_n - a_m| < \frac{\epsilon}{3} \quad \forall\, n \geq m$ …… (iii)

Since l is limit point of $\{a_n\}$, every neighbourhood of l contains infinitely many terms of $\{a_n\}$.

$\Rightarrow a_n \in \left(l - \frac{\epsilon}{3},\ l + \frac{\epsilon}{3}\right)$ for infinitely many values of n.

In particular, we can find a positive integer $k > m$ such that

$$a_k \in \left(l - \frac{\epsilon}{3},\ l + \frac{\epsilon}{3}\right) \quad \text{……. (iv)}$$

i.e. $\quad |a_k - l| < \frac{\epsilon}{3}$ ……. (v)

Also, since $k > m$, therefore, we have $|a_n - a_m| < \frac{\epsilon}{3}$ …… (vi)

Now, $\quad |a_n - l| = |(a_n - a_m) + (a_m - a_k) + (a_k - l)|$

$$\leq |a_n - a_m| + |a_m - a_k| + |a_k - l|$$

$$= |a_n - a_m| + |a_k - a_m| + |a_k - l|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, $\quad |a_n - l| < \epsilon \quad \forall\, n \geq m$

$\Rightarrow \quad \{a_n\}$ converges to l.

Example. 4.14.1. Prove that the sequence whose nth terms are given below are Cauchy sequence.

(i) $\frac{1}{n}$ (ii) $\frac{n}{n+1}$ (iii) $\frac{(-1)^n}{n}$

Sol. (i) Here $\quad a_n = \frac{1}{n}$

$\epsilon > 0$ be given and let $n > m$.

Now $\quad |a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{1}{m} - \frac{1}{n}$

$$< \frac{1}{m} < \epsilon \text{ whenever } m > \frac{1}{\epsilon}$$

$\therefore$ For each $\epsilon > 0$, $\exists$ a $+ve$ integer m such that $|a_n - a_m| < \epsilon \ \forall\, n > m$.

$\Rightarrow \{a_n\}$ is a Cauchy sequence.

(ii) Here $\quad a_n = \frac{n}{n+1}$

$\epsilon > 0$ be given and let $n > m$.

Now $\quad |a_n - a_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right| = \left|\left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{m+1}\right)\right|$

$$= \left|\frac{1}{n+1} - \frac{1}{m+1}\right| = \frac{1}{m+1} - \frac{1}{n+1}$$

$$< \frac{1}{m+1} < \frac{1}{m} < \epsilon \text{ whenever } m > \frac{1}{\epsilon}$$

$\therefore$ For each $\epsilon > 0$, $\exists$ a $+ve$ integer m such that $|a_n - a_m| < \epsilon \ \forall\, n > m$.

$\Rightarrow \{a_n\}$ is a Cauchy sequence.

(iii) Here $a_n = \frac{(-1)^n}{n}$

$\epsilon > 0$ be given and let $n > m$.

Now $|a_n - a_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \left|\frac{(-1)^n}{n}\right| + \left|\frac{(-1)^m}{m}\right|$

$= \frac{1}{n} + \frac{1}{m}$

$< \frac{1}{m} + \frac{1}{m} < \frac{2}{m} < \epsilon$ whenever $m > \frac{2}{\epsilon}$

$\therefore$ For each $\epsilon > 0$, $\exists$ a $+ve$ integer m such that $|a_n - a_m| < \epsilon \ \forall \ n > m$.

$\Rightarrow \{a_n\}$ is a Cauchy sequence.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Limit superior and inferior are equal for sequence $\left\{\frac{1}{n}\right\}$.

Problem 2. The limit point of the sequence $\left\{10 + \frac{1}{n}\right\}$ is 10.

Problem 3. The sequence $\{1 + \sin n\}$ is unbounded.

Problem 4. The set of all limit point of a sequence is open set.

Problem 5. The Sequence $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is convergent.

4.15 *SUMMARY*

1. Range of sequence: The set of all distinct terms of a sequence is called its range.

2. The sequence $\{n\}$ has no limit points.

3. A sequence $\{x_n\}$ is said to be null sequence if it converges to zero i.e. if $\lim_{n\to\infty} x_n = 0$.

4. A sequence is convergent if and only if it is a Cauchy sequence.

5. Bolzano – Weierstrass Theorem for a sequence:

Every Bounded sequence has at least one limit point.

4.16 *GLOSSARY*

Numbers

Intervals

Limit points

Functions

Bounded, Unbounded sets

4.17 *REFERENCES*

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4.18 *SUGGESTED READING*

4. S.C. Malik and Savita Arora, Mathematical Analysis (6th Edition), New Age International Publishers, 2021.
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6. K. A. Ross, Elementary Analysis**,** The Theory of Calculus (2nd edition), Springer, 2013.

4.19 *TERMINAL AND MODEL QUESTIONS*

Q1. Prove that the sequence $\left(1+\frac{1}{n}\right)^{n+1}$ is convergent.

Q2. Prove that the sequence $\left\{\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right\}$ is monotonically increasing.

Q3. If $\{a_n\}$ and $\{b_n\}$ are null sequences, show that $\{a_n+b_n\}$ is also null sequence.

Q4. If $\{a_n\}$ and $\{b_n\}$ are null sequences, show that $\{a_n.b_n\}$ is also null sequence.

Q5. States and prove Bolzano – Weierstrass Theorem for a sequence.

4.20 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. False

CYQ 5. True

UNIT 5: SERIES

5.1 INTRODUCTION

In mathematics, a series is, roughly speaking, the operation of adding infinitely many quantities, one after the other, to a given starting quantity. The study of series is a major part of calculus and its generalization, mathematical analysis. Series are used in most areas of mathematics, even for studying finite structures (such as in combinatorics) through generating functions. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, statistics and finance.

For a long time, the idea that such a potentially infinite summation could produce a finite result was considered paradoxical. This paradox was resolved using the concept of a limit during the 17th century. Zeno's paradox of Achilles and the tortoise illustrates this counterintuitive property of infinite sums: Achilles runs after a tortoise, but when he reaches the position of the tortoise at the beginning of the race, the tortoise has reached a second position; when he reaches this second position, the tortoise is at a third position, and so on. Zeno concluded that Achilles could *never* reach the tortoise, and thus that movement does not exist. Zeno divided the race into infinitely many sub-races, each requiring a finite amount of time, so that the total time for Achilles to catch the tortoise is given by a series. The resolution of the paradox is that, although the series has an infinite number of terms, it has a finite sum, which gives the time necessary for Achilles to catch up with the tortoise.

A brief introduction to Infinite series and some results in infinite series will be discussed.

5.2 OBJECTIVES

After reading this unit learners will be able to

- infinite series
- positive term series

5.3 SERIES

The sum of the terms of a sequence is said to be a **series**. Thus if $y_1, y_2, y_3,$ ....is a sequence then the sum $y_1 + y_2 + y_3 + \cdots$ of all the terms is called an infinite series and is expressed by $\sum_{n=1}^{\infty} y_n$ or $\sum y_n$.

Evidently, we cannot just add up all the infinite number of terms of the series in ordinary way and in fact it is not obvious that this kind of sum has any meaning. Therefore, we start by associating with the given series, a sequence $\{S_n\}$, where S_n denotes the sum of the first n terms of the series.

Hence $S_n = y_1 + y_2 + \cdots . + y_n \quad \forall n$

And this sequence $\{S_n\}$ is said to be the sequence of partial sums of the series.

The **partial sums**

$\boldsymbol{S_1 = y_1};\ \boldsymbol{S_2 = y_1 + y_2};\ \boldsymbol{S_3 = y_1 + y_2 + y_3 + \cdots ..}$ and so on.

The series is **convergent** if the sequence $\{S_n\}$ of partial sums converges and $\lim S_n$ is called the sum of the series.

If $\{S_n\}$ does not tend to a limit then the sum of the infinite series does not exist or we can say that the series does not converges.

An infinite series is converge, diverge or oscillate according as its sequence of partial sums $\{S_n\}$ converges, diverges and oscillates.

Necessary condition of convergences of an infinite series

Theorem: A Necessary condition of convergences of an infinite series $\sum \boldsymbol{y_n}$ is $\lim_{n\to\infty} y_n = 0$.

Proof. Let $S_n = y_1 + y_2 + \cdots \ldots \ldots . + y_n$, so that $\{S_n\}$ is the sequence of partial sums.

It is given that series is converges

Thus, the sequence $\{S_n\}$ is also converges.

Let $\lim_{n\to\infty} S_n = t$. Now $y_n = S_n - S_{n-1}, \quad n > 1$.

Therefore, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = t - t = 0$

Thus $\lim_{n\to\infty} y_n = 0$

NOTE:

A series cannot converge if n^{th} term does not tend to zero.

Cauchy's General Principle of Convergence for Series

Theorem: A necessary and sufficient condition for the convergence of an infinite series $\sum y_n$ is that the sequence of its partial sums $\{S_n\}$ is convergent

Or

An infinite series $\sum y_n$ converges iff for every $\varepsilon > 0$ there exists a positive integer M such that $|y_1 + y_2 + Y_3 + \cdots ... + y_n| < \varepsilon$ whenever

$m \geq n \geq M$

Proof. Let $S_n = \sum y_n = y_1 + y_2 + y_3 + \cdots ... + y_n$ and $S_m = \sum y_m = y_1 + y_2 + y_3 + \cdots ... + y_m$ be the n^{th} and m^{th} partial sum of series respectively, where $m \geq n$.

$\Rightarrow |S_m - S_n| = |(y_1 + y_2 + y_3 + \cdots + y_m) - (y_1 + y_2 + y_3 + \cdots + y_n)|$

$= |y_{m+1} + y_{m+2} + \cdots + y_n|$.

Let $\varepsilon > 0$ and for every ε the series $\sum y_n$ converges iff the sequence of partial sums {Sn} converges

$\Leftrightarrow |S_m - S_n| < \varepsilon \quad \forall\, m \geq n \; for\ some\ M \in \mathbb{N}$

$\Leftrightarrow |y_{m+1} + y_{m+2} + \cdots + y_n| < \varepsilon \quad \forall\, m \geq n \; for\ some\ M \in \mathbb{N}$

Example: Prove that $\sum \frac{1}{n}$ does not converge.

Proof. Let the given series be converges.

Therefore, for any given $\varepsilon > 0$, there exists a positive integer m such that

$\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| < \varepsilon \quad \forall\, n \geq m \;\; and\ p \geq 1.$

If $n = m$ and $p = m$, we get

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+m}$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$

$$> m.\frac{1}{2m} > \frac{1}{2} > \varepsilon$$

i.e. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} > \varepsilon$, a contradiction.

Therefore $\sum \frac{1}{n}$ does not converge

NOTE:

We can see that $\boldsymbol{lim_{n\to\infty}\left(\frac{i}{n}\right) = 0}$ but $\boldsymbol{\sum \frac{1}{n}}$ does not converge

If $\sum y_n = y$ then $\sum cy_n = cy$ independent of n.

Example: If $y_n > 0$ and $\sum y_n$ is convergent with the sum S, then prove that $\frac{y_n}{y_1+y_2+\cdots+y_n} < \frac{2y_n}{S}$, when n is sufficiently large. Also prove that $\sum \frac{y_n}{y_1+y_2+\cdots+y_n}$ is convergent.

Proof. It is given that $\sum y_n$ is convergent with the sum S.

Hence for $\varepsilon > 0 \;\; \exists\, m \in \mathbb{Z}^+$

$|S_n - S| < \varepsilon \;\forall\, n \geq m$ where $S_n = y_1 + y_2 + \cdots + y_n$,

or $\varepsilon < S_n - S < \varepsilon \Rightarrow S - \varepsilon < S_n < S + \varepsilon, \forall\, n \geq m$

For $\varepsilon = \frac{1}{2}S > 0$

$S - \frac{1}{2}S < S_n < S + \frac{1}{2}S \Rightarrow \frac{S}{2} < S_n < \frac{3S}{2} \Rightarrow \frac{2}{S} > \frac{1}{S_n} > \frac{2}{3S}$, $\forall\, n \geq m$

or $\frac{2}{S} > \frac{1}{S_n}, \forall\, n \geq m \Rightarrow \frac{2y_n}{S} > \frac{y_n}{S_n}, \forall\, n \geq m$.

Now $\frac{\boldsymbol{y_{n+1}}}{\boldsymbol{S_{n+1}}} + \frac{\boldsymbol{y_{n+2}}}{\boldsymbol{S_{n+2}}} + \frac{\boldsymbol{y_{n+3}}}{\boldsymbol{S_{n+3}}} + \cdots + \frac{\boldsymbol{y_{n+p}}}{\boldsymbol{S_{n+p}}}$

$$< \frac{\boldsymbol{2}}{\boldsymbol{S}}\left(\boldsymbol{y_{n+1}} + \boldsymbol{y_{n+2}} + \boldsymbol{y_{n+3}} + \cdots + \boldsymbol{y_{n+p}}\right), \forall\, n \geq m, p \geq 1$$

$\Rightarrow \frac{\boldsymbol{y_{n+1}}}{\boldsymbol{S_{n+1}}} + \frac{\boldsymbol{y_{n+2}}}{\boldsymbol{S_{n+2}}} + \frac{\boldsymbol{y_{n+3}}}{\boldsymbol{S_{n+3}}} + \cdots + \frac{\boldsymbol{y_{n+p}}}{\boldsymbol{S_{n+p}}} < \frac{\boldsymbol{2}}{\boldsymbol{S}}(\boldsymbol{S_{n+p}} - \boldsymbol{S_n}), \forall\, n \geq m, p \geq 1.$

As $\sum y_n$ is convergent, then given $\varepsilon > 0$, there exists a positive integer m_1, such that

$\boldsymbol{S_{n+p} - S_n} < \frac{\varepsilon S}{2}, \forall\, n \geq m_1$

Therefore,

$$\frac{\boldsymbol{y_{n+1}}}{\boldsymbol{S_{n+1}}} + \frac{\boldsymbol{y_{n+2}}}{\boldsymbol{S_{n+2}}} + \frac{\boldsymbol{y_{n+3}}}{\boldsymbol{S_{n+3}}} + \cdots + \frac{\boldsymbol{y_{n+p}}}{\boldsymbol{S_{n+p}}} < \frac{\boldsymbol{2}}{\boldsymbol{S}} \frac{\varepsilon S}{2} < \varepsilon, \forall\, n \geq \max(m_1, m)\,, p \geq 1$$

Therefore, by Cauchy's General Principle of convergence, $\sum \frac{y_n}{y_1+y_2+\cdots+y_n}$ is convergent.

5.4 POSITIVE TERM SERIES

Let $\sum y_n$ be an infinite series of positive term series of positive terms ($y_n \geq 0$) and $\{S_n\}$ be the sequence of its partial sums such that

$S_n = y_1 + y_2 + \cdots + y_n \geq 0, \qquad \forall\, n$

$\Rightarrow S_n - S_{n-1} = y_n \geq 0 \Rightarrow S_n \geq S_{n-1}, \ \forall\, n > 1$

Therefore, the sequence $\{S_n\}$ of partial sums of a series of positive terms is a monotonic increasing sequence.

Hence $\{S_n\}$ can either converge or diverge to $+\infty$.

Theorem: A positive term series converges if and only if the sequence of its partial sums is bounded above.

Proof. Let $\sum y_n$ and $\{S_n\}$ be positive term series and a sequence of its partial sums respectively.

$\Rightarrow \{S_n\}$ be a monotonic increasing sequence.

As we know that monotonic increasing sequence converges iff it is bounded above.

Hence $\{S_n\}$ converges if and only if the sequence of its partial sums is bounded above.

Necessary Conditions for convergence of positive term series

Theorem:(Pringsheim's theorem) If a series $\sum y_n$ of positive monotonic decreasing terms converges then $y_n \to 0$ and also $\lim_{n\to\infty} ny_n = 0$.

Proof. Let $\sum y_n$ be the convergent series of positive monotonic decreasing terms.

By the definition of convergent series, for any $\varepsilon > 0$, there exists a positive integer M such that

$$|y_{m+1} + y_{m+2} + \cdots + y_{m+p}| < \frac{\varepsilon}{2}, \quad \forall m \geq M, p \geq 1$$

Let $m + p = n > 2M$ and

$$m = \left[\frac{n}{2}\right] \ \boldsymbol{i.e. m} = greatest\ integer\ not\ greater\ than \frac{n}{2}.$$

Hence

$$y_{m+1} + y_{m+2} + \cdots + y_n < \frac{\varepsilon}{2}$$

But $\sum y_n$ is positive monotonic decreasing.

i.e. $y_{m+1} > y_{m+2} > \cdots > y_n \Rightarrow y_{m+1} + y_{m+2} + \cdots + y_n > \underbrace{y_n + y_n + \cdots .. + y_n}_{(n-m)times}$

$$\Rightarrow y_{m+1} + y_{m+2} + \cdots + y_n > (n-m)y_n$$

Therefore $(n-m)y_n < y_{m+1} + y_{m+2} + \cdots + y_n < \frac{\varepsilon}{2}$

$$\left(n - \frac{n}{2}\right) y_n < \frac{\varepsilon}{2} \qquad because\ m = \left[\frac{n}{2}\right]$$

$$\Rightarrow \frac{n}{2} < \frac{\varepsilon}{2} \Rightarrow \ ny_n < \varepsilon$$

Hence $\lim_{n\to\infty} ny_n = 0$

NOTE:

$\lim_{n\to\infty} ny_n = 0$ is only necessary not sufficient condition. If $\lim_{n\to\infty} ny_n \neq 0$ then the series $\sum y_n$ is obviously divergent..Example $\sum \frac{1}{n}$ diverges because $\lim_{n\to\infty} ny_n = 1 \neq 0$ and positive monotonic decreasing terms.\

Theorem Let $\sum \frac{1}{n^p}$ be positive term series then it is convergent iff $p > 1$.

Proof. Let $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$

Case 1. When $p > 1$

Now

$\frac{1}{1^p} = 1$ …………(1)

$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}}$ ………(2)

$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} < \frac{4}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2$ …….(3)

$\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} < \underbrace{\frac{1}{8^p} + \frac{1}{8^p} + \cdots + \frac{1}{8^p}}_{8\ times} < \frac{8}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3$ …….(4)

… …

… …

… …

$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} < \underbrace{\frac{1}{(2^n)^p} + \frac{1}{(2^n)^p} + \cdots + \frac{1}{(2^n)^p}}_{2^n\ times}$

$= \frac{2^n}{(2^n)^p} = \left(\frac{1}{2^{p-1}}\right)^n$ ..(n)

Adding (1), (2),…..,(n), we get

$\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} = S_{2^{n+1}-1} < 1 + \left(\frac{1}{2}\right)^{p-1} + \cdots + \left(\frac{1}{2^{p-1}}\right)^n$

$= \frac{1\left(1-\left(\left(\frac{1}{2}\right)^{p-1}\right)^{n+1}\right)}{1-\left(\frac{1}{2}\right)^{p-1}} = \frac{2^{p-1}\left(1-\left(\frac{1}{2^{p-1}}\right)^{n+1}\right)}{2^{p-1}-1}$

Therefore

$S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1}-1}, for\ all\ n$

As we know that when n is any positive integer.

$2^{n+1} - 1 > 2^n > n$

Therefore

$S_n < S_{2^n} < S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1}-1}$

Since for a given p, $\frac{2^{p-1}}{2^{p-1}-1}$ is a fixed number.

Hence, the sequence $\{S_n\}$ of partial sums of given positive term series is bounded above.

Therefore, the series converges for $p > 1$.

Case II: When $p \leq 1$

As we know, if n is any positive integer and $p \leq 1$ then

$n^p \leq n$ implies $\frac{1}{n^p} \geq \frac{1}{n}$

Therefore

$1 + \frac{1}{2^p} \geq 1 + \frac{1}{2} > \frac{1}{2}$ ……….(1')

$\frac{1}{3^p} + \frac{1}{4^p} \geq \frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$ …………. (2')

$\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} \geq \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$ ………… (3')

$$\frac{1}{9^p} + \frac{1}{10^p} + \cdots + \frac{1}{16^p} \geq \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}$$

$$\geq \underbrace{\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}}_{8\ times} = \frac{8}{16} = \frac{1}{2} \quad \text{….. (4')}$$

… …

… …

… …

$$\frac{1}{(2^{m-1}+1)^p}+\frac{1}{(2^{m-1}+2)^p}+\cdots+\frac{1}{(2^m)^p}$$

$$\geq \frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+..+\frac{1}{2^m} > \underbrace{\frac{1}{2^m}+\frac{1}{2^m}+\cdots+\frac{1}{2^m}}_{2^{m-1}\,times}$$

$$= \frac{2^{m-1}}{2^m}=\frac{1}{2} \qquad \text{…….(m')}$$

Adding (1'), (2'),…and (m'), we get

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{(2^m)^p}=S_{2^m} > \underbrace{\frac{1}{2}+\frac{1}{2}+\cdots\,....+\frac{1}{2}}_{m\,times}=\frac{m}{2}$$

i.e. $S_{2^m} > \frac{m}{2}$.

Now we try to prove that $\{S_n\}$ is not bounded above.

Let K be any number and there exists $m' \in \mathbb{N}$ such that $\frac{m'}{2} > K$

Let $n > 2^{m'}$

Hence $S_n > S_{2^{m'}} > K$

Therefore, we conclude that the sequence of partial sums $\{S_n\}$ of given series is not bounded above.

Thus, the series diverges for $p \leq 1$.

Therefore, $\sum \frac{1}{n^p}$ converges if $p > 1$.

5.5 COPARISION TEST

Test 1. If $\sum u_n$ and $\sum v_n$ are series of positive terms and $\sum v_n$ is convergent and there is a positive constant k such that $u_n \leq kv_n, \forall\, n$, then $\sum u_n$ is also convergent.

Test 2. If $\sum u_n$ and $\sum v_n$ are series of positive terms and $\sum v_n$ is divergent and there is a positive constant k such that $u_n \geq kv_n, \forall\, n$, then $\sum u_n$ is also divergent.

Test 3. If $\sum u_n$ and $\sum v_n$ are series of positive terms and

(i) if $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$(finite and non-zero), then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

(ii) if $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges, then $\sum u_n$ is also converges.

(iii) if $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ and $\sum v_n$ diverges, then $\sum u_n$ is also diverges.

5.6 D'ALEMBERT'S RATION TEST

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l,$ then

(i) $\sum u_n$ is convergent if $l > 1$.

(ii) $\sum u_n$ is divergent if $l < 1$.

(iii) $\sum u_n$ may converge or diverge if $l = 1$.(i.e. the test fails if $l = 1$).

And if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$, then $\sum u_n$ is convergent.

5.7 CAUCHY'S ROOT TEST

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} (u_n)^{1/n} = l,$ then

(i) $\sum u_n$ is convergent if $l < 1$.

(ii) $\sum u_n$ is divergent if $l > 1$.

(iii) $\sum u_n$ may converge or diverge if $l = 1$.(i.e. the test fails if $l = 1$).

And if $\lim_{n\to\infty} (u_n)^{1/n} = \infty$, then $\sum u_n$ is divergent.

5.8 ALTERNATING SERIES

A series with terms alternately positive and negative is called an alternating series.

Thus, the series $u_1 - u_2 + u_3 - u_4 \dots$ where $u_n > 0$.

For each n, is an alternating series and is briefly written as $\sum(-1)^{n-1}u_n$.

Leibnitz's test on alternating series

The alternating series $\sum(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \dots$ where $u_n > 0 \ \forall \ n$ converges if

(i) $u_n \geq u_{n+1} \forall \ n$ and (ii) $\lim_{n\to\infty} u_n = 0$.

5.9 ABSOLUTE AND CONDITIONAL CONVERGENCE

Definition 1. A series $\sum_{n=1}^{\infty} u_n$ is said to be absolute convergent if the series $\sum_{n=1}^{\infty}|u_n|$ is convergent.

Definition 2. If $\sum_{n=1}^{\infty} u_n$ converges but not absolutely then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Theorem: Every absolutely convergent series is convergent.

Proof. Let $\sum_{n=1}^{\infty} u_n$ be absolutely convergent series, then $\sum_{n=1}^{\infty}|u_n|$ is convergent.

Therefore, By Cauchy's general principle of convergence, given $\epsilon > 0, \exists$ a positive integer m such that $\big||u_{m+1}| + |u_{m+2}| + \cdots + |u_n|\big| < \epsilon \quad \forall \ n > m$

i.e. $|u_{m+1}| + |u_{m+2}| + \cdots + |u_n| < \epsilon \quad \forall \ n > m$ ……. (1)

now, by triangle inequality, we have

$$|u_{m+1} + u_{m+2} + \cdots + u_n| \leq |u_{m+1}| + |u_{m+2}| + \cdots + |u_n|$$

$$< \epsilon \quad \forall\, n > m$$

Therefore, By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Hence $\sum_{n=1}^{\infty} |u_n|$ is convergent $\Longrightarrow$ u_n is convergent.

Note 1. Absolute convergence $\Longrightarrow$ convergent, but convergence need not imply absolute convergence i.e. the convergence of above theorem need not be true.

For Example: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

It is an alternating series. Here $u_n = \frac{1}{n}$. Clearly $u_n > 0\ \forall\, n$

Since $\frac{1}{n} > \frac{1}{n+1}$, $u_n > u_{n+1} \forall\, n$

Also $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$

Therefore, by Leibnitz's test $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Note: If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series, then the series of its positive terms and the series of its negative terms are both convergent.

Note: If $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, then the series of its positive terms and the series of its negative terms are both divergent.

Note: A series with mixed signs cannot converge if the series of its positive terms is convergent (divergent) and the series of its negative terms is divergent (convergent).

Example 1. Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ **(ii)** $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

Sol. (i) Here $u_n = \sin \frac{1}{n}$

$$= \frac{1}{n} - \frac{1}{3!} \cdot \frac{1}{n^3} + \frac{1}{5!} \cdot \frac{1}{n^5} - \cdots$$

$$=\frac{1}{n}\left[1-\frac{1}{3!}\cdot\frac{1}{n^2}+\frac{1}{5!}\cdot\frac{1}{n^4}-\cdots\right]$$

Take $v_n=\frac{1}{n}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[1-\frac{1}{3!}\cdot\frac{1}{n^2}+\frac{1}{5!}\cdot\frac{1}{n^4}-\cdots\right]=1 \text{ which is finite and} \neq 0.$$

Therefore, $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ converges and diverges together.

Since $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ with p = 1.

Therefore, $\sum_{n=1}^{\infty}v_n$ is divergent $\Longrightarrow \sum_{n=1}^{\infty}u_n$ is divergent.

(ii) Here $u_n=\frac{1}{n}\sin\frac{1}{n}$

$$=\frac{1}{n}\left[\frac{1}{n}-\frac{1}{3!}\cdot\frac{1}{n^3}+\frac{1}{5!}\cdot\frac{1}{n^5}-\cdots\right]$$

$$=\frac{1}{n^2}\left[1-\frac{1}{3!}\cdot\frac{1}{n^2}+\frac{1}{5!}\cdot\frac{1}{n^4}-\cdots\right]$$

Take $v_n=\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[1-\frac{1}{3!}\cdot\frac{1}{n^2}+\frac{1}{5!}\cdot\frac{1}{n^4}-\cdots\right]=1 \text{ which is finite and} \neq 0.$$

Therefore, $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ converges and diverges together.

Since $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ with p = 2.

Therefore, $\sum_{n=1}^{\infty}v_n$ is convergent $\Longrightarrow \sum_{n=1}^{\infty}u_n$ is convergent.

Example 2. Discuss the convergence or divergence of the following series:

(i) $\sum_{n=1}^{\infty}\frac{n^2}{n!}$

Sol. Here, $u_n=\frac{n^2}{n!}$

Therefore, $u_{n+1}=\frac{(n+1)^2}{(n+1)!}=\frac{(n+1)^2}{(n+1)n!}=\frac{n+1}{n!}$

Therefore, $\frac{u_n}{u_{n+1}}=\frac{n^2}{n+1}=\frac{1}{\frac{1}{n}+\frac{1}{n^2}}$

Therefore, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{0} = \infty$

Therefore, by D'Alembert's Ration test, $\sum_{n=1}^{\infty} u_n$ is convergent.

Example 3. Test the convergence of the following series:

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}.$$

Sol. Here, $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

Therefore, $(u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^{n} = \left(\frac{n+1}{n}\right)^{-n}$

$$= \left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1}$$

Therefore, $\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} = e^{-1} = \frac{1}{e} < 1$

Therefore, by Cauchy's Root Test, the given series $\sum_{n=1}^{\infty} u_n$ is convergent.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Every absolutely convergent series is convergent.
Problem 2. Every convergent series is absolutely convergent.
Problem 3. The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ is convergent.
Problem 4. A positive term series converges if and only if the sequence of its partial sums is bounded above.
Problem 5. The series $\sum_{n=1}^{\infty} n$ is convergent.

5.10 *SUMMARY*

1. If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series, then the series of its positive terms and the series of its negative terms are both convergent.

2. If $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, then the series of its positive terms and the series of its negative terms are both divergent.

3. A series with mixed signs cannot converge if the series of its positive terms is convergent (divergent) and the series of its negative terms is divergent (convergent).

4. Test 1. If $\sum u_n$ and $\sum v_n$ are series of positive terms and $\sum v_n$ is convergent and there is a positive constant k such that $u_n \leq k v_n, \forall\, n$, then $\sum u_n$ is also convergent.

Test 2. If $\sum u_n$ and $\sum v_n$ are series of positive terms and $\sum v_n$ is divergent and there is a positive constant k such that $u_n \geq k v_n, \forall\, n$, then $\sum u_n$ is also divergent.

Test 3. If $\sum u_n$ and $\sum v_n$ are series of positive terms and

(i) if $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1$(finite and non-zero), then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

(ii) if $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges, then $\sum u_n$ is also converges.

(iii) if $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$ and $\sum v_n$ diverges, then $\sum u_n$ is also diverges.

5.11 GLOSSARY

sequence

limit

5.12 *REFERENCES*

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5.13 *SUGGESTED READING*

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5.14 *TERMINAL AND MODEL QUESTIONS*

Q 1. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n .\ n!}{n^n}$.

Q 2. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Q 3. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(logn)^n}$.

Q 4. Define Cauchy's Roots Test.

Q 5. Test the convergence and absolutely convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+2} .$$

5.15 *ANSWERS*

TQ1. Convergent.

TQ2. Convergent.

TQ3. Convergent.

TQ5. Not convergent

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. True

CYQ 4. True

CYQ 5. False

Course Name: REAL ANALYSIS

Course Code: MT(N) 201

BLOCK-II

FUNCTIONS SINGLE VARIABLE

UNIT 6: LIMITS

CONTENTS:

6.1 *INTRODUCTION*

In Mathematics, the limit of a function is a fundamental concept in calculus and analysis concerning the behaviour of that function near a particular input which may or may not be in the domain of the function.

In previous unit we discussed about sequence and series. In this unit we will be discussed about limit of one variable function.
The limit of a function is defined as the unique real number that the functions take when the variable of the function approaches a particular point. For any given function f(x), and a real number ‘c’, the limit of the function is defined as,

$$\lim_{x \to a} f(x) = L$$

This is read as, “limit of f(x), as x approaches a equals L”

6.2 *OBJECTIVES*

After studying this unit, learner will be able to

(i) Neighborhood

(ii) Interior point

(iii) Open set

(iv) Limit point

6.3 *LIMIT*

■ (ε, δ) definition of Limit.

Let $A \subseteq R$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number l is said to be a limit of f at c if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that
if $x \in A$ and $0 < |x - c| < d$, then $|f(x) - l| < \varepsilon$.

Note: If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.

Proof: Let l and l' be the limits of function f.

For any $\varepsilon > 0$, there exists $\delta_1\left(\frac{\varepsilon}{2}\right) > 0$ such that if $x \in A$ and

$0 < |x - c| < \delta_1\left(\frac{\varepsilon}{2}\right)$, then $|f(x) - l| < \frac{\varepsilon}{2}$.

and there exists $\delta_2\left(\frac{\varepsilon}{2}\right) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_2\left(\frac{\varepsilon}{2}\right)$, then $|f(x) - l'| < \frac{\varepsilon}{2}$.

Let $\delta = \inf\left\{\delta_1\left(\frac{\varepsilon}{2}\right), \delta_2\left(\frac{\varepsilon}{2}\right)\right\}$. Then if $x \in A$ and $0 < |x - c| < \delta$.

The Triangle Inequality implies that

$|l - l'| = \left|l + \left(-f(x) + f(x)\right) - l'\right| \le |l - f(x)| + |f(x) - l'|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

Because $\varepsilon > 0$ is arbitrary. Therefore, $l - l' = 0 \Rightarrow l = l'$.

Example 6.3.1. Prove that $\lim_{x \to a} c = c$

Sol. Let $f(x) = c \; for \; all \; x \in \mathbb{R}$.

Now we will try to prove that $\lim_{x \to a} f(x) = c$.

Let $\varepsilon > 0$ and $\delta = 1$.

Then if $0 < |x - a| < 1$, we have

$|f(x) - c| = |c - c| = 0 < \varepsilon$.

As $\varepsilon > 0$ is arbitrary, by definition of limit we get

$\lim_{x \to a} f(x) = c$.

Example 6.3.2. Prove that $\lim_{x \to b} x^2 = b^2$

Sol. Let $f(x) = \boldsymbol{x^2} \; for \; all \; x \in \mathbb{R}$.

Now we will try to prove that $\lim_{x \to a} f(x) = b^2$.

Now we try to prove that $|f(x) - b^2| = |x^2 - b^2|$ less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to b.

Now

$x^2 - b^2 = (x - b)(x + b)$.

If $|\, x - b| < 1$, then

$|x| < |b| + 1$

Hence $|x + b| \le |x| + |b| < |b| + 1 + |b| < 2|b| + 1$

Thus, if $|x - b| < 1$ then

$|x^2 - b^2| \le |x - b||x + b| < (2|b| + 1)|x - b|$ ……(1)

Let $|x - b| < \frac{\varepsilon}{2|c|+1}$ and we choose $\delta(\varepsilon) = \inf\left\{1, \frac{\varepsilon}{2|c|+1}\right\}$, Then if $0 < |x - b| < \delta(\varepsilon)$,

Now if $|x - b| < 1$, then equation (1) is valid.

If $|x - b| < \frac{\varepsilon}{2|b|+1}$ then

$|x^2 - b^2| < (2|b| + 1)|x - b| < (2|b| + 1).\frac{\varepsilon}{2|c|+1} < \varepsilon$

As we have choice to choose $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$.

We deduce that $\lim_{x\to b} x^2 = b^2$

Example 6.3.3. Prove that $\lim_{x\to b} \frac{1}{x} = \frac{1}{b}$ $if\ b > 0$

Proof. Let $f(x) = \frac{1}{x}$ $for\ all\ x > 0$ and assume $b > 0$

Now we will try to prove that $\lim_{x\to a} f(x) = \frac{1}{b}$.

Therefore, we will try to prove that the difference $\left|f(x) - \frac{1}{b}\right| = \left|\frac{1}{x} - \frac{1}{b}\right|$ less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $b > 0$.

Now

$\left|\frac{1}{x} - \frac{1}{b}\right| = \left|\frac{1}{bx}(b - x)\right| = \frac{1}{bx}|x - b|$ $for\ x > 0$.

Now if $|\,x - b| < \frac{1}{2}b$ then

$-\frac{1}{2}b < x - b < \frac{1}{2}b \Rightarrow \frac{1}{2}b < x < \frac{3}{2}b \Rightarrow \frac{1}{2}b^2 < bx \Rightarrow \frac{2}{b^2} > \frac{1}{bx}$.

Therefore

$0 < \frac{1}{bx} < \frac{2}{b^2}$ for $|\,x - b| < \frac{1}{2}b$

Hence, for these values of x we have

$\left|f(x) - \frac{1}{b}\right| < \frac{2}{b^2}|x - b|$ ....(1)

In order to make this last term less than ε it suffices to take

$|x - b| < \frac{1}{2}b^2\varepsilon$. Consequently, if we choose $\delta(\varepsilon) = \inf\left\{\frac{1}{2}b, \frac{1}{2}b^2\varepsilon\right\}$,

Then if $0 < |x - b| < \delta(\varepsilon)$,

Now if $|x - b| < \frac{1}{2}b$, then equation (1) is valid.

Therefore, since $|x - b| < \frac{1}{2}b^2\varepsilon$, that

$\left|f(x) - \frac{1}{b}\right| = \left|\frac{1}{x} - \frac{1}{b}\right| < \varepsilon$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we conclude that

$\lim_{x\to b} \frac{1}{x} = \frac{1}{b}$ $if\ b > 0$.

Theorem 6.3.1. Let $X \subseteq \mathbb{R}$ and $f, g: X \to \mathbb{R}$ and let $b \in \mathbb{R}$ be a cluster point of X and $\in \mathbb{R}$.

(a) If $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} g = l_2$, then

(i) $\lim_{x\to b} f + g = l_1 + l_2$ (ii) $\lim_{x\to b} f - g = l_1 - l_2$

(iii) $\lim_{x\to b} fg = l_1 l_2$

(b) If $h: X \to \mathbb{R}$ and $h(x) \neq 0$ for all $x \in X$, if $\lim_{x\to b} h = l_3 \neq 0$, then $\lim_{x\to b} \frac{f}{h} = \frac{l_1}{l_3}$

Proof. (a) It is given that $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} g = l_2$, Hence for any $\varepsilon > 0$ there exists a positive number δ_1 and δ_2 such that

$|f(x) - l_1| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta_1$ and $|g(x) - l_2| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$, then

$|f(x) - l_1| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta$ ....(1)

and

$|g(x) - l_2| < \frac{\varepsilon}{2}$ when $0 < |x - b| < \delta$ ....(2)

Now, when $0 < |x - b| < \delta$

$$|(f+g)(x) - (l_1 + l_2)| = |f(x) - l_1 + g(x) - l_2|$$
$$\leq |f(x) - l_1| + |g(x) - l_2|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore

$|(f+g)(x) - (l_1 + l_2)| < \varepsilon$ when $0 < |x - b| < \delta$

Thus, $\lim_{x\to b} f + g = l_1 + l_2$

(ii) When $0 < |x - b| < \delta$

$$|(f-g)(x) - (l_1 - l_2)| = |f(x) - l_1 + g(x) - l_2|$$
$$\leq |f(x) - l_1| + |l_2 - g(x)|$$
$$= |f(x) - l_1| + |g(x) - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(From (1) and (2))

$|(f-g)(x) - (l_1 - l_2)| < \varepsilon$ when $0 < |x - b| < \delta$

Thus, $\lim_{x\to b} f - g = l_1 - l_2$

(iii) $|(fg)(x) - (l_1 l_2)| = |f(x)g(x) - l_1 l_2|$

$$= |f(x)g(x) - f(x)l_2 + f(x)l_2 - l_1 l_2|$$
$$= |f(x)(g(x) - l_2) + l_2(f(x) - l_1)|$$
$$\leq |f(x)||g(x) - l_2| + |l_2||f(x) - l_1| \quad .....(3)$$

As we know that $\lim_{x\to b} f = l_1$. Hence for any $\varepsilon = 1$ there exists a positive number δ'_1 such that

$|f(x) - l_1| < 1$ when $0 < |x - b| < \delta'_1$.

Now

$|f(x)| = |f(x) - l_1 + l_1| \leq |f(x) - l_1| + |l_1|$

$< 1 + |l_1|$, when $0 < |x - b| < \delta'_1$ ....(4)

$\lim_{x\to b} g = l_2$, there exists a positive number δ'_2 such that

$|g(x) - l_2| < \frac{\frac{\varepsilon}{2}}{1+|l_1|}$ when $0 < |x - b| < \delta'_2$ ....(5)

Again $\lim_{x\to b} f = l_1$, there exists a positive number δ'_3 such that

$|f(x) - l_1| < \frac{\frac{\varepsilon}{2}}{|l_2|}$ when $0 < |x - b| < \delta'_3$ ......(6)

Let $\delta' = \min\{\delta'_1, \delta'_2, \delta'_3\}$. Then from (3), (4), (5) and (6),
when $0 < |x - b| < \delta'$

$|(fg)(x) - (l_1 l_2)| < (1 + |l_1|)\frac{\frac{\varepsilon}{2}}{1+|l_1|} + |l_2|\frac{\frac{\varepsilon}{2}}{|l_2|} < \varepsilon.$

Hence $\lim_{x\to b} fg = l_1 l_2$.

(b) $\lim_{x\to b} h = l_3 \neq 0$ therefore for $\varepsilon = \frac{|m|}{2} > 0$ there exists $\delta_3 > 0$ such that

$|h(x) - l_3| < \frac{|m|}{2}$ when $0 < |x - b| < \delta_3$

Now

$|l_3| = |l_3 - h(x) + h(x)| \leq |l_3 - h(x)| + |h(x)| = |h(x) - l_3| + |h(x)|$ or

or $|l_3| < \frac{|l_3|}{2} + |h(x)| \Rightarrow |h(x)| > |l_3| - \frac{|l_3|}{2} = \frac{|l_3|}{2}. or \ \frac{1}{|h(x)|} < \frac{2}{|l_3|}$

It implies that there exists a deleted neighbourhood of b on which $h(x)$ does not vanish.

Now, when $0 < |x - b| < \delta_3$

$$\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| = \left|\frac{f(x)}{h(x)} - \frac{l_1}{l_3}\right| = \left|\frac{f(x)l_3 - h(x)l_1}{h(x)l_3}\right|$$

$$= \left|\frac{f(x)l_3 - l_1 l_3 + l_1 l_3 - h(x)l_1}{h(x)l_3}\right|$$

$$= \left|\frac{l_3(f(x)-l_1) + l_1(l_3 - h(x))}{h(x)l_3}\right|$$

$$\leq \frac{1}{|h(x)|}|f(x) - l_1| + \frac{|l_1|}{|l_3||h(x)|}|h(x) - l_3|$$

$$< \frac{2}{|l_3|} . |f(x) - l_1| + \frac{2}{|l_3|}\frac{|l_1|}{|l_3|}|h(x) - l_3|$$

$$= \frac{2}{|l_3|} . |f(x) - l_1| + \frac{2|l_1|}{|l_3|^2}|h(x) - l_3| \qquad ....(7)$$

Let $\varepsilon > 0$ be given.

It is given that $\lim_{x\to b} f = l_1$ and $\lim_{x\to b} h = l_3$, hence there exists positive numbers δ''_1 and δ''_2 such that

$|f(x) - l_1| < \frac{1}{4}\varepsilon|l_3|$, when $0 < |x - b| < \delta''_1$ ....(8)

$|h(x) - l_3| < \frac{1}{4}\varepsilon \frac{|l_3|^2}{|l_1|}$, when $0 < |x - b| < \delta''_2$ .....(9)

Let $\delta'' = \min\{\delta_3, \delta''_1, \delta''_2\}$. Then from (7), (8) and (9), we get

$\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| < \frac{2}{|l_3|} . \frac{1}{4}\varepsilon|l_3| + \frac{2|l_1|}{|l_3|^2} . \frac{1}{4}\varepsilon \frac{|l_3|^2}{|l_1|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ when $0 < |x - b| < \delta''$

Therefore $\left|\left(\frac{f}{h}\right)(x) - \frac{l_1}{l_3}\right| < \varepsilon$ when $0 < |x - b| < \delta''$

Hence $\lim_{x\to b} \frac{f}{h} = \frac{l_1}{l_3}$

Ex.6.3.4. Find $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

Proof. It is given that $lim_{x\to 0}\ \frac{\sqrt{4+x}-2}{x}$

$lim_{x\to 0}\ \frac{\sqrt{4+x}-2}{x} = lim_{x\to 0}\ \frac{\sqrt{4+x}-2}{x}.\frac{\sqrt{4+x}+2}{\sqrt{4+x}-2} = lim_{x\to 0}\ .\frac{4+x-4}{\sqrt{4+x}+2} =$
$lim_{x\to 0}\ \frac{x}{\sqrt{4+x}+2} = \frac{1}{4}$.

6.4 VARIABLE

A symbol such as *x* or *y*, used to represent an arbitrary element of a set is called a variable. For example *y* = *f* (*x*).

The symbol *x* which represents an element in the domain is called the independent variable, and the symbol *y* which represent the element corresponding to *x* is called the dependent variable. This is based on the fact that value of *x* can be arbitrary chosen, then *y* has a value which depends upon the chosen value of *x*.

6.5 *LIMIT*

■ Definition of Limit.

f(x) is said to tend to a limit as x tends to 'a' if both the left and right hand limits exist and equal, and their common value is called the limit of the function.

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$ where, $h > 0$ is called left hand limit (L.H.L.)

And

$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$ where, $h > 0$ is called right hand limit (R.H.L.)

If **L.H.L. = R.H.L.** then $\lim_{x\to a} f(x)$ exist.

And if **L.H.L. ≠ R.H.L.** then $\lim_{x\to a} f(x)$ does not exist.

Example 6.5.1. Do the following limits exists? if yes, find them.

(i) $\lim_{x\to 1} \sin\frac{1}{x-1}$

(ii) $\lim_{x\to 0} x \sin\frac{1}{x}$

(iii) $\lim_{x\to 1} 2^{\frac{1}{x-1}}$

(iv) $\lim\limits_{x\to 0}\frac{e^{1/x}}{e^{1/x}+1}$

Sol. (i) L.H.L. = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h)$

$= \lim\limits_{h\to 0}\sin\frac{1}{1-h-1} = -\lim\limits_{h\to 0}\sin\frac{1}{h}$.

Now as $h \to 0$, $\sin\frac{1}{h}$ is finite and oscillates between -1 and 1, so it does not tend to any unique and definite value as $h \to 0$. Hence L.H.L. does not exist.

Similarly, the right-hand limit also does not exist as $x \to 1$.

Thus $\lim\limits_{x\to 1}\sin\frac{1}{x-1}$ does not exist.

(ii) L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0}(0-h)\sin\frac{1}{0-h} = \lim\limits_{h\to 0}\text{h}\sin\frac{1}{h}$

$= 0 \times$ a finite quantity between -1 and 1

= 0.

Similarly, R.H.L. = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0}(0+h)\sin\frac{1}{0+h} = \lim\limits_{h\to 0}\text{h}\sin\frac{1}{h}$

$= 0 \times$ a finite quantity between -1 and 1

= 0.

Thus L.H.L. and R.H.L. both exist and are equal, and hence $\lim\limits_{x\to 0} x\sin\frac{1}{x}$ exists and is equal to zero.

$\therefore \quad \lim\limits_{x\to 0} x\sin\frac{1}{x} = 0.$

(iii) L.H.L. = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h)$

$= \lim\limits_{h\to 0} 2^{\frac{1}{1-h-1}} = \lim\limits_{h\to 0} 2^{\frac{1}{-h}} = 2^{-\infty} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0.$

R.H.L. = $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h)$

$= \lim\limits_{h\to 0} 2^{\frac{1}{1+h-1}} = \lim\limits_{h\to 0} 2^{\frac{1}{h}} = 2^{\infty} = \infty.$

Since L.H.L.$\neq$ R.H.L.

$\therefore \quad \lim\limits_{x\to 1} 2^{\frac{1}{x-1}}$ does not exist.

(iv) L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0}\frac{e^{\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+1} = \lim\limits_{h\to 0}\frac{e^{-\frac{1}{h}}}{e^{-\frac{1}{h}}+1} = \frac{0}{0+1} = 0$

R.H.L. = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0}\frac{e^{\frac{1}{0+h}}}{e^{\frac{1}{0+h}}+1} = \lim\limits_{h\to 0}\frac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}+1} = \lim\limits_{h\to 0}\frac{1}{1-e^{-\frac{1}{h}}}$

$= \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1.$

Since L.H.L.≠ R.H.L.

∴ $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$ does not exist.

Example 6.5.2. Find $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} \frac{x^2}{a} - a & for\ 0 < x < a \\ a - \frac{a^3}{x^2} & for\ x > a \end{cases}$.

Sol. L.H.L. = $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

$$= \lim_{h\to 0}\left[\frac{(a-h)^2}{a} - a\right] = \frac{a^2}{a} - a = a - a = 0$$

R.H.L. = $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

$$= \lim_{h\to 0}\left[a - \frac{a^3}{(a+h)^2}\right] = a - \frac{a^3}{a^2} = a - a = 0.$$

Therefore, L.H.L. and R.H.L. both exist and each equal to 0.

∴ $\lim_{x\to 0} f(x) = 0.$

6.6 INFINITE LIMITS

A function $f(x)$ is said to approach + ∞ or - ∞ as $x \to a$, if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) > \varepsilon \text{ or } f(x) < -\varepsilon \text{ whenever } 0 < |x-a| < \delta .$$

Then in other words, $\underset{x\to a}{Limit}\ f(x) = \infty$ or $\underset{x\to a}{Limit}\ f(x) = -\infty$.

Example 6.6.1. Find $\underset{x\to 0}{Limit} \frac{\sin x}{x}$.

Solution. Let $f(x) = \frac{\sin x}{x}$ Here

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit} \frac{\sin h}{h}$$

$$= \underset{h\to 0}{Limit} \frac{h - h^3/3! + h^5/5! - h^7/7! + ...}{h}$$

$$= \underset{h\to 0}{Limit}\, 1 - h^2/3! + h^4/5! - h^6/7! + ... = 1$$

And

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h) = \underset{h\to 0}{Limit}\ \frac{\sin(-h)}{-h}$$

$$= \underset{h\to 0}{Limit}\ \frac{\sin(h)}{h} = 1.$$

Since $f(0+0) = f(0-0) = 1$ and hence $\underset{h\to 0}{Limit}\ \frac{\sin x}{x} = 1.$

Example 6.6.2.. Find $\underset{x\to\infty}{Limit}\ \frac{\sin x}{x}$.

Solution. Let $f(x) = \frac{\sin x}{x}$. Put $x = 1/y$ so as $x \to\infty$, $y\to 0$. Then

$$\underset{x\to\infty}{Limit}\ \frac{\sin x}{x} = \underset{y\to 0}{Limit}\ \frac{\sin(1/y)}{1/y} = \underset{y\to 0}{Limit}\ y\sin\left(\frac{1}{y}\right)$$

Let $g(y) = y\sin\left(\frac{1}{y}\right)$. Then, right hand limit is

$$g(0+0) = \underset{h\to 0}{Limit}\ g(0+h) = \underset{h\to 0}{Limit}\ g(h)$$

$$= \underset{h\to 0}{Limit}\ h\sin\left(\frac{1}{h}\right)$$

$$= 0\times \text{finite quantity which lies between -1 and +1}$$

$$= 0$$

and the left hand limit is

$$g(0-0) = \underset{h\to 0}{Limit}\ g(0-h) = \underset{h\to 0}{Limit}\ g(-h)$$

$$= \underset{h\to 0}{Limit}\ h\sin\left(\frac{1}{h}\right) = 0$$

Since $g(0+0) = g(0-0) = 0$ therefore $\underset{y\to 0}{Limit}\ y\sin\left(\frac{1}{y}\right) = 0$ and hence

$\underset{x\to\infty}{Limit}\ \frac{\sin x}{x} = 0.$

Example 6.6.3. Find $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$.

Solution. Let $f(x) = \sin\left(\frac{1}{x}\right)$. Here

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit}\ \sin\left(\frac{1}{h}\right)$$

As $h \to 0$, the value of $\sin\left(\frac{1}{h}\right)$ oscillates between -1 and +1 passing through zero. Hence there is no definite number l to which $\sin\left(\frac{1}{h}\right)$ tends to as $h \to 0$. Therefore right hand limit does not exist. Similarly left hand limit $f(0-0)$ also does not exists.

Thus $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$ does not exist.

Example 6.6.4. Find $\underset{x\to 0}{Limit}\ (1+x)^{1/x}$.

Solution. Let $f(x) = \underset{x\to 0}{Limit}\ (1+x)^{1/x}$. Now right hand limit is

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit}\ (1+h)^{1/h}$$

$$= \underset{h\to 0}{Limit}\left[1+\frac{1}{h}.h+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2!}.h^2+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)\left(\frac{1}{h}-2\right)}{3!}.h^3+....\right]$$

$$= \underset{h\to 0}{Limit}\left[1+\frac{1}{1!}+\frac{1.(1-h)}{2!}+\frac{1.(1-h)(1-2h)}{3!}+....\right]$$

$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+....\infty = e$$

Similarly, the left hand limit is

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h) = \underset{h\to 0}{Limit}\ (1-h)^{-1/h} = e$$

Thus both f (0 + 0) and f (0 – 0) exists and equal to e. Hence $\underset{x\to 0}{Limit}\,(1+x)^{1/x}=e$.

Example 6.6.5. Show that $\underset{x\to 2}{Limit}\,\frac{|x-2|}{(x-2)}$ does not exist.

Solution. Let $f(x)=\underset{x\to 2}{Limit}\,\frac{|x-2|}{(x-2)}$. Now right hand limit is

$$f(2+0)=\underset{h\to 0}{Limit}\,f(2+h)=\underset{h\to 0}{Limit}\,\frac{|2+h-2|}{(2+h-2)}$$

$$=\underset{h\to 0}{Limit}\,\frac{|h|}{(h)}=\underset{h\to 0}{Limit}\,\frac{h}{h}=1$$

and the left hand limit is

$$f(2-0)=\underset{h\to 0}{Limit}\,f(2-h)=\underset{h\to 0}{Limit}\,\frac{|2-h-2|}{(2-h-2)}$$

$$=\underset{h\to 0}{Limit}\,\frac{|-h|}{(-h)}=\underset{h\to 0}{Limit}\,\frac{h}{-h}=-1$$

Since f (2 + 0) ≠ f (2 – 0). Hence $\underset{x\to 2}{Limit}\,\frac{|x-2|}{(x-2)}$ does not exist.

Example 6.6.6. Find $\underset{x\to 0}{Limit}\,\frac{1}{x}e^{1/x}$.

Solution. Let $f(x)=\underset{x\to 0}{Limit}\,\frac{1}{x}e^{1/x}$. Then

$$f(0+0)=\underset{h\to 0}{Limit}\,f(0+h)=\underset{h\to 0}{Limit}\,f(h)=\underset{h\to 0}{Limit}\,\frac{1}{h}e^{1/h}$$

$$=\infty\ (\text{since } \frac{1}{h}\to\infty \text{ and } e^{1/h}\to\infty \text{ as } h\to 0)$$

and

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h) = \underset{h\to 0}{Limit}\ -\frac{1}{h}e^{-1/h}$$

$$= \underset{h\to 0}{Limit}\ -\frac{1}{he^{1/h}}$$

$$= \underset{h\to 0}{Limit}\ \frac{-1}{h\left[1+\frac{1}{h}+\frac{1}{2!}\frac{1}{h^2}+\frac{1}{3!}\frac{1}{h^3}+....\infty\right]} = 0$$

Since $f(0+0) \neq f(0-0)$. Hence $\underset{x\to 0}{Limit}\ \frac{1}{x}e^{1/x}$ does not exist.

6.7 *L HOSPITAL RULE*

L'Hospital's rule is totally different from the quotient law of differentiation. There is a solid logical base that why we only differentiate numerator and denominator directly, instead of using quotient law of differentiation.

(2) It must be clearly remembered that L'Hospital's method be used only in the situations of $\frac{0}{0}$ and $\frac{\infty}{\infty}$ not in other cases.

(3) In L'Hospital's rule, numerator *f*(*x*) and denominator *g*(*x*) are to be differentiated separately.

(4) It may be helpful for students that $\log_e 1 = 0, \log_e 0 = -\infty, \log_e \infty = +\infty, e^0 = 1, e^{-\infty} = 0$, $e^{\infty} = \infty$.

Example 6.7.1. Evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x}$.

Sol. Clearly, $\lim\limits_{x\to 0} \frac{\sin x}{x}$ is a $\frac{0}{0}$ form.

L'Hospital'sRule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Note: In second method, dash (′) above $\sin x$ and x represents the first derivative with respect to x (variable with respect to the limit has been taken).

Example 6.7.2. Evaluate $\lim_{x \to 0} \frac{\log x}{x-1}$.

Sol. Clearly, $\lim_{x \to 0} \frac{\log x}{x-1}$ is a $\left(\frac{0}{0}\right)$ form.

L'Hospital's Rule:

$$\lim_{x \to 1} \frac{\log x}{x-1} = \lim_{x \to 1} \frac{(\log x)'}{(x-1)'} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{1} = 1.$$

Example 6.7.3. Find $\lim_{x \to 0} \frac{x - \sin x}{x^2}$

Sol. L'Hospital's Rule:

$$\lim_{x \to 0} \frac{(x - \sin x)}{x^3} = \lim_{x \to 0} \frac{(x - \sin x)'}{(x^3)'}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)}{3x^2}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)'}{(3x^2)'}$$

$$= \lim_{x \to 0} \frac{\{-(-\sin x)\}}{6x}$$

$$= \lim_{x \to 0} \frac{(\sin x)'}{(6x)'}$$

$$= \lim_{x \to 0} \frac{(\cos x)}{6}$$

$$= \frac{1}{6}.$$

Example 6.7.4. Find $\lim_{x \to 0} \frac{e^x - 1}{x}$

Sol. L'Hospital's Method:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{(e^x - 1)'}{(x)'} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

Example 6.7.5. Evaluate $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$

Sol. L'Hospital's Rule:

$$\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2} \qquad \mathbf{\left(\frac{0}{0} form\right)}$$

$$= \lim_{x \to 0} \frac{\{x \cos x - \log(1+x)\}'}{(x^2)'}$$

$$= \lim_{x\to 0} \frac{1.\cos x - x(\sin x) - \frac{1}{(1+x)}}{2x} \qquad (\tfrac{0}{0}\textbf{form})$$

$$= \lim_{x\to 0} \frac{(\cos x - x\sin x - \frac{1}{(1+x)})'}{(2x)'}$$

$$= \lim_{x\to 0} \frac{-\sin x - (1.\sin x + x.\cos x) + \frac{1}{(1+x)^2}}{2}$$

$$= \lim_{x\to 0} \frac{-2\sin x - x.\cos x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{1}{2}.$$

Note: Observe that L'Hospital's rule is sometimes easier than the algebraic method. We will explain next examples only by L'Hospital's rule.

Example 6.7.6. Find $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

Sol. $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to 0} \frac{\{(1+x)^n - 1\}'}{\{x\}'}$

$$= \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}$$

$$= n$$

Example 6.7.7. Evaluate $\lim_{x\to 0} \frac{a^x - x^a}{x^x - a^a}$

Sol. $\lim_{x\to a} \frac{a^x - x^a}{x^x - a^a} = \lim_{x\to a} \frac{(a^x - x^a)'}{(x^x - a^a)'} = \lim_{x\to a} \frac{a^x \log a - a.x^{a-1}}{x^x(\log x + 1)}$

$$= \frac{a^a \log a - a.a^{n-1}}{a^a \log a + a^a} = \frac{a^a(\log a - 1)}{a^a(\log a + 1)} = \frac{\log a - 1}{\log a + 1}$$

Note: The first derivate of x^x in above example calculated as follows:

$y = x^x$

Taking logarithms

$\log y = x \log x$

Now differentiating both sides with respect to x

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \log x + 1$$

$$\left(\frac{dy}{dx}\right) = \text{y}(\log x + 1) = x^x(\log x + 1).$$

Example 6.7.8. Evaluate $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$

Sol. $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x}$

$$= \lim_{x\to 0} \frac{(5\sin x - 7\sin 2x + 3\sin 3x)'}{(\tan x - x)'}$$

$$= \lim_{x\to 0} \frac{5\cos x - 7\times 2\cos 2x + 3\times 3\cos 3x}{sec^2 x - 1}$$

$$= \lim_{x\to 0} \frac{(5\cos x - 14\cos 2x + 9\cos 3x)'}{(sec^2 x - 1)'}$$

$$= \lim_{x\to 0} \frac{-5\sin x + 14.2\sin 2x - 9.3\sin 3x}{2\sec x\tan x}$$

$$= \lim_{x\to 0} \frac{-5\cos x + 28.2\cos 2x - 27.3\cos 3x}{2(\sec x\, sec^2 x + \sec x\tan x\tan x)}$$

$$= \frac{-5+56-81}{2} = \frac{-30}{2} = -15\,.$$

Example 6.7.9. Evaluate $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

Sol. $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} \frac{(e^x - e^{\sin x})'}{(x - \sin x)'}$

$$= \lim_{x\to 0} \frac{e^x - \cos x . e^{\sin x}}{1 - \cos x}$$

$$= \lim_{x\to 0} \frac{(e^x - \cos x . e^{\sin x})'}{(1 - \cos x)'}$$

$$= \lim_{x\to 0} \frac{e^x - \{\cos x . \cos x . e^{\sin x} + (-\sin x). e^{\sin x}\}}{\sin x}$$

$$= \lim_{x\to 0} \frac{\{e^x - \cos^2 x . e^{\sin x} + \sin x . e^{\sin x}\}'}{(\sin x)'}$$

$$= \lim_{x\to 0} \frac{e^x - \left\{2\cos x\,.(-\sin x).\,e^{\sin x} + \cos^3 x\,.\,e^{\sin x}\right\} + \{\cos x.\ e^{\sin x} + \sin x\cos x.\,e^{\sin x}\}}{\cos x}$$

$$= \lim_{x\to 0} \frac{e^x + 3\sin x\cos x.\ e^{\sin x} - \cos^3 x . e^{\sin x} + \cos x.\ e^{\sin x}}{\cos x}$$

$= \frac{1-1+1}{1}$

$= 1.$

Case II: Form $\frac{\infty}{\infty}$

Example 6.7.10. Evaluate $lim_{x\to 0}\ \frac{n^2+5}{n^2+4n+3}$.

Sol. Clearly, $\lim_{n\to\infty} \frac{n^2+5}{n^2+4n+3}$ is a $\frac{\infty}{\infty}$ form.

Algebraic Method:

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4n+3} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{5}{n^2}\right)}{n^2\left(1+\frac{4}{n}+\frac{3}{n^2}\right)} = \lim_{n\to\infty} \frac{\left(1+\frac{5}{n^2}\right)}{\left(1+\frac{4}{n}+\frac{3}{n^2}\right)} = 1 .$$

L'Hospital's Method:

$\lim_{n\to\infty} \frac{n^2+5}{n^2+4n+3}$

(Again $\frac{\infty}{\infty}$ form)

$= \lim_{n\to\infty} \frac{(2n)'}{(2n+4)'}$

$= \lim_{n\to\infty} \frac{2}{2}$

$= 1 .$

Example 6.7.11. Evaluate $\lim_{x\to 0} \frac{\log x}{\cot x}$

Sol. This is of the form $\frac{\infty}{\infty}$. We have therefore,

$$\lim_{x\to 0} \frac{log\ x}{cot\ x} = \lim_{x\to 0} \frac{(log\ x)'}{(cot\ x)'} = \lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{-cosec^2 x} \left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to 0} \frac{-\ sin^2 x}{x} \left(\frac{0}{0}\ form\right)$$

$= \lim_{x\to 0} \frac{-2sin\ x\ cos\ x}{1} = 0.$

Example 6.7.12. Find $\lim_{x\to\frac{\pi}{2}} \frac{\log(x-\frac{\pi}{2})}{\tan x}$

Sol. $\lim_{x\to\frac{\pi}{2}} \frac{\log(x-\frac{\pi}{2})}{\tan x}$ is a $\frac{\infty}{\infty}$ form.

We have,

$\lim_{x\to\frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{[\log\left(x-\frac{\pi}{2}\right)]'}{(\tan x)'} =$

$\lim_{x\to\frac{\pi}{2}} \frac{\left(\frac{1}{\left(x-\frac{\pi}{2}\right)}\right)}{sec^2 x}$ $\left(\frac{\infty}{\infty}\ form\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{cos^2 x}{\left(x-\frac{\pi}{2}\right)}$ $\left(\frac{0}{0}\ form\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{(cos^2 x)'}{\left(x-\frac{\pi}{2}\right)'}$

$= \lim_{x\to\frac{\pi}{2}} \frac{-2\cos x \sin x}{1}$

$= 0.$

Example 6.7.13. Evaluate $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)}$

Sol. $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-e^a)} \left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to a} \frac{(\log(x-a))'}{(\log(e^x-e^a))'}$

$= \lim_{x\to a} \frac{\left(\frac{1}{x-a}\right)}{\left(\frac{1}{e^x-e^a}\right)e^x}$

$= \lim_{x\to a} \frac{e^x-e^a}{(x-a)e^x} \left(\frac{0}{0}\ form\right)$

$= \lim_{x\to a} \frac{(e^x-e^a)'}{[(x-a)e^x]'}$

$= \lim_{x\to a} \frac{e^x}{(x-a)e^x+e^x}$

$= \lim_{x\to a} \frac{e^x}{[(x-a)+1]e^x}$

$= \lim_{x\to a} \frac{1}{[(x-a)+1]}$

$= 1$

Example 6.7.14. Find $\lim_{x\to\infty} \frac{e^x + 3x^3}{4e^x + 4x}$

Sol.

$\lim_{n\to\infty} \frac{e^x+3x^3}{4e^x+4x}$ $\left(\frac{\infty}{\infty}\ form\right)$

$= \lim_{n\to\infty} \frac{(e^x+3x^3)'}{(4e^x+4x)'}$

$= \lim_{n\to\infty} \frac{e^x+9x^2}{4e^x+4}$ $\left(\frac{\infty}{\infty}\ form\right)$

$= \lim_{n\to\infty} \frac{(e^x+9x^2)'}{(4e^x+4)'}$

$= \lim_{n\to\infty} \frac{e^x+18x^1}{4e^x}$ $\left(\frac{\infty}{\infty}\ form\right)$

$= \lim_{n\to\infty} \frac{(e^x+18x^1)'}{(4e^x)'}$

$= \lim_{n\to\infty} \frac{(e^x+18)}{4e^x}$ $\left(\frac{\infty}{\infty}\ form\right)$

$= \lim_{n\to\infty} \frac{e^x}{4e^x}$

$= \frac{1}{4}.$

Example 6.7.15. Evaluate $\lim_{x\to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$

Sol. We have,

$$\lim_{x\to 0} \frac{\log(tan^2 2x)}{\log(tan^2 x)} \quad \left(\frac{\infty}{\infty} form\right)$$

$= \lim_{x\to 0} \frac{2\log(\tan 2x)}{2\log(\tan x)}$ $\left(\frac{\infty}{\infty} form\right)$

$= \lim_{x\to 0} \frac{(\log(\tan 2x))'}{(\log(\tan x))'} = \lim_{x\to 0} \frac{\left(\frac{1}{\tan 2x}\right).2\ sec^2 2x}{\left(\frac{1}{\tan x}\right).\ sec^2 x}$

$= \lim_{x\to 0} \frac{2\tan x\ cos^2 x}{\tan 2x\ cos^2 2x} = \lim_{x\to 0} \frac{2\sin x \cos x}{\sin 2x \cos 2x}$

$= \lim_{x\to 0} \frac{\sin 2x}{\sin 2x \cos 2x} = \lim_{x\to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1.$

Example 6.7.16. Evaluate $\lim_{x\to 0} \frac{\log(\sin x)}{\cot x}$

Sol. We have,

$$\lim_{x\to 0}\frac{\log(\sin x)}{\cot x}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to 0}\frac{(\log(\sin x))'\prime}{(\cot x)\prime}$$

$$= \lim_{x\to 0}\frac{\frac{1}{\sin x}.\cos x}{-cosec^2 x} = \lim_{x\to 0}\left(-\frac{\cos x}{\sin x}.sin^2 x\right)$$

$$= \lim_{x\to 0}(-\cos x.\sin x) = 0.$$

Example 6.7.17. Find $\lim_{n\to\infty}\frac{x^n}{e^x}$, where n is a positive integer.

Sol. We have,

$$\lim_{x\to\infty}\frac{x^n}{e^x}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to\infty}\frac{(x^n)'}{(e^x)'} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to\infty}\frac{\left(nx^{n-1}\right)'}{(e^x)\prime} = \lim_{n\to\infty}\frac{n(n-1)x^{n-2}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{n\to\infty}\frac{\left(n(n-1)x^{n-2}\right)'}{(e^x)\prime} = \lim_{n\to\infty}\frac{n(n-1)(n-2)x^{n-3}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$

Repeating this process, we get

$$= \lim_{n\to\infty}\frac{(n(n-1)(n-2)\dots n\ factors)}{e^x}$$

$$= \lim_{n\to\infty}\frac{n!}{e^x} = \frac{n!}{e^\infty} = \frac{n!}{\infty} = 0.$$

Example 6.7.18. Find $\lim_{x\to 0}\frac{\log\sin 2x}{\log\sin x}$

Sol. We have,

$$\lim_{x\to 0}\frac{\log\sin 2x}{\log\sin x} \qquad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to 0}\frac{(\log\sin 2x)\prime}{(\log\sin x)\prime}$$

$$= \lim_{x\to 0}\frac{\left(\frac{2}{\sin 2x}.\cos 2x\right)}{\left(\frac{1}{\sin x}.\cos x\right)}$$

$$= \lim_{x\to 0}\frac{2\cot 2x}{\cot x} \qquad \left(\frac{\infty}{\infty}form\right)$$

$$= \lim_{x\to 0}\frac{(2\cot 2x)\prime}{(\cot x)\prime}$$

$$= \lim_{x\to 0}\frac{-4\ cosec^2\ 2x}{-cosec^2\ x} \qquad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to 0} \frac{4\, sin^2 x}{sin^2 2x}$$

$$= \lim_{x\to 0} \frac{4\, sin^2 x}{(2\sin x \cos x)^2}$$

$$= \lim_{x\to 0} \frac{1}{cos^2 x} = 1.$$

Example 6.7.19. Find $\lim_{x\to\infty} \frac{\log x}{a^x},\ a > 1.$

Sol. We have,

$$\lim_{x\to\infty} \frac{\log x}{a^x}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to\infty} \frac{(\log x)'}{(a^x)'}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{a^x \log a}$$

$$= \frac{1}{\log a} \lim_{x\to\infty} \frac{1}{x\, a^x}$$

$$= \frac{1}{\log a} \times 0 = 0.$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. $\lim_{x\to 0} \frac{\sin x}{x}$ is 0.

Problem 2. $\lim_{x\to 0} \frac{\log x}{x-1}$ is 1.

Problem 3. $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$ does not exist.

Problem 4. $lim_{x\to 0}\ \frac{\sqrt{4+x}-2}{x}$ is 1.

Problem 5. $lim_{x\to 0}\ \frac{n^2+5}{n^2+4n+3}$ is 1.

6.8 *SUMMARY*

1. $\lim_{x \to a} f(x) = L$
This is read as, "limit of f(x), as x approaches a equals L".

2. A symbol such as *x* or *y*, used to represent an arbitrary element of a set is called a variable. For example $y = f(x)$.

3. L'Hospital's method be used only in the situations of $\frac{0}{0}$ and $\frac{\infty}{\infty}$ not in other cases.

6.9 *GLOSSARY*

Numbers

Intervals

Limit points

Functions

Bounded, Unbounded sets

6.10 *REFERENCES*

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6.12 *TERMINAL AND MODEL QUESTIONS*

Q1. Prove that $lim_{x\to 0}\ \frac{5n^2-5}{n^2+4n+3}$ is 5.

Q2. Prove that $lim_{x\to 0}\ \sin\left(\frac{1}{x^2}\right)$ does not exist.

Q3. Prove that $lim_{x\to 0}\ x^2 \sin\left(\frac{1}{x^2}\right)$ exist and equal to 0.

Q4. Prove that $\lim\limits_{x\to 1} 9^{\frac{1}{x-1}}$ does not exist.

Q5. Prove that $\lim\limits_{x\longrightarrow\infty} \frac{15\log x}{a^x},\ a > 1$ is 0.

6.13 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. False

CYQ 2. True

CYQ 3. True

CYQ 4. False

CYQ 5. True

UNIT 7: CONTINUITY

CONTENTS:

7.1 *INTRODUCTION*

In mathematics, a **continuous function** is a function such that a small variation of the argument induces a small variation of the value of the function. This implies there are no abrupt changes in value, known as *discontinuities*. More precisely, a function is continuous if arbitrarily small changes in its value can be assured by restricting to sufficiently small changes of its argument. A **discontinuous function** is a function that is *not continuous*. Until the 19th century, mathematicians largely relied on intuitive notions of continuity and considered only continuous functions.

Continuity is one of the core concepts of calculus and mathematical analysis, where arguments and values of functions are real and complex numbers.

7.2 *OBJECTIVES*

After studying this unit, learner will be able to

(i) Continuity

(ii) Discontinuity

(iii) Type of Discontinuity

(iv) Uniformly continuous

7.3 CONTINUITY (ϵ, δ) DEFINITION

Definition 1. A real valued function $f(x)$ defined on an interval I is said to be continuous at $x = a \in I$ if and only if for any arbitrarily chosen positive number ε, however small, we can find a corresponding number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

We say that $f(x)$ is continuous if it is continuous at every $x \in I$.

Or

$f(x)$ is continuous at $x = a$ is given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

7.4 CONTINUITY FROM LEFT AND CONTINUITY FROM RIGHT

A function $f(x)$ is said to be continuous from left at $x = a$ if $\underset{x \to a-0}{Limit} f(x)$ exists and equal to $f(a)$ *i.e.*,

$$\underset{h \to 0}{Limit} f(a - h) = f(a)$$

Similarly, $f(x)$ is said to be continuous from right at $x = a$ if $\underset{x \to a+0}{Limit} f(x)$ exists and equal to $f(a)$ *i.e.*,

$$\underset{h \to 0}{Limit} f(a + h) = f(a)$$

and $f(x)$ is continuous at $x = a$ iff

$$\underset{x \to a-0}{Limit} f(x) = \underset{x \to a+0}{Limit} f(x) = f(a)$$

$$\underset{h \to 0}{Limit} f(a - h) = \underset{h \to 0}{Limit} f(a + h) = f(a)$$

7.5 DISCONTINUITY

If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of the function.

e. g. The function $f(x) = \dfrac{1}{x-a}$ does not exists at $x = a$ so $f(x)$ is not continuous at $x = a$.

7.6 TYPES OF DISCONTINUITY

(1) Removable discontinuity:

A function $f(x)$ is said to have a removable discontinuity at a point $x = a$ if $\underset{x \to a}{Limit}\, f(x)$ exist but is not equal to $f(a)$ *i.e.*, if

$$f(a-0) = f(a+0) \neq f(a)$$

The function can be made continuous by defining it in such a way that

$$\underset{x \to a}{Limit}\, f(x) = f(a).$$

(2) Discontinuity of the first kind:

A function $f(x)$ is said to have a discontinuity of the first kind or ordinary discontinuity at $x = a$ if $f(a + 0)$ and $f(a - 0)$ both exist but not equal. The point $x = a$ is said to be a point of discontinuity from the left or right according as

$$f(a-0) \neq f(a) = f(a+0) \text{ or } f(a-0) = f(a) \neq f(a+0).$$

(3) Discontinuity of the second kind:

A function $f(x)$ is said to have a discontinuity of the second kind at $x = a$ if none of the limits $f(a + 0)$ and $f(a - 0)$ exist. The point $x = a$ is said to be a point of discontinuity of second kind from the left or right according as $f(a-0)$ or $f(a+0)$ does not exist.

(4) Mixed discontinuity:

If a function f has a mixed discontinuity at 'a' if either

(i) $\lim_{x \to a^-} f(x)$ does not exist and $\lim_{x \to a^+} f(x)$ exists, however $\lim_{x \to a^+} f(x)$ may or may not equal to $f(a)$.

(ii) $\lim_{x \to a^+} f(x)$ does not exist and $\lim_{x \to a^-} f(x)$ exists, however $\lim_{x \to a^-} f(x)$ may or may not equal to $f(a)$.

(5) infinite discontinuity:

A function $f(x)$ is said to have an infinite discontinuity at $x = a$ if $f(a + 0)$ or $f(a - 0)$ is $+\infty$ or $-\infty$ *i.e.*, if $f(x)$ is discontinuous at $x = a$ and $f(x)$ is unbounded in every neighbourhood of $x = a$.

(6) Piecewise continuous function:

A function $f: A \longrightarrow \mathbb{R}$ is said to be piecewise continuous on A if A can be divided into a finite number of parts so that f is continuous on each part.

Clearly, in such a case, f has a finite number of discontinuities and the set A is divided at the points of discontinuities.

Note: Jump of a function at a point.

If both $f(a + 0)$ and $f(a - 0)$ exists, then the jump in the function at $x = a$ is defined as the non-negative difference $f(a-0) \sim f(a+0)$.

A function having a finite number of jumps in a given interval is called piecewise continuous.

Illustrative Examples

Example 1. Test the continuity of $f(x)$ at $x = 1$ when

$$f(x) = \begin{cases} x^2 + 2 & if \;\; x > 1 \\ 2x + 1 & if \;\; x = 1 \\ 3 & if \;\; x < 1 \end{cases}$$

Solution. Here $f(1) = 2.1 + 1 = 3$

$$f(1+0) = \underset{h \to 0}{Limit}\ f(1+h) = \underset{h \to 0}{Limit}\ (1+h)^2 + 2$$

$$= \underset{h \to 0}{Limit}\ 1 + h^2 + 2h + 2 = 3 \text{ as } 1+h > 1.$$

$$f(1-0) = \underset{h \to 0}{Limit}\ f(1-h) = \underset{h \to 0}{Limit}\ (1-h)^2 + 2$$

$$= \underset{h \to 0}{Limit}\ 1 + h^2 - 2h + 2 = 3 \text{ as } 1-h < 1.$$

So $f(1) = f(1+0) = f(1-0)$ .Hence $f(x)$ is continuous at $x = 1$.

Example 2. Discuss the continuity of the function $f(x) = \dfrac{1}{1 - e^{-1/x}}$ when $x \neq 0$ and $f(0) = 0$ for all values of x.

Solution. Test the continuity at $x = 0$

$$f(0+0) = \underset{h \to 0}{Limit}\ f(0+h) = \underset{h \to 0}{Limit}\ f(h)$$

$$= \underset{h \to 0}{Limit}\ \frac{1}{1 - e^{-1/h}} = 1$$

$$f(0-0) = \underset{h \to 0}{Limit}\ f(0-h) = \underset{h \to 0}{Limit}\ f(-h)$$

$$= \underset{h \to 0}{Limit}\ \frac{1}{1 - e^{1/h}} = 0$$

Thus we have $f(0 + 0) \neq f(0 - 0) = f(0)$. So $f(x)$ is not continuous at $x = 0$ and it is a discontinuity of first kind *i.e.*, $f(x)$ is continuous on the left and has a discontinuity of first kind on right at $x = 0$.

Now test the continuity at $x = a \neq 0$

$$f(a) = \frac{1}{1 - e^{-1/a}}$$

$$f(a+0) = \underset{h \to 0}{Limit}\ f(a+h) = \underset{h \to 0}{Limit}\ \frac{1}{1 - e^{-1/(a+h)}}$$

$$= \frac{1}{1 - e^{-1/a}} = f(a)$$

$$f(a-0) = \underset{h\to 0}{Limit}\ f(a-h) = \underset{h\to 0}{Limit}\ \frac{1}{1-e^{-1/(a-h)}}$$

$$= \frac{1}{1-e^{-1/a}} = f(a)$$

Thus we have $f(a + 0) = f(a - 0) = f(a)$. Hence $f(x)$ is continuous at every point except $x = 0$.

Example 3. Test the continuity of the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Solution. Here

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ h \sin \frac{1}{h} = 0$$

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ (-h)\sin\left(-\frac{1}{h}\right) = \underset{h\to 0}{Limit}\ h \sin \frac{1}{h} = 0$$

Thus we have $f(0 + 0) = f(0 - 0) = f(0)$. Hence $f(x)$ is continuous at $x = 0$.

Note. 1. If we check the continuity at $x = c \neq 0$ of the above function, then we see that

$$\underset{x\to c}{Limit}\ f(x) = \underset{x\to c}{Limit}\ x \sin \frac{1}{x}$$

$$= c \sin \frac{1}{c} = f(c)$$

So $f(x)$ is continuous at $x = c$. Thus f (x) is continuous for all $x \in R$ *i.e.*, $f(x)$ is continuous on the whole real line.

Note 2. If we take $f(0) = 2$, in the above function, then $f(0 + 0) = f(0 - 0) \neq f(0)$. The function becomes discontinuities at $x = 0$ and has a removable discontinuity at $x = 0$.

Example 4. If a function $f(x)$ is defined by $f(x) = x - [x]$, where x is a positive variable and $[x]$ denotes the integral part of x. Show that it is discontinuous for integral values of x and continuous for all others. Draw the graph.

Solution. From the definition of the function f (x) we have

$$f(x) = \begin{cases} x-(n-1) & \text{for } n-1<x<n \\ 0 & \text{for } x=n \\ x-n & \text{for } n<x<n+1 \end{cases} \quad \text{where } n \text{ is an integer}$$

First we test the continuity of $f(x)$ at $x = n$. We have $f(n) = 0$.

$$f(n+0) = \underset{h\to 0}{\text{Limit}}\ f(n+h) = \underset{h\to 0}{\text{Limit}}\ (n+h) - n$$

$$= \underset{h\to 0}{\text{Limit}}\ h = 0 \quad [\text{as } n < n + h < n+1]$$

$$f(n-0) = \underset{h\to 0}{\text{Limit}}\ f(n-h) = \underset{h\to 0}{\text{Limit}}\ (n-h) - \overline{n-1}$$

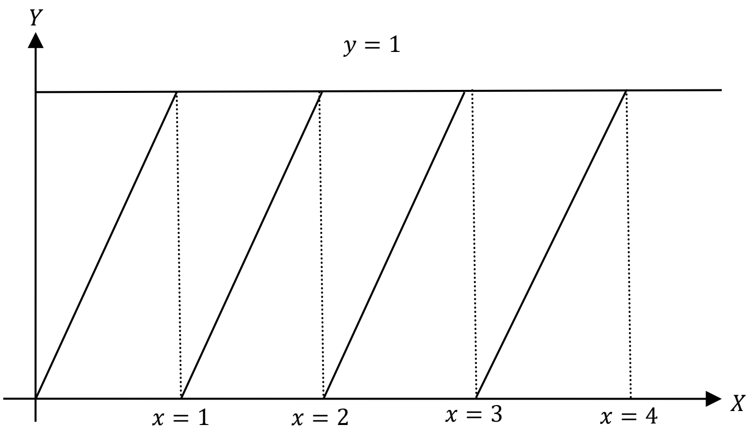
$$= \underset{h\to 0}{\text{Limit}}\ 1-h = 1 \quad [\text{ as } n-1 < n-h < n]$$

Since $f(n-0) \neq f(n+0)$, so the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. it is obviously continuous for all other values of x.

Since x is a positive variable putting = 1, 2, 3, 4, 5, …., we see that graph of the function consists of the following straight lines.

$$y = f(x) = \begin{cases} x & \text{when } 0 < x < 1 \\ 0 & \text{when } x = 1 \\ x-1 & \text{when } 1 < x < 2 \\ 0 & \text{when } x = 2 \\ x-2 & \text{when } 2 < x < 3 \\ 0 & \text{when } x = 3 \\ x-3 & \text{when } 3 < x < 4 \\ 0 & \text{when } x = 5 \end{cases}$$

and so on.



It is clear from the graph that

(1) The function is discontinuous for all integral values of x but continuous for other values of x.

(2) The function is bounded between 0 and 1 in every domain which includes an integer.

(3) The lower bound 0 is attained but upper bound 1 is not attained since $f(x) \neq 1$ for any value of x.

Example 5. Show that the function $f(x) = [x] + [-x]$ has a removable discontinuity for integral values of x.

Solution. We see that $f(x) = 0$ when x is an integer and $f(x) = -1$ when x is not an integer. Hence if n is an integer then

$$f(n-0) = f(n+0) = -1 \text{ and } f(n) = 0.$$

So the function $f(x)$ has a removable discontinuity at $x = n$, where n is an integer.

Example 6. Prove that the function $f(x) = \frac{|x|}{x}$ for $x \neq 0$ and $f(0) = 0$, is continuous at all the points except $x = 0$.

Solution. If $x > 0$ then, $f(x) = \frac{x}{x} = 1$ and if $x < 0$ then, $f(x) = \frac{-x}{x} = -1$.

Therefore the given function can define as:

$$f(x)=\begin{cases}-1, & if\ x<0\\ 0, & if\ x=0\\ 1, & if\ x>0\end{cases}$$

If $x < 0$, then $f(x) = -1$ *i.e.*, $f(x)$ is a constant function and a constant function is always continuous at each point of its domain. This implies that $f(x)$ is continuous for all $x < 0$.

Similarly, we can show that $f(x)$ is continuous for all $x > 0$. Now we see the continuity at $x = 0$.

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h),\ h>0$$
$$= \underset{h\to 0}{Limit}\ 1 = 1$$

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h),\ h>0$$
$$= \underset{h\to 0}{Limit}\ -1 = -1$$

Here $f(0+0) \neq f(0-0) \neq f(0)$. Hence $f(x)$ is not continuous at $x = 0$.

Example 7. Discuss the continuity of the following functions at $x = 0$ of the function $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0\end{cases}$.

Sol.

7.7 UNIFORM CONTINUITY

A function f: X → Y is said to be uniformly continuous on A ⊆ X if for every ε > 0, there exists δ > 0 such that x, y ∈ A, $|x-y| < \delta$ implies

$|f(x) - f(y)| < \varepsilon$.

Note: Uniform continuity is domain base property i.e. Uniform continuity is defined on a set.

Example 7.6.1 Prove that the function f (x) = 3x + 1 is uniformly continuous on ℝ.

Solution: Since |f (x) − f (y)| = |(3x + 1) − (3y + 1)| = 3|x − y|

so, given ε > 0, we choose δ = ε / 3. Then, |x − c| < δ implies

|f (x) − f (c)| = 3|x − c| < 3(ε / 3) = ε.

Hence the function f (x) = 3x + 1 is uniformly continuous on ℝ.

Example 7.6.2 Prove that the function f (x) = x^2 is not uniformly continuous on ℝ.

Solution: for any $\epsilon > 0$and for any x, y ∈ ℝ we have a δ such that |x−y| < δ ⟹ |f(x)−f(y)| < ϵ. Our claim here is now $|x^2 - y^2| < \epsilon$. That is the distance between x^2 and y^2 is at most ϵ everywhere on the vertical axis as long as we keep x and y at most δ away from each other. The problem arises if we let x and y getting larger and larger.

So for the given ϵ we have fixed δ and we may play with x and y values. Let's make them large enough to satisfy $\frac{\delta}{2} < |x - y| < \delta$ and $x > \frac{\epsilon}{\delta}$, $y > \frac{\epsilon}{\delta}$.

Then $|x - y| < \delta$

But $|x^2 - y^2| = |x - y||x + y| > \frac{\delta}{2}\frac{2\epsilon}{\delta}$ $\Longrightarrow |x^2 - y^2| > \delta$

which is a contraction with the definition which is assumed true at the beginning. Therefore f (x) = x^2 can not be uniformly continuous.

Note: Every Uniformly continuous function is continuous function.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The function $f(x) = \sin\frac{1}{x}$ at $x = 0$ has a discontinuity of second kind.

Problem 2. The function $f(x) = e^x$ is not continuous at $x = 0$.

Problem 3. Every uniformly continuous function is continuous function.

Problem 4. Every polynomial function is uniformly continuous on $\mathbb{R}$.

Problem 5. Every continuous function uniformly continuous.

7.8 *SUMMARY*

1. If $\underset{h\to 0}{Limit}\ f(a-h) = \underset{h\to 0}{Limit}\ f(a+h) = f(a)$ then function f(x) is continuous at $x = a$.

2. A function f: X → Y is said to be **uniformly continuous** on A ⊆ X if for every ε > 0, there exists δ > 0 such that x, y ∈ A, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

7.9 *GLOSSARY*

Numbers

Intervals

Sets

Functions

Limits

7.10 *REFERENCES*

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7.12 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that every uniform continuous function is continuous.

Q 2. Prove that every polynomial function is continuous.

Q 3. Prove that $f(x) = \frac{1}{x}$ is not uniformly continuous in (0, 1).

Q 4. Prove that $f(x) = x^2$ is uniformly continuous on [-2, 2].

Q 5. Prove that $f(x) = 2^{1/x}$ is not continuous at 0.

7.13 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. True

CYQ 4. False

CYQ 5. False

UNIT 8: DIFFERENTIATION

Contents

8.1 INTRODUCTION

In previous unit we discussed about limit and continuity. In this unit we will discussed about Differentiability and Mean Value theorem.
Prior to the seventeenth century, a curve was generally described as a locus of points satisfying some geometric condition, and tangent lines were obtained through geometric construction. This viewpoint changed dramatically with the creation of analytic geometry in the 1630s by Rene Descartes (1596–1650) and Pierre de Fermat (1601–1665).
In this new setting geometric problems were recast in terms of algebraic expressions, and new classes of curves were defined by algebraic rather than geometric conditions. The concept of derivative evolved in this new context. The problem of finding tangent lines and the seemingly unrelated problem of finding maximum or minimum values were first seen to have a connection by Fermat in the 1630s. And the relation between tangent lines to curves and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Newton's theory of ''fluxions,'' which was based on an intuitive idea of limit, would be familiar to any modern student of differential calculus once some changes in terminology and notation were made.
But the vital observation, made by Newton and, independently, by Gottfried Leibniz in the 1680s, was that areas under curves could be calculated by reversing the differentiation process. This exciting technique, one that solved previously difficult area problems with ease, sparked enormous interest among the mathematicians of the era and led to a coherent theory that became known as the differential and integral calculus.
In this Unit we will develop the theory of differentiation. Integration theory, including the fundamental theorem that relates differentiation and integration, will be the subject of the next chapter. Consequently, we will concentrate on the mathematical aspects of the derivative and not go into its applications in geometry, physics, economics, and so on.

8.2 OBJECTIVES

In this Unit, we will Discussed about

- Improper integral
- Test of convergence
- Absolute integral

8.3 DERIVATIVE

We begin with the definition of the derivative of a function.

Derivative: Let $I \subseteq \mathbb{R}$ be an interval, $f:(x,y) \to \mathbb{R}$ and $b \in I$. Then $l \in \mathbb{R}$ is said to be derivative of f at b if for any given $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $x \in I$ then

$$\left|\frac{f(x)-f(b)}{x-b} - l\right| < \varepsilon \quad \text{whenever } 0 < |x-b| < \delta.$$

We can also say that f is differentiable at b, and we write $f'(c)$

Or

The derivative of f at b is given by $f'(c) = \lim_{h\to 0}\frac{f(x)-f(b)}{x-h}$ provided this limit exists.

Note: We now show that continuity of f at a point b is a necessary

(but not sufficient) condition for the existence of the derivative at b.

Theorem: If $f: I \to \mathbb{R}$ *has a derivative at* $b \in I$, *then* f *is continuous at* b.

Proof. We have

$$f(x) - f(b) = \left(\frac{f(x)-f(b)}{x-b}\right)(x-b) \qquad \text{For all } x \in I;\ x \neq b$$

Because $f'(b)$ exists, Therefore

$$\lim_{x\to b}\big(f(x)-f(b)\big) = \lim_{x\to b}\left(\left(\frac{f(x)-f(b)}{x-b}\right)(x-b)\right)$$

$$= \lim_{x\to b}\left(\frac{f(x)-f(b)}{x-b}\right)\lim_{x\to b}(x-b)$$

$$= f'(b).0 = 0$$

Therefore, $\lim_{x\to b}(f(x)-f(b)) = 0 \Rightarrow \lim_{x\to b} f(x) - \lim_{x\to b} f(b) \Rightarrow \lim_{x\to b} f(x) = f(b)$

Hence f is continuous at b.

NOTE: The continuity of $f : I \to \mathbb{R}$ at a point does not promise the existence of the derivative at that point.

Theorem: Let $I \subseteq \mathbb{R}$ be an Interval and $f, g: X \to \mathbb{R}$ be functions that are differentiable at $b \in \mathbb{R}$ Then

(i) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at b and

$$(\alpha f)'(b) = \alpha f'(b)$$

(ii) The function f+g is differentiable at b and

$$(f+g)'(b) = f'(b) + g'(b)$$

(iii) The function f and g is differentiable at b and

$$(fg)'(b) = f'(b)g(b) + f(b)g'(b)$$

(iv) If $g(b) \neq 0$, then the function f and g is differentiable at b and

$$\left(\frac{f}{g}\right)'(b) = \frac{f'(b)g(b) - f(b)g'(b)}{(g(b))^2}$$

Proof. (i) Let $h_1 = \alpha f$, then for $x \in I$ and $x \neq b$, we have

$$\frac{h_1(x) - h_1(b)}{x-b} = \frac{(\alpha f)(x) - (\alpha f)(b)}{x-b} = \alpha\frac{f(x)-f(b)}{x-b}$$

Since f is differentiable at b implies $f'(b)$ exists. Therefore

$$\lim_{x\to b}\frac{h_1(x)-h_1(b)}{x-b} = \lim_{x\to b}\alpha\frac{f(x)-f(b)}{x-b} = \alpha\lim_{x\to b}\frac{f(x)-f(b)}{x-b} = \alpha f'(c)$$

Hence $(\alpha f)'(b) = \alpha f'(b)$

(ii) Let $h_2 = f + g$, then for $x \in I$ and $x \neq b$, we have

$$\frac{h_2(x)-h_2(b)}{x-b} = \frac{(f+g)(x)-(f+g)(b)}{x-b} = \frac{f(x)+g(x)-f(b)-g(b)}{x-b} = \frac{f(x)-f(b)+g(x)-g(b)}{x-b}$$

$$= \frac{f(x)-f(b)}{x-b} + \frac{g(x)-g(b)}{x-b}$$

Since f and g are differentiable at b implies $f'(b)$ and $g'(b)$ exists. Therefore

$$\lim_{x\to b}\frac{h_2(x)-h_2(b)}{x-b} = \lim_{x\to b}\frac{f(x)-f(b)}{x-b} + \lim_{x\to b}\frac{g(x)-g(b)}{x-b} = f'(b) + g'(b)$$

Hence $(f+g)'(b) = f'(b) + g'(b)$

(iii) Let $h_3 = fg$, then for $x \in I$ and $x \neq b$, we have

$$\frac{h_3(x) - h_3(b)}{x - b} = \frac{(fg)(x) - (fg)(b)}{x - b} = \frac{f(x)g(x) - f(b)g(b)}{x - b}$$
$$= \frac{f(x)g(x) - f(b)g(x) + f(b)g(x) - f(b)g(b)}{x - b}$$

$$= \frac{g(x)(f(x)-f(b))+f(b)(g(x)-g(b))}{x-b} = g(x)\frac{f(x)-f(b)}{x-b} + f(b)\frac{g(x)-g(b)}{x-b}.$$

It is given that f and g is differentiable at b and

g is differentiable at b $\Rightarrow$ g is continuous i.e. $\lim_{x\to b} g(x) = g(b)$ (by previous theorem)

Therefore

$$\lim_{x\to b}\frac{h_3(x)-h_3(b)}{x-b} = \lim_{x\to b}\left\{g(x)\frac{f(x)-f(b)}{x-b} + f(b)\frac{g(x)-g(b)}{x-b}\right\}$$

$$= \lim_{x\to b} g(x)\frac{f(x)-f(b)}{x-b} + \lim_{x\to b} f(b)\frac{g(x)-g(b)}{x-b} = \lim_{x\to b} g(x)\lim_{x\to b}\frac{f(x)-f(b)}{x-b} +$$
$$f(b)\lim_{x\to b}\frac{g(x)-g(b)}{x-b} = f'(b)g(b) + f(b)g'(b)$$

Hence $(fg)'(b) = f'(b)g(b) + f(b)g'(b)$

(iv) Let $h_4 = \frac{f}{g}$, since g is differentiable at b $\Rightarrow$ since g is continuous at b.

It is given that $g(b) \neq 0$, therefore there exists an interval $I_1 \subseteq I$ with $b \in I_1$ such that

$g(x) \neq 0$ for all $x \in I_1$.

Now for $x \in I_1, x \neq b$, we get

$$\frac{h_4(x) - h_4(b)}{x - b} = \frac{\frac{f}{g}(x) - \frac{f}{g}(b)}{x - b} = \frac{\frac{f(x)}{g(x)} - \frac{f(b)}{g(b)}}{x - b} = \frac{f(x)g(b) - f(b)g(x)}{g(b)g(b)(x - b)}$$

$$= \frac{f(x)g(b) - f(b)g(x)}{g(b)g(b)(x - b)} = \frac{f(x)g(b) - f(b)g(b) + f(b)g(b) - f(b)g(x)}{g(b)g(b)(x - b)}$$

$$\frac{(f(x)-f(b))g(b)-f(b)(g(x)-g(b))}{g(x)g(b)(x-b)} = \frac{1}{g(x)g(b)}\left[\frac{f(x)-f(b)}{x-b}.g(b) - f(b).\frac{g(x)-g(b)}{x-b}\right]$$

Therefore

$$\lim_{x\to b}\frac{h_4(x)-h_4(b)}{x-b} = \lim_{x\to b}\frac{1}{g(x)g(b)}\left[\frac{f(x)-f(b)}{x-b}.g(b) - f(b).\frac{g(x)-g(b)}{x-b}\right]$$

$$= \lim_{x\to b} \frac{1}{g(x)g(b)} \left[\lim_{x\to b} \left(\frac{f(x)-f(b)}{x-b}\right). g(b) - \right.$$

$$\left. f(b). \lim_{x\to b} \left(\frac{g(x)-g(b)}{x-b}\right)\right]$$

$$= \frac{1}{g'^2(b)}. [f'(b)g(b) - f(b)g'(b)]$$

Hence

$$\left(\frac{f}{g}\right)'(b) = \frac{f'(b)g(b)-f(b)g'(b)}{\left(g(b)\right)^2}$$

8.4 MEAN VALUE THEOREM

The Mean Value Theorem, which relates the values of a function to values of its derivative,

is one of the most useful results in real analysis

We begin by looking at the relationship between the relative extrema of a function and

the values of its derivative.

Relative Maximum*:* The function $f: I \to \mathbb{R}$ is said to have a relative maximum at $b \in I$ if there exists a neighborhood $V = V_\delta(b)$ of b such that $f(x) \le f(b)$, for all x in $V \cap I$.

Relative Minimum: The function $f: I \to \mathbb{R}$ is said to have a relative minimum at $b \in I$ if there exists a neighborhood $V' = V'_{\delta'}(b)$ of b such that $f(x) \ge f(b)$, for all x in $V' \cap I$.

Relative Extremum: f has a relative extremum at $b \in I$ if it has either a relative maximum or a relative minimum at b.

Interior Extremum Theorem

Theorem: Let b be an interior point of the interval I at which

$f: I \to R$ has a relative extremum. If the derivative of f at b exists, then

$f'(b) = 0$.

Proof. If $f'(b) > 0$, Then there exists a neighborhood $V \subseteq I$ of b such that

$\frac{f(x)-f(b)}{x-b} > 0$ for $x \in V, x \neq b$

If $x \in V, x > b$, then we get

$f(x) - f(b) = (x - b).\frac{f(x)-f(b)}{x-b} > 0$

But this contradicts the hypothesis that f has a relative maximum at b.

Hence, we cannot have $f'(b) > 0$.

Similarly, we cannot have $f'(b) < 0$.

Therefore, $f'(b) = 0$.

Rolle's Theorem

Theorem: Consider that f is continuous on a closed interval $I = [a, b]$ and

the derivative $f'(0)$ exists at every point of the open interval (a, b), and $f(a) = f(b) = 0$.

Then there exists at least one point c in (a, b) such that $f'(c) = 0$

Proof. If $f(x) = 0$ for all x in I or vanishes identically on I, then any c in (a, b) will satisfy the result of the theorem.

Hence Let f does not vanish identically or $f \neq 0$.

Now replacing f by $(-f)$ and consider f assumes some positive values.

So by the Maximum Minimum Theorem,

The function f attains the value $\sup\{f(x): x \in I\} > 0$ at some point c in I.

Since $f(a) = f(b) = 0$. the point c must lie in (a, b).

Hence $f'(c)$ exists.

Since f has a relative maximum at c.

By the Interior Extremum Theorem, we get

$f'(c) = 0$

Geometrical Representation of Rolle's theorem

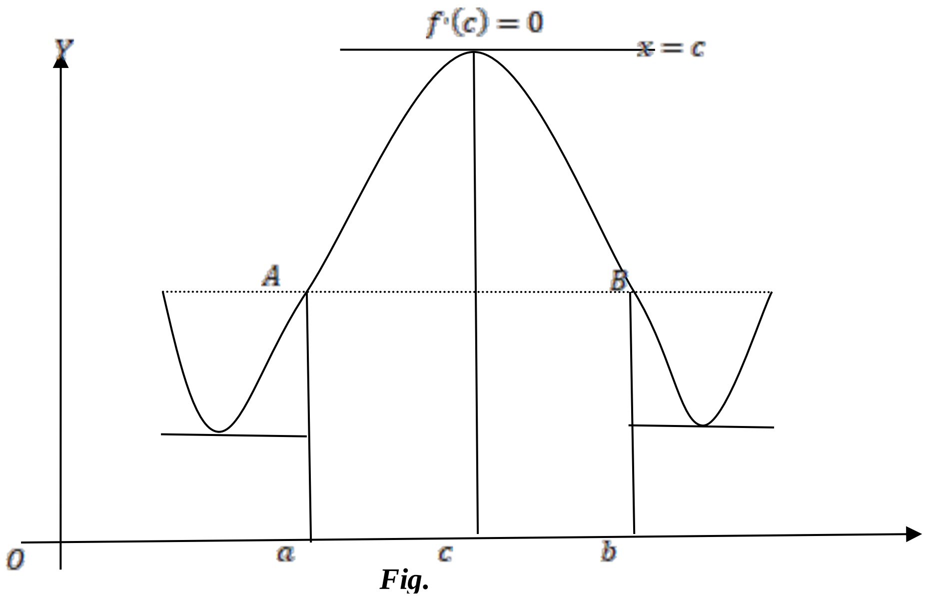


Fig.

In the given graph, the curve $y = f(x)$ is continuous between $x = a$ and $x = b$ and

at every point, within the interval, it is possible to draw a tangent and ordinates corresponding to the abscissa and are equal then there exists at least one tangent to the curve which is parallel to the x-axis.

Algebraically, this theorem tells us that if f (x) is representing a polynomial function in x and the two roots of the equation $f(x) = 0$ are $x = a$ and $x = b$, then there exists at least one root of the equation $f'(x) = 0$ lying between these values.

The converse of Rolle's theorem is not true and it is also possible that there exists more than one value of x, for which the theorem holds good but there is a definite chance of the existence of one such value.

NOTE:

- **Rolle's theorem does not hold good if**

(i) f (x) is discontinuous in the closed interval [a, b].

(ii) f (x) does not exist at some point in (a, b).

(iii) $f(a) \neq f(b)$.

Example: Rolle's Theorem can be used for the location of roots of a function.

For, if a function g can be identified as the derivative of a function f, then between any two roots of f there is at least one root of g.

For example: let $g(x) = \cos x$ then g is known to be

the derivative of $f(x) = \sin x$. Hence, between any two roots of $\sin x$ there is at least one

root of $\cos x$.

On the other hand, $g'(x) = -\sin x = -f(x)$.

Another application of Rolle's Theorem informed us that between any two roots of cos there is at least one root of sin.Therefore, we conclude that the roots of sin and cos interlace each other

Mean Value Theorem:

Suppose that f is continuous on a closed interval $I = [a, b]$and f has a derivative in the open interval (a, b). Then there exists atleast one point c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$

Proof. Assume the function Φ defined on I such that

$$\Phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$$

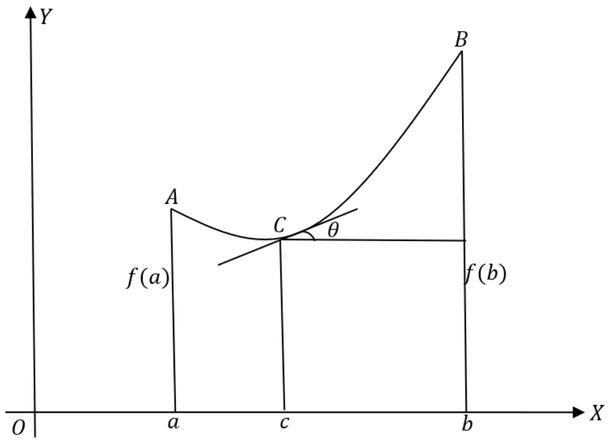
We can easily see that The Conditions of Rolle's Theorem are satisfied by Φ since Φ is continuous on $[a, b]$, differentiable on (a, b), and $\Phi(a) = \Phi(b)$.

Therefore, there exists a point b in (a, b) such that

$$0 = \Phi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}.$$

Therefore $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow f(b) - f(a) = f'(c)(b - a)$

Geometrical Interpretation



The geometric view of the Mean Value Theorem is that there is some point on the curve $y = f(x)$ at which the tangent line is parallel to the line segment through the points $(a, f(a))$ and $(b, f(b))$. Thus it is easy to remember the statement of the Mean Value

Theorem by drawing appropriate diagrams. While this should not be discouraged, it tends to suggest that its importance is geometrical in nature, which is quite misleading. In fact the

Mean Value Theorem is a wolf in sheep's clothing and is the Fundamental Theorem of Differential Calculus.

Cauchy Mean Value Theorem:

Theorem: Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g(x) \neq 0$ for all x in (a, b). Then there exists c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Since $g'(x) \neq 0$ for all x in (a, b), therefore

Using Rolle's Theorem, we get

$g(a) \neq g(b)$.

For x in $[a, b]$, now new define

$$\varphi(x) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x) - g(a)\big) - \big(f(x) - f(a)\big)$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and

$\varphi(a) = \varphi(b) = 0$.

Therefore, According, to Rolle's Theorem

there exists a point c in (a, b) such that

$0 = \varphi'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) - f'(c)$

As we know $g'(c) \neq 0$, we obtain required result that is

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Strictly Increasing: A function f is said to be strictly increasing on an interval I if for any points x_1 and x_2 in I such that $x_1 < x_2$, we have $(x_1) < f(x_2)$.

Strictly decreasing: A function f is said to be strictly increasing on an interval I if for any points x_1 and x_2 in I such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Derivatives can be used to determine whether a function is increasing, decreasing or constant on an interval:

f(x) is increasing if derivative f ′ (x) > 0,

f(x) is decreasing if derivative f ′ (x) < 0,

f(x) is constant if derivative f ′ (x) = 0.

A critical number, c, is one where f ′ (c) = 0 or f ′ (c) does not exist; a critical point is (c, f(c)).

After locating the critical number(s), choose test values in each interval between these critical numbers, then calculate the derivatives at the test values to decide whether the function is increasing or decreasing in each given interval.

(In general, identify values of the function which are discontinuous, so, in addition to critical numbers, also watch for values of the function which are not defined, at vertical asymptotes or singularities ("holes").)

8.5 TAYLOR'S THEOREM

Theorem 1 (Taylor's Theorem) *Let $a < b$, $n \in \mathbb{N} \cup \{0\}$, and $f : [a, b] \to \mathbb{R}$. Assume that $f^{(n)}$ exists and is continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Let $\alpha \in [a, b]$ and define the Taylor polynomial of degree n with expansion point α to be*

Then, for all $x \in [a, b]$,

$$f(x) = P_n(x) + R_n(\alpha, x)$$

where the error term $R_n(\alpha, x)$ is given by

(a) *(integral form)* $R_n(\alpha, x) = \int_\alpha^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\, dt$, *if $f^{(n+1)}$ is integrable.*

(b) *(Lagrange form)* $R_n(\alpha, x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-\alpha)^{n+1}$ *for some c strictly between α and x.*

(c) *(Cauchy form)* $R_n(\alpha, x) = \frac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-\alpha)$ *for some c strictly between α and x.*

(d) *If $r \in \mathbb{N}$, then* $R_n(\alpha, x) = \frac{1}{rn!} f^{(n+1)}(c)\,(x-c)^{n-r+1}(x-\alpha)^r$ *for some c strictly between α and x.*

(e) $R_n(\alpha, x) = \frac{1}{n!} Q^{(n)}(\alpha)(x-\alpha)^{n+1}$ *where*

$$Q(t) = \begin{cases} \frac{f(t)-f(x)}{t-x} & \text{if } t \neq x \\ f'(x) & \text{if } t = x \end{cases}$$

Proof: Fix any evaluation and expansion points $x, \alpha \in [a, b]$. Define the function $S(t)$ by

$$f(x) = f(t) + (x-t)f'(t) + \tfrac{1}{2}(x-t)^2 f''(t) + \cdots + \tfrac{1}{n!}(x-t)^n f^{(n)}(t) + S(t) \qquad (1)$$

Observe that substituting $t = \alpha$ into (1) gives $f(x) = P_n(x) + S(\alpha)$. So we wish to find

$$R_n(\alpha, x) = S(\alpha)$$

The function $S(t)$ is determined by its derivative and its value at one point. Finding a value of $S(t)$ for one value of t is easy. Substitute $t = x$ into (1) to yield $S(x) = 0$. To find $S'(t)$, apply $\frac{d}{dt}$ to both sides of (1). Recalling that x is just a constant parameter,

$$\begin{aligned} 0 &= f'(t) + \big[-f'(t) + (x-t)f''(t)\big] + \big[-(x-t)f''(t) + \tfrac{1}{2}(x-t)^2 f^{(3)}(t)\big] \\ &\qquad + \cdots + \big[-\tfrac{1}{(n-1)!}(x-t)^{n-1} f^{(n)}(t) + \tfrac{1}{n!}(x-t)^n f^{(n+1)}(t)\big] + S'(t) \\ &= \tfrac{1}{n!}(x-t)^n f^{(n+1)}(t) + S'(t) \end{aligned}$$

so that

$$S'(t) = -\tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n$$

(a) By the fundamental theorem of calculus

$$S(\alpha) = -\big[S(x) - S(\alpha)\big] = -\int_\alpha^x S'(t)\,dt = \int_\alpha^x \tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,dt$$

(c) By the mean value theorem, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) &= S(\alpha) - S(x) = S'(c)\ (\alpha - x) = -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(\alpha - x) \\ &= \tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-\alpha) \end{aligned}$$

...

(b) By the generalized mean value theorem (see the notes entitled "The Mean Value Theorem") with $g(t) = (x-t)^{n+1}$, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) &= S(\alpha) - S(x) = \tfrac{S'(c)}{g'(c)}\big(g(\alpha) - g(x)\big) \\ &= -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n \tfrac{1}{-(n+1)(x-c)^n}\ (x-\alpha)^{n+1} \\ &= \tfrac{1}{(n+1)!} f^{(n+1)}(c)(x-\alpha)^{n+1} \end{aligned}$$

Don't forget, when computing $g'(c)$, that g is a function of t with x just a fixed parameter.

(d) By the generalized mean value theorem with $g(t) = (x-t)^r$, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) &= S(\alpha) - S(x) = \tfrac{S'(c)}{g'(c)}\big(g(\alpha) - g(x)\big) \\ &= -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n \tfrac{1}{-r(x-c)^{r-1}}\ (x-\alpha)^r \\ &= \tfrac{1}{rn!} f^{(n+1)}(c)(x-c)^{n-r+1}(x-\alpha)^r \end{aligned}$$

(e) We'll only consider the case that $x \neq \alpha$. For $x = \alpha$, the error $R_n(\alpha, x)$ is obviously zero and we'll just take it as a convention that $\frac{1}{n!}Q^{(n)}(\alpha)(x-\alpha)^{n+1} = 0$ even if $Q^{(n)}(\alpha)$ is not defined. Since f is n times differentiable, so is $Q(t)$, at least for all $t \neq x$. In particular $Q(t)$ is n times differentiable at $t = \alpha$. From the definition of Q we have that

$$\begin{aligned} f(t) &= f(x) + (t-x)\,Q(t) & &\implies & f(\alpha) &= f(x) - (x-\alpha)\,Q(\alpha) \\ f'(t) &= Q(t) + (t-x)\,Q'(t) & &\implies & f'(\alpha) &= Q(\alpha) - (x-\alpha)\,Q'(\alpha) \\ f^{(2)}(t) &= 2Q'(t) + (t-x)\,Q^{(2)}(t) & &\implies & f^{(2)}(\alpha) &= 2Q'(\alpha) - (x-\alpha)\,Q^{(2)}(\alpha) \\ &\vdots & & & &\vdots \\ f^{(k)}(t) &= kQ^{(k-1)}(t) + (t-x)\,Q^{(k)}(t) & &\implies & f^{(k)}(\alpha) &= kQ^{(k-1)}(\alpha) - (x-\alpha)\,Q^{(k)}(\alpha) \end{aligned}$$

for $k \le n$. So

$$f(x) - P(x) = f(x) - f(\alpha) - \sum_{k=1}^{n} \tfrac{1}{k!} f^{(k)}(\alpha)\,(x-\alpha)^k$$
$$= (x-\alpha)Q(\alpha) - \sum_{k=1}^{n} \left\{ \tfrac{1}{(k-1)!} Q^{(k-1)}(\alpha)\,(x-\alpha)^k - \tfrac{1}{k!} Q^{(k)}(\alpha)\,(x-\alpha)^{k+1} \right\}$$

The sum telescopes leaving

$$f(x) - P(x) = \tfrac{1}{n!} Q^{(n)}(\alpha)\,(x-\alpha)^{n+1}$$

as desired.

Example 1. Find the first 4 terms of the Taylor series for the following functions:

(a) $f(x) = log x$ centered at $a = 1$.

(b) $f(x) = \frac{1}{x}$ centered at $a = 1$.

(c) $f(x) = sinx$ centered at $a = \frac{\pi}{4}$.

Sol. (i)

$f(x) = \ln x$. So $f^{(1)}(x) = \dfrac{1}{x}$, $f^{(2)}(x) = -\dfrac{1}{x^2}$, $f^{(3)}(x) = \dfrac{2}{x^3}$,

$$\ln x = \ln 1 + (x-1)\times 1 + \frac{(x-1)^2}{2!}\times(-1) + \frac{(x-1)^3}{3!}\times(2) + \frac{(x-1)^4}{4!}\times(-6) + \ldots$$
$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

(ii)

$f(x) = \dfrac{1}{x}$. So $f^{(1)}(x) = -\dfrac{1}{x^2}$, $f^{(2)}(x) = \dfrac{2}{x^3}$, $f^{(3)}(x) = -\dfrac{6}{x^4}$ and so

$$\frac{1}{x} = 1 + (x-1)\times(-1) + \frac{(x-1)^2}{2!}\times(2) + \frac{(x-1)^3}{3!}\times(-6) + \cdots$$
$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 - \cdots$$

(iii)

$f(x) = \sin x$. So $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$ and so

$$\sin x = \frac{\sqrt{2}}{2} + (x - \frac{\pi}{4}) \times (\frac{\sqrt{2}}{2}) + \frac{(x-\frac{\pi}{4})^2}{2!} \times (-\frac{\sqrt{2}}{2}) + \frac{(x-\frac{\pi}{4})^3}{3!} \times (-\frac{\sqrt{2}}{2}) + \cdots$$

$$= \frac{\sqrt{2}}{2}\left(1 + (x - \frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{2} - \frac{(x-\frac{\pi}{4})^3}{6} + \cdots\right)$$

Example 2.

Find the Taylor series for the function $x^4 + x - 2$ centered at a=1.

Sol.

$f(x) = x^4 + x - 2$. $f^{(1)}(x) = 4x^3 + 1$, $f^{(2)}(x) = 12x^2$, $f^{(3)}(x) = 24x$, $f^{(4)}(x) = 24$ and all other derivatives are zero. Thus

$$x^4 + x - 2 = 0 + (x-1) \times 5 + \frac{(x-1)^2}{2!} \times 12 + \frac{(x-1)^3}{3!} \times 24 + \frac{(x-1)^4}{4!} \times 24$$

$$= 5(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

Example 3.

Find the first 4 terms in the Taylor series for $(x-1)e^x$ near x=1.

Sol.

Either find the Taylor series for e^x and then multiply by $(x-1)$:

$f(x) = e^x \Rightarrow f(1) = e$

$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(1) = e$

$f^{(2)}(x) = e^x \Rightarrow f^{(2)}(1) = e$

$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(1) = e$

so that

$$(x-1)e^x \approx (x-1)\left(e + (x-1)e + \frac{(x-1)^2}{2!}e + \frac{(x-1)^3}{3!}e\right)$$

$$\approx (x-1)e + (x-1)^2 e + \frac{(x-1)^3}{2}e + \frac{(x-1)^4}{6}e$$

or with a bit more work,

$$f(x) = (x-1)e^x \Rightarrow f(1) = 0$$
$$f^{(1)}(x) = (x-1)e^x + e^x \Rightarrow f^{(1)}(1) = e$$
$$f^{(2)}(x) = (x-1)e^x + 2e^x \Rightarrow f^{(2)}(1) = 2e$$
$$f^{(3)}(x) = (x-1)e^x + 3e^x \Rightarrow f^{(3)}(1) = 3e$$
$$f^{(4)}(x) = (x-1)e^x + 4e^x \Rightarrow f^{(4)}(1) = 4e$$

so that

$$(x-1)e^x \approx 0 + (x-1)e + \frac{(x-1)^2}{2!}2e + \frac{(x-1)^3}{3!}3e + \frac{(x-1)^4}{4!}4e$$
$$\approx (x-1)e + (x-1)^2 e + \frac{(x-1)^3}{2}e + \frac{(x-1)^4}{6}e$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The derivatives of $f(x) = \sin x$ is 1 at $x = 0$.

Problem 2. The derivatives $f(x) = e^x$ is 1 at $x = 0$.

Problem 3. Every Differentiable function is continuous function.

Problem 4. Rolle's Theorem can be used for the location of roots of a function.

Problem 5. Every continuous function is Differentiable.

8.6 *SUMMARY*

1. **Theorem:** Let b be an interior point of the interval I at which

$f: I \to R$ has a relative extremum. If the derivative of f at b exists, then

$f'(b) = 0$.

2. **Rolle's Theorem**

Theorem: Consider that f is continuous on a closed interval $I = [a, b]$ and the derivative $f'(0)$ exists at every point of the open interval (a, b), and $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$

8.7 *GLOSSARY*

Numbers

Intervals

Continuity function

Functions

Limits

8.8 *REFERENCES*

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3. W. Rudin, Principles of Mathematical Analysis (3rd Edition), McGraw-Hill Publishing, 1976.

8.9 *SUGGESTED READING*

4. S.C. Malik and Savita Arora, Mathematical Analysis (6th Edition), New Age International Publishers, 2021.

5. Shanti Narayan, A course of Mathematical Analysis (29th Edition), S. Chand and Co., 2005.

6. K. A. Ross, Elementary Analysis**,** The Theory of Calculus (2nd edition), Springer, 2013.

8.10 *TERMINAL AND MODEL QUESTIONS*

Q 1. Using Taylor's theorem, show that $cosx \geq 1 - \frac{x^2}{2} \forall\, x \in \mathbb{R}$.

Q 2. Using Taylor's theorem, show that $x - \frac{x^3}{3!} < sinx < x, x > 0$.

Q 3. Prove that $f(x) = \frac{1}{x}$ is not differentiable at 0**.**

Q 4. Prove that $f(x) = x^2$ is differentiable.

8.11 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. True

CYQ 5. False

Course Name: REAL ANALYSIS

Course Code: MT(N) 201

BLOCK-III

RIEMANN INTEGRAL AND IMPROPER INTEGRAL

UNIT 9: RIEMANN INTEGRAL I

Contents

9.1 INTRODUCTION

In the branch of mathematics known as real analysis, the Riemann integral, created by Bernhard Riemann, was the first rigorous definition of the integral of a function on an interval. It was presented to the faculty at the University of Göttingen in 1854, but not published in a journal until 1868.[1] For many functions and practical applications, the Riemann integral can be evaluated by the fundamental theorem of calculus or approximated by numerical integration, or simulated using Monte Carlo integration.

9.2 OBJECTIVES

In this Unit, we will Discussed about

- Upper Riemann Sums
- Lower Riemann Sums
- Riemann Integral
- Construct mean value theorem of calculus

9.3 RIEMANN INTEGRAL

Now we will discuss the definition of Riemann integral of a function f on an interval $[a, b]$.
We first define some basic terms that will be frequently used.

Partition of I: If $I = [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a partition of I is a finite, ordered set $P = (x_0, x_1, \dots, x_{n-1}, x_n)$ of points in I such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The points of P are used to divide $I = [a, b]$ into non-overlapping subintervals

$I_1 = [x_0, x_1],\ I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$

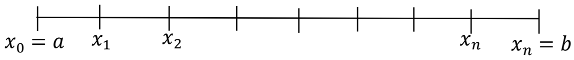


Fig. Partition of $I = [a, b]$

Let f be a bounded real function on $[a, b]$. Obviously f is bounded on each sub-interval corresponding to each partition P. Let M_i and m_i be the supremum and infimum respectively of f in Δx_i. Then

Upper Darboux Sums:

$$\boldsymbol{U(P,f) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n} = \sum_{i=1}^{n} \boldsymbol{M_i}\, \Delta x_i$$

is called Upper Darboux Sums of f corresponding to the partition P.

Lower Darboux Sums:

$$\boldsymbol{L(P,f) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n} = \sum_{i=1}^{n} \boldsymbol{m_i}\, \Delta x_i$$

is called Lower Darboux Sums of f corresponding to the partition P.

Note: Let M and m are the bounds of f in $[a, b]$. Then

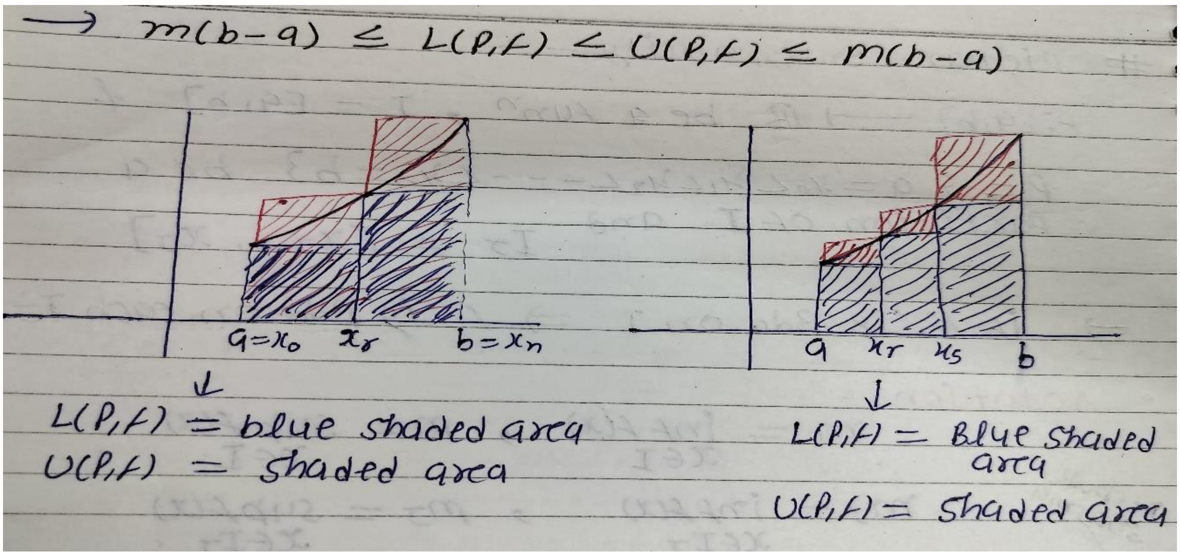
$m \le m_i \le M_i \le M \Rightarrow m\Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le M \Delta x_i$

$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le \sum_{i=1}^{n} M \Delta x_i$

$\Rightarrow m \sum_{i=1}^{n} \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le M \sum_{i=1}^{n} \Delta x_i$

$\Rightarrow \mathbf{m(a-b) \le L(P,f) \le U(P,f) \le M(a-b)}$

Note:



Therefore U(P, f) is increasing L(P, f) is decreasing function.

Example 1. Compute L(P, f) and U(P, f) if $f(x) = x$ For $x \in [0, 3]$ and let $P = \{0, 1, 2, 3\}$ be the partition of [0, 3].

Solution: Partition P divides the interval [0, 3] into sub-intervals

I_1 =[0, 1], I_2 =[1, 2], I_3 =[2, 3]

The length of these intervals are given by

$\delta_1 = 1 - 0 = 1$, $\delta_2 = 2 - 1 = 1$, $\delta_3 = 3 - 2 = 1$.

Also, if M_r and m_r be respectively the l.u.b. and g.l.b. of the function f in $[x_{r-1}, x_r]$, then here we get

$M_1 = 1$, $m_1 = 0$, $M_2 = 2$, $m_2 = 1$ and $M_3 = 3$, $m_3 = 2$

Therefore, U(P, f) = $\sum_{r=1}^{3} M_r \delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3$

= $1.1 + 2.1 + 3.1 = 6$.

And L(P, f) = $\sum_{r=1}^{3} m_r \delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3$

= $0.1 + 1.1 + 2.1 = 3$.

Upper Integral: The infimum of the set of upper sums is called Upper Integral.

i.e. $\int_a^{-b} f\, dx = \inf U = \inf\ \{U(P, f) : P \text{ is a partition of } [a, b]\}$

Lower Integral: The supremum of the set of lower sums is called Lower Integral.

i.e. $\int_{-a}^{b} f\, dx = \sup L = \sup\ \{L(P, f) : P \text{ is a partition of } [a, b]\}$

Darboux's condition of integrability:

When Upper integral and lower integral are equal then f is said to be Riemann Integral over $[a, b]$.

$\int_a^b f\, dx = \int_a^{-b} f\, dx = \int_{-a}^{b} f\, dx$

Another definition of Riemann Integrable: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if P' is any tagged partition of $[a, b]$ with $||P'|| > 0$, then $|S(f, P') - L| < \varepsilon$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $R[a, b]$.

Example 2. Show that a constant function α is integrable and

$\int_a^b dx = \alpha(b - a)$.

Proof. Let P be any partiion of the interval [a,b], then

$L(P,f) = \alpha\Delta x_1 + \alpha\Delta x_2 + \cdots + \alpha\Delta x_n$

$= \alpha\,(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = \alpha(b-a)$

Similarly, $U(P,f) = \alpha\Delta x_1 + \alpha\Delta x_2 + \cdots + \alpha\Delta x_n = \alpha(b-a)$

Therefore

$\int_{-a}^{b} \alpha dx = \sup L(P,f) = \alpha(b-a)$ and

$\int_{-a}^{b} \alpha dx = \inf U(P,f) = \alpha(b-a)$

$\Rightarrow \int_{-a}^{b} \alpha dx = \int_{-a}^{b} \alpha dx = \alpha(b-a)$

Therefore, the constant function is R-integrable and $\int_a^b \alpha dx = \alpha(b-a)$.

Example 3. Prove that function f defines as

$f(x) = \begin{cases} 0, \text{when } x \text{ is rational} \\ 1, \text{ when } x\ is \text{ irrational} \end{cases}$ is not integrable on any interval.

Proof. Let P be any partiion of the interval [a,b], then

$L(P,f) = \sum_{i=1}^{n} m_i\,\Delta x_i = 0\Delta x_1 + 0\Delta x_2 + \cdots + 0\Delta x_n = 0$

Similarly, $U(P,f) = \sum_{i=1}^{n} M_i\,\Delta x_i = 1\Delta x_1 + 1\Delta x_2 + \cdots + 1\Delta x_n = b-a$

Therefore

$\int_{-a}^{b} \alpha dx = \sup L(P,f) = 0$ and

$\int_{-a}^{b} \alpha dx = \inf U(P,f) = b-a$

$\Rightarrow \int_{-a}^{b} \alpha dx \neq \int_{-a}^{b} \alpha dx$

Therefore, the given function is not R-integrable on any interval.

Example 4. Show that function $f(x) = x^3$ is integrable on any interval $[0,b]$.

Proof. Let P be any partiion of the interval $[0,b]$ obtained by dividing interval into n −equal parts. i.e. $P = [\frac{0}{n} = 0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \ldots, \frac{nb}{n} = b]$

Let lower bounds of function in $\Delta x_i = \left(\frac{(i-1)k}{n}\right)^3$ and Upper bounds of function in $\Delta x_i = \left(\frac{ik}{n}\right)^3$

Therefore

$L(P,f) = \sum_{i=1}^{n} m_i \, \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$

$= 0.\frac{b}{n} + \left(\frac{b}{n}\right)^3 . \frac{b}{n} + \left(\frac{2b}{n}\right)^3 . \frac{b}{n} + \cdots + \left(\frac{b(n-1)}{n}\right)^3 . \frac{b}{n} = \frac{b^4}{n^4}[1^3 + 2^3 + \cdots + (n-1)^3]$

$= \frac{b^4(n-1)^2 n^2}{4n^4} = \frac{b^4}{4}\left(1 - \frac{1}{n}\right)^2$

Similarly

$U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$

$= \left(\frac{b}{n}\right)^3 . \frac{b}{n} + \left(\frac{2b}{n}\right)^3 . \frac{b}{n} + \left(\frac{2b}{n}\right)^3 . \frac{b}{n} + \cdots + \left(\frac{bn}{n}\right)^3 . \frac{b}{n} = \frac{b^4}{n^4}[1^3 + 2^3 + \cdots + n^3]$

$= \frac{b^4 n^2 (n+1)^2}{4n^4} = \frac{b^4}{4}\left(1 + \frac{1}{n}\right)^2$

Therefore

$\int_{-0}^{b} \alpha dx = \sup L(P,f) = \frac{b^4}{4}$ and

$\int_{-0}^{b} \alpha dx = \inf U(P,f) = \frac{b^4}{4}$

$\Rightarrow \int_{-0}^{b} \alpha dx = \int_{-0}^{b} \alpha dx = \frac{b^4}{4}$

Therefore, the given function is R-integrable and $\int_0^b \alpha dx = \frac{b^4}{4}$.

Example 5. Show that the function $f(x) = sinx$ is integrable in $\left[0, \frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}} sinx \, dx = 1$.

Solution: Let any partition of $\left[0, \frac{\pi}{2}\right]$ be

$$P = \left\{0 = \frac{0\pi}{2n}, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{r\pi}{2n}, \dots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

Which dissects $\left[0, \frac{\pi}{2}\right]$ into n equal parts.

The length of each subinterval = $\frac{\pi}{2n}$ and the r^{th} sub - interval is

$$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right].$$

As $f(x) = sinx$ is increasing in $\left[0, \frac{\pi}{2}\right]$, so we have

$m_r = \frac{sin(r-1)\pi}{2n}$ and $M_r = \frac{sinr\pi}{2n}, r = 1,2,3, \dots, n$

Therefore, $U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i = \sum_{i=1}^{n} \left(sin\frac{r\pi}{2n}\right).\frac{\pi}{2n}$

$$= \frac{\pi}{2n}\left[sin\frac{\pi}{2n} + sin\frac{2\pi}{2n} + \cdots + sin\frac{n\pi}{2n} \right]$$

$$= \frac{\pi}{2n}\left[\frac{\sin\left(\frac{\pi}{2n}+\frac{n-1}{n}.\frac{\pi}{2n}\right) sin\frac{n\pi}{4n}}{sin\frac{\pi}{4n}}\right]$$

$\because \ \sin a + \sin(a+d) + \cdots + \sin[a+(n-1)d]$

$$= \frac{\sin\left(a+\frac{n-1}{2}.d\ \right) sin\frac{nd}{2}}{\sin(d/2)}$$

Or $U(P,f) = \frac{\pi}{2n}\left[\left\{\sin\frac{(n+1)\pi}{4n}\sin\left(\frac{\pi}{4}\right)\right\}/\sin\left(\frac{\pi}{4n}\right)\right]$

$$= \frac{\pi}{2n}\left[\left\{\sin(\frac{\pi}{4}+\frac{\pi}{4n})\frac{1}{\sqrt{2}}\right\}/\sin\left(\frac{\pi}{4n}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}n}\left[\left\{\sin\frac{\pi}{4}cos\frac{\pi}{4n} + \cos\frac{\pi}{4}sin\frac{\pi}{4n}\right\}/\sin\left(\frac{\pi}{4n}\right)\right]$$

$$= \frac{\pi}{4n}\left(\cot\frac{\pi}{4n}+1\right)$$

Similarly, $L(P,f) = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n}-1\right)$

Now, Riemann lower integral = $\lim_{n\to\infty} L(P,f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}-1\right)$

$$= \lim_{n\to\infty}\frac{\pi/4n}{\tan(\frac{\pi}{4n})} - \lim_{n\to\infty}\frac{\pi}{4n}$$

= 1 – 0 = 1 **…… (1)**

And Riemann upper integral = $\lim_{n\to\infty} U(P,f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}+1\right)$

= 1 **……. (2)**

From equation (1) and (2) we get

$f(x) = sinx$ is integrable in $\left[0,\frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}} sinx\, dx = 1$.

9.4 INEQUALITIES FOR INTEGRALS

We already prove that

$m(b-a) \leq \int_a^b f\,dx \leq M(b-a)\ when\ b \geq a$ …………………..(I)

If $b < a$, so that $a > b$ and

$m(a-b) \leq \int_b^a f\,dx \leq M(a-b)$ $\Rightarrow -m(a-b) \leq -\int_b^a f\,dx \leq -M(a-b)$

$\Rightarrow m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$ when $b < a$………………(II)

Deduction 1: If f is bounded and integrable on $[a,b]$, then there exists a number k lying between bounds of f such that $\int_b^a f\,dx = k(b-a)$

Deduction 2: If f is continuous and integrable on $[a,b]$, then there exists a number c lying between a and b such that $\int_b^a f\,dx = f(c)(b-a)$

Deduction 3: If f is bounded and integrable on $[a,b]$, and $\alpha > 0$ is a number such that $|f(x)| \leq \alpha$ for all $x \in [a,b]$, then

$\left|\int_b^a f\,dx\right| \leq \alpha|b-a|$.

Proof. Let M and m be the upper bounds and lower bounds of $f(x)$ respectively.

Let $\alpha > 0$ is a number such that $|f(x)| \leq \alpha$ for all $x \in [a,b]$

Hence for $b > a, -\alpha \leq f(x) \leq \alpha$

$\Rightarrow -\alpha \leq m \leq f(x) \leq M \leq \alpha$

$\Rightarrow -\alpha(b-a) \leq m(b-a) \leq \int_a^b f(x) \leq M(b-a) \leq \alpha(b-a)$

$\Rightarrow \left|\int_a^b f(x)\right| \leq \alpha(b-a)$

If $a > b$, we have

$\left|\int_a^b f(x)\right| \leq \alpha(a-b)$

Therefore $\left|\int_a^b f(x)\right| \leq \alpha|b-a|$.

The result is trivial for $a = b$.

Deduction 4: If f is bounded and integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then

$\int_a^b f\, dx \geq 0$ when $b \geq a$ and $\int_a^b f\, dx \leq 0$ when $b \leq a$

Proof. Because $f(x) \geq 0$ for all $x \in [a, b]$, then the lower bound of $f(x)\ i.e. m \geq 0$

From Inequality (I) and (II) , we get

$\int_a^b f\, dx \geq 0$ when $b \geq a$ and $\int_a^b f\, dx \leq 0$ when $b \leq a$

Deduction 5 : If f and g are bounded and integrable on $[a, b]$, such that $f(x) \geq g(x)$., for all $x \in [a, b]$.then

$\int_a^b f\, dx \geq \int_a^b g\, dx$ when $b \geq a$ and $\int_a^b f\, dx \leq \int_a^b f\, dx$ when $b \leq a$

Proof. It is given that $f \geq g$ then $f - g \geq 0$ for all $x \in [a, b]$.

Using deduction 4, we have

$\int_a^b (f - g) dx \geq 0\ \ if\ b \geq a$

$\Rightarrow \int_a^b f dx \geq \int_a^b g dx\ \ if\ b \geq a$

Similarly

$\int_a^b f dx \leq \int_a^b g dx\ \ if\ b \leq a$

9.5 REFINEMENT OF PARTITIONS AND TAGGED PARTITIONS

Norm: The norm (or mesh) of P to be the number

$\mu(P) = \max\ \{x_1 - x_0,\ x_2 - x_1, \ldots, x_n - x_{n-1}\}$

OR

the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a, b]$.

Refinement: A partition P^* is said to be a refinement of P if $P^* \supseteq P$ i.e. every point of P is a point of P^*.

Or we can say that P^* refines P or P^* is finer than P.

If P_1 and P_2 are two partitions, then $P^* = P_1 \cup P_2$.

Theorem 9.5.1. Suppose that $f : [a, b] \to R$ is bounded and P and P^* be partitions of [a, b] and refinement of P respectively. Then

(i) $L(P, f) \leq L(P^*, f)$

(ii) $U(P^*, f) \leq U(P, f)$

Proof. Let P be partition of $[a, b]$ and $\boldsymbol{P}^*$ contains just one more point $'\alpha'$ than P.

Let $\alpha \in \Delta x_i$ $i.e$ $x_{i-1} < \alpha < x_i$.

It is given that the function f is bounded over the interval $[a, b]$.

⇒ It is bounded in every subinterval Δx_i.

Let β_1, β_2 and m_i be the infimum of f in the interval $[x_{i-1}, \alpha], [\alpha, x_i]$ and $[x_{i-1}, x_i]$ respectively.

Obviously $m_i \leq \beta_1$ and $m_i \leq \beta_2$.

Hence

$L(P^*, f) - L(P, f) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + \beta_1(\alpha - x_{i-1}) + \beta_1(x_i - \alpha) + m_{i+1}\Delta x_{i+1} + \cdots + m_n \Delta x_n - (m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_i \Delta x_n + m_n \Delta x_n)$

$$= \beta_1(\alpha - x_{i-1}) + \beta_2(x_i - \alpha) - m_i(x_i - x_{i-1})$$

$$= \beta_1 \alpha - \beta_1 x_{i-1} + \beta_2 x_i - \beta_2 \alpha - m_i x_i + m_i x_{i-1}$$

$$= \beta_1 \alpha - \beta_1 x_{i-1} - m_i \alpha + m_i \alpha + \beta_2 x_i - \beta_2 \alpha - m_i x_i + m_i x_{i-1}$$

$$= \alpha(\beta_1 - m_i) - x_{i-1}(\beta_1 - m_i) - m_i (x_i - \alpha) + \beta_2(x_i - \alpha)$$

$$= (\alpha - x_{i-1})(\beta_1 - m_i) + (\beta_2 - m_i)(x_i - \alpha)$$

$x_i > \alpha > x_{i-1}$ and $\beta_1, \beta_2 \geq m_i \Rightarrow (\alpha - x_{i-1}),\ (x_i - \alpha), (\beta_1 - m_i)$ and $(\beta_2 - m_i)$ are positive.

Therefore, $L(P^*, f) - L(P, f) \geq 0$

If P^* contains p points more than P, we repeat the above reasoning p times and conclude that

$L(P^*, f) \geq L(P, f)$

Similarly, we can prove that $U(P^*, f) \leq U(P, f)$

Corollary If a refinement P^* of P contains k points more than P and $|f(x)| \leq K$, for all $x \in [a, b]$, then

(i) $L(P, f) \leq L(P^*, f) \leq L(P, f) + 2kK\mu$

(ii) $U(P, f) \geq U(P^*, f) \geq U(P, f) - 2kK\mu$

Proof. . Let P be partition of $[a, b]$ and P^* contains just one more point $'\alpha'$ than P.

Let $\alpha \in \Delta x_i$ $i.e$ $x_{i-1} < \alpha < x_i$.

It is given that the function f is bounded over the interval $[a, b]$.

⇒ It is bounded in every subinterval Δx_i.

Let β_1, β_2 and m_i be the infimum of f in the interval $[x_{i-1}, \alpha], [\alpha, x_i]$ and $[x_{i-1}, x_i]$ respectively.

Obviously $m_i \leq \beta_1$ and $m_i \leq \beta_2$.

Hence

$$L(P^*, f) - L(P, f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + \beta_1(\alpha - x_{i-1}) +$$

$$\beta_1(x_i - \alpha) + m_{i+1}\Delta x_{i+1} + \cdots + m_n\Delta x_n - (m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_i\Delta x_n + m_n\Delta x_n)$$

$$= \beta_1(\alpha - x_{i-1}) + \beta_2(x_i - \alpha) - m_i(x_i - x_{i-1})$$

$$= \beta_1\alpha - \beta_1 x_{i-1} + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$$

$$= \beta_1\alpha - \beta_1 x_{i-1} - m_i\alpha + m_i\,\alpha + \beta_2 x_i - \beta_2\alpha - m_i x_i + m_i x_{i-1}$$

$$= \alpha(\beta_1 - m_i) - x_{i-1}(\beta_1 - m_i) - m_i\,(x_i - \alpha) + \beta_2(x_i - \alpha)$$

$$= (\alpha - x_{i-1})(\beta_1 - m_i) + (\beta_2 - m_i)\,(x_i - \alpha)$$

It us given that $|f(x)| \leq K$ for all $x \in [a, b]$, therefore

$-K \leq m_i \leq \beta_1 \leq K \;\Rightarrow\; K \geq -m_i$ and $K \geq \beta_1 \Rightarrow\; 2K \geq \beta_1 - m_i$ or $2K \geq \beta_1 - m_i \geq 0$

Similarly

$2K \geq \beta_2 - m_i \geq 0$

Therefore

$L(P^*, f) - L(P, f) \leq 2K(\alpha - x_{i-1}) + 2K\ (x_i - \alpha) = 2K(\alpha - x_{i-1} + x_i - \alpha) = 2K(x_i - x_{i-1})$

Therefore

$L(P^*, f) - L(P, f) \leq 2K\Delta x_i$

Let μ be the norm of P, hence

$L(P^*, f) - L(P, f) \leq 2K\mu$

Let each additional point is introduced one by one, by repeating the above reasoning k times, we get

$L(P^*, f) - L(P, f) \leq 2Kk\mu \Rightarrow L(P^*, f) \leq L(P, f) + 2Kk\mu$

Also, $L(P, f) \leq L(P^*, f)$

Hence $L(P, f) \leq L(P, f) + 2Kk\mu$

Similarly, we can prove that $U(P, f) \geq U(P^*, f) \geq U(P, f) - 2kK\mu$

CHECK YOUR PROGRESS

True or false/MCQ Questions

Problem 1. The function $f(x) = sinx$ is integrable in $\left[0, \frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}} sinx\ dx = 1$.

Problem 2. If the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f(x) = [x]$ where, $[.]$ represent the greatest integer function then

(a) $f(x)$ is continuous function on $\mathbb{R}$.

(b) $f(x)$ is Differential function on $\mathbb{R}$.

(c) $f(x)$ is Riemann integrable.

(d) $f(x)$ is not Riemann integrable.

Problem 3. Every Riemann integrable function is continuous function.

Problem 4. Every polynomial function is Riemann integrable on $\mathbb{R}$.

Problem 5. $\mathrm{U(P, f)}$ is increasing $\mathrm{L(P, f)}$ is decreasing function.

9.6 *SUMMARY*

1. If $\underset{h\to 0}{Limit}\ f(a-h) = \underset{h\to 0}{Limit}\ f(a+h) = f(a)$ then function f(x) is continuous at $x = a$.

2. A function f: X → Y is said to be **uniformly continuous** on A ⊆ X if for every ε > 0, there exists δ > 0 such that x, y ∈ A, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

9.7 *GLOSSARY*

integration

continuity

Functions

Limits

9.8 *REFERENCES*

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9.10 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that every constant function is Reimann integrable.

Q 2. Prove that every polynomial function is Riemann integrable.

Q 3. Show that the function $f(x) = sinx$ is integrable in $\left[0, \frac{\pi}{2}\right]$

Q 4. Using Riemann integration prove $\int_0^1 x\,dx = \frac{1}{2}$.

Q 5. Define upper and lower Riemann sums.

9.11 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. (c)

CYQ 3. False

CYQ 4. True

CYQ 5. True

UNIT 10: RIEMANN INTEGRAL II

Contents

10.1 INTRODUCTION

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems.

In the 1850s, Bernhard Riemann adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of ‘‘integration’’ could be defined: the class of ‘‘integrable’’ functions. The Fundamental Theorem of Calculus became a result that held only for a restricted set of integrable functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue’s theory of integration. But there have been some advances made in more recent times that extend even the Lebesgue theory to a considerable extent.

10.2 OBJECTIVES

In this Unit, we will

- Discussed about Riemann Integral
- Construct mean value theorem of calculus

10.3 DARBOUX THEOREM

Darboux Theorem

Theorem 10.3.1. If f is bounded function on $[a, b]$ then to every

$\varepsilon > 0$, there corresponds $\delta > 0$ such that

(i) $U(P, f) < \int_a^{-b} f\, dx + \varepsilon$

(ii) $L(P, f) > \int_{-a}^{b} f\, dx - \varepsilon$

For every partition P of $[a, b]$ with norm $\mu(P) < \delta$

Proof. It is given that f is bounded on $[a,b]$. Hence there exists $\alpha > 0$ such that

$f(x) \leq \alpha \quad for\ all\ x \in [a,b]$

Now

$\int_a^{-b} f\, dx = \inf U = \inf \{U(P,f): P \text{ is a partition of } [a,b]\}$

Hence for every $\varepsilon > 0$ there exists a partition $P' = \{x_0, x_1, x_3, \ldots, x_k\}$ of $[a,b]$ such that

$U(P_1, f) < \int_a^{-b} f\, dx + \frac{1}{2}\varepsilon$ ………………………..(1)

Also partition P' contains $k-1$ points other than a and b.

Let δ be a positive number such that

$2(k-1)\alpha\delta = \frac{1}{2}\varepsilon$ ………………………………….(2).

Let P be any partition such that $P = \{x_0, x_1, x_3, \ldots, x_n\}$ with norm

$\mu(P) < \delta.$

Assume P^* be a refinement of P and P' such that $P^* = P \cup P'$

P^* be a refinement of $P \Rightarrow P^*$ have $p-1$ more point than P and also $f(x) \leq \alpha$

Therefore

$U(P,f) \geq U(P^*,f) \geq U(P,f) - 2(p-1)\alpha\delta$ (Using previous corollary)

$\Rightarrow U(P,f) - 2(p-1)\alpha\delta \leq U(P^*,f)$

$\leq U(P',f)$

$< \int_a^{-b} f\, dx + \frac{1}{2}\varepsilon$ (Using eq (1))

Therefore

$U(P,f) < \int_a^{-b} f\, dx + \frac{1}{2}\varepsilon + 2(p-1)\alpha\delta$

Using equation (2), we get

$U(P,f) < \int_a^{-b} f\, dx + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon < \int_a^{-b} f\, dx + \varepsilon$

Similarly, we can prove that $L(P,f) > \int_{-a}^{b} f\, dx - \varepsilon$

Note:

- **Tags:** If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$ for $i = 1,2,..,n$, then the points are called tags of the subintervals I_i.
- **Tagged Partition of I:** A set of ordered $P = \{([x_{i-1}, x_i], t_i); i = 1,2,\dots,n\}$ of subintervals and corresponding tags is called a tagged partition of I.

10.4 CONDITION OF INTEGRABILITY AND SOME PROPERTIES OF INTEGRABLE FUNCTIONS

We already discussed that the bounded function is integrable if upper and lower integral are equal. Now we try to study the necessary and sufficient condition for integrability of a function.

FIRST FORM

Theorem 10.4.1. The necessary and sufficient condition for integrability of a bounded function f is for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition P of $[a,b]$with norm $\mu(P) < \delta$ and $U(P,f) - L(P,f) < \varepsilon$

Proof. Necessary condition

Let f be a bounded function and integrable over interval $[a,b]$,

Hence $\int_{-a}^{b} f\,dx = \int_{a}^{-b} f\,dx = \int_{a}^{b} f\,dx$

Let $\varepsilon > 0$ be any positive number.

By Darbaux's Theorem there exists a positive number δ such that foe every partition P with norm $\mu(P) < \delta$

$U(P,f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon$ ……………………..(1)

$L(P,f) > \int_{-a}^{b} f\,dx - \frac{1}{2}\varepsilon$ ……………………(2)

$\Rightarrow -L(P,f) < -\int_{-a}^{b} f\,dx + \frac{1}{2}\varepsilon$………………………………..(3)

By adding inequality (1) and (3), we get

$U(P,f) - L(P,f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon - \int_{-a}^{b} f\,dx + \frac{1}{2}\varepsilon = \varepsilon$

Hence for every partition P of $[a, b]$with norm $\mu(P) < \delta$

$U(P, f) - L(P, f) < \varepsilon$

Sufficient Condition

Assume for every partition P of $[a, b]$with norm $\mu(P) < \delta$ and

$U(P, f) - L(P, f) < \varepsilon$……………………………………………(4)

for any partition P of $[a, b]$, we have

$U(P, f) \geq \int_a^{-b} f\, dx \Rightarrow \int_a^{-b} f\, dx \leq U(P, f)$…………………………..(5)

$L(P, f) \leq \int_{-a}^{b} f\, dx \Rightarrow - \int_{-a}^{b} f\, dx \leq -L(P, f)$………………………..(6)

Adding inequality (5) and (6), we get

$\int_a^{-b} f\, dx - \int_{-a}^{b} f\, dx \leq U(P, f) - L(P, f)$

Using inequality (4), we get

$\int_a^{-b} f\, dx - \int_{-a}^{b} f\, dx < \varepsilon$

Because ε is any arbitrary positive number and also we know that a non negative number is less than every positive number.

Therefore it should be equal to 0.

i.e. $\int_a^{-b} f\, dx - \int_{-a}^{b} f\, dx < \varepsilon = 0$

Therefore $\int_a^{-b} f\, dx = \int_{-a}^{b} f\, dx$ which implies that f is integrable over interval $[a, b]$.

SECOND FORM

Theorem 10.4.2. A bounded function f is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$

Proof. Necessary condition

Let f be a bounded function and integrable over interval $[a, b]$,

Hence $\int_{-a}^{b} f\, dx = \int_a^{-b} f\, dx = \int_a^{b} f\, dx$

Let $\varepsilon > 0$ be any positive number.

As we know that the

$\int_{-a}^{b} f\,dx$ =supremum of lower sums and $\int_{a}^{-b} f\,dx =$ infimum of upper sums

Hence there exists a partition P' and P'' such that

$U(P',f) < \int_{a}^{-b} f\,dx + \frac{1}{2}\varepsilon$

$\Rightarrow U(P',f) < \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon$ …………..(1)

$L(P'',f) > \int_{-a}^{b} f\,dx - \frac{1}{2}\,\varepsilon$

$\Rightarrow L(P'',f) > \int_{a}^{b} f\,dx - \frac{1}{2}\,\varepsilon$

$\Rightarrow \int_{a}^{b} f\,dx < L(P'',f) + \frac{1}{2}\,\varepsilon$ …………(2)

Assume P be the commom refinement of partitions P' and P'' i.e.

$P = P' \cup P''$

Therefore

$U(P,f) \leq U(P',f) < \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon$ (using inequality (1))

$\Rightarrow U(P,f) < L(P'',f) + \frac{1}{2}\,\varepsilon + \frac{1}{2}\,\varepsilon = L(P'',f) + \varepsilon$

Therefore, $U(P,f) - L(P,f) < \varepsilon$ for a partition P.

Sufficient Condition

Assume $\varepsilon < 0$ be any positive number. Consider P be a partitions such that

$U(P,f) - L(P,f) < \varepsilon$…………………………………………..(3)

Now for any partition P of $[a,b]$, we have

$U(P,f) \geq \int_{a}^{-b} f\,dx \Rightarrow \int_{a}^{-b} f\,dx \leq U(P,f)$…………………………..(4)

$L(P,f) \leq \int_{-a}^{b} f\,dx \Rightarrow -\int_{-a}^{b} f\,dx \leq -L(P,f)$………………………..(5)

Adding inequality (4) and (5), we get

$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx \leq U(P,f) - L(P,f)$

Using inequality (4), we get

$\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon$

Because ε is any arbitrary positive number and also we know that a non negative number is less than every positive number.

Therefore, it should be equal to 0.

i.e. $\int_{a}^{-b} f\,dx - \int_{-a}^{b} f\,dx < \varepsilon = 0$

Therefore $\int_{a}^{-b} f\,dx = \int_{-a}^{b} f\,dx$ which implies that f is integrable over interval $[a, b]$.

Integrability of the sum and difference of Integrable functions

Theorem 10.4.3. Let f_1 and f_2 are two bounded and integrable function on $[a, b]$ then $f = f_1 + f_2$ is also integrable on $[a, b]$ and $\int_{a}^{b} f\,dx = \int_{a}^{b} f_1\,dx + \int_{a}^{b} f_2\,dx$

Proof. Let f_1 and f_2 are two bounded $\Rightarrow f = f_1 + f_2$ is bounded on $[a, b]$.

Let P be any partition P of $[a, b]$ such that

$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.

Let M_i' and m_i' are the upper and lower bound of f_1 respectively and M_i'' and m_i'' are the upper and lower bound of f_2 respectively in Δx_i.

Assume M_i and m_i are the upper and lower bound of f respectively in Δx_i.

Therefore

$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$ ……………(1)

Multiplying inequality (1) by Δx_i, we get

$(m_i' + m_i'')\Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le (M_i' + M_i'')\Delta x_i$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$$\sum_{i=1}^{n} (m_i' + m_i'')\Delta x_i \le \sum_{1=1}^{n} m_i \Delta x_i \le \sum_{1=1}^{n} M_i \Delta x_i$$

$$\le \sum_{1=1}^{n} (M_i' + M_i'')\Delta x_i$$

$\Rightarrow L(P, f_1) + L(P, f_2) \le L(P, f) \le U(P, f) \le U(P, f_1) + U(P, f_2)$

$U(P, f) \le U(P, f_1) + U(P, f_2)$ ……….(2)

$L(P, f_1) + L(P, f_2) \le L(P, f)$

$-L(P, f) \le -(L(P, f_1) + L(P, f_2))$ ………….(3)

Let $\varepsilon > 0$ be any positive number.

It is given that f_1 and f_2 are integrable. Hence for any partition P there exists $\delta > 0$ such that the norm $\mu(P) < \delta$, we have

$U(P, f_1) - L(P, f_1) < \frac{1}{2}\varepsilon$………………………….(4)

$U(P, f_2) - L(P, f_2) < \frac{1}{2}\varepsilon$…………………………...(5)

From (2),(3),(4) and (5), we get

$U(P, f) - L(P, f) \le U(P, f_1) + U(P, f_2) - \big(L(P, f_1) + L(P, f_2)\big)$

$= U(P, f_1) - L(P, f_1) + U(P, f_2) - L(P, f_2) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$

Therefore

$U(P, f) - L(P, f) < \varepsilon$.

Hence the function f is integrable.

f_1 and f_2 are integrable and $\varepsilon > 0$ is any positive number.

Using Darboux's theorem, there exists $\delta > 0$ such that for all partitions P whose norm $\mu(P) < \delta$, we have

$U(P, f_1) < \int_a^b f_1\, dx + \frac{1}{2}\varepsilon$ ……………(6)

And

$U(P, f_2) < \int_a^b f_2\, dx + \frac{1}{2}\varepsilon$ ……………(7)

Using inequality (2), we get

$\int_a^b f\, dx \le U(P, f) \le U(P, f_1) + U(P, f_2)$

Using inequalities (6) and (7), we get

$\int_a^b f\, dx < \int_a^b f_1\, dx + \frac{1}{2}\varepsilon + \int_a^b f_2\, dx + \frac{1}{2}\varepsilon = \int_a^b f_1\, dx + \int_a^b f_2\, dx + \varepsilon$

As we know ε is arbitrary, therefore

$\int_a^b f\,dx \le \int_a^b f_1\,dx + \int_a^b f_2\,dx$ ……………..(8)

Now replacing f_1 and f_2 with $(-f_1)$ and $(-f_2)$ respectively, we get

$\int_a^b (-f)\,dx \le \int_a^b (-f_1)\,dx + \int_a^b (-f_2)\,dx$

i.e. $\int_a^b f\,dx \ge \int_a^b f_1\,dx + \int_a^b f_2\,dx$ …………..(9)

From inequality (8) and (9), we get

$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$

Theorem 10.4.4. Let f_1 and f_2 are two bounded and integrable function on $[a,b]$ then $f = f_1 - f_2$ is also integrable on $[a,b]$ and $\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$

Proof. Let f_1 and f_2 are two bounded $\Rightarrow f = f_1 + (-f_2)$ is bounded on $[a,b]$.

Let P be any partition P of $[a,b]$ such that $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$.

Let M_i' and m_i' are the upper and lower bound of f_1 respectively and M_i'' and m_i'' are the upper and lower bound of f_2 respectively in Δx_i.

$\Rightarrow -M_i''$ and $-m_i''$ are the upper and lower bound of $(-f_2)$ respectively in Δx_i.

Assume M_i and m_i are the upper and lower bound of f respectively in Δx_i.

Therefore

$m_i' + (-m_i'') \le m_i \le M_i \le M_i' + (-M_i'')$

$\Rightarrow m_i' - M_i' \le m_i \le M_i \le M_i'' - m_i''$ ………..(1)

Multiplying inequality (1) by Δx_i, we get

$(m_i' - M_i'')\Delta x_i \le m_i\Delta x_i \le M_i\Delta x_i \le (M_i' - m_i'')\Delta x_i$

Adding all these inequalities for $i = 1,2,3,\dots,n$, we get

$$\sum_{i=1}^{n}(m_i' - M_i')\Delta x_i \le$$

$$\sum_{1=1}^{n} m_i\Delta x_i \le \sum_{1=1}^{n} M_i\Delta x_i \le \sum_{1=1}^{n}(M_i' - m_i'')\Delta x_i$$

$\Rightarrow L(P, f_1) - U(P, f_2) \leq L(P, f) \leq U(P, f) \leq U(P, f_1) - L(P, f_2)$

$U(P, f) \leq U(P, f_1) - L(P, f_2)$ …………(2)

$L(P, f_1) - U(P, f_2) \leq L(P, f)$

$-L(P, f) \leq U(P, f_2) - L(P, f_1)$ …………….(3)

Let $\varepsilon > 0$ be any positive number.

It is given that f_1 and f_2 are integrable. Hence for any partition P there exists $\delta > 0$ such that the norm $\mu(P) < \delta$, we have

$U(P, f_1) - L(P, f_1) < \frac{1}{2}\varepsilon$ ……………….(4)

$U(P, f_2) - L(P, f_2) < \frac{1}{2}\varepsilon$ ………………...(5)

From (2), (3), (4) and (5), we get

$U(P, f) - L(P, f) \leq U(P, f_1) - L(P, f_2) + U(P, f_2) - L(P, f_1)$

$= U(P, f_1) - L(P, f_1) + U(P, f_2) - L(P, f_2)$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$

Therefore

$U(P, f) - L(P, f) < \varepsilon$.

Hence the function f is integrable.

f_1 and f_2 are integrable and $\varepsilon > 0$ is any positive number.

Using Darboux's theorem, there exists $\delta > 0$ such that for all partitions P whose norm $\mu(P) < \delta$, we have

$U(P, f_1) < \int_a^b f_1\, dx + \frac{1}{2}\varepsilon$ ……………(6)

And

$L(P, f_2) > \int_a^b f_2\, dx + \frac{1}{2}\varepsilon$

$\Rightarrow -L(P, f_2) < -\int_a^b f_2\, dx + \frac{1}{2}\varepsilon$ ……………(7)

Using inequality (2), we get

$\int_a^b f\, dx \leq U(P, f) \leq U(P, f_1) - L(P, f_2)$

Using inequalities (6) and (7), we get

$\int_a^b f\,dx < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon - \int_a^b f_2\,dx + \frac{1}{2}\varepsilon = \int_a^b f_1\,dx - \int_a^b f_2\,dx + \varepsilon$

As we know ε is arbitrary, therefore

$\int_a^b f\,dx \le \int_a^b f_1\,dx - \int_a^b f_2\,dx$ …………………………..(8)

Now replacing f_1 and f_2 with $(-f_1)$ and $(-f_2)$ respectively, we get

$\int_a^b (-f)\,dx \le \int_a^b (-f_1)\,dx - \int_a^b (-f_2)\,dx$

i.e. $\int_a^b f\,dx \ge \int_a^b f_1\,dx - \int_a^b f_2\,dx$…………………………..(9)

From inequality (8) and (9), we get

$\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$.

Oscillation: The oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)|: x_1, x_2 \in [a, b]\}$ of numbers.

Let M and m be the upper and lower bounds of f on $[a, b]$ respectively.

$\Rightarrow m \le f(x_1) \le M$ and $m \le f(x_2) \le M$ for all $x_1, x_2 \in [a, b]$

$\Rightarrow |f(x_1) - f(x_2)| \le M - m$ for all $x_1, x_2 \in [a, b]$ ……….(1)

$\Rightarrow M - m$ is an upper bound of $\{f(x_1) - f(x_2)$, for all $x_1, x_2 \in [a, b]\}$

Let $\varepsilon > 0$ be any positive number, because M is supremum of f.

Therefore, there exists $y \in [a, b]$ such that

$f(y) > M - \frac{1}{2}\varepsilon$ …………(2)

Similarly, there exists $z \in [a, b]$ such that

$f(z) > m + \frac{1}{2}\varepsilon$ …………(3)

From inequalities (2) and (3), we conclude that there exist $x, y \in [a, b]$ such that

$f(y) - f(z) > M - \frac{1}{2}\varepsilon - m - \frac{1}{2}\varepsilon = M - m - \varepsilon$

Or $|f(y) - f(z)| > M - m - \varepsilon$ …………(4)

From inequalities (1) and (4), we conclude that

$M - m$ is an upper bound and also number less than $M - m$ cannot be upper bound of given set.

Hence $M - m = \sup\{\,|f(y) - f(z)|: y, z \in [a, b]\}$ ....(A)

10.5 SOME IMPORTANT THEOREM

Theorem 10.5.1. If f and g are two bounded and integrable functions on $[a, b]$ then the product fg is also bounded and integrable on $[a, b]$.

Proof. It is given that f and g are two bounded therefore there exists α such tha

$|f(x)| \leq \alpha$ and $|g(x)| \leq \alpha$ for all $x \in [a, b]$

$\Rightarrow |fg(x)| = |f(x)||g(x)| \leq \alpha.\alpha \leq \alpha^2$

It implies that fg is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any partition of $[a, b]$.

Let M_i' and m_i' are the upper and lower bound of f respectively and M_i'' and m_i'' are the upper and lower bound of g respectively in Δx_i.

Assume M_i and m_i are the upper and lower bound of fg respectively in Δx_i.

Now for all $x, x' \in \Delta x_i$,

$$(fg)(x') - (fg)(x) = f(x')g(x') - f(x)g(x)$$

$$= f(x')g(x') - f(x)g(x') + f(x)g(x') - f(x)g(x)$$

$$= g(x')(f(x') - f(x)) + f(x)(g(x') - g(x))$$

It implies that

$$|(fg)(x') - (fg)(x)| = |g(x')\big(f(x') - f(x)\big) + f(x)\big(g(x') - g(x)\big)|$$

$$\leq |g(x')||f(x') - f(x)| + |f(x)||\,g(x') - g(x)|$$

Hence, From inequality (A), we get

$M - m \leq \alpha(M' - m') + \alpha(M'' - m'')$ ……………………(1)

Let $\varepsilon > 0$ be given number and it is given that f and g integrable on interval $[a, b]$.

Therefore there exists a positive number $\delta > 0$ such that for any partition P with norm $\mu(P) < \delta$

$U(P, f) - L(P, f) \leq \frac{\varepsilon}{2\alpha}$ …………………………………..(2) and

$U(P, g) - L(P, g) \leq \frac{\varepsilon}{2\alpha}$ …………………………………..(3)

Now multiply inequality (1) with Δx_i, we get

$$(M - m)\Delta x_i \leq \alpha(M' - m')\Delta x_i + \alpha(M'' - m'')\Delta x_i$$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$$\sum_{1=1}^{n}(M - m)\Delta x_i \leq \sum_{1=1}^{n} \alpha(M' - m')\Delta x_i + \sum_{1=1}^{n} \alpha(M'' - m'')\Delta x_i$$

$$\Rightarrow \sum_{1=1}^{n} M\Delta x_i - \sum_{1=1}^{n} m\Delta x_i \leq \alpha\left(\sum_{1=1}^{n} M'\Delta x_i - \sum_{1=1}^{n} m'\Delta x_i\right) + \alpha\left(\sum_{1=1}^{n} M''\Delta x_i - \sum_{1=1}^{n} m''\Delta x_i\right)$$

$$\Rightarrow U(P, fg) - L(P, fg) \leq \alpha\big(U(P, f) - L(P, f)\big) + \alpha\big(U(P, g) - L(P, g)\big)$$

$$\leq \alpha\frac{\varepsilon}{2\alpha} + \alpha\frac{\varepsilon}{2\alpha}$$

Therefore $U(P, fg) - L(P, fg) \leq \varepsilon$

Hence, we conclude that fg is integrable on $[a, b]$.

Theorem 10.5.2. If f and g are two bounded and integrable functions on $[a, b]$ and there exists a positive number k such that $|g| \geq k$ for all $x \in [a, b]$ then the f/g is also bounded and integrable on $[a, b]$.

Proof. It is given that f and g are two bounded therefore there exists α such that

$|f(x)| \leq \alpha$ and $k \leq |g(x)| \leq \alpha \Rightarrow \frac{1}{k} \geq \frac{1}{|g(x)|} \geq \frac{1}{\alpha}$ for all $x \in [a, b]$

$$\Rightarrow |(f/g)(x)| = |f(x)|/|g(x)| \leq \alpha.\frac{1}{k} \leq \frac{\alpha}{k}$$

It implies that fg is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any partition of $[a, b]$.

Let M_i' and m_i' are the upper and lower bound of f respectively and M_i'' and m_i'' are the upper and lower bound of g respectively in Δx_i.

Assume M_i and m_i are the upper and lower bound of f/g respectively in Δx_i.

Now for all $x, x' \in \Delta x_i$,

$$\left|\left(\frac{f}{g}\right)(x') - \left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(x')}{g(x')} - \frac{f(x)}{g(x)}\right| = \left|\frac{f(x')g(x)-f(x)g(x')}{g(x)g(x')}\right|$$

$$= \left|\frac{f(x')g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x')}{g(x)g(x')}\right|$$

$$= \left|\frac{g(x)\big(f(x')-f(x)\big)-f(x)\big(g(x')-g(x)\big)}{g(x)g(x')}\right|$$

$$\leq \alpha\frac{|f(x')-f(x)|}{|g(x)g(x')|} + \alpha\frac{|g(x')-g(x)|}{|g(x)g(x')|}$$

Hence, From inequality (A), we get

$$M - m \leq \alpha.(M' - m').\frac{1}{k^2} + \alpha.(M'' - m'').\frac{1}{k^2}$$

Hence

$$M - m \leq \frac{\alpha}{k^2}(M' - m') + \frac{\alpha}{k^2}(M'' - m'') \qquad \text{.........(1)}$$

Let $\varepsilon > 0$ be given number and it is given that f and g integrable on interval $[a, b]$.

Therefore, there exists a positive number $\delta > 0$ such that for any partition P with norm

$\mu(P) < \delta$

$$U(P, f) - L(P, f) \leq \frac{\varepsilon k^2}{2\alpha} \qquad \text{........(2) and}$$

$$U(P, g) - L(P, g) \leq \frac{\varepsilon k^2}{2\alpha} \qquad \text{...........(3)}$$

Now multiply inequality (1) with Δx_i, we get

$$(M - m)\Delta x_i \leq \frac{\alpha}{k^2}(M' - m')\Delta x_i + \frac{\alpha}{k^2}(M'' - m'')\Delta x_i$$

Adding all these inequalities for $i = 1,2,3, \dots, n$, we get

$$\sum_{1=1}^{n}(M - m)\Delta x_i \leq \sum_{1=1}^{n}\frac{\alpha}{k^2}(M' - m')\Delta x_i + \sum_{1=1}^{n}\alpha(M'' - m'')\Delta x_i$$

$\Rightarrow \sum_{1=1}^{n} M\Delta x_i - \sum_{1=1}^{n} m\Delta x_i \leq \frac{\alpha}{k^2}(\sum_{1=1}^{n} M'\Delta x_i - \sum_{1=1}^{n} m'\Delta x_i) + \frac{\alpha}{k^2}(\sum_{1=1}^{n} M''\Delta x_i - \sum_{1=1}^{n} m''\Delta x_i)$

$\Rightarrow U(P,fg) - L(P,fg) \leq \frac{\alpha}{k^2}\big(U(P,f) - L(P,f)\big) + \frac{\alpha}{k^2}\big(U(P,g) - L(P,g)\big)$

$$\leq \frac{\alpha}{k^2}\frac{\varepsilon k^2}{2\alpha} + \frac{\alpha}{k^2}\frac{\varepsilon k^2}{2\alpha}$$

Therefore $U(P,fg) - L(P,fg) \leq \varepsilon$

Hence we conclude that f/g is integrable on $[a,b]$.

Theorem 10.5.3. If f ís bounded and integrable functions on $[a,b]$ then $|f|$ is also bounded and integrable on $[a,b]$ and also $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$.

Proof. It is given that f is bounded therefore there exists α such that

$|f(x)| \leq \alpha$ for all $x \in [a,b]$

It implies that the function $|f|$ is bounded.

Since f is integrable, for a given positive number $\varepsilon > 0$ there exists a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a,b]$ and such that

$U(P,f) - L(P,f) < \varepsilon$ …………………..(1)

Let M_i and m_i are the upper and lower bound of f respectively and M_i' and m_i' are the upper and lower bound of g respectively in Δx_i.

Now for all $x, x' \in \Delta x_i$,

$\big||f|(x) - |f|(y)\big| = \big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$

$\Rightarrow M_i' - m_i' \leq M - m$……………………….(2)

Now multiply inequality (2) with Δx_i, we get

$(M_i' - m_i')\Delta x_i \leq (M_i - m_i)\Delta x_i$

Adding all these inequalities for $i = 1,2,3,\dots,n$, we get

$\sum_{1=1}^{n}(M_i' - m_i')\Delta x_i \leq \sum_{1=1}^{n}(M_i - m_i)\Delta x_i$

$\Rightarrow \sum_{1=1}^{n} M_i'\Delta x_i - \sum_{1=1}^{n} m_i'\Delta x_i \leq \sum_{1=1}^{n} M_i\,\Delta x_i - \sum_{1=1}^{n} m_i\Delta x_i$

$\Rightarrow U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f)$

Using inequality (1), we get

$U(P,|f|) - L(P,|f|) < \varepsilon$. Hence $|f|$ is integrable on $[a,b]$.

We Know that if f and g are bounded and integrable on $[a,b]$ such that $f \geq g$ then

$\int_a^b f\, dx \leq \int_a^b g\, dx$ when $b \leq a$

Hence $\int_a^b f\, dx \leq \int_a^b |f|\, dx$

and $-\int_a^b f\, dx = \int_a^b (-f)\, dx \leq \int_a^b |f|\, dx$

$\Rightarrow \left|\int_a^b f\, dx\right| \leq \int_a^b |f|\, dx$

Note: The Converse of the above theorem is not true. For example, the function

$$f(x) = \begin{cases} 1, & when\ x\ is\ rational \\ -1, & when\ x\ is\ irrational \end{cases}$$

Here $\int_a^{-b} f\, dx = b - a$ but $\int_a^{-b} f\, dx = a - b$

It implies that f is not integrable.

But $|f(x)| = 1\ \ for\ all\ x$, therefore $\int_a^b |f|\, dx$ exists and equal to $b - a$.

Here we observe that $|f|$ is integrable.

Theorem 10.5.4. Every Monotonic function f is Riemann integrable.

Proof: Let us suppose that the function f is monotonically increasing function on $[a,b]$.

Now for a given positive number ε, there exists a partition P = $\{a = x_0, x_1, x_2, \dots, x_n = b\}$ of

$[a,b]$ such that the length of each sub – interval $< \frac{\varepsilon}{[f(a)-f(b)+1]}$

i.e. $(x_r - x_{r-1}) < \frac{\varepsilon}{[f(a)-f(b)+1]}$ for $r = 1, 2, \dots, n$ ……. (1)

Again, the function $f$being monotonically increasing on $[a,b]$, it is bounded and monotonically increasing on each sub – interval $[x_{r-1}, x_r]$.

Let the bounds of function f on the sun – interval $[x_{r-1}, x_r]$ be M_r and m_r, then

$M_r = f(x_r)$ and $m_r = f(x_{r-1})$ ……. (2)

Therefore, for this partition P, we find that

$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_r - m_r)(x_r - x_{r-1})$

$< \frac{\varepsilon}{[f(a)-f(b)+1]}\sum_{i=1}^{n}[f(x_r) - f(x_{r-1})]$

Therefore, $U(P,f) - L(P,f) < \frac{\varepsilon}{[f(a)-f(b)+1]}\sum_{i=1}^{n}[f(x_n) - f(x_0)]$

Therefore, $U(P,f) - L(P,f) < \frac{\varepsilon}{[f(a)-f(b)+1]}\sum_{i=1}^{n}[f(b) - f(a)]$

Therefore, $U(P,f) - L(P,f) < \varepsilon$

Therefore, Every Monotonically increasing function f is Riemann integrable.

Similarly, we can prove that Every Monotonically decreasing function f is Riemann integrable.

Therefore, every monotonic function is Riemann integrable.

10.6 RIEMANN SUM

Riemann Sum: Let P' is the tagged partition then the Riemann sum of a function $f : [a,b] \to \mathbb{R}$ corresponding to P' can be defined as

$S(f,P') = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})$

If the function f is positive on $[a,b]$, then the Riemann Sum is the sum of the areas of n rectangles whose bases are the subintervlas $I_1 = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$. See Fig 5.1.

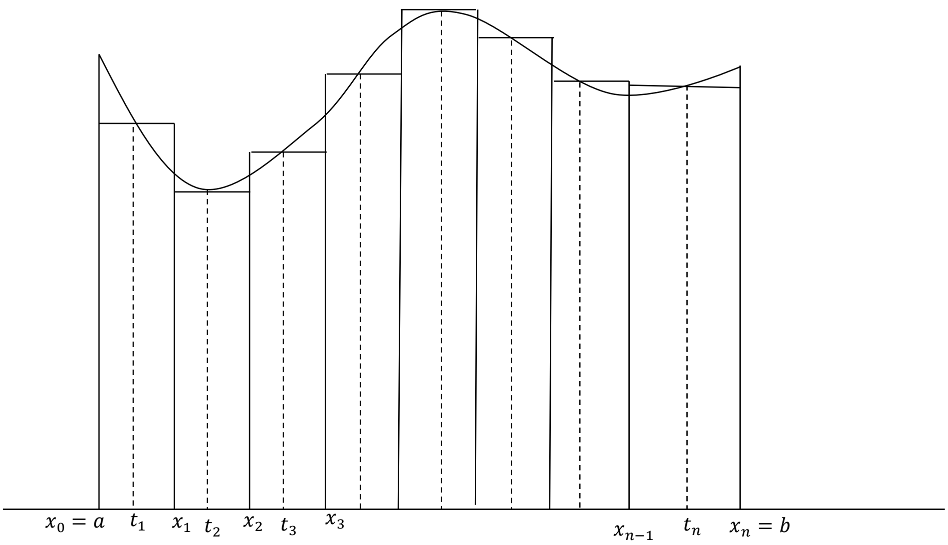

Fig 5.1. A Riemann Sum

Theorem. If $f: [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Proof. Let $\varepsilon > 0$ be given.

Now f is continuous on $[a, b] \Rightarrow$ It is also uniformly continuous.

Therefore, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$.

For any large integer N we assume an equally spaced partition

$x_k = a + kh$, with $h = \frac{b-a}{N}$ and $k = 0,1, \dots, N$. We choose N so large that $\frac{b-a}{N} < \delta$.

Now function f is continuous on any of the intervals $[x_{k-1}, x_k]$,

Hence there must exist points $c_k, d_k \in [x_{k-1}, x_k]$ where f attains its minimum and maximum, respectively, i.e.

$f(c_k) \leq f(x) \leq f(d_k)\ for\ all\ x \in [x_{k-1}, x_k]$.
Let $s, t: [a, b] \to \mathbb{R}$ are two step functions such that on each interval $[x_{k-1}, x_k)$
$s(x) = f(c_k)$ and $t(x) = f(d_k)$.
Therefore, we conclude that $s(x) \leq f(x) \leq t(x)$ for some
$x \in [x_{k-1}, x_k)$
Since $|c_k - d_k| \leq \frac{b-a}{N} < \delta$ then for any $x \in [x_{k-1}, x_k)$
$t(x) - s(x) = f(d_k) - f(c_k) < \frac{\varepsilon}{b-a}$.

This also holds for each interval $[x_{k-1}, x_k)$ $(k = 1,2, \dots, N)$

Hence we shown that $0 \leq t(x) - s(x) < \frac{\varepsilon}{b-a}$ for all $x \in [a, b]$

Now compare the integrals of t and s and since $t \leq s + \frac{\varepsilon}{b-a}$

Then $\int_a^b t(x)dx \leq \int_a^b \left(s(x) + \frac{\varepsilon}{b-a}\right) dx = \int_a^b s dx + \varepsilon.$

Fundamental Theorem of Calculus

Theorem. A function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

Proof. It is given that $F' = f$ is bounded and integrable on $[a, b]$.

Therefore, for every given $\varepsilon > 0$ there exists a positive number δ such that for every partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$, with norm $\mu(P) < \delta$.

$\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)dx\right| < \varepsilon$ ……………………….(1)

For every choice of points t_i in Δx_i.

Because we have freedom in the selection of points t_i in Δx_i., we choose them in a particular way as follows:

By Lagrange Mean value theorem, we have

$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i \qquad (i = 1,2, \dots, n)$

Hence $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$

It implies that $\sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a).$

From inequality (1), we get

$\int_a^b f(x)dx = F(b) - F(a)$

This theorem is also known as the Second Fundamental theorem of Integral Calculus.

First Mean Value theorem

Theorem. A function f is continuous on $[a, b]$, then there exists a number k in $[a, b]$ uch that $\int_a^b f\, dx = f(k)(b - a)$.

Proof. It is given that f is continuous on $[a, b]$, therefore f is Riemann Integrable on $[a, b]$.

Let M and m are the upper and lower bound of f on $[a, b]$ respectively.

As we know that

$m(b-a) \leq \int_a^b f\, dx \leq M(b-a)$

Hence there exists a real number $\gamma \in [m, M]$ such that

$\int_a^b f\, dx = \gamma(b-a)$

Because f is continuous on $[a,b]$, it attains every value between m and M.

Hence, there exists a number $k \in [a,b]$ such that $f(k) = \gamma$.

Therefore, $\int_a^b f\, dx = f(k)(b-a)$

CHECK YOUR PROGRESS

True or false/MCQ Questions

Problem 1.

Let f(x) is defined by f(x) = $\begin{cases} x^2, x \text{ is rational} \\ x^3, x \text{ is irrational} \end{cases}$, x ∈ [0, 1]

then

(a) f is Riemann integrable on [0, 1] and $\int_0^a f(x)dx$ = 1/3

(b) f is Riemann integrable on [0, 1] and $\int_0^a f(x)dx$ = 1/4

(c) f is not Riemann integrable on [0, 1].

(d) None of the above.

Problem 2. Every continuous function on closed interval is not Riemann integrable.

Problem 3. Every continuous function on closed interval is Riemann integrable.

Problem 4. For every polynomial function Riemann upper integral is equal to Riemann lower integral.

Problem 5. U(P, f) is decreasing, L(P, f) is decreasing function.

10.7 *SUMMARY*

1. Darboux Theorem

If f is bounded function on $[a, b]$ then to every

$\varepsilon > 0$, there corresponds $\delta > 0$ such that

(i) $U(P, f) < \int_a^{-b} f\, dx + \varepsilon$

(ii) $L(P, f) > \int_{-a}^{b} f\, dx - \varepsilon$

For every partition P of $[a, b]$ with norm $\mu(P) < \delta$

2. Every Monotonic function f is Riemann integrable.

3. If $f: [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

10.8 *GLOSSARY*

integration

continuity

Functions

Limits

10.9 *REFERENCES*

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10.10 *SUGGESTED READING*

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5. Shanti Narayan, A course of Mathematical Analysis (29th Edition), S. Chand and Co., 2005.

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10.11 *TERMINAL AND MODEL QUESTIONS*

Q 1. Prove that every constant function is Reimann integrable.

Q 2. Prove that every polynomial function is Riemann integrable.

Q 3. Show that the function $f(x) = sinx$ is integrable in $\left[0, \frac{\pi}{2}\right]$

Q 4. Using Riemann integration prove $\int_0^1 x\,dx = \frac{1}{2}$.

Q 5. Define upper and lower Riemann sums.

10.12 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. (c)

CYQ 2. False

CYQ 3. True

CYQ 4. True

CYQ 5. False

UNIT 11: SEQUENCE AND SERIES OF FUNCTION

Contents

11.1 INTRODUCTION

Mathematics allows us to create sequences and series not only for real numbers but also for functions. This article will give you a deeper understanding of how to construct sequences and series for real-valued functions. We will also delve into the concept of convergence in sequences and series of functions. To solidify these concepts, we will include some solved problems on sequences and series of functions.

11.2 OBJECTIVES

In this Unit, we will Discussed about

- Sequence of functions
- Series of functions
- Abel's test
- Dirichlet's test

11.3 SEQUENCE OF FUNCTION

Let f_n be a real – valued function defined on an interval I and for each $n \in \mathbb{N}$. Then
$< f_1, f_2, f_3, \ldots, f_n, \ldots >$ is called a sequence of real valued function.
Denoted by $\{f_n\}$ or $\langle f_n \rangle$.
Example: $\{f_n\} = \{x^n, 0 \le x \le 1\}$ and $\left\{\frac{sinnx}{n}, 0 \le x \le 1\right\}$ are sequence of functions.

11.4 POINTWISE CONVERGENCE

For each $n \in$ N, let $f_n: A \to$ R be a real-valued function on A. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $(f_n(x))$ converges to the real number $f(x)$.

We often write lim $n\to\infty$ fn $(x) = f$ (x) or lim $n\to\infty$ $f_n = f$

Example 1. $f_n(x) = \frac{x^2 + nx}{n} = \frac{x^2}{n} + x$

$$\lim_{n\to\infty}\left(\frac{x^2}{n} + x\right) = 0 + x = x.$$

If $f(x) = x$, then $f_n \to f$ as $n \to \infty$. In this case, the functions f_n are everywhere continuous and differentiable, and the limit function is also everywhere continuous and differentiable.

Example 2. Let $f_n(x) = x^n$ on the set [0, 1].

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}.$$

In this case, the functions $f_n(x)$ are continuous on [0, 1], but the limit function f(x) is not continuous at every point of [0, 1].

Note: Suppose $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$. then

(i) If each f_n is continuous on A, then f is continuous on A.

(ii) if each f_n is differentiable on A, then f is differentiable on A.

11.5 UNIFORM CONVERGENCE

Let (f_n) be a sequence of functions defined on $A \subseteq$ R. We say that (f_n) converges uniformly on A to the limit function f defined on A if for every $\epsilon > 0$ there exists an N ∈ ℕ such that $|f_n(x) - f(x)| < \epsilon$

for all $x \in A$, whenever $n \ge N$.

Note: In the definition, the value of N is independent of x.

Example1: $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$

Therefore, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n^2}} = |x|$.

So, $f_n(x) \to f(x) = |x|$ pointwise.

Let $\epsilon > 0$ be given. Choose $N \in$ N large enough such that 1 $N < \epsilon$. Then for any $x \in$ R and $n \ge N$ we have

$$|f_n(x) - f(x)| = \left|\sqrt{x^2 + \frac{1}{n^2}} - |x|\right| = \left|\sqrt{x^2 + \frac{1}{n^2}} - |x|\right| \left(\frac{\left|\sqrt{x^2+\frac{1}{n^2}} + |x|\right|}{\left|\sqrt{x^2+\frac{1}{n^2}} + |x|\right|}\right)$$

$$= \frac{\frac{1}{n^2}}{\left|\sqrt{x^2+\frac{1}{n^2}} + |x|\right|} \leq \frac{\frac{1}{n^2}}{\left|\sqrt{0+\frac{1}{n^2}} + 0\right|} = \frac{1}{n} < \epsilon$$

This shows that $(f_n) \to f$ uniformly on R. Note that each $f_n\ (x)$ is both continuous and differentiable on R, but $f\ (x) = |x|$ is continuous on R and not differentiable at $x = 0$.

Example2: $f_n(x) = \begin{cases} nx & if\ 0 \leq x \leq 1\ \frac{1}{n} \\ 2 - nx & if\ \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & otherwise \end{cases}$

Therefore, $\lim_{n\to\infty} f_n(x) = f(x) = 0$

If f(x) = 0, then $(f_n) \to g$ pointwise.

Let $\epsilon = 1/2$ and $x_n = \frac{1}{n}$ then

$|\ f_n\ (x_n) - f\ (x_n)| = \quad |\ 1- 0| = 1 > \epsilon = 1/2.$

So, it is not true that for all $\epsilon > 0$, there exist an $N \in \mathbb{N}$ large enough such that $n \geq N$ implies $|\ f_n\ (x_n) - f\ (x_n)| <$ for all x.

So $f_n(x)$ does not converge to $f(x)$ uniformly.

11.6 CAUCHY CRITERION FOR UNIFORM CONVERGENCE

A sequence of functions (f_n) defined on a set $A \subseteq$ R converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|\ f_n\ (x) - f_m\ (x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

Proof: (⇒) Assume the sequence (f_n) converges uniformly on A to a limit function f. Let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ such that

$|\ f_n\ (x) - f\ (x)| < \frac{\epsilon}{2}$,

whenever $n \geq N$ and $x \in A$. Then if $n, m \geq N$ and $x \in A$, we have

$$|\ f_n\ (x) - f_m\ (x)| = |\ f_n\ (x) - f\ (x) + f\ (x) - f_m\ (x)|$$
$$\leq |\ f_n\ (x) - f\ (x)| + |\ f_m\ (x) - f\ (x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(⇐) Conversely, assume that for every $\epsilon > 0$ there exists an $N \in$ N such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$. This hypothesis implies that, for each $x \in A$, $(f_n(x))$ is a Cauchy sequence. By Cauchy's Criterion, this sequence converges to a point, which we will call $f(x)$. So, the uniformly Cauchy sequence converges pointwise to the function $f(x)$. We must show that the convergence is also uniform. For the value of ϵ given above, we use the corresponding N. Then for $n, m \geq N$ and all $x \in A$,

$|f_n(x) - f_m(x)| < \epsilon$

Taking the limit as $m \to \infty$ gives

$|f_n(x) - f_m(x)| \leq \epsilon$ for all $x \in A$, which shows that (f_n) converges uniformly to f on A.

This completes the proof.

11.7 CONTINUOUS LIMIT THEOREM

Let (f_n) be a sequence of functions defined on $A \subseteq$ R that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let $\epsilon > 0$ be given. Fix $c \in A$. Since $f_n \to f$ uniformly, there exists an $N \in \mathbb{N}$ such that

$|f_N(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in A$.

Since f_N is continuous at c, there exists $\delta > 0$ such that

$|f_N(x) - f_N(\text{c})| < \frac{\epsilon}{3}$ whenever $|x - c| < \delta$.

If $|x - c| < \delta$, then

$$|f(x) - f(\text{c})| = |f(x) + f_N(x) - f_N(x) - f_N(\text{c}) + f_N(\text{c}) - f(\text{c})|$$

$$\leq |f(x) - f_N(\text{c})| + |f_N(x) - f_N(\text{c})| + |f_N(\text{c}) - f(\text{c})|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

The first and third $\epsilon/3$ are due to uniform convergence and the choice of N. The second $\epsilon/3$ is due to the choice of δ. This shows that f is continuous at c, as desired.

11.8 UNIFORM CONVERGENCE AND DIFFERENTIATION

- **Differentiable Limit Theorem**

  Let $(f_n) \to f$ pointwise on the closed interval $[a, b]$ and assume each f_n is differentiable. If $(f_n') \to g$ uniformly on $[a, b]$, then f is differentiable and $f' = g$.

Proof: Fix $c \in [a, b]$ and let $\epsilon > 0$. We'll show there exists $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \epsilon,$$

whenever $0 < |x - c| < \delta$ and $x \in [a, b]$.

For $x \neq c$, consider the following:

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \le \underbrace{\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right|}_{\text{(iii)}} + \underbrace{\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right|}_{\text{(ii)}} + \underbrace{\left|f_n'(c)-g(c)\right|}_{\text{(i)}}. \qquad (*)$$

Since $\lim_{n\to\infty} f_n'(c) = g(c)$, there exists $N_1 \in \mathbb{N}$ such that

$$\left|f_n'(c) - g(c)\right| < \frac{\epsilon}{3} \qquad \text{(i)}$$

for all $n \geq N_1$.

From Cauchy's Criterion for uniform convergence, since the sequence ($f'n$) converges uniformly to g, there exists an $N_2 \in \mathbb{N}$ such that

$$\left|f_n'(x) - f_m'(x)\right| < \frac{\epsilon}{3}$$

whenever $m, n \geq N_2$ and $x \in [a, b]$. Set $N = \max\{N_1, N_2\}$

The function f_N is differentiable at c. So there exists $\delta > 0$ such that

$$\left|\frac{f_N(x) - f_N(c)}{x - c} - f'_N(c)\right| < \frac{\epsilon}{3} \qquad \text{(ii)}$$

whenever $0 < |x - c| < \delta$ and $x \in [a, b]$. We'll show this δ will suffice.

Suppose $0 < |x - c| < \delta$ and $m \geq N$. By the Mean Value Theorem applied to $f_m - f_N$ on the interval $[c, x]$ (if $x < c$ the argument is the same) there exists $\alpha \in (c, x)$ such that

$$f'_m(\alpha) - f'_N(\alpha) = \frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x - c}.$$

By our choice of N,

$$|f'_m(\alpha) - f'_N(\alpha)| < \frac{\epsilon}{3}$$

and so

$$\left|\frac{[f_m(x) - f_N(x)] - [f_m(c) - f_N(c)]}{x - c}\right| < \frac{\epsilon}{3}.$$

Since $f_m \to f$ as $m \to \infty$, by the Algebraic Order Limit Theorem,

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| \leq \frac{\epsilon}{3}. \qquad \text{(iii)}$$

Combining inequalities (*), (i), (ii), and (iii), we obtain for $0 < |x - c| < \delta$ and $x \in [a, b]$

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves that $f = \lim_{n\to\infty} f_n$ is differentiable and that $f' = g = \lim_{n\to\infty} f'_n$. □

11.9 SERIES OF FUNCTION

Let $\{f_n\}$ is a sequence of real valued functions on an interval I,

then $f_1 + f_2 + \cdots + f_n + \cdots$ is called a series of real valued function defined on I.

this series is denoted by $\sum_{n=1}^{\infty} f_n$.

Examples: $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ (This equals $e\ x$ for all $x \in$ R.)

Note: Let f and f_n for $n \in$ N be functions defined on a set $A \subseteq$ R.

(a) The infinite series $\sum f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$ converges pointwise on $\boldsymbol{A}$ to $\boldsymbol{f}$ $(\boldsymbol{x})$ if the sequence of partial sums

$s_k(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_k(x)$ converges pointwise to f (x) on A.

(b) The infinite series converges uniformly on $\boldsymbol{A}$ to $\boldsymbol{f}$ $(\boldsymbol{x})$ if the sequence of partial sums converges uniformly on A to f (x).

Note: Since an infinite series of functions is defined in terms of the limit of a sequence of partial sums, everything we already know about sequences applies to series. For the sum $\sum_{n=1}^{\infty} f_n(x)$, we merely restate all of the previous theorems for the sequence of k^{th} partial sums $s_k(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_k(x)$.

Note: Let f_n be continuous functions defined on a set $A \subseteq$ R, and assume $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to a function f. Then f is continuous on A.

Note: Term-by-term Differentiability Theorem

Suppose the following three statements:

(i) Let f_n be differentiable functions defined on an interval $A = [a, b]$.

(ii) Assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A.

(iii) There exists a point $x_0 \in [a, b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function f (x) satisfying $f'(x) = g(x)$ on A. In other words,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and } f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

11.10 CRITERION FOR UNIFORM CONVERGENCE OF SERIES

1. Cauchy Criterion for Uniform Convergence of Series:

A series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A \subseteq$ R if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon$ whenever $n > m \geq N$ and $x \in$ A.

2. Weierstrass $\boldsymbol{M}$-Test:

For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq$ R, and let $M_n >$ 0 be a real number satisfying $|f_n(\mathrm{x})| < M_n$ for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

Examples 1. The continuous functions $\frac{\cos 2^n x}{2^n}$ for $n \in \{0, 1, 2, 3, \ldots\}$ satisfy

$$\left|\frac{\cos 2^n x}{2^n}\right| \leq M_n = \frac{1}{2^n}$$

for all $x \in$ R and $n \in \{0, 1, 2, 3, \ldots\}$. Since

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-(\frac{1}{2})} = 2 < \infty,$$

by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} \frac{\cos 2^n x}{2^n}$ converges uniformly to a continuous function

$$\mathrm{g(x)} = \sum_{n=0}^{\infty} \frac{\cos 2^n x}{2^n}.$$

Examples 2. Define $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{1+x^{2n}}$. Find the values of x where the series converges and show that we get a continuous function on this set.

Sol. If $|x| < 1$, then by the Comparison Test, the series converges as follows:

$$0 \leq \sum_{n=0}^{\infty} \frac{x^{2n}}{1+x^{2n}} \leq \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} < \infty.$$

If $|x| \geq 1$, then the series diverges by the Divergence Test since

$$\lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}} = \begin{cases} \frac{1}{2} & \text{if } x = \pm 1, \\ 1 & \text{if } |x| > 1. \end{cases}$$

Now let $0 < K < 1$. Then on the interval $[-K, K]$ we have

$$\left|\frac{x^{2n}}{1+x^{2n}}\right| \leq x^{2n} \leq M_n = K^{2n}.$$

Since

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} K^{2n} = \frac{1}{1-K^2} < \infty,$$

the series converges uniformly on $[-K, K]$ to a continuous function. Since K was arbitrary, the series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}}$$

is a continuous function on $(-1, 1)$.

What about the derivative? Consider

$$g(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{2n}}{1 + x^{2n}}\right) = \sum_{n=1}^{\infty} \frac{nx^{2n-1}}{(1 + x^{2n})^2}.$$

For $0 < K < 1$, we apply the Weierstrass M-test on the interval $[-K, K]$. For $x \in [-K, K]$,

$$\left|\frac{nx^{2n-1}}{(1 + x^{2n})^2}\right| \le |nx^{2n-1}| \le N_n = nK^{2n-1}.$$

The sum $\sum_{n=0}^{\infty} N_n = \sum_{n=0}^{\infty} nK^{2n-1}$ converges and by the Weierstrass M-test the series of derivatives converges uniformly on the interval $[-K, K]$. By the Differentiable Limit Theorem, $f'(x) = g(x)$ for $x \in [-K, K]$. Since K was arbitrary, $f'(x) = g(x)$ for $x \in (-1, 1)$. That is,

$$\frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}} = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{2n}}{1 + x^{2n}}\right) \quad \text{for } x \in (-1, 1).$$

11.11 ABEL'S TEST

Let **(i)** the series of functions $\sum_{n=1}^{\infty} f_n(x)$ be uniformly convergent on $[a, b]$

And **(ii)** The sequence of functions $(g_n(x))$ be monotonic for every $x \in [a, b]$ and uniformly bounded on $[a, b]$.

Then the series $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ is uniformly convergent on $[a, b]$.

Example: Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$.

Sol. Let $g_n(x) = \frac{x^n}{(1+x^n)}$, $x \in [0, 1]$

Then $g_{n+1} - g_n = \frac{x^n(x-1)}{(1+x^n)(1+x^{n+1})} \leq 0$ for all $x \in [0, 1]$.

For each $x \in [0, 1]$, the sequence (g_n) is monotonic and for all $x \in [0, 1]$, $|g_n| < 1$ for all $n \in \mathbb{N}$.

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent series of real numbers and therefore it is uniformly convergent on $[0, 1]$.

By Abel's test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n(1+x^n)}$ is uniformly convergent on $[0, 1]$.

11.12 DIRICHLET'S TEST

Let (i) the sequence of partial sums (s_n) of the series of functions $\sum_{n=1}^{\infty} f_n(x)$ be uniformly bounded on $[a, b]$.

(ii) The sequence of functions $(g_n(x))$ be monotonic for every $x \in [a, b]$ And

(iii) The sequence of functions $(g_n(x))$ is uniformly convergent to 0 on $[a, b]$.

Then Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ is uniformly convergent on $[a, b]$.

Example: Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+n)}{n^2}$ is uniformly convergent in any closed and bounded interval $[0, 1]$.

Sol. Let $f_n(x) = (-1)^n$, $g_n(x) = \frac{(x^2+n)}{n^2}$, $x \in [a, b]$.

Let $s_n = f_1 + f_2 + \cdots + f_n$. Then the sequence (s_n) is bounded.

Then $g_{n+1} - g_n = \frac{x^2+n+1}{(n+1)^2} - \frac{x^2+n}{n^2} < 0$ for all $x \in [a, b]$.

This shows that (g_n) is a monotone decreasing sequence for each x in $[a, b]$.

$\lim_{n \to \infty} g_n(x) = 0$ each x in $[a, b]$.

Thus, the sequence of functions (g_n) is such that each g_n is continuous on $[a, b]$, the sequence converges to a continuous function on $[a, b]$ and (g_n) is monotone decreasing sequence on $[a, b]$.

Therefore, by Dirichlet's test $\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+n)}{n^2}$ is uniformly convergent.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Every pointwise convergent is uniform convergent.
Problem 2. Every uniform convergent is pointwise convergent.
Problem 3. The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $0 \leq x \leq 1$ is pointwise convergent to 0.
Problem 4. The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $0 \leq x \leq 1$ is uniformly convergent to 0.

11.13 *SUMMARY*

1. A sequence of functions (f_n) defined on a set $A \subseteq$ R converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

2. Let **(i)** the series of functions $\sum_{n=1}^{\infty} f_n(x)$ be uniformly convergent on $[a, b]$

And **(ii)** The sequence of functions $(g_n(x))$ be monotonic for every $x \in [a, b]$ and uniformly bounded on $[a, b]$.

Then the series $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ is uniformly convergent on $[a, b]$.

11.14 GLOSSARY

sequence

series

11.15 REFERENCES

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3. W. Rudin, Principles of Mathematical Analysis (3rd Edition), McGraw-Hill Publishing, 1976.

11.16 SUGGESTED READING

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5. Shanti Narayan, A course of Mathematical Analysis (29th Edition), S. Chand and Co., 2005.
6. K. A. Ross, Elementary Analysis**,** The Theory of Calculus (2nd edition), Springer, 2013.

11.17 TERMINAL AND MODEL QUESTIONS

Q 1. Prove that the $x = 0$ is a point of non – uniform convergence of the sequence of functions $\langle f_n \rangle$ Where, $f_n(x) = \frac{nx}{1+n^2x^2}$.

Q 2. Prove that the sequence of functions $\langle f_n \rangle$ Where, $f_n(x) = \frac{n^2x}{1+n^2x^2}$ is non-uniformly convergent on [0, 1].

Q 3. Prove that the sequence of functions $\langle f_n \rangle$ Where, $f_n(x) = \frac{nx}{1+n^2x^2}$ is uniformly convergent on [a, b], a > 0 but is only pointwise convergent on [0, b].

11.18 *ANSWERS*

CHECK YOUR PROGRESS

CYQ 1. False

CYQ 2. True

CYQ 3. True

CYQ 4. True

UNIT 12: IMPROPER INTEGRAL I

Contents

12.1 INTRODUCTION

In mathematical analysis, an improper integral is an extension of the notion of a definite integral to cases that violate the usual assumptions for that kind of integral. In the context of Riemann integrals (or, equivalently, Darboux integrals), this typically involves unboundedness, either of the set over which the integral is taken or of the integrand (the function being integrated), or both. It may also involve bounded but not closed sets or bounded but not continuous functions. While an improper integral is typically written symbolically just like a standard definite integral, it actually represents a limit of a definite integral or a sum of such limits; thus improper integrals are said to converge or diverge. If a regular definite integral (which may metonymically be called a proper integral) is worked out as if it is improper, the same answer will result. The concept of Riemann integrals as developed in previous chapter requires that the range of integration is finite and the integrand remains bounded on that domain. if either (or both) of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral.

12.2 OBJECTIVES

In this Unit, we will Discussed about

- Improper integral
- Proper integral
- Type of improper integral

12.3 PROPER INTEGRAL

The definite integral $\int_a^b f(x)dx$ is called a proper integral if

(i) the interval of integration $[a, b]$ is finite or bounded.

(ii) the integrand f is bounded on $[a, b]$.

If F(x) is an indefinite integral of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

12.4 IMPROPER INTEGRAL

The definite integral $\int_a^b f(x)dx$ is called a improper integral if either or both the above conditions are not satisfied. Thus $\int_a^b f(x)dx$ is an improper integral if either the interval of integration $[a, b]$ is not finite or f is not bounded on $[a, b]$ or neither the interval $[a, b]$ is finite nor f is bounded over it.

(i) In the definite integral $\int_a^b f(x)dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)dx$ is called an **improper integral of the first kind.**

For example: $\int_1^\infty \frac{dx}{\sqrt{x}}$, $\int_{-\infty}^0 e^{2x}dx$ are improper integral of the first kind.

(ii) In the definite integral $\int_a^b f(x)dx$, if both a and b are finite so that the interval of integration is finite but f has one or more point of infinite discontinuity i.e. f is not bounded on $[a, b]$, then $\int_a^b f(x)dx$ is called an **improper integral of the second kind.**

For example: $\int_1^2 \frac{1}{x^2}dx$, $\int_1^2 \frac{1}{2-x}dx$ are improper integral of second kind.

(iii) In the definite integral $\int_a^b f(x)dx$, if the interval of integration is unbounded and f is also unbounded, $\int_a^b f(x)dx$ is called an **improper integral of the third kind.**

For example: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$ is an improper integral of third kind.

12.5 IMPROPER ITEGRAL AS THE LIMIT OF A PROPER INTEGRAL

(a) When the improper integral is of the first kind, either a or b or both a and b are infinite but f is bounded. We define

(i) $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx, \ (t > a)$

The improper integral $\int_a^\infty f(x)dx$ is said to be **convergent** if the limit of right-hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

If the integral is neither convergent or divergent, then it is said to be **oscillating**.

(ii) $\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{a} f(x)dx, \ \ (t < b)$

The improper integral $\int_{-\infty}^{b} f(x)dx$ is said to be **convergent** if the limit of right-hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

(iii) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$ where c is any real number

$$= \lim_{t_1\to-\infty} \int_{t_1}^{c} f(x)dx + \lim_{t_2\to\infty} \int_{c}^{t_2} f(x)dx$$

The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is said to be **convergent** if both the limits on the right-hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note: $\int_{-\infty}^{\infty} f(x)dx \neq \lim_{t\to\infty} [\int_{-t}^{c} f(x)dx + \int_{c}^{t} f(x)dx]$.

(b) When the improper integral is second kind, both a and b are finite but f has one points of infinite discontinuity on $[a, b]$.

(i) If $f(x)$ becomes infinite at $x = b$ only, we define

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon\to0+} \int_{a}^{b-\epsilon} f(x)dx.$$

The improper integral $\int_{a}^{b} f(x)dx$ is said to be convergent if the limit on the right- hand side exists finitely and the interval is said to be divergent if the limit is $+\infty$ or $-\infty$.

(ii) If $f(x)$ becomes infinite at $x = a$ only, we define

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon\to0+} \int_{a+\epsilon}^{c} f(x)dx.$$

The improper integral $\int_{a}^{b} f(x)dx$ is said to be convergent if the limit on the right- hand side exists finitely, otherwise it is said to be divergent.

(iii) If $f(x)$ becomes infinite at $x = c$ only where $a < c < b$, we define

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

$$= \lim_{\epsilon_1\to0+} \int_{a}^{c-\epsilon_1} f(x)dx + \lim_{\epsilon_2\to0+} \int_{c+\epsilon_2}^{b} f(x)dx.$$

The improper integral $\int_a^b f(x)dx$ is said to be convergent if both the limits on the right- hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note: (1) if $f(x)$ has finite discontinuity at the end point of the interval of integration, then the point of discontinuity is approached from within the interval.

Thus, if the interval of integration is $[a, b]$ and

(i) f has infinite discontinuity at $'a'$, we consider $[a + \epsilon, b]$ as $\epsilon \longrightarrow 0+$.

(ii) f has infinite discontinuity at $'b'$, we consider $[a, b - \epsilon]$ as $\epsilon \longrightarrow 0+$.

Note: (2) A proper integral is always convergent.

Note: (3) If $\int_a^b f(x)dx$ is convergent, then

(i) $\int_a^b kf(x)dx$ is convergent, $k \in \mathbb{R}$,

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$ and each interval or right-hand side is convergent.

Note: (4) For any c between a and $b, i.e. a < c < b$, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If $\int_c^b f(x)dx$ is a proper integral, then the two integrals $\int_a^b f(x)dx$ and $\int_a^c f(x)dx$ converges or diverge together.

Thus, while testing the interval $\int_a^b f(x)dx$ convergence at a it may be replaced by $\int_a^c f(x)dx$ for any convenient c such that $a < c < b$.

ILLUSTRATIVE EXAMPLES

Example 1. Examine the convergence of the following improper integrals:

(i) $\int_0^\infty \frac{1}{x} dx$ **(ii)** $\int_1^\infty \frac{1}{\sqrt{x}} dx$ **(iii)** $\int_1^\infty \frac{1}{x^{3/2}} dx$ **(iv)** $\int_0^\infty \frac{1}{1+x^2} dx$

Sol. (i) By definition, $\int_0^\infty \frac{1}{x} dx = \lim_{t \to \infty} \int_0^t \frac{1}{x} dx$

$$= \lim_{t \to \infty} [logx]_1^t = \lim_{t \to \infty} logt = \infty$$

Therefore, $\int_0^\infty \frac{1}{x} dx$ is divergent.

(ii) By definition, $\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty} \int_1^t x^{-1/2} dx$

$$= \lim_{t\to\infty} [2\sqrt{x}]_1^t = \lim_{t\to\infty} (2\sqrt{t} - 2) = \infty$$

Therefore, $\int_0^\infty \frac{1}{\sqrt{x}} dx$ is divergent.

(iii) By definition, $\int_1^\infty \frac{1}{x^{3/2}} dx = \lim_{t\to\infty} \int_1^t x^{-3/2} dx$

$$= \lim_{t\to\infty} \left[\frac{x^{-1/2}}{-1/2}\right]_1^t = \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}} + 2\right) = 0 + 2 = 2, \text{ which is finite.}$$

Therefore, $\int_1^\infty \frac{1}{x^{3/2}} dx$ is convergent and its value is 2.

(iv) By definition, $\int_0^\infty \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx$

$$= \lim_{t\to\infty} (\tan^{-1} t)_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0)$$

$$= \frac{\pi}{2} \text{ which is finite.}$$

Therefore, $\int_0^\infty \frac{1}{1+x^2} dx$ is convergent and its value is $\frac{\pi}{2}$.

Example 2. Examine the convergence of the following improper integrals:

(i) $\int_0^\infty e^{-mx} dx \quad (m > 0)$ **(ii)** $\int_a^\infty \frac{x}{1+x^2} dx$ **(iii)** $\int_0^\infty \frac{1}{(1+x)^3} dx$

(iv) $\int_0^\infty sinx\, dx$

Sol. (i) By definition, $\int_0^\infty e^{-mx} dx = \lim_{t\to\infty} \int_0^t e^{-mx} dx$

$$= \lim_{t\to\infty} -\frac{1}{m}(e^{-mt} - 1)$$

$$= -\frac{1}{m}(0 - 1) = \frac{1}{m} \text{ which is finite.}$$

$\Rightarrow \int_0^\infty e^{-mx} dx$ is convergent and its value is $\frac{1}{m}$.

(ii) By definition, $\int_a^\infty \frac{x}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{x}{1+x^2} dx$

$$= \lim_{t\to\infty} \int_0^t \frac{1}{2} (\frac{2x}{1+x^2}) dx$$

$$= \lim_{t \to \infty} \left(\frac{1}{2} \log(1 + x^2) \right)_a^t$$

$$= \lim_{t \to \infty} \frac{1}{2} [\log(1 + t^2) - \log(1 + a^2)] = \infty$$

Therefore, $\int_a^\infty \frac{x}{1+x^2} dx$ is divergent.

(iii) By definition, $\int_0^\infty \frac{1}{(1+x)^3} dx = \lim_{t \to \infty} \int_0^t (1 + x)^{-3} dx$

$$= \lim_{t \to \infty} \left[\frac{(1+x)^{-2}}{-2} \right]_0^t$$

$$= \lim_{t \to \infty} -\frac{1}{2} \left[\frac{1}{(1+t)^2} - 1 \right] = -\frac{1}{2}(0 - 1) = \frac{1}{2} \text{ which is finite.}$$

Therefore, $\int_0^\infty \frac{1}{(1+x)^3} dx$ is convergent and its value is $\frac{1}{2}$.

(iv) By definition, $\int_0^\infty sinx\, dx = \lim_{t \to \infty} \int_0^t sinx\, dx = \lim_{t \to \infty} [-cos\, x]_0^t$

$$= \lim_{t \to \infty} (1 - cost)$$

Which does not exist uniquely since cost oscillates between -1 and +1 when $t \longrightarrow \infty$.

Therefore, $\int_0^\infty sinx\, dx$ oscillates.

Example 3. Examine for convergence the integrals:

(i) $\int_1^\infty xe^{-x}\, dx$ (ii) $\int_0^\infty x^2 e^{-x}\, dx$

Sol. (i) $\int_1^\infty xe^{-x}\, dx = \lim_{t \to \infty} \int_1^t xe^{-x}\, dx$

$$= \lim_{t \to \infty} [-xe^{-x} - e^{-x}]_1^t$$

$$= \lim_{t \to \infty} [-te^{-t} - e^{-t} + e^{-1} + e^{-1}]$$

$$= \lim_{t \to \infty} \left(\frac{-t}{e^t} \right) - \lim_{t \to \infty} (e^{-t}) + \frac{2}{e} \text{ (applying L' Hospital rule to first limit)}$$

$$= \lim_{t \to \infty} \left(\frac{-1}{e^t} \right) - 0 + \frac{2}{e} = 0 + \frac{2}{e} = \frac{2}{e} \text{ which is finite.}$$

Therefore, $\int_1^\infty xe^{-x}\, dx$ is convergent and its value is $\frac{2}{e}$.

(ii) $\int_0^\infty x^2 e^{-x}\, dx = \lim_{t \to \infty} \int_0^t x^2 e^{-x}\, dx$

$$= \lim_{t \to \infty} [-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}]_0^t$$

$$= \lim_{t\to\infty}[-t^2e^{-t} - 2te^{-t} - 2e^{-1} + 2]$$

$$= \lim_{t\to\infty}\left(\frac{-t^2}{e^t}\right) - 2\lim_{t\to\infty}\left(\frac{t}{e^t}\right) - 0 + 2 \text{ (Applying L' Hospital rule)}$$

$$= \lim_{t\to\infty}\left(\frac{-2t}{e^t}\right) - 2\lim_{t\to\infty}\left(\frac{t}{e^t}\right) + 2$$

(Again, applying L' Hospital rule)

$$= \lim_{t\to\infty}\left(\frac{-2}{e^t}\right) - 2\times 0 + 2 = 0 + 2 = 2 \text{ which is finite.}$$

Therefore, $\int_0^\infty x^2e^{-x}\,dx$ is convergent and its value is 2.

Example 4. Examine for convergence of the integrals:

(i) $\int_{-\infty}^0 e^{2x}\,dx$ (ii) $\int_{-\infty}^0 \frac{1}{p^2+q^2x^2}\,dx$

Sol. (i) $\int_{-\infty}^0 e^{2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\,dx$

$$= \lim_{t\to-\infty}\left[\frac{e^{2x}}{2}\right]_t^0 = \lim_{t\to-\infty}\frac{1}{2}(1-e^{2t}) = \frac{1}{2}(1-0) = \frac{1}{2} \text{ which is finite.}$$

Therefore, $\int_{-\infty}^0 e^{2x}\,dx$ is convergent and its value is $\frac{1}{2}$.

(ii) $\int_{-\infty}^0 \frac{1}{p^2+q^2x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{p^2+q^2x^2}\,dx$

$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{q^2(\frac{p^2}{q^2}+x^2)}\,dx$$

$$= \lim_{t\to-\infty}\left[\frac{1}{q^2}.\frac{1}{p/q}\tan^{-1}\frac{x}{p/q}\right]_t^0$$

$$= \lim_{t\to-\infty}\frac{1}{pq}\left[0-\tan^{-1}\frac{qt}{p}\right] = -\frac{1}{pq}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2pq} \text{ which is finite.}$$

Therefore, $\int_{-\infty}^0 \frac{1}{p^2+q^2x^2}\,dx$ is convergent and its value is $\frac{\pi}{2pq}$.

Example 5. Examine for convergence of the integrals:

(i) $\int_{-\infty}^\infty e^{-x}\,dx$ **(ii)** $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Sol. (i) $\int_{-\infty}^\infty e^{-x}\,dx = \int_{-\infty}^0 e^{-x}\,dx + \int_0^\infty e^{-x}\,dx$

$$= \lim_{t_1\to-\infty}\int_{t_1}^0 e^{-x}\,dx + \lim_{t_2\to\infty}\int_0^{t_2} e^{-x}\,dx$$

$$= \lim_{t_1\to-\infty}[-e^{-x}]_{t_1}^0 + \lim_{t_2\to-\infty}[-e^{-x}]_0^{t_2}$$

$$= \lim_{t_1 \to -\infty} (-1 + e^{-t_1}) + \lim_{t_2 \to -\infty} (-e^{-t_2} + 1)$$

$$= (-1 + \infty) + (0 + 1) = \infty$$

Therefore, $\int_{-\infty}^{\infty} e^{-x}\, dx$ is divergent to ∞.

(ii) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{1}{1+x^2}\, dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{1}{1+x^2}\, dx$$

$$= \lim_{t_1 \to -\infty} [\tan^{-1} x]_{t_1}^{0} + \lim_{t_2 \to -\infty} [\tan^{-1} x]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} [-\tan^{-1} t_1] + \lim_{t_2 \to \infty} [-\tan^{-1} t_2]$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \text{ which is finite.}$$

Therefore, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ is convergent and its value is π.

Example 6. Examine for convergence of the integrals:

(i) $\int_0^1 logx\, dx$ **(ii)** $\int_0^{1/e} \frac{1}{x(logx)^2}\, dx$

Sol. (i) 0 is only point of infinite discontinuity of the integrand on $[a, b]$.

Therefore, $\int_0^1 logx\, dx = \lim_{\epsilon \to 0+} \int_{0+\epsilon}^{1} (logx).1dx$

Integration by parts

$$= \lim_{\epsilon \to 0+} [x\, logx - x]_{\epsilon}^{1} = \lim_{\epsilon \to 0+} (-1 - \epsilon \log \epsilon + \epsilon)$$

$$= -1 \text{ which is finite.} \quad \left[since \lim_{\epsilon \to 0} x^n logx = 0, n > 0\right]$$

Therefore, $\int_0^1 logx\, dx$ is convergent and its value is -1.

(ii) since $\lim_{x \to 0} x(logx)^n = 0, n > 0$, therefore, 0 is the only point of infinite discontinuity of the integrand on $[0, \frac{1}{e}]$.

Therefore, $\int_0^{1/e} \frac{1}{x(logx)^2}\, dx = \lim_{\epsilon \to 0+} \int_{0+\epsilon}^{1/e} (logx)^{-2} \frac{1}{x}\, dx$

$$= \lim_{\epsilon \to 0+} \left[\frac{(logx)^{-1}}{-1}\right]_{\epsilon}^{1/e}$$

$$= \lim_{\epsilon \to 0+} \left[\frac{1}{\log\frac{1}{e}} - \frac{1}{\log \epsilon}\right] = -[-1-0] = 1$$

which is finite.

Therefore, $\int_0^{1/e} \frac{1}{x(logx)^2}\, dx$ is convergent and its value is 1.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Every proper integral is always convergent.
Problem 2. $\int_0^1 \frac{1}{x^2}\, dx$ is convergent.
Problem 3. $\int_0^1 \frac{1}{x^2}\, dx$ is divergent to $+\infty$.
Problem 4. $\int_0^e \frac{1}{x(logx)^3}\, dx$ is convergent to $-\frac{1}{2}$.
Problem 5. $\int_1^2 \frac{1}{x(logx)^3}\, dx$ is divergent to $+\infty$.

12.6 *SUMMARY*

1. In the definite integral $\int_a^b f(x)dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)dx$ is called an **improper integral of the first kind.**

2. In the definite integral $\int_a^b f(x)dx$, if both a and b are finite so that the interval of integration is finite but f has one or more point of infinite discontinuity i.e. f is not bounded on $[a, b]$, then $\int_a^b f(x)dx$ is called an **improper integral of the second kind.**

3. In the definite integral $\int_a^b f(x)dx$, if the interval of integration is unbounded and f is also unbounded, $\int_a^b f(x)dx$ is called an **improper integral of the third kind.**

4. (i) $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx, \ \ (t > a)$

The improper integral $\int_a^\infty f(x)dx$ is said to be **convergent** if the limit of right-hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

If the integral is neither convergent or divergent, then it is said to be **oscillating**.

(ii) $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx, \ \ (t < b)$

The improper integral $\int_{-\infty}^b f(x)dx$ is said to be **convergent** if the limit of right-hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

(iii) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ where c is any real number

$$= \lim_{t_1\to-\infty} \int_{t_1}^c f(x)dx + \lim_{t_2\to\infty} \int_c^{t_2} f(x)dx$$

The improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be **convergent** if both the limits on the right-hand side exist finitely and independent of each other, otherwise it is said to be divergent.

5. For any c between a and $b, i.e. a < c < b$, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If $\int_c^b f(x)dx$ is a proper integral, then the two integrals $\int_a^b f(x)dx$ and $\int_a^c f(x)dx$ converges or diverge together.

Thus, while testing the interval $\int_a^b f(x)dx$ convergence at a it may be replaced by $\int_a^c f(x)dx$ for any convenient c such that $a < c < b$.

12.7 GLOSSARY

sequence

series

12.8 *REFERENCES*

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12.9 *SUGGESTED READING*

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12.10 *TERMINAL AND MODEL QUESTIONS*

Q 1. Examine for convergence of the integral $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$.

Q 2. Examine for convergence of the integral $\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x+2}\, dx$.

Q 3. Examine for convergence of the integral $\int_{0}^{1} \frac{1}{x^2-3x+2}\, dx$.

Q 4. Examine for convergence of the integral $\int_{0}^{\pi} \frac{1}{sinx}\, dx$.

Q 5. Examine for convergence of the integral $\int_{0}^{\pi} \frac{1}{1+cosx}\, dx$.

12.11 *ANSWERS*

TQ1. Convergent to $\frac{\pi}{2}$.

TQ2. Convergent to π.

TQ3. Divergent to ∞.

TQ4. Divergent to ∞.

TQ5. Divergent to ∞.

CHECK YOUR PROGRESS

CYQ 1. True

CYQ 2. False

CYQ 3. True

CYQ 4. True

CYQ 5. True

UNIT 13: IMPROPER INTEGRAL II

Contents

13.1 INTRODUCTION

In mathematical analysis, an improper integral is an extension of the notion of a definite integral to cases that violate the usual assumptions for that kind of integral. In this unit we discussed tests for convergence of $\int_a^b f(x)dx$ at 'a', general test for convergence and absolute convergence of some functions, also tests for convergence of $\int_a^b f(x)dx$ at '∞'.

13.2 OBJECTIVES

In this Unit, we will Discussed about

- Improper integral
- Test of convergence
- Absolute integral

13.3 TEST FOR CONVERGENCE OF $\int_a^b f(x)dx$ AT a

Let a be the only point of infinite discontinuity of $f(x)$ on $[a,b]$. The case when b is the only point of infinite discontinuity can be dealt with in the same way.

Without any loss of generality, we assume that $f(x)$ is positive (or non-negative) on $[a,b]$.

In case $f(x)$ is negative, we can replace it by $(-f)$ for testing the convergence of $\int_a^b f(x)dx$.

Theorem: A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ at 'a' where f is positive on $(a, b]$, is that there exists a positive number M, independent of $\epsilon > 0$ such that

$$\int_{a+\epsilon}^{b} f(x)dx < M \ \forall \ \epsilon \text{ in } (0, b-a)$$

Proof: Since a is the only point of infinite discontinuity of f on $[a, b]$, therefore, f is continuous on $(a, b]$.

Also f is positive on $(a, b]$.

$\Rightarrow$ For $a < a+\epsilon < b$ i.e. for $0 < \epsilon < b-a$, f is positive and continuous on $[a+\epsilon,\ b]$.

$\Rightarrow \int_{a+\epsilon}^{b} f(x)dx = A(\epsilon)$ represents the area bounded by f on $[a+\epsilon,\ b]$ and x −axis.

$\Rightarrow$ As $\epsilon \longrightarrow 0+$, i.e. as ϵ decrease, $A(\epsilon)$ increases since the length of the interval increases.

$\Rightarrow \lim_{\epsilon \longrightarrow 0+} A(\epsilon) = \lim_{\epsilon \longrightarrow 0+} \int_{a+\epsilon}^{b} f(x)dx$ will exist finitely iff $A(\epsilon)$ is bounded above.

$\Rightarrow \int_a^b f(x)dx$ will converge iff $\exists$ a real number $M > 0$ and independent of ϵ such that $A(\epsilon) < M$

$\Rightarrow \int_a^b f(x)dx$ converges iff $\int_{a+\epsilon}^{b} f(x)dx < M \ \forall \ \epsilon$ in (0, b - a).

Note: If for every $M > 0$ and some ϵ in (0, b - a). $A(\epsilon) > M$, then $\int_{a+\epsilon}^{b} f(x)dx$ is not bounded above.

Therefore, $\int_{a+\epsilon}^{b} f(x)dx$ tend to $+\infty$ as ϵ tend to 0+ and hence, the improper integral $\int_a^b f(x)dx$ diverges to $+\infty$.

13.4 COMPARISION TEST I

If f and g are two positive functions with $f(x) \leq g(x)$ for all x in $(a, b]$ and a is only point of infinite discontinuity on $[a, b]$, then

(i) $\int_a^b g(x)dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

13.5 COMPARISION TEST II (LIMIT FORM)

If f and g are two positive functions on $(a, b]$, a being the only point of infinity discontinuity, and $\lim_{x \to a+} \frac{f(x)}{g(x)} = 1$ where l is non-zero finite number, then two $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ converge or diverge together.

Note: let f and g be two positive functions on $(a, b]$, a being the only point of infinite discontinuity. Then

(i) $\lim_{x \to a+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x) dx$ converges $\Rightarrow \int_a^b f(x) dx$ is converges

(ii) $\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x) dx$ diverges $\Rightarrow \int_a^b f(x) dx$ is diverges.

Note:

(i) The improper integral $\int_a^b \frac{1}{(x-a)^n} dx$ is convergent

if and only if $n < 1$.

(ii) The improper integral $\int_a^b \frac{1}{(b-x)^n} dx$ is convergent

if and only if $n < 1$.

Note:

(i) if a is the only point of infinite discontinuity of f on $[a, b]$ and $\lim_{x \to a+} (x - a)^\mu f(x)$ exists and non-zero finite, then $\int_a^b f(x) dx$ converges if and only if $\mu < 1$.

(ii) if b is the only point of infinite discontinuity of f on $[a, b]$ and $\lim_{x \to b+} (b - x)^\mu f(x)$ exists and non-zero finite, then $\int_a^b f(x) dx$ converges if and only if $\mu < 1$.

- **ILLUSTRATIVE EXAMPLES:**

Example 1: Examine the convergence of the integrals.

(i) $\int_0^1 \frac{1}{\sqrt{x^2+x}}\mathrm{d}x$ **(ii)** $\int_1^2 \frac{1}{(1+x)\sqrt{2-x}}\mathrm{d}x$

Sol. (i) Here $f(x) = \frac{1}{\sqrt{x^2+x}} = \frac{1}{\sqrt{x}\sqrt{x+x}}$

0 is the point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{\sqrt{x}}$, then $\lim\limits_{x\to 0+} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0+} \frac{1}{\sqrt{x+1}} = 1$ which is

non-zero and finite.

Therefore, By comparison test, $\int_0^1 f(x)\mathrm{d}x$ and $\int_0^1 g(x)\mathrm{d}x$ converge or diverge together.

But $\int_0^1 g(x)\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x$

(From, $\int_a^b \frac{1}{(x-a)^n}\mathrm{d}x$ with a = 0 converges.

since n = ½ < 1)

Therefore, $\int_0^1 f(x)\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x+1}}\mathrm{d}x$ is convergent.

(ii) Here $f(x) = \frac{1}{(1+x)\sqrt{2-x}}$

2 is the point of infinite discontinuity of f on $[1, 2]$.

Take $g(x) = \frac{1}{\sqrt{2-x}}$, then $\lim\limits_{x\to 2-} \frac{f(x)}{g(x)} = \lim\limits_{x\to 2-} \frac{1}{x+1} = \frac{1}{3}$ which is

non-zero and finite.

Therefore, by comparison test, $\int_1^2 f(x)\mathrm{d}x$ and $\int_1^2 g(x)\mathrm{d}x$ converge or diverge together.

But $\int_1^2 g(x)\mathrm{d}x = \int_1^2 \frac{1}{\sqrt{2-x}}\mathrm{d}x$

(From, $\int_a^b \frac{1}{(b-x)^n}\mathrm{d}x$ with b = 2 converges.)

since n = ½ < 1)

Therefore, $\int_1^2 f(x)\mathrm{d}x = \int_0^1 \frac{1}{(1+x)\sqrt{2-x}}\mathrm{d}x$ is convergent.

Example 2: Examine the convergence of the integral.

$$\int_0^1 \frac{1}{x^3(2+x^2)^5}\mathrm{d}x$$

Sol. Here $f(x) = \frac{1}{x^3(2+x^2)^5}$

0 is the point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{x^3}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{(2+x^2)^5} = \frac{1}{32}$ which is

non-zero and finite.

Therefore, by comparison test, $\int_0^1 f(x)\mathrm{d}x$ and $\int_0^1 g(x)\mathrm{d}x$ converge or diverge together.

But $\int_0^1 g(x)\mathrm{d}x = \int_0^1 \frac{1}{x^3}\mathrm{d}x$

(From, $\int_a^b \frac{1}{(x-a)^n}\mathrm{d}x$ with a = 0 diverges.

since n = 3 > 1)

Therefore, $\int_0^1 f(x)\mathrm{d}x = \int_0^1 \frac{1}{x^3(2+x^2)^5}\mathrm{d}x$ is divergent.

Example 3: Examine the convergence of the integral.

$$\int_0^1 \frac{\log x}{1+x}\mathrm{d}x$$

Sol. Since $\frac{\log x}{1+x}$ is negative on $(0, 1]$, we take $f(x) = -\frac{\log x}{1+x}$

0 is the point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{x^n}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^n \log x}{1+x} = 0$ if $n > 0$.

Taking n between o and 1, the integral $\int_0^1 g(x)\mathrm{d}x$ is convergent.

Therefore, by comparison test, $\int_0^1 f(x)\mathrm{d}x$ is convergent.

Example 4: Examine the convergence of the integral.

$$\int_0^{\pi/2} \frac{\sin x}{x^p}\mathrm{d}x$$

Sol. If p is negative or zero, the given integral is a proper integral and hence convergent when $p \le 0$.

When p > 0, the only point of discontinuity is 0.

Let $f(x) = \frac{\sin x}{x^p}$

Take $f(x) = \frac{1}{x^{\mu}}$ then

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} x^{\mu-p} sinx = \lim_{x\to 0+} x^{\mu-p+1}(\frac{sinx}{x})$$

$= 1$ if $\mu - p + 1 = 0$

$= 0$ if $\mu - p + 1 > 0$

$= \infty$ if $\mu - p + 1 < 0$

By taking $0 < \mu < 1$ and also $\mu = p - 1$ so that

$0 < p - 1 < 1$ i.e. $1 < p < 2$.

Therefore, $\int_0^{\pi/2} g(x)\mathrm{d}x$ is convergent and hence $\int_0^{\pi/2} f(x)\mathrm{d}x$ is convergent.

By taking $0 < \mu < 1$ and also $\mu > p - 1$ so that

$-1 < p - 1 < \mu < 1$ i.e. $0 < p < 2$.

Therefore, $\int_0^{\pi/2} g(x)\mathrm{d}x$ is convergent and hence $\int_0^{\pi/2} f(x)\mathrm{d}x$ is convergent.

Hence $\int_0^{\pi/2} \frac{\sin x}{x^p}\mathrm{d}x$ is convergent if $p < 2$ and divergent if $p \geq 2$.

Example 5: Show that $\int_0^{\pi/2} \frac{sin^m x}{x^n} dx$ exists if and only if $n < m + 1$.

Sol. Here $f(x) = \frac{sin^m x}{x^n} = \left(\frac{sinx}{x}\right)^n . \frac{1}{x^{n-m}}$

$$\lim_{x\to 0+} f(x) = \begin{cases} 0 \ if \ n - m < 0 \\ 1 \ if \ n - m = 0 \\ \infty \ if \ n - m > 0 \end{cases}$$

Therefore, the given integral is proper integral if $n - m \leq 0 \ i.e.$

$n \leq m$ and an improper integral if $n - m > 0$; 0 is only the point of infinite discontinuity of f on $[0, \frac{\pi}{2}]$.

When $n - m > 0$, take $g(x) = \frac{1}{x^{n-m}}$

Therefore, $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \left(\frac{sinx}{x}\right)^n = 1$ which is non zero and finite.

Also $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^{n-m}}\,dx$ is convergent iff $n-m<1$ i.e. $n<m+1$.

Therefore, by comparison test, the given interval is convergent iff $n-m<1$.

13.6 GENERAL TEST FOR CONVERGENCE

This test for convergence of an improper integral (finite limits of integration but discontinuous integrand) holds whether or not the integrand keeps the same sign.

- **Cauchy's test.** The improper integral $\int_a^b f(x)\,dx$, a being the only point of infinite discontinuity, converges at a if and only if to each $\epsilon>0$, there corresponds a $\delta>0$ such that $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)dx\right|<\epsilon$ for all $0<\lambda_1,\lambda_2<\delta$.

13.7 ABSOLUTE CONVERGENCE

Definition: The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

Theorem: Every absolutely convergent integral is convergent.

or $\int_a^b |f(x)|\,dx$ exits $\Longrightarrow \int_a^b f(x)\,dx$ exists.

Proof. Since $\int_a^b |f(x)|\,dx$ exists, therefore by Cauchy's test , for every $\varepsilon>0$, there corresponds a $\delta>0$ such that.

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} |f(x)|dx\right|<\varepsilon, \forall\ 0<\lambda_1,\lambda_2<\delta \quad \ldots..(1)$$

Also, we know that $$\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)dx\right| \le \left|\int_{a+\lambda_1}^{a+\lambda_2} |f(x)|dx\right| \quad \ldots\ldots(2)$$

From (1) and (2), we have $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)dx\right|<\varepsilon,, \forall 0<\lambda_1,\lambda_2<\delta$

$\therefore$ By Cauchy's test $\int_a^b f(x)\,dx$ is exists.

Note: Since $|f(x)|$ is always positive, the comparison tests can be applied for examining the convergence of $\int_a^b |f(x)|\,dx$, i.e., absolute convergence $\int_a^b f(x)\,dx$.

Note 2: The converse of the above theorem is not true. Every convergent integral is not absolutely convergent. A convergent integral which is not absolutely convergent is called a **conditionally Convergent Integral.**

Example 1. Test the convergence of $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$.

Sol. Let f(x) = $\frac{\sin\frac{1}{x}}{\sqrt{x}}$

Clearly, f does not keep the same sign in a neighborhood of 0.

Now $|f(x)| = \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| = \frac{\left|\sin\frac{1}{x}\right|}{|\sqrt{x}|} \leq \frac{1}{\sqrt{x}}, \forall\, x \in (0,1]$

But $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent at 0. (Since $n = \frac{1}{2} < 1$)

Therefore, by comparison test, $\int_0^1 |f(x)|dx$ is convergent at 0.

Since absolute convergence ⟹ convergence.

Therefore, $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

Example 2. Show that $\int_0^1 \frac{\sin\frac{1}{x}}{x^p}dx$, p > 0, converges absolutely for p < 1.

Sol. Let $f(x) = \frac{\sin\frac{1}{x}}{x^p}$ p > 0

Clearly, f does not keep the same sign in a neighborhood of 0.

Now, $|f(x)| = \left|\frac{\sin\frac{1}{x}}{x^p}\right| = \frac{\left|\sin\frac{1}{x}\right|}{|x^p|} \leq \frac{1}{x^p}, \forall\, x \in (0,1]$

Also $\int_0^1 \frac{1}{x^p}dx$ is convergent iff p < 1.

Therefore, by convergent test, $\int_0^1 |f(x)|dx$ converges if p < 1.

Hence $\int_0^1 f(x)dx$ converges absolutely for p < 1.

13.8 CONVERGENT AT ∞

Theorem: A necessary and sufficient condition for convergence of $\int_a^\infty f(x)dx$, where $f(x) > 0 \ \forall \ x \in [x,t]$, is that there exists a positive number M, independent of t, such $\int_a^t f(x)dx <$ M $\forall \ t \geq a$.

Proof. Let F(t) = $\int_a^t f(x)dx$

Since f is positive in $[a,t]$, the function F(t) monotonically increases with t and will therefore, tend to a finite limit if and only if it is bounded above, i.e. there exists a positive number M, independent of t, such that $F(t)$ < M $\forall \ t \geq a$

$\Longrightarrow \int_a^t f(x)dx < M \ \forall \ t \geq a$

Note: if no such number M exists, then the monotonic increasing function F(t) is unbounded above and therefore tends to ∞ as $t \longrightarrow \infty$.

Therefore, $\int_a^t f(x)dx$ diverges to ∞.

- **Comparison test I.**
  If f and g are two functions such that
  $0 < f(x) \leq g(x) \ \forall \ x \in [a,\infty)$, then
  (i) $\int_a^\infty g(x)dx$ is convergent $\Rightarrow \int_a^\infty f(x)dx$ is convergent
  (ii) $\int_a^\infty f(x)dx$ is divergent $\Rightarrow \int_a^\infty g(x)dx$ is divergent.

- **Comparison test II.**
  If f and g are two positive functions on $[a,\infty)$ and $\lim\limits_{x\longrightarrow\infty} \frac{f(x)}{g(x)} = l$
  then **(i)** if l is non-zero finite, the two integrals
  $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converges or diverges together.
  (ii) if $l = 0$ and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
  (iii) if $l = \infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

Note: A useful comparison integral.

The improper integral $\int_a^\infty \frac{1}{x^n}dx \ (a > 0)$ convergent if and only if $n > 1$.

Example 1. Examine the convergence of the following integral

(i) $\int_1^\infty \frac{x^3}{(1+x)^5} dx$ (ii) $\int_1^\infty \frac{1}{(2+x)\sqrt{x}} dx$

Sol. (i) Let f(x) $= \frac{x^3}{(1+x)^5} = \frac{x^3}{x^5\left(1+\frac{1}{x}\right)^5} = \frac{1}{x^2\left(1+\frac{1}{x}\right)^5}$

Take g(x) $= \frac{1}{x^2}$

Therefore, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)^5}$ =1 which is non-zero and finite.

By comparison test, the two integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together.

But $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2} dx$ is convergent (since n = 2 >1)

Therefore, $\int_1^\infty f(x)dx = \int_1^\infty \frac{x^3}{(1+x)^5} dx$ is convergent.

(ii) Let f(x) $= \frac{1}{(2+x)\sqrt{x}} = \frac{1}{x^{\frac{3}{2}}\left(1+\frac{2}{x}\right)}$

Take g(x) $= \frac{1}{x^{3/2}}$

Therefore, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{2}{x}}$ =1 which is non-zero and finite.

By comparison test, the two integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together.

But $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^{3/2}} dx$ is convergent (since n = $\frac{3}{2} > 1$)

Therefore, $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{(2+x)\sqrt{x}} dx$ is convergent.

Example 2. Examine the convergence of the following integral

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx \qquad m, n > 0$$

Sol. $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \int_0^a \frac{x^{2m}}{1+x^{2n}} dx + \int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$ where, $a > 0$

The first integral on the right is a proper integral and therefore, convergent. The given integral will be convergent or divergent according as $\int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$ is convergent or divergent.

Let $f(x) = \frac{x^{2m}}{1+x^{2n}} = \frac{x^{2m}}{x^{2n}\left(1+\frac{1}{x^{2n}}\right)} = \frac{1}{x^{2n-2m}\left(1+\frac{1}{x^{2n}}\right)}$

Take $g(x) = \frac{1}{x^{2n-2m}}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^{2n}}} = 1$ (Since n > 0)

Which is non-zero and finite.

Therefore, by comparison test, $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

But $\int_0^\infty g(x)dx = \int_0^\infty \frac{1}{x^{2n-2m}} dx$ converges if and only if $2n - 2m > 1$ i.e. $n - m > \frac{1}{2}$.

Therefore, $\int_a^\infty f(x)dx$ converges if and only if $n - m > \frac{1}{2}$. Hence the given integral converges if and only if $n - m > \frac{1}{2}$.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Every improper integral is always convergent.
Problem 2. $\int_a^\infty \frac{1}{x^n} dx$ (a > 0) is convergent if n > 1.
Problem 3. $\int_2^\infty \frac{1}{\sqrt{x^2-1}} dx$ is divergent.
Problem 4. $\int_1^\infty \frac{logx}{x^2} dx$ is convergent.
Problem 5. Every absolute convergent is convergent.

13.9 *SUMMARY*

1. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ at 'a' where f is positive on $(a, b]$, is that there exists a positive number M, independent of $\epsilon > 0$ such that

$$\int_{a+\epsilon}^b f(x)dx < M \ \forall \ \epsilon \text{ in } (0, b-a).$$

2. Comparison test: If f and g are two positive functions with

$f(x) \leq g(x)$ for all x in $(a,b]$ and a is only point of infinite discontinuity on $[a,b]$, then

(i) $\int_a^b g(x)dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

3. if a is the only point of infinite discontinuity of f on $[a,b]$ and $\lim_{x \to a+}(x-a)^{\mu} f(x)$ exists and non-zero finite, then $\int_a^b f(x)dx$ converges if and only if $\mu < 1$.

13.10 GLOSSARY

sequence

series

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13.13 *TERMINAL AND MODEL QUESTIONS*

Q 1. Examine for convergence of the integral $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$.

Q 2. Prove that every absolute convergent integral is convergent.

Q 3. Examine for convergence of the integral $\int_0^\infty \frac{x^3+1}{x^4}\,dx$.

Q 4. Examine for convergence of the integral $\int_0^\infty \frac{cosx}{1+x^2}\,dx$.

Q 5. Examine for convergence of the integral $\int_e^\infty \frac{1}{x(logx)^{n+1}}\,dx$.

13.14 *ANSWERS*

TQ1. Convergent if $0 < p < 1$ and divergent if $p \geq 1$.

TQ3. Convergent.

TQ4. Convergent.

TQ5. Divergent if $n \leq 0$, convergent if n < 0.

CHECK YOUR PROGRESS

CYQ 1. False

CYQ 2. True

CYQ 3. True

CYQ 4. True

CYQ 5. True

UNIT 14: DIRICHLET AND ABEL'S TEST FOR IMPROPER INTEGRALS

Contents

14.1 INTRODUCTION

In mathematics, Abel's test (also known as Abel's criterion) is a method of testing for the convergence of an infinite series. The test is named after mathematician Niels Henrik Abel, who proved it in 1826.[1] There are two slightly different versions of Abel's test – one is used with series of real numbers, and the other is used with power series in complex analysis. Abel's uniform convergence test is a criterion for the uniform convergence of a series of functions dependent on parameters.

In mathematics, there are several integrals known as the Dirichlet integral, after the German mathematician Peter Gustav Lejeune Dirichlet, one of which is the improper integral of the sine function over the positive real line.

14.2 OBJECTIVES

In this Unit, we will Discussed about

- Improper integral
- Abel's Test
- Dirichlet's Test

14.3 ABEL'S TEST

If $\int_a^\infty f(x)dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \geq a$, then $\int_a^\infty f(x)g(x)dx$ is convergent at ∞.

Or

An infinite integral which converges (not necessarily absolutely) will remain convergent after the insertion of a factor which is bounded and monotonic.

Proof: Since g is monotonic on $[a, \infty)$, it is integrable on $[a, t]$, for all

$t \geq a$.

Also, since f is integrable on $[a, t]$, we have by second mean value theorem.

$\int_{t_1}^{t_2} f(x)g(x)dx = g(t_1)\int_{t_1}^{p} f(x)dx = g(t_2)\int_{p}^{t_2} f(x)dx$ .... (1)

Where $a < t_1 \le p \le t_2$

Since g is bounded on $[a, \infty)$, there exists a positive number k such that

$$|g(x)| \le k \ \forall \ x \ge a$$

In particular $|g(t_1)| \le k$, $|g(t_2)| \le k$ .... (2)

Let $\in > 0$ be given,

Since $\int_a^{\infty} f(x)dx$ is convergent, there exists a number t_0 such that

$\left|\int_{t_1}^{t_2} f(x)dx\right| \le \frac{\epsilon}{2k} \ \forall \ t_1, t_2 \ge t_0$ .... (3)

Let the number t_1, t_2 in (1) be $\ge t_0$ so that the number p which lies between t_1 and t_2, is also $\ge t_0$. Hence from (3),

$\left|\int_{t_1}^{p} f(x)dx\right| \le \frac{\epsilon}{2k}, \left|\int_{p}^{t_2} f(x)dx\right| \le \frac{\epsilon}{2k}$ .... (4)

From (1), (2) and (4), it follows that a positive number t_0 exists such that for all $t_1, t_2 \ge t_0$.

$$\left|\int_{t_1}^{t_2} f(x)g(x)dx\right| = \left|g(t_1)\int_{t_1}^{p} f(x)dx + g(t_2)\int_{p}^{t_2} f(x)dx\right|$$

$$\le |g(t_1)|\left|\int_{t_1}^{p} f(x)dx\right| + |g(t_2)|\left|\int_{p}^{t_2} f(x)dx\right| < k.\frac{\epsilon}{2k} + k.\frac{\epsilon}{2k} = \epsilon$$

Hence, by Cauchy's test, $\int_a^{\infty} f(x)g(x)dx$ is convergent at ∞.

14.4 DRICHLET'S TEST

If $\int_a^t f(x)dx$ is bounded for all $t \ge a$ and $g(x)$ is a bounded and monotonic function for $x \ge a$, tending to 0 as $x \longrightarrow \infty$, then $\int_a^{\infty} f(x)g(x)dx$ is convergent at ∞.

Or

An infinite integral which oscillates finitely becomes convergent after the insertion of a monotonic factor which tends to zero as limit.

Proof. Since g is monotonic on $[a, \infty)$, it is integrable on $[a, t]$, for all

$t \geq a$.

Also, since f is integrable on $[a, t]$, we have by second mean values theorem,

$$\int_{t_1}^{t_2} \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x = g(t_1)\int_{t_1}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x = g(t_2)\int_{p}^{t_2} \mathrm{f}(x)\mathrm{d}x \quad \text{.......... (1)}$$

Where $a < t_1 \leq p \leq t_2$

Since $\int_{\mathrm{a}}^{\mathrm{t}} \mathrm{f}(x)\mathrm{d}x$ is bounded for all $t \geq a$, there exists a positive number k such that

$$\left|\int_{t_1}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x\right| \leq k \ \forall\, t \geq a \quad \text{......... (2)}$$

Now,
$$\left|\int_{t_1}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x\right| = \left|\int_{t_2}^{\mathrm{a}} \mathrm{f}(x)\mathrm{d}x + \int_{\mathrm{a}}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x\right|$$

$$= \left|\int_{\mathrm{a}}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x - \int_{\mathrm{a}}^{t_1} \mathrm{f}(x)\mathrm{d}x\right|$$

$$\leq \left|\int_{\mathrm{a}}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x\right| + \left|\int_{\mathrm{a}}^{t_1} \mathrm{f}(x)\mathrm{d}x\right|$$

$$\leq k + k = 2k \quad \forall\, t_1, p \geq a \quad \text{........ (3)}$$

Similarly,
$$\left|\int_{p}^{t_2} \mathrm{f}(x)\mathrm{d}x\right| \leq 2k \ \ \forall\, t_2, p \geq a \quad \text{........ (4)}$$

Let $\epsilon > 0$ be given

Since $\lim_{x \to \infty} g(x) = 0$, there exists a number t_0 such that

$|g(x)| < \frac{\epsilon}{4k} \forall\, x \geq t_0$

Let the number t_1, t_2 in (1) be $\geq t_0$, then

$|g(t_1)| < \frac{\epsilon}{4k}$ and $|g(t_2)| < \frac{\epsilon}{4k}$ .... (5)

From (1), (3), (4) and (5) it follows that a positive number t_0 exists such that for all t_1, $t_2 \geq t_0$

$$\left|\int_{t_1}^{t_2} \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x\right| = \left|g(t_1)\int_{t_1}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x + g(t_2)\int_{\mathrm{p}}^{t_2} \mathrm{f}(x)\mathrm{d}x\right|$$

$$\leq |g(t_1)|\left|\int_{t_1}^{\mathrm{p}} \mathrm{f}(x)\mathrm{d}x\right| + |g(t_2)|\left|\int_{p}^{t_2} \mathrm{f}(x)\mathrm{d}x\right| < \frac{\epsilon}{4k}.2k + \frac{\epsilon}{4k}.2k = \epsilon$$

Hence, by Cauchy's test, $\int_{\mathrm{a}}^{\infty} \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x$ is convergent at ∞.

Examples 1. Examine the convergence of the integrals:

(i) $\int_0^\infty \frac{sinx}{x} \mathrm{d}x$ **(ii)** $\int_0^\infty \frac{sinx}{\sqrt{x}} \mathrm{d}x$ **(iii)** $\int_0^\infty \frac{sinx}{x^{3/2}} \mathrm{d}x$

(iv) $\int_{\mathrm{a}}^\infty \frac{sinx}{x^m} \mathrm{d}x$ where a and m both are positive.

Sol. (i) Since $\lim_{x\to\infty} \frac{sinx}{x} = 1$, therefore 0 is not a point of infinite discontinuity.

Now, $\int_0^\infty \frac{sinx}{x} \mathrm{d}x = \int_0^1 \frac{sinx}{x} \mathrm{d}x + \int_1^\infty \frac{sinx}{x} \mathrm{d}x$

Also $\int_0^1 \frac{sinx}{x} \mathrm{d}x$ is a proper integral. Let us examine the convergence of $\int_1^\infty \frac{sinx}{x} \mathrm{d}x$ at ∞.

Let $f(x) = sinx$ and $g(x) = \frac{1}{x}$

Since $\left|\int_1^{\mathrm{t}} \mathrm{f}(x)\mathrm{d}x\right| = \left|\int_1^{\mathrm{t}} \sin x \, \mathrm{d}x\right|$

$$= |cos1 - cost| \leq |cos1| + |cost| \leq 2$$

Therefore, $\int_1^{\mathrm{t}} \mathrm{f}(x)\mathrm{d}x$ is bounded for all $t \geq 1$.

Also $g(x)$ is bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

By Dirichlet's test, $\int_1^\infty \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x = \int_1^\infty \frac{sinx}{x} \mathrm{d}x$ is convergent.

Hence, from (1), $\int_0^\infty \frac{sinx}{x} \mathrm{d}x$ is convergent.

(ii) Since $\lim_{x\to\infty} \frac{sinx}{\sqrt{x}} = \lim_{x\to\infty} \frac{sinx}{x}.\sqrt{x} = 1 \times 0 = 0.$

Therefore, 0 is not a point of infinite discontinuity.

Now $\int_0^\infty \frac{sinx}{\sqrt{x}} \mathrm{d}x = \int_0^1 \frac{sinx}{\sqrt{x}} \mathrm{d}x + \int_1^\infty \frac{sinx}{\sqrt{x}} \mathrm{d}x$ .... (1)

Also $\int_0^1 \frac{sinx}{\sqrt{x}} \mathrm{d}x$ is a proper integral. So, let us examine the convergence of $\int_1^\infty \frac{sinx}{\sqrt{x}} \mathrm{d}x$ at ∞.

Let $f(x) = sinx$ and $g(x) = \frac{1}{\sqrt{x}}$

Since $\left|\int_1^{\mathrm{t}} \mathrm{f}(x)\mathrm{d}x\right| \leq 2$ [see part (i)]

Therefore, $\int_1^{\mathrm{t}} \mathrm{f}(x)\mathrm{d}x$ is bounded for all $t \geq 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \longrightarrow \infty$.

By Dirichlet's test, $\int_1^\infty \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x = \int_1^\infty \frac{sinx}{\sqrt{x}}\mathrm{d}x$ is convergent.

Hence, from (1), $\int_0^\infty \frac{sinx}{\sqrt{x}}\mathrm{d}x$ is convergent.

(iii) $\int_0^\infty \frac{sinx}{x^{3/2}}\mathrm{d}x = \int_0^1 \frac{sinx}{x^{3/2}}\mathrm{d}x + \int_1^\infty \frac{sinx}{x^{3/2}}\mathrm{d}x$ **……… (1)**

For the integral $\int_0^1 \frac{sinx}{x^{3/2}}\mathrm{d}x$, 0 is a point of infinite discontinuity.

Let $f(x) = \frac{sinx}{x^{3/2}} = \frac{sinx}{x}.\frac{1}{\sqrt{x}}$

Take $g(x) = \frac{1}{\sqrt{x}}$

Therefore, $\lim_{x\longrightarrow 0+} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow 0+} \frac{sinx}{x} = 1$ which is non-zero and finite.

Since $\int_0^1 g(x)\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x$ is convergent.

∴ By comparison test, $\int_0^1 f(x)\mathrm{d}x = \int_0^1 \frac{sinx}{x^{3/2}}\mathrm{d}x$ is convergent.

Convergence of $\int_1^\infty \frac{sinx}{x^{3/2}}\mathrm{d}x$ at ∞.

Let $f(x) = sinx$ and $g(x) = \frac{1}{x^{3/2}}$

Since $\left|\int_1^t \mathrm{f}(x)\mathrm{d}x\right| \leq 2$ [see part (i)]

∴ $\int_1^t \mathrm{f}(x)\mathrm{d}x$ is bounded for all $t \geq 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \longrightarrow \infty$.

By Dirichlet's test, $\int_1^\infty \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x = \int_1^\infty \frac{sinx}{x^{3/2}}\mathrm{d}x$ is convergent.

Hence, from (1), $\int_0^\infty \frac{sinx}{x^{3/2}}\mathrm{d}x$ is convergent.

(iv) Let $f(x) = sinx$ and $g(x) = \frac{1}{x^m}$, m > 0

Since $\left|\int_a^t \mathrm{f}(x)\mathrm{d}x\right| = \left|\int_a^t \sin x \,\mathrm{d}x\right|$

$$= |\cos a - \cos t| \leq |\cos a| + |\cos t| \leq 2$$

$\therefore \int_a^t f(x)dx$ is bounded for all $t \geq 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \longrightarrow \infty$ for m > 0.

By Dirichlet's test, $\int_a^\infty f(x)g(x)dx = \int_a^\infty \frac{sinx}{x^m}dx$ where m and a are both positive, is convergent.

Examples 2. Examine the convergence of the integrals:

(i) $\int_0^\infty \sin x^2 \, dx$ **(ii)** $\int_0^\infty \frac{x}{1+x^2} \sin x \, dx$

Sol. (i) We have $\int_0^\infty \sin x^2 \, dx = \int_0^1 \sin x^2 \, dx + \int_1^\infty \sin x^2 \, dx$ …… (1)

But $\int_0^1 \sin x^2 \, dx$ is a proper integral and therefore convergent.

Convergence of $\int_1^\infty \sin x^2 \, dx$ at ∞.

$\int_1^\infty \sin x^2 \, dx = \int_1^\infty (2x \sin x^2) . \frac{1}{2x} dx$

Let $f(x) = 2x \sin x^2$ and $g(x) = \frac{1}{2x}$

Since, $\left|\int_1^t f(x)dx\right| = \left|\int_1^t 2x \sin x^2 \, dx\right| = |\{-\cos x^2\}_1^t|$

$$= |\cos a - \cos t^2| \leq |\cos a| + |\cos t^2| \leq 2$$

$\therefore \int_1^t f(x)dx$ is bounded for all $t \geq 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \longrightarrow \infty$.

By Dirichlet's test, $\int_1^\infty f(x)g(x)dx = \int_1^\infty \sin x^2 \, dx$ is convergent.

Hence, from (1) $\int_0^\infty \sin x^2 \, dx$ is convergent.

(ii) We have

$\int_0^\infty \frac{x}{1+x^2} \sin x \, dx$

$= \int_0^1 \frac{x}{1+x^2} \sin x \, dx + \int_1^\infty \frac{x}{1+x^2} \sin x \, dx$ …… (1)

But $\int_0^1 \frac{x}{1+x^2} \sin x \, dx$ is a proper integral and therefore convergent.

Convergence of $\int_1^\infty \frac{x}{1+x^2} \sin x \, dx \, \infty$.

Let $f(x) = sin\, x$ and $g(x) = \frac{x}{1+x^2}$

Since $\left|\int_1^t f(x) dx\right| \leq 2$

$\therefore$ $\int_1^t f(x) dx$ is bounded for all $t \geq 1$.

Also, $\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x}{1+x^2} = 0$

$g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \longrightarrow \infty$.

By Dirichlet's test, $\int_1^\infty f(x) g(x) dx$ = $\int_1^\infty \frac{x}{1+x^2} \sin x \, dx$ is convergent.

Hence, from (1) $\frac{x}{1+x^2} \sin x$ is convergent.

Examples 3. Examine the convergence of the integrals:

(i) $\int_0^\infty e^{-ax} \frac{sinx}{x} dx, a \geq 0$ **(ii)** $\int_a^\infty e^{-x} \frac{sinx}{x^2} dx, a > 0$

Sol. (i) Let $f(x) = \frac{sinx}{x}$ and $g(x) = e^{-ax}$, $a \geq 0$.

Since $\int_0^\infty f(x) dx$ is convergent and $g(x)$ is bounded and monotonically decreasing function of x for $x > 0$.

$\therefore$ By Abel's test, $\int_0^\infty f(x) g(x) dx = \int_0^\infty e^{-ax} \frac{sinx}{x} dx$ is convergent.

(ii) Let $f(x) = \frac{sinx}{x^2}$ and $g(x) = e^{-x}$

Since $|f(x)| = \left|\frac{\boldsymbol{sin\, x}}{x^2}\right| \leq \frac{1}{x^2}$ and $\int_a^\infty \frac{1}{x^2} dx$ is convergent.

Therefore, $\int_a^\infty f(x) dx$ is also convergent.

Again $g(x)$ is monotonic decreasing and bounded function for $x > a$.

Therefore, by Abel's test, $\int_a^\infty f(x)\, g(x) dx$ = $\int_a^\infty e^{-x} \frac{sinx}{x^2} dx, a > 0$ is convergent.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Improper integral is not convergent.

Problem 2. If $\int_a^\infty f(x)dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \geq a$, then $\int_a^\infty f(x)g(x)dx$ is convergent at ∞.

Problem 3. $\int_0^\infty \cos x^3 \ dx$ is divergent.

Problem 4. $\int_1^\infty \frac{logx}{x^2} \ dx$ is convergent.

Problem 5. Every absolute convergent need not be convergent.

14.5 *SUMMARY*

1. If $\int_a^\infty f(x)dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \geq a$, then $\int_a^\infty f(x)g(x)dx$ is convergent at ∞. (This is **Abel's Test**).

2. If $\int_a^t f(x)dx$ is bounded for all $t \geq a$ and $g(x)$ is a bounded and monotonic function for $x \geq a$, tending to 0 as $x \longrightarrow \infty$, then $\int_a^\infty f(x)g(x)dx$ is convergent at ∞. (This is **Dirichlet's Test**).

14.6 GLOSSARY

Proper integral

Improper integral

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14.9 *TERMINAL AND MODEL QUESTIONS*

Q 1. Test the convergence of the integral $\int_0^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$.

Q 2. Define Abel's Test with example.

Q 3. Examine for convergence of the integral $\int_e^\infty \frac{\log x \sin x}{x}\,dx$.

Q 4. Test the convergence of the integral $\int_a^\infty (1-e^{-x}).\frac{cosx}{x^2}\,dx\ , a > 0$.

Q 5. Define Dirichlet's Test with example.

14.10 *ANSWERS*

TQ1. Convergent.

TQ3. Convergent.

TQ4. Convergent.

CHECK YOUR PROGRESS

CYQ 1. False

CYQ 2. True

CYQ 3. False

CYQ 4. True

CYQ 5. False

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