BACHELOR OF SCIENCE/ BACHELOR OF ARTS
(ACCORDING TO NEP 2020)

MT(N)-120
THREE DIMENSIONAL GEOMETRY

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: THREE DIMENSIONAL GEOMETRY

COURSE CODE: MT(N)-120

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India,
263139

Editor

Prof. A. S. Uniyal,
Nodal Officer RUSA
Govt. of Uttarakhand

Dr. Kamlesh Bisht
Department of Mathematics
School of Sciences,
Uttarakhand OpenUniversity

Dr. Shivangi Upadhyay
Assistant Professor (AC)
Department of Mathematics
Uttarakhand Open University

Unit Writer	Block	Unit
Dr. Mahendra Singh Assistant Professor Department of Mathematics RH Govt. P.G College, Kashipur	I & III	1, 2, 3, 7 & 8
Dr. Narendra Kumar Associate Professor Department of Mathematics Govt. P.G College, Manila	II	4, 5 & 6
Dr. Dhirendra Singh Assistant Professor Department of Mathematics PNG Govt. P.G College, Ramnagar	IV	9, 10 &11
Dr. Shivangi Upadhyay Department of Mathematics School of Sciences, Uttarakhand Open University	V	12, 13 & 14

Course Title and Code : Three Dimensional Geometry (MT(N)-120)
Copyright : Uttarakhand Open University
Edition : 2023
Published By : Uttarakhand Open University, Haldwani, Nainital- 263139
Printed By : Premier Prinitng Press Qty : 47

CONTENTS

THREE DIMENSIONAL GEOMETRY
MT(N)-120

COURSE INFORMATION

The present self learning material **"Three Dimensional Geometry"** has been designed for BA/BSc (First Semester) learners of Uttarakhand Open University, Haldwani. This self-study material was created to increase learners access to excellent learning materials. There are 14 units in this course. System of coordinate in 3D is the focus of the first and second units. Projection, Direction ratio's and Direction Cosine's are covered in Unit 3. Plane and straight line are the main topics of Unit 4, 5 and 6. Units 7 and 8 each provided an explanation of of tetrahedron and change of axis and Sphere. Cone and the Cylinder and Paraboloids is the topic of Units 9 and10. The concepts of the Central conicoids is presented in Units 11.

Discussion of tracing of Conics and Polar equation of a conic in the units 12, 13 and 14. Simple, succinct, and clear explanations of the fundamental ideas and theories have been provided. The right amount of relevant examples and exercises have also been added to help learners to understand the material.

BLOCK I

SYSTEM OF COORDINATES AND PROJECTIONS

UNIT 1: SYSTEM OF COORDINATES IN THREE DIMENSIONS-I

CONTENTS

1.1 INTRODUCTION

In the previous classes, you should have learnt and studied by now that there exits two types of objects-

1. Which has only length and breadth e.g. rectangle, square etc. These types of objects are called $2D$ objects i.e. study of plane analytical geometry. In general 2-D geometry is the study of two independents variable.
2. Which has length , breadth and height e.g. cube, cuboid, cone, cylinder etc. These types of objects are called 3-D objects i.e. study of solid geometry. In general 3-D is the study of three independents variable.

Dimension is a property of shapes which tells whether the shape has the height (or depth) or not. A three dimensional shape also called as a solid shape is a shape which has three measurements length, breadth and height. They look different by observing them from different places and different angles. But whereas $2D$- shapes have only two measurements length and breadth.

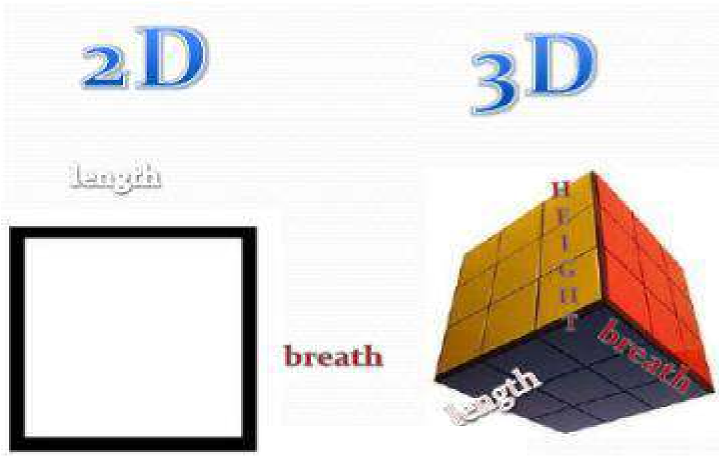


Fig 1.1.1

We have different three dimensional shapes. Here we have to study about the 3D coordinates of a point, which do not lie in a plane, but needs space.

A polyhedron is a $3D$-shape whose faces are polygons where as a non-polyhedron contains no polygon shaped faces. Examples of polyhedrons can be cubes, cuboids, pyramids and prisms etc.

Polygonal faces are the regions by which the polyhedron is bounded. For example, the polygonal faces of the cube are the squares.

An edge of a polyhedron is a line segment where any two of its faces meet.

<u>REMARK</u>: $F + V = E + 2$ is a formula called Euler's formula where F, V and E represents the number of faces, vertices and edges respectively.

<u>For example in a cube:</u>

In a cube $HGDABEFC$, if $F = 6, V = 8, E = 12$

Then, $F + V = E + 2$, i.e. 8+6 = 12+2,

It can verify this by looking your room 6 faces, 8 vertices and 12 edges.

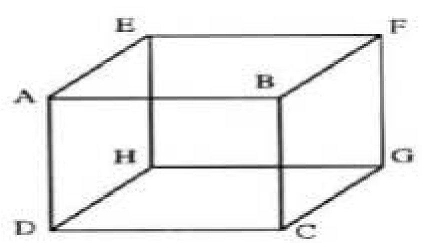


Fig 1.1.2

Analytical geometry is that branch of mathematics which treats geometry algebraically.

A regular polyhedron is a polyhedron whose faces are congruent to each other. Otherwise it is said to be irregular polyhedron. An example for a regular polyhedron is a cube because it has all congruent faces (squares).

A convex polyhedron is a polyhedron where the line segment joining any two points on it lies completely in it.

A concave polyhedron is a polyhedron in which the plane sections are concave.

A prism is a polyhedron where two of its bases are parallel and the side faces are parallelograms.

A pyramid is a polyhedron where the base is a polygon and the side faces are triangles with a common vertex.

$2D$ space:

Two-dimensional space is represented with the $X$and Y-axes. $3D$ animation adds depth, or the Z-axis. This is seen in figures given below. If two straight lines intersect each other at right angle at a point O then lines are said to be axes and intersection point is called origin. The plane which contains these two lines is called co-ordinate plane in $2D$ space.

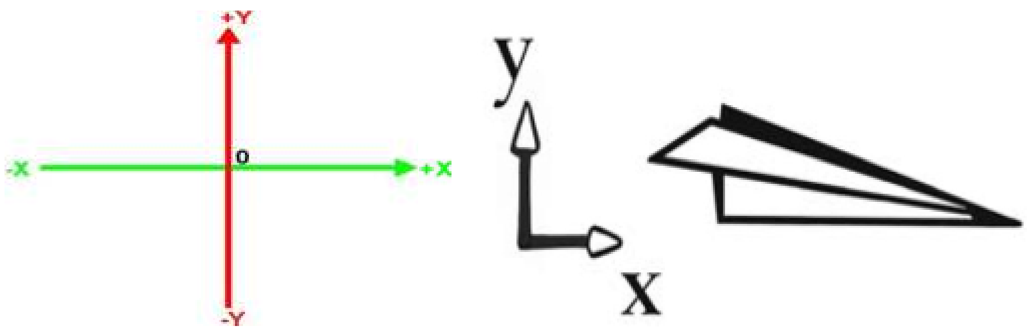


Fig. 1.1.3

3D space:

This three axes reference system is known as the Cartesian coordinate system. It is used to define many properties of a $3D$ object, including its position, rotation, and scale. For example, an object's position can be represented as 12, 0, 10 — meaning that it is 12 units to the right of center on the X-axis, 0 units from center along the Y-axis, and 10 units along the Z-axis

In the coordinate geometry of two dimensions or in plane analytic geometry, the position of a point in a plane is referred to two intersecting lines (in the same plane as that of the point) called the axes of reference and their point of intersection is called the origin of co-ordinates.

If three straight lines intersect each other at right angle at a point O then three lines are said to be co-ordinate axes and point of intersection of three lines is called origin. Also plane containing any two lines are called co-ordinates planes.

Fig 1.1.4

If the axes are at right angles, then these are called rectangular axes otherwise they are called oblique axes. In both the cases (rectangular axes or oblique axes) they divide the plane into four quadrants called the first, second, third and fourth respectively. But it is not always possible to determine the positions of all the points with respect to the above coordinate system for example any five vertices of a cube because these points do not lie in the same plane. Such points are called points in space. We can demonstrate a point in space as follows Consider your classroom and let it is a rectangular parallelepiped in shape. Now the position of any particular point in air on the ceiling fan hanging from the roof of the classroom represents a point in space. The geometry in which we study such points in space is called as –“Analytical geometry” or “Solid geometry” or “Volumetric geometry” or “$3D$ geometry”.

Hence $3D$ geometry is an extension of $2D$ geometry with one extra dimension (in coordinate denoted by z). If we take the value of the third dimension z to be zero, then the $3D$ system reduces to $2D$ system.

1.2 OBJECTIVES

After studying this unit, you should be able to :

- Understand the difference between the 3D coordinate and the 2D coordinate systems.
- Understand if the given point lies either in a space or in a plane.
- Understand the different forms of coordinates of a point in space.
- Understand the relationship between the different forms of coordinates of a point in space.
- Check in which octant the given point lies.
- Find the ordered and equality properties of the coordinates of two points.
- Find the coordinates of a point when origin is shifted to another point and vice-versa.

- Find the equation of the same surface in two different forms namely – Cartesian form and spherical polar form.
- Know that which three dimensional objects are generally exist around us.
- Differentiate between the rectangular and oblique coordinate systems.

1.3 DEFINITION OF ORIGIN, COORDINATE AXES AND COORDINATE PLANES

A 2D Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numericalcoordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a coordinate axis or just axis (plural axes) of the system and the point where they meet is its origin, at ordered pair (0,0). The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.

One can use the same principle to specify the position of any point in three-dimensional space by three Cartesian coordinates, its signed distances to three mutually perpendicular planes (or, equivalently, by its perpendicular projection onto three mutually perpendicular lines). In general, n Cartesian coordinates (an element of real n-space) specify the point in an n-dimensional Euclidean space for any dimension n. These coordinates are equal, up to sign, to distances from the point to n mutually perpendicular hyperplanes.Cartesian coordina1te system with a circle of radius 2 centered at the origin marked above. The equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$ where a and b are the coordinates of the center (a, b) and r is the radius.

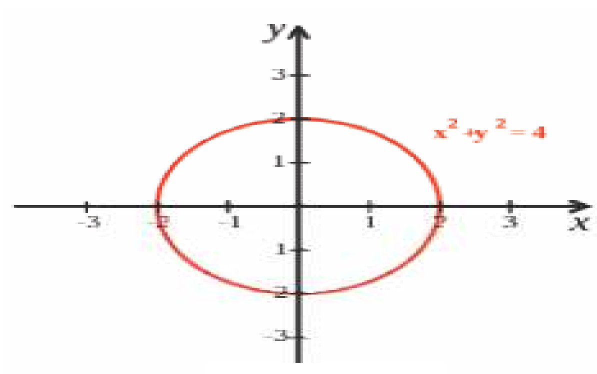


Fig. 1.3.1

The invention of Cartesian coordinates in the 17th century by René Descartes (Latinized name: Cartesius) revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra. Using the Cartesian coordinate system, geometric shapes (such as curves) can be described by Cartesian equations: algebraic equations involving the coordinates of the points lying on the shape. For example, a circle of radius 2, centered at the origin of the plane, may be described as the set of all points whose coordinates x and y satisfy the equation $x^2 + y^2 = 4$.

Cartesian coordinates are the foundation of analytic geometry, and provide enlightening geometric interpretations for many other branches of mathematics, such as linear algebra, complex analysis, differential geometry, multivariate calculus, group theory and more. A familiar example is the concept of the graph of a function. Cartesian coordinates are also essential tools for most applied disciplines that deal with geometry, including astronomy, physics, engineering and many more. They are the most common coordinate system used in computer graphics, computer-aided geometric design and other geometry-related data processing.

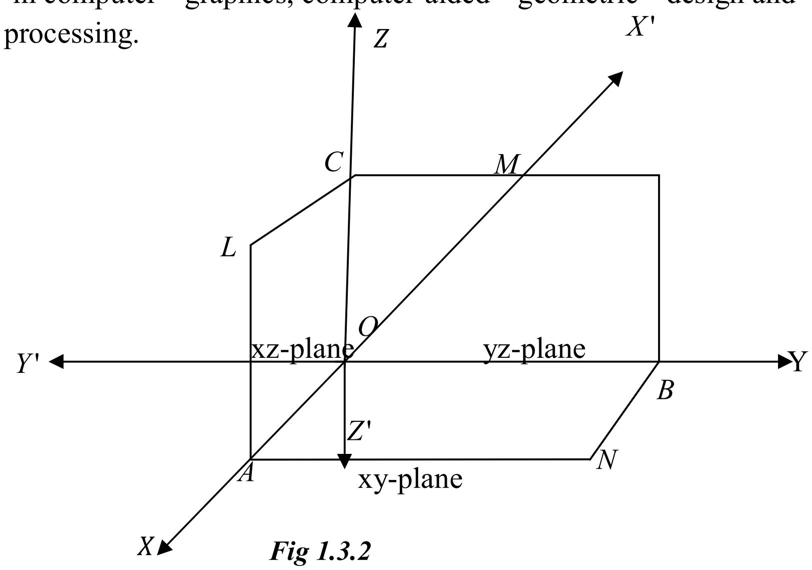


Fig 1.3.2

Let O be the point of intersection of two mutually perpendicular straight lines $X'OX$ and $Y'OY$ drawn in the plane of paper. Now consider a third straight line $Z'OZ$ passing through O and perpendicular to the plane of paper. Hence OZ is perpendicular to both OX and OY. Here the direction of OZ is taken such that OX, OY and OZ form a right handed system that is (i.e.) OZ points in the direction in which a right handed screw will translate if rotated from OX to OY. Such a system of three mutually perpendicular lines namely $X'OX$, $Y'OY$ and $Z'OZ$ is called a right handed system of three dimensional rectangular coordinate axes.

<u>**ORIGIN**</u>: It is the intersection point of axes and planes. Equation of origin in 2-D is

$$X = 0, \qquad Y = 0$$

Equation of origin in $3-D$

$$X=0, \qquad Y=0, \qquad Z=0$$

Everything starts from origin, for example age of a person starts from birth. In other way, the point O, which is the point of intersection of three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ is called the origin of the $3D$ coordinate system.

COORDINATE AXES:

The three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ are called the coordinate axes namely x-axis, y-axis and z-axis respectively. OX, OY and OZ are taken to be positive directions whereas OX', OY' and OZ' are taken to be negative directions of x-axis, y-axis and z-axis respectively.

Note: East and North directions are source of positive energy therefore if we travel towards East and North from origin are said to be positive direction and West and South directions are said to be negative directions.

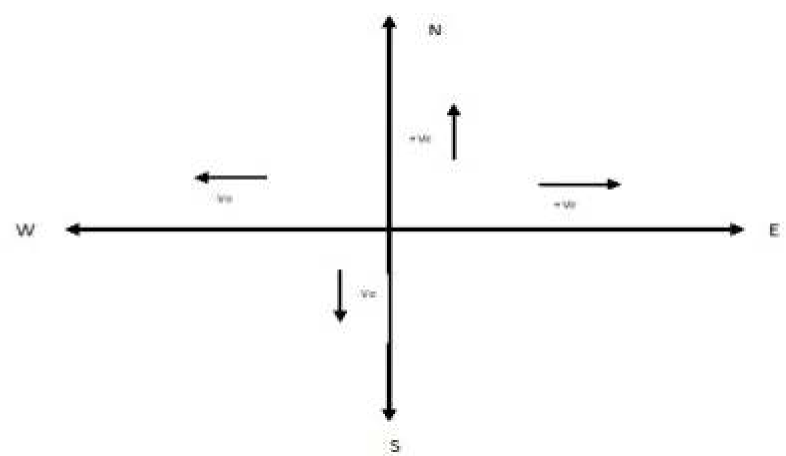


Fig 1.3.3

Similarly in 3-D from earth upper direction is positive and lower directions are said to be negative directions. i.e. in 3-D

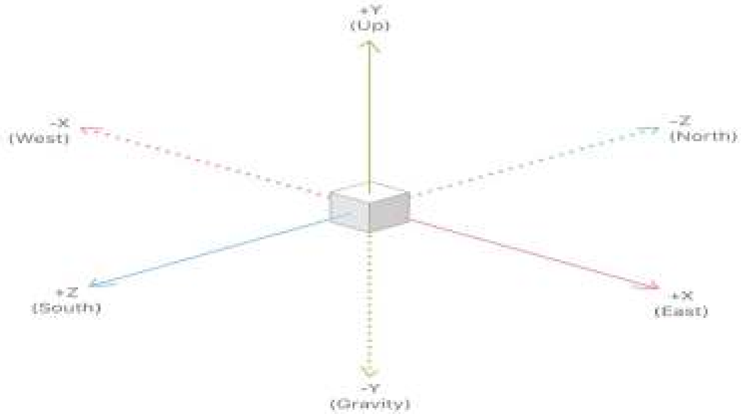


Fig 1.3.4

Basic Concept: Sun rise from East

Sun set in West

Rivers origin North

Rivers ending South

Things increases upwards

Things decreases downwards

Equation of axis in 2-D: For x-axis; $y = 0$

For y-axis; $x = 0$

Equation of axis in 3-D: For x-axis; $y = 0 = z$

For y-axis; $x = 0 = z$

For z-axis; $x = 0 = y$

COORDINATE PLANES

If any two of the three coordinate axes $X'OX$, $Y'OY$ and $Z'OZ$ are taken in pair, we get three planes XOY, YOZ and ZOX namely xy-plane, yz-plane and zx-plane respectively. These planes are called coordinate planes. Co-ordinate planes are planes containing the Axes. Therefore there are three co-ordinate planes. In XY plan i.e. plane containing x and y-axis and its equation is $z = 0$. Similarly, for YZ plane $x = 0$ and XZ plane $y = 0$.

1.4 PLANES PARALLEL TO THE COORDINATE PLANES

(i) The equation of the plan x = 0 represents the yz-plane and equation of the plane x = a represents the plane parallel to yz-plane at a distance a unit right/left to the yz-plane, according as a is positive or negative. Now, we draw a plane parallel to yz- plane at a distance a unit right/left to the yz-plane.

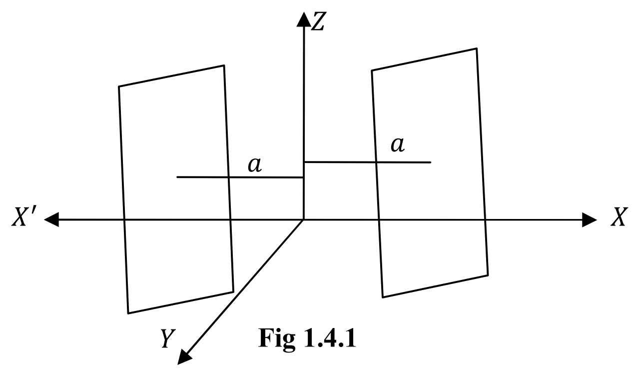


Fig 1.4.1

Parallel to XY plane: It is a plane at a height a from xy-plane and its is given by z = a and co-ordinate on this plane will be $(\propto, \beta, a)$, where $\propto, \beta$are variables and a is constant.

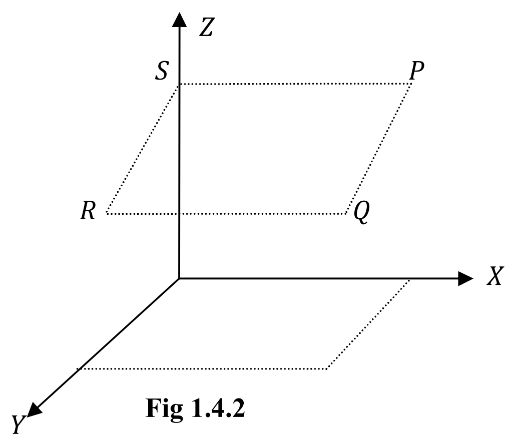


Fig 1.4.2

(ii) The equation of the plane y = 0 represents the xz plane and the equation of the plane y = b represents the plane parallel to xz plane at a distance b unit above/below to the xz plane , according as b is positive or negative.

Parallel to YZ plane: It is a plane at a distance b in x-axis parallel to YZ plane and its equation is x=b. Co-ordinates on this plane will be $(b, \beta, \gamma)$where β an γvariables and b is constant.

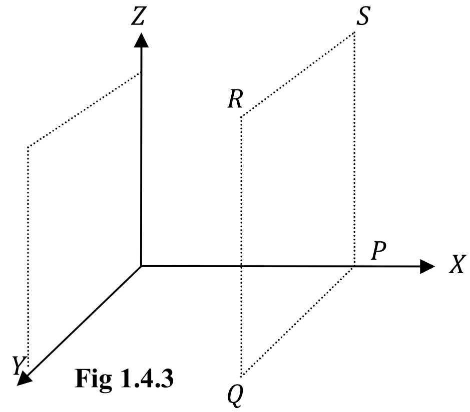


Fig 1.4.3

(iii) The equation of the plane z = 0 represents the xy-plane and z = c represents the plane parallel to xy-plane at a distance c unit above/below to the xy-plane , according as c is positive or negative.

Parallel to YZ plane: It is a plane at a distance c in y-axis parallel to XZ plane and its equation is $y = c$. Co-ordinates on this plane will be (α, c, γ)whereα and γvariables and c is constant.

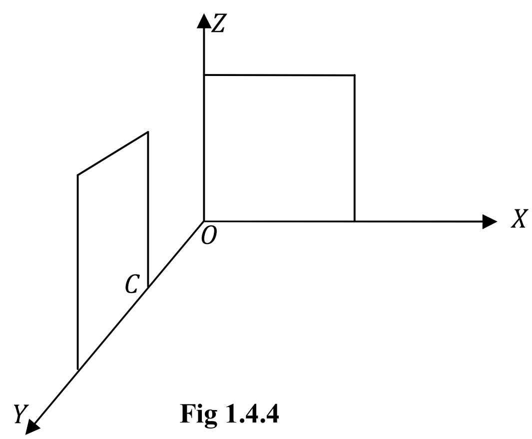


Fig 1.4.4

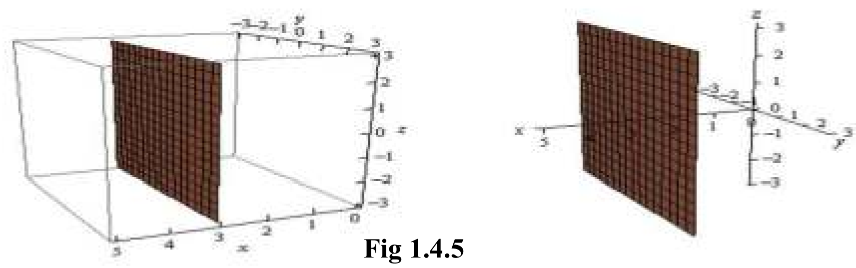


Fig 1.4.5

Finally, here is the graph of $x = 3$ in R^3. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system that we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.

1.5 COORDINATES OF A POINT IN SPACE

Let P be any arbitrary point in the space. Through P, we draw planes parallel to the coordinate planes. Clearly these planes are also perpendicular to the coordinate axes. Let these planes cut the coordinate axes in A, B and C respectively. If OA = x, OB = y and OC = z then numbers x, y and z are determined by the point P, when taken with proper sign are called the coordinates of the point P.

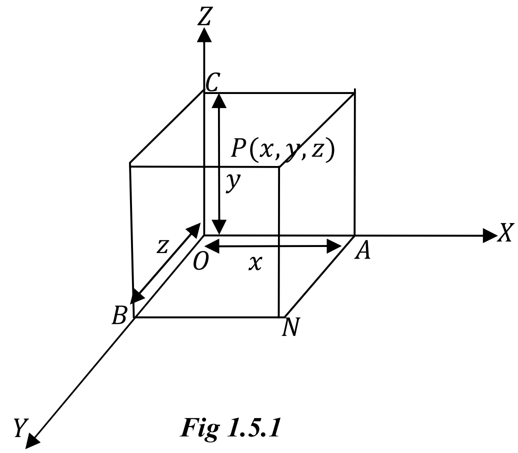


Fig 1.5.1

In other words we can defined as draw perpendicular PN through P on xy-plane. Through N draw a line NA parallel to OY intersecting the line OX (x-axis) at point A. Then if OA =x, AN = y and PN = z, then (x, y, z) are the coordinates of point P.

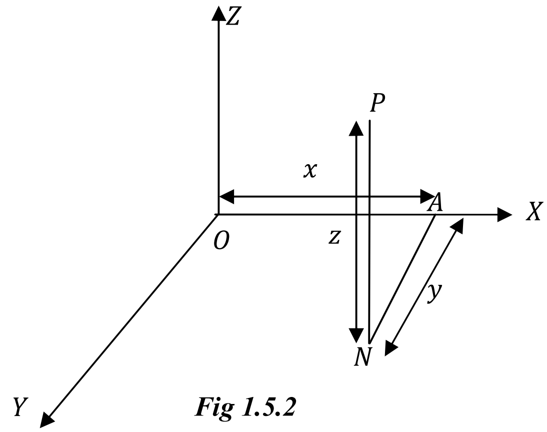


Fig 1.5.2

Here it is remarkable that a point P in the space has one and only one set of coordinates referred to one set of rectangular coordinate axes for example the coordinates of origin are given by (0, 0, 0).

Thus the perpendicular distances of a point P with proper signs from the three coordinate planes respectively are the coordinates of the point P.

The general point $P\,(a, b, c)$ is shown on the 3D graph below. The point N is directly below P on the x-y plane.

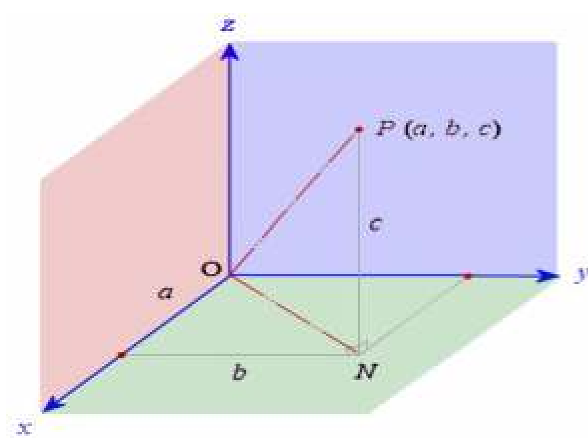


Fig 1.5.3

x co-ordinate of a point in space: It is the length of perpendicular from the point to the YZ plane.

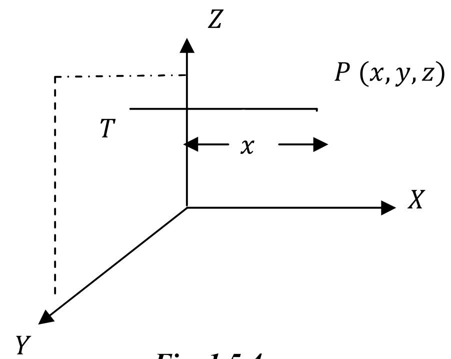


Fig. 1.5.4

y co-ordinate of a point in space: It is the length of perpendicular from the point to XZ plane.

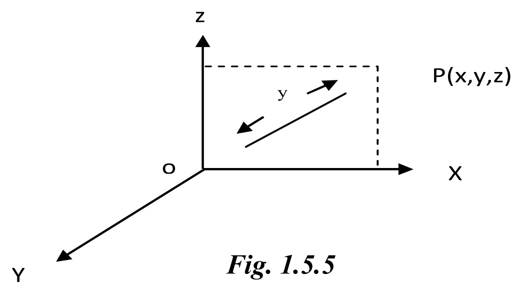


Fig. 1.5.5

z co-ordinate of a point in space: It is the length of perpendicular from the point to XY plane.

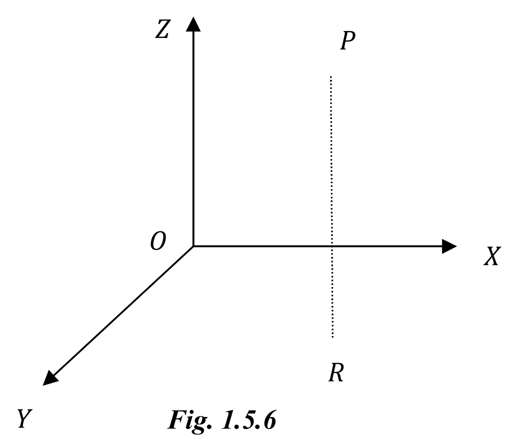


Fig. 1.5.6

Therefore a point $P(x, y, z)$ in $3D$ can be defined that point p is at a perpendicular distance x from YZ plane, y from XZ plane and z from XY plane.

Simply first move x on X −axis then y parallel to y axis and then z parallel to z axis we will reach to point $P(x, y, z)$.

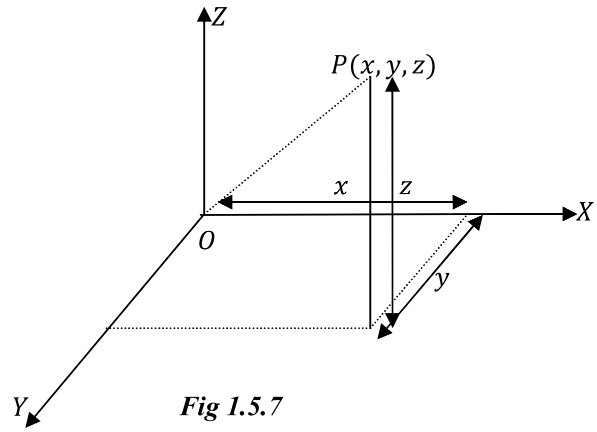


Fig 1.5.7

A three-dimensional Cartesian coordinate system is formed by a point called the origin (denoted by O) and a basis consisting of three mutually perpendicular vectors. These vectors define the three coordinate axes: the $\vec{x}, \vec{y}$ and $\vec{z}$ axis. They are also known as the abscissa, ordinateand applicate axes respectively. The coordinates of any point in space are determined by three real numbers: $x,\ y,\ z$.

REMARKS

- If a point lies on the x-axis then its y and z coordinates will always be zero and vice-versa.
- If a point lies on the y-axis then its x and z coordinates will always be zero and vice-versa.
- If a point lies on the z-axis then its y and x coordinates will always be zero and vice-versa.
- If a point lies on the xy-plane then its z coordinate will always be zero and vice-versa.
- If a point lies on the yz-plane then its x coordinate will always be zero and vice-versa.
- If a point lies on the xz-plane then its y coordinate will always be zero and vice-versa.
- Origin is the only point in the space whose all coordinates are zero.
- In case of rectangular coordinate system the three planes xy-plane, yz-plane and zx-plane are mutually orthogonal.
- The intersection of two coordinate planes gives a coordinate axis (i.e. a straight line) e.g. the intersection of two coordinate planes namely xy-plane and yz-plane give y- coordinate axis. Hence the equation of y-axis will be, x = 0 = z. Similarly the equations of the x-axis and z-axis are given by, y= 0 = z and x = 0 = y respectively.

1.6 POSITION VECTOR OF A POINT RELATED TO ITS COORDINATES

Let $\vec{r}$ be the position vector of the point P in the space with coordinates (x, y, z), that is $\overrightarrow{OP} = \vec{r}$. Let $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ be the unit vectors in the direction of OX, OY and OZ respectively. Now draw a perpendicular PN from the point P on xy-plane such that $PN = z$ with proper sign and $\overrightarrow{NP} = z\hat{k}$. Now through the point N draw two lines NA and NB parallel to y-axis and x-axis respectively.

If with proper signs we have, $OA = x$ and $OB = y$ then, $\overrightarrow{OA} = x\,\hat{\imath}$ and $\overrightarrow{OB} = y\,\hat{j}$.

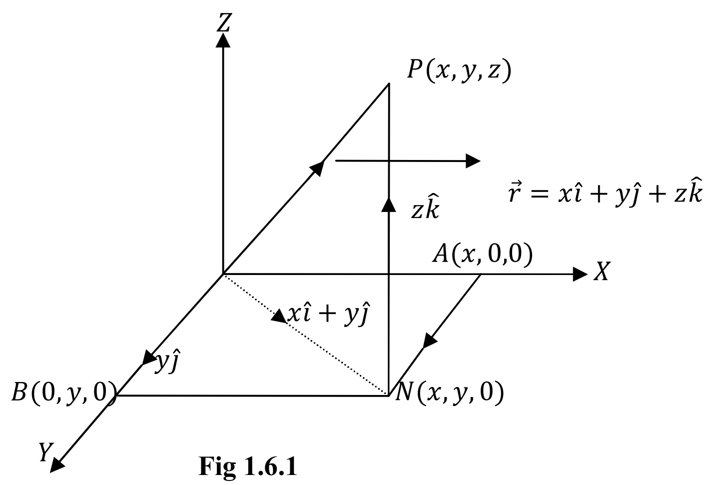


Fig 1.6.1

Now using Vector Analysis we have,

$\overrightarrow{OP} = \vec{r} = \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OA} + \overrightarrow{AN} + \overrightarrow{NP}$ [using addition law of vectors]

Hence $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

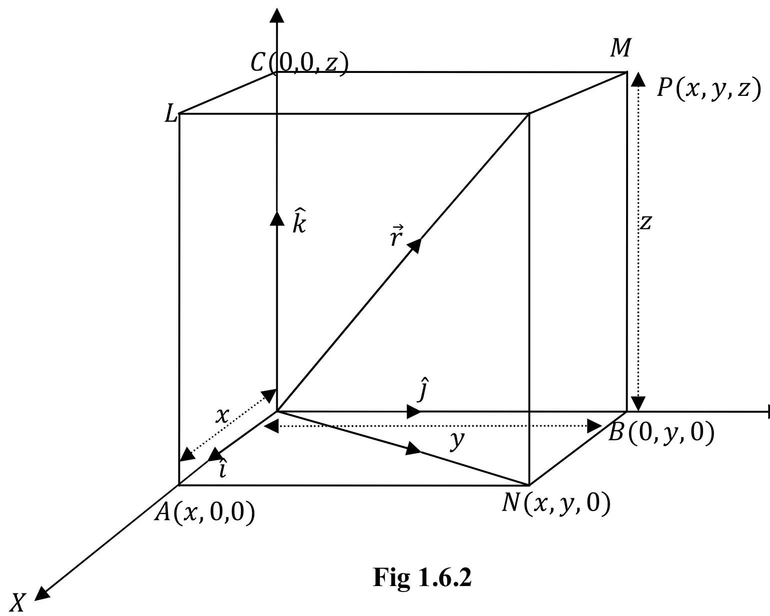


Fig 1.6.2

Hence it is concluded that if the coordinates of the point P are (x, y, z) ,then the position vector $\vec{r}$ of point P is given by, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

1.7 OCTANTS

We have learnt by now that there exists the three coordinate planes namely xy-plane, yz-plane and zx-plane. These three coordinate planes divide the space into eight parts called octants. The point P belongs to which octant is determined by the signs of the coordinates of the point P. The octant OXYZ in which all the coordinates of a point are positive is called the first octant or positive octant

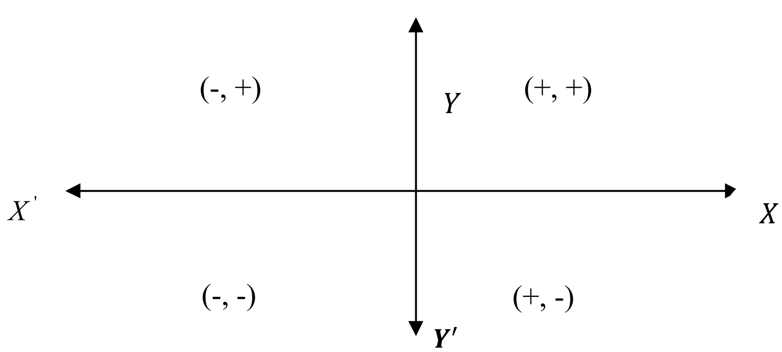


Fig. 1.7.1

In 3-D, upper side Z will be positive and lower side Z will be negative. Thus in upper octant will be (+,+,+), (-,+,+), (-,-,+) and (+,-,+)

Similarly, lower octant will be (+,+,-), (-,+,-), (-,-,-) and (+,-,-). On the basis of this rule we can form the sign convention table.

SIGN CONVENTION The signs of the coordinate of a point P in various octants can be determined with the help of the following table —

Octant	OXYZ	OX'YZ	OXY'Z	OXYZ'	O$X'Y'$Z	OX$Y'Z'$	OX'YZ'	O$X'Y'Z'$
x	+	-	+	+	-	+	-	-
y	+	+	-	+	-	-	+	-
z	+	+	+	-	+	-	-	-

Note: These octant can be seen directly by

$$\{+,-\}\times\{+,-\}\times\{+,-\} = \{(+,+),(+,-),(-,+),(-,-)\}\times\{+,-\}$$
$$= \{(+,+,+),(+,-,+),(-,+,+),(-,-,+),(+,+,-),(+,-,-),(-,+,-),(-,-,-)\}$$

SOLVED EXAMPLES

Ex. 1. Find the position of the following points in space.

1. (2,-3, 4)
2. (1, 2, -1)
3. (0, 0, -5)
4. (3, 0 , 0)
5. (2, 5, 1)
6. (-4, 6, 0)

Sol-Here we use the concept of sign convention and coordinates of a point in space.

1. Clearly (2, -3, 4) is a point in the octant OXY'Z and its distances from the coordinate planes yz, zx and xy are 2, 3 and 4 respectively.
2. (1, 2, -1) is a point in the octant OXYZ' and its distances from the coordinate planes yz, zx and xy are 1, 2 and 1 respectively.
3. (0, 0, -5) is a point on OZ' that is on the negative side of z-axis and its distance from the origin is 5 .
4. (3, 0, 0) is a point on the positive side of x-axis that is on OX and its distance from origin is 3.
5. Clearly (2, 5, 1) is a point in the octant OXYZ that is in the first or positive octant and its distances from the coordinate planes yz, zx and xy are 2, 5 and 1 respectively.

6. Clearly (-4, 6, 0) is a point in the octant OX'YZ and its distances from the coordinate planes yz, and zx are 4 and 6 respectively. This point lies in the xy-plane because its z coordinate is zero.

Ex.2. Find the locus of the point,

(i) whose y coordinate is 7.
(ii) whose x coordinate is 4 and y coordinate is 5.
(iii) whose z coordinate is 3.

Sol- (i)The required locus of the point is y = 7 which represents a plane parallel to ZOX plane or zx-plane. The distance of this plane from zx-plane is 7.

(ii) The required locus is x = 4 and y = 5 which represent a line parallel to z axis.
(iii) The required locus of the point is z = 3 which represents a plane parallel to XOY plane or xy-plane. The distance of this plane from xy-plane is 3.

Ex. 3. State the common property of the coordinates of points lying on (i) x-axis (ii) y-axis (iii)z-axis (iv) xy-plane (v) yz-plane (vi) xz-plane.

Sol.(i) The y and z coordinates of all the points lying on x-axis are zero that is y = 0 and z = 0.

(ii) The x and z coordinates of all the points lying on y-axis are zero that is x = 0 and z = 0.

(iii) The x and y coordinates of all the points lying on z-axis are zero that is x = 0 and y = 0.

(iv) The z coordinate of all the points lying on xy-plane is zero that is z = 0. Hence the required common property is z = 0.

(v) The x coordinate of all the points lying on yz-plane is zero that is x = 0. Hence the required common property is x = 0.

(vi) The y coordinate of all the points lying on zx-plane is zero that is y = 0. Hence the required common property is y = 0.

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ -1) The point (8, -5, -2) lies in the octant

(a) OXY'Z (b) O$X'Y'$Z (c)OX$Y'Z'$ (d) OX'YZ'

(SCQ -2) The number of octants in 3D coordinate system is

(a) 8 (b) 3 (c) 4 (d) Infinite

(SCQ -3) The angle between the axes in 3D rectangular Cartesian system is ,

(a) 30° (b) 90° (c) 60° (d) 45°

(SCQ -4) The coordinates of the origin in 3D rectangular Cartesian system generally are,

(a) (–1,-1,-1) (b) (1 ,1, 1) (c) (1,-1 ,1) (d) (0, 0,0)

(SCQ - 5) For any point lying on the y-axis, we have

(a) x = 0 and z = 0 (b) y = 0 and z = 0 (c) x = 0 and y = 0 (d) None of these

(SCQ - 6) The equation of the xz-plane is

(a) y = 0 (b) x = 0 (c) z = 0 (d) None of these

(SCQ - 7) The equation of the plane parallel to yz-plane is,

(a) x = 10
(b) z = 5
(c) y = 8
(d) None of these

(SCQ – 8) The equation, x =0 = y represents a

(a) plane (b) straight line (c) point (d) None of these

(SCQ – 9) The distance of the point (4, 0, 7) from x-axis is

(a) 4 (b) 0 (c) 7 (d) None of these

(SCQ – 10) The positive octant is,

(a) $OX'Y'Z'$ (b) OXYZ (c) OX'YZ' (d) OXY'Z

1.8 CHANGE OF ORIGIN

To change the origin O=(0,0,0) of the coordinate system to another point O'=(α, β, γ), whereas the directions of the axes remain the same.

<u>DERIVATION</u> Let O=(0,0,0) be the origin of the coordinate system with OX, OY, OZ as the original coordinate axes. Let O'= (α, β, γ) be the new origin of the transferred coordinate system, the coordinates of O' are taken with respect to OX, OY, OZ as coordinate axes. Draw three lines $O'X'$, $O'Y'$ and $O'Z'$ through O' parallel to and in the same directions of OX, OY and OZ respectively.

Let P be any arbitrary point in the space whose coordinates are (x, y, z) with respect to the original coordinate axes OX, OY, OZ. Also suppose that the coordinates of the same point P with respect to the new axes $O'X'$, $O'Y'$ and $O'Z'$ are (X', Y', Z').

From the point O' draw a perpendicular O'L to OZ .From the point P draw a perpendicular PM to yz-plane , which meets at point N to $x'y'$-plane. Then from the above figure we have

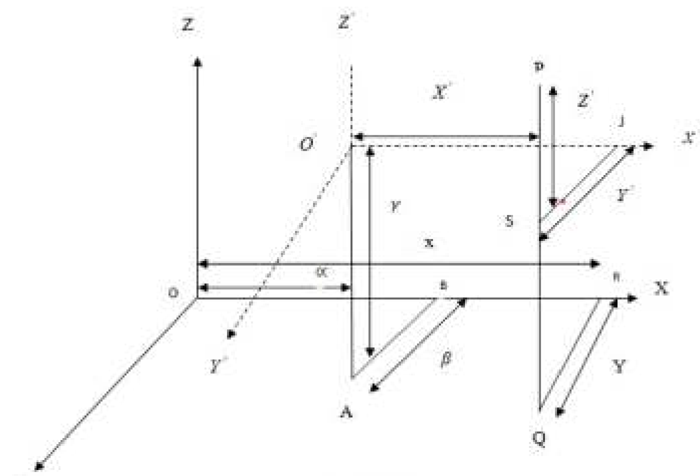


Fig. 1.8.1

O'L= α =NM, Now PM= PN+NM (Since PM=x and PN = x') hence we get,

x = $x' + \alpha$

Similarly we get, y = $y' + \beta$ and z = $z' + \gamma$ Combining all the results we get,

$$x = x' + \alpha$$
$$y = y' + \beta$$
$$z = z' + \gamma$$

Also from the above transformations we make the following conclusion,

$$x = x' - \alpha$$
$$y = y' - \beta$$
$$z = z' - \gamma$$

REMARKSHere from the above discussion, we make the following conclusions—

(1) SHIFTING OF ORIGIN FROM O=(0,0,0) TO O'=(α,β,γ) ---

If we shift the origin from O=(0,0,0) to another point O'=(α, β, γ) , then we have to replace x by x + α , y by y + β and z by z + γ. Then the transformed equation of a space curve , surface etc. is obtained w. r. t. (α, β, γ) as the new origin .Here it is remarkable from equation (1), that we should replace x by x' + α, y by y' + β and z by z' + γ but we replace the same by x + α , y + β and z + γ respectively just for shake convenience so that the transformed equation is also in current coordinates (x, y, z).

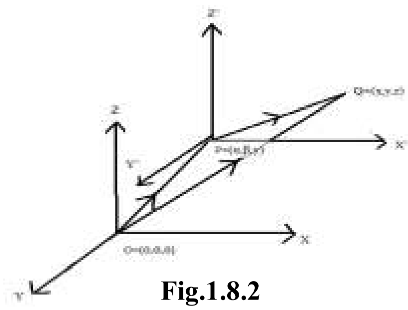


Fig.1.8.2

(2) **SHIFTING THE ORIGIN BACK FROM THE POINT O'=(α,β,γ) TO THE POINT O=(0,0,0)** Sometimes it is required to shift the new origin back to the original origin. For this we replace x (i.e. x') by x-α, y (i.e. y') by y-β and z (i.e. z') by z-γ, in the transformed equation referred to the new origin O' to get the corresponding equation referred to the original origin O. Here we do so such that the transformed equation is also in the current coordinates.

ANOTHER METHOD OR VECTOR METHOD (CHANGE OF ORIGIN)

Let OX, OY and OZ be a set of three mutually orthogonal coordinate axes. Let the coordinates of two points P and Q w.r.t. these three mutually orthogonal axes are (α, β, γ) and (x, y, z) respectively. Now suppose we shift the origin from O= (0,0,0) to the point P i.e. (α, β, γ) and find the coordinates of the point Q w.r.t. the point P as origin.

Draw the lines PX', PY' and PZ' through the point P, parallel and in the same directions of OX, OY and OZ respectively.
The position vectors of the points P and Q w.r.t. O as origin are given by,

$\vec{OP}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ $\vec{OQ}=x\hat{i}+y\hat{j}+z\hat{k}$

Now the position vector of Q with respect to P as origin is given by,

$\vec{PQ}=\vec{PO}+\vec{OQ}=-\vec{OP}+\vec{OQ}=\vec{OQ}-\vec{OP}$, hence we get, $\vec{PQ}=(x-\alpha)\hat{i}+(y-\beta)\hat{j}+(z-\gamma)\hat{k}$

Hence the coordinates of the point Q with respect to P as origin are given by, (x-α, y-β, z-γ).

1.9 SPHERICAL POLAR CO-ORDINATES

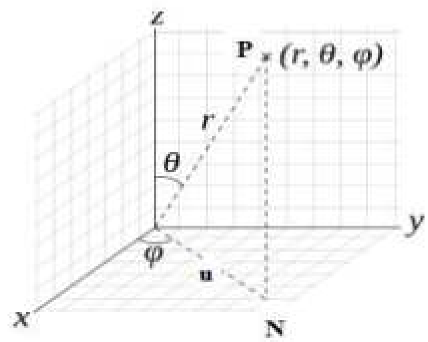


Fig. 1.9.1

Spherical polar coordinates (r, θ, φ) , commonly used in Mathematics as well as in Physics, includes radial distance r, polar angle θ (theta), and azimuthal angle φ (phi).

In mathematics, a spherical polar coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. It can be seen as the three-dimensional version of the polar coordinate system.

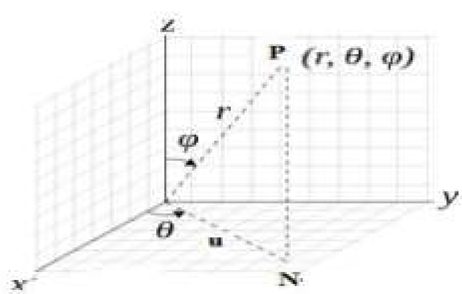


Fig. 1.9.2

The radial distance is also called the radius or radial coordinate. The polar angle may be called colatitudes, zenith angle, normal angle, or inclination angle.

The use of symbols and the order of the coordinates differ between sources. In one system frequently encountered in physics (r, θ, φ) gives the radial distance, polar angle, and azimuthal angle, whereas in another system used in many mathematics books (r, θ, φ) gives the radial distance, azimuthal angle, and polar angle.

To define a spherical coordinate system, one must choose two orthogonal directions, the zenith and the azimuth reference, and an origin point in space. These choices determine a reference plane that contains the origin and is perpendicular to the zenith. The spherical coordinates of a point *P* are then defined as follows:

- The radius or radial distance is the Euclidean distance from the origin *O* to *P*.
- The inclination (or polar angle) is the angle between the zenith direction and the line segment *OP*.
- The azimuth (or azimuthal angle) is the signed angle measured from the azimuth reference direction to the orthogonal projection of the line segment *OP* on the reference plane.

The sign of the azimuth is determined by choosing what a positive sense of turning about the zenith is. This choice is arbitrary, and is part of the coordinate system's definition.

The elevation angle is 90 degrees (π/2 radians) minus the inclination angle.

If the inclination is zero or 180 degrees (π radians), the azimuth is arbitrary. If the radius is zero, both azimuth and inclination are arbitrary.

In linear algebra, the vector from the origin *O* to the point *P* is often called the position vector of *P*.

The angles are typically measured in degrees (°) or radians, where 360° = 2π radians. Degrees are most common in geography, astronomy, and engineering, whereas radians are commonly used in mathematics and theoretical physics. The unit for radial distance is usually determined by the context.

When the system is used for physical three-space, it is customary to use positive sign for azimuth angles that are measured in the counter-clockwise sense from the reference direction on the reference plane, as seen from the zenith side of the plane.

Let $X^{'}OX$, $Y^{'}OY$ and $Z'OZ$ be the set of rectangular axes. Let P be any point in the space .Draw a perpendicular PN from the point p to the xy-plane. The position of P is determined if the length of op, angles ZOP and XON are known. Let us suppose OP = r, < ZOP = θ and < XON = φ, measured positively in the directions shown by arrowsin the figure. The quantities r, θ, φ defined as above, are called the spherical polar coordinates of the point P and are written as (r, θ, φ). Suppose the Cartesian coordinates of the point P are (x, y, z). Now we shall find relations between Cartesian coordinates and the spherical polar coordinates of the point P.

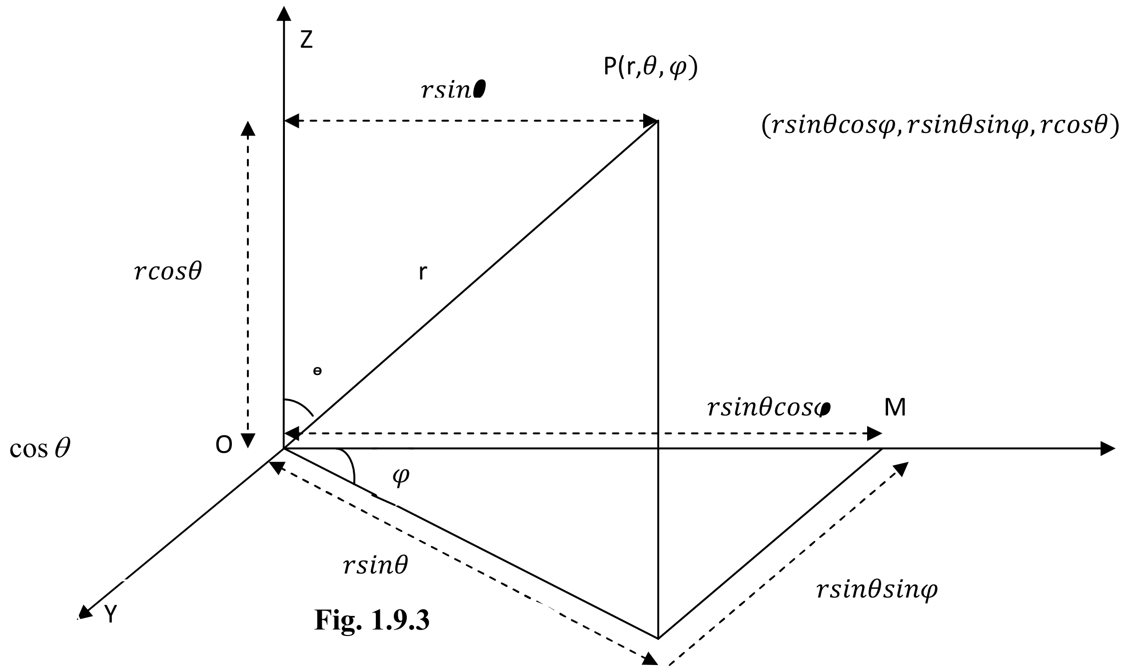


Fig. 1.9.3

Since PN is perpendicular to the xy-plane and the line ON lies in xy-plane, hence <PNO is 90°. Now in right angle triangle PNO we have

$$\cos\theta = \frac{PN}{OP} = \frac{z}{r}$$

Hence, $z = r\cos\theta$ ---------------- (1)

Also, $\sin\theta = \frac{ON}{OP}$ => ON = OP $\sin\theta$

Hence, ON = $r\cos\theta$ -------------- (2)

Since <ONP = 90°, we get $\cos\phi = \frac{x}{ON}$

⇨ $x = r\sin\theta\cos\phi$ [using equ.-(2)] -------------------- (3)

Similarly we get, y = $r\sin\theta\sin\phi$ ------------------------ (4)

Here equations (1), (3) and (4) gives the required results. That is,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Now squaring and adding the equations (1), (3) and (4) we get $x^2 + y^2 + z^2 = r^2$

Dividing equation (4) by (3) we get $\tan\phi = \frac{y}{x}$

Squaring and adding equations (3) and (4) we get $x^2 + y^2 = r^2(\sin\theta)^2$

i.e. $r\sin\theta = \sqrt{x^2 + y^2}$ ----------------------(5)

Now dividing equation (5) by (1) we have, $\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$

Hence the required relations between spherical polar coordinates (r, θ, φ) and Cartesian coordinates (x, y, z) are given by,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan\phi = \frac{y}{x}$$

With the help of these relations we can convert the cartesian coordinates into spherical polar coordinates and vice-versa.

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ-11) If the line OP of length r makes an angle θ with the x-axis and lies in the xy-plane plane then the coordinates of P will be,

(a) (rcos θ ,r sin θ,0) (b) (rcos θ ,0,r sin θ) (c) (r sin θ,r cos θ,0) (d) None of these

(SCQ-12) The polar equation of the sphere $x^2 + y^2 + z^2 = 36$ will be,

(a) r = 15 (b) r = 4 (c) r = 6 (d) r = 10

(SCQ-13) Equation of the plane perpendicular to the x-axis is,

(a) y = 0 (b) x+ λ y = 0 (c)) x = λ (d) z = λ

where λ is any arbitrary constant.

(SCQ-14) The point of intersection of three coordinate axes is called

(a) oblique (b) origin (c) section (d) centroid

(SCQ-15) The points (2, 5, 9) and (2k, λ, μ) will represent the same point if

(a) k = 3 (b) μ = 1 (c) λ = 2 (d) k = 1

(SCQ-16) If the Cartesian coordinates of a point are (2, 2, -4) then the value of φ will be,

(a) π/4 (b) π/2 (c) π/3 (d) π

(SCQ-17) The curve of intersection of two planes is,

(a) a parabola (b) a circle (c) a straight line (d) a curved line

(SCQ-18) General equation of the plane parallel to zx-planeis,

(a) x + y = λ (b) y= 0 (c) y = λ (d) z = λ

where λ is any arbitrary constant

(SCQ – 19) For cylindrical coordinates we have the following relation,

(a) $u^2 = x^2 - y^2$ (b) $u^2 = x^2 + y^2$ (c) u = x + y (d) u = x - y

(SCQ – 20) If the origin is shifted from (0,0,0) to (α,β,γ) ,then we have to replace x by,

(a) x+α (b) x-α (c) x+β (d) none of these

FILL IN THE BLANKS

(SCQ – 21) The point (6, -8, 3) lies in the octant -------------------------------.

(SCQ – 22) The equation of the plane parallel to yz-plane and at distance 3 units in the direction of OX' will be ---------------------------.

(SCQ – 23) In case of spherical polar coordinates, $\tan\theta = \frac{\quad}{x}$

(SCQ – 24) If the coordinate axes are at right angles then the system of coordinate axes is called --- otherwise it is called ---------------------------------

(SCQ – 25) In cylindrical coordinate system the------coordinates remains the same.

(SCQ – 26) If (8, 7, 6) = (8, 7, 2k) then k = ------.

1.10 SUMMARY

In this unit, we have learned about the difference between the plane (2D) coordinate system and three dimensional (3D) coordinate system. Mainly in three dimensional (3D)l coordinate system there is inclusion of z-axis which is perpendicular to the xy-plane i.e. the plane in which both the axes (x-axis and y-axis) lies. Also we have learned that there exists three mutually perpendicular axes (XO X', YO Y' and ZO Z'),three coordinate planes (xy-plane, yz-plane and xz-planes which divide the whole space into eight parts called octants. Here we found the equations of surfaces like as sphere, cone, cylinder etc. in different forms such as Cartesian form and polar form . Also we have learned to differentiate whether a point lies in a space or in a plane. Also we have learnt the method to convert the coordinates of a point in two different forms namely-Cartesian and polar coordinate systems.. We have derived the expression for the relationships between the two coordinate systems. We have studied the method to find the changed coordinates of a point when the origin is shifted to any other point and vice-versa. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics . To check your progress, self check questions (SCQ's) are given place to place.

1.11 GLOSSARY

i. 3D object or a solid – an object that occupies space.
ii. Face – surface of a solid.
iii. Edge – where two faces meet.
iv. Vertex – where two edges meet.
v. Mutually – shared by two or more.
vi. Coplanar- lying in a plane.
vii. Eliminate – To remove or to omit or to neglect that is not wanted or needed or required.
viii. Perpendicular – right angle or at an angle of 90° or pointing straight up.
ix. Origin – the point from which we start generally at O = (0,0.0).
x. Position vector - situation or location of a point with respect to origin with direction.
xi. Axes - plural of axis.
xii. e.g. – for example.
xiii. i.e. - that is.
xiv. w.r.t. – with respect to.
xv. Tetrahedron - a Pyramid/Kyra with triangular base having 4 surfaces (3 lateral & 1 base), 4 vertices, 6 edges etc.
xvi.

1.12 REFERENCES

1. Geometry of two and three Dimensions - Lata S. , Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. 3 D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd. , New Delhi, India.
6. Leonard E. Dickson :The First course in the Geometry.
7. Burnside,William Show, Panton &Arthur William :Volumetric Theory.
8. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series ,McGraw Hill.
9. P.N .Pandey :Polar Coordinate Geometry, Sharda Academic Publishing House, Allahabad.
10. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.
11. Anton, Howard (1994), Elementary Linear Algebra (7th ed.), John Wiley & Sons, ISBN 978-0- 471-58742-2.
12. Arfken, George B. and Hans J. Weber. Mathematical Methods For Physicists, Academic Press; 6 edition (June 21, 2005). ISBN 978-0-12-059876-2.
13. Brannan, David A.; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press, ISBN 978-0-521-59787-6

1.14 SUGGESTED READINGS

1. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
3. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

1.15 TERMINAL QUESTIONS

(TQ -1) Derive the relations between the Cartesian coordinates (x, y, z) and spherical polar coordinates(r, θ, φ) of a point.

(TQ -2) Find the polar coordinates of the point (3, 4 ,5), so that r may be positive .

(TQ -3) Find the polar coordinates of a point whose Cartesian coordinates are (-2, 1, -2) .

(TQ -4) Derive the relations between the Cartesian coordinates of a point P when the origin is shifted from (0, 0, 0) to the point (α, β, γ) and coordinates of P are (x, y, z) and (x', y', z') w.r.t. (0, 0, 0) and (α, β, γ) as origin.

(TQ -5) Derive the relation between the Cartesian coordinates of a point and its position vector. What will be the position vector of the point whose Cartesian coordinates are (7, 0, -3).

1.16 ANSWERS

SELF CHECK QUESTIONS (SCQ'S)

(SCQ – 1) c (SCQ – 2) a (SCQ – 3) b (SCQ – 4) d

(SCQ – 5) a (SCQ – 6) a (SCQ – 7) a (SCQ – 8) b

(SCQ – 9) c (SCQ – 10) b (SCQ – 11) a (SCQ – 12) c

(SCQ – 13) c (SCQ – 14) b (SCQ – 15) d (SCQ – 16) a

(SCQ – 17) c (SCQ – 18) c (SCQ – 19) b (SCQ – 20) a

(SCQ – 21) OXY'Z (SCQ – 22) x = -3 (SCQ – 23) $\sqrt{x^2 + y^2}$ (SCQ – 24) rectangular, oblique (SCQ-25) z (SC Q – 26) 3.

TERMINAL QUESTIONS (TQ'S)

(TQ -1)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan\phi = \frac{y}{x}$$

(TQ – 2) $[5\sqrt{2}, \frac{\pi}{4}, \tan^{-1}(\frac{4}{3})]$

(TQ – 3) $[3, \tan^{-1}(-\frac{\sqrt{5}}{2}), \tan^{-1}(-\frac{1}{2})]$

(TQ-4)

$$x = x^2 + \alpha \qquad y = y^2 + \beta \qquad z = z^2 + \gamma$$

(TQ – 5) $7\hat{i} - 3\hat{k}$

UNIT 2: SYSTEM OF COORDINATES IN THREE DIMENSIONS -II

CONTENTS

2.1 INTRODUCTION

In the previous unit, you should have learnt and studied by now that there exits two types of objects-

1. which has only length and breadth e.g. rectangle, square etc. These types of objects are called $2D$ objects.
2. which has length , breadth and height e.g. cube, cuboid, cone, cylinder etc. These types of objects are called $3D$ objects.

Also we have learnt by now that there exits two types of spaces- $2D$ space and $3D$ space.

2D space Two-dimensional space is represented with the $X-$ and $Y-$axes. $3D$ animation adds depth, or the $Z-$axis.

3D space This three-axes reference system is known as the Cartesian coordinate system. It is used to define many properties of a $3D$ object, including its position, rotation, and scale. For example, an object's position can be represented as $12, 0, 10$ — meaning that it is 12 units to the right of center on the X −axis, 0 units from center along the Y-axis, and 10 units along the Z −axis.

Also we learnt by now that there exist three coordinate planes namely xy, yz and zx-planes. These three coordinate planes divide the whole space into eight parts called octants. According to a point in the space lies only in one octant. Also we have discussed about the planes parallel to the coordinate planes or planes perpendicular to the coordinate axes, the equations of these planes and intersection of the two planes.

We have derived the relations between the Cartesian coordinates and spherical polar coordinates of a point. We have discussed the method how the Cartesian coordinates can be converted into polar coordinates and vice-versa.

2.2 OBJECTIVES

After studying this unit, you should be able to –

Understand the different forms of coordinates of a point in space.

- Understand the relationship among the different forms of coordinates of a point in space.
- Find the distance between two given points.
- Find the section of the joining of the two given points.
- Calculate the centroid of a triangle and a tetrahedron.
- Check in which octant the given point lies.
- Calculate the distances of a point from the coordinate axes.
- Find whether the give point intersects the join of the two given points internally or externally.
- Find the ordered and equality properties of the coordinates of two points.

2.3 CYLINDRICAL CO-ORDINATES SYSTEM

ANOTHER METHOD TO FIND THE POSITION OF A POINT IN SPACE: A cylindrical coordinate system with origin O, polar axis A, and longitudinal axis L. The dot is the point with radial distance ρ, angular coordinate φ, and height z.

A cylindrical coordinate system is a three-dimensional coordinate system that specifies point positions by the distance from a chosen reference axis, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis. The latter distance is given as a positive or negative number depending on which side of the reference plane faces the point.

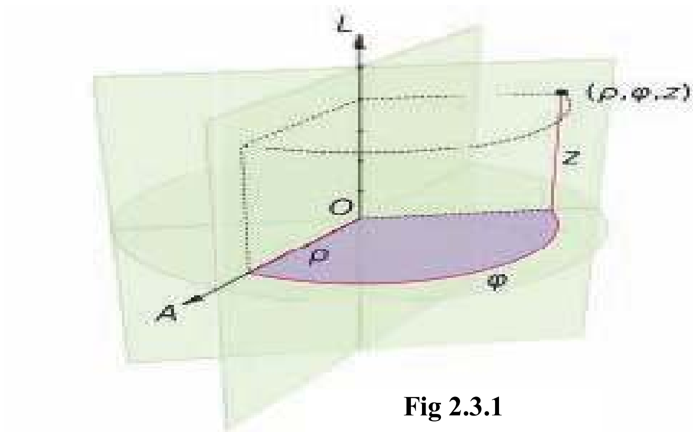


Fig 2.3.1

The origin of the system is the point where all three coordinates can be given as zero. This is the intersection between the reference plane and the axis.

The axis is variously called the cylindrical or longitudinal axis, to differentiate it from the polar axis, which is the ray that lies in the reference plane, starting at the origin and pointing in the reference direction.

The distance from the axis may be called the radial distance or radius, while the angular coordinate is sometimes referred to as the angular position or as the azimuth. The radius and the azimuth are together called the polar coordinates, as they correspond to a two-dimensional polar coordinate system in the plane through the point, parallel to the reference plane. The third coordinate may be called the height or altitude (if the reference plane is considered horizontal), longitudinal position or axial position.

Cylindrical coordinates are useful in connection with objects and phenomena that have some rotational symmetry about the longitudinal axis, such as water flow in a straight pipe with round cross-section, heat distribution in a metal cylinder, electromagnetic fields produced by an electric current in a long, straight wire, accretion disks in astronomy, and so on.

They are sometimes called "cylindrical polar coordinates" and "polar cylindrical coordinates", and are sometimes used to specify the position of stars in a galaxy ("galactocentric cylindrical polar coordinates").

The three coordinates $(\rho,\ \varphi,\ z)$ of a point P are defined as:

- The axial distance or radial distance ρ is the Euclidean distance from the z-axis to the point P.
- The azimuth φ is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane.
- The axialcoordinate or height z is the signed distance from the chosen plane to the point P.

Unique cylindrical coordinates

As in polar coordinates, the same point with cylindrical coordinates (ρ, φ, z) has infinitely many equivalent coordinates, namely $(\rho, \varphi \pm n \times 360°, z)$ and $(-\rho, \varphi \pm (2n+1) \times 180°, z)$, where n is any integer. Moreover, if the radius ρ is zero, the azimuth is arbitrary.

In situations where someone wants a unique set of coordinates for each point, one may restrict the radius to be non-negative $(\rho \geq 0)$ and the azimuth φ to lie in a specific interval spanning 360°, such as $(-180°, +180°]$ or $[0, 360°)$.

Let P be any point in the space. The position of P can also be determined if ON (where N is the foot of perpendicular from point P to the xy-plane), $\angle XON$ and NP are known.

Now suppose that $ON = u$, $\angle XON = \varphi$ and $NP = z$. The quantities u, φ, z are called the cylindrical coordinates of the point P and are written as (u, φ, z).

Suppose the Cartesian coordinates of the point P are given by (x, y, z). Now since the foot of perpendicular N from the point P lies in the xy-plane, hence the Cartesian coordinates of the point N will be $(x, y, 0)$.

Now from above figure given in the previous article, we have

$x = u \cos\varphi$, $y = u \sin\varphi$ and $z = z$.

Also, $u^2 = x^2 + y^2$ and $\tan\varphi = \frac{y}{x}$

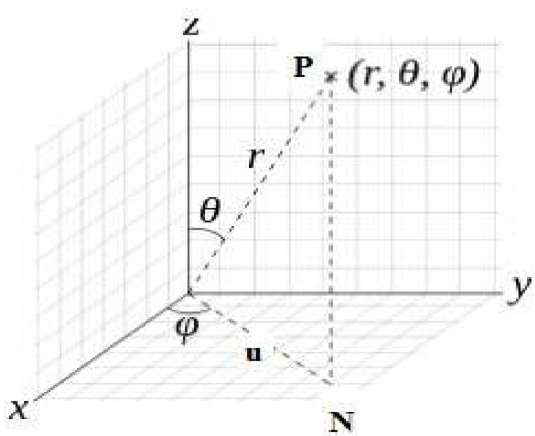


Fig 2.3.2

Here it is remarkable that the z-coordinate remains the same in both the coordinate systems namely Cartesian coordinate system and cylindrical coordinate system.

2.4 CURVILINEAR CO-ORDINATE SYSTEM

Although Cartesian orthogonal coordinates are very intuitive and easy to use, it is often found more convenient to work with other coordinate systems. Being able to change all variables andexpression involved in a given problem, when a different coordinate system is chosen, is one ofthose skills a physicist, and even more a theoretical physicist, needs to possess.
In this lecture a general method to express any variable and expression in an arbitrary curvilinearcoordinate system will be introduced and explained. We will be mainly interested to find out general expressions for the gradient, the divergence and the curl of scalar and vector fields. Specific applications to the widely used cylindrical and spherical systems will conclude in this lecture.
The cartesian orthogonal coordinate system is very intuitive and easy to handle. Once an originhas been fixed in space and three orthogonal scaled axis are anchored to this origin, any pointin space is uniquely determined by three real numbers, its cartesian coordinates. A curvilinearcoordinate system can be defined starting from the orthogonal cartesian one. If x, y, z are thecartesian coordinates, the curvilinear ones (u, v, w) can be expressed as smooth functions of x, y, z according to:

$$\left.\begin{aligned} u &= u(x, y, z) \\ v &= v(x, y, z) \\ w &= w(x, y, z) \end{aligned}\right\} \qquad (1)$$

These functions can be inverted to give x, y, z-dependency on u, v, w:

$$\left.\begin{aligned} x &= x(x, y, z) \\ y &= y(x, y, z) \\ z &= z(x, y, z) \end{aligned}\right\} \qquad (2)$$

There are infinitely many curvilinear systems that can be defined using equations (1) and (2).
We are mostly interested in the so-called orthogonal curvilinear coordinate systems, defined asfollows. Any point in space is determined by the intersection of three planes:
$u = \text{const}, v = \text{const}\,; w = \text{const}$
We could call these surfaces as coordinate surfaces. Three curves, called coordinate curves, areformed by the intersection of pairs of these surfaces. Accordingly, three straight lines can becalculated as tangent lines to each coordinate curve at the space point. In an orthogonal curvedsystem these three tangents will be orthogonal for all points in space (see Figure 1).
In order to express differential operators, like the gradient or the divergence, in curvilinear coordinates it is convenient to start from the infinitesimal increment in cartesian coordinates,
In this generic orthogonal curved coordinate system three coordinate surfaces meet ateach point P in space. Their mutual intersection gives rise to three coordinate curves which arethemselves perpendicular in P.

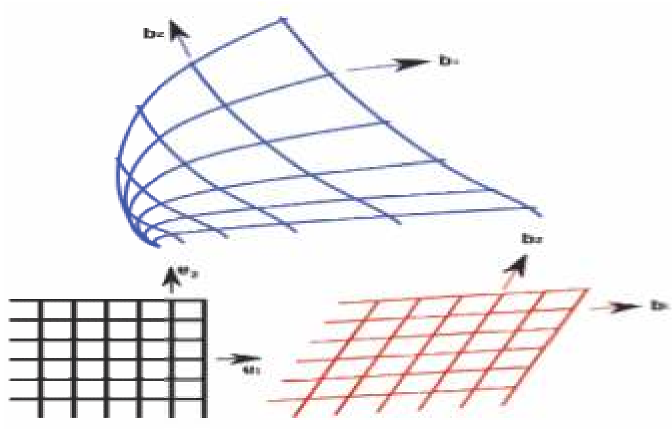


Fig 2.4.1

Curvilinear (top), affine (right), and Cartesian (left) coordinate in two-dimensional space.

In geometry, curvilinear coordinates are a coordinate system for Euclidean space in which the coordinate lines may be curved. Commonly used curvilinear coordinate systems include: rectangular, spherical, and cylindrical coordinate systems. These coordinates may be derived from a set of Cartesian coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. This means that one can convert a point given in a Cartesian coordinate system to its curvilinear coordinates and back. The name curvilinear coordinates, coined by the French mathematician Lame, derives from the fact that the coordinate surfaces of the curvilinear systems are curved.

Well-known examples of curvilinear coordinate system in three-dimensional Euclidean space R^3 are Cartesian, cylindrical and spherical polar coordinates. A Cartesian coordinate surface in this space is a coordinate plane; for example $z = 0$ defines the xy plane. In the same space, the coordinate surface $r = 1$ in spherical polar coordinates is the surface of a unit sphere, which is curved. The formalism of curvilinear coordinates provides a unified and general description of the standard coordinate systems.

Curvilinear coordinates are often used to define the location or distribution of physical quantities which may be, for example, scalars, vectors, or tensors. Mathematical expressions involving these quantities in vector calculus and tensor analysis (such as the gradient, divergence, curl, and Laplacian) can be transformed from one coordinate system to another, according to transformation rules for scalars, vectors, and tensors. Such expressions then become valid for any curvilinear coordinate system.

Depending on the application, a curvilinear coordinate system may be simpler to use than the Cartesian coordinate system. For instance, a physical problem with spherical symmetry defined in R^3 (for example, motion of particles under the influence of central forces) is usually easier to solve in spherical polar coordinates than in Cartesian coordinates.Equations with boundary conditions that follow coordinate surfaces for a particular curvilinear coordinate system may be easier to solve in that system. One would for instance describe the motion of a particle in a rectangular box in Cartesian coordinates, whereas one would prefer spherical coordinates for a particle in a sphere. Spherical coordinates are one of the most used curvilinear coordinate systems in such fields as Earth sciences, Cartography, and Physics (in particular Quantum mechanics, Relativity), and engineering.

2.5 ORTHOGONAL CURVILINEAR CO-ORDINATE SYSTEM:

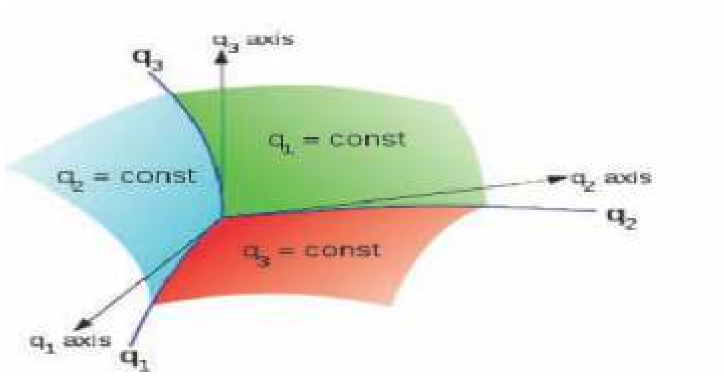


Fig 2.5.1

Fig. 2.5.1 of Coordinate surfaces, coordinate lines, and coordinate axes of general curvilinear coordinates.

For now, consider $3D$ space. A point P in $3D$ space (or its position vector $\boldsymbol{r}$) can be defined using Cartesian coordinates $(x,\ y,\ z)$ [equivalently written as $(x_1,\ x_2,\ x_3)$].

It can also be defined by its curvilinear coordinates $(q_1,\ q_2,\ q_3)$ if this triplet of numbers defines a single point in an unambiguous way. The relation between the coordinates is then given by the invertible transformation functions:

The surfaces q_1 = constant, q_1 = constant, q_1=constant are called the coordinate surfaces and the space curves formed by their intersection in pairs are called the coordinate curves. The coordinate axes are determined by the tangents to the coordinate curves at the intersection of three surfaces. They are not in general fixed directions in space, which happens to be the case for simple Cartesian coordinates, and thus there is generally no natural global basis for curvilinear coordinates.

In the Cartesian system, the standard basis vectors can be derived from the derivative of the

location of point P with respect to the local coordinate

Applying the same derivatives to the curvilinear system locally at point P defines the natural basis vectors:

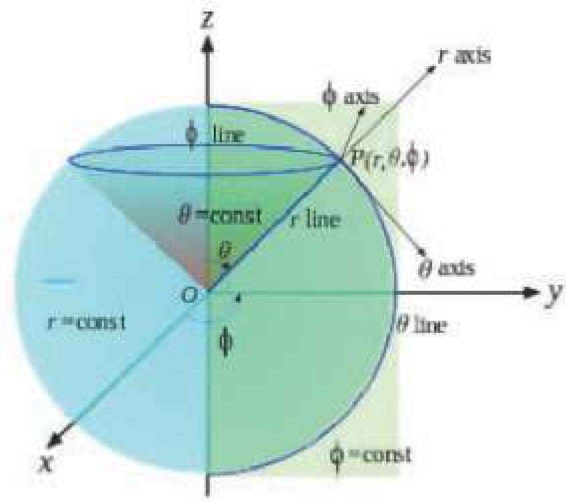


Fig 2.5.2

Fig. 2.5.2 of Coordinate surfaces, coordinate lines, and coordinate axes of spherical coordinates. Surfaces: r - spheres, θ - cones, φ - half-planes; Lines: r - straight beams, θ - vertical semicircles, φ - horizontal circles; Axes: r - straight beams, θ - tangents to vertical semicircles, φ - tangents to horizontal circles.

Such a basis, whose vectors change their direction and/or magnitude from point to point is called a local basis. All bases associated with curvilinear coordinates are necessarily local. Basis vectors that are the same at all points are global bases, and can be associated only with linear or affine coordinate systems.

Note: for this article **e** is reserved for the standard basis (Cartesian) and h or b is for the curvilinear basis.

These may not have unit length, and may also not be orthogonal. In the case that they are orthogonal at all points where the derivatives are well-defined, we define the Lame coefficients (after Gabriel Lame)

By and the curvilinear orthonormal basis vectors by-

It is important to note that these basis vectors may well depend upon the position of P; it is therefore necessary that they are not assumed to be constant over a region. (They technically form a basis for the tangent bundle of at P, and so are local to P.)

In general, curvilinear coordinates allow the natural basis vectors h_i not all mutually perpendicular to each other and not required to be of unit length: they can be of arbitrary magnitude and direction. The use of an orthogonal basis makes vector manipulations simpler than for non-orthogonal. However, some areas of physics and engineering, particularly fluid mechanics and continuum mechanics, require non-orthogonal bases to describe deformations and fluid transport to account for complicated directional dependences of physical quantities. A discussion of the general case appears later in higher classes.

SOLVED EXAMPLES

Ex.1. Find the spherical polar coordinates of the point whose Cartesian coordinates are given by (1, 2, 3).

Sol. The Cartesian coordinates of the given point are $(1, 2, 3)$ that is,

$$x = 1, \quad y = 2, \quad z = 3$$

Let (r, θ, φ) be the spherical polar coordinates of the above point. Now we know that the required relations between spherical polar coordinates (r, θ, φ) and Cartesian coordinate (x, y, z) are given by

$r = \sqrt{x^2 + y^2 + z^2}$	$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$	$\tan\varphi = \frac{y}{x}$

With the help of these relations we can convert the Cartesian coordinates into spherical polar coordinates.

(Putting the values of x, y & z and on simplifying we get),

$$r = \sqrt{14} \quad , \tan\theta = \frac{\sqrt{5}}{3} \quad => \quad \theta = tan^{-1}(\frac{\sqrt{5}}{3}) \quad , \qquad \tan\varphi = 2 \Rightarrow \varphi = tan^{-1}(2)$$

Hence the spherical polar coordinates of the point whose Cartesian coordinates are (1, 2, 3) will be,

$$\{\sqrt{14}, tan^{-1}(\frac{\sqrt{5}}{3}), tan^{-1}(2)\}.$$

Ex.2. Find the spherical polar coordinates of the point whose Cartesian coordinates are given by $(-2, 1, -2)$.

Sol. The Cartesian coordinates of the given point are $(-2, 1, -2)$ that is,

$$x = -2, \quad y = 1, \qquad z = -2$$

Let $(r,\ \theta,\ \varphi)$ be the spherical polar coordinates of the above point. Now we know that the required relations between spherical polar coordinates (r, θ, φ) and Cartesian coordinates (x, y, z) are given by,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

$$\tan\varphi = \frac{y}{x}$$

With the help of these relations we can convert the Cartesian coordinates into spherical polar coordinates.

(Putting the values of x, y & z and on simplifying we get),

$$r = 3,\ tan\,\theta = \frac{\sqrt{5}}{-2} \Rightarrow \theta = tan^{-1}(\frac{\sqrt{5}}{-2}) \ , \quad tan\,\varphi = \frac{-1}{2} \Rightarrow \varphi = tan^{-1}(\frac{-1}{2})$$

Hence the spherical polar coordinates of the point whose Cartesian coordinates are $(-2, 1, -2)$ will be $\{3, tan^{-1}(\frac{\sqrt{5}}{-2}), tan^{-1}(\frac{-1}{2})\}$.

****Ex.3.**Find the equation of a cylinder for which the axis is OZ and radius is 2, in (i) Cartesian , (ii) spherical polar and (iii) cylindrical coordinates.

SOLUTIONLet $P = (x_1, y_1, z_1)$ be any point on the surface of the cylinder, then coordinates of point P always satisfy the relation given by-

$$x_1{}^2 + y_1{}^2 = PM^2 = 4$$

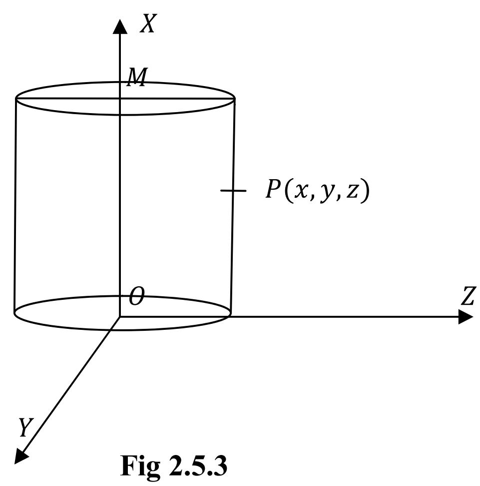


Fig 2.5.3

Hence the required equation of the Cylinder in Cartesian form is

$x^2 + y^2 = 4$ ……………………………………………………………………...(1)

Now putting, $x = u \cos \varphi$ and

$y = u \sin \varphi$, in equation (1) we get the equation in cylindrical form as,

$(u \cos \varphi)^2 2 + (u \sin \varphi)^2 = 4$

Simplifying we get, $u = 2$ ……………………………………………………….(2)

This is the required equation of the cylinder in cylindrical form.

Now substituting, $x = r \sin \theta \cos \varphi$ and $y = r \sin \theta \sin \varphi$ in equation (1) we get the equation of cylinder in spherical form as,

$$(r \sin \theta \cos \varphi)^2 + (r \sin \theta \sin \varphi)^2 = 4$$

Simplifying we get, $r \sin \theta = 2$

This is the required equation of the cylinder in spherical polar form.

**EXAMPLE -(4) Find the equations of the right circular cone whose vertex is origin O, axis is OZ i. e. z −axis and semi- vertical angle is α.

SOLUTION

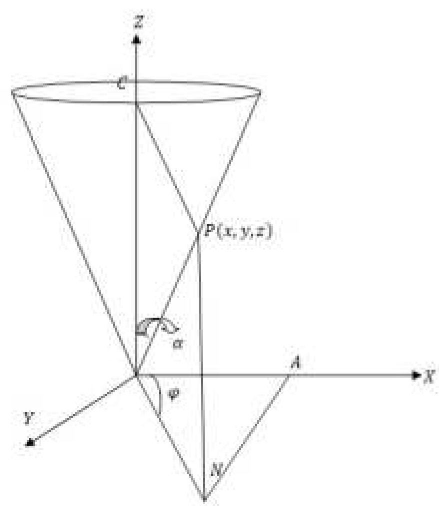


Fig. 2.5.4

Let $P = (x, y, z)$ be any point on the surface of the cone. PN and PC are perpendiculars drawn from the point P to the xy- plane and z-axis respectively. From the point N draw a perpendicular NA to the x-axis i.e. OX.

CARTESIAN EQUATION

Given that $\angle POC = \alpha, z = PN = OC$

and $PC = ON = \sqrt{OA^2 + NA^2} = \sqrt{x^2 + y^2}$

Now in the triangle PCO, we get $tan\ \alpha = \frac{PC}{OC}$

$$PC = OC\ \ tan\,\alpha \Rightarrow \sqrt{x^2 + y^2} = z\ \ tan\,\alpha$$

Squaring we get, $x^2 + y^2 = z^2 \tan^2 \alpha$

This is the required equation of the cone in Cartesian coordinates.

CYLINDRICAL EQUATION Let $P = (u, \varphi, z)$ be any point on the surface of the cone. Then as in case of Cartesian equation discussed above we have,

$PC = OC\ \ tan\,\alpha$ (But since $ON = PC = u$, and $OC = z$ hence we get)

$$u = z\ \ tan\,\alpha$$

This is the required equation of the cone in cylindrical coordinates.

SPHERICAL POLAR EQUATION Let $P = (r, \theta, \varphi)$ be any point on the surface of the cone. Then from the figure it is clear that $\theta = < POC = \alpha$.

Hence θ = α is the required equation of the cone in spherical coordinates.

EXAMPLE-(5) What is the polar equation of the plane which is parallel to the xy-plane and is at a distance c from it? Also find the cylindrical equation of the plane which is parallel to the zx −plane and is at a distance b from it.

SOLUTION CASE-I Since we know that the equation of any plane parallel to the xy-plane is given by, $z = \pm$(distance of the plane from the xy-plane). Here + or – sign is taken according as the plane is in the positive or negative direction of z −axis.

Hence in this case the required equation of the plane will be, $z = c$.

Now we know that the relations between the Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, φ) are given by,-

$$x = r\sin\theta\cos\theta$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Using the third relation $z = r\cos\theta$, the required equation of the plane becomes-

$$r\cos\theta = c$$

CASE-II Since we know that the equation of any plane parallel to the zx-plane is given by,

$y = \pm$(distance of the plane from the zx-plane)

Here $+$ or $-$ sign is taken according as the plane is in the positive or negative direction of y-axis.

Hence in this case the required equation of the plane will be, $y = b$.

Now we know that the relations between the Cartesian coordinates (x, y, z) and cylindrical polar coordinates (u, φ, z) are given by,-

$$x = \text{u}\cos\varphi, \qquad y = u\sin\varphi, \qquad z = z$$

Using the second relation $y = u\ sin\,\varphi$, the required equation of the plane becomes-

$$u\sin\varphi$$

EXAMPLE-(6) Find the equation of a sphere in (i) cylindrical form (ii) polar form whose centre is origin and radius is c.

SOLUTION We know that the equation of a sphere with centre at origin and radius c in Cartesian coordinates is given by, $x^2 + y^2 + z^2 = c^2$ ------------------------------------(1)

CASE-I Now we know that the relations between the Cartesian coordinates (x, y, z) and cylindrical polar coordinates (u, φ, z) are given by,-

$$x = \text{u}\cos\varphi, \qquad y = u\sin\varphi, \qquad z = z$$

Using these relations the required equation of the sphere becomes-

$$(u\ cos\varphi)^2 + (u\ sin\varphi)^2 + z^2 = c^2$$

Simplifying we get,

$$u^2 + z^2$$

This is the required equation of the sphere in cylindrical coordinates.

CASE-II We know that the required relations between spherical polar coordinates (r, θ, φ) and Cartesian coordinates (x, y, z) are given by,

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan\varphi = \frac{y}{x}$$

With the help of these relations we can convert the Cartesian equation of the sphere into spherical polar coordinates. Using the first relation we get ,

$$r = c$$

is the required equation of the sphere in spherical polar coordinates (r, θ, φ).

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ-1) In cylindrical coordinates the value of $\tan\varphi$ is,

(b) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{u}{z}$ (d) $\frac{z}{y}$

(SCQ-2) The equation of the sphere $x^2 + y^2 + z^2 = 8$ in cylindrical coordinates is,

(a) $r = 2\sqrt{2}$ (b) $r = 8$ (c) $u^2 + z^2 = 8$ (d) $r^2 + z^2 = 8$

(SCQ-3) Equation of the sphere passing through the origin and radius R in cylindrical coordinates is,

(a) $u^2 + z^2 = R^2$ (b)) $u^2 + z^2 = R$ (c)) $u^2 + z^2 = 0$ (d) $u^2 + z^2 = \sqrt{R}$

(SCQ-4) Cylindrical coordinates of the point whose Cartesian coordinates are (1, 1, 1) will be-

(a) $[\sqrt{3}, \frac{\pi}{4}, tan^{-1}\sqrt{2}]$ (b) $[\sqrt{2}, \frac{\pi}{4}, 1]$ (c) $[\sqrt{2}, \frac{\pi}{4}, tan^{-1}\sqrt{2}]$ (d) None of these

(SCQ-5) If the axes are curves, the coordinate system is called

(a) Cartesian (b) polar (c) cylindrical (d) curvilinear

2.6 DISTANCE BETWEEN TWO GIVEN POINTS

The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$in the xy-plane is given by the distance formula

$$d(P_1, P_2) = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

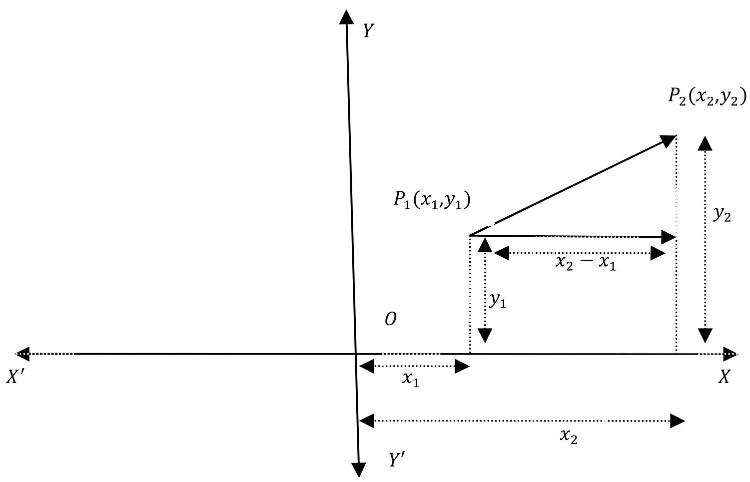


Fig. 2.6.1

Similarly, the distance between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in xyz space is given by the following generalization of the distance formula

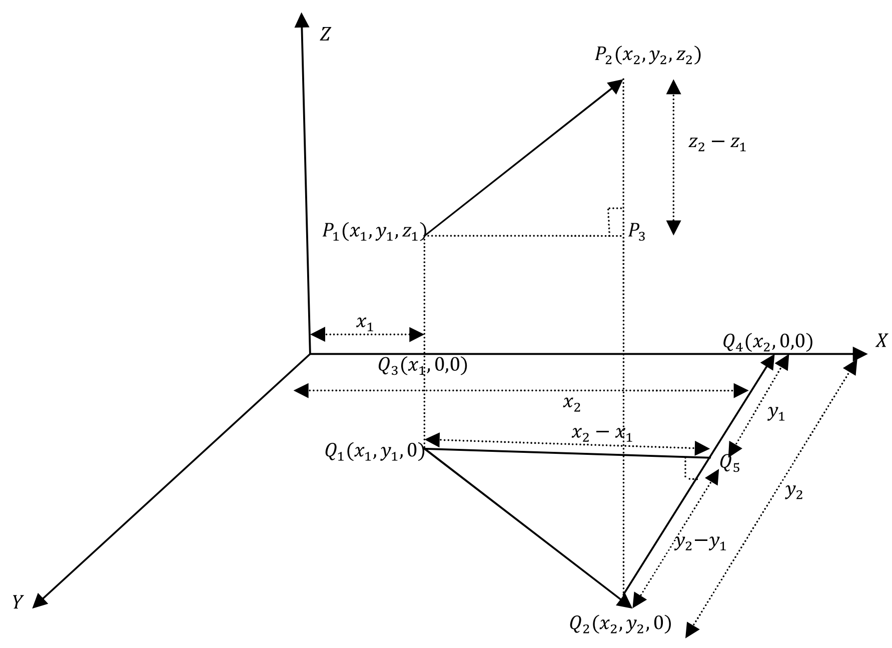


Fig. 2.6.2

$Q_1Q_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$Q_1Q_2 = P_1P_3$
Or $(P_1P_2)^2 = (P_1P_3)^2 + (P_2P_3)^2$
$= (Q_1Q_2)^2 + (P_2P_3)^2$
$\Rightarrow {P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$
$\Rightarrow d(P_1P_2) = P_1P_2 =$
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$………………………..(2.6.1)

This can be proved by repeated application of the Pythagorean Theorem.

Example The distance between $P_1 = (2,3,1)$ and $P_2 = (8, -5, 0)$ is

$$d(P_1, P_2) = \sqrt{(8-2)^2 + (-5-3)^2 + (0-1)^2} = \sqrt{101}$$

REMARKS (1) Distance of any point $P = (x, y, z)$ from origin $O = (0, 0, 0)$ is given by,

$$\boldsymbol{OP = \sqrt{x^2 + y^2 + z^2}}$$

(2) Distance between two points remains unchanged or invariant under change of origin .

2.7 DISTANCES OF A POINT FROM COORDINATE AXES

To find the perpendicular distances of the point (x, y, z) from the coordinate axes.

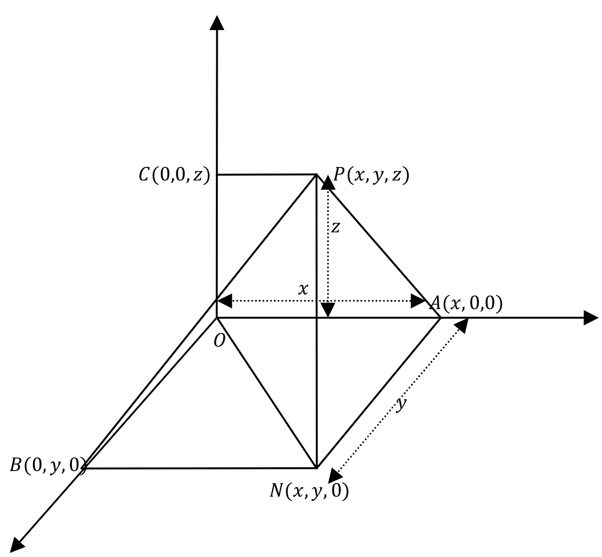


Fig. 2.7.1

Let $P = (x, y, z)$ be any point in the space. From the point P draw perpendicular PN to the xy −plane. Now from the point N we draw Perpendiculars NA and NB to The coordinate axes respectively. Draw the lines to join PA, PB and ON. Now draw a perpendicular PC from the point P to the z −axis.

From the figure it is clear that $OA = x = NB, NA = y = OB, PN = OC = z$

Now PA = length of perpendicular from the point P on x-axis = $\sqrt{NA^2 + PN^2}$

$$PA = \sqrt{y^2 + z^2}$$

This is the distance of point P from x-axis.

Similarly we can derived the expression for the distances of point P from x and z-axes.

PB = length of perpendicular from the point P on y-axis = $\sqrt{NB^2 + PN^2}$

$$PB = \sqrt{x^2 + z^2}$$

This is the distance of point P from y-axis.

PC = length of perpendicular from the point P on z-axis = $\sqrt{NA^2 + OA^2}$

$$PA = \sqrt{y^2 + x^2}$$

This is the distance of point P from z-axis.

Note-: Directly one can find PA i.e. distance between two points $P(x, y, z)$ and $A(x, 0,0)$

i.e. $PA = \sqrt{(x - x)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{y^2 + z^2}$.

Similarly PB and PC.

2.8 SECTION FORMULAE

To find the coordinates of the point which divides the line joining two given points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$in the given ratio $m : n$.

DERIVATION

Suppose the point R, having the coordinates (x, y, z) divides the line joining two given points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ internally in the given ratio $m : n$.

Let the planes PNA, QMB and RKC through the points P, Q and R and parallel to the yz-plane meet the x-axis at the points A, B and C respectively.

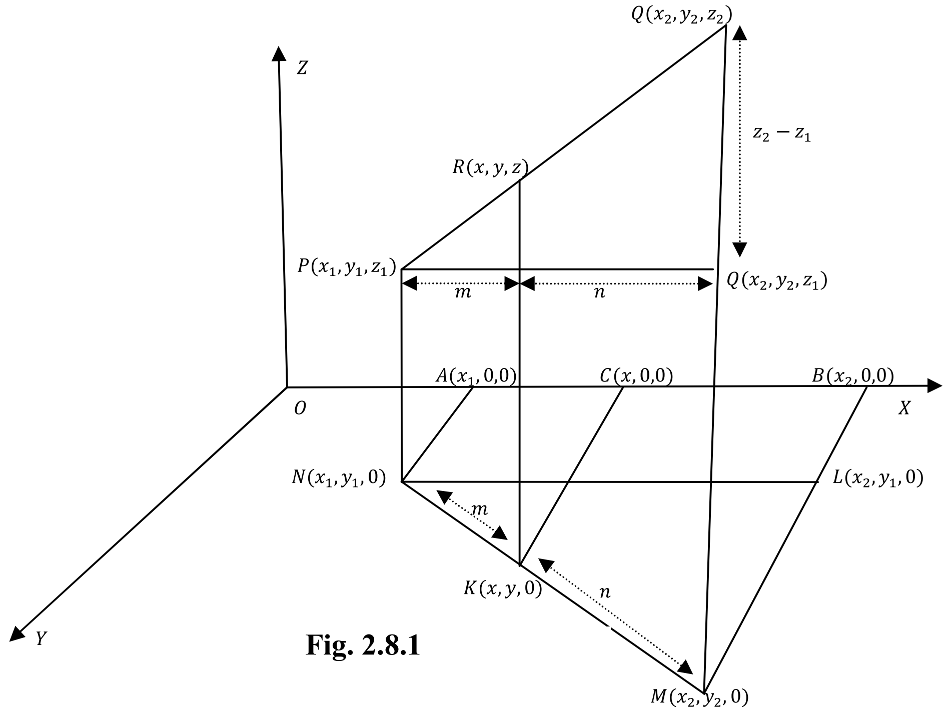


Fig. 2.8.1

Then these planes divide any two straight lines proportionality, hence we have

$$\frac{AC}{CB} = \frac{PR}{RQ} = \frac{m}{n} \text{ or } \frac{OC-OA}{OB-OC} = \frac{m}{n}$$

$$\Rightarrow \frac{x-x_1}{x_2-x} = \frac{m}{n}$$

Cross multiplying and then simplifying we get ,

$$x = \frac{mx_2 + nx_1}{m+n}$$

Similarly we get,

$$y = \frac{my_2 + ny_1}{m+n}$$

$$z = \frac{mz_2 + nz_1}{m+n}$$

REMARKS-

(1) If the point R divides the line joining the points P and Q externally in the ratio $m : n$ then we take $-n$ in place of n in the above results.

(2)If the point R is the mid point of the line joining the two given points points $P = (x_1, y_1, z_1)$and $Q = (x_2, y_2, z_2)$, then clearly $m = n = k$ (let). Putting these values in the results of internal division

$$x = \frac{mx_2+nx_1}{m+n}$$

$$y = \frac{my_2+ny_1}{m+n}$$

$$z = \frac{mz_2+nz_1}{m+n}$$

we get coordinates of point R as,

$x = \frac{x_1 + x_2}{3}$	$y = \frac{y_1 + y_2}{3}$	$z = \frac{z_1 + z_2}{3}$

2.9 CENTROID OF A TRIANGLE AND A TETRAHEDRON

CENTROID OF A TRIANGLE To find the coordinates of the centroid of a trianglewhose vertices are (x_i, y_i, z_i) where $i = 1, 2, 3$.

DERIVATION

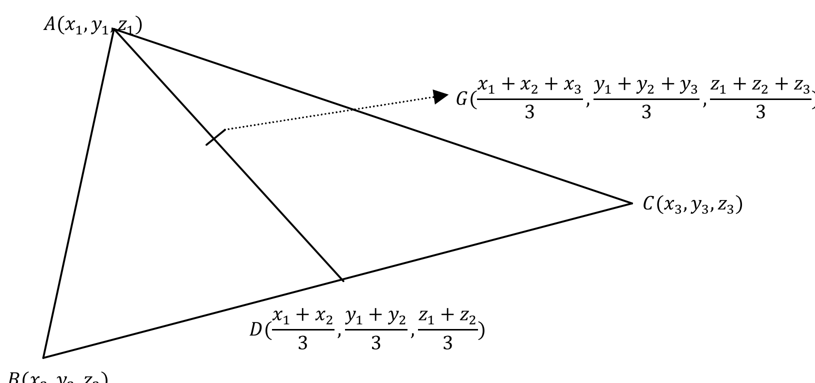


Fig 2.9.1

Suppose D is the middle point of BC. Then the coordinates of the point D are given by, i.e. $D = \left(\frac{x_1+x_2}{3}, \frac{y_1+y_2}{3}, \frac{z_1+z_2}{3}\right)$.

We know that centroid G of a triangle is the point of intersection of their medians and it divides any of its median internally in the ratio 1 : 2 from the base.
Now applying the results,

$$x = \frac{mx_2+nx_1}{m+n}$$

$$y = \frac{my_2+ny_1}{m+n}$$

$$z = \frac{mz_2+nz_1}{m+n}$$

we get

x-coordinate of G $= \frac{2.\left(\frac{x_2+x_3}{3}\right)+1.x_1}{2+1} = \frac{x_1+x_2+x_3}{3}$

Similarly,

y-coordinate of G $= \frac{2.\left(\frac{y_2+y_3}{3}\right)+1.y_1}{2+1} = \frac{y_1+y_2+y_3}{3}$

z-coordinate of G $= \frac{2.\left(\frac{z_2+z_3}{3}\right)+1.z_1}{2+1} = \frac{z_1+z_2+z_3}{3}$

x-coordinate of G $= \frac{x_1+x_2+x_3}{3}$

y-coordinate of G $= \frac{y_1+y_2+y_3}{3}$

z-coordinate of G $= \frac{z_1+z_2+z_3}{3}$

CENTROID OF A TETRAHEDRON

To find the coordinates of the centroid of a tetrahedron whose vertices are (x_i, y_i, z_i) where $i = 1, 2, 3, 4$.

DERIVATION

Let $ABCD$ be a tetrahedron and G_1 be the centroid of the trianglear base ABC.

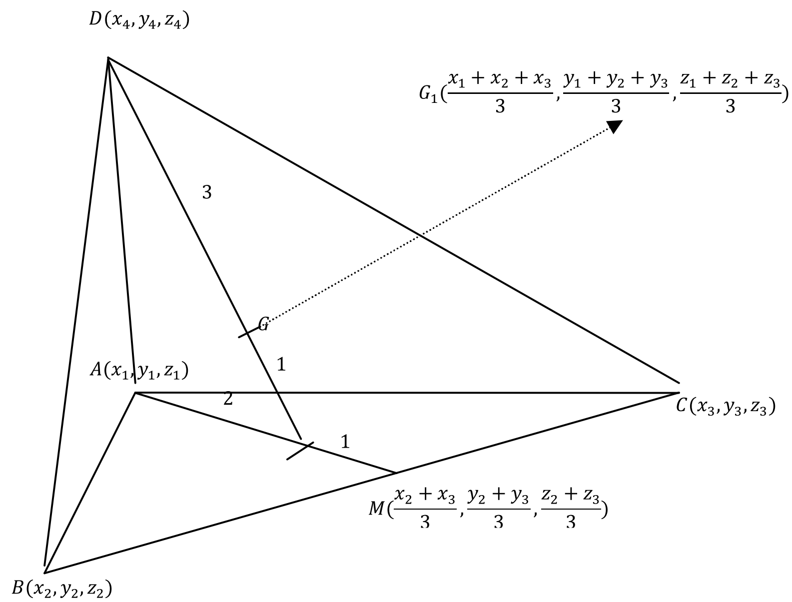


Fig 2.9.2

Then as we have discussed above in case of centroid of a triangle we have,

x-coordinate of $G_1 = \dfrac{x_1+x_2+x_3}{3}$

y-coordinate of $G_1 = \dfrac{y_1+y_2+y_3}{3}$

z-coordinate of $G_1 = \dfrac{z_1+z_2+z_3}{3}$

The centroid G of the tetrahedron $ABCD$ divides the line DG_1 internally in the ratio $1 : 3$ from G_1. Hence using the results,

$x = \frac{mx_2+nx_1}{m+n}$	$y = \frac{my_2+ny_1}{m+n}$	$z = \frac{mz_2+nz_1}{m+n}$

We have,

$$x\text{-coordinate of } G = \frac{3\left(\frac{x_1+x_2+x_3}{3}\right)+1.x_4}{3+1} = \frac{x_1+x_2+x_3+x_4}{4}$$

Similarly,

$$y\text{-coordinate of } G = \frac{3\left(\frac{y_1+y_2+y_3}{3}\right)+1.y_4}{3+1} = \frac{y_1+y_2+y_3+y_4}{4}$$

$$z\text{-coordinate of } G = \frac{3\left(\frac{z_1+z_2+z_3}{3}\right)+1.y_4}{3+1} = \frac{z_1+z_2+z_3+z_4}{4}$$

Hence the centroid of the tetrahedron is given by,

$$G = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ-6) Find the distance of the point $(3,0,-4)$ from origin,

(a) 25 (b) 5 (c) -1 (d) $\sqrt{5}$

(SCQ-7) Three points A, B and C will be collinear if,

(a) $AB + BC = AC$ (b) $AB + BC > 0$ (c) $AB + BC < 0$ (d) $AB + BC = 0$

(SCQ-8) Distance of the point $(-5, 12, -7)$ from z-axis is,

(a) 7 (b) 12 (c) 13 (d) 5

(SCQ–9) If the vertices of a triangle are $(5, -1, 0), (6, 2, 4)$ and $(-5, 3, 7)$ then the y-coordinate of its centroid will be,

(a) 1 (b) -2 (c) $\frac{4}{3}$ (d) 0

(SCQ–10) The point which divides the join of $(2, 3, 4)$ and $(3, -4, 7)$ externally in the ratio $1 : 2$ is

(a) $(7,2,4)$ (b) $(1, -5,6)$ (c) $(1,2, -6)$ (d) (1,10,1)

(SCQ–11) The centroid of a triangle is the point of intersection of its,

a) normal's from vertices (b) medians (c) sides (d) none of these

(SCQ –12) A Pyramid with base triangle is called a

(a) tetrahedron (b) prism (c) cone (d) cylinder

FILL IN THE BLANKS

(SCQ–13) The ratio in which the yz-plane divides the line joining of the points $(-2, 4, 7)$ and $(3, -5, 8)$ is ------------------.

(SCQ–14) The coordinates of the midpoint of the line joining of (3, 0, 7) and (-5, 4, -1) will be ---------------------.

(SCQ–15) The base of a tetrahedron is always a ----------------------.

(SCQ–16) If the ratio of division comes out to be negative then the division is called ------------------ division.

2.10 SUMMARY

In this unit, we have learned about the different methods to locate a point in the space such as cylindrical coordinate system, curvilinear coordinate system, and orthogonal curvilinear coordinate system and to convert the Cartesian coordinates of a point into cylindrical coordinate system with the help of relations between them. Also we have learned that there exists a coordinate system in which the axes are curves called curvilinear coordinate system. In this system if the three axes are mutually perpendicular then the system is called orthogonal curvilinear coordinate system. Here we found the distance between two points. With the help of this formula we can find whether the given three points are collinear or not also we can determine whether the given points are vertices of a right angled triangle, isosceles triangle, equilateral triangle, rectangle, square, rhombus etc. Also we have learned to differentiate whether a point divides the join of two points internally or externally and to find the coordinates of the point of division. Also we have learnt the method to convert the coordinates of a point in three different forms namely-Cartesian, polar and cylindrical coordinate systems.. We have derived the expression for the relationships between the coordinate systems. We have studied the method to find the coordinates of centroid of a triangle and also of a tetrahedron. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics. To check your progress, self check questions (SCQ's) are given place to place.

2.11 GLOSSARY

1. Line segment – a line of finite length having two end points .
2. Vertex – where two edges meet.
3. Mutually – shared by two or more.
4. Coplanar- lying in a plane.
5. Eliminate – To remove or to omit or to neglect that is not wanted or needed or required.
6. Perpendicular – right angle or at an angle of $90°$ or pointing straight up.
7. Origin – the point from which we start generally at $O = (0,0,0)$.
8. Position vector - situation or location of a point with respect to origin with direction.
9. Axes - plural of axis.
10. e.g. – for example.
11. i.e. - that is.
12. w.r.t.– with respect to
13. Collinear – lying on a straight line.
14. Tetrahedron - a Pyramid/Kyra with triangular base having 4 surfaces (3 lateral & 1 base), 4 vertices, 6 edges etc.

2.12 REFERENCES

1. Geometry of two and three Dimensions- S. Lata, Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. 3 D Geometry -R.K. Pandey, Krishna Prakashan media (P) Ltd., Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal, S.J. Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd., New Delhi, India.
6. Leonard E. Dickson: The First course in the Geometry.
7. Burnside, William Show, Panton & Arthur William: Volumetric Theory.
8. Murray R. Spiege l: Vector Analysis, Schaum's Outline Series, McGraw Hill.
9. P.N.Pandey: Polar Coordinate Geometry, Sharda Academic Publishing House, Allahabad.
10. P.K. Jain and Khalil Ahmed: A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

11. Anton, Howard (1994), Elementary Linear Algebra (7th ed.), John Wiley & Sons, ISBN 978-0- 471-58742-2.
12. Arfken, B. George and Hans J. Weber. Mathematical Methods For Physicists, Academic Press; 6 edition (June 21, 2005). ISBN 978-0-12-059876-2.
13. Brannan, A. David; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press, ISBN 978-0-521-59787-6

2.13 SUGGESTED READINGS

1. Analytical geometry– Nazrul Islam, Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley, Hardy Robert and Sons.
3. Engineering Mathematics- R.D. Sharma, New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

2.14 TERMINAL QUESTIONS

(TQ -1) Derive the relations between the Cartesian coordinates (x, y, z) and cylindrical coordinates $(u,\ \varphi, z)$ of a point.

(TQ -2) Show that the points $(0,7,10), (-1,6,6)$ and $(-4,9,6)$ form an isosceles right angled triangle.

(TQ -3) Find the point which is equidistant from the points $O = (0,0,0)$, $A = (a,0,0), B = (0,b,0)$ and $C = (0,0,c)$.

(TQ -4) Find the ratio in which the yz-plane divides the line joining the points $(3,5,-7)$ and $(-2,1,8)$. Also find the coordinates of the point of division.

(TQ -5) The midpoints of the sides of a triangle are $(1,5,-1), (0,4,-2)$ and $(2,3,4)$. Find the coordinates of its vertices.

(TQ -6) Show that the points $A = (1,2,3), B = (2,3,1)$ and $C = (3,1,2)$ are the vertices of an equilateral triangle.

(TQ -7) A point P moves in the space such that the ratio of its distances from the two fixed points $A = (-2,2,3)$ and $B = (13,-3,13)$ respectively is in the ratio $2:3$ i.e. $PA : PB = 2 : 3$. Find the locus of the point P.

(TQ -8) If the axes are rectangular and the coordinates of the points A and B are $(3,4,5)$ and $(-1,3,-7)$ respectively then find the locus of a variable point P which moves such that (i) $PA = PB$ (ii) $PA^2 - PB^2 = 2k^2$.

2.15 ANSWERS

SELF CHECK QUESTIONS (SCQ'S)

(SCQ –1) b (SCQ–2) c (SCQ – 3) a (SCQ – 4) b

(SCQ – 5) d (SCQ – 6) b (SCQ – 7) a (SCQ – 8) c

(SCQ – 9) c (SCQ – 10) d (SCQ – 11) b (SCQ – 12) a

(SCQ – 13) 2 : 3 (SCQ–14) (-1, 2, 3) (SCQ–15) triangle (SCQ – 16) external.

TERMINAL QUESTIONS (TQ'S)

(TQ – 1) $x = u \cos\varphi, y = u \sin\varphi$ and $z = z$

Also, $u^2 = x^2 + y^2$ and $\tan\varphi = \frac{y}{x}$

(TQ – 2) Show that $AB = BC$ and $AB^2 + BC^2 = AC^2$

(TQ – 3) $\{\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\}$ (TQ – 4) $2 : 3, (0, \frac{13}{5}, 2)$

(TQ – 5) $(1, 2, 3), (3, 4, 5)$ and $(-1, 6, -7)$ (TQ – 6) Prove that $AB = BC = CA$

(TQ – 7) $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

(TQ – 8) (i) $8x + 2y + 24z + 9 = 0$ (ii) $8x + 2y + 24z + 9 + 2k^2 = 0$

UNIT 3: PROJECTION, DIRECTION RATIOS AND DIRECTION COSINES

CONTENTS

3.1 INTRODUCTION

In the previous unit, you should have learnt and studied by now that there exits two types of objects-

1. Which has only length and breadth e.g. rectangle, square etc. These types of objects are called 2-D objects. These types of objects can be drawn in a plane of paper.
2. Which has length, breadth and height e.g. cube, cuboid, cone, cylinder etc. These types of objects are called 3-D objects. These types of objects cannot be drawn in a plane but need space.

Also we have learnt by now that there exits two types of spaces-$2D$ space and $3D$ space.

<u>$2D$ space</u> Two-dimensional space is represented with the $X-$ and $Y-$axes. $3D$ animation adds depth, or the Z-axis.

<u>$3D$ space</u> This three-axes reference system is known as the Cartesian coordinate system. It is used to define many properties of a $3D$ object, including its position, rotation, and scale. For example, an object's position can be represented as $12, 0, 10$ — meaning that it is 12 units to the right of center on the X-axis, 0 units from center along the $Y-$axis, and 10 units along the $Z-$axis.

Also we learnt by now that there exists three coordinate planes namely xy, yz and zx-planes. These three coordinate planes divide the whole space into eight parts called octants. According to the sign convention a point in the space lies only in one octant. Also we have discussed about the planes parallel to the coordinate planes or planes perpendicular to the coordinate axes, the equations of these planes and intersection of the two planes.

We have derived the relations between the Cartesian coordinates and spherical polar coordinates and cylindrical coordinates of a point. We have discussed the method how the Cartesian coordinates can be converted into polar coordinates, cylindrical coordinates and vice-versa. We also learned about a coordinate system in which the axes are curves such type of coordinate system are called curvilinear coordinate system. Also in addition if these curves are right angled then the coordinate system is called orthogonal curvilinear coordinate system. Also we have learnt to calculate the distance between two given points and the distances of a point from the coordinate axes. We discussed to find the coordinates of a point dividing internally or externally in a given ratio. We derived the expressions for the centroid of a triangle and of a tetrahedron.

3.2 OBJECTIVES

After studying this unit, you should be able to -

- Understand the concept of non-coplanar lines in space and the angle between them.
- Understand the relationship between the dr's and the dc's of a line.
- Differentiate between the dr's and the dc's of a line.
- Understand the concept of Lagrange's identity.
- Find the angle between two given lines.
- Find the dr's and the dc's of the join of the two given points.
- Calculate the perpendicular distance of a point from a given line.
- Find the projection of a given point on a given line segment
- Find the projection of a given line on a given line segment.

3.3 ANGLE BETWEEN TWO NON-COPLANAR (NON-INTERSECTING)LINES

We know that the angle between two intersecting lines or coplanar lines means the acute angle $(0 < \theta < 90°)$ between them. But if the lines do not lie in the same plane then the lines do not intersect such lines are called skew or non-coplanar lines. The angle between these lines is defined as follows"The angle between two non-coplanar or non-intersecting lines is equal to the angle drawn through any point in the space and parallel to them." e.g. the angle between the two non-coplanar lines LM and RS is equal to the angle POQ between the lines OP and OQ drawn through the point O and parallel to LM and RS respectively.

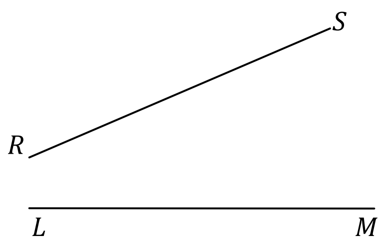


Fig 3.3.1 (a)

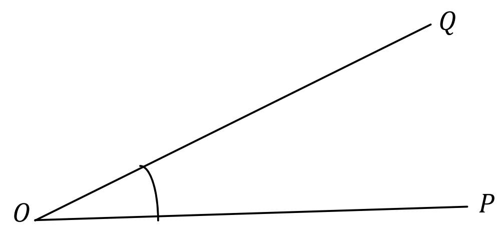


Fig 3.3.1 (b)

3.4 DIRECTION ANGLES

If a line makes angle α with x-axis, angle β with y-axis and angle γ with z-axis, then α, β and γ are said to be direction angles.

3.4.1 TRIGNOMETRICAL RATIO OF ANGLES

1. <u>DIRECTION COSINES OF LINE</u>

If a line has direction angle α, β and γ then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ respectively are said to be direction cosines of the line and represented by l, m, n respectively.

i.e. $l = cos\,\alpha,\ m = cos\,\beta, n = cos\,\gamma$ and expressed by (l, m, n).

Similarly, $(\sin\alpha, \sin\beta, \sin\gamma)$ are direction sines.

$(\tan\alpha, \tan\beta, \tan\gamma)$ are direction tangents and so on.

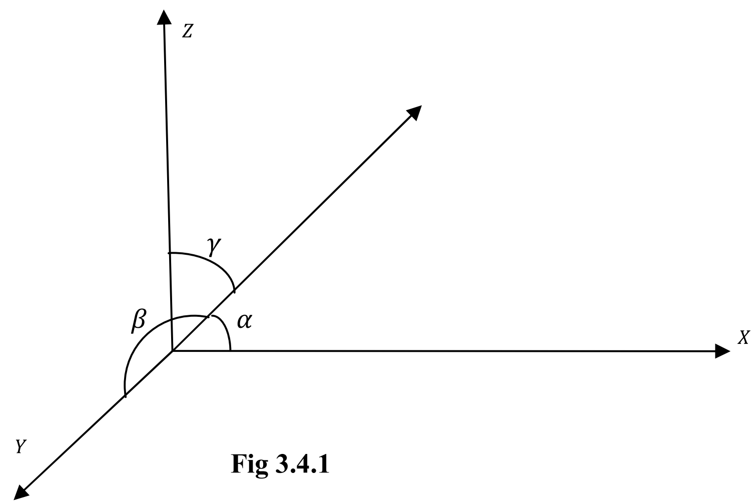


Fig 3.4.1

REMARKS

- Since $l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$ and we know that $-1 < \cos x < 1\ \forall\ x \in R$, so l, m and n are real numbers with values varying between -1 to 1. So, dc's $\in [-1,1]$.
- The angles made by the x −axis with the coordinate axis are $0°, 90°$ and $90°$. Hence, the direction cosines are $\cos 0°$, $\cos 90°$ and $\cos 90°$ i.e. $1, 0, 0$.
- The dc's of x, y and z axis are $(1,0,0), (0,1,0)$ and $(0,0,1)$.
- The direction cosines of a line parallel to any coordinate axis are equal to the direction cosines of the corresponding axis.
- The dc's are associated by the relation $l^2 + m^2 + n^2 = 1$.
- If the given line is reversed, then the direction cosines will be $\cos(\pi - \alpha)$, $\cos(\pi - \beta), \cos(\pi - \gamma)$or $-\cos\alpha, -\cos\beta, -\cos\gamma$.
- Thus, a line can have two sets of dc's according to its direction.
- The direction cosines of two parallel lines are always the same.
- Direction ratios are proportional to direction cosines and hence for a given line, there can be infinitely many direction ratios.
- If a, b and c are the direction ratios, then the direction cosines l, m and n are given by

  direction cosines are $l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$.

Also, since the angles α, β and γ are not coplanar but α, β, and γ lie in the space, hence it is not necessary that

□ $\alpha + \beta + \gamma = 360°$

- We know that for x-axis, we have $l = 1$ and $m = n = 0$. Hence,
  - Projection of PQ on $x -$ axis= $1(x_2 - x_1) + 0(y_2 - y_1) + 0(z_2 - z_1) = (x_2 - x_1)$
  - Projection of PQ on $y -$ axis = $0(x_2 - x_1) + 1(y_2 - y_1) + 1(z_2 - z_1) = (y_2 - y_1)$
  - Projection of PQ on $z -$ axis = $0(x_2 - x_1) + 0(y_2 - y_1) + 1(z_2 - z_1) = (z_2 - z_1)$.
- If a, b, c are the projections of a line segment on the coordinate axis then the length of the segment = $\sqrt{a^2 + b^2 + c^2}$

3.4.2 RELATION BETWEEN THE DIRECTION COSINES

Let OP be any line through the origin O which has direction cosines l, m, n. Let P be the point having coordinates (x, y, z) and $OP = r$ then $OP^2 = x^2 + y^2 + z^2 = r^2$ …. (3.4.1.1)

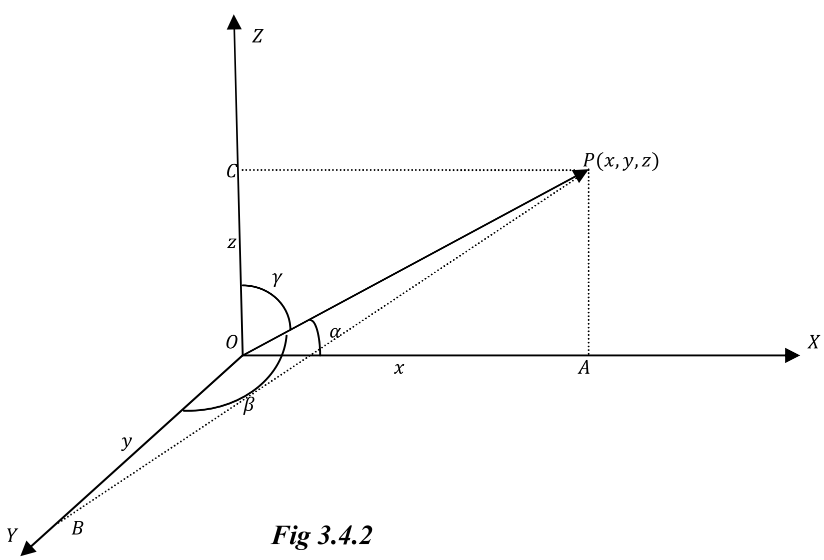


Fig 3.4.2

From P draw PA, PB, PC perpendicular on the coordinate axes, so that

$$OA = x, OB = y, OC = z.$$

Also, $\angle POA = \alpha$, $\angle POB = \beta$ and $\angle POC = \gamma$.

From triangle AOP, $l = cos\,\alpha = \frac{x}{r} \Rightarrow x = lr$

Similarly, $y = mr$ and $z = nr$

Hence from (3.4.1.1)

$r^2 (l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2.$

Hence, we get

$$l^2 + m^2 + n^2 = 1$$

Note-:

A line in the space exist if its satisfies the direction cosine identity

i.e. $l^2 + m^2 + n^2 = 1$

For example line which makes angles 30° with $x-$axis 45° with $y-$axis and 30° with $z-$axis does not exist because $\cos^2 30^\circ + \cos^2 45^\circ + \cos^2 30^\circ \neq 1$.

3.4.3 DIRECTION COSINES OF A LINE JOINING TWO POINTS

If we have two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then the dc's of the line segment joining these two points are

$(x_2 - x_1)/PQ,\ (y_2 - y_1)/PQ\ ,\ (z_2 - z_1)/PQ$

i.e. $(x_2 - x_1)/\sqrt{\Sigma(x_2 - x_1)^2},\ (y_2 - y_1)/\sqrt{\Sigma(y_2 - y_1)^2},\ (z_2 - z_1)/\sqrt{\Sigma(z_2 - z_1)^2}$

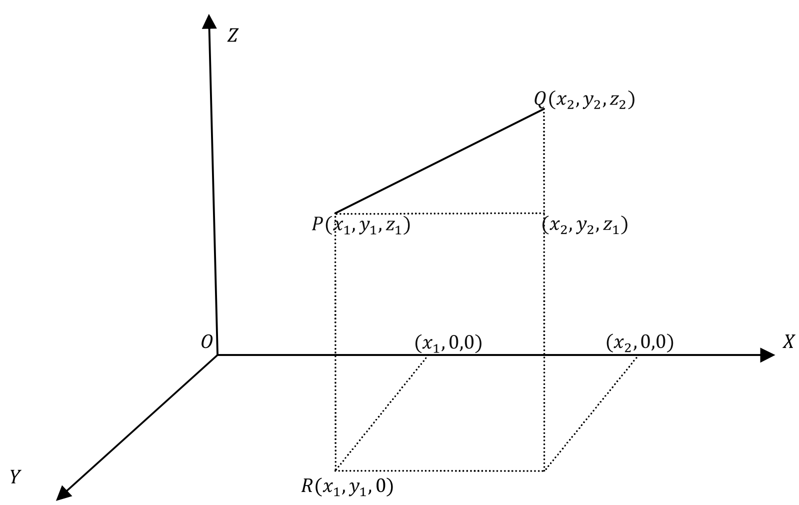


Fig 3.4.3

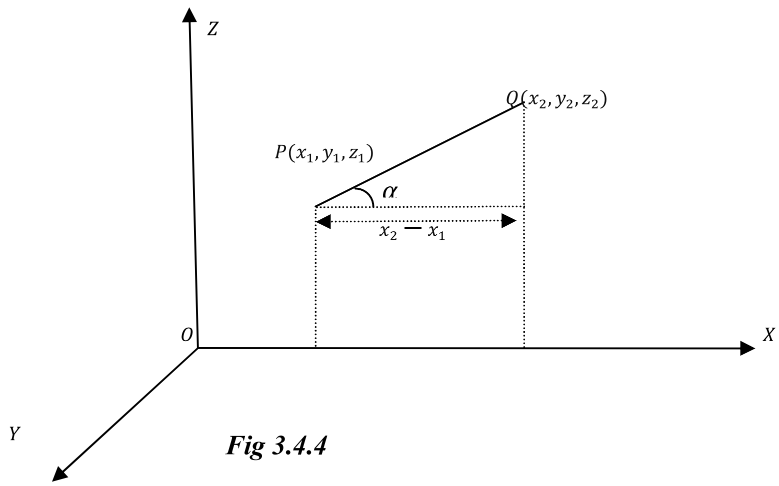


Fig 3.4.4

$\cos\alpha = \frac{x_2 - x_1}{PQ}$

Similarly, $\cos\beta = \frac{y_2 - y_1}{PQ}$

and $\cos\gamma = \frac{z_2 - z_1}{PQ}$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Therefore, Direction cosines of PQ are $\left(\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}\right)$

and Direction ratio of PQ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

For example direction cosine of the line joining the points $(2, 1, 5)$ to $(3, 5, 4)$ are $\left(\frac{3-1}{\sqrt{(3-1)^2+(5-1)^2+(4-5)^2}}, \frac{5-1}{\sqrt{(3-1)^2+(5-1)^2+(4-5)^2}}, \frac{4-5}{\sqrt{(3-1)^2+(5-1)^2+(4-5)^2}}\right) = \left(\frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}\right)$

And direction ratios are $(2, -4, 1)$.

3.5 PROJECTION OF A POINT ON A GIVEN LINE

Projection of a given point on a given line segment is also a point. Suppose P is any given point in the space and AB is a given line segment, then the projection of P on AB is the foot of perpendicular M from the point P on the line segment AB.

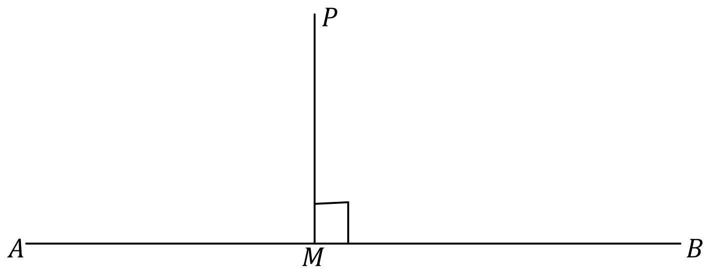


Fig. 3.5.1

3.6 PROJECTION OF A GIVEN LINE SEGMENT ON ANOTHER GIVEN LINE

To find the projection of the line joining two points $P(x_1, y_1, z_1)$and $Q(x_2, y_2, z_2)$ on another line whose dc's are l, m, n.

DERIVATION

Let AB be a given straight line whose direction cosines are l, m and n. Let $\vec{a}$ is a unit vector along AB, then $\vec{a} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$.

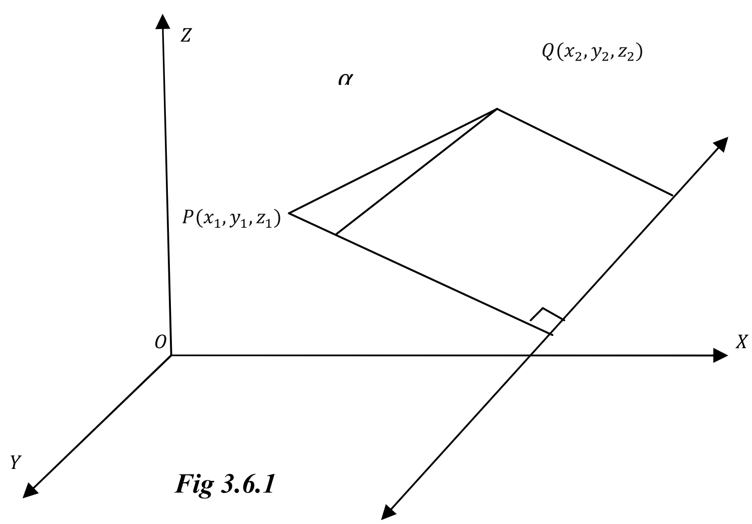


Fig 3.6.1

Let AB be a given straight line whose direction cosines are l, m and n. If $\vec{a}$ is the unit vector along the line AB, then $\vec{a} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$. Let $P(x_1, y_1, z_1)$and $Q(x_2, y_2, z_2)$ are any two points. From the points $P(x_1, y_1, z_1)$and $Q(x_2, y_2, z_2)$, we draw perpendiculars on the line AB

which meet the line AB at the points M and N respectively. Then MN is the projection of the line segment PQ on the line AB. Now from the point P we draw a line parallel to the line AB which meets the perpendicular QN at point R.

Then clearly, $MN = PR$.

Suppose θ is the angle between the line segment PQ and the line AB. Now since PR is parallel to the line AB, hence $\angle QPR = \theta$.

Now, projection of the line segment PQ on the line $AB = MN = PR = \boldsymbol{PQ} \cos\theta$

$$= |\overrightarrow{PQ}\,| \cos\theta$$

$$= \overrightarrow{PQ}\;.\;\vec{a} = (\overrightarrow{OQ} - \overrightarrow{OP})\;.\;\vec{a},$$

where $\vec{a}$ is the unit vector in the direction of the given line AB or in the direction of PR.

$$\cos\theta = \frac{(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n}{|\overrightarrow{PQ}|}$$

$$\Rightarrow |\overrightarrow{PQ}| \cos\theta = \{(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})\}.(l\hat{\imath} + m\hat{\jmath} + n\hat{k})$$

Hence, $MN = PR = \{(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})\}.(l\hat{\imath} + m\hat{\jmath} + n\hat{k})$

$$= \{(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}\}.(l\hat{\imath} + m\hat{\jmath} + n\hat{k})$$

$$= (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$

Hence the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line AB whose dc's are (l, m, n) is given by $(x_2 - x_1)\;l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$.

REMARKS

- If the projection of a line AB on another line CD is zero then the two lines will be mutually perpendicular i.e. $AB \perp CD$ and vice-versa.
- The projection of the line joining two points $P(x_1, y_1, z_1)$and $Q(x_2, y_2, z_2)$ on another line AB whose dr's are (a, b, c) is given by $\frac{(x_2 - x_1)\,a + (y_2 - y_1)\,b + (z_2 - z_1)\,c}{\sqrt{a^2 + b^2 + c^2}}$.

SOLVED EXAMPLES

EXAMPLE (1)

If a line makes equal angles with the coordinate axes i.e equally inclined with the coordinate axes then find the dc's of this line.

SOLUTION

If α, β and γ are the angles made by the line with the coordinate axes then the direction cosines of the line are given by l, m and n.

i.e.$l = \cos\alpha, m = \cos\beta, \ n = \cos\gamma$

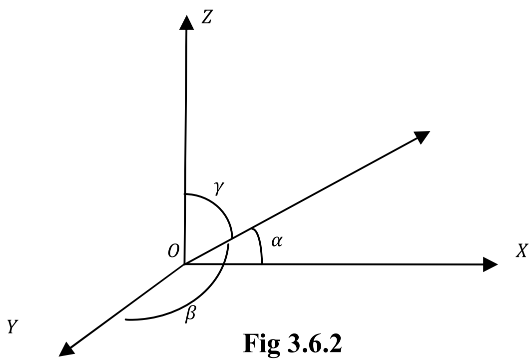


Fig 3.6.2

But here α, β and γ are equal i.e. $\alpha = \beta = \gamma$

Also we know that the **relation between the Direction Cosines of a line is given by,**

$$l^2 + m^2 + n^2 = 1$$

Hence we get, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$ i.e. $3\cos^2\alpha = 1$

$$\cos^2\alpha = \frac{1}{3} \text{i.e. } \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Hence the dc's of the required line are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.

EXAMPLE-(2)

A line with positive direction cosines passes through the point $P(2,-1,2)$ and makes equal angles with the coordinate axis. The line meets the plane $2x + y + z = 9$ at point Q. Find the length of the line segment PQ.

SOLUTION

The direction cosines are $l = m = n = \frac{1}{\sqrt{3}}$.

Hence, the equation of the required line is

$$\frac{(x-2)}{1/\sqrt{3}} = \frac{(y+1)}{1/\sqrt{3}} = \frac{(z-2)}{1/\sqrt{3}}$$

Hence, this gives $x-2 = y+1 = z-2 = r$

Hence, any point on the line is $Q = (r+2, r-1, r+2)$.

Since Q lies on the plane $2x + y + z = 9$

Therefore $2(r+2) + (r-1) + (r+2) = 9$

This yields $4r + 5 = 9$ or $r = 1$.

Hence the coordinates of Q are $(3,0,3)$.

Hence, $PQ = \sqrt{(3-2)^2 + (0+1)^2 + (3-2)^2} = \sqrt{3}$

EXAMPLE-(3)

Find the direction cosines l, m, n of two lines connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.

SOLUTION

In order to compute the values of l, m and n from the given relations, we shall first solve these equations.

We have $l - 5m + 3n = 0 \Rightarrow l = 5m - 3n$

Substituting this value in the second equation we have

$$7(5m-3n)^2 + 5m^2 - 3n^2 = 0$$

Hence, $30(2m-n)(3m-2n) = 0$ i.e. $2m = n$ and $3m = 2n$.

Therefore, $\frac{m}{1} = \frac{n}{2} = \frac{(5m-3n)}{5} - 2.3 = \frac{l}{(-1)} = \frac{1}{\sqrt{6}}$

and $\frac{m}{2} = \frac{n}{3} = \frac{(5m-3n)}{(5.2-3.3)} = \frac{l}{1} = \frac{1}{\sqrt{14}}$

Hence, the required dc's of the line are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ and $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

EXAMPLE(4)

If the direction ratios of a line are 0, 2 and -3, then what are the direction cosines?

SOLUTION

We know that if a, b and c are the direction ratios, then the direction cosines l, m and n are given by

Direction cosines are $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

Hence, the direction cosines are

$$\frac{0}{\sqrt{(0)^2+(2)^2+(-3)^2}}, \frac{2}{\sqrt{(0)^2+(2)^2+(-3)^2}}, \frac{-3}{\sqrt{(0)^2+(2)^2+(-3)^2}}$$

which gives $\left(0, \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right)$.

EXAMPLE -(5)

Find the length of a line segment of a line whose projections on the axes are $2, 3, 6$.

SOLUTION

SupposePQ is a segment of a line whose direction cosines are l, m, n. Then we have,

2 $=$ the projection of PQ on the x $-$axis $=$ $l.PQ$……………………..(1)

3 $=$ the projection of PQ on the y $-$axis $=$ $m.PQ$ …………………...(2)

6 $=$ the projection of PQ on the z $-$axis $=$ $n.PQ$ …………………....(3)

Squaring the equations (1), (2) and (3) and then adding we get,

$$4+9+36 = (l^2+m^2+n^2).PQ^2$$

$$\Rightarrow 49 = 1.PQ^2$$

$$\Rightarrow PQ = 7$$

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ-1) If l, m and n are the dc's of a line then,

(a) $l + m + n = 1$ (b) $l^2 + m^2 + n^2 = 1$ (c) $l + m = n^2$ (d) $1 + m + n = 3$

(SCQ-2) The dc's of x $-$axis are

(a) $(0, 0, 0)$ (b) $(0, 0, 1)$ (c) $(0, 1, 0)$ (d) $(1, 0, 0)$

(SCQ-3) If the given line AB is reversed, then the direction cosines of the line BA will be

(a) same in magnitude but opposite in sign (b) same in magnitude and sign

(c) opposite in sign and reversed in magnitude (d) None of these

(SCQ-4) Projection of a point on a given straight line will be a

(a) point (b) straight line (c) curve (d) None of these

(SCQ-5) The projection of the line joining the points $(7,5,0)$ and $(-5,2,7)$ on $y-$ axis is

(a) -3 (b) 3 (c) 9 (d) 25

(SCQ-6) If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the dc's of a straight line, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ will be equal

(a) 1 (b) 0 (c) 2 (d) 3

(SCQ-7) The dc's of a line cannot be

(a) $(1,1,0)$ (b) $(0,0,1)$ (c) $(0,1,0)$ (d) $(1,0,0)$

(SCQ-8) If the given line AB is reversed, then the direction cosines of the line BA will be

(a) same in magnitude but opposite in sign (b) same in magnitude and sign

(c) opposite in sign and reversed in magnitude (d) None of these

(SCQ-9) Projection of a line on a given straight line in shape will be a

(a) point (b) straight line (c) curve (d) None of these

(SCQ-10) The projection of the line joining the points $(9,5,10)$ and $(-8,2,1)$ on $z-$ axis is

(a) -3 (b) 3 (c) 9 (d) 25

(SCQ-11) The projection of the line joining the points $(9,15,10)$ and $(-8,-10,1)$ on y −axis is

(a) -3 (b) 3 (c) 9 (d) 25

(SCQ-12) If O be the origin and $P = (3,8,0)$ and $Q = (8,k,6)$ are two points such that OP is perpendicular to OQ, then the value of k is

(a) -3 (b) 3 (c) 9 (d) 5

3.7 LAGRANGE'S IDENTITY

By actual multiplication and regrouping, it can be easily proved that-

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2$$
$$= (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$$

This identity is called "Lagrange's Identity". With the help of this identity the calculation becomes easier and is found very useful in simplification of many results.

NOTE-:

1. Angle between two vectors

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

2. Angle between unit vectors

$$\cos\theta = \hat{a}.\hat{b}$$

3. If l, m and n be dc's or a line then unit vector in the direction of line is given by

$$\hat{a} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$$

3.8 ANGLE BETWEEN TWO LINES OR VECTORS

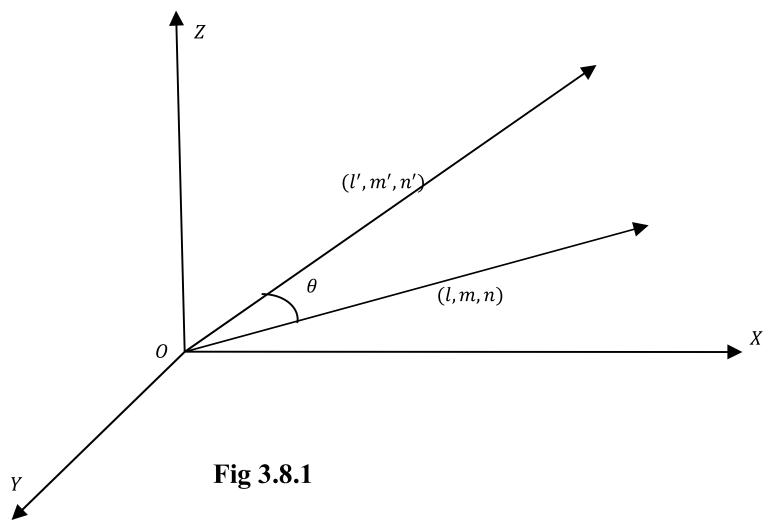


Fig 3.8.1

Angle between two lines is the sum as angle between two unit vectors in the directions of these lines.

Therefore, $\cos\theta = \widehat{OP}.\widehat{OQ}$

$$= ll' + mm' + nn'$$

Let L_1 and L_2 represent two lines having the direction ratios as a_1, b_1, c_1 and a_2, b_2, c_2 respectively such that they are passing through the origin. Let us choose a random point A on the line L_1 and B on the line L_2. Considering the directed lines OA and OB as shown in the figure given below, let the angle between these lines be θ.

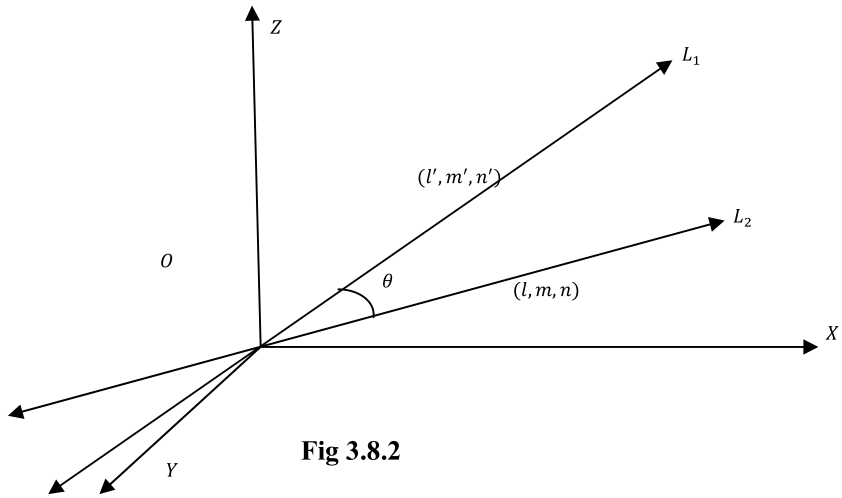


Fig 3.8.2

Using the concept of direction cosines and direction ratios, the angle θ between L_1 and L_2 is given by:

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

In terms of $\sin\theta$, this expression can be rewritten as follows:

$\sin\theta = \sqrt{1 - (\cos\theta)^2}$ hence

$$\sin\theta = \sqrt{1 - \left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)^2}$$

$$= \sqrt{\frac{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$$

Therefore $\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$ (Using Lagrange's identity given in the article 3.7)

Special Cases

- If L_1 and L_2 having the direction ratios as a_1, b_1, c_1 and a_2, b_2, c_2 respectively are perpendicular to each other, then $\theta = 90^\circ$. Therefore,

$$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

If L_1 and L_2 having the direction ratios as $a_1, b_1, c_1$and a_2, b_2, c_2 respectively are parallel to each other, then then $\theta = 0^\circ$. Therefore,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

3.9 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The distance (or perpendicular distance) from a point to a line is the shortest distance from a fixed point to any point on a fixed infinite line in Euclidean geometry. It is the length of the line segment which joins the point to the line and is perpendicular to the line. The formula for calculating it can be derived and expressed in several ways.

Knowing the shortest distance from a point to a line can be useful in various situations—for example, finding the shortest distance to reach a road, quantifying the scatter on a graph, etc. In Deming regression, a type of linear curve fitting, if the dependent and independent variables have equal variance this results in orthogonal regression in which the degree of imperfection of the fit is measured for each data point as the perpendicular distance of the point from the regression line.

Let AB be the straight line passing through the point $A(a, b, c)$ and having direction cosines $l, m,$ and n. Now, if AN is assumed to be the projection of line AP on the straight line AB then we have

$$AN = l(x - a) + m(y - b) + n(z - c)$$

and $AP = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$

Therefore, perpendicular distance PN of point P from the line passing through the point A is given by

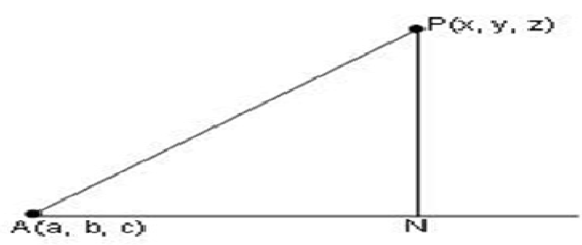


Fig. 3.9.1

Here $PN = \sqrt{AP^2 - AN^2}$

3.10 RELATIONS BETWEEN THE DIRECTION COSINES OF THREE MUTUALLY PERPENDICULAR LINES

Let OX, OY and OZ be the given set of mutually orthogonal coordinate axes and OX', OY' and OZ' are three mutually perpendicular lines through O. Let (l_1, m_1, n_1), $(l_2, m_2, n_2)$and (l_3, m_3, n_3) are the direction cosines of OX' , OY' and OZ' respectively with respect to OX, OY and OZ. Then we have the following six relations-

$$l_1^2 + m_1^2 + n_1^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1$$
$$l_3^2 + m_3^2 + n_3^2 = 1$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$
$$l_3 l_2 + m_3 m_2 + n_3 n_2 = 0$$

Also the direction cosines of OX, OY and OZ with respect to OX', OY' and OZ' are given by (l_1, m_1, n_1), $(l_2, m_2, n_2)$and(l_3, m_3, n_3) respectively. Hence we also have the following six relations

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$m_1 n_1 + \ m_2 n_2 + m_3 n_3 = 0$$
$$l_1 n_1 + \ l_2 n_2 + l_3 n_3 = 0$$

REMARKS

a. If three lines are mutually perpendicular, then we can find the direction cosines of one line in terms of the other two lines.

DERIVATION

If the lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are mutually perpendicular, then we have

$$\begin{aligned} {l_1}^2 + {m_1}^2 + {n_1}^2 &= 1 \\ {l_2}^2 + {m_2}^2 + {n_2}^2 &= 1 \\ {l_3}^2 + {m_3}^2 + {n_3}^2 &= 1 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 &= 0 \\ l_3 l_2 + m_3 m_2 + n_3 n_2 &= 0 \end{aligned}$$

Solving the equations,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

for l_1, m_1, n_1 by Cross multiplication method, we get

$$\frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{l_3 n_2 - l_2 n_3} = \frac{n_1}{l_2 m_3 - l_3 m_2}$$

$$= \frac{\sqrt{{l_1}^2 + {m_1}^2 + {n_1}^2}}{\sqrt{(m_2 n_3 - m_3 n_2)^2 + (l_3 n_2 - l_2 n_3)^2 + (l_2 m_3 - l_3 m_2)^2}}$$

$$= \frac{\sqrt{1}}{\sqrt{\sin 90^\circ}} = \pm 1$$

(By the definition of angle between two straight lines , as we have learnt in the unit of straight line), Hence we have

$$\begin{aligned} l_1 &= \pm(m_2 n_3 - m_3 n_2) \\ m_1 &= \pm(l_3 n_2 - l_2 n_3) \\ n_1 &= (l_2 m_3 - l_3 m_2) \end{aligned}$$

Similarly, we can find the direction cosines of other two lines in terms of the remaining one by using the determinant given below

$$D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Where the dc's of any constituent in D = ±(its cofactor).

b. To prove that $D = \pm 1$,where D is defined as in remark (2).

PROOF we have

$$D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Squaring above we get,

$$D^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} . \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

(Multiplying the determinants in row by row pattern, we get

$$D^2 = \begin{vmatrix} {l_1}^2 + {m_1}^2 + {n_1}^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 & {l_2}^2 + {m_2}^2 + {n_2}^2 & l_3 l_2 + m_3 m_2 + n_3 n_2 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 & l_3 l_2 + m_3 m_2 + n_3 n_2 & {l_3}^2 + {m_3}^2 + {n_3}^2 \end{vmatrix}$$

Now,

$${l_1}^2 + {m_1}^2 + {n_1}^2 = 1$$

$${l_2}^2 + {m_2}^2 + {n_2}^2 = 1$$

$${l_3}^2 + {m_3}^2 + {n_3}^2 = 1$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

$$l_3 l_2 + m_3 m_2 + n_3 n_2 = 0$$

Now using the results given above,we get,

$$D^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Solving we get, $D^2 = 1$.

Taking square root on both sides we get, $D = \pm 1$.Hence proved

SOLVED EXAMPLES

EXAMPLE -(1)

Find the perpendicular distance of point $P(0, 2, 3)$ from straight line passing through $A(\ 1, -3,2)$ given that the direction ratios are $(1, 2, 2)$.

SOLUTION

We know that if we have the direction ratios, then we can easily compute the direction cosines.

If 'a', 'b', and 'c' are the direction ratios, then the direction cosines are given by

$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$.

Here, we have the direction ratios as $a = 1, b = 2, c = 2$.

Hence, the direction cosines are $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Therefore, $PN = l(x - a) + m(y - b) + n(z - c)$

$$= \frac{1}{3}(0 - 1) + \frac{2}{3}(2 + 3) + \frac{2}{3}(3 - 2)$$

$$= -\frac{1}{3} + \frac{10}{3} + \frac{2}{3} = \frac{11}{3}$$

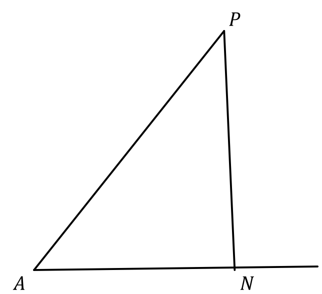


Fig. 3.10.1

$$AP = \sqrt{(0-1)^2 + (2 + 3)^2 + (3-2)^2} = \sqrt{27}$$

Therefore, Perpendicular distance $PN = \sqrt{AP^2 - PN^2} = \sqrt{27 - \frac{121}{9}} = \sqrt{\frac{122}{9}} = \frac{\sqrt{122}}{3}$.

EXAMPLE-(2)

Show that the equation to the right circular cone whose vertex is at the origin, whose axes has direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ and whose semi-vertical angle is θ, is given by'

$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = \sin^2\theta\ (x^2 + y^2 + z^2)$

SOLUTION

Let$P = (x, y, z)$ is any point on the surface of the cone. Then

$OP^2 = x^2 + y^2 + z^2$……………………………………………………..(1)

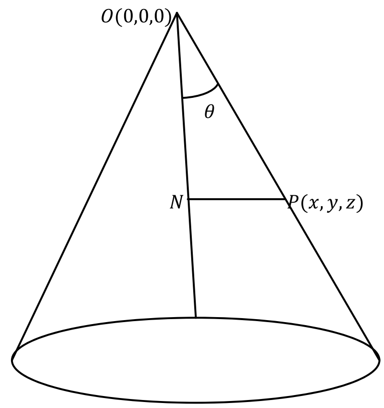


Fig. 3.10.2

ON = Projection of OP on the axis of cone

$$= x\cos\alpha + y\cos\beta + z\cos\gamma$$

Now PN = perpendicular distance of P from the axis of the cone,

$$= \sqrt{OP^2 - ON^2}$$

Hence

$$PN^2 = (x^2 + y^2 + z^2) - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$$

$$= (x^2 + y^2 + z^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$$

(Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$)

Now, using Lagrange's identity, i.e.

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2$$

$$= (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$$

we get,

$$PN^2 = (y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 \quad \ldots\ldots\ldots\ldots(2)$$

But $PN = OP\ \sin\theta$

Squaring above and using equations (1) and (2) we get,

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = \sin^2\theta\ (x^2 + y^2 + z^2)$$

This is the required equation of the cone.

SELF CHECK QUESTIONS

CHOOSE THE CORRECT OPTION.

(SCQ-13) The identity

$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2 = (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (b_2 - a_2b_1)^2$ is given by

(a) Newton (b) Euler (c) Lagrange (d) Pythagoras

(SCQ-14) If the vertices of a triangle are $A = (1, 4, 2)$, $B = (-2, 1, 2)$ and $C = (2, -3, 4)$ then the angle B is equal to

(a) $\cos^{-1}(\sqrt{3})$ (b) $\cos^{-1}(\frac{1}{\sqrt{3}})$ (c) $\cos^{-1}(\frac{2}{5})$ (d) $\frac{\pi}{2}$

(SCQ-15) The direction cosines of a line which is equally inclined to the positive direction of coordinate axes are

(a) [1, 0, 0] (b) $[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ (c) $[\frac{1}{3,}, \frac{1}{3,}, \frac{1}{3,}]$ (d) None of these

(SCQ-16) If $P(x, y, z)$ is any point in the space and the line OP makes an angle β with y −axis then

(a) $r^2 + y^2 = sin\,\beta$ (b) $r^2 + y^2 = cos\,\beta$ (c) $y = r\,cos\,\beta$ (d) $y = r\,sin\,\beta$

(SCQ-17) If the direction cosines of a line are (k, k, k) then the value of k is

(a) 1 (b) $\pm\frac{1}{\sqrt{3}}$ (c) $\frac{1}{3}$ (d) $\pm\frac{1}{2}$

(SCQ-18) If the projections of a line on the coordinate axes are $3, 6, 6$ then the length of the line will be

(a) 5 (b) 15 (c) 81 (d) 9

(SCQ-19) $l = m = n = 1$ represent the direction cosines of

(a) x-axis (b) y-axis (c) z-axis (d) None of these

FILL IN THE BLANKS

(SCQ-20) If the lines with dr's (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular then……………..

(SCQ-21) If the projection of a line AB on another line CD is zero, then the two lines will be…………………

(SCQ-22) If the lines with dr's (a_1,b_1,c_1) and (a_2,b_2,c_2) are parallel then………………………

(SCQ-23) The projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line AB whose dr's are (a, b, c) is given by………………………………….

(SCQ-24) The identity given by

$(a_1^2+b_1^2 + c_1^2)\,(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2 =$

$(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$is known as…….

3.11 SUMMARY

In this unit, we have learned about the difference between the direction ratios and the direction cosines of a line in three dimensional coordinate system. Also we have learned about the dc's of three mutually perpendicular axes (XOX', $YOY'$and ZOZ') as a special case. Here we found that the direction cosines of a line are unique but direction ratios can be written in infinite many ways. Also we have learned to check whether the given numbers are the direction cosines or direction ratios of a line by using the property that sum of squares of the direction cosines of a line is always unity but this property is not satisfied by direction ratios. Also we have learnt the method to convert the direction ratios of a line into direction cosines and the method to find the angle between two non-coplanar or coplanar lines. Here we learnt a new identity known as the "Lagrange's identity". We have derived the expression for the projection of a point and a line segment on a given line. We have studied the relations between the dc's of three mutually perpendicular lines. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics. To check your progress, self check questions (SCQ's) are given place to place.

3.12 GLOSSARY

1. Line segment – a line of finite length having two end points.
2. dr's – direction ratios or numbers proportional to direction cosines.
3. dc's – direction cosines.
4. Vertex – where two edges meet.
5. Mutually – shared by two or more.
6. Coplanar- lying in a plane.
7. Eliminate – To remove or to omit or to neglect that is not wanted or needed or required.
8. Perpendicular – right angle or at an angle of $90°$ or pointing straight up.
9. Origin – the point from which we start generally at O = (0,0.0).
10. Position vector - situation or location of a point with respect to origin with direction.
11. Axes - plural of axis.
12. e.g. – for example.
13. Tetrahedron - a Pyramid/Kyra with triangular base having 4 surfaces (3 lateral & 1 base), 4 vertices, 6 edges etc.
14. i.e. - that is.
15. w.r.t. – with respect to
16. Collinear – lying on a straight line.
17. Identity – an equation true for all values of the variables.

3.13 REFERENCES

1. Geometry of two and three Dimensions- Lata S. , Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. $3D$ Geometry- R.K. Pandey, Krishna Prakashan media (P) Ltd., Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd., New Delhi, India.
6. Leonard E. Dickson-The First course in the Geometry.
7. Burnside,William Show, Panton &Arthur William-Volumetric Theory.
8. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series,McGraw Hill.
9. P.N .Pandey :Polar Coordinate Geometry, Sharda Academic Publishing House, Allahabad.
10. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.
11. *Anton, Howard (1994), Elementary Linear Algebra (7th ed.), John Wiley & Sons,* ISBN 978-0- 471-58742-2.

12. Arfken, George B. and Hans J. Weber. Mathematical Methods For Physicists, Academic Press; 6 edition (June 21, 2005). ISBN 978-0-12-059876-2.
13. *Brannan, David A.; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press,* ISBN 978-0-521-59787-6

3.14 SUGGESTED READINGS

1. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
3. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

3.15 TERMINAL QUESTIONS

(TQ-1) If α, β and γ are the angles which a straight line makes with the positive direction of the axes then prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(TQ-2) If a line makes angles α, β, γ and δ with the four diagonals of a cube, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

(TQ-3) If two pairs of opposite edges of a tetrahedron be at right angles, then show that so is the third.

(TQ-4) Find the angle between any two diagonals of a cube.

(TQ-5) Show that the acute angle between the lines whose direction cosines are given by the equations, $l + m + n = 0, l^2 + m^2 - n^2 = 0$ is $\frac{\pi}{3}$.

(TQ-6) Find the projection of CD on AB, where the coordinates of the points A, B, C and D are given by $A = (3,4,5), B = (4,6,3), C = (-1,2,4)$ and $D = (1,0,5)$.

(TQ-7) If a variable line in two adjacent positions has direction cosines (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$. Show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$

(TQ-8) If the lines with direction cosines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are mutually perpendicular, then show that the line with direction ratios $l_1 + m_1 + n_1$, $l_2 + m_2 + n_2$ and $l_3 + m_3 + n_3$ makes equal angle with them.

(TQ-9) Find the direction cosines of the line whose direction ratios are $(1,1,1)$.

(TQ-10) Show that the three points $A = (2,-1,3), B = (4,3,1)$ and $C = (3,1,2)$ are collinear by using the concept of direction ratios.

3.16 ANSWERS

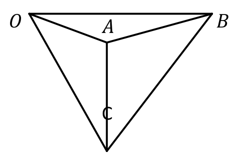
SELF CHECK QUESTIONS (SCQ'S)

(SCQ-1) b (SCQ-2) d (SCQ-3) a (SCQ-4) a
(SCQ-5) b (SCQ-6) c (SCQ-7) a (SCQ-8) a
(SCQ-9) b (SCQ-10) c (SCQ-11) d (SCQ-12) a
(SCQ-13) c (SCQ-14) d (SCQ-15) b (SCQ-16) c
(SCQ-17) b (SCQ-18) d (SCQ-19) d
(SCQ-20) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (SCQ-21) perpendicular
(SCQ-22) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$(SCQ-23) $\frac{(x_2-x_1)a + (y_2-y_1)\, b + (z_2-z_1)c}{\sqrt{a^2+b^2+c^2}}$

(SCQ-24) Lagrange's identity

TERMINAL QUESTIONS (TQ'S)

(TQ-3)



[Here $(OA, BC), (OB, CA)$ and (OC, AB) are pairs of opposite edges. Consider any two pairs mutually perpendicular and prove the third one.]

(TQ – 4) $\cos^{-1}(\frac{1}{3})$

(TQ – 6) - $\frac{4}{3}$

(TQ – 9) $[\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}]$

(TQ – 10) Prove that dr's of AB and AC are proportional

BLOCK II
PLANE AND STRAIGHT LINE

UNIT 4: THE PLANE I

CONTENTS

4.1 INTRODUCTION

Plane- A plane is a surface such that every straight line joining any two points on it lies wholly on it.

OR

Plane is the locus of all point such that line joining to these points is always perpendicular to a fixed line vector, locus of point is called plane and perpendicular line is called normal to the plane.

In previous unit we analyzed Projection, Direction ratio's and Direction Cosine's. In this unit we will discussed about plane and different form of the equation of a plane.

4.2 OBJECTIVES

The main objective of this unit is to learn following contents.

- General equation of a plane.
- Equation of a plane passing through origin.
- Equation of a plane passing through one point.
- Equation of a plane passing through three points.
- Normal form of the equation of a plane
- To reduce the general equation of the plane to the normal form
- Equation of a plane in intercept form
- Equation of coordinate plane
- Equation of the plane parallel two the coordinate planes
- Equation of a plane through a given point and perpendicular to a given line.
- Equation of a plane parallel to axis

4.3 GENERALEQUATION OF A PLANE

In this section we will learn about the equation of a plane.

4.3.1 EQUATION OF A PLANE PASSING THROUGH A POINT AND PERPENDICULAR TO A GIVEN LINE

Let plane is passing through a fixed point A(x_1, y_1, z_1) and normal vector to the plane is $\bar{n}\{a, b, c\}$ where a, b, c are direction ratio to the $\bar{n}$.

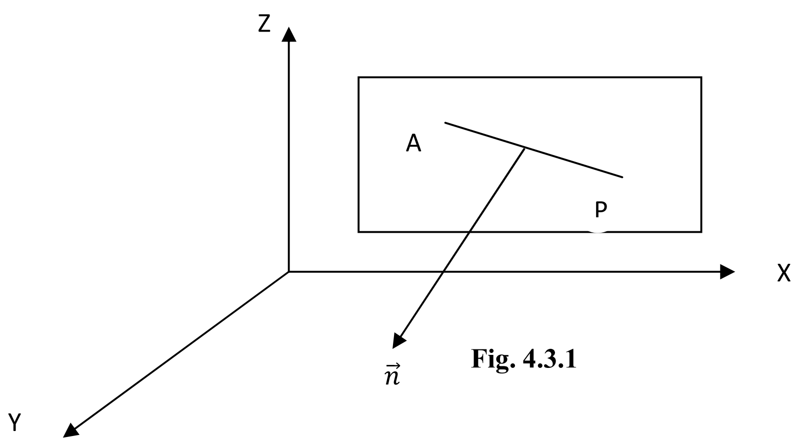


Fig. 4.3.1

Let $P(x, y, z)$ be the general point on the plane then direction ratio of line $AP = \{x - x_1, y - y_1, z - z_1\}$

as $\bar{n}$ is perpendicular to $\vec{AP}$

$\Rightarrow \vec{AP}.\bar{n} = 0$

i.e. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow ax + by + cz = ax_1 + by_1 + cz_1$

$\Rightarrow ax + by + cz + d = 0$

4.3.2 GENERAL EQUATION OF PLANE

Statement: The linear equation in three variables that is ax + by + cz + d = 0 always represent a plane, where a, b, c, d are constants.

Proof Suppose (x_1, y_1, z_1) and $(x_2, y_2\ z_2)$ are two points lying on the plane,

ax + by + cz + d = 0, therefore points (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfying the equation of the plane

$ax_1 + by_1 + cz_1 + d = 0$ .... (1)

$ax_2 + by_2 + cz_2 + d = 0$ .... (2)

Multiplying equation (1) by $\dfrac{1}{1+m}$ and (2) by $\dfrac{m}{1+m}$

then adding equation (1) and (2), we get

$$a\left(\frac{mx_2 + x_1}{m+1}\right) + b\left(\frac{my_2 + y_1}{m+1}\right) + c\left(\frac{mz_2 + z_1}{m+1}\right) + d\left(\frac{1}{1+m} + \frac{m}{1+m}\right) = 0$$

$$a\left(\frac{mx_2 + x_1}{m+1}\right) + b\left(\frac{my_2 + y_1}{m+1}\right) + c\left(\frac{mz_2 + z_1}{m+1}\right) + d = 0 \qquad \text{.................. (3)}$$

Equation (3) implies that point $\left(\dfrac{mx_2 + x_1}{m+1}, \dfrac{my_2 + y_1}{m+1}, \dfrac{mz_2 + z_1}{m+1}\right)$

lies on the plane. This point is internal point of straight line joining the point (x_1, y_1, z_1) and (x_2, y_2, z_2), which divide the straight line in the ratio m:1, for any value of m except m = 1.

Note: Number of constant in general equation of plane

General equation of plane is

ax + by + cz + d = 0

If $d \neq 0$, dividing above equation by d then we get

$$\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + \frac{d}{d} = 0$$

Ax + By + Cz + 1 = 0

which implies that there are three constant in the equation of a plane. Therefore three conditions are necessary to find these three constants.

4.4 EQUATION OF A PLANE PASSING THROUGH ORIGIN

General equation of a plane is

ax + by + cz + d = 0

If above plane passes through origin then (0, 0, 0) satisfying the equation of the plane.

$a.0 + b.0 + c.0 + d = 0 \Rightarrow d = 0$

Putting d = 0 in general equation of the plane, we get

ax + by + cz = 0,

which is required equation of plane passes through origin

4.5 EQUATION OF A PLANE PASSING THROUGH ONE POINT

Let equation of plane be

ax + by + cz + d = 0 .... (1)

If it passes through (x_1, y_1, z_1) ,then

$ax_1 + by_1 + cz_1 + d = 0$ .... (2)

Subtracting (2) from (1), we get required equation of plane, that is

$a(x- x_1) + b(y - y_1) + c(z - z_1) = 0$

4.6 EQUATION OF A PLANE PASSING THROUGH THREE POINTS

Equation of plane passing through the points $A\,\vec{a}(x_1, y_1, z_1),\ B\,\vec{b}(x_2, y_2, z_2),\ C\,\vec{c}(x_3, y_3, z_3)$

Where $\vec{a}$, $\vec{b}$, $\vec{c}$ are position vectors of *A*, *B* and *C* respectively.

Then $\vec{AB} = \vec{b} - \vec{a} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\vec{AC} = \vec{c} - \vec{a} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$

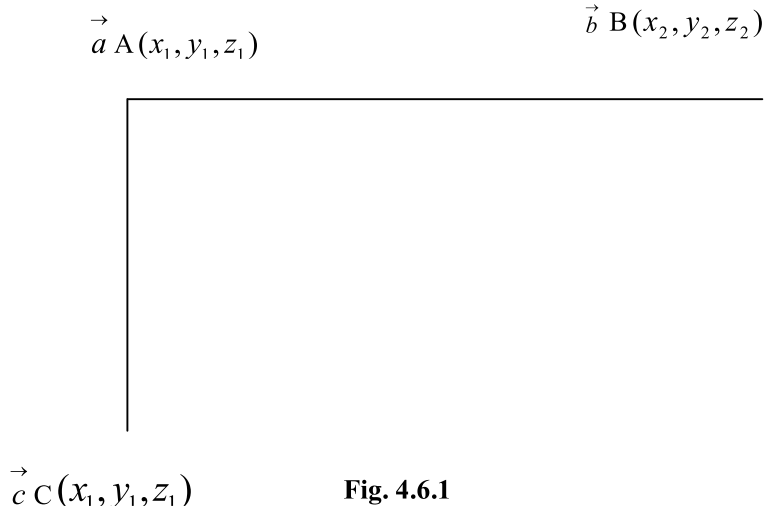


Fig. 4.6.1

Normal vector to the plane will be

$$\vec{n} = \vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$\therefore$ Equation of the plane is

$$(\vec{r} - \vec{a}).\vec{n} = 0$$

i.e. $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Alternate Method:

To find the equation of a plane passes through three points whose coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let the general equation of the plane be

$ax + by + cz + d = 0$ .... (1)

If the equation of plane passes through the given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) the coordinates of these points satisfying equation (1) of the plane, so that we have-

$ax_1 + by_1 + cz_1 + d = 0$ .... (2)

$ax_2 + by_2 + cz_2 + d = 0$ .... (3)

$ax_3 + by_3 + cz_3 + d = 0$ .... (4)

Eliminating a, b, c and d from equations (1), (2), (3) and (4), the equation of the required plane is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad .... (5)$$

Note If four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are coplanar that is lying in the same the plane then we have

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

4.7 NORMAL FORM OF THE EQUATION OF A PLANE

General equation of the plane is given by

$$\vec{r}\,.\,\vec{n}=\vec{a}\,.\,\vec{n}$$

As $\vec{a}\,.\,\vec{n}$ = constant = k

$$\vec{r}\,.\,\vec{n}=k$$

If we write $\vec{n}$ in terms of $\hat{n}$ (unit normal) then write side constant is p.

i.e. $\vec{r}\,.\,\hat{n}\left|\vec{n}\right|=k$

$$\Rightarrow \vec{r}\,.\,\hat{n}=\frac{k}{\left|\vec{n}\right|}=\frac{\vec{a}.\vec{n}}{|\vec{n}|}=\frac{\vec{a}}{\left|\vec{n}\right|}.\vec{n}=\vec{a}\,.\,\hat{n}$$

$$\vec{r}\,.\,\hat{n}=p$$

p is called $\perp$ distance from origin to the plane.

In other way, lx+ my+ nz = p, where {l, m, n} are direction cosine to the normal vectors of the plane.

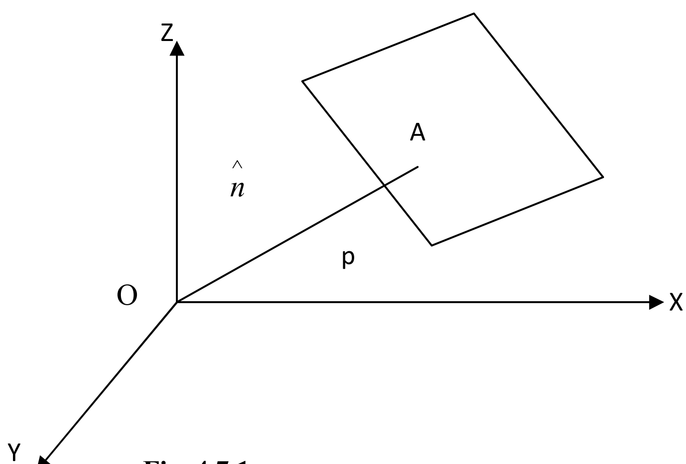


Fig. 4.7.1

Note: Equation ax+ by+ cz+ d = 0 can be written as $\vec{r}\,.(a\hat{i}+b\hat{j}+c\hat{k})=-d$

Or $\vec{r}.\dfrac{(-a\hat{i}-b\hat{j}-c\hat{k})}{\sqrt{a^2+b^2+c^2}}=\dfrac{d}{\sqrt{a^2+b^2+c^2}}=p$

Anothernormal form of the equation of the plane

To find the equation of a plane whose perpendicular distance from the origin is p and cosα , cosβ , cosγ are direction cosines of this perpendicular.

Let P (x, y, z) be any point on the plane. Let ON be perpendicular drawn from origin to the plane.

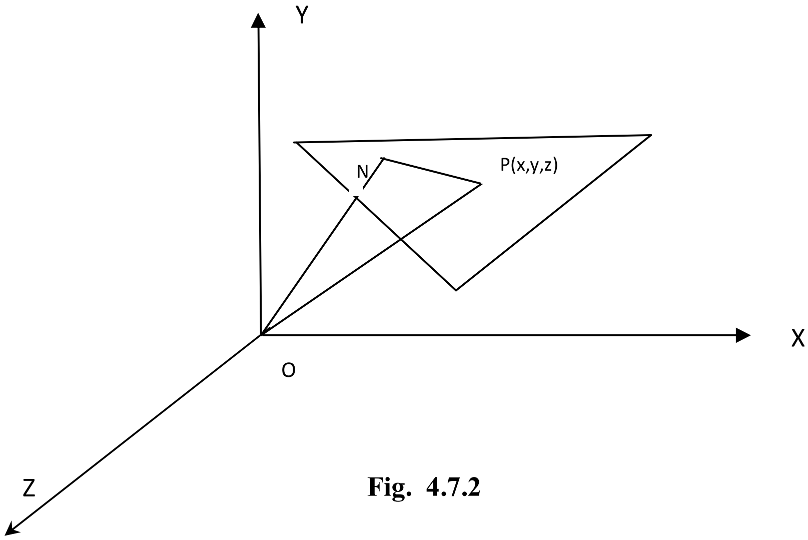


Fig. 4.7.2

Given that ON = p

N is the point on the plane, where perpendicular ON touches the plane. ON makes angle α with x-axis, β with y-axis, γ with z- axis therefore coordinates of N are

$(p \cos \alpha , p \cos \beta , p \cos \gamma)$.

Direction ratios of the line PN are $(x - p \cos \alpha ,\ y- p \cos \beta , z- p \cos \gamma)$.

Since ON and PN are perpendicular to each other, therefore.

$\cos \alpha\, (x- p \cos \alpha) + \cos \beta\, (y- p \cos \beta) + \cos \gamma\, (z- p \cos \gamma) = 0$

$x \cos \alpha + y \cos \beta + z \cos \gamma = p\, (\cos^2 \alpha + \cos^2 \beta + \cos^2\gamma)$

$x \cos \alpha + y \cos \beta + z \cos \gamma = p$

If we take l, m, n are direction cosines of the normal to the plane then equation of the plane is-

$\ell x + my + nz = p$

This is known as normal form of the equation of the plane.

4.8 TO REDUCE THE GENERAL EQUATION OF THE PLANE TO THE NORMAL FORM

Let the general equation of the plane be

$ax + by + cz + d = 0$ .... (1)

If ℓ, m, n, are the direction cosines of the normal to the plane, then the equation of the plane in the normal form is

$\ell x + my + nz = p$ .... (2)

where p is the length of normal drawn from origin on plane.

If (1) and (2) represent the same plane then equation (1) and (2) are identical therefore, we have

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d} = \pm\frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

and $p = \pm\dfrac{-d}{\sqrt{a^2 + b^2 + c^2}}$,

substituting these values of ℓ, m, n and p in equation (2), we get the equation of the plane (1) in the normal form.

4.9 INTERCEPTS FORM OF THE EQUATION OF A PLANE

If a plane cuts intercept a, b and c with x-axis, y-axis and z-axis respectively then equation of plane in terms a, b, c is called intercept form.

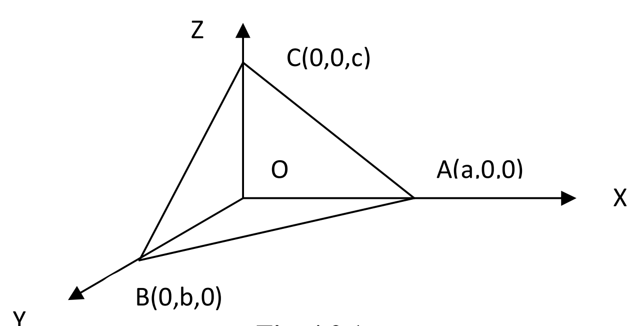


Fig. 4.9.1

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$= bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k}$$

Equation of the plane $\vec{r}.\vec{n} = \vec{OA}.\vec{n}$

$$\vec{r}.(bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k}) = a\,\hat{i}.\,(bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k})$$

$$\vec{r}.(bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k}) = abc$$

$$\vec{r}.(\frac{\hat{i}}{a} + \frac{\hat{j}}{b} + \frac{\hat{k}}{c}) = 1$$

i.e. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Another intercepts form of the plane

To find the equation of a plane which makes intercepts a , b and c on the axis of x, y and z respectively.

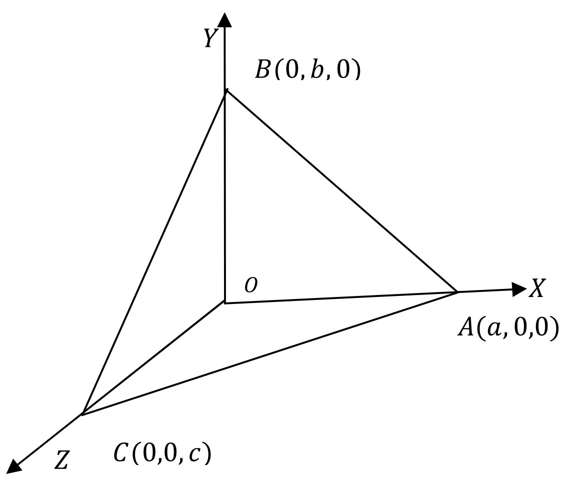


Fig. 4.9.2

Let the general equation of plane be

$$Ax + By + Cz + D = 0 \qquad \text{..................} \quad (1)$$

where, $D \neq 0$, because the plane does not passes through origin.

Plane passes through the points (a, 0, 0,) , (0, b, 0) and (0, 0, c), therefore these points satisfy above equation of the plane

$$A\,a + D = 0 \;\Rightarrow\; A = \frac{-D}{a}$$

Similarly, we get $B = \frac{-D}{b}$ and $C = \frac{-D}{c}$

Putting these values in equation (1) we get the required equation of plane

$$-\frac{D}{a}x - \frac{D}{a}y - \frac{D}{a}z + D = 0$$

Or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

4.10 EQUATIONS OF THE COORDINATE PLANES

(i) Equation of the yz – plane

The x – coordinate of any point lying on the yz- plane is given by x = 0,therefore the equation of yz plane is given by x =0.

Similarly the equation of zx – plane and xy – plane are given by y = 0 and z = 0 respectively.

(ii) Equations to the planes parallel to the coordinate planes-

The equation of the plane parallel to the yz-plane at a distance ‘a’ from it. The x- coordinate of any point lying on plane parallel to yz- plane at a distance ‘a’ from it is equal to a, therefore the equation of the plane parallel to the yz- plane is x = a.

Similarly the plane parallel to xz- plane at a distance ‘b’ from it ,is given by y = b and the plane parallel to xy-plane at a distance ‘c’ form it, is given by z=c.

4.11 EQUATION OF THE PLANE THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN LINE

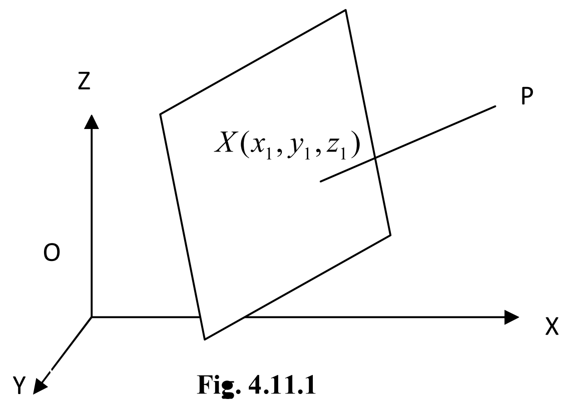


Fig. 4.11.1

Here, normal vector to the plane = {a, b, c}

∴ Equation of the plane is given by

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

Alternative Method

We know, the equation of plane through a point (x_1, y_1, z_1) is

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$................. (1)

If it is perpendicular to a line, whose direction ratios are ℓ, m, n. This line and normal to the plane are parallel. The direction ratios of normal to the plane are a, b, c.

Using the condition when two lines are parallel i.e. their direction ratios are proportional.

Therefore

$$\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n} = \lambda \ (say)$$

$a = \ell\lambda$, $b = m\lambda$, $c = n\lambda$

Putting these values of a, b, c, in equation (1), we get

$\ell(x - x_1) + m(y - y_1) + n(z - z_1) = 0$

which is required equation of plane.

4.12 EQUATION OF A PLANE PARALLEL TO AXES

In this section we will discuss about the equation of plane

Equation of a plane passing through a point

Equation of a plane passing through a point A $\vec{a}(x_1, y_1, z_1)$ and parallel to a plane $\vec{r}.\vec{n_1} = \alpha_1$
When two planes are parallel then both planes have same normal

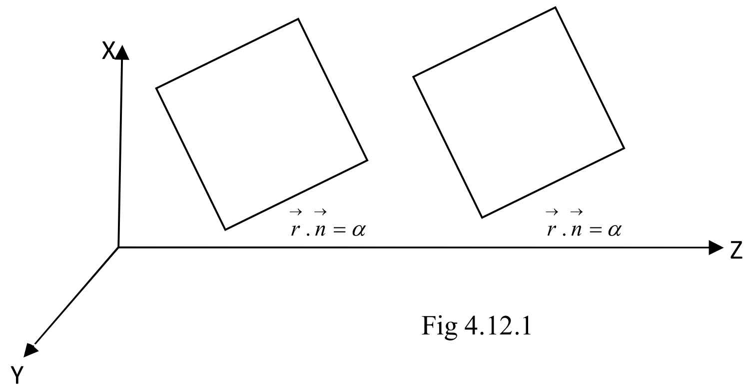


Fig 4.12.1

$\therefore$ Equation of the required plane will be $\vec{r}.\vec{n_1} = \vec{a}.\vec{n_1}$

In Cartesian form equation of the plane parallel to $ax + by + cz + d = 0$ will be

$ax + by + cz = \lambda$

As it passes through (x_1, y_1, z_1) then $\lambda = ax_1 + by_1 + cz_1$

Therefore equation of the plane will be

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of a plane parallel to axis

To find the equation of the plane parallel to x-axis

Let general equation of the plane be

ax + by + cz + d = 0 .... (1)

where a, b, c are direction ratio of normal to the plane.

The x-axis and normal to the plane are at right angle ,therefore

a × 1 + b × 0 + c × 0 = 0

since d.c'.s of the x-axis are 1, 0, 0 i.e. a = 0

Putting a = 0 in the equation (1), we get the required equation of the plane ie plane parallel to x-axis is

b y + c z + d = 0

Similarly, we get the equation of plane parallel to y-axis is ax + cz + d = 0 and plane parallel to z- axis is ax = by + d = 0.

4.13 SUMMARY

1. The linear equation ax + by + cz + d = 0, always represent a plane. It is known as general equation of plane. In vector form it is represented as $\vec{r}.(a\hat{i}+b\hat{j}+c\hat{k})+d=0$
2. Equation of a plane passing through origin is ax + by + cz = 0. In vector form it is represented as $\vec{r}.(a\hat{i}+b\hat{j}+c\hat{k})=0$
3. Equation of a plane passes through point (x_1, y_1, z_1) is a(x – x_1) + b(y – y_1) + c (z - z_1) = 0. In vector form it is represented as $(\vec{r}-\vec{r_1}).\vec{n}=0$
4. Equation of the Normal form of the plane is ℓx + my + nz = p
   where ℓ, m, n are direction cosines normal to the plane and *p* is perpendicular distance from origin to the plane. In vector form it is represented as $\vec{r}.\hat{n}=p$
5. Equation of the Intercept form of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a, b, c are intercepts with coordinate axes. In vector form it is represented as $\vec{r}.(\frac{1}{a}\hat{i}+\frac{1}{b}\hat{j}+\frac{1}{c}\hat{k})=1$
6. x = 0, represent a point in one dimensional space.
7. x = 0, represent a line ie equation of y-axis, in two dimensional space.
8. x = 0, represent a line in three dimensional space. x = 0 is equation of yz plane.
9. x = constant, represent a plane parallel to yz = plane.
10. Plane parallel to x-axis is
    by + cz + d = 0
11. Equation of xy plane is z = 0
12. Equation of plane parallel to xy plane is z = constant
13. Equation of plane parallel to y-axis is
    ax + cz + d = 0
14. Equation of xz plane is y = 0
15. Equation of plane parallel to xz plane is y = constant
16. Equation of plane parallel to z- axis is

ax + by + d = 0

17 Equation of a plane passes through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3)

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

18. Four or more points are said to be coplanner if they all lies in the same plane.

4.14 GLOSSARY

1. Plane- a surface such that every straight line joining any two points lies wholly on it
2. Coplanar- lying in a plane.
3. Collinear – lying on a straight line.
4. Intercept- points where plane\stright line cross the respective axes

4.15 REFERENCES

1. Geometry of two and three Dimensions- Lata S. , Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. $3D$ Geometry- R.K. Pandey, Krishna Prakashan media (P) Ltd., Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd., New Delhi, India.
6. Leonard E. Dickson-The First course in the Geometry.
7. Burnside,William Show, Panton &Arthur William-Volumetric Theory.
8. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series,McGraw Hill.
9. Anton, Howard (1994), Elementary Linear Algebra (7th ed.), John Wiley & Sons, *ISBN 978-0- 471-58742-2*.
10. Arfken, George B. and Hans J. Weber. Mathematical Methods For Physicists, Academic Press; 6 edition (June 21, 2005). ISBN 978-0-12-059876-2.
11. Brannan, David A.; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press, ISBN 978-0-521-59787-6

4.16 SUGGESTED READINGS

1. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
3. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.
6.

4.17 SOLVED EXAMPLE

Ex 1. Find the equation of a plane passing through origin.

Sol : - See 4.4

Ex 2. Find the intercepts of the plane $2x + 3y + 2z = 12$ on the coordinate axis.

Sol.- The given equation can be written as

$\frac{x}{6} + \frac{y}{4} + \frac{z}{6} = 1$, dividing both side by 12

$\therefore$ The required intercepts are x = 6, y = 4, z =6

Ex 3. Find the direction cosines of the normal to the plane

$2x + 3y + 2z = 12$

Sol.- The direction ratios of the normal to the given planes are coefficients of x, y, z in the equation of the plane that is 2, 3, 2.

$\therefore$ The direction cosines of the normal are –

$$\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}$$

Ex. 4. Find the equation of plane passing through the points (a, 0, 0), (0, b, 0), (0, 0, c).

Sol.- From the coordiantes of the given points, we see that intercept made by given plane on the axes are a, b and c.

Therefore required equation of plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Ex. 5. Find the equation of a plane passes through (2, 0, 0), (0, 3, 0) , (0, 0, 5)

Sol.- The required equation of plane is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{5}=1 \qquad \text{(see example 4)}$$

Ex.6. Find the equation of a plane passes through the point (2, 3, 4).

Sol.- We know the equation of plane passes through point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Therefore required equation of plane is

$a(x - 2) + b(y - 3) + c(z - 4) = 0$

where a, b, c are direction ratios normal to this plane.

Ex.7 Find the equation of a plane which passes through the point (1, 2, 3) and direction ratios normal to the plane are 2, 5, 7.

Sol.- Required equation of plane is

$2(x - 1) + 5(y - 5) + 7(z - 3) = 0$

$2x + 5y + 7z = 48$

Ex.8 Reduce the plane $2x + y + 2z = 5$ into normal form

Sol.- We know the normal form of a plane is

$\ell x + my + nz = p$

where ℓ, m, n are direction cosines normal to the plane and p is length of perpendicular drawn form origin

Direction ratio's normal to the plane are 2, 1, 2

Therefore direction cosines normal to the plane are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

Dividing given equation of plane by 3, we get

$$\frac{2}{3}x+\frac{1}{3}y+\frac{2}{3}z=\frac{5}{3}$$

which is required normal form of given equation of plane.

Ex.9 Find the equation of plane passes through the points- (1, 1, 1) , (1, 2, 1) and (0, 2, 3).

Sol.- Let the equation of plane be

$ax = by + cz + d = 0$

If plane passes through the point (1, 1, 1), then a + b + c + d = 0 .... (1)

If plane passes through the point (1, 2, 1) then.

a + 2b + c + d = 0 .... (2)

If plane passes through the point (0, 2, 3), then

2b + 3c + d = 0 .... (3)

From (1) and (2), we get b = 0

From (2) and (3), we get

a – 2c = 0

a = 2c or c = (a/2)

From (1) and (3), we get

3a + 3b + 3c + 3d – 2b – 3c – d = 0

3a + b + 2d = 0

Put b = 0

3a – 2d = 0

$$d = \left(\frac{3}{2}\right)a$$

Putting value of b, c, d in general equation of plane, we get

$$ax + 0.y + \left(\frac{a}{2}\right)z - \left(\frac{3}{2}\right)a = 0$$

$$ax + \left(\frac{a}{2}\right)z - \left(\frac{3}{2}\right)a = 0$$

2x + z – 3 = 0

which is required equation of plane.

Alternate Solution:

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 1-1 & 2-1 & 9-1 \\ 0-1 & 2-1 & 3-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 2x + z - 3 = 0$$

Ex.10 Find the equation of the plane passes through the points (1, 2, 1), (2, 3, 4) and (-1, 2, -1).

Sol.- Let the equation of plane be ax + by + cz + d = 0

If the plane passes through the point (1, 2, 1) then

a + 2b + c + d = 0 .... (1)

If the plane passes through the point (2, 3, 4) and (- 1, 2, -1) then we get equations (2) and (3) respectively

2a + 3b + 4c + d = 0 .... (2)

-a + 2b – c + d = 0 .... (3)

From (1) and (3), we have

4b + 2d = 0 $\Rightarrow$ d = -2b or b = $\left(-\frac{d}{2}\right)$

From (1) and (2), we have

2a + 4b + 2c + 2d – 2a – 3b – 4c – d = 0

b – 2c + d = 0 .... (4)

From (2) and (3), we have

2a + 3b + 4c + d – 2a + 4b – 2c + 2d = 0

7b + 2c + 3d = 0 .... (5)

From (4) and (5) , we have

- 14c + 7d – 2c – 3d = 0

- 16 c + 4d = 0 $\Rightarrow$ c = $\frac{1}{4}d$

Put value of b and c in equation (3), we get

a = 2b – c + d = - d - $\frac{1}{4}d$ + d = -$\frac{1}{4}d$

Putting values of a, b, c in general equation of the plane

$$-\frac{1}{4}dx - \frac{d}{2}y + \frac{1}{4}dz + d = 0$$

- x – 2y + z + 4 = 0

x + 2y – z – 4 = 0

Alternate Solution:

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ 2-1 & 3-2 & 4-1 \\ -1-1 & 2-2 & -1-1 \end{vmatrix} = 0$$

⇒ x + 2y – z – 4 = 0

Ex.11 Prove that four points (1, 2, 1) (2, 3, 4) (-1, 2, -1) and (3, 1, 1) lying in a plane ie. Co-planner

Sol.- The equation of plane passes through the points (1, 2, 1), (2, 3, 4), (-1, 2, -1) is

x + 2y – z – 4 = 0 (see example 10)

If given four points laying in the plane then point (3, 1, 1) satisfy the above equation of plane.

1×3+2×1-1×1- 4 = 0

3+2-1-4 = 0

0 = 0

which shows four points lying in the same plane.Hence coplanner

Ex.12 Find the intercepts made on the co-ordinate axes by the plane x + 2y + 2z = 9. Find also the direction cosines of the normal to the plane.

Sol.- The given equation can be rewritten as

$$\frac{x}{9} + \frac{y}{\left(\frac{9}{2}\right)} + \frac{z}{\left(\frac{9}{2}\right)} = 1$$

∴ The required intercepts are 9, 9/2 and 9/2 . The direction ratios of the normal to the given plane are the coefficients of x, y, z in the equation of the plane that is 1, 2, 2.

∴ The direction cosines of the normal are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$

Ex.13. Find the perpendicular distance from the origin to the plane 2x + y + 2z = 3

Sol.- The equation of the plane is

$2x + y + 2z = 3$

To reduce it to the normal form, we divide it by 3 then get

$$\left(\frac{2}{3}\right) x + \left(\frac{1}{3}\right) y + \left(\frac{2}{3}\right) z = 1$$

where $\frac{2}{3}, \frac{1}{3}, \frac{2}{2}$ is the direction cosines and 1 is the perpendicular distance from the origin to the plane.

Ex.14 A plane meets the coordinate axes in A, B, C such that the centroid of triangle ABC is the point (p, q, r). Show that the equation of the plane is

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

Sol.- Let the equation of the plane in intercepts from be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where the coordinates of the point A,B,C are (a, 0, 0) , (0, b, 0), (0, 0, c) respectively.

So the centroid of the triangle ABC is the point $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

But it is given that the centroid of the triangle ABC is the point (p, q, r), therefore

$$\frac{a}{3} = p, \ \frac{b}{3} = q, \ \frac{c}{3} = r$$

or $a = 3p, \ b = 3q, \ c = 3r$

Putting these value of a, b, c, in above equation of plane, we get

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

or $$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

which is required equation of plane.

Ex.15 A variable plane is at a constant distance p from the origin and cuts the axis at A,B,C. Show that the locus of the centroid of triangle ABC is

$$x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$$

Sol.- Let the equation of the variable plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{..................} \quad (1)$$

where a, b, c are variables

The above plane is at a constant distance p from the origin that is the length of the perpendicular drawn from the origin to the plane is always p, whatever a, b, c may be. Therefore

$$p^2 = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

or $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \qquad \text{..................} \quad (2)$$

The plane cuts the axis at the points A(a, 0, 0) ,B(0, b, 0), C (0, 0, c).

Let (x, y, z) be the coordinate of the centroid of triangle ABC then point (x, y, z) is variable point because a, b, c are variable,therefore

$$x = \frac{a}{3},\ y = \frac{b}{3},\ z = \frac{c}{3}$$

or a = 3x, b = 3y, c = 3z

Putting these values of a, b, c, in equation (2), we get

$$x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$$

which is required equation of locus of the centroid of the triangle ABC.

Ex.16 A plane meets the cordinate axes at A,B and C such that the centroid of triangle ABC is (1, -2, 3). Find the equation of the plane.

Sol. – Let the intercept form of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where the coordinates of A, B, C are (a, 0, 0) (0, b, 0), (0, 0, c) respectively.

The centroid of the triangle ABC is

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) \text{,that is } \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

But , given centriod of triangle ABC is (1, -2, 3)

Therefore $\frac{a}{3} = 1$, $\frac{b}{3} = -2$, $\frac{c}{3} = 3$

or a = 3, b = - 6, c = 9

Putting these values of a, b, c in above equation of the plane, we get required equation of plane.

That is $\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$

or $\frac{x}{3} - \frac{y}{6} + \frac{z}{9} = 1$

Ex.17 A variable plane moves so that the sum of receiprocals of its intercepts on the three coordinate axes is constant. Show that it passes through, a fixed point.

Sol.- Let the equation of variable plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{..................} \quad (1)$$

where a, b, c are variables.

The intercepts on the coordinate axes are a, b and c

Given that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \text{constant} = \frac{1}{m} \text{(say)}$$

or $\frac{m}{a} + \frac{m}{b} + \frac{m}{c} = 1$

Above equation imply that the fixed point (m, m, m), from which the plane passes .

Ex.18 Find the equation of the plane passing through the point P (1, 2, 3) and perpendicular to OP, where O is the origin.

Sol.- The equation of the plane passing through (1, 2, 3) is

$$a(x – 1) + b(y – 2) + c (z-3) = 0 \qquad \text{..................} \quad (1)$$

Since (1) is perpendicular to OP. therefore the normal to plane and line OP are parallel. Therefore their direction ratios are proportional. The direction ratio of line OP is

1 – 0, 2 – 0, 3 – 0 ie. 1, 2, 3

Direction ratios normal to plane (1) are a, b, c a, b, c are proportional to 1, 2, 3

$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$ = m (say), where m ≠ 0

a = m, b = 2m , c = 3m

Putting these values of a, b, c, in equation (1), we get

m (x – 1) + 2m (y – 2) + 3m (z – 3) = 0

or (x – 1) + 2 (y – 2) + 3(z – 3) = 0

x + 2y + 3z – 1 – 4 – 9 = 0

x + 2y + 3z = 14

which is required equation of the plane.

Ex.19 Find the equation of the plane passing through the points (1, 2, 3), (2, 3, 4) and 3x + 4y + 5z = 9

Sol.- The plane passing through point (1, 2, 3) is

a(x – 1) + b(y – 2) + c(z – 3) = 0 .... (1)

If the plane (1) also passes through (2, 3, 4)

Then

a (2 - 1) + b(3 – 2) + c(4 – 3) = 0

or a + b + c = 0 .... (2)

If the plane (1) is perpendicular to the plane 3x + 4y + 5z = 9, then (two planes are perpendicular if their normals are perpendicular to each other)

3a + 4b + 5c = 0 .... (3)

From (2) and (3) by cross multiplication

$$\frac{a}{5-4} = \frac{b}{3-5} = \frac{c}{4-3}$$

$\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$ = K (say) .... (4)

Putting the values of a, b, c from (4) into (1), we get

(x – 1) – 2(y – 2) + (z – 3) = 0

x – 2y + z = 0

which is required equation of plane.

Self Check Questions

1. The equation ax + by + d = 0 represent a plane

   a) Parallel to x-axis b) Parallel to y- axis

   c) Parallel to z-axis d) None of these

2. The equation 3y + 4z = 5 represent a plane parallel to

   a) x – axis b) y – axis c) z – axis d) None of these

3. If the equation of a plane in normal form is $\ell x + my + nz = p$, then $\ell^2 + m^2 + n^2$ is equal to

   a) – 1 b) 0 c) 2 d) 1

4. The equation of the plane through (1, 0, 0), (0, 1, 0), (0, 0, 1) is

   a) $x + y + z = 1$ b) $x - y - z = 1$

   c) $x + y - z = 1$ d) $x - y + z = 1$

5. The equation of the plane through (0, 0, 0) , (0, 1, 0), (1, 0, 0) is

   a) z = 0 b) y = 0 c) x = 0 d) None of these

4.18 TERMINAL QUESTIONS

EXERCISE- 4.2

Fill in the blanks-

1. The general equation of the plane is ....
2. The normal form of the plane is ....
3. The intercept from of the plane is ....
4. A plane cuts intercepts of length 2, 3, 5 on coordinate axes, then the equation of plane is ....
5. The direction ratios of the normal to the plane ax + by + cz + d = 0 are ....
6. The intercept from of the plane 2x + 2y + 5z = 10 is ....
7. In the plane $\ell x + my + nz = p$, the perpendicular distance from origin to this plane is ....
8. The equation of the plane through the point (x_1 , y_1, z_1) is ....
9. The plane ax + by + d = 0 is a plane parallel to ....

10. In a plane $ax + by + cz + d = 0$, $a^2 + b^2 + c^2$ is not equal to ....
11. The equation of xy-plane is ....
12. The plane parallel to xy- plane is ....
13. The equation $x = 0$ represent a .... plane.
14. The equation of yz -plane is ....
15. The equation of zx -plane is ....

True or False-

16. The number of arbitrary constants in the general equation of a plane is 3.
17. The direction cosines normal to the plane $2x + 3y + 5z = 1$ is 2, 3, 5.
18. The equation x = constant represent a plane parallel to yz- plane.
19. The equation of a plane passes through the origin is $ax + by + cz = 0$.
20. The equation $ax + by + d = 0$ is a plane parallel to the z- axis.
21. The perpendicular distance from origin to the plane $ax + by + cz + d = 0$ is d.
22. The normal from of the plane is $\ell x + my + nz = p$.
23. The plane $ax + by + cz + d = 0$ will pass through the origin if $d = 0$.
24. The perpendicular distance from origin to the plane $x + 2y + 2z = 9$ is 3.
25. The direction cosines normal to the plane $x + 2y + 2z = 9$ is $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

Multiple choice questions

Choose the correct answer for each of the following questions.

26. The intercept on x-axis of the plane

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1 \text{ is}$$

a) 2 b) 3 c) 4 d) 1

27. The perpendicular distance from the origin to the plane $2x + 4y + 4z = 36$ is

a) 4 b) 5 c) 6 d) 7

28. The direction cosines normal to the plane $2x + 4y + 4z = 36$ is

a) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ b) 2, 4, 4 c) 1, 2, 2 d) None of these

29. The equation of xy – plane is

a) x = 0 b) y = 0 c) z = 0 d) None of these

30. The equation of a plane parallel to xz – plane is

a) x = a b) y = a c) z = a d) y = 0

EXERCISE- 4.2

1. Reduce the equation of the plane 2x + 3y + 5z = 10 to normal from and find the perpendicular distance of the plane from the origin.
2. Reduce the equation of plane 2x + 4y + 6z = 12 to intercept from.
3. Find the equation of the plane in intercept from which cuts coordinate axis in 2, 3, 5
4. Find the equation of the plane which passes through (4, 5, 6) and direction ratios of a line normal to this plane are 2, 3, 5.
5. A plane meets the coordinate axes in A,B,C, such that the centroid of triangle ABC is the point (6, 7, 9). Show that the equation of the plane is
$$\frac{x}{6} + \frac{y}{7} + \frac{z}{9} = 3$$
6. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (2, 4, 5) and (4, 6, 7).
7. Find the equation of the plane passing through the points (2, 2, - 1), (3, 4, 2) and (1, 0, 2).
8. Find the equation of the plane passing through (2, 0, 0), (0, 3, 0), (0, 0, 5). Find also the perpendicular distance form the origin.
9. A variable plane which remains at a constant distance 3p from the origin cuts the coordinate axis at A, B, C. Show that the locus of the centroid of triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.
10. Find the equation to the plane through the points (1, 1, 0), (1, 2, 1), (-2, 2, -1)
11. Show that the four points (0, -1, -1), (4, 5, 1) (3, 9, 4) and (-4, 4, 4) are coplanar.
12. Show that the four points (0, -1, 0), (2, 1, -1) (1, 1, 1) and (3, 3, 0) are co-planar and also show that the equation of the plane passing through these points is
$$4x – 3y + 2z = 3$$
13. Find the equation of the plane passing through the point P (2, 3, - 1) and perpendicular to OP, where O is origin.
14. Find the equation of the plane passing through the points (1, -1, 2) and (2, -2, 2) and

which is perpendicular to the plane $6x - 2y + 2z = 9$.

4.19 *ANSWER*

Answer of self check question:

1. c
2. a
3. d
4. a
5. c

Answer of exercise 4.1:

1. $ax + by + cz + d = 0$
2. $\ell x + my + nz = p$
3. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
4. $\frac{x}{2} + \frac{y}{3} + \frac{z}{5} = 1$
5. a, b, c
6. $\frac{x}{5} + \frac{y}{5} + \frac{z}{2} = 1$
7. p
8. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
9. z- axis
10. zero
11. $z = 0$
12. z = constant
13. yz
14. $x = 0$
15. $y = 0$
16. T,
17. F
18. T
19. T
20. T
21. F
22. T
23. T
24. T
25. T
26. a
27. c
28. a
29. c
30. b

UNIT 5: THE PLANE II

CONTENTS

5.1 INTRODUCTION

In previous unit we studied about plane and various forms of plane. This unit is continuation of previous unit, in which we learn more about plane.

5.2 *OBJECTIVES*

The main objectives of this unit is to learn following topics about plane

- Angle between two planes.
- Distance of a point from a plane.
- Parallel planes.
- Perpendicular planes.

- Distance between two planes.
- Position of a point with respect to a plane.
- Plane through the intersection of two given planes.
- Planes bisecting the angles between the given planes.
- Equation of pair of planes.

5.3 *ANGLE BETWEEN TWO PLANES*

Angle between two planes is the angle between their normal vectors i.e. if $\vec{r}.\vec{n}_2 = \alpha_2$ be two planes then angle between them θ is given by $\cos\theta = \dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$

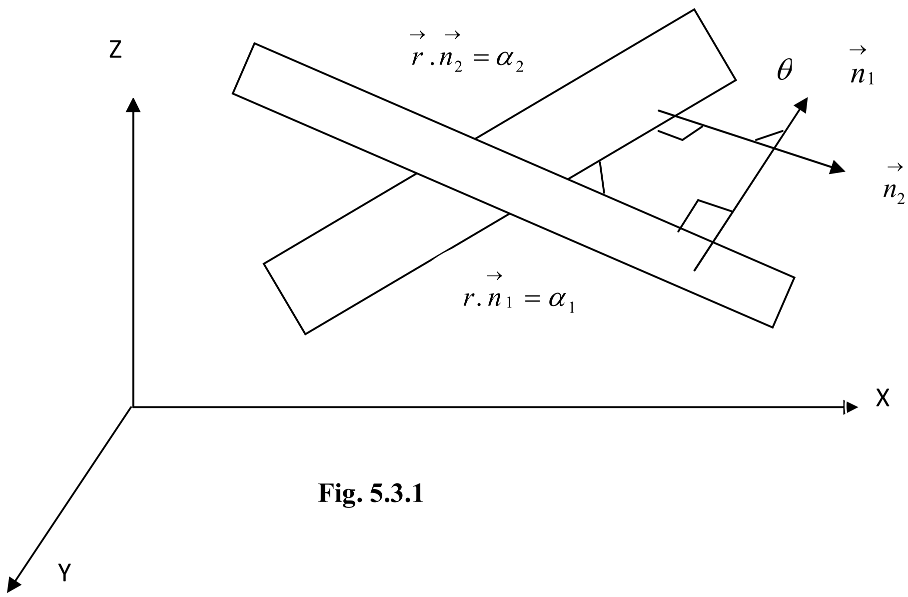


Fig. 5.3.1

Another method

Angle between two planes is equal to the angle between their normal.

Let the equation of two planes be

$a_1x + b_1y + c_1z + d_1 = 0$ .... (1)

$a_2x + b_2y + c_2z + d_2 = 0$ .... (2)

The direction ratio's of the normal to the plane (1) are a_1, b_1, c_1 and the direction ratio's of the normal to the plane (2) are a_2, b_2, c_2. If θ is the angle between planes (1) and (2) then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{...............} \quad (3)$$

5.3.1 CONDITION OF PERPENDICULARITY OF TWO PLANES

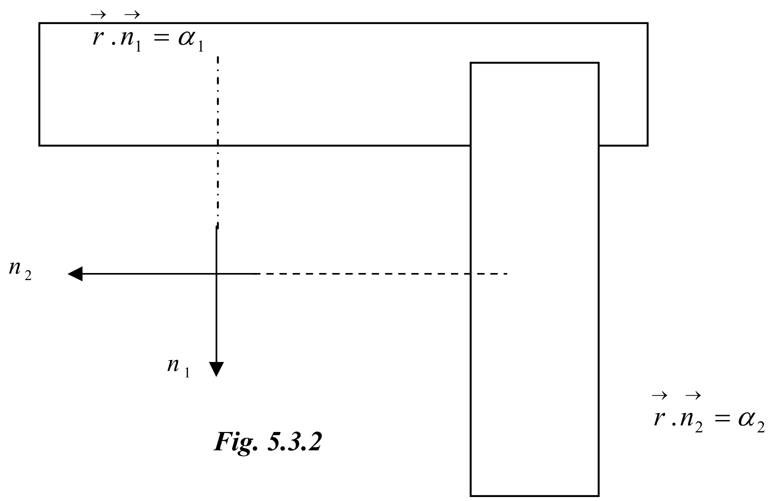


Fig. 5.3.2

Two planes are perpendicular if the angle between their normal's are at right angle.

$\cos 90^\circ = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

Or, $\vec{n}_1.\vec{n}_2 = 0$ is the condition for perpendicularity of two planes.

5.3.2 CONDITION FOR PARALLEL OF TWO PLANES

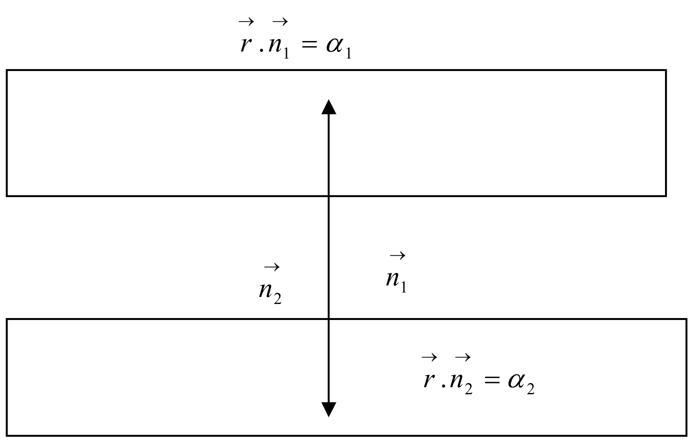


Fig. 5.3.2

Two planes are parallel then their normal are also parallel and we know the condition, when two line are parallel

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

which is required condition.

Or if $\vec{r}.\vec{n_1} = \alpha_1$ and $\vec{r}.\vec{n_2} = \alpha_2$ are parallel then $\vec{n_2} = \lambda \vec{n_1}$ is the condition for perpendicularity of two planes.

5.4 *PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE*

To find perpendicular distance from the point (x_1, y_1, z_1) to a given plane.

Let P (x_1, y_1, z_1) be any point and let the general equation of the plane be-

$$ax + by + cz + d = 0 \quad .... \quad (1)$$

Normal form of plane (1) is given by

$$\frac{a}{\pm\sqrt{a^2+b^2+c^2}}x + \frac{b}{\pm\sqrt{a^2+b^2+c^2}}y + \frac{c}{\pm\sqrt{a^2+b^2+c^2}}z = -\frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

or $\quad lx + my + nz = p \quad .... \quad (2)$

Where

l= $\frac{a}{\pm\sqrt{a^2+b^2+c^2}}$, m = $\frac{b}{\pm\sqrt{a^2+b^2+c^2}}$, n = $\frac{c}{\pm\sqrt{a^2+b^2+c^2}}$ and

p = $-\frac{d}{\pm\sqrt{a^2+b^2+c^2}}$,

Now consider the plane passing through (x_1, y_1, z_1) and parallel to (2) is

$$lx_1 + my_1 + nz_1 = p_1 \quad .... \quad (3)$$

where p_1 is the perpendicular distance of (3) from origin

The perpendicular distance from (x_1, y_1, z_1) to (1) is equal to

$= p_1 - p$

$= lx_1 + my_1 + nz_1 - p_1$

Putting value of ℓ, m, n, and p in above equation, we get

$$= \frac{ax_1 + by_1 + cz_1 + d}{\pm\sqrt{a^2 + b^2 + c^2}}$$

which is required perpendicular distance from the point (x_1, y_1, z_1) to plane (1)

Note -1- If the equation of plane is given in normal from that is $lx + my + nz = p$.

Then the perpendicular distance from (x_1, y_1, z_1) to this plane is given by

$$= lx_1 + my_1 + nz_1 - p_1$$

Note- 2- To find perpendicular distance between two parallel planes, we choose any point on one of the plane and then find its perpendicular distance from other.

Another method

Perpendicular distance from a point $A\vec{a}(x_1, y_1, z_1)$ on a plane

$\vec{r}.\vec{n} = \alpha$ … (1)

We first construct a plane through $A(\vec{a})$ and parallel to $\vec{r}.\vec{n} = \alpha$

Equation will be $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ … (2)

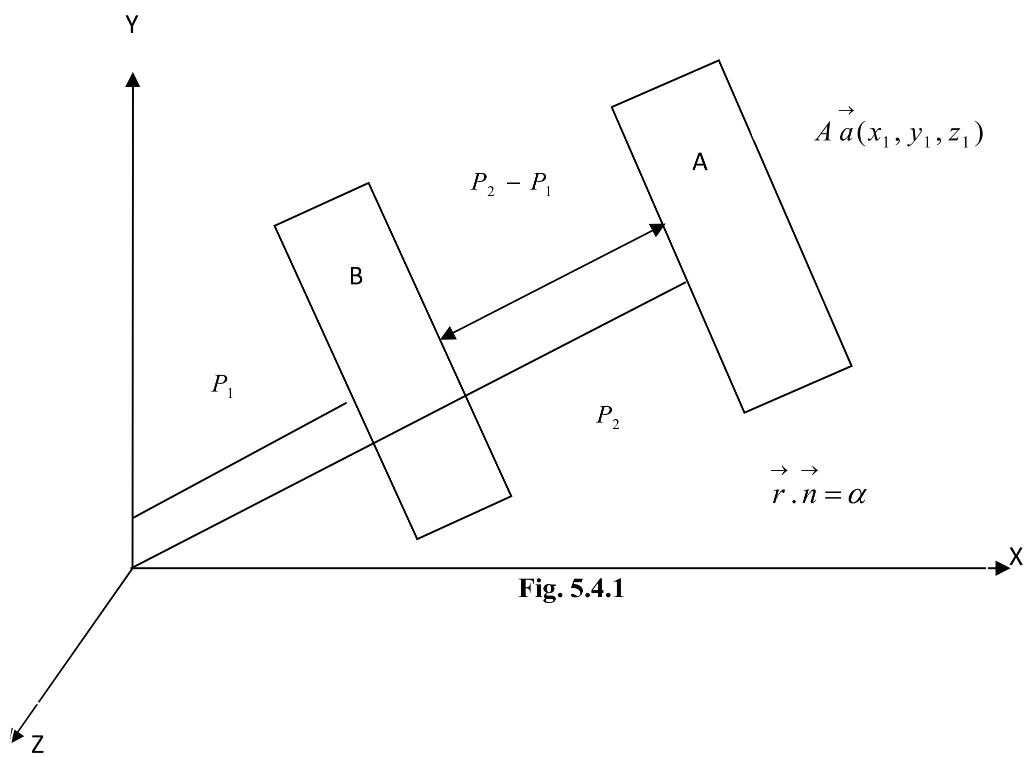


Fig. 5.4.1

Now $AB = |P_2 - P_1|$

where $P_2 =$ Perpendicular distance from origin to plane $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ and

$P_1 =$ Perpendicular distance from origin to plane $\vec{r}.\vec{n} = \alpha$

Here, $P_1 = \frac{\alpha}{|\vec{n}|}, P_2 = \frac{\vec{a}.\vec{n}}{|\vec{n}|}$

Hence, $AB = |P_2 - P_1| = \left|\frac{\vec{a}.\vec{n} - \alpha}{|\vec{n}|}\right|$

5.5 CONDITION FOR TWO GIVEN POINTS TO LIE ON THE PLANE

To find the condition for two given points to lie on the same or opposite sides of a plane

Let the equation of a given plane be

$ax + by + cz + d = 0$................. (1)

and $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be any two points.

Suppose the line PQ meets the plane in a point R, where R divides PQ in the ratio λ: 1, then the coordinates of R are$\left(\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right), \left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right), \left(\frac{\lambda z_2 + z_1}{\lambda + 1}\right)\right)$

Point R lie on the plance (1) therefore,we have

$$a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c\left(\frac{\lambda z_2 + z_1}{\lambda + 1}\right) + d = 0$$

$$\lambda(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0$$

$$\lambda = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} \quad .... (2)$$

λ is positive or negative according as R divided PQ internally or externally, that is according as P and Q are on the opposite or same, side of the plane (1)

λ is positive if $(ax_1 + by_1 + cz_1 + d)$and $(ax_2 + by_2 + cz_2 + d)$have opposite sign. In this case point P and Q lie on opposite side of the plane.

Again from equation (2), we see that λ is negative if $ax_1 + by_1 + cz_1 + d$and $ax_2 + by_2 + cz_2 + d$ have same sign. In this case P and Q lie on the same side of the plane.

Hence the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$lie on the same or opposite sides of the plane $ax + by + cz + d = 0$ according as the expressions$ax_1 + by_1 + cz_1 + d$and $ax_2 + by_2 + cz_2 + d$are of the same or opposite sign.

5.6 *POSITION OF A POINT WITH RESPECT TO GIVEN PLANE*

If $A\vec{a}(x_1, y_1, z_1)$ is point and $ax + by + cz + d = 0$ or $\vec{r}.\vec{n} = \alpha$ be a plane. Then we find perpendicular distance from A to the plane

i.e. $p = \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$ in Cartesian from and $\dfrac{\alpha - \vec{a}.\vec{n}}{|\vec{n}|}$ in vector from

(i) If p is zero then point lies on the plane.
(ii) If $P > 0$ then point lies upper side of the plane.
(iii) If $P < 0$ then point lies lower side of the plane.

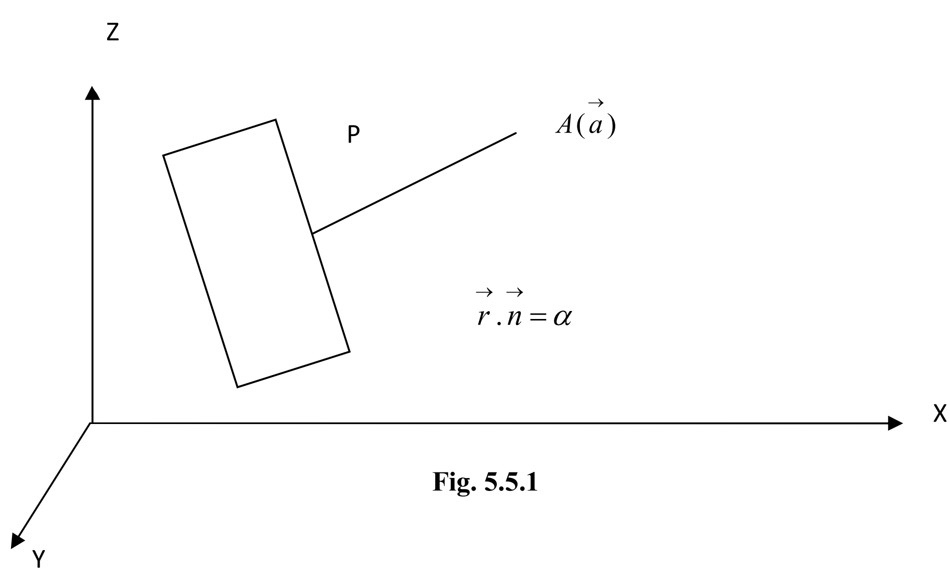


Fig. 5.5.1

5.7 *PLANE THROUGH THE INTERESCTION OF TWO GIVEN PLANES*

To find equation of any plane passing through the line of intersection of two give planes $P \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $Q \equiv a_2x + b_2y + c_2z + d_2 = 0$ is P + λ Q = 0, whereλ is a parameter.

Proof- The equation P + λ Q = 0 is $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0$

Above equation is the equation of a plane because every liner equation in the three variables x, y, z always represent a plane.

All the points which satisfy both the equations P = 0 and Q = 0, also satisfy the equationP + λ Q = 0.

Hence P + λ Q = 0 is the equation of that plane which is passes through the line of intersection of the planes P = 0 and Q = 0

5.7.1 PLANES BISECTING THE ANGLES BETWEEN THE GIVEN PLANES

Let the equation of the given planes be

$a_1x + b_1y + c_1z + d_1 = 0$ .... (1)

$a_2x + b_2y + c_2z + d_2 = 0$ .... (2)

Let (x, y, z) be any point on the plane bisecting the angle between the given planes, then this point (x, y, z) must be at equal distance from the given planes (1) and (2) therefore , we have

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad .... (3)$$

Equation (3) is the required equation of the planes bisecting the angles between planes (1) and (2).

Note 1 To find the equation of bisecting plane in which the origin lies.First we write the equations (1) and (2) in such a way that the constant terms d_1 and d_2 are of the same sign. then the equations of the plane bisecting the angle in which the origin lies is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and the equation of the plane bisecting the angle in which the origin does not lie is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1 + b_1^2 + c_1^2}} = - \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note 2 To find the angle between the given planes in which the origin lies is acute or obtuse.

First write the equation (1) and (2) in such a way that the constant terms d_1 and d_2 are of same sign.

The angle between the two planes in which the origin lies is acute or obtuse according as the angle between the normals to the two planes drawn from the origin is obtuse or acute. Therefore, we have

1. If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then the angle between the planes, in which the origin lies acute.
2. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then the angle between the planes, in which the origin lies is obtuse.

5.7.2 EQUATION OF BISECTOR PLANES BETWEEN TWO PLANES

Bisector planes are the locus of all points which are equidistance between two planes.

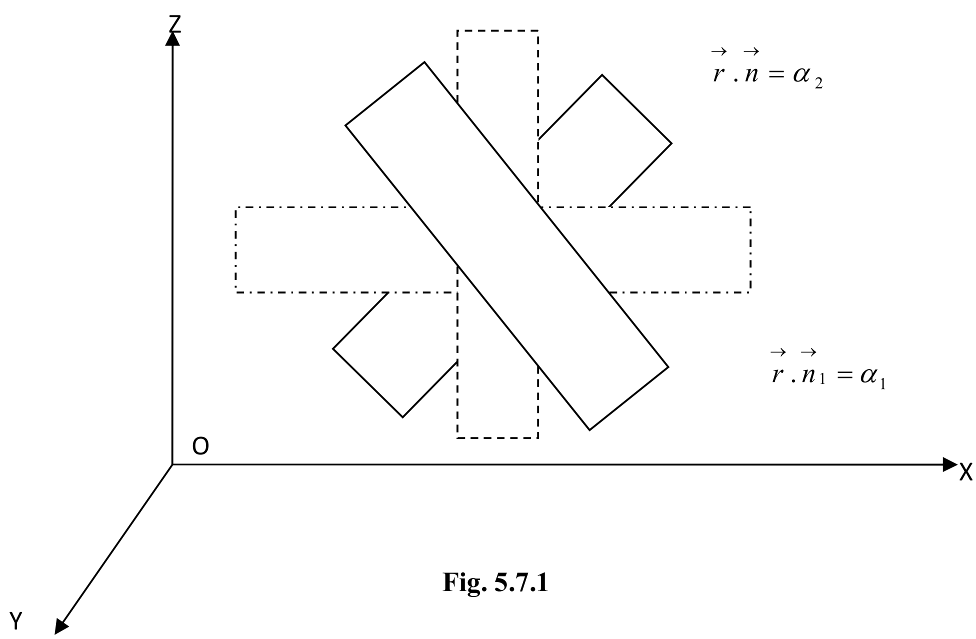


Fig. 5.7.1

Let $\vec{r}.\vec{n}_1 = \alpha_1$ and $\vec{r}.\vec{n}_2 = \alpha_2$ be two planes then equation of bisector planes are given by

$\frac{\vec{r}.\vec{n}_1 - \alpha_1}{\left|\vec{n}_1\right|} = \pm \frac{\vec{r}.\vec{n}_2 - \alpha_2}{\left|\vec{n}_2\right|}$ $a_2x + b_2y + c_2z + d_2$Or $\vec{r}.\hat{n}_1 \pm \vec{r}.\vec{n}_2 = \frac{\alpha_1}{\left|\vec{n}_1\right|} \mp \frac{\alpha_2}{\left|\vec{n}_2\right|}$

5.8 THE EQUATION OF PAIR OF PLANES

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be the equations of the two planes, then the equation $(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$ represent a pair of planes as it is satisfied by all points which lie on either plane

$a_1x + b_1y + c_1z + d_1 = 0$ or $a_2x + b_2y + c_2z + d_2 = 0$

To find the condition that the general homogeneous equation of second degree in x, y and z, that is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ .... (1)

represents a pair of planes. Also find the angle between them.

Let the equation of the planes represented by (1) be

$\ell_1x + m_1y + n_1z = 0$ and $\ell_2x + m_2y + n_2z = 0$, these equations do not contain constant term otherwise their product will not be homogeneous equation. Therefore, we have

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (\ell_1x + m_1y + n_1z)(\ell_2x + m_2y + n_2z) = 0$

Now comparing the coefficients of x^2, y^2, z^2, yz, zx, xy in above equation,we get

$\ell_1\ell_2 = a$; $m_1m_2 = b$; $n_1n_2 = c$; $m_1n_2 + m_2n_1 = 2f$; $n_1\ell_2 + n_2\ell_1 = 2g$ and $\ell_1m_2 + \ell_2m_1 = 2h$ .... (2)

The required condition is obtained by eliminating ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 from the relations (2)

Now consider the product of two zero determinants

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$$

Multiplying above determinants, we get

$$\begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_1m_2 + l_2m_1 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ l_1n_2 + l_2n_1 & m_1n_2 + m_2n_1 & 2n_1n_2 \end{vmatrix} = 0$$

Using relations (2) in above determinant, we get

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, which is the required condition.

Angles between the planes- If θ is the angle between the planes represented by (1)

Then, we have

$$\tan\theta = \frac{\left[\sum (m_1n_2 - m_2n_1)^2\right]^{1/2}}{\ell_1\ell_2 + m_1m_2 + n_1n_2} \qquad \text{.......... (3)}$$

$(m_1n_2 – m_2n_1)^2 = (m_1n_2 + m_2n_1)^2 – 4m_1m_2n_1n_2 = (2f)^2 – 4bc = 4(f^2 – bc)$

$\sum(m_1n_2 – m_2n_1)^2 = \sum 4(f^2 – bc) = 4 (f^2 + g^2 + h^2 – bc – ca – ab)$

Putting this value in equation (3), we get

$$\tan\theta = \frac{2(f^2 + g^2 + h^2 - bc - ca - ab)^{1/2}}{(a+b+c)}$$

$$\text{or } \theta = \tan^{-1}\frac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a+b+c} \qquad \text{......... (4)}$$

Condition of perpendicularity

If two planes. given by (1) are perpendicular then $\theta = \frac{\pi}{2}$

or $\tan\theta = \infty$

From (4), we get $a + b + c = 0$

Thus the two planes given by (1) will be perpendicular if $a + b + c = 0$

5.9 *SUMMARY*

1. Angle between planes $a_1x + b_1y + c_1z + d_1= 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

   $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

   In vector form it is represented as follows:

   $$\vec{r}\cdot\vec{n}_1 = \alpha_1,\quad \vec{r}\cdot\vec{n}_2 = \alpha_2 \text{ and } \cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$$

2. Two planes are perpendicular if their normals are perpendicular, that is

   $a_1a_2 + b_1b_2 + c_1c_2 = 0$

   In vector form condition of perpendicularity represented as $\vec{n}_1\cdot\vec{n}_2 = 0$

3. Two planes are parallel if their normals are parallel, that is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

In vector form condition of parallel of two planes represented as $\vec{n}_1 \times \vec{n}_2 = 0$ or $\vec{n}_1 = k\,\vec{n}_2$

4. Perpendicular distance of a plane

ax + by + cz + d = 0 from the point (x_1, y_1, z_1) is

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

In vector form it is represented as $= \left| \frac{\vec{r}.\vec{n} - \vec{a}.\vec{n}}{|\vec{n}|} \right|$

5. Equation of plane passing through the line of intersection of two given planes P = 0 and Q = 0 is P +λQ = 0 , where λ is a parameter.

6. Planes bisecting the angles between the given planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Let $\vec{r}.\vec{n}_1 = \alpha_1$ and $\vec{r}.\vec{n}_2 = \alpha_2$ be two planes then equation of bisector planes is given by $\frac{\vec{r}.\vec{n}_1 - \alpha_1}{|\vec{n}_1|} = \pm \frac{\vec{r}.\vec{n}_2 - \alpha_2}{|\vec{n}_2|}$

7. The general homogeneous equation of second degree in x,y,z i.e.
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes if a b c + 2 f g
8. $h - af^2 - bg^2 - ch^2 = 0$ and angles between pair of plains is given by

$$\theta = \tan^{-1} \frac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a + b + c}$$

5.10 GLOSSARY

1. Perpendicular plane: normal of respective plane perpendicular to each other
2. Parallel plane: normal of respective plane parallel to each other
3. Angle between plane: angle between their respective normal

5.11 REFERENCES

1. Geometry of two and three Dimensions- Lata S., Himalaya Publishing House, Mumbai.Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
2. 3*D* Geometry- R.K. Pandey, Krishna Prakashan media (P) Ltd., Meerut.
3. Brannan, David A.; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press, ISBN 978-0-521-59787-6

5.12 SUGGESTED READING

1. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
2. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
3. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons

5.13 SOLVED EXAMPLE

Ex. 1 Find the angle between the planes 3x – 4y + 5z = 9 and 2x – y – 2z = 6

Sol.- The required angle θ between the given planes is the angle between their normals. The direction ratios of the normals of the given planes are 3, -4, 5 and 2, -1, -2 respectively.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{3.2 + (-4)(-1) + 5(-2)}{\sqrt{3^2 + (-4)^2 + 5^2}\sqrt{2^2 + (-1)^2 + 2^2}}$$

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

That is given planes are at right angles.

Ex. 2 Find the equation of the planes through the point (1, 2, 1) and parallel to the plane 2x

+ 3y + 4z = 5

Sol.- Equation of any plane parallel to the given plane is

$2x + 3y + 4z = \lambda$ .... (1)

If this passes through (1, 2, 1), then we have

$2(1) + 3(2) + 4(1) = \lambda$

$\Rightarrow \lambda = 12$

Putting this value of λ in equation (1), we get the required equation of the plane, that is $2x + 3y + 4z = 12$

Ex. 3- Find the equation of the plane through (2, 4, 5) and parallel to the plane $x + y + 2z = 5$

Sol.- Equation of any plane parallel to the given plane is

$x + y + 2z = \lambda$ .... (1)

It this plane passed through the point (2, 4, 5) we get $\lambda = 16$.

Putting this value of λ in equation (1), we get the required equation of the plane $x + y + 2z = 16$

Ex. 4- Find the angle between the planes $2x + 3y + 5z = 4$ and $x - 4y + 2z = 7$.

Sol.- Angle between given planes is the angle between their normals. The direction ratios of their normal are 2, 3, 5 and 1, -4, 2 respectively.

$$\cos\theta = \frac{2.1 + 3(-4) + 5(2)}{\sqrt{2^2 + 3^2 + 5^2}\ \sqrt{1^2 + (-4)^2 + 2^2}}$$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \dfrac{\pi}{2}$

Hence the given planes are orthogonal that is perpendicular.

Ex. 5- Find the equation of the plane through the point (2, 3,1) and (1, 5, 2), and perpendicular to the plane $2x + 4y + z = 9$.

Sol.- Let the equation of the plane be

$ax + by + cz + d = 0$ .... (i)

If it passes through (2, 3, 1), then

$2a + 3b + c + d = 0$ .... (ii)

If it passes through (1, 5, 2), then

$a + 5b + 2c + d = 0$ .... (iii)

The direction ratios of normal to given plane are 2, 4, 1 and direction ratios of normal to plane (i) is a, b, c. Both plane are perpendicular

Therefore

$2a + 4b + c = 0$ .... (iv)

Subtracting (iii) from (ii), we get

$a - 2b - c = 0$ .... (v)

Adding (iv) and (v), we get

$3a + 2b = 0$

$\Rightarrow 3a = -2b$

$\Rightarrow a = -\frac{2}{3}b$

Putting this value of a in (v), we get

$-\frac{2}{3}b - 2b - c = 0$

$\Rightarrow c = -\frac{8}{3}b$

Putting value of a and c in (ii)

$$2\left(\frac{-2}{3}b\right) + 3b - \frac{8}{3}b + d = 0$$

$$\Rightarrow -\frac{4}{3}b + 3b - \frac{8}{3}b + d = 0$$

$$\Rightarrow d = \frac{3}{3}b = b$$

Putting value of a, c, d in equation (i),we get

$$\frac{-2}{3}bx + by - \frac{8}{3}bz + b = 0$$

$-2x + 3y - 8z + 3 = 0$

$2x - 3y + 8z - 3 = 0$

which is required equation of the plane.

Ex.6. Show that the angle between the planes $2x + y + z = 7$ and $x - y + 2z = 5$ is $\frac{\pi}{3}$

Sol.- The direction ratios of normal of given planes are 2, 1, 1 and 1, - 1 respectively. Let angle between given planes be θ

$$\cos\theta = \frac{2\times 1 + 1\times -1 + 1\times 2}{\sqrt{4+1+1}\ \sqrt{1+1+4}}$$

$$\Rightarrow \cos\theta = \frac{2-1+2}{6}$$

$$\Rightarrow \cos\theta = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Ex.7. Find the equation of the plane through the points (1, 1, 1) and (2, 3, 4) and perpendicular to the plane $x + y + z = 0$

Sol.- Let the equation of the plane be

$ax + by + cz + d = 0$ .... (i)

If it passes through the points (1, 1, 1) and (2, 3, 4) then we get equation (ii) and (iii) respectively

$a + b + c + d = 0$ .... (ii)

$2z + 3b + 4c + d = 0$ .... (iii)

Plane (i) is perpendicular to given plane therefore we have-

$a + b + c = 0$ .... (iv)

Performing (ii) – (iii), we get $d = 0$

Putting value of $d = 0$ in equation (iii) we get

$2a + 3b + 4c = 0$ .... (v)

Multiplying (iv) by 2, then subtracting from (v), we get

$b + 2c = 0$

$\Rightarrow b = - 2c$

Again multiplying (iv) by 3, then subtracting from (v), we get

$- a + c = 0$

$\Rightarrow a = c$

Putting value of a, b, d in equation (i), we get

$cx – 2cy + cz + 0 = 0$

$\Rightarrow x – 2y + z = 0$

which is required equation of the plane.

ALTERNATE METHOD:

Now, $\vec{n} \perp AB$ and $\vec{n} \perp \vec{n}_1$

$\therefore n = \vec{n} \times AB$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

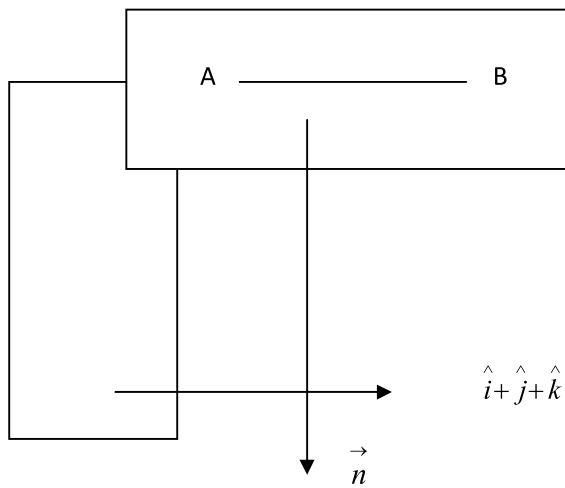


Fig 5.13.1

$= -\hat{i} + 2\hat{j} - \hat{k}$

$\equiv \hat{i} - 2\hat{j} + \hat{k}$

$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = (\hat{i} + \hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + \hat{k})$

$\Rightarrow x - 2y + z = 1 - 2 + 1 = 0$

Ex.8- Find the equation of the plane through (2, 3, 4) and (1, 1, 3) and parallel to the x-axis

Sol.- The equation of the plane through the point (2, 3, 4) is

$a(x – 2) + b(y – 3) + c(z – 4) = 0$ .... (i)

If this plane passes through the point (1, 1, 3) then we have

$a(1 - 2) + b(1 – 3) + c(3 – 4) = 0$

$\Rightarrow - a – 2b – c = 0$

$\Rightarrow a + 2b + c = 0$ .... (ii)

If the plane (i) is parallel to x – axis, then it is perpendicular to yz plane.

The equation of yz – plane is x = 0

$1.x + 0.y + 0.z = 0$

By condition of perpendicularity of two planes.

We have

$a.1 + b.0 + c.0 = 0$

$\Rightarrow a = 0$

Put a = 0 in (ii), we get

$c = -2b$

Putting value of a and c in equation (i), we get

$b(y - 3) - 2b(z - 4) = 0$

$\Rightarrow y - 2z + 5 = 0$, which is required equation of the plane.

Ex. 9 Find the equation of the plane through (2, 3, 4) and perpendicular to the line joining (3, 4, 5) and (5, 6, 7).

Sol.- The equation of the plane through the point (2, 3, 4) is

$a(x - 2) + b(y - 3) + c(z - 4) = 0$ .... (i)

The direction ratios of the line joining the points (3, 4, 5) and (5, 6, 7) are 2, 2,2.

This line is perpendicular to plane (i) therefore normal to the plane is parallel to this line.

By condition of parallel lines, we have

$$\frac{a}{2} = \frac{b}{2} = \frac{c}{2} = k\,(say)$$

$\Rightarrow a = 2k, \; b = 2k, \; c = 2k$

Putting values of a, b, c, is equation (i), we have

$2k\,(x - 2) + 2k\,(y - 3) + 2k\,(z - 4) = 0$

$\Rightarrow x + y + z = 9$ which is required equation of the plane.

Ex.10 Find the distance of the point (1, 2, 3) from the plane 2x + 3y + 4z = 9

Sol.- The given plane is 2x + 3y + 4z = 9

$\therefore$ The required distance is

$$= \frac{2.1 + 3.2 + 4.3 - 9}{\sqrt{4+9+16}}$$

$$= \frac{2 + 6 + 12 - 9}{\sqrt{29}}$$

$$= \frac{11}{\sqrt{29}}$$

Ex.11 Find the perpendicular distance from the origin to the plane 2x + 3y + 4z = 9. Find also the direction cosines of the normal to the plane.

Sol.- The equation of the plane is

$2x + 2y + z = 9$ .... (i)

Reducing the above form of the plane to normal form.

For this dividing above equation of plane by $\sqrt{4+4+1} = \sqrt{9} = 3$

$$\left(\frac{2}{3}\right)x + \left(\frac{2}{3}\right)y + \left(\frac{1}{3}\right)z = \frac{9}{3} = 3 \qquad \text{............ (ii)}$$

Comparing this equation to normal form, that is

$\ell x + my + nz - p$, we get perpendicular distance = p = 3

From equation (ii), we see the direction cosines of the normal to the plane are

$\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$.

Ex.- 12 Find the distance between the parallel planes- 2x - y + 3z – 5 = 0 and 4x – 2y + 6z + 12 = 0

Sol.- **First Method-**

Let point $P(x_1, y_1, z_1)$ be in the plane 2x – y + 3z – 5 = 0, then we have

$2x_1 - y_1 + 3z_1 - 5 = 0$ .... (i)

Now find the perpendicular distance from point $P(x_1, y_1, z_1)$ to plane 4x – 2y + 6z + 12 = 0

$$= \frac{4x_1 - 2y_1 + 6z_1 + 12}{\sqrt{4^2 + (-2)^2 + 6^2}}$$

$$= \frac{4x_1 - 2y_1 + 6z_1 + 12}{\sqrt{16 + 4 + 36}}$$

$$= \frac{2(2x_1 - y_1 + 3z_1) + 12}{\sqrt{56}}$$

$$= \frac{2(5)+12}{\sqrt{56}} = \frac{10+12}{\sqrt{56}} = \frac{22}{\sqrt{56}} = \frac{11}{\sqrt{14}}$$

which is required distance between given planes.

Second Method-

Reduce the given equations of both the planes to normal from, we get

$$\frac{2}{\sqrt{14}}x - \frac{1}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z = \frac{5}{\sqrt{14}} \quad \text{............ (i)}$$

Dividing second equation of plane that is 4x – 2y + 6z + 12= 0 by 2, we get

2x – y + 3z + 6 = 0

Now Reduce it to normal form.

$$\frac{2}{\sqrt{14}}x - \frac{1}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z = \frac{-6}{\sqrt{14}} \quad \text{............ (ii)}$$

From (i) and (ii), we get

$p_1 = \frac{5}{\sqrt{14}}$ and $p_2 = \frac{-6}{\sqrt{14}}$

Therefore required distance between given planes is

$$p_1 - p_2 = \frac{5}{\sqrt{14}} + \frac{6}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

Ex.- 13 Find the distance between the planes ax +by + cz + d = 0 and ax + by + cz + e= 0

Sol.- The given plane are parallel planes as their equation differ only in constant term.

Reducing both the plane into normal form by dividing $\sqrt{a^2 + b^2 + c^2}$, we get $p_1 = \frac{-d}{\sqrt{a^2+b^2+c^2}}$, $P_2 = \frac{-e}{\sqrt{a^2+b^2+c^2}}$

Distance between given planes are $\frac{-d+e}{\sqrt{a^2+b^2+c^2}}$

Ex.- 14 Find the equations of the planes parallel to the planex + 2y + 3z – 5 = 0 which are at a unit distance from the point (1, 2, 2).

Sol.- The equation of any plane parallel to the plane x + 2y + 3z – 5 = 0 is

x + 2y + 3z + λ = 0 .... (i)

Given that the distance of plane (i) from the point (1, 2, 2) is 1.

$$\Rightarrow \frac{1.1 + 2.2 + 3.2 + \lambda}{\sqrt{1^2 + 2^2 + 3^2}} = 1$$

$$\Rightarrow \frac{1 + 4 + 6 + \lambda}{\sqrt{14}} = 1$$

$$\Rightarrow \frac{\lambda + 11}{\sqrt{14}} = 1$$

$$\Rightarrow \lambda = \sqrt{14} - 11$$

Putting this value of λ in equation (i), we get required equation of plane.

Ex.15- Find the equations of the plane passes through (1, 0, -2) and perpendicular to both the planes 2x + y – z – 2 = 0 and x – y – z = 3

Sol.- The equation of the plane passes through the point (1, 0, -2) is

a(x – 1) + b(y – 0) + c(z + 2) = 0

ax + by + cz – a + 2c = 0 .... (i)

If the plane (i) is perpendicular to the plane 2x + y – z – 2 = 0, then we have

2a + b – c = 0 .... (ii)

If the plane (i) is perpendicular to the planex – y – z = 3, then we have

a – b – c = 0 .... (iii)

Adding (ii) and (iii), we get

3a – 2c = 0

$$\Rightarrow a = \left(\frac{2}{3}\right)c$$

Putting this value of a in (iii), we get

$$\frac{2}{3}c - b - c = 0$$

$$\Rightarrow - b - \frac{1}{3}c = 0$$

$$\Rightarrow b = - \left(\frac{1}{3}\right)c$$

Putting value of a and b in equation (i), we get

$$\left(\frac{2}{3}\right)cx - \left(\frac{1}{3}\right)cy + cz - \left(\frac{2}{3}\right)c + 2c = 0$$

$2x - y + 3z - 2 + 6 = 0$

$2x - y + 3z + 4 = 0$ which is required equation of the plane.

Ex. 16 Show that the points (1, 1, 1) and (-1, 1, 1) are on opposite sides of the plane

$2x + 2y + z - 4 = 0$

Sol. The value of the expression $2x + 2y + z - 4$ at the point (1, 1, 1) is 1, which is positive , the value of the expression $2x + 2y + z - 4$ at the point (- 1, 1, 1) is – 3, which is negative.

Hence the expression $2x + 2y + z - 4$ have opposite sign at given points therefore given points lie on opposite side of the plane.

Ex. 17- Find the equation of the plane through the line of intersection of the planes $2x + y + 3z + 9 = 0$, $x + 2y + z - 3 = 0$ and passing through the origin.

Sol.- The Equation of any plane passing through the line of intersection of the given planes is

$(2x + y + 3z + 9) + \lambda(x + 2y + z - 3) = 0$ .... (1)

Given that plane (1) is also passes through origin. Therefore, we have

$9 - 3\lambda = 0$ or $\lambda = 3$

Putting value of λ in equation (1), we get

$(2x + y + 3z + 9) + 3(x + 2y + z - 3) = 0$ or $5x + 7y + 6z = 0$

which is required equation of the plane.

Ex. 18- Show that the origin lies in the acute angle between the planes $x + 2y + 2z = 6$ and $4x - 3y + 12z + 12 = 0$. Find the planes bisecting the angles between them and the one which bisects the acute angle.

Sol.- The given planes are $x + 2y + 2z - 6 = 0$ and $4x - 3y + 12z + 12 = 0$

Making the constant term positive in above equations of planes

$-x - 2y - 2z + 6 = 0$ .... (i)

$4x - 3y + 12z + 12 = 0$ .... (ii)

Now evaluating $a_1a_2 + b_1b_2 + c_1c_2 = -4 + 6 - 24 = -22$ == Negative

Hence origin lies in the acute angle

The equation of planes bisecting the angles between the given planes is

$$\frac{-x - 2y + 2z + 6}{3} = \pm \frac{4x - 3y + 12z + 12}{13}$$

$\Rightarrow 13(-x - 2y - 2z + 6) = \pm 3(4x - 3y + 12z + 12)$

$\Rightarrow$- 13x – 26y – 26z + 78 = $\pm$ (12x - 9y + 36z + 36)

$\Rightarrow$-13x – 26y – 26z + 78 = 12x – 9y + 36 z + 36

$\Rightarrow$- 25x – 17 y – 62z + 42 = 0

$\Rightarrow$25 x + 17 y + 62 z – 42 = 0 and - 13x – 26y – 26z + 78 = - 12x + 9y -36z – 36
$\Rightarrow$- x – 35y + 10z + 114 = 0

The required bisecting planes are 25x + 17 y + 62z – 42 = 0 and x + 35y – 10z – 114 = 0

Let θ be the angle between (1) and 25x + 17y + 62z – 42 = 0, then

$$\cos\theta = \frac{-25-34-124}{3\sqrt{25^2+17^2+62^2}} = \frac{-183}{3\sqrt{4758}}$$

Ex. 19- Prove that the equation

$x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$

represents a pair of planes. Find the angle between them.

Sol.- Comparing the given equation with the equation

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

we get a = 2 ; b = - 6 ; c = - 12 ; f = 9 ; g = 1 ; h = ½

We know that if homogeneous equation of second degree represent pair of planes, then $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

In this case

$abc + 2fgh - af^2 - bg^2 - ch^2 = 144 + 9 - 162 + 6 + 3 = 162 - 162 = 0$

Hence the given equation represents a pair of planes.

If θ be the angle between the planes, then we have

$$\tan\theta = \frac{2\sqrt{(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c} = \frac{\sqrt{185}}{16}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{185}}{16}\right)$$

SELF CHECK QUESTIONS

1. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular if

a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ b) $a_1a_2 + b_1b_2 + c_1c_2 = 0$

c) $\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} = 0$ d) None of these

2. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if

a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ b) $a_1a_2 + b_1b^2 + c_1c_2 = 0$

c) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = 0$ d) None of these

3. The equation of the plane passes through origin and parallel to plane $ax + by + cz + d = 0$ is

a) $ax + by + cz + d = 0$ b) $ax + by + cz = 0$

c) $ax - by + cz = 0$ d) $ax + by - cz = 0$

4. The plane parallel to the plane $x + y + z + 5 = 0$ and through (a, b, c) is

a) $ax + by + cz = 0$ b) $x + y + z = a + b + c$

c) $x + y + z + a + b + c = 0$ d) None of these

5. The points (1,3,2) and (2,1,3) lie on the ------- of the plane $x - 2y + z + 5 = 0$.

a) Same side b) Opposite side

6. The angle between the planes $x - 2y + z - 3 = 0$ and $2x + 3y + 5z + 6 = 0$, in which the origin lies is

a) acute b) obtuse

5.14 *TERMINAL QUESTIONS*

EXERCISE- 5.1

Fill in the blanks-

1. If two planes are parallel, then direction ratios of their normals are __________
2. The equation of the plane through the point (2, 3, 5) and parallel to the plane $x + y + z + 5 = 0$ is
3. xy- plane and yz – plane are _________
4. If a plane is parallel to x- axis, the coefficient of x in its equation must be ________
5. The distance between parallel planes $x + 2y + 2z = 9$ and $x + 2y + 2z = 6$ is _____
6. The two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$are perpendicular if _______________
7. The distance from (1, 2, 3) to the plane $2x + 3y + 5z - 6 = 0$ is ___________

8. The angle between the planes $x - 2y + 2z + 10 = 0$ and $4x + 3y + z + 15 = 0$ is __________
9. The perpendicular distance from (0, 0, 0) to the plane $ax + by + cz + d = 0$ is _____
10. The points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) lie on the same or opposite sides of the plane $ax + by + cz + d = 0$ according as the expressions $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of _________
11. The plane through the line of intersection of planes $x+2y+z+9 = 0$; $3x+y+z-3 = 0$ and passes through origin is ___________
12. The equation of planes bisecting the angles of planes $2x+2y+z+5 = 0$ and $4x - 4y + 2z +3 = 0$ is __________
13. Equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of Planes if__________.
14. Equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of Planes, then these planes are perpendicular if______________

True or False-

1. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
2. The plane $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
3. The perpendicular distance from (0, 0, 0) to the plane $ax + by + cz + d = 0$ is d.
4. Angle between two planes are the angle between their normals.
5. Two parallel planes are differ only by constant term.
6. The x-axis is perpendicular to the yz-plane.
7. Angle between the planes $x + y + z = 0$ and $x - 2y + z = 0$ is $\frac{\pi}{2}$.
8. Distance from the point (2, 3, 5) to plane $2x + 2y + z = 0$ is 5.
9. Angle between the planes $2x + 4y + 4z = 15$ and $3x + 4y + 7z = 9$ is $\frac{\pi}{3}$.
10. The points (1,2,3) and (3,2,1) lie on the same side of the plane $x + y + z - 7 = 0$.
11. Equation $x^2 + y^2 + z^2 = 0$ represents a pair of planes.
12. Equation $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 0$ represents a pair of planes.

Multiple choice questions

1. The perpendicular distance from (a, b, c) to the plane $ax + bx + cz = 0$ is

   a) $\sqrt{a^2 + b^2 + c}$ b) $\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

   c) $a + b + c$ d) None of these

2. The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$ is

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{6}$

3. Two planes are perpendicular if angle between their normals are-

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{6}$

4. The perpendicular distance from origin to the plane $\ell x + my + nz = p$ is

a) p b) 0 c) 2p d) None of these

5. The equation of the plane through the origin and parallel to the plane

$4x + 7y + 9z + 14 = 0$ is

a) $4x + 7y + 9z = 0$ b) $4x - 7x + 9y = 0$

c) $4x + 7x - 9z = 0$ d) None of these

6. The planes $2x + 3y + 4z = 15$ and $4x + 6y + 8z = 9$ are

a) Perpendicular b) Parallel

c) Same d) None of these

EXERCISE- 5.2

1. Find the angle between the planes $3x + 4y - 5z = 15$ and $2x + 6y + 6z = 9$
2. Find the equation of the plane parallel to the plane $3x - 6y - 2z = 4$ at a distance 3 from the origin.
3. Find the distance of the point P(1, 2, 1) from the plane $x - 2y + 4z = 5$
4. Find the distance between parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 10 = 0$
5. Find the equation of the plane parallel to the plane $x + 2y - 2z + 4 = 0$ which are at a distance of 3 units from the point (2, 1, 1).
6. Show that the planes $4x + 3y - 5z = 10$ and $6x + 2y + 6z = 15$ are at right angle.
7. Find the equation of the plane parallel to the plane $2x - 2y - z + 1 = 0$ and at distance 2 units from the point (1, 2, 1).
8. Find the locus of a point such that the sum of squares of whose distance from the planes. $x + y + z = 0$, $x - y + z = 0$ is 5.
9. Show that the points (1,2,3) and (2,3,1) are on opposite side of the plane $3x + 4y - 8z + 10 = 0$
10. Find the planes bisecting the angles between planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$.
11. Find thebisector of the acute angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$.
12. Find the equation of the plane passing through the line of intersection of the planes $2x + y + 2z = 7$; $4x - 5y - 2z = 3$ and the point (1,1,1).
13. Prove that the equation $4x^2 + 8y^2 + z^2 - 6yz + 5zx - 12xy = 0$ represents a pair of planes

and find the angle between them.

5.15 *ANSWERS*

Answer of self cheque questions:

1.	b	2.	a	3.	a
4.	b	5.	a	6.	a

Exercise 5.1

Fill in the blanks

1. Proportional
2. x + y + z – 10 = 0
3. Perpendicular
4. Zero
5. 1
6. $a_1a_2 + b_1b_2 + c_1c_2 = 0$
7. $\frac{17}{\sqrt{38}}$
8. $\frac{\pi}{2}$
9. $\frac{-d}{\pm\sqrt{a^2+b^2+c^2}}$
10. Same or Opposite sign
11. 10x+5y+4z = 0
12. 8y +7 = 0 and 8x + 4z +13 = 0
13. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
14. a + b + c = 0

True or False

1. T
2. T
3. F
4. T
5. T
6. T
7. T
8. T
9. F
10. T
11. F
12. T

Multiple choice question

1. a
2. b
3. a
4. a
5. a
6. b

Exercise 5.2

1. $\frac{\pi}{2}$
2. $3x - 6y - 2z \pm 21 = 0$
3. $\frac{4}{\sqrt{21}}$
4. $\frac{2}{3}$
5. $x + 2y - 2z + 7 = 0$
6. $2x - 2y - z + 9 = 0$
8. $x^2 + y^2 + z^2 + 4xz = 15$
10. $x + 35y - 10z - 156 = 0$; $25x + 17y + 62z - 78 = 0$
11. $23x - 13y + 23z + 45 = 0$
12. $x + 4y + 4z - 9 = 0$.
13. $\tan^{-1}\left(\frac{\sqrt{29}}{13}\right)$

UNIT 6: THE STRAIGHT LINE

CONTENTS

6.1 *INTRODUCTION*

In the previous unit, we studied that every equation of the first degree in x, y, z represents a plane. Also two planes intersect in a line, therefore the two equations together represent that line. Thus equations $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d = 0$ represent the line of intersection of these planes. These equations are called general equation or unsymmetrical form of straight line.

6.2 *OBJECTIVES*

The main objective of this unit is to learn following contents.

- Equation of the straight line in symmetrical form.
- Transformation of General form into symmetrical form
- Plane through a given line
- Plane through a given line and parallel to another line
- Foot and length of perpendicular from a point to a line.
- Intersection of three planes.
- Shortest distance between two lines.

6.3 *EQUATION OF THE STRAIGHT LINE*

In this section we will discuss about the equation of a straight line.

6.3.1 EQUATION OF THE STRAIGHT LINE IN CARTESIAN FORM

Straight line is the locus of all points such that direction vector of it from O fixed point is always same.

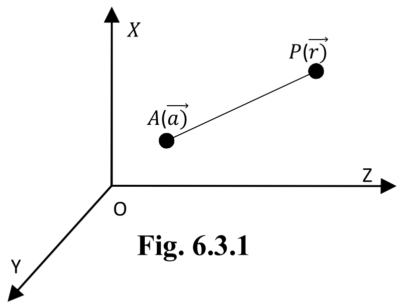


Fig. 6.3.1

$\overrightarrow{AP} = \vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{a}$ is fixed point and $\vec{b}$ is fixed direction.

In Cartesian form

$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} + \lambda(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$

$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda$.

6.3.2 EQUATION OF THE STRAIGHT LINE IN TERMS OF DIRECTION RATIOS

To find the equation of a straight line passing through a given point $A\ (\alpha, \beta, \gamma)$ and having direction cosines l, m, n.

Let $P(x, y, z)$ be any point on the line, which is at a distance r from A i.e. $AP = r$. Projection of AP on the $x-$axis is $x - \alpha$.

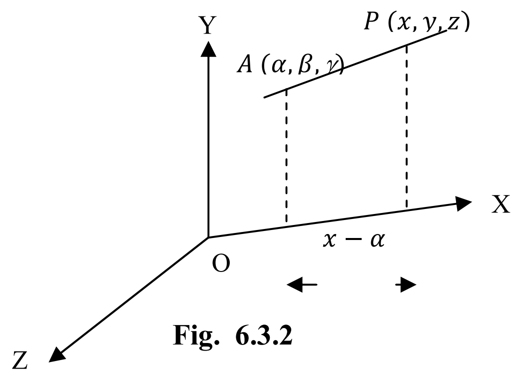


Fig. 6.3.2

Also it is lr, Hence we have

$x - \alpha = lr$ or $\frac{x-\alpha}{l} = r$

Similarly we get

$y - \beta = m\,r$ or $\frac{y-\beta}{m} = r$

and

$z - \gamma = n\,r$ or $\frac{z-\gamma}{n} = r$

$$\therefore \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \ldots\ldots\ldots\ldots \quad (6.3.1)$$

Equation (6.3.1) is symmetrical form of the straight line.

If direction ratios are a, b, c and we know the fact that direction cosines and direction ratios are proportional, that is

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \ldots\ldots\ldots\ldots \quad (6.3.2)$$

$$l = ka, \quad m = kb, \quad n = kc$$

Putting values of l, m, n in equation (6.3.1) we get

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$$

which is the equation of straight line in terms of direction ratios.

In vector form, $\vec{r} = \alpha\hat{\imath} + \beta\hat{\jmath} + \gamma\hat{k} + \lambda(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$

6.4 *LINE THROUGH TWO POINTS-*

Two find the equation of straight line through two given points.

Let two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Then direction ratios of the line

PQ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Therefore, Equation of the line PQ through the point $P(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

In vector form, $\vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1)$

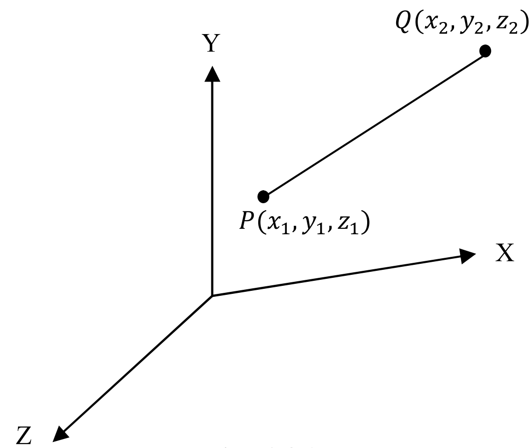


Fig. 6.4.1

6.5 TRANSFORMATION OF STRAIGHT LINE FROM GENERAL FORM TO SYMMETRICAL FORM -:

Let the general form of the straight line be given by the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.1)$$

In order to transform the general form of the straight line given by (6.5.1), we find out the direction cosines or direction ratios of the straight line and some point on the line.

Let l, m, n be the direction ratios of the line.

This line lies on both the planes, therefore it is perpendicular to the normal of both the planes. The direction ratios of the normal to the planes given by (6.5.1) are a_1, b_1, c_1 and a_2, b_2, c_2 respectively, therefore we have

$$la_1 + mb_1 + nc_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.2)$$

$$la_2 + mb_2 + nc_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.5.3)$$

Solving equation (6.5.2) and (6.5.3) for l, m, n, we have

$$\frac{\ell}{b_1 c_2 - b_2 c_1} = \frac{m}{c_1 a_2 - c_2 a_1} = \frac{n}{a_1 b_2 - a_2 b_1} \quad \ldots\ldots\ldots\ldots (4)$$

Equation (6.5.4) shows the direction ratios of the line are $b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$.

Now we find the coordinate of any point on the line in many ways. One of them is that we choose the point as the one where the line intersect the xy- plane that is the plane z = 0.
Putting $z = 0$ in equation (1), we get

$$a_1x + b_1y + d_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$a_2x + b_2y + d_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Solving (4) and (5), we get

$$\frac{x}{b_1d_2 - b_2d_1} = \frac{y}{a_2d_1 - a_1d_2} = \frac{1}{a_1b_2 - a_2b_1}$$

Or $x = \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$; $y = \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}$

Therefore, the coordinates of the point where the line meets the plane z = 0 is

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right)$$

The required equation of straight line (1) in symmetrical form is

$$\frac{x - \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right)}{b_1c_2 - b_2c_1} = \frac{y - \left(\dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}\right)}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

Note If $a_1b_2 - a_2b_1 = 0$, then we should choose the point where the line meets the plane yz i.e. $x = 0$ or plane zx i.e. $y = 0$.

In vector form:

Let $\vec{r}.\overrightarrow{n_1} = \alpha_1$ and $\vec{r}.\overrightarrow{n_2} = \alpha_2$ be the two planes then line vector is given $\overrightarrow{n_1} \times \overrightarrow{n_2}$ and let $\vec{a}$ be the fixed point on the line that we can find by putting $z = k$. We solve two equation and find x, y in terms of k i.e. $x = f(k), y = g(k),\ z = k$.

Thus equation of straight line in vector form will be

$\vec{r} = \vec{a} + \lambda(\overrightarrow{n_1} \times \overrightarrow{n_2})$ where $\vec{a} = f(k)\hat{\imath} + g(k)\hat{\jmath} + k\hat{k}$

6.6 *GENERAL POINT ON A LINE*

Let $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be the equation of a line then general point on the line can be written as $x = a + \lambda x_1, y = b + \lambda y_1, z = c + \lambda z_1$.

Similarly in vector form

$\vec{r} = \vec{a} + \lambda\vec{b}$ for different value of λ we get different points.

6.7 *INTERSECTION POINT OF A LINE AND A PLANE*

Let $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ be a straight line and $Ax+By+Cz+D=0$ be the plane to find the point of intersection we take a general point on the line and for intersection it lies on the plane

$x=a+rx_1, y=b+ry_1, z=c+rz_1$…………………………………….. (6.7.1)

Putting in plane

$$A(a+rx_1)+B(b+ry_1)+C(c+rz_1)+D=0$$

$$\Rightarrow r(Aa+Bb+Cc)=-[Ax_1+By_1+Cz_1+D]$$

$$\Rightarrow r=\frac{-[Ax_1+By_1+Cz_1+D]}{Aa+Bb+Cc}$$

Putting r in equation (6.7.1), we get (x,y,z) i.e. point of intersection of a line and a plane.

Similarly in vector form we put value of $\vec{r}=\vec{a}+\lambda\vec{b}$in the equation of the plane $\vec{r}.\vec{n}=\alpha$, we get

$$(\vec{a}+\lambda\vec{b}).\vec{n}=\alpha$$

$$\Rightarrow \lambda=\frac{\alpha-\vec{a}.\vec{n}}{\vec{b}.\vec{n}}$$

Putting this value of λ in straight line, we get point of intersection of a line and a plane.

i.e., $\vec{r}=\vec{a}+\left(\frac{\alpha-\vec{a}.\vec{n}}{\vec{b}.\vec{n}}\right)\vec{b}$.

SOLVED EXAMPLE

Ex 1- Find the equation of a straight line passes through (2, 3, 5) and whose direction ratio are 1, 2, 3. Also write vector equation of the line.

Sol.- We know that equation of straight line passes through point (α,β,γ) and whose direction ratios are isl,m,n

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Therefore the required equation of straight line is

$$\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-5}{3}$$

In vecror form $\vec{r}=2\hat{\imath}+3\hat{\jmath}+5\hat{k}+\lambda(\hat{\imath}+2\hat{\jmath}+3\hat{k})$.

Ex 2- Find the equation of a straight line through two points $(2,3,4)$ and $(4,6,9)$. Also write vector equation of the line.

Sol.- The required equation of straight line is

$$\frac{x-2}{4-2}=\frac{y-3}{6-3}=\frac{z-4}{9-4}$$

or $$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{5}$$

In vecror form $\vec{r}=2\hat{\imath}+3\hat{\jmath}+4\hat{k}+\lambda(2\hat{\imath}+3\hat{\jmath}+5\hat{k})$.

Ex 3- Find the coordinates of the point of intersection of the line .

$\frac{x-1}{1}=\frac{y+3}{2}=\frac{z-2}{3}$ with the plane $2x+3y+4z=21$

Sol.- Suppose $\frac{x-1}{1}=\frac{y+3}{2}=\frac{z-2}{3}=r$ (say)

Any point on the given line is $(r+1,2r-3,3r+2)$.

If the point $(r+1,2r-3,3r+2)$ lie on the plane, then

$\Rightarrow 2(r+1)+3(2r-3)+4(3r+2)=21$

$\Rightarrow 2r+2+6r-9+12r+8=21$

$\Rightarrow 20r=20$

$\Rightarrow r=1$

Putting the value of r in $(r+1,2r-3,3r+2)$, we get the required point, that is

$(2, -1, 5)$.

Ex 4- Find the ratio is which the line joining the points $(2, 4, 5)$ and $(4, 5, 7)$ is cut by the plane $x + y - 2z + 6 = 0$.

Sol.- The line joining the points $(2, 4, 5)$ and $(4, 5, 7)$ is

$$\frac{x-2}{4-2} = \frac{y-4}{5-4} = \frac{z-5}{7-5}$$

$$\frac{x-2}{2} = \frac{y-4}{1} = \frac{z-5}{2} = r \text{ (say)}$$

Any point on the line is $(2r + 2, r + 4, 2r + 5)$.

If this point, intersect with the plane, then

$$(2r + 2) + (r + 4) - 2(2r + 5) + 6 = 0$$

$$\Rightarrow -r + 2 = 0$$

$\Rightarrow r = 2$

Putting this value of r in $(2r + 2, r + 4, 2r + 5)$, we get the point $(6, 6, 9)$.

Suppose point $(6, 6, 9)$ divide the join of $(2, 4, 5)$ and $(4, 5, 7)$ in the ratio $m{:}\,n$, then

$$\frac{4m+2n}{m+n} = 6; \ \frac{5m+4n}{m+n} = 6; \ \frac{7m+5n}{m+n} = 9$$

which gives $\frac{m}{n} = \frac{2}{-1}$ i.e. $m : n = 2 : -1$

Minus sign indicate that point $(6, 6, 9)$ divide the join of $(2, 4, 5)$ and $(4, 5, 7)$ externally.

6.8 *PLANE THROUGH A GIVEN LINE*

In this section we will learn about the equation of a plane through a given line.

6.8.1 PLANE THROUGH A GIVEN LINE IN VECTOR FORM

Let $\vec{r} = \vec{a} + \lambda\vec{b}$ be a straight line then plane containing this line will be $\vec{r}.\vec{N} = \vec{a}.\vec{N}$ where $\vec{N}$ is normal vector. On extra conditions will be given for tat we will find $\vec{N}$.

If plane contains a point $\vec{c}$ then $\vec{N}$ will be perpendicular on $\vec{c} - \vec{a}$ asa well as on $\vec{b}$.

Therefore, $\vec{N} = (\vec{c} - \vec{a}) \times \vec{b}$ and hence required plane.

6.8.2 PLANE THROUGH A GIVEN LINE IN GENERAL AND SYMMETRIC FORM

To find the equation of a plane through a given line whose equation are given in

(a) General form (b) Symmetric form

a) Let the equation of the line in the general form be-

$$P \equiv a_1x + b_1y + c_1z + d_1 = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$Q \equiv a_2x + b_2y + c_2z + d_2 = 0$$

Then $P + \lambda Q = 0$ represents the plane through the line (1)

b) Let the equation of the line in symmetrical form be

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

As the required plane passes through line (2) so it passes through point (α, β, γ), which is a fixed point on this line.

The equation of plane through the point (α, β, γ) is

$a\,(x - \alpha) + b\,(y - \beta) + c\,(z - \gamma) = 0$…….................. (3)

The plane (3) is passes through the line (2), therefore line and normal to the plane are perpendicular. Consequently, we get

$al + bm + cn = 0$…………………………….............. (4)

Hence the required equation of the plane through the line (2) is given by (3) and (4), together.

6.9 PLANE THROUGH A GIVEN LINE AND PARALLEL TO ANOTHER LINE

To find the equation of a plane through a given line and parallel to another line .

In (6.7), we see that the equation of the plane through the line.

$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$ is

$a\,(x - \alpha_1) + b\,(y - \beta_1) + c\,(z - \gamma_1) = 0$ ……............ (1)

where $al_1 + bm_1 + cn_1 = 0$……………………............... (2)

If the plane (1) is parallel to another line

$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$ ……………………………................ (3)

Then normal to the plane given by (1) is perpendicular to the line given by (3).

Therefore

$al_2 + bm_2 + cn_2 = 0$……………………………............... (4)

Eliminating a, b, c from equation (1), (2) and (4), we get the required equation of plane, i.e.

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

IN VECTOR FORM

Equation of a plane passing through a given line $\vec{r} = \vec{a} + \lambda\vec{b}$ and parallel to another line $\vec{r_1} = \vec{a_1} + \lambda_1\vec{b_1}$.

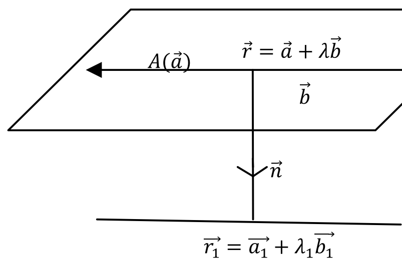


Fig 6.9.1

In this case plane passing through $\vec{a}$ and perpendicular to $\vec{b}$ and $\vec{b_1}$.

Therefore $\vec{n} = \vec{b} \times \vec{b_1}$.

Therefore, equation of required plane will be $\vec{r}.(\vec{b} \times \vec{b_1}) = \vec{a}.(\vec{b} \times \vec{b_1})$.

6.10 COPLANAR LINES

Lines lie in the same plane is called coplanar line.

In (6.9) if line (1) and (2) are coplanar, then point (α_2, β_2, γ_2) also lies in plane (5), therefore we have

$$\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

which is required condition for line (1) and (2) are coplanar.

In vector form:

Let $\vec{r_1} = \vec{a_1} + \lambda_1\vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \lambda_2\vec{b_2}$ be two lines. For coplanar normal vector to the plane will be $\overrightarrow{b_1} \times \overrightarrow{b_2}$, $\vec{n} = \overrightarrow{b_1} \times \overrightarrow{b_2}$ and $\overrightarrow{AB} = \overrightarrow{a_2} - \overrightarrow{a_1}$.

In case of coplanarity

$$\overrightarrow{AB}.\vec{n} = 0$$

$\Rightarrow(\overrightarrow{a_2} - \overrightarrow{a_1}).\overrightarrow{b_1} \times \overrightarrow{b_2} = 0$

$$\Rightarrow [\overrightarrow{a_2} - \overrightarrow{a_1}, \overrightarrow{b_1}, \overrightarrow{b_2}] = 0$$

In cartesian form

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

SOLVED EXAMPLE

Ex 1- Find the condition that the line

$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ may be

4. Perpendicular to the plane

ii) Parallel to the plane

iii) Lie on the plane.

Where the plane is $2x + 3y + 4z = 10$.

Sol.- i) If the line is perpendicular to the plane then line is parallel to the normal of the given plane and hence we have the condition.

$$\frac{\ell}{2} = \frac{m}{3} = \frac{n}{4}$$

ii)If the line is parallel to the plane then line is perpendicular to the normal of the given plane therefore we have the condition.

$2\ell + 3m + 4n = 0$ and $2\alpha + 3\beta + 4\gamma - 10 \neq 0$

iii) If the line lie on the plane then point (α, β, γ) lie on the plane and line is perpendicular to the normal of the given plane. Therefore we have the condition.

$2\ell + 3m + 4n = 0$ and $2\alpha + 3\beta + 4\gamma - 10 = 0$

Ex 2- Prove that the line

$\frac{x-1}{1} = \frac{y-2}{3} = \frac{z-3}{3}$ is parallel tothe plane $3x - 2y + z = 15.$

Sol.- The direction ratios of given line are $1, 3, 3$ and direction ratios of normal to the plane are $3, -2, 1$.

We know that if the line is parallel to the plane then line is perpendicular to the normal of given plane therefore , we have

$1(3) + 3(-2) + 3(1) = 3 - 6 + 3 = 0$

Hence the given line is parallel to the given plane.

Ex 3- Find the equation of the plane through the point $(2, -3, 4)$ and the line

$2x + 3y - z + 10 = 0 = x - 2y + z + 6$

Sol.- The equation of plane through the given line is

$(2x + 3y - z + 10) + \lambda(x - 2y + z + 6) = 0$………….................. (1)

If this plane passes through the point $(2, -3, 4)$, then we have

$(2(2) + 3(-3) - 4 + 10) + \lambda(2 - 2(-3) + 4 + 6) = 0$

$\Rightarrow (4 - 9 - 4 + 10) + \lambda(2 + 6 + 4 + 6) = 0$

$\Rightarrow 1 + \lambda(18) = 0$

$\Rightarrow \lambda = -\frac{1}{18}$

Putting value of λ in equation (1), we get

$36x + 54y - 18z + 180 - x + 2y - z - 6 = 0$

$$\Rightarrow 35x + 56y - 19z + 174 = 0$$

which is required equation of plane.

Ex 4- Find the equation of the plane through the line

$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-5}{4}$ and parallel to y–axis.

Sol.- The given line is $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-5}{4}$

Given line is passes through the point $(1, 3, 5)$ and direction ratios of given line are $2, 3, 4$.

Equation of any plane through the given line is

$a(x - 1) + b(y - 3) + c(z - 5) = 0$.. (1)

where $2a + 3b + 4c = 0$.. (2)

If the plane (1) is parallel to y –axis, whose direction ratio's are $0, 1, 0$, then normal to this plane is right angle with y – axis, therefore we have

$a(0) + b(1) + c(0) = 0$

$\Rightarrow b = 0$

Putting value of $b = 0$ in equation (2), we get

$2a + 4c = 0$

$\Rightarrow a = -2c$

Putting value of a and b in equation (1), we get

$-2c(x - 1) + 0(y - 3) + c(z - 5) = 0$

$\Rightarrow -2c(x - 1) + c(z - 5) = 0$

$\Rightarrow -2x + 2 + z - 5 = 0$

$\Rightarrow 2x - z + 3 = 0$

which is required equation of the plane.

6.11 FOOT AND LENGTH OF PERPENDICULAR FROM A POINT TO A LINE

To find the perpendicular distance of a point P(x,y,z) from a given line, when its equation are given in symmetric form

Let the equation of the line be

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r(say) \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Any point on the line (1) is $N(\alpha + \ell r, \beta + mr, \gamma + nr)$

If this point N is foot of the perpendicular from $P(x_1, y_1, z_1)$, then the line PN is perpendicular to (1). The direction ratios of the line PN are

$\alpha + \ell r - x_1, \beta + mr - y_1, \gamma + nr - z_1.$

Given line (1) and PN are perpendicular, so we have

$$\ell(\alpha + \ell r - x_1) + m(\beta + mr - y_1) + n(+nr - z_1) = 0$$

$$\Rightarrow (\ell^2 + m^2 + n^2)r = (\ell x_1 + my_1 + nz_1) - (\ell\alpha + m\beta + n\gamma)$$

$$\Rightarrow r = \frac{\ell(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{(\ell^2+m^2+n^2)}$$

Substituting this value of r in $(\alpha + \ell r, \beta + mr, \gamma + nr)$.We can find the coordinates of N, the foot of perpendicular and perpendicular distance PN.

Second Method-

In right angled triangle PNA,

we have

$PA^2 = AN^2 + PN^2$

$PN^2 = PA^2 - AN^2$

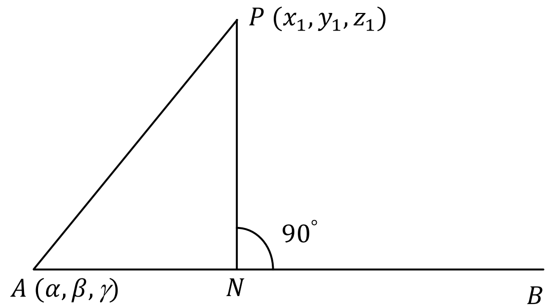


Fig. 6.11.1

where $PA =$ distance between points $P(x_1, y_1, z_1)$ and $A(\alpha, \beta, \gamma)$

$= \sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2}$

And AN $=$ Projection of AP in AB

$$= (x_1 - \alpha)\ell + (y_1 - \beta)m + (z_1 - \gamma)n$$

In vector form:

Foot of perpendicular and length of perpendicular a point $\vec{a_1}$ on a line $\vec{r} = \vec{a} + \lambda\vec{b}$,

$\overrightarrow{AP} = \vec{r} - \vec{a_1}$as $\overrightarrow{AP}$is perpendicular on the line.therefore,

$\overrightarrow{AP}.\vec{b} = 0$

i.e. $(\vec{r} - \vec{a_1}).\vec{b} = 0$.

$\Rightarrow (\vec{r} - \vec{a_1}).\vec{b} = 0$

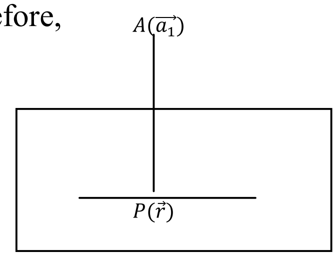


Fig. 6.11.2

$$\Rightarrow (\vec{a} + \lambda\vec{b} - \vec{a_1}).\vec{b} = 0$$

$$\Rightarrow \lambda = \frac{(\vec{a} - \vec{a_1}).\vec{b}}{|\vec{b}|^2}$$

Putting λ in $\vec{r} = \vec{a} + \lambda\vec{b}$

$= \vec{a} + \frac{(\vec{a}-\vec{a_1}).\vec{b}}{|\vec{b}|^2}.\vec{b}$ is the foot of perpendicular and

Length of perpendicular $=|\vec{r} - \vec{a_1}|$

6.12 *INTERSECTION OF THREE PLANES –*

Let the equation of three planes be given by

$P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ .... (1)

$P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ .... (2)

$P_3 \equiv a_3x + b_3y + c_3z + d_3 = 0$ .... (3)

Now if we take two equations at a time, we get three lines of intersection of the planes P_1, P_2, P_3. The following three cases arise

(1) Three lines of intersection of three planes may coincide. In this case, the three given planes have a common line of intersection.

(2) Three lines of intersection of three planes may be parallel to each other and no two of them coincide. In this case the three given planes form a triangular prism.

(3) Three lines of intersection of three planes may intersect in a point. In this case the three planes intersect in a point.

Rectangular array formed by the coefficient of equations (1), (2), (3) is $D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$

Case – 1 – The three planes intersect in a common line.

The equation of any plane through the line of intersection of planes (1) and (2) is given by $P_1 + \lambda P_2 = 0$ i.e.

$(a_1 + \lambda a_2)\, x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0$ .... (4)

If the three plane planes intersect in a common line, then for some value of λequation (4) should represent the plane (3). Therefore comparing the coefficients of equation (3) and equation (4), we get

$$\frac{a_1+\lambda a_2}{a_3} = \frac{b_1+\lambda b_2}{b_3} = \frac{c_1+\lambda c_2}{c_3} = \frac{d_1+\lambda d_2}{d_3} = m\ (say)$$

or $a_1 + \lambda a_2 - ma_3 = 0$

$0b_1 + \lambda b_2 - mb_3 = 0$

$$c_1 + \lambda c_2 - mc_3 = 0$$

$$c_1 + \lambda d_2 - md_3 = 0$$

Now eliminating λ and m from any three equations out of four equations given above, we get the conditions

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

i.e. $D_4 = D_1 = D_2 = D_3 = 0$

which is required conditions for the three planes P_1, P_2, P_3 intersect in a common line.

Case-2 – The three planes from a triangular prism.

The three planes will form a triangular priam if the line of intersection of any two planes is parallel to third plane and does not lie in it.

The line of intersection of the plane P_1, P_2 is

$$\frac{x-\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1}=\frac{y-\left(\dfrac{a_2d_1-a_1d_2}{a_1b_2-a_2b_1}\right)}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1} \text{ where } a_1b_2-a_2b_1\neq 0$$

The direction ratios of above line are $b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$. If this line is parallel to the plane P_3, then the normal to the plane P_3 and line are perpendicular therefore we have

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

which in determinant form $D_4 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

The above line does not lie in plane P_3, therefore

$$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)+c_3(0)+d_3\neq 0$$

$$a_3(b_1d_2 - b_2d_1) + b_3(d_1a_2 - d_2a_1) + d_3(a_1b_2 - a_2b_1) \neq 0$$

which in determinant form. $D_3 = \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$

Case-3-The three planes intersect in a point. The three planes P_1, P_2, P_3 intersect in a point. Solving equation (1), (2), (3) by Crammer's rule, we get

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\frac{x_1}{D_1}=\frac{-y}{D_2}=\frac{z}{D_3}=\frac{-1}{D_4}$$

$$x = -\frac{D_1}{D_4}, \quad y = \frac{D_2}{D4}, \qquad z = \frac{D_3}{D4}$$

We get a point if $D_4 \neq 0$, therefore $D_4 \neq 0$ is the required condition.

6.13 *SHORTEST DISTANCE BETWEEN TWO LINES*

Skew lines- Two lines are said to be skew lines or non intersecting lines if they do not lie in the same plane.

Shortest distance – The straight line which is perpendicular to each of the two skew lines is called the line of shortest distance. The length of this line intercepted between the given lines is called the length of shortest distance.

Method – 1 – Length and equation of shortest distance.

Let the equation of the given lines be

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} = r_1(say) \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} = r_2(say) \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Any point on line (1) is $P(\alpha_1 + \ell_1 r_1, \beta_1 + m_1 r_1, \gamma_1 + n_1 r_1)$ and any point on line (2) is $Q\,(\alpha_2 + \ell_2 r_2, \beta_2 + m_2 r_2, \gamma_2 + n_2 r_2)$. If the line PQ is line of shortest distance, then PQ is perpendicular to line (1) as well as line (2). Therefore, by condition of two perpendicular lines, we have

$$\ell_1(\alpha_2 + \ell_2 r_2 - \alpha_1 - \ell_1 r_1) + m_1(\beta_2 + m_2 r_2 - \beta_1 - m_1 r_1) + n_1(\gamma_2 + n_2 r_2 - \gamma_1 - n_1 r_1) = 0$$

$$(\ell_1\ell_2 + m_1 m_2 + n_1 n_2)r_2 - (\ell_1^2 + m_1^2 + n_1^2)r_1 + \ell_1(\alpha_2 - \alpha_1) + m_1(\beta_2 - \beta_1) + n_1(\gamma_2 - \gamma_1) = 0 \ldots (3)$$

$$(\ell_2^2 + m_2^2 + n_2^2)r_2 - (\ell_1\ell_2 + m_1 m_2 + n_1 n_2)r_1 + \ell_2(\alpha_2 - \alpha_1) + m_2(\beta_2 - \beta_1) + n_2(\gamma_2 - \gamma_1) = 0$$

(4)

Solving equation (3) and (4), to evaluate r_1 and r_2. After obtaining r_1 and r_2, we find coordinates of points P and Q, then obtain the equation of shortest distance line PQ and shortest distance PQ.

Method – 2 – Let PQ be the line which is perpendicular to both the given line (1) and (2). Let ℓ, m, n be the direction ratios of the line PQ.

PQ is perpendicular to line (1) and (2), so we have

$$\ell \ell_1 + m m_1 + n n_1 = 0 \qquad \text{.................... (5)}$$

$$\ell \ell_2 + m m_2 + n n_2 = 0 \qquad \text{.................... (6)}$$

Solving (5) and (6), we get

$$\frac{\ell}{m_1 n_2 - m_2 n_1} = \frac{m}{\ell_2 n_1 - \ell_1 n_2} = \frac{n}{\ell_1 m_2 - \ell_2 m_1}$$

Taking $L = m_1 n_2 - m_2 n_1,\ M = \ell_2 n_1 - \ell_1 n_2,\ N = \ell_1 m_2 - \ell_2 m_1$

Therefore the direction cosines of the line PQ is

$$\frac{L}{\sqrt{L^2+M^2+N^2}}, \frac{M}{\sqrt{L^2+M^2+N^2}}, \frac{N}{\sqrt{L^2+M^2+N^2}}$$

Shortest distance PQ = Projection of the line AB

$$= \frac{L(\alpha_2-\alpha_1) + M(\beta_2-\beta_1) + N(\gamma_2-\gamma_1)}{\sqrt{L^2+M^2+N^2}}$$

$$= \frac{1}{\sqrt{L^2+M^2+N^2}} \begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

Note If given lines are coplanar then $S.D. = 0$.

IN VECTOR FORM -

Let $\overrightarrow{r_1} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\overrightarrow{r_2} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ be the two lines

$$\overrightarrow{AB} = \overrightarrow{r_2} - \overrightarrow{r_1}$$

$$= \overrightarrow{a_2} - \overrightarrow{a_1} + \mu\overrightarrow{b_2} - \lambda\overrightarrow{b_1}$$

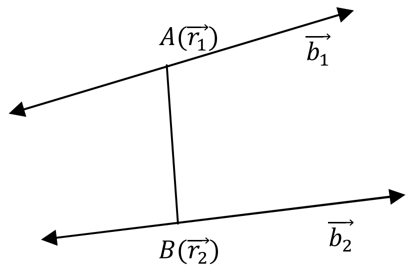


Fig.6.13.1

As $\overrightarrow{AB}$ perpendicular to $\overrightarrow{b_1}$ as well as $\overrightarrow{AB}$ perpendicular to $\overrightarrow{b_2}$.

$\Rightarrow \overrightarrow{AB}.\overrightarrow{b_1} = 0$ and $\overrightarrow{AB}.\overrightarrow{b_2} = 0$.

We get two equations in λ and μ

i.e. $(\overrightarrow{a_2} - \overrightarrow{a_1}).\overrightarrow{b_1} + \mu\overrightarrow{b_2}.\overrightarrow{b_1} - \lambda\left|\overrightarrow{b_1}\right|^2 = 0$ and $(\overrightarrow{a_2} - \overrightarrow{a_1}).\overrightarrow{b_2} + \mu\left|\overrightarrow{b_2}\right|^2 - \lambda\,\overrightarrow{b_2}.\overrightarrow{b_1} = 0$

Putting value in AB, we get

Shortest distance $= \left|\overrightarrow{AB}\right|$ and equation of shortest distance is $\vec{r} = \overrightarrow{r_1} + k(\overrightarrow{b_1} \times \overrightarrow{b_2})$

6.14 SUMMARY

1. General equation of a straight line is $ax + by + cz\, l_1\, d = 0 = a_1x + b_1y + c_1z + d_1$.
2. Symmetrical form of a straight line passing through (α, β, γ) with direction ratios $\ell\ m, n$ is $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

3. Equation of straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

4. Equation of a plane through a line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is

$a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$, where $a\ell + bm + cn = 0$

5. Condition, when a line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ lies on the plane

$ax + by + cz + d = 0$ is $a\alpha + b\beta + c\gamma + d = 0$ and $a\ell + bm + cn = 0$

6. Equation of a plane through a line $\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}$ and parallel to another

line $\frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ is

$$(m_1n_2 - m_2n_1)(x - \alpha_1) + (\ell_2n_1 - \ell_1n_2)(y - \beta_1) + (\ell_1m_2 - \ell_2m_1)(z - \gamma_1) = 0$$

7. Coplanar lines are those lines which lies on the same plane, the S.D. between coplanar lines are zero.

8. S.D. between the lines $\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}$ and $\frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$

$$\frac{1}{\sqrt{\Sigma(m_1n_2-m_2n_1)}}\begin{vmatrix}\alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}$$

9. Three planes $a_ix + b_iy + c_iz + d_i = 0;\ i = 1, 2, 3$ intersect in a point if $D_4 \neq 0$

10. Three planes intersect in a line if $D_4 = 0 = D_1 = D_2 = D_3$

11. Three planes intersect to form a triangular prism if $D_4 = 0$ and $D_1 \neq 0, D_2 \neq 0, D_3 \neq 0$

6.15 *GLOSSARY*

1. Straight line: locus of all points such that direction vector of it from O fixed point is always same.
2. Coplanar line: Lines lie in the same plane is called coplanar line.
3. Skew lines- lines do not lie in the same plane.
4. Shortest distance – straight line which is perpendicular to each of the two skew lines

6.16 *REFERENCES*

1. Geometry of two and three Dimensions- Lata S., Himalaya Publishing House, Mumbai.Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
2. $3D$ Geometry- R.K. Pandey, Krishna Prakashan media (P) Ltd., Meerut.
3. Brannan, David A.; Esplen, Matthew F.; Gray, Jeremy J. (1999), Geometry, Cambridge University Press, ISBN 978-0-521-59787-6

6.17 *SUGGESTED READING*

1. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
2. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
3. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons

6.18 *SOLVED EXAMPLE*

SOLVED EXAMPLE- 6.1

Ex 1- Find the distance of the point $(2, 3, 5)$ from the plane $2x + 2y + 3z - 34 = 0$, measured parallel to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}.$$

Sol.- The line through $(2,3,5)$ and parallel to the given line is

$$\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-5}{1}= r \text{ (say)}$$

Any point on the line is $(r + 2, 2r + 3, r + 5)$.

If this point lie on the given plane, then

$\Rightarrow 2(r + 2) + 2(2r + 3) + 3(r + 5) - 34 = 0$

$\Rightarrow 9r + 25 - 34 = 0$

$\Rightarrow 9r - 9 = 0$ or $r = 1$

The given line intersect the given plane on the point $(3,5,6)$

The distance between point $(2,3,5)$ and $(3,5,6)$ is $\sqrt{6}$.

Ex 2- Find symmetrical form the equation of the line given by

$x - 2y + 3z = 4$; $2x - 3y + 4z = 5$

Sol.- Let ℓ, m, n be the direction ratios of the line. Then as this line on both the given planes, so it is perpendicular to the normal of these planes, therefore we have

$\ell - 2m + 3n = 0$

$2\ell - 3m + 4n = 0$

Solving above equations, we get

$$\frac{\ell}{-8+9}=\frac{m}{6-4}=\frac{n}{-3+4}$$

$$\frac{\ell}{1}=\frac{m}{2}=\frac{n}{1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Putting $z = 0$ in the given equation of line, we get

$x - 2y = 4$

$2x - 3y = 5$

Solving these, we get $x = -2$ and $y = -3$.

$\therefore$ The required line meets the plane $z = 0$at $(-2, -3, 0)$.

The required equation of straight line in symmetrical form is

$$\frac{x+2}{1} = \frac{y+3}{2} = \frac{z-0}{1}$$

Or $x + 2 = \frac{y+3}{2} = z$

Ex 3- Find symmetrical form of the equation of the line given by

$x = ay + b\,;\, z = cy + d$

Sol.- The equations of the given planes can be written as

$x - ay - b = 0\ ;\ z - cy - d = 0$

The direction ratios normal to these planes are $1, -a, 0$ and $0, -c, 1$ respectively.

Let $\ell,\ m,\ n$ be the direction ratios of the line in symmetrical form

We know that normals of these planes are perpendicular to the line, therefore we have

$\ell.1 + m(-a) + n.0 = 0$

$\Rightarrow \ell.0 + m(-c) + n.1 = 0$

$\Rightarrow \ell - a\,m = 0$

$\Rightarrow -cm + n = 0$

$\Rightarrow \frac{\ell}{a} = \frac{m}{1} = \frac{n}{c}$ .... (1)

Putting $y = 0$ in the given equations of planes,

We get $x = b$ and $z = d$

Therefore, the required line meets the plane $y = 0$ at the point $(b, 0, d)$

The required equation of straight line in symmetrical form is

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

Ex 4- Find the equation of the line through the point $(1, 2, 3)$ and parallel to the line

$x - y + 2z = 4$; $3x + y + z = 5$

Sol.- Let ℓ, m, n be the direction ratios of the line

The line through the point $(1, 2, 3)$ is

$$\frac{x-1}{\ell} = \frac{y-2}{m} = \frac{z-3}{n} \quad \ldots\ldots\ldots (1)$$

Line (1) is parallel to the line $x - y + 2z = 4;\ 3x + y + z = 5$. i.e. line (1) is perpendicular to the naormals of the planes x – y + 2z = 4 and 3x + y + z = 5. Therefore, we have

$\ell - m + 2n = 0$

$3\ell + m + n = 0$

On solving above equations, we get

$$\frac{\ell}{-1-2} = \frac{m}{6-1} = \frac{n}{1+3}$$

$$\Rightarrow \frac{\ell}{-3} = \frac{m}{5} = \frac{n}{4} \quad \ldots\ldots\ldots (2)$$

From (1) and (2) we get required equation of line

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

Ex 5- Prove that the lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$

Sol.- The given equations of lines are

$$\left.\begin{array}{l} x = ay + b \\ z = cy + d \end{array}\right\} \ldots\ldots(1)$$

and

$$\left.\begin{array}{l} x = a'y + b' \\ z = c'y + d' \end{array}\right\} \ldots\ldots(2)$$

From equation (1), we have

$$\frac{x-b}{a} = y, \quad \frac{z-d}{c} = y$$

$$\Rightarrow \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c} \ldots\ldots (3)$$

which is symmetrical form of straight line (1) (also see example 7)

Similarly from equation (2), we have

$$\frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'} \ldots\ldots(4)$$

which is symmetrical form of straight line (2). From (3) and (4), we see the direction ratios of the lines are $a, 1, c$ and $a', 1, c'$.

If line are perpendicular, then, we have

$aa' + 1 + cc' = 0$ or $aa' + cc' + 1 = 0$

Hence proved .

The line through the point (1, 2, 3) is

$$\frac{x-1}{\ell} = \frac{y-2}{m} = \frac{z-3}{n} \ldots\ldots (5)$$

SOLVED EXAMPLE- 6.2

Ex 1- Find the equation of the plane determined by the parallel lines

$$\frac{(x+1)}{2} = \frac{(y-2)}{3} = \frac{z}{1} \text{ and } \frac{(x-3)}{2} = \frac{(y+4)}{3} = \frac{z-1}{1}.$$

Sol.- To find the plane by the two parallel lines, means to find the plane through two points i.e. points $(-1, 2, 0)$ and $(3, -4, 1)$, also normal to the plane is perpendicular to given lines .

The plane through the point $(-1, 2, 0)$ is

$\Rightarrow a(x + 1) + b(y - 2) + c(z - 0) = 0$... (1)

If it passes through $(3, -4, 1)$, then we have

$a(3 + 1) + b(-4 - 2) + c(1) = 0$

$4a - 6b + c = 0$... (2)

Direction ratios of normal to the plane are a, b, c, and direction ratios of given line are 2, 3, 1, both are perpendicular, therefore we have

$2a + 3b + c = 0$... (3)

Solving (2) and (3), for a, b, c, we have

$$\frac{a}{-6-3} = \frac{b}{2-4} = \frac{c}{12+12}$$

$$\Rightarrow \frac{a}{-9} = \frac{b}{-2} = \frac{c}{24}$$

$$\Rightarrow \frac{a}{9} = \frac{b}{2} = \frac{c}{-24} = k(say) \qquad \text{.................. (4)}$$

Substituting the value of a, b, c, in equation (1), we get required equation of plane

$$9x + 2y - 24z + 5 = 0$$

Ex 2- Show that the lines $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{4}$ and $\frac{x-5}{3} = \frac{y-7}{2} = \frac{z-9}{1}$ are coplanar and find the equation of the plane containing them.

Sol.- The equation of plane containing given lines is

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x - 1)(3 - 8) - (y - 1)(2 - 12) + (z - 1)(4 - 9) = 0$

$\Rightarrow -5(x - 1) + 10(y - 1) - 5(z - 1) = 0$

$\Rightarrow -5x + 10y - 5z = 0$

$\Rightarrow x - 2y + z = 0$ ……………….. (1)

If the given lines are coplanar, then point $(5, 7, 9)$ also lies in the plane (1).

$5 - 2 \times 7 + 9 = 5 - 14 + 9 = 0$

Hence given lines are coplanar .

Ex 3- Show that the lines $x = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ are coplanar and find the point of intersection.

Sol.- If given lines are coplanar then the value of the determinant

$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$ should be zero.

$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$ (Since two rows are identical)

Hence the given lines are coplanar

$\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3} = r_1$ (say)

$\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4} = r_2$ (say)

Any point on line first is $P(r_1, 2r_1 + 2, 3r_1 - 3)$ and second is $Q(2r_2 + 2, 3r_2\ 6, 4r_2 + 3)$.

If the given lines intersect then points P and Q coincide for some value of r_1 and r_2.

Therefore , we have

$r_1 = 2r_2 + 2$ or $r_1 - 2r_2 = 2$ .... (1)

$2r_1 + 2 = 3r_2 + 6$ or $2r_1 - 3r_2 = 4$ .... (2)

$3r_1 - 3 = 4r_2 + 3$ or $3r_1 - 4r_2 = 6$ .... (3)

Solving (1) and (2), we get $r_1 = 2$ and $r_2 = 0$, which also satisfy equation (3), therefore the required point of intersection is (2, 6, 3).

SOLVED EXAMPLE- 6.3

Ex 1- Find the equation of the perpendicular from the point $(3, 1, 2)$ to the line

$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-8}{1}$. Find also the coordinates of the foot of perpendicular. Hence find length of perpendicular.

Sol.- The equation of given line is

$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-8}{1} = r$ (say)... (1)

Any point on the line is $N(r + 1, 2r + 2, r + 8)$.

Let this point N be the foot of the perpendicular from the point P $(3, 1, 2)$ to the line (1). The direction ratios of the perpendicular PN are $r - 2, 2r + 1, r + 6$. The direction ratios of the given line (1) are $1, 2, 1$. Line PN is perpendicular to the line (1), therefore we have

$1(r - 2) + 2(2r + 1) + 1(r + 6) = 0$

$\Rightarrow r - 2 + 4r + 2 + r + 6 = 0$

$\Rightarrow 6r + 6 = 0$

$\Rightarrow r = -1$

Putting this value of r, we get the coordinates of point N ie. $(0, 0, 7)$ and the direction ratios of PN are $-3, -1, 5$.

Hence the equation of the perpendicular PN through point P

$$\frac{x-3}{-3} = \frac{y-1}{-1} = \frac{z-2}{5}$$

The length of the perpendicular $PN = \sqrt{9+1+25}$

$$= \sqrt{35}$$

Ex 2- Prove that the equation of the perpendicular from the point $(1, 6, 3)$ to the line

$x = \frac{y-1}{2} = \frac{z-2}{3}$ are $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$ and coordinates of the foot of the perpendicular are $(1, 3, 5)$.

Sol.- The equation of the given line is

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r \text{ (say)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Any point on the line (1) are $N(r, 2r + 1, 3r + 2)$.

Let this point N be the foot of the perpendicular from the point $P(1, 6, 3)$ to the line (1). The direction ratios of the perpendicular PN are $r - 1, 2r - 5, 3r - 1$. The direction ratios of the given line (1) are $1, 2, 3$. Line (1) is perpendicular to PN, therefore we have

$$1(r - 1) + 2(2r - 5) + 3(3r - 1) = 0$$

$$\Rightarrow r + 4r + 9r - 1 - 10 - 3 = 0$$

$$\Rightarrow 14r - 14 = \Rightarrow r = 1$$

Putting this value of r, we get the coordinates of point N are $(1, 3. 5)$ and direction ratios of the line PN are $0, -3, 2$.

Equation of the line PN through the point $P\ (1, 6, 3)$ is

$$\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$$

Hence proved.

Ex 3- Find the nature of the intersection of the sets of planes.

$x - y + z = 3\,,\ 2x + 5y + 3z = 0\,,\ 3x - 2y - 6z + 1 = 0$

Sol.- The equation of the given planes are

$x - y + z - 3 = 0$.. (1)

$2x + 5y + 3z = 0$.. (2)

$3x - 2y - 6z + 1 = 0$.. (3)

The rectangular array formed by coefficients & constant terms of above equations is

$$\begin{vmatrix} 1 & -1 & 1 & -3 \\ 2 & 5 & 3 & 0 \\ 3 & -2 & -6 & 1 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now evaluating determinant D_4

$$D_4 = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 5 & 3 \\ 3 & -2 & -6 \end{vmatrix} = 1\,(-30 + 6) + 1(-12 - 9) + 1(-4 - 15)$$

$= -24 - 21 - 19 = -64 \neq 0$

Hence the given planes intersect in a point.

Ex 4- Show that the planes $2x + 4y + 2z + 7 = 0, 5x + y - z + 9 = 0,$ $x - y - z + 6 = 0$ form a triangular prism.

Sol.- The equation of the given planes are

$2x + 4y + 2z + 7 = 0$.. (1)

$5x + y - z + 9 = 0$... (2)

$x - y - z + 6 = 0$.. (3)

The rectangular array formed by coefficients & constant terms is

$$\begin{vmatrix} 2 & 4 & 2 & 7 \\ 5 & 1 & -1 & 9 \\ 1 & -1 & -1 & 6 \end{vmatrix}$$ ....(4)

Now evaluating determinant D_4

$$D_4 = \begin{vmatrix} 2 & 4 & 2 \\ 5 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$$

$= 2(-1 - 1) - 4(-5 + 1) + 2(-5 - 1)$

$= -4 + 16 - 12$

$= 0$

Also

$$D_4 = \begin{vmatrix} 4 & 2 & 7 \\ 1 & -1 & 9 \\ -1 & -1 & 6 \end{vmatrix}$$

$= 4(-6 + 9) - 2(6 + 9) + 7(-1 - 1)$

$= 12 - 30 - 14 = -32 \neq 0$

Hence $D_4 = 0$ and $D_1 \neq 0$.

Hence given planes form a triangular prism.

Ex 5- Prove that the three planes

$x + y + z + 2 = 0, x + 2y + 2z + 2 = 0, x + 3y + 3z + 2 = 0,$ intersect in a line.

Sol.- The rectangular array formed by coefficients & constant terms is

$$D = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 2 \end{vmatrix}$$

Evaluating D_4

$$D_4 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix}$$

$$= 1(6 - 6) - 1(3 - 2) + 1(3 - 2)$$

$$= 0$$

Also

$$D_1 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = 0 \text{ (Two columns are identical)}$$

We also get $D_2 = D_3 = 0$. Therefore, $D_4 = 0 = D_1 = D_2 = D_3$.

So the given planes intersect in a line.

Ex 6- Find the shortest distance between the lines

$$\frac{x-3}{1} = \frac{y-5}{2} = \frac{z-2}{1} \text{ and } \frac{x-1}{5} = \frac{y+1}{-3} = \frac{z+1}{1}$$

Sol.- Let ℓ, m, n be the direction ratios of the shortest distance line, then we have

$$\ell + 2m + n = 0$$

$$5\ell - 3m + n = 0$$

Solving above equations , we get

$$\frac{\ell}{2+3} = \frac{m}{5-1} = \frac{n}{-3-10}$$

$\Rightarrow \frac{\ell}{5} = \frac{m}{4} = \frac{n}{-13}$

The direction ratios of S.D. line are $5, 4, -13$ therefore direction cosines of S.D. line are $\frac{5}{\sqrt{210}}, \frac{4}{\sqrt{210}}, \frac{-13}{\sqrt{210}}$

Also as $P(3,5,2)$ and $Q(1,-1,-1)$ are points on the line first and second respectively.

the length of S.D. $=$ Projection of PQ on the S.D. line.

$$= \frac{5}{\sqrt{210}}(3-1) + \frac{4}{\sqrt{210}}(5+1) - \frac{13}{\sqrt{210}}(2+1)$$

$$= \frac{10+24-39}{\sqrt{210}} = \frac{-5}{\sqrt{210}}$$

Hence S.D. $= \frac{5}{\sqrt{210}}$

Ex 7- Find the S.D. between lines.

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1} \;\&\; \frac{x+1}{-1} = \frac{y+2}{1} = \frac{z+1}{2}$$

and also find its equation.

Sol.- The given lines are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1} = r_1 \text{ (say)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ (1)}$$

$$\frac{x+1}{-1} = \frac{y+2}{1} = \frac{z+1}{2} = r_2 \text{ (say)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ (2)}$$

Any point on the line (1) is $P\ (r_1 + 1, 2r_1 + 2, r_1 + 3)$ and on the line (2) is $Q\ (-r_2 - 1, r_2 - 2, 2r_2 - 1)$.

Therefore the direction ratios of the line PQ are

$r_1 + r_2 + 2, 2r_1 - r_2 + 4, r_1 - 2r_2 + 4.$

If PQ is S.D. line, then PQ is perpendicular to both the given lines, therefore we have

$r_1 + r_2 + 2 + 4r_1 - 2r_2 + 8 + r_1 - 2r_2 + 4 = 0$

$\Rightarrow 6r_1 - 3r_2 + 14 = 0$.. (3)

and

$-r_1 - r_2 - 2 + 2r_1 - r_2 + 4 + 2r_1 - 4r_2 + 8 = 0$

$\Rightarrow 3r_1 - 6r_2 + 10 = 0$ .... (4)

Solving (3) and (4), we get

$r_1 = -2$ and $r_2 = \frac{2}{3}$

Hence point P is $(-1, -2, 1)$ and Q is $\left(\frac{-5}{3}, \frac{-4}{3}, \frac{1}{3}\right)$ therefore S.D. is $\frac{2}{\sqrt{3}}$.

Required equation of S.D. is

$$\frac{x+1}{\frac{2}{3}} = \frac{y+2}{\frac{-2}{3}} = \frac{z-1}{\frac{2}{3}}$$

Or $\frac{x+1}{1} = \frac{y+2}{-1} = \frac{z-1}{1}$

Self cheque question

1. If three planes are intersecting, then which one of the following is not possible.
    a) Intersecting in a point
    b) Intersecting in a line
    c) Forming a triangular prism
    d) Intersecting in two points.
2. The point where the line $\frac{x-1}{1} = \frac{y+1}{2} = \frac{z-1}{1}$ meets the plane $x + y + z + 7 = 0$ is

a) (1, 5, 1) b) (-1 , -5, -1)

c) (1, -5, 1) d) None of these

3. The planes $x + y + z = 6, x - y - z = 7, x + y - z = 5$ intersect

a) In a point b) In a line

c) Form triangular prism d) None of these

4. The lines $\frac{x}{\ell_1} = \frac{y}{m_1} = \frac{z}{n_1}$ and $\frac{x}{\ell_2} = \frac{y}{m_2} = \frac{z}{n_2}$ are perpendicular if

a) $\ell_1\ell_2 + m_1m_2 + n_1n_2 \neq 0$ b) $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

c) $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ d) None of these

5. The line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is perpendicular to the plane.

a) $2x + 3y + 4z = 5$ b) $2x - 3y + 4z = 6$

c) $2x + 3y - 4z = 7$ d) $2x - 3y - 4z = 9$

6.19 *TERMINAL QUESTIONS*

EXERCISE- 6.1

1- Find the equation of a straight line passes through the point $(2, 3, 5)$ and whose direction ratios are $1, 2, 3$

2- Find m so that the lines given by the following equations may be perpendicular to each other.

$$\frac{x-1}{2m} = \frac{y-2}{4} = \frac{z-3}{6} \text{ and } \frac{x-3}{5} = \frac{y-4}{7} = \frac{z-5}{-8}$$

3- Find the coordinates of the point of intersection of the line

$\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+2}{-3}$ with the plane $3x + 4y + 5z = 9$

4. Find the equations of the line passes through (α, β, γ) and right angles to the lines

$$\frac{x}{\ell_1}=\frac{y}{m_1}=\frac{z}{n_1} \text{ and } \frac{x}{\ell_2}=\frac{y}{m_2}=\frac{z}{n_2}$$

5. Find the equations of the line joining the point $(2,-3,4)$ and $(3,4,5)$.

6. Find the symmetrical form of straight line $x+y+z+4=0, 4x+y-2z+5=0$.

7. Find the symmetrical form of straight line $x=4y+5;\ z=3y+2$.

EXERCISE- 6.2

1. Find the equation of the plane through the line $\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-4}{5}$ and parallel to the x −axis.

2. Find the equation of the plane through points $(2,-1,0), (3,-4,5)$ and parallel to the line $x=3y=4z$.

3. Find the equation to the plane through the point $(2,-3,1)$ and perpendicular to the line $3x-y+z+6=0=5x+y+3z+4$.

4. Prove that the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ is parallel to the plane

$4x+4y-5z+10=0$.

5. Find the equation of the plane through the line $2x+y-z-3=0=5x-3y+4z+9$ and parallel to the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}$.

6. Prove that lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar and find the equation of the plane containing them.

7. Prove that the lines $\frac{x-5}{4}=\frac{y-7}{7}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ are coplanar.

8. Prove that the two lines $\frac{x-1}{1}=\frac{y-1}{4}=\frac{z-1}{6}$ and $\frac{x-2}{2}=\frac{y-5}{3}=\frac{z-7}{3}$ are coplaner and find their point of intersection.

EXERCISE – 6.3

(SCQ-1) Prove that the equation of the perpendicular from the point (1, 6, 3) to the line $\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ are $\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$ and the coordinates of the foot of the perpendicular are $(1,3,5)$.

(SCQ-2) Find the equation of the perpendicular and coordinates of the foot of perpendicular drawn from the point $(5,9,3)$ to the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$

(SCQ-3) Find the S.D. between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. Find equation of S.D. and the points where it meets the given lines.

(SCQ-4) Find the S.D. between the lines

$$\frac{x-3}{1}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and } \frac{x+3}{1}=\frac{y+7}{2}=\frac{z-6}{4}$$

(SCQ-5) For what value of λ, the following planes $x - y + z + 1 = 0$; $\lambda x + 3y + 2z - 3 = 0$; $3x + \lambda y + z - 2 = 0$

i) Intersect in a point ii) Intersect in a line

iii) Form a triangular prism.

(SCQ-6) Examine the nature of the intersection of the following set of planes

i) $x + y - z = 3,\ 2x + 3y + 5z = 4,\ 3x - 2y - z + 5 = 0$

ii) $x + 2y + 3z = 6,\ 3x + 4y + 5z = 4,\ 5x + 4y + 3z = 8$

iii) $x + y - 5z = 20, x - y + z + 4 = 0, 2x - y - z - 4 = 0$

EXERCISE -6.4

OBJECTIVE QUESTIONS

Fill in the blanks-

1. The equation of $x -$ axis in symmetrical from is ....
2. The equation of straight line passing through the points $(2, 3, 5)$ and $(4, 7, 8)$ ....
3. The straight line passing through the fixed point ....

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{5}$$

4. The equation of the plane through the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{5} \text{ is ....}$$

5. The equation of the plane through the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ is ....
6. Two lines are coplanar if the shortest distance between them are ....
7. The line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ is perpendicular to the plane ax + by + cz + d = 0, if ....
8. The planes $x = 0$; $y = 0$; $z = 0$ intersecting in a ....
9. The three planes $x + y + z = 0$, $x + 2y + 2z = 0$, $x + 3y + 3z = 0$ intersect in a ....

True/False -

10. The equation of y-axis in general form is $x = 0 = z$.
11. The equation of a straight line passing through origin and direction ratios are ℓ, m, n is $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$.
12. Lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y}{4} = \frac{z}{8}$ are parallel.
13. The symmetrical form of the line

$x = ay + b, z = cy + d$ is $\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$

14. The equation of the plane through the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ is $ax + by + cz = 0$ are

$a + b + c \neq 0.$

15. Two or more lines are said to be coplanar if they are lying in the same plane.
16. The three planes $2x + 3y - z = 2, 3x + 3y + z = 4, x - y + 2z = 5$ intersect in a point.
17. Two lines are said to be skew lines if they do not lie in the same plane.
18. Shortest distance between coplanar lines are not equal to zero.

MULTIPLE CHOICE QUESTIONS

19. Coordinates of any point on the line $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (say)

are given by

a) $(\ell r, mr, nr)$ b) $(\alpha + \ell r,\ \beta + mr, \gamma + nr)$

c) $(\alpha - \ell r,\ \beta - mr, \gamma - nr)$ d) None of these

20. The equation of z – axis in symmetrical form is

a) $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ b) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

c) $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ d) None of these

21. The angle between the lines y = 0 = z and x = 0 = y is

a) $\frac{\pi}{3}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{2}$ d) None of these

22. The plane through the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ is

a) $ax + by + cz = 0$ and $a + b + c = 0$

b) $ax + by + cz = 0$ and $a + b + c \neq 0$

c) $ax + by + cz = 0$ and $a + 2b + 3c \neq 0$

d) $ax + by + cz = 0$ and $a + 2b + 3c = 0$

23. The line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ is parallel to the plane

a) $2x - 4y + z + 5 = 0$ b) $x - 2y + z + 9 = 0$

c) $x + 2y + z + 6 = 0$ d) None of these

6.20 *ANSWERS*

Answer of self cheque question:

1. d 2. b 3. a 4. b

5. a

EXERCISE- 6.1

1. $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-5}{3}$

2. m = 2

3. (0, 1, 1)

4. $\frac{x-\alpha}{m_1n_2 - m_2n_1} = \frac{y-\beta}{n_1\ell_2 - n_2\ell_1} = \frac{z-\gamma}{\ell_1 m_2 - \ell_2 m_1}$

5. $\frac{x-2}{1} = \frac{y+3}{7} = \frac{z-4}{1}$

6. $\frac{x+\frac{1}{3}}{1} = \frac{y+\frac{11}{3}}{-2} = \frac{z}{1}$

7. $\frac{x-5}{4} = \frac{y}{1} = \frac{z-2}{3}$

EXERCISE- 6.2

1. $5y - 3z - 3 = 0$
2. $29x - 27y - 22z - 85 = 0$
3. $x + y - 2z + 3 = 0$
5. $7x + 9y - 10z = 27$
6. $x - 2y + z = 0$
8. $(2, 5, 7)$

EXERCISE- 6.3

2. $\frac{x-5}{1} = \frac{y-9}{2} = \frac{z-3}{-2}$; $(3, 5, 7)$
3. S.D. $= 3\sqrt{30}$; $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{1}$;Points of intersection are $(3, 8, 3)$ and $(-3, -7, 6)$
4. $5\sqrt{6}$
5. (i) Intersect in a point if $\lambda \neq 4$ and $\lambda \neq -3$

   (ii) Intersect in a line if $\lambda = -3$

   (iii) Form a triangular prism if $\lambda = 4$
6. (i) Point

   (ii) Triangular prism

   (iii) Line

ANSWER (OBJECTIVE QUESTIONS)

1. $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$
2. $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-5}{3}$
3. $(1, 0, 3)$

4. $a(x-1)+b(y-2)+c(z-3)=0$ & $2a+3b+5c=0$
5. $x-2y+z=0$
6. Zero
7. $\frac{a}{\ell}=\frac{b}{m}=\frac{c}{n}$
8. Point
9. Line

10. T	11. T	12. F
13. T	14. F	15. T
16. T 17. T	18. T	

ANSWER (MCQ)

24. b	25. a	26.c
27. d	28. b	

BLOCK-3:

VOLUME OF TETRAHEDRON AND SPHERE

UNIT 7: VOLUME OF TETRAHEDRON AND CHANGE OF AXES

CONTENTS

7.1 *INTRODUCTION*

In the previous classes we have learnt and studies about many types of 2D & 3D objects, some of which(for examples length, distance etc.) are measured in centimeters or meter, some(for examples mass, weight etc.) in grams or kilograms etc., some of which(for examples water, milk, oil etc.) in milliliters or liters and so on. All of these units in which the objects are measured are called measuring units. There exist some standard measures available in the market which we use to measure these things. Here we have to study about the 3D objects, which do not lie in a plane, but occupy space and we will find method to calculate the capacity of some particular objects. The capacity of a container is defined as the maximum quantity of a liquid that the given container or vessel can hold. It is also called the volume of the container. We have learnt by now the formulae for the volume of some solid shapes, which we use in our daily life as cubical, cuboidal, conical, cylindrical, spherical objects. To remainder the volumes we give the important results as follows

Solid shapes	Examples	Curved /lateral surface area	Total surface area	Volume	Remarks
Cube	Dice	$4a^2$ (Lateral S. Area)	$6a^2$	a^3	a =side of the cube
Cuboid	Book,Pencil-box, brick etc.	2b(l+h) (Lateral S. Area)	2(lb + bh + lh)	lbh	l = length , b = breadth h = height
Cone	Ice-cone	πrl	$\pi r(r+l)$	$\frac{1}{3}\pi r^2 h$	r = radius of base h=height of cone l= slant height also, $l^2=r^2+h^2$
Cylinder	Closed cylinder	$2\pi rh$	$2\pi r(r+h)$	$\pi r^2 h$	r= radius of base h= height of cylinder
Sphere	Football	$4\pi r^2$	$4\pi r^2$	$\frac{4}{3}\pi r^3$	r=radius ofsphere

Here we study the different methods to find the volume of a tetrahedron. A tetrahedron is a special case of Pyramid, whose base is a triangle. In case of a Pyramid the base may be square, rectangle, rhombus, pentagon, hexagon etc. A tetrahedron is a Pyramid/Kyra with

triangular base having 4 surfaces(3 lateral & 1 base),4 vertices, 6 edges etc. Here we give a 2D imagination of the tetrahedron, in which we take a equilateral triangle PQR of side 2a & join all the three midpoints A,B,C mutually to form an equilateral triangle ABC, length of side a inside the triangle PQR. Then we fold the triangles PAC, QAB & RBC upside keeping the triangle ABC, as the base. In such a case the points P, Q, R all meet a point say O. Now OABC represent a tetrahedron, whose edge will be equal to the side of the triangle i.e. a and slant height will be equal to theheight of triangle ABC (as shown here in the figure below).

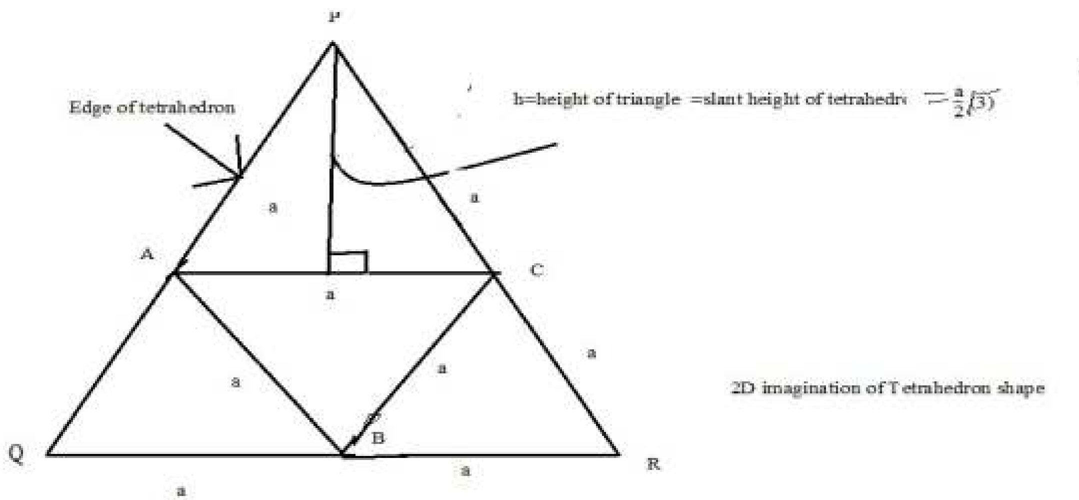


Fig. 7.1.1

The 3D imagination of the above tetrahedron is shown below.

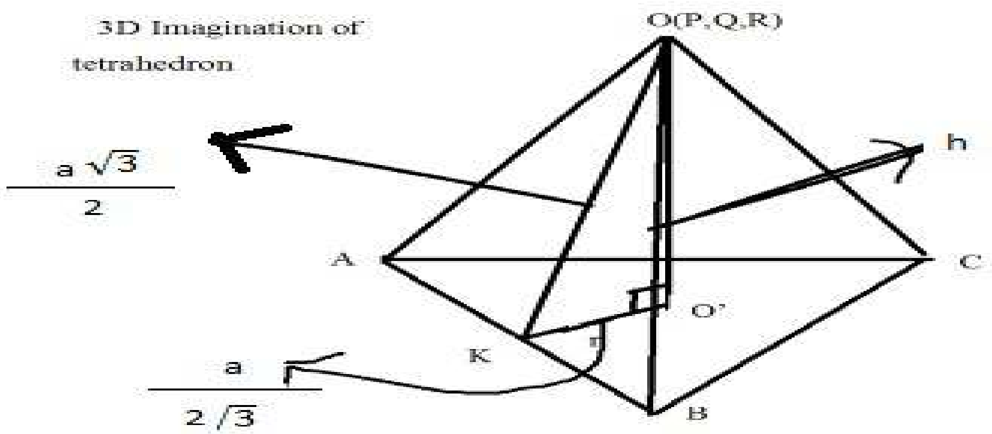


Fig. 7.1.2

In this imagination, OO' is the perpendicular drawn from the vertex O on the triangular base ABC. The point O' will be the incentre or circumcentreor orthocentre or centroidor centre of gravity of the triangular base ABC, because the triangle ABC isan equilateraltriangle.Here OO' will be the height of the tetrahedron, OK will be the slant height of tetrahedron or the height of the equilateral triangle of side a, hence $l = \frac{1}{2}\sqrt{(3)} \times a$ and $O'K$ will be the radius of the incircle of the triangle ABC,i.e.

$O'K$=r $=\frac{1}{2}$ [a/$\sqrt{(3)}$].Now in the right angletriangle $O'OK$,we have, $(OO')^2$=$(OK)^2$-$(O'K)^2$

[Putting the values of OK&$O'K$ and simplifying we get, OO'=a [$\sqrt{(\frac{2}{3}\frac{2}{3})}$]

Hence we can find the volume of this tetrahedron OABC,by using the same result as in case of a cone because there is a similarity between cone & tetrahedron that both starts from some part (as triangle, circle, square etc.) and ends at a poin

Hence we get,

Volume of tetrahedron $=\frac{1}{3}\times$area of base (ΔABC)$\times$(height of the tetrahedron)

[Area of base (ΔABC) & height of the tetrahedron both are known, hence we can find the volume of thetetrahedron]

Further we will discuss the different cases of transformation of coordinates i.e. the effect of, case-(1) shifting of origin,case-(2) rotation of axes i.e. change of directions of axes& case-(3) shifting of origin& rotation of axes i.e. change of directions of axes both together, on the equation of a space curve , theequation of a surface etc. Here the coordinates of a points or the equation of a space curve or theequation of a surface etc. w.r.t. a system of coordinate axes is given and we find the same in another system of coordinates. Here we use the concept of relativity for example if the distance of city A from the city B isa kilometers & the distance of the city C is b kilometers from the city A, then the distance between the cities B & C will be a+b or a- b according as the city C is before or after the city A.Similarlysome rulesarefollowed in case of coordinatesystem.

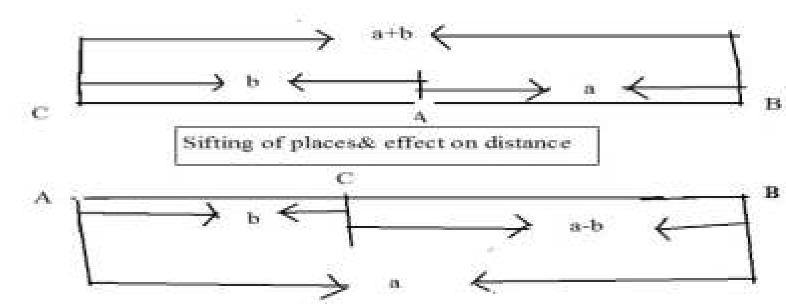


Fig. 7.1.3

Here the distance between the cities B & C depend upon the position of the city C i.e. the relativistic study is done with respect to the city C.

7.2 *OBJECTIVES*

After studying this unit, you should be able to –

- Understand the basic concepts of tetrahedron.
- Calculate the volume of tetrahedron by different methods.
- Understand the concept that the volume of a tetrahedron remains unchanged on calculating by different methods.
- Check whether the given four points are coplanar or not.
- Calculate the area of triangle of the base of the tetrahedron in terms of its volume and height.
- Study the effects of transformation of coordinates on the coordinates of apoint, equation of a space curve, equation of a surface etc.
- Understand the different types of transformation of coordinates.
- Understand the relationships between the direction cosines ofthree mutually perpendicular straight lines.
- Understand the concept of relativity i.e. to study with respect to another existentialreference system.

The main objective of this unit is to make the typical form of space curves and surfaces easier for study purposes.

7.3 *VOLUME OF TETRAHEDRON (VERTICES FORM)*

To find the volume of the tetrahedron, the coordinates of whose vertices are given.

DERIVATION

Let ABCD be a tetrahedron made by four triangular planes ABD, ADC, BDC and ABC as shown here in the figure.Let the coordinatesof the vertices of the tetrahedron ABCD are,

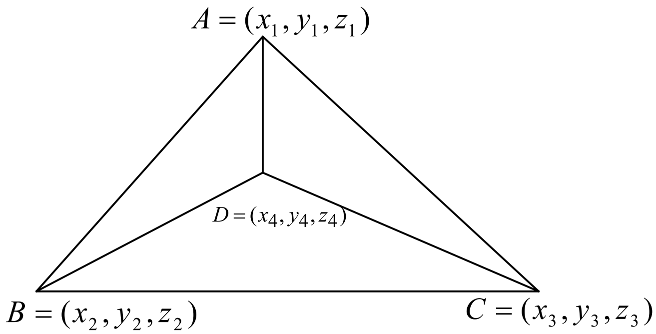


Fig 7.3.1

Suppose V is the volume of the tetrahedron ABCD,

Then applying the formula of volume of tetrahedron (a special case of Pyramid, with triangular base) we get,

$$V=\frac{1}{3}\times area\ of\ base \times height$$

$$=\frac{1}{3}\times area\ of\ \Delta BCD \times height$$

$$=\frac{1}{3}\times area of \Delta BCD \times p \text{ --------------------- } (1)$$

where p is the perpendicular distance of the vertex A from the triangular base BCD.

Now let the equation of the plane BCD is,

$$ax+by+cz+d=0 \text{ ------------------------ } (2)$$

If the plane (2) contains the points $B=(x_2,y_2,z_2)$; $C=(x_3,y_3,z_3)$ and $D=(x_4,y_4,z_4)$ then we get,

$$ax_2+by_2+cz_2+d=0 \text{ --------------------- } (3)$$

$ax_3 + by_3 + cz_3 + d = 0$ --------------------- (4)

$ax_4 + by_4 + cz_4 + d = 0$ --------------------- (5)

Now eliminating the arbitrary constants a, b, c, and d from the equations (2), (3), (4) and (5) we get the equation of the plane BCD, as

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Expanding along R_1 we get,

$$x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} - 1\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0 \text{........}(6)$$

Since for a given tetrahedron the vertices A, B, C and D fixed, hence the determinants with $x, y, z,$ and constant term in equation (6) are constants.

Hence equation (6) is a linear or first degree equation in x, y, z. So it represents the equation of a plane or particularly the general equation of the plane BCD. Now,

P=perpendicular distance of the point $A(x_1, y_1, z_1)$ from the plane BCD [given by equation (6), using the formula of perpendicular distance from a point to a given plane as we have learnt in the plane].

$$P = \frac{x_1\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y_1\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - 1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2}}$$

$$P = \frac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2}} \text{......................(7)}$$

Now we will find the area of the triangle BCD, by projection method as follows---

We have learnt by now in the unit of plane that the projection of an area bounded by a plane curve is the area enclosed by the projection of curve on the given plane.

To find the area of the triangle BCD, the coordinates of whose vertices are $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ and the equation of the triangular plane BCD is,

$$x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - 1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = 0$$

$\Rightarrow lx + my + nz - \lambda = 0$ where, $l = \dfrac{\Delta_1}{\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}}$, $m = \dfrac{\Delta_2}{\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}}$, $n = \dfrac{\Delta_3}{\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}}$ and

$\lambda = \dfrac{\Delta_4}{\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}}$ where, $\Delta_1 = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$ and

$$\Delta_4 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Let [l, m, n] be the direction cosines of the normal to the plane of triangle BCD and let Δ denotes the area of this triangle and B_x, C_x and D_x be the projections of the three vertices B, C and D respectively on the yz-plane. Clearly the coordinates of these points are given by,

$B_x = (0, y_2, z_2), C_x = (0, y_3, z_3)$ and $D_x = (0, y_4, z_4)$.

Let Δ_x denotes the area of the triangle $B_x\ C_x\ D_x$ *i.e.* Δ_x is the area of projection of the area Δ on yz-plane so

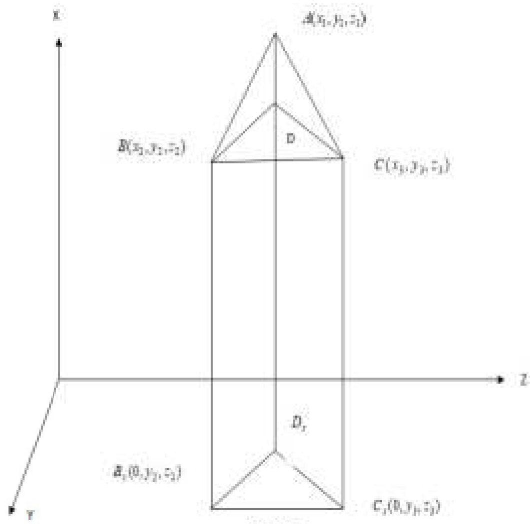


Fig. 7.3.2

that we have, $\Delta_x = \Delta.l$ --------------------(8)

Also by coordinate geometry of 2D the area of triangle B_x, C_x and D_x is given by,

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} \text{------------- (9)}$$

From equations (8) and (9) comparing the values of Δ_x we get,

$$\Delta.l = \frac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}$$

i.e. $2.\Delta.l = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}$........................(10)

Similarly if Δ_y and Δ_z are the areas of projections of the area Δ on zx and xy planes , then as in equation (8) we get, $\Delta_y = \Delta.m$ ----------------------(11)

$\Delta_z = \Delta.n$ ---------------------- (12)

where, $\Delta_y = \frac{1}{2}\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}$..............................(13)

Similarly we can define ,

$\Delta_z = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$..............................(14)

Squaring the equations (9), (13), (14) and then adding we get,

$\Delta_x{}^2 + \Delta_y{}^2 + \Delta_z{}^2 = \Delta^2(l^2 + m^2 + n^2) = \Delta^2$..............(15) (as $l^2 + m^2 + n^2 = 1$)

Putting the values of $\Delta_x, \Delta_y, \Delta_z$ in equation (15), we get the square of the denominator of p i.e. equation (7) as, denominator=$4.\Delta^2$

Also the numerator of p i.e. equation (7) is converted to = $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Putting the values of numerator and denominator in equation (7), the value of p reduces to,

$$P=\frac{\begin{vmatrix}x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix}}{\sqrt{4\Delta^2}}=\frac{\begin{vmatrix}x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix}}{2\Delta}$$

and the area of the triangular base BCD of the tetrahedron is considered as Δ,hence substituting the values of the area of the triangular base BCD of the tetrahedron and p in equation (1) , we get the required volume of the tetrahedron V as—

$$V=\frac{1}{3}\times\Delta\times\frac{\begin{vmatrix}x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix}}{2\Delta}$$

$$V=\frac{1}{6}\begin{vmatrix}x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{vmatrix} \qquad \text{..............................(16)}$$

This is the required volume of the tetrahedron ABCD,the coordinates of whose vertices A, B, C and D are (x_i, y_i, z_i) where i = 1, 2, 3 and 4 respectively.

REMARKS

1. If one vertex of the tetrahedron is at the origin O = (0,0,0) , then the volume of that tetrahedron will be,

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Expanding along R_4 we get,

$$V = -\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

2. If the value of V comes out to be negative , then we neglect the negative sign, as the volume can not be negative.
3. The volume of atetrahedron with vertices (x_i, y_i, z_i) where i = 1, 2, 3 and 4 respectively can also be calculated as,

$$V = \frac{1}{6}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}$$

OR

$$V = \frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}$$

4. Volume of tetrahedron can also be calculated if the area of its base and height are given by using the basic formula as in case of a pyramid.

SOLVED EXAMPLES

Ex.1- Findthe volume of the tetrahedron formed by the planes, $x+y=0; y+z=0; z+x=0$ and $x+y+z=1$.

Sol.First of all we will find the coordinates of vertices of the tetrahedron formed by the given four planes. For this we will take the sets of three-three planes out of the given four and will find the points of intersection such as, -

$x+y$ ---------------(1)

$y+z=0$ ----------------(2)

$z+x=0$ -----------------(3)

$x+y+z=1$ ---------------(4)

Taking the planes;(1), (2) and (3) we get the point A= (0,0,0)

Taking the planes;(1), (2) and (4) we get the point B= (1,-1,1) $=(x_1, y_1, z_1)$

Taking the planes;(1), (3) and (4) we get the point C= (-1,1,1) = (x_2, y_2, z_2)

Taking the planes;(2), (3) and (4) we get the point D= (1,1,-1) = (x_3, y_3, z_3)

These are the four vertices of the tetrahedron.Since one vertex is at origin , hence the volume of the tetrahedron is given by,

$$V=-\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Putting the values in the above determinant and simplifying we get, V = $-\frac{2}{3}$, (here we neglect the negative sign as the volume can not be negative)

Hence the required volume of the tetrahedron V= $\frac{2}{3}$ cube units

7.4 *VOLUME OF TETRAHEDRON (VECTOR FORM)*

To find the volume of the tetrahedron,whose three coterminous edges in right handed system are denoted by the vector quantities $\vec{a}$,$\vec{b}$ and $\vec{c}$.

DERIVATIONLet OABC be a tetrahedron with O as origin and OA, OB and OC as coterminous edges. Let the position vectors of the vertices A, B and C are the vector quantities $\vec{a}$, $\vec{b}$ and $\vec{c}$respectively i.e. OA= $\vec{a}$,OB= $\vec{b}$ and OC =$\vec{c}$.

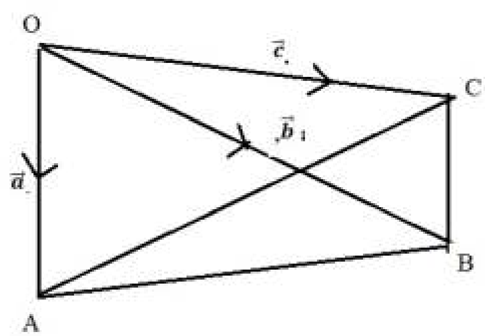


Fig. 7.4.1

Then the volume V of the tetrahedron OABC is given by,

V = $\frac{1}{3}$ × (area of the triangle OBC)×(perpendicular distance of the vertex A from the plane OBC)

= $\frac{1}{3}$ ×(area of the triangle OBC)× p-------------------(1)

We have learnt by now in vector analysis that the area of tringle OBC is given by ,

Area $=\frac{1}{2}|\vec{b}\times\vec{c}|$ -------------------------------(2)

Let $\hat{n}$ be the unit vector perpendicular to the plane of triangle OBC such that $\vec{b}$,$\vec{c}$ and $\hat{n}$ form a right handed system of vectors then ,

$$\hat{n}=\frac{\vec{b}\times\vec{c}}{\left|\vec{b}\times\vec{c}\right|}$$

Since the vectors $\vec{b}$,$\vec{c}$ and $\hat{n}$ form a right handed system, hence the vectors $\vec{b}$, $\vec{c}$ and $\vec{b}\times\vec{c}$ form a right handed system.

Hence the length of perpendicular from the vertex A on the plane OBC = p = the length of projection of

OA on the perpendicular to the plane OBC in the direction of $\hat{n}$ $=\overrightarrow{OA}.\hat{n}=\vec{a}.\frac{\vec{b}\times\vec{c}}{\left|\vec{b}\times\vec{c}\right|}$

Hence, $p=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left|\vec{b}\times\vec{c}\right|}$........................(3)

Putting the values from the equations (2) & (3) in equation (1) we get,

$$V=\frac{1}{3}\times\frac{1}{2}\times\left|\vec{b}\times\vec{c}\right|\times\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left|\vec{b}\times\vec{c}\right|}$$

OR

$$V = \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

This is the required formula for the volume of tetrahedron, whose three coterminous edges in right handed system are denoted by the vector quantities $\vec{a}, \vec{b}$ and $\vec{c}$.

REMARKS

1. If OABC is a tetrahedron, where O = (0,0,0) is origin and the coordinates of the vertices are $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ and $C = (x_3, y_3, z_3)$.Then,

   $$OA = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, OB = \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \text{ and } OC = \vec{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$$

   Now the volume of the tetrahedron OABC with one vertex at the origin becomes,

   $$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad \text{‘OR’} \quad V = \frac{1}{6}[\vec{a}\ \ \vec{b}\ \ \vec{c}]$$

2. If DA, DB and DC are three coterminous edges of the tetrahedron ABCD

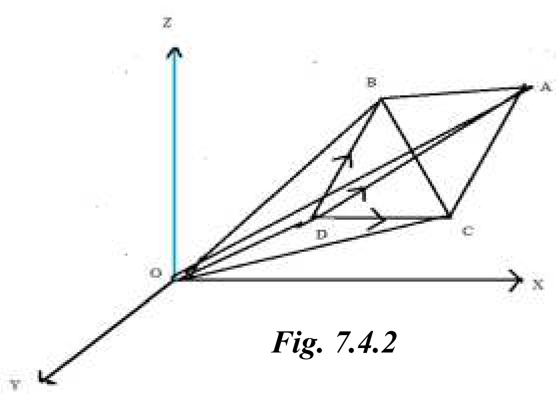


Fig. 7.4.2

then,using vector analysis we get,

$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = (x_1-x_4)\hat{\imath} + (y_1-y_4)\hat{\jmath} + (z_1-z_4)\hat{k}$

Similarly,

$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = (x_2-x_4)\hat{\imath} + (y_2-y_4)\hat{\jmath} + (z_2-z_4)\hat{k}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (x_3-x_4)\hat{\imath} + (y_3-y_4)\hat{\jmath} + (z_3-z_4)\hat{k}$

Now volume of tetrahedron ABCD is given by, $V = \frac{1}{6}[\overrightarrow{DA}\ \overrightarrow{DB}\ \overrightarrow{DC}]$

$$V = \frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}$$

$$V = \frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 & 0 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 & 0 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

On adding R4 to R_1, R_2 and R_3, we get the same form as in case of the vertices form

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

RemarksThe necessary and sufficient condition for the four planes to be coplanar is that the volume of the tetrahedron formed by them as vertices is zero i.e.

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \text{ i.e. } \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

7.5 *VOLUME OF TETRAHEDRON (FACES FORM)*

To find the volume of a tetrahedron when the equations of its four faces are given

DERIVATION

Let the equations of the four planes representing the faces of the tetrahedron are,

$a_1x + b_1y + c_1z + d_1 = 0$ --------------------- (1)

$a_2x + b_2y + c_2z + d_2 = 0$ --------------------- (2)

$a_3x + b_3y + c_3z + d_3 = 0$ --------------------- (3)

$a_4x + b_4y + c_4z + d_4 = 0$ --------------------- (4)

Now we consider the sets of three-three planes, out of the four given planes. Solving the equations of three planes for x, y, z we will get the point of intersection of the planes i.e. the point of intersection of three faces of the tetrahedron i.e. a vertex of the tetrahedron. Taking a set of three planes out of the four, we will get all the four vertices of the tetrahedron. Hence by using the previous article (7.3), we can find the volume of the tetrahedron. But here we will discuss a different method of finding the volume of tetrahedron by using the equations of the faces of the tetrahedron directly.

For this solving the equations (1), (2) and (3) by using Cramer's rule or by the method of determinants, we get

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad \text{....................(5)}$$

Suppose,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Let the capitals letters denote the co-factors of the corresponding small letters in the determinant Δ i.e. $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$-----------etc. denote the cofactors of $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$-----etc. respectively. Then the equation (5) may be rewritten as,

$$\frac{x}{-A_4} = \frac{-y}{B_4} = \frac{z}{-C_4} = \frac{-1}{D_4}$$

Hence the point of intersection of the planes (1), (2) and (3) is given by,

$$\left[\frac{A_4}{D_4}, \frac{B_4}{D_4}, \frac{C_4}{D_4}\right]$$

Similarly solving the other three sets of three planes we get the other three vertices of the tetrahedron as,

$$\left[\frac{A_1}{D_1}, \frac{B_1}{D_1}, \frac{C_1}{D_1}\right], \left[\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right] \text{ and } \left[\frac{A_3}{D_3}, \frac{B_3}{D_3}, \frac{C_3}{D_3}\right]$$

Now the required volume of the tetrahedron is given by,

$$V=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

$$V=\frac{1}{6}\begin{vmatrix} \frac{A_1}{D_1} & \frac{B_1}{D_1} & \frac{C_1}{D_1} & 1 \\ \frac{A_2}{D_2} & \frac{B_2}{D_2} & \frac{C_2}{D_2} & 1 \\ \frac{A_3}{D_3} & \frac{B_3}{D_3} & \frac{C_3}{D_3} & 1 \\ \frac{A_4}{D_4} & \frac{B_4}{D_4} & \frac{C_4}{D_4} & 1 \end{vmatrix}$$

Multiplying R_1 by D_1, R_2 by D_2, R_3 by D_3 and R_4 by D_4, then by rules of determinants D_1, D_2, D_3 and D_4 are taken common in division. Hence we get,

$$V=\frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$V=\frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{vmatrix}$$

[Above we take the transpose of the determinant as $|A|=|A|^{(n-1)}$]

$$V=\frac{1}{6D_1D_2D_3D_4}|adj.\Delta| \quad \text{[By definition of adjoint of a matrix]}$$

[Now we know that, if A is any square matrix of order n then |adj.A| = $|A|^{(n-1)}$,here adj.Δ is a determinant of a square matrix of order 4, hence |adj.Δ| =Δ^3 using the above result in V, we get]

$$V = \frac{1}{6D_1D_2D_3D_4} \times \Delta^3$$

7.6 *VOLUME OF TETRAHEDRON (CONCURRENT EDGES & MUTUAL INCLINATION FORM)*

To find the volume V of a tetrahedron, in terms of the lengths of three concurrent edges and their mutual inclination.

DERIVATION

Let OABC be the tetrahedron with vertex O as origin. Let the lengths of the edges OA, OB and OC are a, b and c; also the angles BOC, COA and AOB are λ, μ and γ respectively. Taking any three mutually perpendicular lines through origin O as coordinate axes, let the direction cosines of the lines OA,OB and OC with respect to these coordinate axes are $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ respectively. Then the coordinates of the vertices A, B and C of the tetrahedron are given by,

$$A = (l_1a, m_1a, n_1a)\ B = (l_2b, m_2b, n_2b)\ C = (l_3c, m_3c, n_3c)$$

Let the position vectors of the vertices A, B and C are denoted by $\vec{a}$,$\vec{b}$ and $\vec{c}$ respectively. Then we have,

$\vec{a}.\vec{b} = ab \text{ Cos } \gamma = l_1a.l_2b+m_1a,m_2b+n_1a.n_2b = ab\ (l_1l_2+m_1m_2+n_1n_2)$

Hence, Cos γ = $(l_1l_2+m_1m_2+n_1n_2)$ -- (1)

Also, $\vec{b}.\ \vec{c} = b\ c \text{ Cos } \lambda = l_2a.l_3b+m_2a,\ m_3b+n_2a.n_3b = b\ c\ (l_2l_3+m_2m_3+n_2n_3)$

Hence, Cos λ = $(l_2l_3+m_2m_3+n_2n_3)$ --- (2)

$\vec{c}.\vec{a} = ca \text{ Cos } \mu = l_1a.l_3b+m_1a,\ m_3b+n_1a.n_3b = c\ a\ (l_1l_3+m_1m_3+n_1n_3)$

Hence, Cos μ = $(l_1l_3+m_1m_3+n_1n_3)$ -- (3)

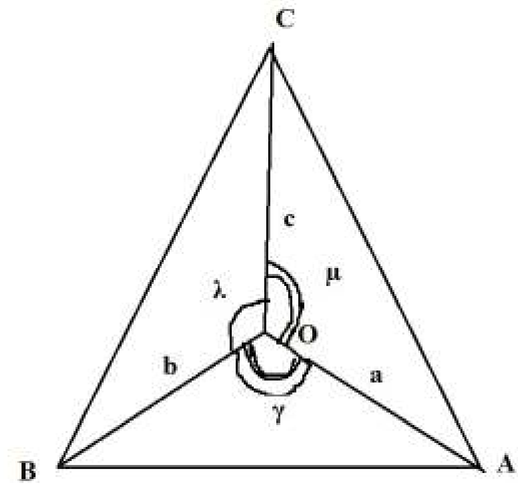


Fig. 7.6.1

Now the volume V of the tetrahedron is given by,

$$V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\text{i.e. } V = \frac{1}{6}\begin{vmatrix} l_1a & m_1a & n_1a \\ l_2b & m_2b & n_2b \\ l_3c & m_3c & n_3c \end{vmatrix}$$

(Taking a, b & c common from R_1, R_2 & R_3 respectively, we get)

$$V = \frac{1}{6}abc\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \text{ [Squaring, we get]}$$

$$V^2 = \frac{1}{36}a^2b^2c^2\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

(Multiplying the determinants by Row by Row pattern we get)

$$V^2 = \frac{1}{36}a^2b^2c^2\begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_3l_2+m_3m_2+n_3n_2 \\ l_1l_3+m_1m_3+n_1n_3 & l_3l_2+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

(Keeping in view the commutativity w.r.t. multiplication of real numbers in above determinant)

$$V = \frac{1}{36}a^2b^2c^2\begin{vmatrix} 1 & \cos\gamma & \cos\mu \\ \cos\gamma & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

[Using equations (1), (2), (3)] [Taking square root on both sides and neglecting the negative sign as the volume can't be negative]

$$V = \frac{1}{6}abc\begin{vmatrix} 1 & \cos\gamma & \cos\mu \\ \cos\gamma & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{\frac{1}{2}}$$

7.7 *AREA OF TRIANGLE OF THE BASE OF A TETRAHEDRON*

We have learnt by now the different methods to find the volume (V) of a tetrahedron also in the unit of plane we have learnt the method of finding the perpendicular distance (p) of a point to a plane. Suppose V is the volume of the given tetrahedron ABCD and p is the perpendicular from the vertex A to the opposite plane BCD, then we have to deduce the area of the triangular plane BCD .Since there exist many types of shapes in 3D Geometry e.g. the shapes which are same from bottom to top as cylinder, cube, cuboid etc. but many shapes e.g. cone, pyramid etc. are such that which start from some 2D shape and ends at a point. Tetrahedron is also a surface of such type.

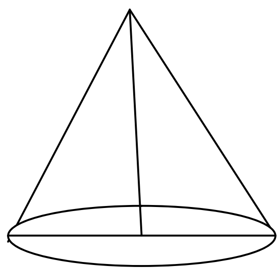


Fig. 7.7.1 (a)

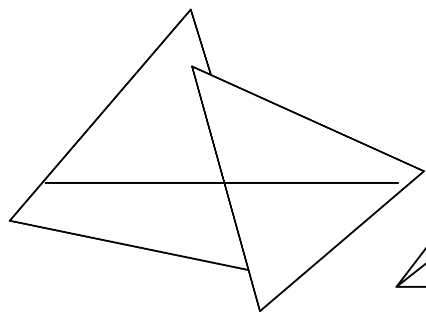


Fig. 7.7.1 (b)

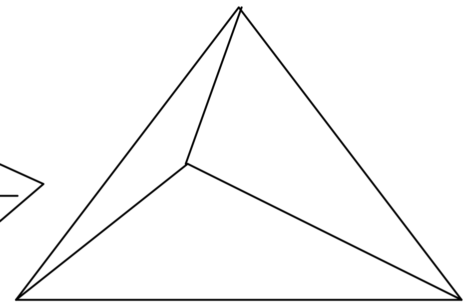


Fig. 7.7.1 (c)

In such type of shapes the formulae for volume's given by,-

$$V = \frac{1}{3} \times area\ of\ base \times height$$

$$= \frac{1}{3} \times area\ of\ \Delta BCD \times height$$

$$= \frac{1}{3} \times area\ of\ \Delta BCD \times \text{p} \text{--------------------- (1)}$$

where p is the perpendicular distance of the vertex A from the triangular base BCD

From this formula area of ΔBCD $= \frac{3V}{P}$; Hence if the values of V and P are known then the value of area of Δ BCD can be evaluated.

e.g. if the volume of a tetrahedron is 48 cube units and its height is 6 units, then the area of the base of the tetrahedron will be $= \frac{3.48}{6} = 24$ square units.

SOLVED EXAMPLES

Ex-2. Prove that the volume of the tetrahedron formed by the planes, $my+nz=0$, $nz+lx=0$

$lx+my=0$ and $lx+my+nz=p$ is $\frac{2}{3}\times\left[\frac{P^3}{lmn}\right]$.

Sol.- The equations of the given four planes are –

$my+nz=0$ -----------------(1)

$nz+lx=0$ ------------------(2)

$lx+my=0$ ----------------(3)

$lx+my+nz=p$ ----------(4)

Clearly the planes (1), (2) and (3) pass through the origin and so the point of intersection of the planes (1) < (2) and (3) is O =(0,0,0) , Similarly solving the other three sets of three planes we get the other three vertices of the tetrahedron as,

$$A=\left\{\frac{p}{1},\frac{p}{m},-\frac{p}{n}\right\}\ B=\left\{\frac{p}{1},-\frac{p}{m},\frac{p}{n}\right\}\ C=\left\{-\frac{p}{1},\frac{p}{m},\frac{p}{n}\right\}$$

Now the volume of the tetrahedron is given by,

$$V=-\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

(Since one vertex of the tetrahedron is at origin, hence we take the coordinates of the points A, B, C as the values of $x_1,y_1,z_1;x_2,y_2,z_2;x_3,y_3$ and z_3 respectively and apply the above formula for finding the volume)

$$V=\frac{1}{6}\begin{vmatrix}\frac{p}{1} & \frac{p}{m} & -\frac{p}{n}\\ \frac{p}{1} & -\frac{p}{m} & \frac{p}{n}\\ -\frac{p}{1} & \frac{p}{m} & \frac{p}{n}\end{vmatrix}$$

(Now taking common $\frac{p}{1}$ from C_1, $\frac{p}{m}$ from C_2 and $\frac{p}{n}$ from C_3 we get)

$$V=\frac{1}{6}.\frac{p^3}{lmn}\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{vmatrix}$$

(Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$, we get)

$$V=\frac{1}{6}.\frac{p^3}{lmn}\begin{vmatrix}0 & 0 & -1\\ 2 & 0 & 1\\ 0 & 2 & 1\end{vmatrix}$$ (Now expanding along R_1, we get)

$$V=\frac{4p^3}{6lmn}\begin{vmatrix}0 & 0 & -1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{vmatrix}$$

$$V=-\frac{4p^3}{6lmn}$$

$$V=-\frac{2p^3}{3lmn}$$

(Simplifying & neglecting the negative sign, we get the required volume)

$$V = \frac{2p^3}{3lmn} \text{....................................(5)}$$

To deduce the area of the triangle Let the area of the triangle formed by the plane $lx + my + nz = 0$ is Δ and the length of perpendicular drawn from the opposite vertex O to this plane is d, then,

d = perpendicular distance from O(0, 0, 0) to the plane $lx + my + nz = p$

$$d = \frac{P}{\sqrt{l^2 + m^2 + n^2}} \text{............(6)}$$

Also volume V of the tetrahedron is given by ,

$$V = \frac{1}{3} \times \Delta \times d \text{...(7)}$$

Putting the values of V and d from equations (5) and (6) in equation (7), we get

$$\Delta = \frac{3V}{d}$$

$$\Delta = \frac{2P^2}{lmn}\sqrt{l^2 + m^2 + n^2}$$

SELF CHECK QUESTIONS

Choose the correct option.

(SCQ-1) The vertices of a tetrahedron are (0, 0, 0), (2a, 0, 0), (0, b, 0) and (0, 0, 3c); then the volume of the tetrahedron is,

(b) 0 (b) a^2b (c) abc (d) $\frac{3}{4} \times$abc

(SCQ-2) The number of edges of a tetrahedron is,

(b) 4 (b) 9 (c) 6 (d) Infinite

(SCQ-3) The tetrahedron is a special case of a

(b) Pyramid (b) Prism (c) Cylinder (d) Cone

(SCQ-4) The coordinates of one vertex of the tetrahedron $x = 0, y = 0, z = 0$ and $x + y + z = 1$ are

(a) (1, 0, 1) (b) (0, 0, 0) (c) (0, -1, 2) (d) (0,0,1)

(SCQ-5) The volume of the tetrahedron with, $x + y = 0, y + z = 0, z + x = 0$ and $x + y + z = 1$ as faces is

(b) 6 (b) 3 (c) $-\frac{2}{3}$ (d) $\frac{2}{3}$

(SCQ-6) The four points will be coplanar, if the volume of the tetrahedron formed by them is,

(b) negative (b) positive (c) zero (d) zero or negative

(SCQ-7) The volume of the tetrahedron passing through the origin and making intercepts of length a, b, c on coordinate axes is,

(e) 2abc (b) 6abc (c) 0 (d) $\frac{1}{6}\times$abc

(SCQ– 8) If the volume of a tetrahedron is zero, then the vertices of the tetrahedron will be

(b) coplanar (b) collinear (c) parallel lines (d) none of these

(SC Q– 9) The base of a tetrahedron is always a

(a) square (b) triangle (c) rectangle (d) rhombus

(SCQ–10) The number of curved faces of a tetrahedron is,

(a) 4 (b) 0 (c) 6 (d) none of these

(SCQ–11) If the volume of a tetrahedron is 20 cube units and the length of perpendicular from vertex to the opposite plane is 6 units, then the area of the base of the tetrahedron (in square units) will be

(a) 20 (b) 120 (c) $\frac{10}{3}$ (d) 10

(SCQ–12) The shape of tetrahedron is similar to the shape of-

(a) cylinder (b) cone (c) sphere (d) cube

(SCQ–13)The number of flat surfaces of a tetrahedron is

(a) 4 (b) 0 (c) 6 (d) 2

(SCQ–14) A tetrahedron is formed such that its each face is an equilateral triangle, then the side of triangle is called its

(a) height (b) slant height (c) edge (d) radius

(SCQ –15) Find the value of k such that the four points (0, 0, 0), (k, 0, 0), (0, 3k, 0) & (9, 0, 0) are coplanar-

(a) 3 (b) 27 (c) 9 (d) for every k

(SCQ--16) The intersection of any three faces of a tetrahedron gives a

(a) Curve (b) point (c) line (d) plane

7.8 *CHANGE OF AXES*

We have learnt by now that in case of 3D Geometry there exists three axes and the point of intersection of these axes is called origin. Here we study the effects of (1) change of origin from O = (0, 0, 0) to another point (2) rotation of axes at an angle, i.e. change of directions of axes, keeping origin fixed (3) when change of origin and rotation of axes both are done together.

7.9 *TRANSFORMATION OF COORDINATES*

We havelearnt by nowin the study of 3D Geometry or Solid Geometry or Analytical Geometry or Volumetric Geometry, the coordinates of a point, the equation of a space curve (intersection of two surfaces), the equation of surfaces (1) plane surfaces –such as surface of a table, surface of a black-board etc. and (2) curved surfaces-surface of a sphere, surface of a ellipsoid etc. In the next unit we will study about the surface of sphere and remaining in the unit of coincides.

The coordinates of a point, the equation of a space curve (intersection of two surfaces) and the equation of surfaces may change in the following cases

Case–(1): When the origin of the coordinate system is changed. This process is also called "Translation of axes". Generally we consider the point O = (0,0,0) as origin and it is transferred to another point say $O' = (\alpha, \beta, \gamma)$.

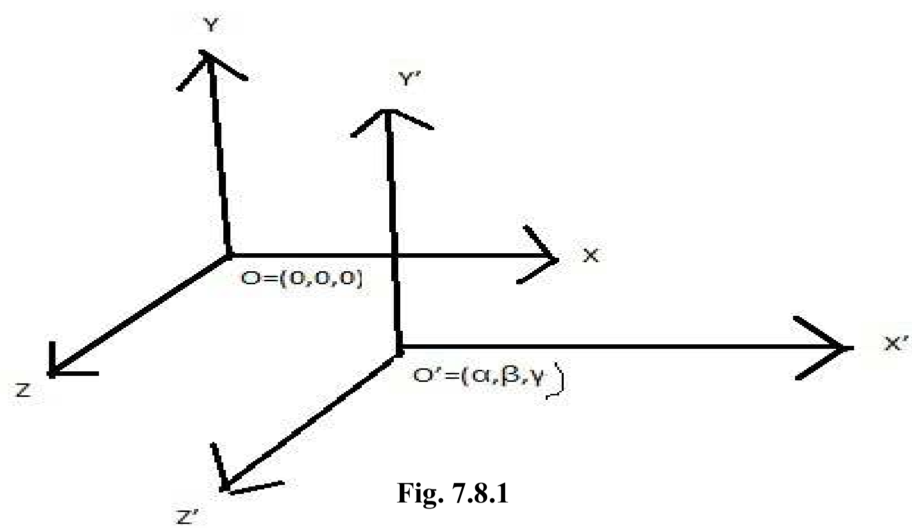


Fig. 7.8.1

By means of this process the study of the coordinates of a point, the equation of a space curve (intersection of two surfaces), the equation of surfaces become easier.

e.g.we have to study the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$,then we can convert it into the equation,

$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$,if we shift the origin to the point =(-*g*, -*f*) then we will study that this equation is converted to,

$x^2 + y^2 = g^2 + f^2 - c$ and the further study becomes easier.

Case-(2): When the directions of the axes are changed, keeping the origin fixed i.e. at the same point.Thisprocess is called the rotation of axes.

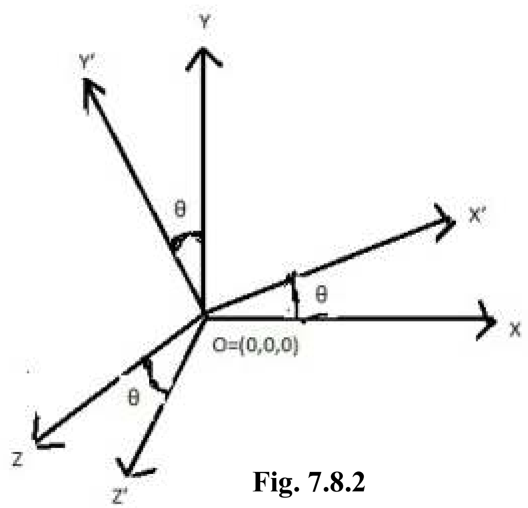


Fig. 7.8.2

Case-(3): When the origin of coordinate axes is changed as well as the directions of the coordinate axes, is changed

Now we are in the position to define the transformation of coordinates.

DEFINITION"The process of changing the coordinates of a point, the equation of a space curve (intersection of two surfaces), the equation of surfaces etc. is called the transformation of coordinates."

This process of the transformation of coordinates will be of great advantages to tackle most of the problems by converting them into easier form (standard form).

Now we will discuss the different cases of the transformation of coordinates one by one in details.

7.9.1 *CHANGE OF ORIGIN (TRANSLATION OF AXES)*

To change the origin O=(0,0,0) of the coordinate system to another point $O' = (\alpha, \beta, \gamma)$, whereas the directions of the axes remain the same.

DERIVATIONLet O=(0,0,0) be the origin of the coordinate system with *OX*, *OY*, *OZ* as the original coordinate axes.Let $O' = (\alpha, \beta, \gamma)$ be the new origin of the transferred coordinate system, the coordinates of O' are taken with respect to *OX*, *OY*and *OZ* as coordinate axes.

Draw three lines $O'X', O'Y'$ and $O'Z'$ through O' parallel to the and in the same directions as *OX*, *OY* and *OZ* respectively.

Let *P* be any arbitrary point in the space whose coordinates are (*x*,*y*,*z*) with respect to the original coordinate axes *OX*, *OY*, *OZ*. Also suppose that the coordinates of the same point *P* with respect to the newaxes $O'X', O'Y'$ and $O'Z'$ are (x', y', z').

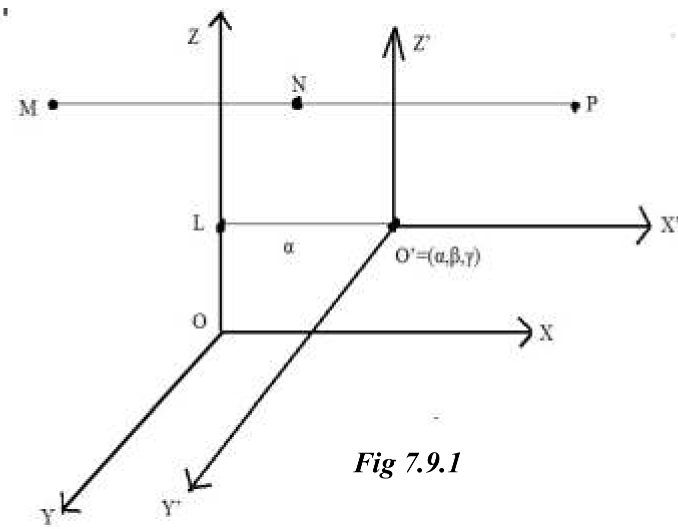


Fig 7.9.1

Fig. 7.9.1

From the point O' draw a perpendicular $O'L$ to OZ. From the point P draw a perpendicular PM to yz-plane, which meets at point N to $y'z'$-plane. Then from the above figure we have,

$O'L = \alpha = NM$, Now $PM= PN+NM$ (Since $PM=x$ and PN = x' hence we get)

$$x = x' + \alpha$$

Similarly we get, $y = y' + \beta$ and $z = z' + \gamma$ combining all the results we get,

$$x = x' + \alpha$$
$$y = y' + \beta \text{ ....(1)}$$
$$z = z' + \gamma$$

Also from the above transformations we make the following conclusion,

$$x' = x - \alpha$$
$$y' = y - \beta \text{ ....(2)}$$
$$z' = z - \gamma$$

<u>REMARKS</u> Here from the above discussion, we make the following conclusions-

(1) Shifting of origin from $O = (0, 0, 0)$ to $O' = (\alpha, \beta, \gamma)$

If we shift the origin from O=(0,0,0) to another point O'=(α,β, γ), then we have to replace x by $x+\alpha$, y by $y+\beta$ and z by $z+\gamma$. Then the transformed equation of a space curve, surface etc. is obtained w.r.t. (α,β, γ) as the new origin.Here it is remarkable from equation (1), that we should replace x by $x' + \alpha$, y by $y' + \beta$ and z by $z' + \gamma$ but we replace the same by $x + \alpha$, $y + \beta$ and $z + \gamma$ respectively just for shake convenienceso that the transformed equation is also in current coordinates (x, y, z).

Shifting the origin back from the point $O' = (\alpha, \beta, \gamma)$ to the point $O = (0, 0, 0)$ Sometimes it is required to shift the new origin back to the original origin. For this we replace x (i.e. x') by $x-\alpha$, y (i.e. y') by $y-\beta$ and z (i.e. z') by $z-\gamma$, in the transformed equation referred to the new origin O' to get the corresponding equation referred to the original origin O . Here we do so such that the transformed equation is also in the current coordinates.

Another method or vector method (change of origin)

Let *OX*, *OY* and *OZ* be a set of three mutually orthogonal coordinate axes. Let the coordinates of two points *P* and *Q* w.r.t. these three mutually orthogonal axes are (α,β,γ) and (x,y,z) respectively. Now suppose we shift the origin from $O=(0,0,0)$ to the point *P* i.e. (α,β,γ) and find the coordinates of the point *Q* w.r.t. the point *P* as origin.

Draw the lines PX', PY' and PZ' through the point *P*, parallel and in the same directions of *OX*, *OY* and *OZ* respectively.

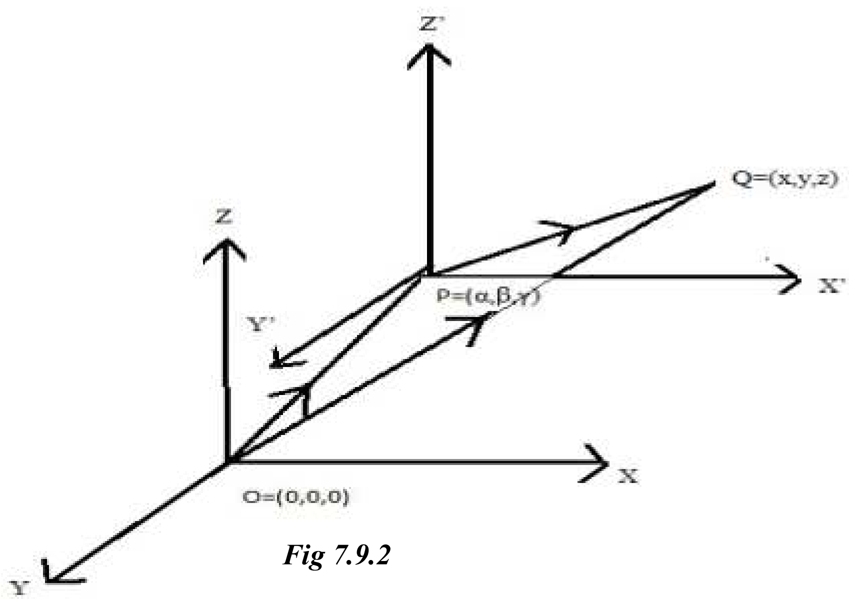


Fig 7.9.2

The position vectors of the points *P* and *Q* w.r.t. *O* as origin are given by,

$\vec{OP} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}, \vec{OQ} = x\hat{i} + y\hat{j} + z\hat{k}$

Now the position vector of Q with respect to P as origin is given by,

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP}$,Hence we get, $\overrightarrow{PQ} = (x-\alpha)\,\hat{\imath} + (y-\beta)\,\hat{\jmath} + (z-\gamma)\,\hat{k}$

Hence the coordinates of the point Q with respect to P as origin are (x-α, y-β, z-γ).

ANOTHER METHOD Let position vector of P w.r.t. origin O be $\vec{r}$ and position vector of O' w.r.t. origin O be $\vec{a}$.

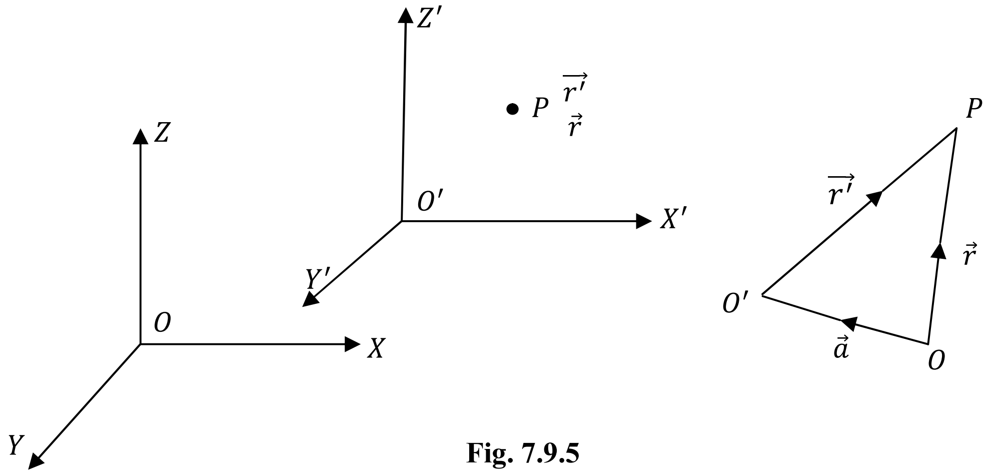


Fig. 7.9.5

Let position vector of P w.r.t. origin O be$\overrightarrow{r'}$,

Then $\vec{r} = \overrightarrow{r'} + \vec{a}$

$\Rightarrow x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \alpha\hat{\imath} + \beta\hat{\jmath} + \gamma\hat{k} + x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k}$

Comparing real and imaginary part, we get

$x = x' + \alpha, y = y' + \beta$ and $\gamma = z' + \gamma$

7.9.2 *ROTATION OF AXES (CHANGE OF DIRECTIONS OF AXES)*

To find the transformed coordinates of a point, when the directions of axes are changed, keeping the origin fixed.

DERIVATION Let O be the origin, also let OX, OY, OZ are the original system of coordinate axes. Let OX', OY' and OZ' be the new system of coordinate axes through the same origin O.

Let $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are the direction cosines of OX', OY' and OZ' respectively with respect to *OX*, *OY* and *OZ*.

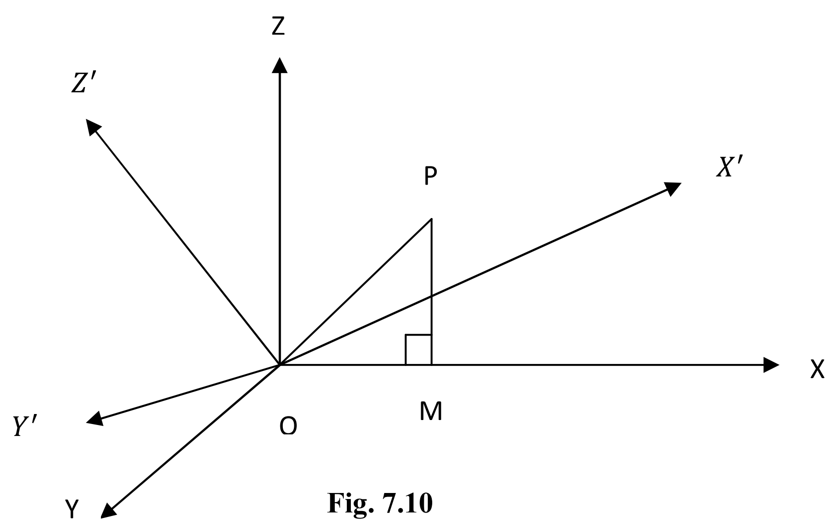


Fig. 7.10

Then direction cosines of *OX*, *OY* and *OZ* with respect to OX', OY' and OZ' will be $[l_1, l_2, l_3]$, $[m_1, m_2, m_3]$ and $[n_1, n_2, n_3]$ respectively.

Let P be any arbitrary point in the space whose coordinates are (x, y, z) with respect to *OX*, *OY*, *OZ* as coordinate axes and (x', y', z') with respect to OX', OY', OZ' as coordinate axes respectively.

Draw a perpendicular PM from the point P on OX.

From figure, it is clear that—

X = OM =Projection of OP on OX

$=\sum corresponding\ multiplication\ of\ the\ dr's\ of\ OP\ \&\ dc's\ of\ OX$ --------------- (1)

Now with respect to the coordinate axes OX', OY' and OZ', the coordinates of the point P are (x', y', z'). Hence the dr's of the line OP are $(x' - 0, y' - 0, z' - 0)$ i.e (x', y', z'). Also the dc's of OX w.r.t. OX', OY' and OZ' are $[l_1, l_2, l_3]$. Hence the equation (1) gives,

$x = x'l_1 + y'l_2 + z'l_3$-- (2)

similarly we get ,

$y = x'm_1 + y'm + z'm_3$--- (3)

$z = x'n_1 + y'n_2 + z'n_3$ --- (4)

Now multiplying the equations (2), (3) and (4) by l1, m_1 and n_1 respectively and then adding we get,

$l_1x + m_1y + n_1z = x'(l_1^2 + m_1^2 + n_1^2) + y'(l_1l_2 + m_1m_2 + n_1n_2) + z'(l_1l_3 + m_1m_3 + n_1n_3)$ ... (5)

Since $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are the direction cosines of OX', OY' and OZ' i.e. dc's of three mutually perpendicular lines, hence using the relations among the dc's of three mutually perpendicular lines [see article 7.11], i.e. $l_1^2 + m_1^2 + n_1^2 = 1$, $l_2^2 + m_2^2 + n_2^2 = 1$, $l_3^2 + m_3^2 + n_3^2 = 1$

$l_1l_2 + m_1m_2 + n_1n_2 = 0$, $l_1l_3 + m_1m_3 + n_1n_3 = 0$, $l_2l_3 + m_2m_3 + n_2n_3 = 0$

in equation (5) we get,

$l_1x + m_1y + n_1z = x'(1) + y'(0) + z'(0) = x'$ 'OR'

$x' = l_1x + m_1y + n_1z$ --------------------------- (6)

similarly, we get, $y' = l_2x + m_2y + n_2z$ ---------------------------(7)

$z' = l_3x + m_3y + n_3z$ --------------------------- (8)

Hence from the above discussion we conclude the following results –

RESULT-(1) If OX, OY, OZ are original coordinate axes and an equation w.r.t. these coordinate axes is given, then to find the transformed equation w.r.t. OX', OY' and OZ' as coordinate axes , we replace x by $x'l_1 + y'l_2 + z'l_3$, y by $x'm_1 + y'm_2 + z'm_3$ and z by $x'n_1 + y'n_2 + z'n_3$.

Just for shake convenience in practice we leave the dashes and hence the required transformed equation is obtained by replacing, x by $l_1x + l_2y + l_3z$, y by $m_1x + m_2y + m_3z$ and z by $n_1x + n_2y + n_3z$; so that the transformed equation is also in current coordinates (x, y, z).

RESULT-(2) When the transformed axes OX', OY' and OZ' are shifted back to the original coordinate axes OX, OY and OZ. If we want to shift the coordinate axes OX', OY' and OZ' back to their original positions OX, OY and OZ then we replace x (i.e.x') by $l_1x + m_1y + n_1z$; y (i.e. y') by $l_2x + m_2y + n_2z$ and z (i.e. z') by $l_3x + m_3y + n_3z$.

Here also we get the transformed equation in current coordinates (x, y, z).

REMARK The above results (1) & (2) of transformations in the article Rotation of axes (Change of directions of axes) from OX, OY, OZ to OX', OY', OZ' and vice-versa can be conveniently remembered by the following transformation table

	x'	y'	z'
X	l_1	l_2	l_3
Y	m_1	m_2	m_3
Z	n_1	n_2	n_3

From this table to find the value of x, we multiply each element of the row of x with the corresponding elements of the top row and then add i.e. we have, $x = x'l_1 + y'l_2 + z'l_3$.

Similarly, we can find the values of y and z. As, y = $x'm_1 + y'm_2 + z'm_3$ and z = $x'n_1 + y'n_2 + z'n_3$.

From this table to find the value of x, we multiply each element of the column of x' with the corresponding elements of the first column and then add i.e. we have, $x' = l_1x + m_1y + n_1z$

Similarly, we can find the values of y' and z'. As, $y' = l_2x + m_2y + n_2z$ & $z' = l_3x + m_3y + n_3z$.

7.10 *RELATIONS BETWEEN THE DIRECTION COSINES OF THREE MUTUALLY PERPENDICULAR LINES*

Let OX, OY and OZ be the given set of mutually orthogonal coordinate axes and OX', OY' and OZ' are three mutually perpendicular lines through O. Let $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are the direction cosines of OX', OY' and OZ' respectively with respect to OX, OY and OZ. Then we have the following six relations-

$$l_1^2 + m_1^2 + n_1^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1$$
$$l_3^2 + m_3^2 + n_3^2 = 1$$
$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$l_1l_3 + m_1m_3 + n_1n_3 = 0$$
$$l_3l_2 + m_2m_3 + n_2n_3 = 0$$

Also we have learnt that the direction cosines of OX, OY and OZ with respect to OX', OY' and OZ' are given by$[l_1, l_2, l_3]$, $[m_1, m_2, m_3]$ and $[n_1, n_2, n_3]$ respectively. Hence we also have the following six relations

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$

$n_1^{\ 2} + n_2^{\ 2} + n_3^{\ 2} = 1$

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$

$l_1 n_1 + l_2 n_2 + l_3 n_3 = 0$

REMARKS

(1) Distance between two given points in the space remains unaltered under the transformation of coordinates i.e. under translation of axes and rotation of axes.

(2) If three lines are mutually perpendicular, then we can find the direction cosines of one line in terms of the other two lines.

DERIVATION

If the lines with direction cosines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are mutually perpendicular, then we have

$l_1^{\ 2} + m_1^{\ 2} + n_1^{\ 2} = 1$

$l_2^{\ 2} + m_2^{\ 2} + n_2^{\ 2} = 1$

$l_3^{\ 2} + m_3^{\ 2} + n_3^{\ 2} = 1$

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$

$l_3 l_2 + m_2 m_3 + n_2 n_3 = 0$

Solving the equations, $l_1 l_2 + m_1 m_2 + n_1 n_2$=0

$l_1 l_3 + m_1 m_3 + n_1 n_3$=0, for l_1, m_1,n_1 by cross multiplication method we get,

$$\frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{l_3 n_2 - l_2 n_3} = \frac{n_1}{l_2 m_3 - l_3 m_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{(m_2 n_3 - m_3 n_2)^2}} = \frac{\sqrt{1}}{\sqrt{sin 90^\circ}} = \pm 1$$

(By the definition of angle between two straight lines , as we have learnt in the unit of straight line)

Hence we get,

$$l_1 = \pm(m_2n_3 - m_3n_2)$$

$$m_1 = \pm(l_3n_2 - l_2n_3)$$

$$n_1 = \pm(l_2m_3 - l_3m_2)$$

Similarly, we can find the direction cosines of other two lines in terms of the remaining one by using the determinant given below,

$$D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Where the dc's of any constituent in D = ±(its cofactor).

(3).To prove that D = ±1 ,where D is defined as in remark (2).

ProofWe have,

$$D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Squaring above we get,

$$D^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

(Multiplying the determinants in row by row pattern, we get)

$$D^2 = \begin{vmatrix} {l_1}^2 + {m_1}^2 + {n_1}^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_1l_3 + m_1m_3 + n_1n_3 \\ l_1l_2 + m_1m_2 + n_1n_2 & {l_2}^2 + {m_2}^2 + {n_2}^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_1l_3 + m_1m_3 + n_1n_3 & l_2l_3 + m_2m_3 + n_2n_3 & {l_3}^2 + {m_3}^2 + {n_3}^2 \end{vmatrix}$$

Now using the results given below,

$${l_1}^2 + {m_1}^2 + {n_1}^2 = 1$$

$${l_2}^2 + {m_2}^2 + {n_2}^2 = 1$$

$${l_3}^2 + {m_3}^2 + {n_3}^2 = 1$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$l_2l_3 + m_2m_3 + n_2n_3 = 0$$

$$l_1l_3 + m_1m_3 + n_1n_3 = 0$$

we get,

$$D^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Solving we get, $D^2 = 1$

Taking square root on both sides we get, $D = \pm 1$

Hence proved.

SOLVED EXAMPLES

Ex.1 If OA, OB and OC are three mutually perpendicular lines through the origin with direction cosines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ respectively and if OA = OB = OC = a.Find the equation of the plane ABC.

Sol If the lines OA, OB and OC are taken as the coordinate axes , then the equation of the plane which makes equal intercepts (each equal to a) on these axes is, --

$\frac{X}{a}+\frac{Y}{a}+\frac{Z}{a}=1$ i.e. $X+Y+Z=a$ --------------------- (1)

Now to find the equation of the plane (1) with respect to original coordinate axes we have to replace X by $l_1x+m_1y+n_1z$, Y by $l_2x+m_2y+n_2z$ and Z by $l_3x+m_3y+n_3z$ in equation (1). Hence the equation of the plane ABC with respect to original coordinate axes is given by,-

$(l_1x+m_1y+n_1z) + (l_2x+m_2y+n_2z) + (l_3x+m_3y+n_3z) = a$ OR

$(l_1+l_2+l_3)x + (m_1+m_2+m_3)y + (n_1+n_2+n_3)z = a$

This is the required equation of the plane.

Ex.2 Show that if the general expression of second degree in x, y and z i.e.$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$, be transformed by change of coordinates from one set of rectangular axes to another with the same origin; the expressions $a + b + c$ and $u^2+v^2+w^2$ remain unaltered.

SolThe given expression is,

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$ ------------------------------------ (1)

Let the given set of axes OX, OY and OZ be transformed to the new set of axes $OX', OY'$and OZ' keeping the same origin. Let $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are the direction cosines of OX', OY' and OZ' respectively with respect to OX, OY and OZ. Then to get the transformed form of the expression (1) with respect to $OX', OY'$and OZ', we have to replace x by $xl_1 + yl_2 + zl_3$, y by $xm_1 + ym_2 + zm_3$ and z by $xn_1 + yn_2 + zn_3$;so that the transformed expression of expression (1) is,

$a(xl_1 + yl_2 + zl_3)^2+b(xm_1 + ym_2 + zm_3)^2+c(xn_1 + yn_2 +zn_3)^2+2f(xm_1 + ym_2 + zm_3)(xn_1 + yn_2 + zn_3) +2g(xl_1 + yl_2 + zl_3)(xn_1 + yn_2 + zn_3) +2h(xm_1 + ym_2 + zm_3)(xl_1 + yl_2 + zl_3) +2u(xl_1 + yl_2 + zl_3) +2v(xm_1 + ym_2 + zm_3) +2w(xn_1 + yn_2 + zn_3) +d$ ----------------------- (2)

The expression (2) can be rewritten as,

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy+2Ux+2Vy+2Wz+D$

Where,

$A = al_1^2+bm_1^2+cn_1^2+2fm_1n_1+2gn_1l_1+2hl_1m_1$

$B = al_2^2+bm_2^2+cn_2^2+2fm_2n_2+2gn_2l_2+2hl_2m_2$

$C = al_3^2+bm_3^2+cn_3^2+2fm_3n_3+2gn_3l_3+2hl_3m_3$

$U = ul_1+vm_1+wn_1$

$V = ul_2+vm_2+wn_2$

$W = ul_3+vm_3+wn_3$

$D = d$

Now, $A+B+C = a[l_1^2+l_2^2+l_3^2] + b[m_1^2+m_2^2+m_3^2] + c[n_1^2+n_2^2+n_3^2] + 2f[m_1n_1+m_2n_2+m_3n_3] + 2g[n_1l_1+n_2l_2+n_3l_3] + 2h[l_1m_1+l_2m_2+l_3m_3]$

Now using the relations of three mutually perpendicular lines i.e.,

$$l_1^2 + m_1^2 + n_1^2 = 1$$

$$l_2^2 + m_2^2 + n_2^2 = 1$$

$$l_3^2 + m_3^2 + n_3^2 = 1$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

$$l_2l_3 + m_2m_3 + n_2n_3 = 0$$

$$l_1l_3 + m_1m_3 + n_1n_3 = 0$$

we get,

$$A + B + C = a.1\, b.1 + c.1 + 2f.0 + 2g.0 + 2h.0 \ = a + b + c$$

Hence, $A + B + C \ = \ a + b + c$ -- (3)

Also, $U^2+V^2+W^2= (ul_1+vm_1+wn_1)^2 + (ul_2+vm_2+wn_2)^2 + (ul_3+vm_3+wn_3)^2 =u^2[l_1^2+l_2^2+l_3^2] + v^2[m_1^2+m_2^2+m_3^2] + w^2[n_1^2+n_2^2+n_3^2] + 2vw[m_1n_1+m_2n_2+m_3n_3] +2uw[n_1l_1+n_2l_2+n_3l_3]+ 2uv[l_1m_1+l_2m_2+l_3m_3]$

Now using the above relations we get,

$U^2+V^2+W^2=u^2.1+v^2.1+w^2.1+2vw.0+2uw.0+2uv.0=u^2+v^2+w^2$

Hence, $U^2+V^2+W^2 = u^2+v^2+w^2$ ----------------------------------- (4)

From the transformed expressions given in (3) & (4), we conclude that the expressions a + b + c and $u^2 + v^2 + w^2$ remain unaltered, hence proved.

SELF CHECK QUESTIONS

(SCQ-17) If we change the directions of axes keeping the origin fixed, then the distance between two points

a) increases b) decreases (c remains the same d) may increase or decrease

(SCQ-18) If we change the directions of axes keeping the origin fixed, then it is also called as

a) rotation of axes b) translation of axes c) change of origin d) none of these

(SCQ-19) If the lines with direction cosines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ and $[l_3, m_3, n_3]$ are mutually perpendicular, then the value of the determinant, $D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ is,

a) -1 b) ± 1 c) 1 d) 0

(SCQ-20) If the origin is shifted from (0, 0, 0) to (α, β, γ), then we have to replace x by,

a) x+α b) x-α c) x+β d) none of these

FILL IN THE BLANKS

(SCQ-21) If the distance between the two points w.r.t. O = (0,0,0) as origin is 5 units, then the distance between them after transferring the origin O to the point (2,3,4) will be------------------ units.

(SCQ-22) The two lines with direction cosines $[l_1, m_1, n_1]$ & $[l_2, m_2, n_2]$ will be mutually perpendicular if $l_1l_2+m_1m_2+n_1n_2=$ ----------------------.

(SCQ-23) The maximum number of relations between the direction cosines of three mutually perpendicular lines is = --------------------------.

(SCQ-24) The translation of axes is also called as --.

7.11 *SUMMARY*

In this unit, we have learned about the difference between the various surfaces, particularly the difference between a tetrahedron, a pyramid and a prism. Also we have learned that a tetrahedron contains 4 fully flat or plane surface having 3 lateral surfaces & 1 base, 4 vertices and 6 edges. Also we have studied that the base of a tetrahedron is always a triangle and it ends at a point while in case of a pyramid, the base may be square, rectangle, rhombus etc. Here we derived the various methods to find the volume of a tetrahedron in different forms such as vertices form, vector form, faces form, concurrent edges and mutual inclination form etc. Also we have learned about the method in which we can check whether the given four points will be coplanar or not. Also we have learnt the method to find the area of the base of a tetrahedron in terms of its volume and height. Later on we have derived the expression for the transformation of coordinates. The transformation of coordinates mainly consists of two parts

(1) Change of origin or translation of axes and

(2) Rotation of axes or change of directions of axes. With the help of these expressions, also by the vector method and by using the concept of relativity, we have studied about the change of coordinates of a point, the equation of a space curve (intersection of two surfaces), the equation of surfaces etc. We also found the conditions when we come back to the original coordinate system from the transformed coordinate system. Here we also revise the relations between the

direction cosines of three mutually perpendicular lines. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics. To check your progress, self check questions (SCQ's) are given place to place.

7.12 *GLOSSARY*

1. 3D object or a solid – an object that occupies space.
2. Face – surface of a solid.
3. Edge – where two faces meet.
4. Vertex – where two edges meet.
5. Mutually – shared by two or more.
6. Slant- to be at an angle, not vertical or horizontal.
7. Eliminate – To remove or to omit or to neglect that is not wanted or needed or required.
8. Perpendicular – right angle or at an angle of $90°$ or pointing straight up.
9. Origin – the point from which we start generally at O= (0, 0, 0).
10. Axes – plural of axis.
11. Volume—the amount of space that something contains or fills.
12. Rotation—movement in circles around a central point.
13. Invariant—not changing or constant.

7.13 *REFERENCES*

1. Geometry of two and three Dimensions - Lata S., Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal , S.J, Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd. , New Delhi, India.
6. Leonard E. Dickson :The First course in the Geometry.
7. Burnside, William Show, Panton &Arthur William: Volumetric Theory.
8. Murray R. Spiege l: Vector Analysis, Schaum's Outline Series, McGraw Hill.
9. P.N. Pandey: Polar Coordinate Geometry, Sharda Academic Publishing House, Allahabad.
10. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

7.14 *SUGGESTED READINGS*

1. Analytical geometry –NazrulIslam , Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
3. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhaniya ,S.Chand Publication, New Delhi.

7.15 *TERMINAL QUESTIONS*

(TQ-1) Prove that the necessary and sufficient condition for the four points to be coplanar is that the volume of the tetrahedron formed by them as its vertices is zero.

(TQ-2) Find the locus of the point P, if the volume of the tetrahedron PABC is 5 cube units; where A=(3,2,1), B=(-2,0,-3) and C=(0,0,-2).

(TQ-3) Show that the distance between two points remains unaltered under translation of axes and rotation of axes.

(TQ-4) A variable plane makes with co-ordinates planes a tetrahedron of constant volume $64k^3$.Find the locus of the centroid of the tetrahedron.

(TQ-5) If the volume of the tetrahedron whose vertices are (a,1,2), (3,0,1), (4,3,6) and(2,3,2) is 6 cube units, find the value of a.

(TQ-6) If A, B, C are three fixed points and a variable point P moves so that the volume of the tetrahedron PABC is constant, show that the locus of P is a plane parallel to the plane ABC.

(TQ-7) Find the volume of the tetrahedron included between the coordinate planes and the plane 2x-3y-z-6=0.

(TQ-8) Find the volume of tetrahedron, whose three coterminous edges in right handed system are denoted by the vector quantities $\vec{a}$,$\vec{b}$ and $\vec{c}$, where $\vec{a} = 2\hat{\imath}+2\hat{\jmath}+6\hat{k}$,$\vec{b} = \hat{\imath}+3\hat{\jmath}+2\hat{k}$ and $\vec{c} = -\hat{\imath}+5\hat{\jmath}+5\hat{k}$

(TQ-9) Show that if the expression of 2^{nd} degree in x, y and z given by $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, be transformed by change of coordinates from one set of rectangular axes to another with the same origin; the expressions a + b + c, $f^2+g^2+h^2$-bc-ca-ab and abc+2fgh-af^2-bg^2-ch^2 remain invariants.

(TQ-10) Suppose the coordinates of two points P and Q with respect to O= (0,0,0) as origin are (x_1,y_1,z_1) and (x_2,y_2,z_2) .If the origin is shifted to the point (α, β, γ), find the coordinates of these points in new coordinate system and check whether the distance between the points in new coordinate system changes or remains invariant.

7.16 *ANSWERS*

SELF CHECK QUESTIONS (SCQ'S)

(SCQ-1) c (SCQ-2) c (SCQ-3) a (SCQ – 4) b

(SCQ-5) d (SCQ-6) c (SCQ-7) d (SCQ – 8) a

(SCQ-9) b (SCQ-10) b (SCQ-11) d (SCQ – 12) b

(SCQ-13) a (SCQ-14) c (SCQ-15) d (SCQ – 16) b

(SCQ-17) c (SCQ-18) a (SCQ-19) b (SCQ-20) a

(SCQ-21) 5 (SCQ – 22) 0 (SCQ – 23) 6 (SCQ – 24) change of origin.

TERMINAL QUESTIONS (TQ'S)

(TQ – 2) 2x+3y-4z=38

(TQ – 4) $xyz = 6k^3$

(TQ – 5) a = 0

(TQ –7) 6 cube units

(TQ – 8) $\frac{22}{3}$ cube

(TQ – 10) $(x_1 + \alpha, y_1 + \beta, z_1 + \gamma)$ &$(x_2 + \alpha, y_2 + \beta, z_2 + \gamma)$; remains invariant.

UNIT 8: THE SPHERE

CONTENTS

8.1 INTRODUCTION

In the previous classes, you should have learnt and studied by now that there exits two types of 3D surfaces -

i. Plane or flat surfaces e.g. surfaces of a cube, surface of a table etc.
ii. Curved surfaces e.g. surface of a cricket ball, surface of a watermelon etc.

In case of a plane surface the line joining any two points on it fully lies in it but this property is not followed by the curved surfaces.

Some examples of solid shapes which we use in our daily life are cubical, cuboidal, conical, cylindrical, spherical objects etc.

Here we have to study about the sphere. In our daily life we use many types of solid shapes or surfaces.

Some important results about our daily life solid shapes are as follows –

Solid Shapes	Examples	Total Surfaces	Plane or Flat Surfaces	Curved Surfaces	Edges	Vertices
Cube	Dice	6	All Plane	0	12	8
Cuboid	Book	6	All Plane	0	12	8
Sphere	Ball	1	0	1	0	0
Cone	Ice- cone	2	1	1	1	1
Cylinder	Gas Cylinder	3	2	1	2	0

Hence a sphere is a fully curved surface with no edge and no vertex.

8.2 OBJECTIVES

After studying this unit, you should be able to -

- Understand the difference between the surfaces (3D) and the plane curves (2D).
- Understand if the given equation of 2nd degree in x, y and z represents a sphere or not.
- Understand the different forms of equation of a sphere.
- Understand the relationship among the different forms of equation of a sphere.
- Find the section of a sphere by a given plane.
- Calculate the length of tangent from a given external point to the sphere.
- Check whether a given line or plane touches the given sphere or not.
- Calculate the angle of intersection of two spheres.
- Find whether the two give spheres do not touch or touch internally or externally.
- Find the coordinates of the point of contact of two spheres.

8.3 DEFINITION OF A SPHERE

A sphere is the locus of a variable point P (say), which moves in the space such that its distance from a fixed point (say C) always remains the same i.e. CP $=$ constant .

The fixed point C is called the centre of the sphere and the constant distance $CP = r$ (say) is called the radius of the sphere.

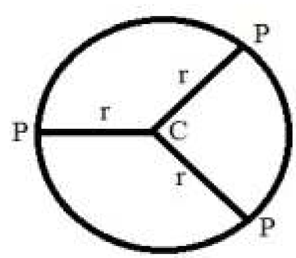


Fig. 8.3.1

8.4 EQUATION OF A SPHERE (CENTRAL FORM)

Here we shall find the equation of a sphere when centre and radius are given by using the concept of its definition given in article (8.3) i.e. distance between two points .

DERIVATION

Let $C = (a, b, c)$ be the coordinates of the centre of the sphere and r be the radius of the sphere.

Let $P = (x, y, z)$ be any arbitrary point on the surface of the sphere. Then by definition of a sphere (8.3) we have,

$CP =$ Constant $= r$

i.e. $CP^2 = r^2$ (squaring above)

Now we get the equation of the sphere,

by using the concept of distance between

two points $C \& P$. i.e.

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ ------------ (1)

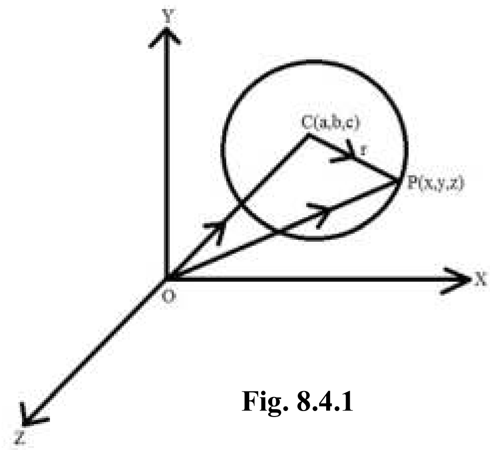


Fig. 8.4.1

This is the required equation of the sphere, whose centre is (a, b, c) and radius is r.

This form is called the Central form.

ANOTHER METHOD

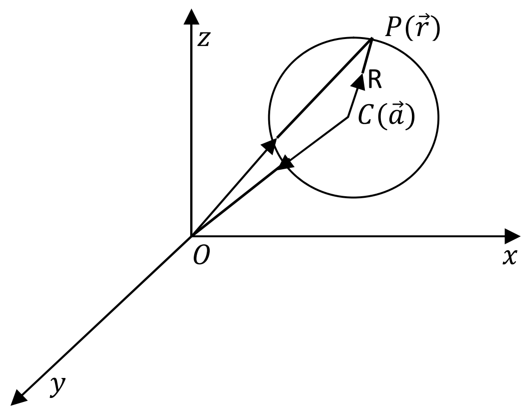


Fig. 8.4.2

Let position vector of centre of sphere be $\vec{a}$ and P (whose position vector be $\vec{r}$) be any vector on the surface of the sphere and R be the radius of the sphere

Then $\left|\overrightarrow{CP}\right| = R$ or $\left|\overrightarrow{CP}\right|^2 = R^2$ i.e. $|\vec{r}-\vec{a}|^2 = R^2$ $(\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC})$

$(\vec{r}-\vec{a}).(\vec{r}-\vec{a}) = R^2$ $\quad (\vec{a}.\vec{a} = |\vec{a}|^2)$

i.e. $\vec{r}.\vec{r} - \vec{a}.\vec{r} - \vec{r}.\vec{a} + \vec{a}.\vec{a} = R^2$

$$\Rightarrow |\vec{r}|^2 - 2\,\vec{a}.\vec{r} + |\vec{a}|^2 = R^2$$

$\Rightarrow |\vec{r}|^2 - 2\,\vec{a}.\vec{r} + |\vec{a}|^2 = R^2$……………………………..(8.4.2)

$$\Rightarrow x^2 + y^2 + z^2 - 2(ax + by + cz) + a^2 + b^2 + c^2 = R^2$$

$\Rightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ which is same as equation (1) and (2) is the vector equation of the sphere.

REMARKS

(1) If the centre of the sphere is at origin that is $(0,0,0)$ and radius is r then the equation (1) of the sphere becomes
$x^2 + y^2 + z^2 = r^2$……………………………….(3)
and vector equation is given by $|\vec{r}| = R$, where R is radius of sphere.

Difference between a circle and a sphere

A circle is a $2D$ plane curve while a sphere is a $3D$ curved surface. Only One tangent can be drawn at a point on a circle while infinite many tangents can be drawn at any point on a sphere. The plane, containing all the tangents at a point of the sphere is called the "Tangent plane" at that point.

Ex.1. Find the equation of the sphere whose centre is $(2, -3, 4)$ and which passes through the point $(1, 2, -1)$.

Sol. Let the centre of the sphere be denoted by C& the given point on the surface of sphere be P then, $C = (2, -3, 4)$ and $P = (1, 2, -1)$.

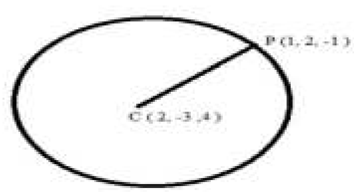


Fig. 8.4.3

Hence, The radius of a the sphere = CP

= the distance between the points C and P

$$= \sqrt{(2-1)^2 + (-3-2)^2 + (4+1)^2}$$

$$= \sqrt{1 + 25 + 25} = \sqrt{51}$$

Hence the required equation of the sphere whose centre is $(2, -3, 4)$ and radius is $\sqrt{51}$ is given by,

$(x-2)^2 + (y+3)^2 + (z-4)^2 = (\sqrt{51})^2$[By using central form of equation of a sphere]

Solving above we get,

$x^2 + y^2 + z^2 - 4x + 6y - 8z = 0 - 22 = 0$, required equation of sphere.

8.5 GENERAL EQUATION OF A SPHERE

A general equation of a sphere means an equation which includes all types of sphere.

DERIVATION

We have learnt by now that the equation of a sphere in central form (8.4) is given by,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

where (a, b, c) is the centre and r is the radius of the sphere.

Expanding and rearranging above we get,

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - r^2 = 0$$

Using substitutions, $a = -u$

$b = -v$

$c = -w$

$a^2 + b^2 + c^2 - r^2 = d$, in above we get,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ……………………………..(8.5.1)

This equation is called the general form of the equation of a sphere. From this equation we make the following observations

1. It is a second degree equation in x, y and z.
2. The coefficients of x^2 , $y^2$and z^2 are equal.
3. The coefficients of xy, yz and zx are zero that is the terms of xy, yz and zx are absent.

If these three conditions are satisfied then the given equation of second degree in x, y and z will represent a sphere.

Now the general equation (1) of sphere can be rewritten as,

$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) + d = u^2 + v^2 + w^2$ (adding u^2,v^2,w^2 on both sides we get)

i. e. $(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$ (making perfect squares & transferring d to R.H.S.)

i.e. $\{x - (-u)\}^2 + \{y - (-v\}^2 + \{z - (-w\}^2 = \left(\sqrt{u^2 + v^2 + w^2 - d}\right)^2$ ....(2)

Comparing (2) with (1), we gets

Centre = $(-u, -v, -w)$
Radius = $\sqrt{u^2 + v^2 + w^2 - d}$

………………………………(3)

REMARKS

1. The most general equation of second degree is given by,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

It represents a sphere if,

a = b = c ≠ 0 and f = g = h = 0.

2. For a sphere the general equation is,
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ whose centre and radius are given by,

Centre = $(-u, -v, -w)$

Radius = $\sqrt{u^2 + v^2 + w^2 - d}$

OR

For finding the centre and radius of the given sphere in general form, we make the coefficients of x^2, y^2 and z^2 unity by dividing the coefficient of x^2, y^2 and z^2 as they are equal to

Centre of sphere = $[-\frac{1}{2}(\text{coeff. of x}) - \frac{1}{2}(\text{coeff. of y}) - \frac{1}{2}(\text{coeff. of z})]$

Radius of sphere =

$$\sqrt{\left(\frac{1}{2}\text{coeff. of x}\right)^2 + \left(\frac{1}{2}\text{coeff. of y}\right)^2 + \left(\frac{1}{2}\text{coeff. of z}\right)^2 - (\text{constant term})}$$

3. If $u^2 + v^2 + w^2 - d < 0$ then the radius of the sphere (8.5.1) is Imaginary, whereas the centre is real, such a sphere is calledthe Pseudo sphere or Virtual sphere.
4. The centre of a sphere does not lie on the sphere, it lies inside the sphere. Hence it does not satisfy the equation of the sphere.
5. In general to find an equation of sphere we need at least four condition because there are four variable u, v, w and d.

Ex. 2. Find the equation of the sphere passing through the origin and making intercepts $a, b, c$with the axes respectively. Also find the centre and radius of the sphere.

Sol. Let the sphere cuts the axes at A, B and C respectively and O be the origin, then

According to question,

$$OA = a, \quad OB = b, \quad OC = c$$

Also $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c), O = (0, 0, 0)$

Let the equation of sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$……………………………….(I)

If this sphere passes through the origin $O = (0, 0, 0)$ we get form equation (I), $d = 0$.

Putting $d = 0$ equation (I) of sphere becomes,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$……………………………………..(II)

If the sphere (II) to passes through the point $A = (a, 0, 0)$ we get,

$$a^2 + 0 + 0 + 2ua + 0 + 0 = 0$$

Simplifying this we get, $u = -\frac{a}{2}$

Similarly, if the sphere (II) passes through the points $B = (0, b, 0)$ and $C = (0, 0, c)$, we get,

$$v = -\frac{b}{2}, \quad w = -\frac{c}{2}$$

Putting the values of u, v and w in equation (II) the required equation of the sphere is,

$x^2 + y^2 + z^2 - a\,x - b\,y - c\,z = 0$ ……………………………………(III)

Now we have two find the centre and radius of sphere (III). For this comparing the equation (III) with the general equation of sphere i.e. $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

We get $u = -\frac{a}{2}, v = -\frac{b}{2}, w = -\frac{c}{2}$

Hence the centre of sphere = $(-u, -v, -w) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ and

radius of the sphere $= \sqrt{u^2 + v^2 + w^2 - d} = \frac{1}{2}\sqrt{a^2 + b^2 + c^2}$ (Putting the values of u, v, w &d and simplifying)

8.6 DIAMETER FORM OF EQUATION OF A SPHERE

To find the equation of a sphere on the line joining the two given points as a diameter.

OR

To find the equation of a sphere when the end points i.e. extremities one of its diameter are given.

DERIVATION

Let $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ are the two given points as end points of diameter of the sphere, whose equation is to be determined.

Let $P = (x, y, z)$ be any arbitrary point on the surface of the sphere, drawn on the line joining A and B as the diameter.

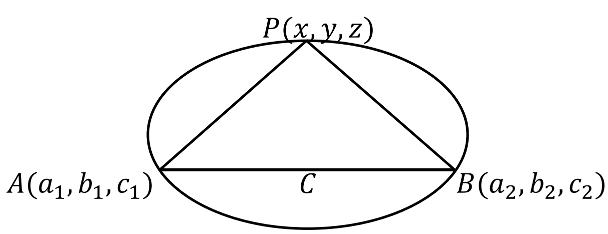


Fig. 8.6.1

Then, AP is perpendicular to BP, since any diameter of a sphere always substends a right angle at any point on the surface of the sphere.

Now the direction ratios of the lines AP and BP are $x - a_1,\ y - b_1, z - c_1$ and $x - a_2,\ y - b_2, z - c_2$ respectively.

As we have learnt in the unit of the straight line if (p_1, q_1, r_1) and (p_2, q_2, r_2) are direction ratios of two mutually perpendicular straight lines then,

$$p_1p_2 + q_1q_2 + r_1r_2 = 0$$

Now as the line AP is perpendicular to the line BP, so applying the above result for these two lines we get the required equation of a sphere in equation as

$$(x - a_1)(x - a_2) + (y - b_1)(y - b_2) + (z - c_1)(z - c_2) = 0 \ldots\ldots(1)$$

which is the required equation of the sphere with AB as diameter.

REMARK Centre C of the sphere with the points $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ joining as diameter will be the midpoint of A and B. Hence the coordinates of the centre C of the sphere are given by,

$$C = \{\frac{1}{2}(a_1 + a_2), \frac{1}{2}(b_1 + b_2), \frac{1}{2}(c_1 + c_2)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Ex. 3. Find the centre of the sphere with the points $(-5, 3, 4)$ and $(1, 7, 2)$ joining as the end points of the diameter.

Sol. The centre of the sphere will be the midpoint of the end points of the diameter.

Hence centre$= \left(\frac{1}{2}(-5 + 1), (3 + 7), \frac{1}{2}(4 + 2)\right) = (-2, 5, 3)$

8.7 EQUATION OF A SPHERE THROUGH FOUR GIVEN POINTS

To find the equation of a sphere, which passes through four given points?

DERIVATION

Let the coordinates of the four given points A, B, C and D are (a_1, b_1, c_1) , (a_2, b_2, c_2) , $(a_3, b_3, c_3)$and (a_4, b_4, c_4)respectively.

Let the equation of the sphere passing through these four points A, B, C and D is,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ …………………………..(1)

(The general equation of a sphere)

If this sphere (8.7.1) passes through the given four points A, B, C and D then equation (8.7.1) must satisfy these four points. Hence we have,

${a_1}^2 + {b_1}^2 + {c_1}^2 + 2ua_1 + 2vb_1 + 2wc_1 + d = 0$…………(2)

${a_2}^2 + {b_2}^2 + {c_2}^2 + 2ua_2 + 2vb_2 + 2wc_2 + d = 0$…………(3)

${a_3}^2 + {b_3}^2 + {c_3}^2 + 2ua_3 + 2vb_3 + 2wc_3 + d = 0$…………(4)

${a_4}^2 + {b_4}^2 + {c_4}^2 + 2ua_4 + 2vb_4 + 2wc_4 + d = 0$…………(5)

Now eliminating the arbitrary constants u, v, w and d from the equations (1), (2), (3), (4) and (5), we get the required equation of the sphere as,

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ {a_1}^2 + {b_1}^2 + {c_1}^2 & a_1 & b_1 & c_1 & 1 \\ {a_2}^2 + {b_2}^2 + {c_2}^2 & a_2 & b_2 & c_2 & 1 \\ {a_3}^2 + {b_3}^2 + {c_3}^2 & a_3 & b_3 & c_3 & 1 \\ {a_4}^2 + {b_4}^2 + {c_4}^2 & a_4 & b_4 & c_4 & 1 \end{vmatrix} = 0 \quad ....(6)$$

REMARKWe put the values of u, v, w and d (by solving (2), (3), (4) and (5) in equation (1) we get the values of u, v, w and d the required sphere.

Ex. 4. Find the equation of the sphere circumscribing the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol. The equations of the given planes that is faces of tetrahedron are ,

$x = 0$.....(1), $y = 0$.....(2), $z = 0$......(3), $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.........(4)

Taking any three -three planes out of the four at a time we get four vertices of the tetrahedron as ,

Solving (1), (2) and (3) we get $O = (0, 0, 0)$

Solving (2), (3) and (4) we get $A = (a, 0, 0)$

Solving (1), (3) and (4) we get $B = (0, b, 0)$

Solving (1), (2) and (4) we get $C = (0, 0, c)$

Now as in the previous example- (B), the equation of the sphere $OABC$ is given by,

$$x^2 + y^2 + z^2 - a\,x - b\,y - c\,z = 0$$

Ex. 5. A sphere of constant radius $2k$ passes through origin and meets the axes in A, B and C. Prove that the locus of the centroid of the tetrahedron $OABC$ is ,

$x^2 + y^2 + z^2 = k^2$.

Sol. If we take $OA = a$, $OB = b$ and $OC = c$, then equation of the sphere $OABC$ (as in example-B) is given by ,

$x^2 + y^2 + z^2 - a\,x - b\,y - c\,z = 0$……………………………..(1)

radius of this sphere $= \frac{1}{2}\sqrt{a^2 + b^2 + c^2} = 2\,k$ (according to the question)

i.e. $\frac{1}{2}\sqrt{a^2 + b^2 + c^2} = 2k$

i.e. $\sqrt{a^2 + b^2 + c^2} = 4k$, Squaring both sides we get , $a^2 + b^2 + c^2 = 16\,k^2$……..(2)

Let (α, β, μ) be the centroid of the tetrahedron $OABC$ then ,

$$\alpha = \frac{1}{2}(0 + a + 0 + 0)$$

i.e. $a = 4\alpha$

Similarly $b = 4\,\beta$, $c = 4\,\mu$. Putting these values of a, b, c in equation (2) we get ,

$16\,\alpha^2 + 16\,\beta^2 + 16\,\mu^2 = 16\,k^2$ or

$$\alpha^2 + \beta^2 + \mu^2 = k^2$$

taking the locus of $(\alpha\,, \beta, \mu)$ we get ,

$x^2 + y^2 + z^2 = k^2$.

This is the required locus of the centroid of tetrahedron, which is clearly equation of sphere with centre at origin i.e. at $O = (0, 0, 0)$ and radius k.

SELF CHECK QUESTIONS

Choose the correct option:

(SCQ-1) The radius of the sphere $9x^2 + 9y^2 + 9z^2 = 16$, is

a) 3 b) 4 c) $\frac{4}{3}$ d) $\frac{3}{4}$

(SCQ -2) The number of spheres that can be made to pass through the three given points is,

a) 3 b) 9 c) 1 d) Infinite

(SCQ -3) The angle substended by any diameter of a sphere at any point on surface of sphere is

a) $30°$ b) $90°$ c) $60°$ d) $45°$

(SCQ -4) The coordinates of the centre of the sphere

$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ are,

a) $(-u, -v, -w)$ b) (u, v, w) (c) $(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a})$ (d) $(\frac{u}{a}, \frac{v}{a}, \frac{w}{a})$

(SCQ-5) The centre of the sphere $(x - a)(x + u) + (y - v)(y - w) + (z - p)(z + q) = 0$ is

a) $\left(\frac{a-u}{2}, \frac{v+w}{2}, \frac{p-q}{2}\right)$ (b) $\left(\frac{u-a}{2}, \frac{v+w}{2}, \frac{q-p}{2}\right)$

(c) $\left(\frac{a+u}{2}, \frac{v+w}{2}, \frac{p+q}{2}\right)$ (d) $\left(\frac{a+v}{2}, \frac{u+w}{2}, \frac{p+q}{2}\right)$

(SCQ-6) The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere iff $u^2 + v^2 + w^2 - d$ is

a) negative (b) positive (c) zero (d) zero or negative

(SCQ -7) The equation of the sphere passing through the origin and making intercepts of length a, b, c on coordinate axes is,

(a) $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0$
(b) $x^2 + y^2 + z^2 + ax + by + cz = 0$
(c) $x^2 + y^2 + z^2 - ax - by - cz = 0$
(d) $x^2 + y^2 + z^2 = a + b + c$

(SCQ-8) The equation, $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere if

a) $u = v = w$ b) $a = b = c$ c) $a = b = c \neq 0$ d) $u + v + w = 0$

8.8 PLANE SECTION OF A SPHERE

To prove that the section of a sphere by a plane is a circle and to find the centre and radius of the circle so obtained.

DERIVATION

Let the equations of sphere and plane respectively are,

$S = ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$………………(1)

$P = lx + my + nz - p = 0$……………………………………… (2)

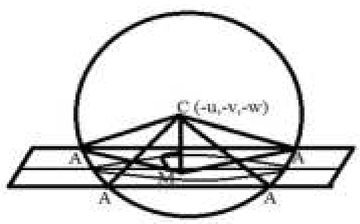


Fig. 8.8.1

We have learnt by now that if C is the centre of the sphere (8.8.1) then,

$$C = (-u, -v, -w)$$

Let M is the foot of perpendicular CM from the centre C of the sphere (1) to the plane (2). Also let A is any arbitrary point on the section of the sphere (1) by the plane (2).

Join M to A. Since CM is perpendicular to the plane (2) and the line AM lies in the plane (2), hence wherever the point A, on the section of (1) and (2) is taken AM is always perpendicular to CM.

Now, AC = the radius of the sphere = $\sqrt{u^2 + v^2 + w^2 - d}$……………….(3)

CM = length of perpendicular from $C = (-u, -v, -w)$ to the plane $lx + my + nz - p = 0$

$CM = \frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{l^2 + m^2 + n^2}}$ ……………………………………………………(4)

after taking modulus given by equation (8.8.4), $CM = \frac{lu + mv + nw + p}{\sqrt{l^2 + m^2 + n^2}}$

Now using the Pythagoras theorem, in the right angle triangle AMC we get,

$(AM)^2 = (CA)^2 - (CM)^2$…………………………………………………………..(5)

Since the radius of the sphere (1) i.e. AC and the length of perpendicular from the centre C of the sphere (1) to the plane (2) i.e. CM both are constants, hence from equation (5), AM will always be constant.

CONCLUSION

From the above discussion we conclude that, the distance AM of any arbitrary point A, {on the section of sphere (1) by the plane (2)}, from the fixed point M, {the foot of perpendicular from the centre C of the sphere (1) to the plane (2) } is always constant. Hence the locus of the point A is a circle with centre M and radius AM.

Where $AM = \sqrt{(CA)^2 - (CM)^2}$

and values of AC and CM are given by equations (8.8.3) and (8.8.4) respectively .

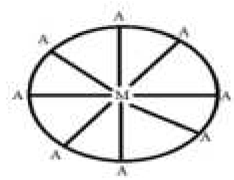


Fig. 8.8.2

EQUATION OF THE CIRCLE SO OBTAINED We have learnt by now that the intersection of the sphere (1) and the plane (2) is a circle with centre M and radius AM. Hence in general, the equation of the circle consists of the equations of the sphere (1) and that of the plane (2) taken together i.e. S = 0 = P,represents the required equations of the circle, the square of whose radius is $AM^2 = (CA)^2 - (CM)^2$, which can be easily found out.

To find the coordinates of M i.e. centre of the circle

The equation of the line CM through $C(-u, -v, -w)$& perpendicular to the plane (2) is given by

$\frac{x-(-u)}{l} = \frac{y-(-v)}{m} = \frac{z-(-w)}{n} = r$ (let)

Where (l, m, n) are dr's of the normal (perpendicular) to the plane and CM is perpendicular to this plane.

Hence l, m, n are proportional to the dr's of CM.

Now the above equation of straight line gives,

$$\left.\begin{aligned} x &= l r - u \\ y &= mr - v \\ z &= nr - v \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(.6)$$

Since r is arbitrary, hence these are coordinates of any point on the perpendicular CM. Let for some value of r this point represents M, then (6) must satisfy equation of plane (2), by which we can obtained value of r. Putting the values of l, m, n, r, u, v, w in equation (6), we get the required coordinates of the centre of the circle .

REMARKS

(1) GREAT CIRCLE As in $2D$ geometry in case of a circle the chord will be maximum, if it passes through the centre of circle i.e. diameter is the maximum chord in case of circle. Similarly, the plane section of a sphere will be greatest if the plane passes through the centre of sphere. The section of a sphere passing through the centre of a sphere is known as "great circle", its centre and radius would be the same as that of the given sphere.

(2) Equation of the sphere passing through the circle $S = 0 = P$ is given by,

$S + \boldsymbol{\mu} P = 0$, (where $\boldsymbol{\mu}$ is any arbitrary constant).

Ex.6. Find the radius of the circle given by the equations,

$$3x^2 + 3y^2 + 3z^2 + x - 5y - 2 = 0, x + y = 2$$

Sol. The equations of the given circle are

$x^2 + y^2 + z^2 + \frac{1}{3}x - \frac{5}{3}y - \frac{2}{3} = 0$ ....(1)

$x + y = 2$ ……………………………………(2)

The centre of the sphere (1) is,

$C = \left(\frac{-1}{6}, \frac{5}{6}, 0\right)$ and its radius $AC = \sqrt{u^2 + v^2 + w^2 - d}$

(Putting the values of u, v, w& d and on simplifying we get), $AC = \frac{5}{6}\sqrt{2}$

Also

$CM =$ length of perpendicular from the centre C of the sphere (1) to the plane $x + y - 2 = 0$

$$= \frac{1}{6}(4\sqrt{2})$$

(Applying the formula of length of perpendicular from a point to a plane and neglecting negative sign)

Hence the radius of the circle = AM

$$= \sqrt{(CA)^2 - (CM)^2}$$

Solving we get radius of circle = $\frac{5}{6}\sqrt{2}$

8.9 INTERSECTION OF TWO SPHERES

To show that the curve of intersection of two spheres a circle.

Let the equations of two given spheres are

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$…………………(1)

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$………………...(2)

If the two given spheres intersect, then there must be some points, which will lie on both the spheres. These points, common to both the spheres, satisfy both the equations, $S_1 = 0$ and $S_2 = 0$, hence these points also satisfy the equation, $S_1 - S_2 = 0$ In other words, the equation of the points of intersection of the spheres will be, $S_1 - S_2 = 0$ i.e.

$2(u_1 - u_2)\,x + 2\,(v_1 - v_2)\,y + 2(w_1 - w_2)\,z + (d_1 - d_2) = 0$………(3)

Clearly equation (3) is a linear or first degree equation in x, y and z. Hence it represents a plane.

Hence the curve of intersection of two spheres $S_1 = 0$ and $S_2 = 0$ will be the same curve as the curve of intersection of any one of the given sphere i.e. $S_1 = 0$ or $S_2 = 0$ and the plane, $S_1 - S_2 = 0$.

Also we have learnt in the previous article (8.8) that the plane section of a sphere is a circle, hence the intersection of two spheres is also a circle i.e. the equation of two spheres taken together represents a circle given by $S_1 = 0 = S_2$.

REMARKS To find the centre and radius of the circle obtained by the intersection of two spheres $S_1 = 0$ and $S_2 = 0$, we can take either (1) the sphere $S_1 = 0$ and the plane $S_1 - S_2 = 0$ or (2) the sphere $S_2 = 0$ and the plane $S_1 - S_2 = 0$.

Then using the same method as in case of previous article (plane section of a sphere), we can find centre and radius of the required circle.

1. The equation of the sphere passing through the circle $S_1 = 0 = S_2$ is given by $S_1 + \boldsymbol{\mu} S_2 = 0$, where $\boldsymbol{\mu}$ is any arbitrary constant.

Ex.7. Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0,\ 5y + 6z + 1 = 0$ ……(1) and

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,\ x + 2y - 7z = 0$…….(2)

lie on the same sphere and find its equation.

Sol. The equation of any sphere passing through the circle (1) is given by,

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0 + \lambda_1 (5y + 6z + 1) = 0$……(3)

Similarly, the equation of any sphere passing through the circle (2) is given by,

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0 + \lambda_2 (x + 2y - 7z) = 0$ (4)

If the two given circles (1) & (2) lie on the same sphere then for some values of λ_1 & λ_2 equations (3) & (4) must be identical .

Comparing the coefficients of x, y, z and constant terms in (3) & (4) we get,

$-2 = -3 + \lambda_2$……………………………………..(5)

$3 + 5\,\lambda_1 \; = -4 + 2\;\lambda_2$……………………………(6)

$4 + 6\;\lambda_1 \; = 5 - 7\;\lambda_2$ ………………………………..(7)

$-5 + \;\lambda_1 \; = \; -6$ …………………………………… (8)

Clearly the equation (5) gives, $\lambda_2 = 1$ and

the equation (8) gives, $\lambda_1 = \; -1$

Here we observe that these values of λ_1& λ_2 also satisfies the equations (6) & (7).

Hence the given circles (1) & (2) lie on the same sphere, whose equation is obtained by putting the values of $\lambda_1$in equation (3) or by putting the value of $\lambda_2$in equation(4).

Hence the required equation of the sphere is,

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

8.10 POWER OF A POINT WITH RESPECT TO A SPHERE

First we shall discuss the intersection of a straight line and a sphere.

We have learnt by now that the equation of a straight line in the symmetrical form is given by

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$……………………………………………………………(1)

Where (α, β, γ) is the point through which the straight line passes and (l, m, n) are its direction ratios of this line i.e. $(l, m, n\,)$are proportional to its direction cosines of the line.

Then any point on the line (8.10.1) is given by $P = ((l\,r + \alpha), (m\,r + \beta), (n\,r + \gamma))$, Since r is arbitrary hence the point P is any arbitrary point on the line (8.10.1). Let the equation of the sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$……………………………….(2)

Now suppose that the line (1) meets the sphere (2) at point P, then P must lie on the sphere (2) . Hence the point P must satisfy the equation of the sphere (2). So we get,

$(l\,r + \alpha)^2 + (m\,r + \beta)^2 + (n\,r + \gamma)^2 + 2u(l\,r + \alpha) + 2v(m\,r + \beta) + 2w(n\,r + \gamma) + d = 0.$

Simplifying this equation we get,

$(l^2 + m^2 + n^2)r^2 + 2[l(\alpha + u) + m(\beta + v) + n(\gamma + w)]r + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0$ ……………………………………………………………..(3)

Clearly equation (8.10.3) is a quadratic (second degree) equation in r, hence it gives two values of r say r_1 and r_2.

Then there are two points of intersection of line (1) and sphere (2) given by,

$(l\,r_1 + \alpha, m r_1 + \beta, n\,r_1 + \gamma)$ and $(l\,r_2 + \alpha, m\,r_2 + \beta, n\,r_2 + \gamma)$.

Hence a straight line intersects a sphere atmost in two points which may be real and distinct, coincident or imaginary depending upon the nature of values of r_1 and r_2.

Now suppose that (l, m, n) are the actual direction cosines(dc's) of the line (1) , then $l^2 + m^2 + n^2 = 1$ ……………………………………………………(.4)

Using the identity (4), the equation (3) becomes ,

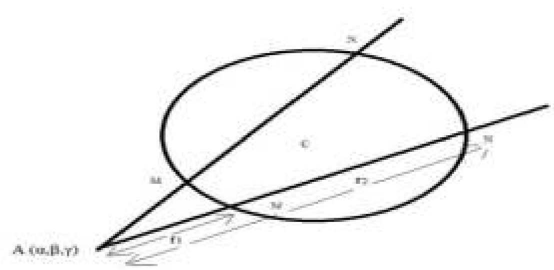


Fig. 8.8.2

$r^2 + 2[l(\alpha + u) + m(\beta + v) + n(\gamma + w)]r + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 (5)$

If r_1 and r_2 are the roots of this equation (5) ,then the distance of the point $A = (\alpha, \beta, \gamma)$ outside the sphere from the points of intersections of the line (1) and sphere (2) say M and N results $AM = r_1$, $AN = r_2$ then

$AM.AN = r_1.r_2$ = product of the roots of equation (5) = $\frac{constantterms}{coeff.of r^2}$

$$= \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = constant \ldots (6)$$

Hence it is concluded that if lines are drawn from a fixed point A to intersect a given sphere in points M and N (both arbitrary), then the product $AM.AN$ is always constant. This constant product is called the power of the point A, w. r. t. to the given sphere.

REMARKS

1. SPECIAL CASE: - To find the length of tangent from a given point to the sphere.

DERIVATION

Here we consider a special case of the previous article 8.10 (power of a point w. r. t. a sphere), suppose that the points M and N are coincident, i.e. $AM = AN$.

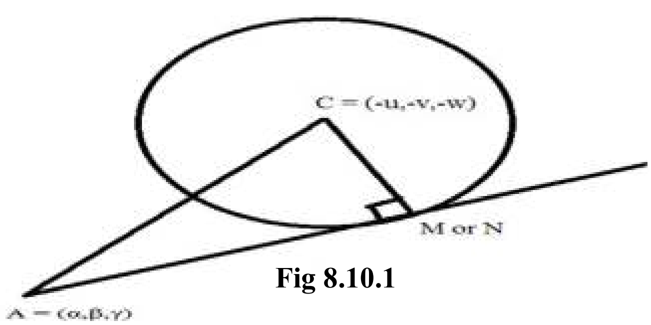


Fig 8.10.1

In this case the line AM or AN will be a tangent to the sphere (2) at point M or N.

Now we will find the length of this tangent putting AM =AN in equation (6) to get,

$$(AM)^2 = (AN)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$$

i.e. $AM = AN = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d}$

(Taking positive sign only as length can't be negative)

2. THEORETICAL METHOD

For the sphere (2), $C = (-u, -v, -w)$

$$AC = \sqrt{(\alpha + u)^2 + (\beta + v)^2 + (\gamma + w)^2}$$

CM = radius of the sphere

$$= \sqrt{u^2 + v^2 + w^2 - d}$$

Now in the right angle triangle AMC, $(AM)^2 = (AC)^2 - (CM)^2$

$$= (\alpha + u)^2 + (\beta + v)^2 + (\gamma + w)^2 - (u^2 + v^2 + w^2 - d)$$

Simplifying this we get,

$$(AM)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$$

i.e. $AM = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d}$

3. SHORT CUT METHOD TO FIND THE LENGTH OF TANGENT

To find the power of a point $A = (\alpha, \beta, \gamma)$ w. r. t. the given sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

we should replace x by α , y by β , and z by γ in the equation of the sphere then,

power of the point A w. r. t. the sphere = square of the length of tangent from A to the sphere

$$= \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$$

Here it is remarkable that the coefficients of x^2, y^2 and z^2 in the equation of a sphere must be unity .

Ex. 8. Find the equation of a sphere for which the circle,

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, \ 2x + 3y + 4z = 8 \text{ is a great circle.}$$

Sol. The equation of any sphere passing through the given circle is

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$$

$$x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda)y + (-2 + 4\lambda)z + (2 - 8\lambda) = 0 \ldots\ldots\ldots\ldots(1)$$

The centre of the sphere (1), is given by $C = \left(-\lambda, \frac{-1}{2}(7 + 3\lambda), 1 - 2\lambda\right)$. Since λ is arbitrary, hence for some value of λ the equation (1) represents the sphere, for which the given circle is a great circle.

If the given circle is a great circle of the sphere (1), then centre C of the sphere (1) must lie on the given plane $2x + 3y + 4z = 8$. Since great circle is the plane section of a sphere through its centre. Hence the point C satisfies the equation of plane,

i.e. $2(-\lambda) + 3\{\frac{-1}{2}(7 + 3\lambda)\} + 4(1 - 2\lambda) = 8$

i.e. $-2\lambda - \frac{3}{2}(7 + 3\lambda) + 4 - 8\lambda = 8$

i.e. $-2\lambda - \frac{21}{2} - \frac{9\lambda}{2} + 4 - 8\lambda = 8$

i.e. $\lambda(-2 - \frac{9}{2} - 8) = 8 + \frac{21}{2} - 4$

Simplifying we get $\lambda = -1$.

Substituting this value of λ in equation (1), we get the equation of the required sphere as,

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$.

Ex. 9. Find the equation of the circle whose centre is (α, β, γ) and which lies on the sphere $x^2 + y^2 + z^2 = a^2$.

Sol. The equation of the given sphere is,

$x^2 + y^2 + z^2 = a^2$……………………………………………..(1)

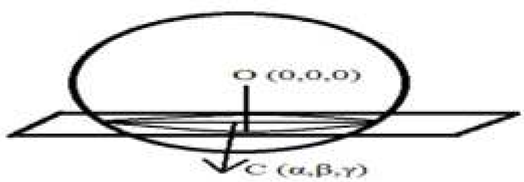


Fig 8.10.1

The centre of the sphere (1) is $O = (0,0,0)$ and the centre of the circle is given to be the point $C = (\alpha, \beta, \gamma)$.

The direction ratios of the line OC are $(\alpha - 0, \beta - 0, \gamma - 0)$ i.e. (α, β, γ).

Therefore the equation of the plane passing through C and perpendicular to OC is given by,

$$a(x-\alpha) + \beta\,(y-\beta) + \gamma\,(z-\gamma) = 0$$

i.e. $\alpha\, x + \beta\, y + \gamma\, z = \alpha^2 + \beta^2 + \gamma^2$……………………………(2)

The equations (1) & (2) together represent the required equation of the circle.

SELF CHECK QUESTIONS

Choose the correct option.

(SCQ-9) The section of a sphere by a plane is

(a) a parabola (b) a circle (c) an ellipse (d) a hyperbola

(SCQ-10) The power of the point $(1, 2, 1)$ w .r. t. the sphere

$x^2 + y^2 + z^2 - x + y + 3z + 5 = 0$ is

(a) 15 (b) 4 (c) 6 (d) 10

(SCQ-11)Equation of the sphere passing through the circle S = 0 =P is,

(a) $S - \lambda P = 0$ (b) $S + \lambda P = 0$ (c) $P + \lambda S = 0$(d) all of these

where λ is any arbitrary constant.

(SCQ-12) The points of intersection of two spheres will lie on

(a) a circle (b) a plane (c) an ellipse (d) a straight line

(SCQ-13) The maximum number of points in which a straight line intersects a sphere is,

(a) 3 (b) 1 (c) 2 (d) Infinite

(SCQ-14)If the power of a point w. r. t a given sphere is λ, then the length of tangent from that point to the given sphere will be

(a) λ^2 (b) 2λ (c) $\frac{\lambda}{2}$ (d) $\sqrt{\lambda}$

(SCQ-15) The curve of intersection of two spheres is

(a) a plane (b) a circle (c) a straight line (d) a curved line

(SCQ-16) General equation of the sphere passing through the circle $S_1 = 0 = S_2$ is

(a) $S_1 - S_2 = 0$(b) $S_1 + S_2 = 0$ (c) $S_1 + \lambda S_2 = 0$ (d) $\frac{S_1}{S_2} = 0$

where λ is any arbitrary constant.

8.11 EQUATION OF TANGENT PLANE OF A SPHERE AT A POINT

Let $S = x^2 + y^2 + z^2 + 2ux + 2uy + 2wz + d = 0$…………………………(1) be the equation of a sphere and $A(x_1, y_1, z_1)$ be any point on the sphere then we have to find the equation of tangent plane at $A(x_1, y_1, z_1)$ to the sphere, centre of the sphere (1) is $(-u, -v, -w)$, so the direction ratio of the normal line to the tangent plane is $(x_1 + u, y_1 + v, z_1 + w)$

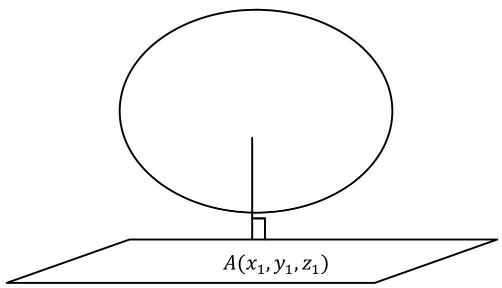


Fig 8.11.1

Therefore, equation of tangent plane is

$(x_1 + u)(x - x_1) + (y_1 + v)(y - y_1) + (z_1 + w)(z - z_1) = 0$ …………….(2)

$\Rightarrow xx_1 + yy_1 + zz_1 - ({x_1}^2 + {y_1}^2 + {z_1}^2 + ux_1 + vy_1 + wz_1) + ux + vy + wz = 0$ ... (3)

As $A(x_1, y_1, z_1)$ lies on the sphere, therefore

${x_1}^2 + {y_1}^2 + {z_1}^2 + 2ux_1 + 2uy_1 + 2wz_1 + d = 0$……………………………….….(4)

Putting values of $x_1^2+y_1^2+z_1^2$ from (4) in equation (3), we get

$$xx_1+yy_1+zz_1+ux+vy+wz+(2ux_1+2vy_1+2wz_1)-ux_1+vy_1+wz_1+d=0$$

$\Rightarrow x(x_1+u)+y(y_1+v)+z(z_1+w)+\ ux_1+vy_1+wz_1+d=0$…………..(5)

Ex. 10. Find the equation of tangent plane to the sphere

$x^2+y^2+z^2+2x+4y+6z-7=0$ at a point $(1,-1,1)$.

Sol. Here, $u=1,\ v=2,w=3$

Therefore, $x_1=1,y_1=-1,z_1=1$

Equation of the tangent plane is

$x(1+1)+y(-1+2)+z(1+3)+\ 1.1+2.(-1)+3.1-7=0$

$\Rightarrow 2x+y+4z-5=0$

8.12 A CONDITION OF TANGENCY

In this section we will discuss about the condition of tangency

8.12.1 *CONDITION FOR THE GIVEN LINE TO TOUCH THE GIVEN SPHERE*

Let the given straight line in symmetrical form be

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$…………………………………………………………………..(1)

and the equation of the sphere be,

$x^2+y^2+z^2+2ux+2uy+2wz+d=0$……………………………………………..(2)

If the line (8.12.1) touches the sphere (8.12.2), then it intersects the sphere (8.12.2) at one point only i.e. the roots of equation (3) of the article (8.10-power of a point w. r. t. a sphere),

$$\Rightarrow (l^2 + m^2 + n^2)r^2 + 2[l(\alpha + u) + m\ (\beta + v) + n\,(\gamma + w)]\,r + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad (3)$$

must be equal or coincident .

i.e. $$[l(\alpha + u) + m\ (\beta + v) + n\,(\gamma + w)^2 = +(l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d)$$

This is the required condition of tangency for the line (1) to touch the sphere (2).

(Here we used the result that the roots of the quadratic equation $ax^2 + bx + c = 0$ will be equal iff $b^2 = 4ac$)

8.12.2 *CONDITION FOR THE GIVEN PLANE TO TOUCH THE GIVEN SPHERE*

Let the equation of given plane is

$l\,x + m\,y + n\,z = p$……………………………….(4)

and the equation of the given sphere is,

$x^2 + y^2 + z^2 + 2ux + 2uy + 2wz + d = 0$……………………………………….(5)

We have learnt by now that the equation (5) is the general equation of the sphere with centre $C = (-u, -v, -w)$ and radius $= \sqrt{u^2 + v^2 + w^2 - d}$.

If the plane (8.12.4) touches the sphere (8.12.5) at P then the perpendicular distance from the centre C of the sphere (8.12.5) to the point P must be equal to the radius of the sphere (8.12.5), i.e. CP = radius of the sphere (as shown here in the fig. Below)

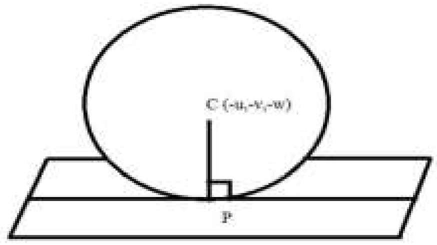


Fig 8.12.1

i.e. $CP = \left|\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}}\right|$

i.e. $CP = \left|\frac{lu+mv+nw+p}{\sqrt{l^2+m^2+n^2}}\right| = \sqrt{u^2+v^2+w^2-d}$

Squaring and cross multiplying we get.

$(lu+mv+nw+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d)$……………………….(6)

This is the required condition of tangency for the plane (1) to touch the sphere (2).

8.13 THE ANGLE OF INTERSECTION OF TWO SPHERES

DEFINITION

The angle of intersection of two spheres is the same as the angle between their tangent planes at a common point of intersection of the two spheres.

Since the radii of the spheres to the common point of intersection are perpendicular to the tangent planes at that point, so the angle between the radii of spheres at the common point of intersection is equal to the angle between the tangent planes i.e. the angle of intersection of the two given spheres.

TO FIND THE FORMULA FOR ANGLE OF INTERSECTION OF TWO SPHERES

Let P be a common point of intersection of the two spheres,

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$............................(7)

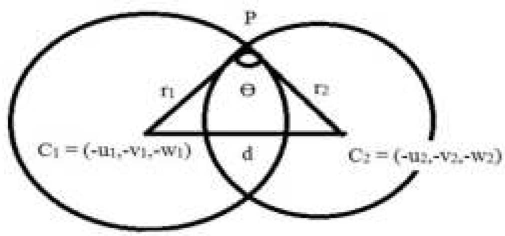


Fig 8.13.1

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$............................(8)

whose centres are $C_1 = (-u_1, -v_1, -w_1)$ and $C_2 = (-u_2, -v_2, -w_2)$ and radii are

$r_1 = \sqrt{{u_1}^2 + {v_1}^2 + {w_1}^2 - d_1}$ and $r_2 = \sqrt{{u_1}^2 + {v_1}^2 + {w_1}^2 - d_1}$ respectively.

Let θ be the angle of intersection of the two spheres at P i.e. $< C_1PC_2 = \theta$.

Also let $C_1C_2 = d$.

Now applying the Cosine formula for the triangle C_1PC_2 wet get, $2r_1r_2\cos\theta = r_{12} + r_{22} - d_2$

Substituting the values of r_1, r_2 and d, we can get the required angle of intersection of the two spheres. The angle of intersection is always taken to be acute angle i.e. $0 < \theta < 90°$.

Ex.11. Find the angle of intersection of the spheres,

$$x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$$

$$x^2 + y^2 + z^2 - 6x - 2y + 2z + 2 = 0$$

Sol. Let C_1, C_2&r_1, $r_2$represent the centres and radii of the given spheres respectively then$C_1 = (1,2,3)$and$C_2 = (3,1,-1)$

$$\text{Let } d = C_1C_2 = \sqrt{(3-1)2 + (1-2)2 + (-1-3)2}$$

$$= \sqrt{4 + 1 + 16}$$

$$= \sqrt{21}$$

$r_1 = \sqrt{1+4+9-10} = 2$ and$r_2 = \sqrt{9+1+1-2} = 3$

Let θ be the angle of intersection of the two spheres then,

$$2r_1r_2 \cos\theta = {r_1}^2 + {r_2}^2 - d^2$$

i.e. $2.2.3 \cos\theta = 4 + 9 - 21$ (putting the values of r_1, r_2&d)

Simplifying we get, $\cos\theta = \frac{2}{3}$(taking positive sign as we consider the angle θ as acute angle)

8.14 ORHOGONAL SPHERES

If the angle of intersection of two spheres is a right angle i.e.90 ° i.e. $\frac{\pi}{2}$ then the two spheres are said to be orthogonal spheres.

Condition for orthogonality of two spheres

In this section we will find the condition, when the two given spheres,

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$…………………………..(1)

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$…………………………..(2)

will be orthogonal i.e. the angle of intersection of these two spheres will be a right angle i.e. $\frac{\pi}{2}$.

From the previous article (8.13) the angle of intersection of two spheres is given by,

$2r_1r_2 \cos\theta = r_{12} + r_{22} - d_2$……………………………………………………..(3)

where r_1, r_2 are the radii of the spheres and d is the distance between their centres.

But for orthogonal spheres, $\theta = \frac{\pi}{2}$

Hence, $\cos\theta = \cos\frac{\pi}{2} = 0$

Hence equation (8.14.1) reduces to,

$2r_1r_2 \cos\frac{\pi}{2} = r_{12} + r_{22} - d$……………………………………………………..(4)

i.e. $2r_1r_2 .0 = r_{12} + r_{22} - d$

i.e. $0 = {r_1}^2 + {r_2}^2 - d^2$

i.e. ${r_1}^2 + {r_2}^2 = d^2$

Substituting the values,

$${r_1}^2 = {u_1}^2 + {v_1}^2 + {w_1}^2 - d_1$$

${r_2}^2 = {u_2}^2 + {v_2}^2 + {w_2}^2 - d_2$ and $d^2 = (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2$

in equation (8.14.2) and simplifying we get,

$$2\,(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2$$

This is the required condition for the two spheres given by the equations (8.14.1) and (8.14.2) to be orthogonal.

Ex.12. Two spheres of radii $r_1$and r_2 cut orthogonally. Prove that the radius of common circle is $\frac{r_1r_2}{\sqrt{{r_1}^2+{r_2}^2}}$.

Sol. Let the equation of the common circle is -

$x^2 + y^2 = a^2, \; z = 0$ ………………………………(1)

Clearly a is the radius of this circle, which is to be found out.

The equations of the two spheres through the circle (1) are given by ,

$(x^2 + y^2 - a^2) + 2\lambda_1 z = 0$…………………………………………………………..(2)

$(x^2 + y^2 - a^2) + 2\lambda_2 z = 0$…………………………………………………………(3)

where λ_1&$\lambda_2$are arbitrary constants.

The radii r_1 and r_2 of these two spheres (2) & (3) are given by,

$$r_1{}^2 = \lambda_1{}^2 + a^2$$

$$r_2{}^2 = \lambda_2{}^2 + a^2$$

Since the spheres (2) & (3) cut orthogonally, hence applying the condition of orthogonality we get,

$$2\,(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2$$

Substituting the values we get,

$$0 + 0 + 2\lambda_1\lambda_2 = (-a^2) + (-a^2)$$

i.e. $2\lambda_1\lambda_2 = -2a^2$

i.e. $\lambda_1\lambda_2 = -a^2$

Squaring we get, $\lambda_1{}^2\lambda_2{}^2 = a^4$

{Using equations (4) &(5) in above, we get}

i.e. $(r_1{}^2 - a^2)\,.\,(r_2{}^2 - a^2) = a^4$

i.e. $r_1{}^2r_2{}^2 - a^2\,(r_1{}^2 + r_2{}^2) = 0$

(Simplifying and taking only positive value of a as radius can't be negative, we get)

$a = \frac{r_1 r_2}{\sqrt{{r_1}^2+{r_2}^2}}$ which is the required radius of the common circle.

8.15 TOUCHING SPHERES

There may be two types of the touching sphere-

(1) Externally touching spheres

Two spheres with centres C_1 and C_2 and radii r_1 and r_2 respectively are said to touch externally if the distance between their centers i.e. C_1 and $C_2$is equal to the sum of their radii.

i.e. $C_1C_2 = r_1 + r_2$.

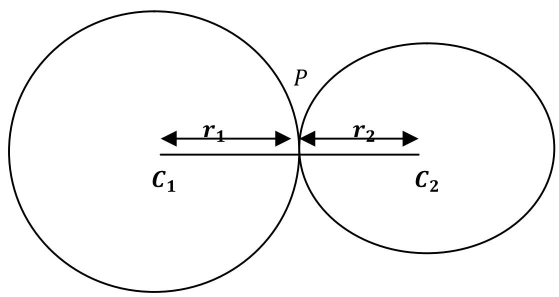


Fig 8.15.1

In this case the point of contact (say P) of the two sphere divides the line C_1C_2 , joining the centres of the spheres internally in the ratio if their radii. Clearly, $PC_1 : PC_2 = r_1 : r_2$

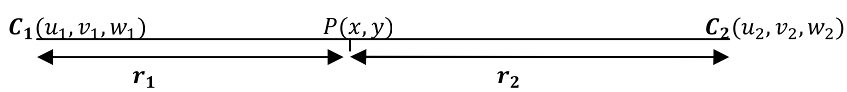


Fig 8.15.2

Hence we can also find the coordinates of point P as,

$x = \frac{r_1(-u_2)+r_2(-u_1)}{r_1+r_2}$, $y = \frac{r_1(-v_2)+r_2(-v_1)}{r_1+r_2}$, $z = \frac{r_1(-w_2)+r_2(-w_1)}{r_1+r_2}$

(2) Internally touching spheres

Two spheres with centres C_1 and C_2 and radii r_1 and r_2 respectively are said to touch internally if the distance between their centres i.e. C_1 and C_2 is equal to the difference of their radii i.e. $C_1C_2 = r_1 \sim r_2 = |r_1 - r_2|$

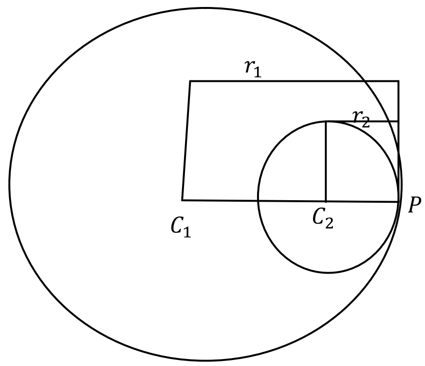


Fig 8.15.3

In this case the point of contact (say P) of the two spheres divides the line C_1C_2, joining the centres of the spheres externally in the ratio if their radii.

Clearly $PC_1 : PC_2 = r_1 : r_2$

Hence we can also find the coordinates of point P as.

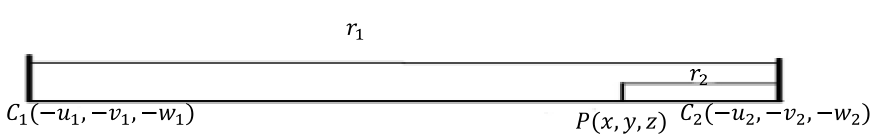


Fig 8.15.4

$$x = \frac{-u_2 r_1 - u_1 r_2}{r_1 - r_2}$$

$$y = \frac{-v_2 r_1 - v_1 r_2}{r_1 - r_2}$$

$$z = \frac{-w_2 r_1 - w_1 r_2}{r_1 - r_2}$$

Ex. 13. The variable sphere $x2 + y2 + z2 + 2ux + 2vy + 2wz + 1 = 0$ cuts the sphere $3x^2 + 3y^2 + 3z^2 - 6x + 10y + z - 8 = 0$ orthogonally. Show that the point (u, v, w) moves on a fixed plane.

Sol. The equations of the given sphere are,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + 1 = 0$ ……………...(1)

$x^2 + y^2 + z^2 - 2x + \frac{10}{3}y + \frac{1}{3}z - \frac{8}{3} = 0$……………………(2) (Dividing each term by 3)

if the spheres (1) & (2) cut orthogonally, then by condition of orthogonality, we have--

$$2u(-1) + 2v\left(\frac{5}{3}\right) + 2w\left(\frac{1}{6}\right) = 1 - \frac{8}{3}$$

$$-2u + \frac{10v}{3} + \frac{w}{3} = \frac{-5}{3}$$

Simplifying we get,

$$6u - 10v - w - 5 = 0$$

Replacing u by x, v by y&w by z the locus of the point (u, v, w) is

$6x - 10y - z - 5 = 0$, which is a fixed plane

Ex. 14. Find the equations of the spheres which pass through the circle $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0,\ 2x + y + z = 4$ and touch the plane $3x + 4y - 14 = 0$

Sol. The equation of any sphere passing through the given circle is

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$$

or

$$x^2 + y^2 + z^2 + (-2 + 2\lambda)x + (2 + \lambda)y + (4 + \lambda)z - 3 - 4\lambda = 0$$

If the sphere (1) touches the plane $3x + 4y - 14 = 0$…………(2)

Then length of perpendicular from the centre C of the sphere (1) to the plane (2) = radius of the sphere

i.e. $p = R$

Here $C = \left(1 - \lambda, (-2 - \lambda), \frac{1}{2}(-4 - \lambda)\right)$

$$(\text{Radius of the sphere})^2 = (1 - \lambda)^2 + \{\frac{1}{2}(-2 - \lambda)^2 + \frac{1}{2}(4 - \lambda)^2\} = R^2$$

Simplifying we get, $R^2 = \frac{3}{2}\lambda^2 + 5\lambda + 9$…………………………………..(4)

Length of perpendicular from C to the plane (2),

$p = \left|\frac{1}{5}(-5\lambda - 15)\right| = (\lambda + 3)$………………………………………………(5)

Using (4) & (5) in (3) and squaring, we get

$$\lambda^2 + 6\lambda + 9 = \frac{3}{2}\lambda^2 + 5\lambda + 9$$

Simplifying this we get two values of λ as , either $\lambda = 0$ or $\lambda = 2$.

Putting these values of λ in equation (1) the required equations of the spheres are

$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$ and $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$

SELF CHECK QUESTIONS

(SCQ-17) If two spheres of radii 3 and 4 cut orthogonally then the radius of the common circle is

(a) 12 (b) $\frac{12}{5}$ (c) $\frac{4}{3}$ (d) $\frac{5}{12}$

(SCQ-18) The point of contact of the spheres $x^2 + y^2 + z^2 + 2x - 4y - 4z - 7 = 0$ and $x^2 + y^2 + z^2 + 2x - 4y - 16z + 65 = 0$ is

(a) $(1,2,6)$ (b) $(1,-2,6)$ (c) $(1,2,-6)$ (d) $(-1,2,6)$

(SCQ-19) If the plane 3x+4y-9z = k touches the sphere $x^2 + y^2 + z^2 = 25$ then the value of k is

(a) 5 (b) 25 (c) constant (d) none of these

(SCQ-20) Find the value of d so that the sphere $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 - d\,x + 9 = 0$ cut orthogonally

(a) 0 (b) 18 (c) 81 (d) always

FILL IN THE BLANKS

(SCQ-21) If the distance between the centres of two spheres is equal to the difference of their radii then the two spheres touch…………………………

(SCQ-22) The two spheres,

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$……………..(1)

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ …...………..(2)

cut orthogonally if $2(u_1.u_2 + v_1.v_2 + w_1.w_2) =$…………………………...

(SCQ-23) The plane $3x + 4y - z = 9$ touches the sphere $x^2 + y^2 + z^2 = k$, if $k =$……………………………..

(SCQ-24) If θ is the angle of intersection of two sphere with radii r_1& r_2 and d is the distance between their centers then, $2r_1r_2\cos\theta =$………………………

8.16 SUMMARY

In this unit, we have learned about the difference between the plane curves and surfaces, particularly the difference between a circle and a sphere. Also we have learned that sphere is a fully curved surface with one face, no vertex and no edge. Here we found the equations of a sphere in different forms such as central form, general form, diameter form and four point form by different methods. Also we have learned whether a second degree equation in x, y, z represents a sphere or not. We found that the section of a sphere with a plane is a circle and the intersection of two spheres is also a circle. Also we have learnt the method to find the centre and radius of the circle so obtained. We have derived the expression for the power of a point with respect to a sphere and found the length of tangent from an external point to the sphere with the help of this expression, also by theoretical method and by short-cut method. We have studied about the angle of intersection of two spheres and condition to find whether the two given spheres are orthogonal or not. We also found the condition of tangency for a given line or a given plane to touch the given sphere and the condition for the two spheres to be touching spheres i.e. whether the two given spheres touch internally or externally. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics . To check your progress, self check questions (SCQ's) are given place to place.

8.17 GLOSSARY

1. $3D$ object or a solid – an object that occupies space.
2. Face – surface of a solid.
3. Edge – where two faces meet.
4. Vertex – where two edges meet.
5. Mutually – shared by two or more.
6. Diameter – a straight line that goes from one side to the another of a circle or sphere, passing through the centre.

7. Eliminate – To remove or to omit or to neglect that is not wanted or needed or required.
8. Perpendicular – right angle or at an angle of $90°$ or pointing straight up.
9. Origin – the point from which we start generally at $O = (0,0.0)$.
10. Section – a view or drawing as if the 1[st] was cut from the 2[nd], so that you can see the inside.

8.18 REFERENCES

1. Geometry of two and three Dimensions - Lata S. , Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. 3 D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
4. Geometry of two and three Dimensions- P. K. Mittal, S.J, Publications, Chhipi tank, Meerut.
5. Encyclopedia Britannica (India) Pvt. Ltd. , New Delhi, India.
6. Leonard E. Dickson :The First course in the Geometry.
7. Burnside,William Show, Panton &Arthur William :Volumetric Theory.
8. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series ,McGraw Hill.
9. P.N .Pandey :Polar Coordinate Geometry, Sharda Academic Publishing House, Allahabad.
10. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

8.19 SUGGESTED READINGS

1. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
2. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
3. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
4. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
5. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

8.20 TERMINAL QUESTIONS

(TQ-1) A point moves such that the ratio of its distances from two fixed point is always constant. Show that its locus is a sphere.

(TQ-2) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9, x + 3y + 4z = 5$ and the point (1,2,3).

(TQ-3) Show that the spheres $x^2 + y^2 + z^2 - 4x - 2y + 2z - 3 = 0$ and $x^2 + y^2 + z^2 - 8x - 8y - 10z + 41 = 0$ touch externally.

(TQ-4) Find the equation of the sphere that passes through the circle $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$, $3x - 4y + 5z - 15 = 0$ and cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

(TQ-5) Find the centre and radius of the sphere $4(x - 7)(x + 2) + 4(y - 4)(y + 9) + 4(z + 1)(z - 1) + 175 = 0$.

(TQ-6) Find the equation of the sphere described on the line joining $(1, -2, 3)$&$(-2, 3, -1)$ as diameter.

(TQ-7) Find the equation of the sphere which passes through the points $(1, -3, 4)$, $(1, -5, 2)$, $(1, -3, 0)$ and whose centre lies on the plane $x + y + z = 0$

(TQ-8) Find the equation of the sphere passing through the circles $y^2 + z^2 = 9, x = 4$ and $y^2 + z^2 = 36, x = 1$.

(TQ-9) Find the condition that the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ may cut orthogonally.

(TQ-10) Find the value of r, if the plane 2x-2y+z = 9 touches the sphere, $x^2 + y^2 + z^2$.

8.21 ANSWERS

SELF CHECK QUESTIONS (SCQ'S)

(SCQ –1) c (SCQ – 2) d (SCQ – 3) b (SCQ – 4) c

(SCQ –5) a (SCQ – 6) b (SCQ – 7) c (SCQ – 8) c

(SCQ – 9) b (SCQ – 10) a (SCQ – 11) d (SCQ – 12) b

(SCQ – 13) c (SCQ – 14) d (SCQ – 15) b (SCQ – 16) c

(SCQ – 17) b (SCQ – 18) d (SCQ – 19) a (SCQ – 20) d

(SCQ – 21) internally (SCQ – 22) $d_1 + d_2$ (SCQ – 23) 81 (SCQ – 24) ${r_1}^2 + {r_2}^2 - d^{\,2}$

TERMINAL QUESTIONS (TQ'S)

(TQ – 2) $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$

(TQ – 4) $5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$

(TQ – 5) centre = $(\frac{5}{2}, -\frac{5}{2}, 0)$& radius = $\frac{1}{2}\sqrt{(79)}$

(TQ – 6) $x^2 + y^2 + z^2 + x - y - 2z - 11 = 0$

(TQ – 7) $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$

(TQ – 8) $x^2 + y^2 + z^2 + 4x - 41 = 0$(TQ – 9) $a^2 = d$

(TQ – 10) $r = 3$

BLOCK-4:

THE CONE AND CENTRAL CONICOID

UNIT 9: THE CONE

CONTENTS

9.1 *INTRODUCTION*

The locus of a variable line passing through a fixed point and satisfying one more condition such as intersects a given conic or touches a given surface, is called a cone.

The fixed point is called the vertex and the variable line, the generator of the cone.

The given curve is called guiding curve or base curve of the cone and the degree of this curve is known as the degree of cone.

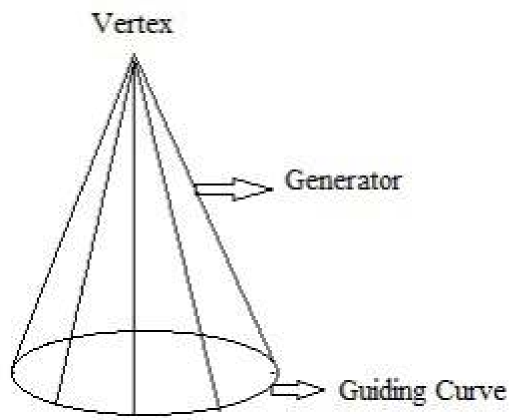


Fig 9.1.1

9.2 *OBJECTIVES*

The objectives of the cone is to provide a scientific basis to the decision makers for solving the problem involving the equation of the cone with vertex as origin, tangent plane, condition of tangency of a plane and a cone, right circular cone, enveloping cone of a system to give solution.

9.3 *EQUATION OF THE CONE WITH VERTEX AT THE ORIGIN*

To show that the equation of the cone whose vertex is origin is homogenous in *x, y, z.*

Let the equation of the cone whose vertex is at origin

$f(x, y, z) = 0$ (1)

Let P(x_1, y_1, z_1) be a point on the cone, then

$f(x_1, y_1, z_1) = 0$ (2)

Also, equation of the generator OP is

$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} = r\ (say)$ (3)

Any point on OP is (rx_1, ry_1, rz_1). Since the generator completely lies on the cone, the point (rx_1, ry_1, rz_1) must lie on the cone, i.e.,

$f(rx_1, ry_1, rz_1) = 0$ (4)

For all values of *r*.

This will be true only if $f(x, y, z) = 0$ is homogenous in *x, y, z.*

To show that every homogeneous equation in *x, y, z* represents a cone whose vertex is origin

Let the homogeneous equation in *x, y, z* be

$f(x, y, z) = 0$ (1)

If P(x_1, y_1, z_1) be a point on the above surface, then

$f(x_1, y_1, z_1) = 0$ (2)

Also, since the equation is homogeneous, we have

$f(rx_1, ry_1, rz_1) = 0$ (3)

where, r is any number

But (rx_1, ry_1, rz_1) is any point on the line OP. Thus, every point of the line OP lies on the surface (1). Thus, surface is being generated by lines through O, and therefore represents a cone with vertex at origin.

Ex. 1. Show that the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$ is

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

Sol. The equation of the cone will be obtained by making

$ax^2 + by^2 + cz^2 = 1$ (1)

Homogenous with the help of $lx + my + nz = p$ (2)

Solving the equation (2)

$\frac{lx+my+nz}{p} = 1$ (3)

Putting the value of equation (3) in equation (1) to make homogenous, the required cone becomes

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2.$$

Ex. 2. Find the equation of the cone with vertex at the origin and which passes through the curve $x^2 + by^2 = 2z$, $lx + my + nz = p$.

Sol. The equation of the cone will be obtained by making

$ax^2 + by^2 = 2z$ (1)

Homogenous with the help of $lx + my + nz = p$ (2)

Solving the equation (2)

$\frac{lx+my+nz}{p} = 1$ (3)

Making the equation of the given surface homogeneous with the help of the equation of the given plane, we have

$$ax^2 + by^2 = 2z\left(\frac{lx + my + nz}{p}\right)$$

or

$$p(ax^2 + by^2) = 2z(lx + my + nz)$$

or

$$apx^2 + bpy^2 - 2nz^2 - 2lzx - 2myz = 0$$

9.4 *TANGENT PLANE*

Let the equation of the cone be

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (1)$$

And that of a line through any point $P(\alpha, \beta, \gamma)$ be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad (2)$$

Any point on (2) is (α+lr, β+mr, γ+nr). The point in which the line (2) meets the cone (1) are determined by

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 + 2f(\beta + mr)(\gamma + nr) + 2g(\gamma + nr)(\alpha + lr) + 2gh(\alpha + lr)(\beta + mr) = 0$$

$$\Rightarrow r^2[al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm] + 2r[(a\alpha + h\beta + g\gamma)l + (h\alpha + b\beta + f\gamma)m + (g\alpha + f\beta + c\gamma)n] + f(\alpha, \beta, \gamma) = 0(3)$$

This being a quadratic in r, the line (2), in general meets the cone (1) in two points. If (α, β, γ) lies on (1), then f (α, β, γ)=0 and consequently from (3) one value of r is zero. Further if

$$l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma) = 0$$

then the other root of (3) is also zero, i.e., the two points of intersection coincide at P.

Hence in this case the line is a tangent to the cone at P. Here (4) is the condition for the line (2) to be a tangent line at P (α,β,γ) to the cone (1).

The locus of all such tangent line at P, i.e., the tangent plane to the cone at P is obtained by eliminating l,m,n between (4) and (2). Thus we have

$$(x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma)+(z-\gamma)(g\alpha+f\beta+c\gamma)=0$$

Or $x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0$

Since $f(\alpha,\beta,\gamma)=0$

$a\alpha x+b\beta y+c\gamma z+f(\gamma y+\beta z)+g(\alpha z+\gamma x)+h(\beta x+\alpha y)=0$

this is the equation of the tangent plane to cone at (α,β,γ).

9.5 *CONDITION OF TANGENCY OF A PLANE AND A CONE*

To find the condition that plane $ux+vy+wz=0$ may touch the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

The plane $ux+vy+wz=0$ will touch the cone if the cone of the angle between the lines the lines of intersection of the plane and the cone is zero.

We know that, if θ be the angle between the lines of intersection of plane and the cone, then

$$tan\theta=\pm\frac{2P\,(u^2+v^2+w^2)}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}$$

Now the plane will touch the cone if $\theta=0$, i.e.,

$tan\,\theta=0$

or$P^2=\begin{vmatrix} a & h & g\,u \\ h & b & f\,v \\ g & f & c\,w \end{vmatrix}=0\Rightarrow|u \quad vw \quad 0|$

i.e., $Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0$ (1)

where A,B,C,F,G,H are co-factor of a,b,c,f,g,h in

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2$$

i.e., $A = \frac{\partial \Delta}{\partial a} = bc - f^2, \; B = \frac{\partial \Delta}{\partial a} = ca - g^2$,

$$C = \frac{\partial \Delta}{\partial c} = ab - h^2, \qquad F = \frac{1}{2}\frac{\partial \Delta}{\partial f} = gh - af \,,$$

$$G = \frac{1}{2}\frac{\partial \Delta}{\partial g} = hf - bg, \qquad H = \frac{1}{2}\frac{\partial \Delta}{\partial h} = fg - ch \,.$$

Thus (1) is the required condition.

Ex. 3. Prove that the equation $\sqrt{(fx)} \pm \sqrt{(gy)} \pm \sqrt{(hz)} = 0$ represents a cone that touches the coordinate planes.

Sol. We have

$\sqrt{(fx)} \pm \sqrt{(gy)} \pm \sqrt{(hz)} = 0$……………………………….(1)

$\Rightarrow \sqrt{(fx)} \pm \sqrt{(gy)} = \mp\sqrt{(hz)}$

Now squaring both side, we get

$$\left(\sqrt{(fx)} \pm \sqrt{(gy)}\right)^2 = \left(\mp\sqrt{(hz)}\right)^2$$
$$\Rightarrow fx + gy \pm 2\sqrt{fgxy} = hz$$
$$\Rightarrow fx + gy - hz = \mp 2\sqrt{fgxy}$$

Now again squaring both sides, we get

$$(fx + gy - hz)^2 = \left(\mp 2\sqrt{fgxy}\right)^2$$
$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 + 2fgxy - 2ghyz - 2hfzx = 4fgxy$$
$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 + 2fgxy - 2ghyz - 2hfzx - 4fgxy = 0$$
$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2fgxy - 2ghyz - 2hfzx = 4fgxy$$

SELF CHECK QUESTIONS

Choose the correct option.

(SCQ-1) The surface generated by a moving straight line which passes through fixed point and intersects a given curve or surface

(a) a parabola (b) a circle (c) a cone (d) a hyperbola

(SCQ-2) The general equation of second degree of a cone which passes through the coordinate axes is

(a) $x^2 + fyz + gzx + hxy = 0$ (b) $fyz + gzx + hxy = 0$

(c) $z^2 - fyz + gzx + hxy = 0$ (d) $fyz - gzx - hxy = 0$

(SCQ-3) The tangents plane at any point of a cone passes through

(a) Vertex (b) generator (c) origin (d) intersecting point

(SCQ-4) Any straight line lying on the surface of a cone is called

(a) Coordinate axis (b) normal (c) vertical line (d) generator

9.6 *RIGHT CIRCULAR CONE*

A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through that fixed point.

The fixed point is called the **vertex**, the fixed line the axis, and the constant angle the **semi-vertical** angle of the cone.

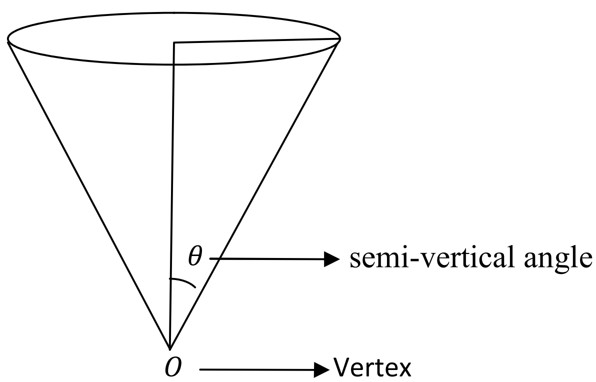


Fig. 9.5.1

9.7 *EQUATION OF RIGHT CIRCULAR CONE*

To find the equation of the right circular cone with its vertex at (α, β, γ), its axis the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

and its semi vertical angle θ.

Let $A\,(\alpha,\beta,\gamma)$ be the vertex of the cone. Let $P\,(x,y,z)$ be any point on the surface of the cone, so that the line joining it to the vertex A makes angle θ with the axis AO.

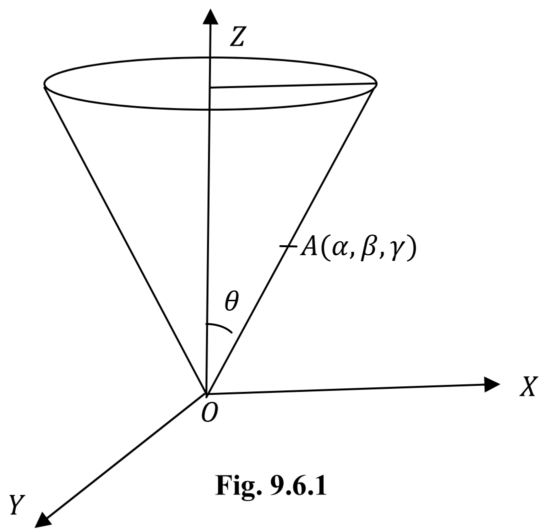


Fig. 9.6.1

Direction-cosines of AP are, therefore proportional to $x-\alpha, y-\beta, z-\gamma$. Also the d.r.'s of AO are l, m, n.

$$cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2(y-\beta)^2+(z-\gamma)^2}}$$

Hence, the required equation of the cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$
$$=(l^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}cos^2\theta.$$

Corollary 1 If the vertex of the cone be at the origin, the equation to the cone becomes

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)\{(x^2 + y^2 + z^2)cos^2\theta$$

Corollary 2 If the vertex of the cone be at the origin, and the axis be the z-axis, then the equation of the cone on taking *l=0, m=0, n=1 and α=0, β=0, γ=0* becomes

$$z^2 = (x^2 + y^2 + z^2)cos^2\theta$$

or

$$x^2 + y^2 = z^2 tan^2\theta$$

Ex. 4. Find the equation to the right circular cone whose vertex is $(2, -3,5)$, axis makes equal angles with the co-ordinate axes and semi-vertical angle is $30°$.

Sol. The vertex of given cone is$(2, -3, 5)$.

First, we will find the direction ratios of the axis i.e. l, m and n.

Let α be tha angle make by axis with respect to x-axis and also it is given that axis makes equal angles with the co-ordinate axes.

Therefore, $l = \cos\alpha\,, m = \cos\alpha$ and $n = \cos\alpha$

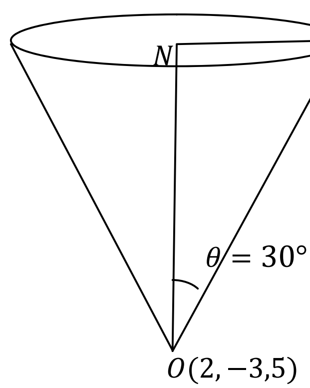


Fig. 9.6.2

Hence, the required equation of the right circular cone is

$$[\cos\alpha\,(x-2)+\cos\alpha\,(y+3)+\cos\alpha\,(z-5)]^2 = (\cos^2\alpha+\cos^2\alpha + \cos^2\alpha)\{(x-2)^2+(y+3)^2 + (z-5)^2\}cos^2 30$$

$$\Rightarrow \cos^2\alpha\,[(x-2)+(y+3)+(z-5)]^2 = (3\cos^2\alpha)\{(x-2)^2+(y+3)^2+(z-5)^2\}\frac{3}{2}$$

$$\Rightarrow (x+y+z-4)^2 = \frac{9}{4}(x^2-4x+4+y^2+6y+9+z^2-10z+25)$$

$$\Rightarrow 4\big(x^2+y^2+(z-4)^2+2xy+2y(z-4)+2x(z-4)\big) = 9(x^2+y^2+z^2 - 4x+6y - 10z+38)$$

$$\Rightarrow 4(x^2+y^2+z^2-8z+16+2xy+2yz-8y+2zx-8x) = 9x^2+9y^2+9z^2 -36x+54y-90z+342$$

$$\Rightarrow 4x^2-9x^2+4y^2-9y^2+4z^2-9z^2+8xy+8yz+8zx-32x+36x-32y - 54y-32z+90z+64-342=0$$

$$\Rightarrow -5x^2-5y^2-5z^2+8xy+8yz+8zx+4x-86y+58z-278=0$$

$$\Rightarrow 5(x^2+y^2+z^2)-8(xy+yz+zx)-4x+86y-58z+278=0$$

9.8 *ENVELOPING CONE*

The locus of the tangent lines drawn from a given point to a given surface is called the **enveloping cone** of that surface with the given point as vertex. This cone is also known as the tangent cone of the surface.

9.9 *EQUATION OF THE ENVELOPING CONE OF A SPHERE*

To find the equation of the enveloping cone of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

with its vertex at (α, β, γ).

Let the equation of any line through the point A (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (1)$$

The coordinates of the point on it whose distance from A is r are $(\alpha + lr, \beta + mr, \gamma + nr)$. If the line (1) meets the given sphere in this point, then

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$$

or

$$r^2(l^2 + m^2 + n^2) + 2r(\alpha l + \beta m + \gamma n + lu + mv + nw) + (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \qquad (2)$$

This is a quadratic equation in *r*. It gives two values of r corresponding to two points on the sphere where the line (1) meets it. In case the line (1) touches the sphere, the two points must coincide and for this, the roots of the equation (2) should be equal. Hence

$$(\alpha l + \beta m + \gamma n + lu + mv + nw)^2 = (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d)$$

Or

$$[(\alpha + u)l + (\beta + v)m + (\gamma + w)n]^2 = (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2u\beta + 2w\gamma + d) \qquad (3)$$

Eliminating l, m, n between (1) and (3), the equation of the required enveloping cone is

$[(\alpha+u)(x-\alpha)+(\beta+v)(y-\beta)+(\gamma+w)(z-\gamma)]^2=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2][\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+\ d]$ (4)

This condition is conveniently simplified by using the notations.

$$S=x^2+y^2+z^2+2ux+2vy+2wz+d,$$

$S_1=\alpha^2+\beta^2+\gamma^2+2u\alpha+2u\beta+2w\gamma+d$ and

$$T=\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d$$

Then the equation (4) of the enveloping cone can be written as

$$(T-S_1)^2=(S-2T+S_1)S_1$$

which on simplification takes the form

$$SS_1=T^2$$

Ex. 5. Show that the equation to the right circular cone whose vertex is O, axis OZ and semi-vertical angle α is $x^2+y^2=z^2tan^2\alpha$

Sol. Let $P\ (x,y,z)$ be any point on the cone. From P draw PN perpendicular to z- axis.

Then

$PN=OM=\sqrt{(x^2+y^2)}$ and $ON = z$.

From the adjoining figure it is evident that

$$\tan\alpha=\frac{PN}{ON}$$

$\Rightarrow \tan\alpha=\frac{\sqrt{(x^2+y^2)}}{z}\Rightarrow x^2+y^2=z^2tan^2\alpha$

SELF CHECK QUESTIONS

(SCQ-5) The equation of right circular cone with vertex at origin, axis the z-axis and semi vertical angle θ is

(a) $x^2 + y^2 = z^2 \tan^2 \theta$ (b) $x^2 + y^2 = -z^2 \tan^2 \theta$

(c) $y^2 + z^2 = x^2 \tan^2 \theta$ (d) $x^2 + z^2 = y^2 \tan^2 \theta$

(SCQ-6) The equation of right circular cone with vertex at origin, axis the y-axis and semi vertical angle θ is

(a) $x^2 + y^2 = z^2 \tan^2 \theta$ (b) $x^2 + y^2 = -z^2 \tan^2 \theta$

(c) $y^2 + z^2 = x^2 \tan^2 \theta$ (d) $x^2 + z^2 = y^2 \tan^2 \theta$

(SCQ-7) If the y- axis is a generator of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, then the value of b is

(a) 1 (b) -1 (c) 0 (d) none of these

FILL IN THE BLANKS

(SCQ-8) Any straight line lying on the surface of a cone is called its ……………………….

(SCQ-9) Two cones which are such that each is the locus of the normals through the vertex to the tangent planes to the other are called…………….

(SCQ-10) The condition that the plane $ux + vy + wz = 0$ cuts the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

in perpendicular lines is $(a + b + c)(u^2 + v^2 + w^2) =$………………

(SCQ-11) The locus of the tangents from a given point to a given surface of a cone called the…………..

(SCQ-12) The section of a right circular cone by any plane perpendicular to its axis is a…….

9.10 *SUMMARY*

In this unit, we have learned about cone. Also we have learned that cone is a surface generated by a moving straight line which passes through a fixed point and intersects a given curve. We found that the every homogeneous equation of second degree in x, y and z represents a cone whose vertex at origin and find the general equation of a cone which passes through coordinate axes. We also find the equation of tangent plane and condition of tangency of a plane and a cone. We also learnt about right circular cone and equation of right circular cone with given vertex and given direction ratios of the axes. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics . To check your progress, self check questions (SCQ's) are given place to place.

9.11 *GLOSSARY*

1. $3D$ object or a solid – an object that occupies space.
2. Cone- surface generated by moving straight line passes through fixed point and intersect curve
3. Generator- moving straight line
4. Vertex – fixed point from which generator passes
5. Semi-verticle angle- constant angle made by generator with axis of cone
6. Mutually – shared by two or more.

9.12 *REFERENCES*

1. Geometry of two and three Dimensions - Lata S. , Himalaya Publishing House, Mumbai.
2. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
3. 3 D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
4. Encyclopedia Britannica (India) Pvt. Ltd. , New Delhi, India.
5. Leonard E. Dickson :The First course in the Geometry.
6. Burnside,William Show, Panton &Arthur William :Volumetric Theory.

7. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series ,McGraw Hill.
8. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

9.13 *SUGGESTED READINGS*

6. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
7. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
8. Engineering Mathematics-R.D. Sharma,New Age Era International Publication, New Delhi.
9. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
10. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

9.14 *TERMINAL QUESTIONS*

(TQ-1) Determine the equation of the cone whose vertex is the point (1, 0, -1) and whose generating lines passes through the ellipse.

(TQ-2) To find the conditions that the general equation of second degree may represent a cone.

(TQ-3) The equation of cone is given as$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26xy + 2z - 17 = 0$. Find the vertex of cone.

(TQ-4) Find the equation to the cones with vertex at the origin and which passes through the curves given by:

(i) $\alpha x^2 + \beta y^2 = 2z;\ ax + by + cz = d$

(ii) $x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0;$

$$x^2 + y^2 + z^2 + 2x - 3y + 4z - 5 = 0$$

(TQ-5) Show that the tangent planes to the cone $fyz + gzx + hxy = 0$ are perpendicular to the generators of the cone $f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0$

(TQ-6) Find the equation of the right circular cone with vertex (1,-2,-1), semi vertical angle 60° and the axis $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$.

(TQ-7) Find the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y = 2$ with its vertex at (1,1,1).

(TQ-8) Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the paraboloid $ax^2 + by^2 = 2cz$.

(TQ-9) Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2y + 6z + 2 = 0$ with its vertex at (1,1,1).

9.15 *ANSWERS*

SELF CHECK QUESTIONS (SCQ'S)

(SCQ-1) c (SCQ-2) b (SCQ-3) a (SCQ-4) d

(SCQ-5) a (SCQ-6) d (SCQ-7) c

(SCQ-8) generator

(SCQ-9) reciprocal cone

(SCQ-10) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

(SCQ-11) enveloping cone

(SCQ – 12) Circle

TERMINAL QUESTIONS (TQ'S)

(TQ-1)

(TQ-2)

(TQ-3) $(1, -2,2)$

(TQ-4) (i)$d(\alpha x^2 + \beta y^2) = 2z(ax + by + cz)$

(ii) $2x^2 + y^2 - 5xy - 3yz + 4zx = 0$

(TQ – 5) ------------

(TQ – 6) $7x^2 + 7y^2 - 25z^2 + 80yz - 60zx + 48xy + 22x + 4y + 170z + 78 = 0$

(TQ – 7) $3x^2 - y^2 + 4zx - 10x + 2y - 4z + 6 = 0$

(TQ-8) $ab(x^2 + y^2) - 2c(a + b)z = c^2$

(TQ-9) $8x^2 + 9y^2 - 7z^2 - 8zx - 8x - 18y + 22z + 2 = 0$

UNIT 10: THE CYLINDER AND PARABOLOIDS

CONTENTS

10.1 INTRODUCTION

A cylinder is a surface generated by a variable straight line which moves in such a way that it is always parallel to a fixed straight line and satisfies one more condition, e.g., it may intersect a given curve or may touch a given surface.The line which generate the surface of the cylinder is called its generator, the fixed straight line is called the axis or the guiding line of the cylinder, and the given curve of the surface is called the guiding curve or the guiding surface as the case may be.

Definition 1. A cylinder is the set of lines which intersect a given curve and which are parallel to a fixed line which does not lie in the plane of the curve. The fixed line is called the **axis** of cylinder and the curve is called the base **curve (or directrix)** of the cylinder.

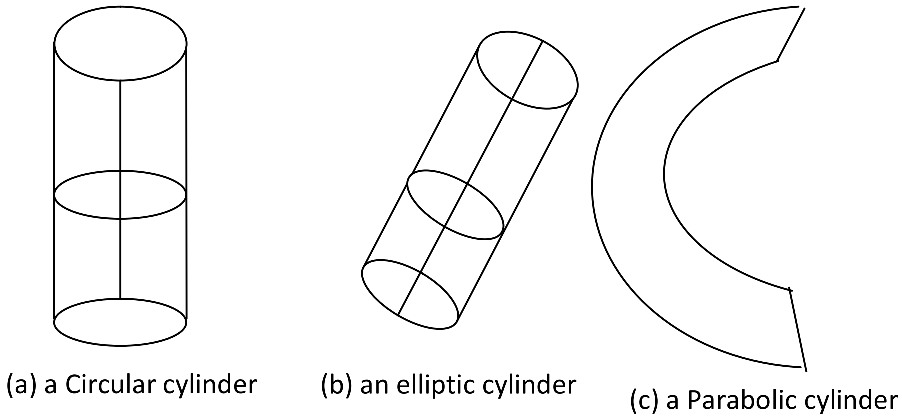


Fig 10.2.1

Definition 2. A cylinder whose base curve is circle and whose axis passes through the centre of the base curve and perpendicular to the plane of the base curve is called a **right circular cylinder**.

Note: In this section cylinder means a right circular cylinder

10.2 OBJECTIVES

The objective of the cylinder and paraboloids is

- To help students to develop the basic concept of three dimensional coordinate geometry,
- To solve the problems of equation of cylinder and paraboloids as enveloping cylinder, equation of the enveloping cylinder of sphere, right circular cylinder, hyperbolic paraboloid, intersection of a line with paraboloid, tangent line, tangent plane and condition of tangency.

10.3 EQUATION OF A CYLINDER

To find the equation of a cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$(1)

and whose guiding curve is the conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \ \ z = 0$ (2)

Let (α, β, γ) be any point on the surface of the cylinder so that the equation of the generators through the point are

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ (3)

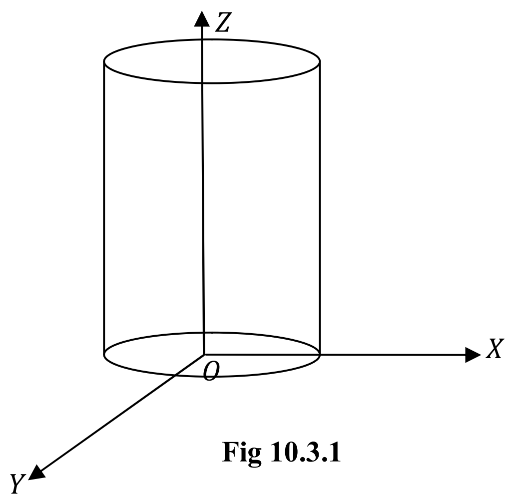


Fig 10.3.1

This line meets the plane $z = 0$at the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}\;\; i.e., \left(\alpha-\frac{ly}{n},\beta-\frac{my}{n},0\right)$$

Since the generator (3) intersects the guiding curve, therefore this point should satisfy the equation of the conic (2), i.e.,

$$a\left(\alpha-\frac{ly}{n}\right)^2+2h\left(\alpha-\frac{ly}{n}\right)\left(\beta-\frac{my}{n}\right)+b\left(\beta-\frac{my}{n}\right)^2+2g\left(\alpha-\frac{ly}{n}\right)+2f\left(\beta-\frac{my}{n}\right)+c=0$$

$$\Rightarrow a(n\alpha-l\gamma)^2+2h(n\alpha-l\gamma)(n\beta-m\gamma)+b(n\beta-m\gamma)^2+2gn(n\alpha-l\gamma)+2fn(n\beta-m\gamma)+cn^2=0$$

Hence the locus of the point (α,β,γ) is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2+2gn(nx-lz)+2fn(ny-mz)+cn^2=0$$

This is the required equation of the cylinder.

Ex.1. Find the equation of a cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and passing through the curve $x^2+y^2=16,\;\; z=0$.

Sol. We have,

the equation of guiding curves are $x^2+y^2=16,\;\; z=0$..............................(1)

and the equation of given line is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$...(2)

Let $K(\alpha,\beta,\gamma)$ be a point on the cylinder. The equation of the generator through the point $K(\alpha,\beta,\gamma)$ which is a line parallel to the given line (2) are

$\frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{z-\gamma}{3}$ ....(3)

The generator (3) meets the plane $z=0$ in the point given by

$$\frac{x-\alpha}{1}=\frac{0-\gamma}{3},\frac{y-\beta}{2}=\frac{0-\gamma}{3},z=0$$

therefore, the point is$(x,y,z)=(\alpha-\frac{\gamma}{3},\ \beta-\frac{2\gamma}{3},0)$

Since the generator (3) meets the conic (1), hence the given point $(\alpha-\frac{\gamma}{3},\ \beta-\frac{2\gamma}{3},0)$ will satisfy the equation of conic given by (1),

Hence, $(\alpha-\frac{\gamma}{3})^2+(\beta-\frac{2\gamma}{3})^2=16$

$$\Rightarrow (3\alpha-\gamma)^2+(3\beta-2\gamma)^2=16.9 \Rightarrow (3\alpha-\gamma)^2+(3\beta-2\gamma)^2=144$$

Byreplacing $\alpha=x,\beta=y$ and $\gamma=z$, we get the required equation of cylinder which is given by

$$(3x-z)^2+(y-2z)^2=16.9 \Rightarrow (3x-z)^2+(3y-2z)^2=144$$

$$\Rightarrow 9x^2-6xz+z^2+9y^2-12yz+4z^2-144=0$$

$$\Rightarrow 9x^2+9y^2+5z^2-6xz-12yz-144=0$$

10.4 EQUATION OF THE ENVELOPING CYLINDER OF A SPHERE

The locus of the lines drawn in a given direction or parallel to a given line so as to touch a given surface is called the enveloping cylinder of the surface.

To find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ whose generators are parallel to the line$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Let (α,β,γ) be any point on the surface of the cylinder so that the equation of the generators through the point are $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\ (say)$ (1)

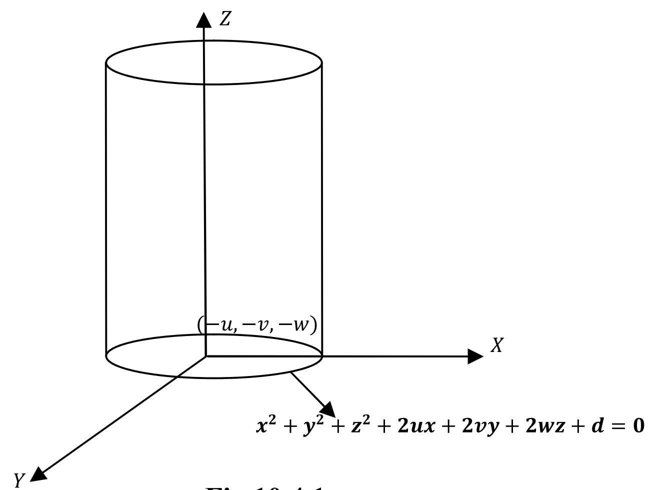


Fig 10.4.1

Any point on the line is $(\alpha+lr,\beta+mr,\gamma+nr)$

If it lies on the given sphere, we have

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$$

or

$$(l^2+m^2+n^2)+2u(l\alpha+m\beta+n\gamma+lu+mv+nw)+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0 \quad (2)$$

Since the line (1) is tangent to the given sphere, the roots of the quadratic equation (2) must be identical, i.e.,

$$(l\alpha+m\beta+n\gamma+lu+mv+nw)^2 = (l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)$$

Hence the locus of (α,β,γ) is

$$(lx+my+nz+lu+mv+nw)^2=(l^2+m^2+n^2)(x^2+y^2+z^2+2ux+2vy+2wz+d)$$

or

$[l(x+u)+m(y+v)+n(z+w)]^2=(l^2+m^2+n^2)(x^2+y^2+z^2+2ux+2vy+2wz+d)$.

This is the required equation of the enveloping cylinder of the given sphere.

Ex.2. Find the equation of a the enveloping cylinder of the sphere

$x^2+y^2+z^2-2x+4y-1=0$ having its generators parallel to the line $x=y=z$.

Sol. We have,

the given sphere whose equation is $x^2+y^2+z^2-2x+4y-1=0$ ....(1)

and the generators of the enveloping cylinder are parallel to the line

$x=y=z$ ....(2)

Let $M(x_1,y_1,z_1)$ be any point on the enveloping cylinder.

Now, the equations of the generator through the point $M(x_1,y_1,z_1)$which is a line parallel to the given line (2) are

$\frac{x-x_1}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}=r$ (say)..(3)

Any point is$(x,y,z)=(x_1+r,\ x_2+r,\ x_3+r)$

Using the value of given point (x,y,z) in the equation of sphere we get the points of intersection of generator with the sphere given by

$$(x_1+r)^2+(x_2+r)^2+(x_3+r)^2-2(x_1+r)+4(x_2+r)-1=0$$

$$\Rightarrow {x_1}^2+2rx_1+r^2+{x_2}^2+2rx_2+r^2+{x_3}^2+2rx_3+r^2-2x_1-2r+4x_2+4r-1=0$$

$\Rightarrow 3r^2+2r(x_1+x_2+x_3+1)+({x_1}^2+{x_2}^2+{x_3}^2-2x_1+4x_2-1)=0$ ......(4)

Since the generator (3)is a tangent to the given sphere (1), therefore the two values of r are equal or we can say that roots of r in equation (4) are equal

Therefore, $(2(x_1+x_2+x_3+1))^2-4.3.(x_1{}^2+x_2{}^2+x_3{}^2-2x_1+4x_2-1)=0$

[Using $B^2-4AC=0$]

$\Rightarrow 4(x_1+x_2+x_3+1)^2-12(x_1{}^2+x_2{}^2+x_3{}^2-2x_1+4x_2-1)=0$

$$\Rightarrow x_1{}^2+x_2{}^2+(x_3+1)^2+2x_1x_2+2x_2(x_3+1)+2x_1(x_3+1)-3x_1{}^2-3x_2{}^2-3x_3{}^2+6x_1-12x_2+3=0$$

$$\Rightarrow x_1{}^2+x_2{}^2+x_3{}^2+1+2x_3+2x_1x_2+2x_2x_3+2x_2+2x_1x_3+2x_1-3x_1{}^2-3x_2{}^2-3x_3{}^2+6x_1-12x_2+3=0$$

$$\Rightarrow -2x_1{}^2-2x_2{}^2-2x_3{}^2+2x_1x_2+2x_2x_3+2x_1x_3+8x_1-10x_2+2x_3+4=0$$

$$\Rightarrow x_1{}^2+x_2{}^2+x_3{}^2-x_1x_2-x_2x_3-x_1x_3-4x_1+5x_2-x_3-2=0$$

Byreplacing$x_1=x$, $x_2=y$ and $x_3=z$, we get the required equation of enveloping cylinder which is given by

$$\Rightarrow x^2+y^2+z^2-xy-yz-xz-4x+5y-z-2=0$$

10.5 RIGHT CIRCULAR CYLINDER

A right circular cylinder is a surface generated by a straight line which intersects a given circle and is perpendicular to its plane. The circle is called the **guiding circle of the cylinder**.

The normal to the plane of the guiding circle through its centre is called the **axis of the cylinder**.

Section of a right circular cylinder by any plane perpendicular to its axis is a circle and is called a **normal section**.

Obviously, all the normal section of the circular cylinder is circles having the same radius which is also called the **radius of the cylinder**. The length of the perpendicular from any point on a right circular cylinder to its axis is equal to its radius.

A right circular cylinder may also be defined as a surface generated by a straight line which moves parallel to a fixed straight line and always remains at a constant distance from it. The fixed line is called the axis and the constant distance the radius of the cylinder.

10.6 EQUATION OF RIGHT CIRCULAR CYLINDER

To find the equation of right circular cylinder whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and whose radius is r.

Let $P\ (x, y, z)$ be any point on the surface of the cylinder, then the perpendicular distance of P from the axis is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is equal to $PN\ =\ r$

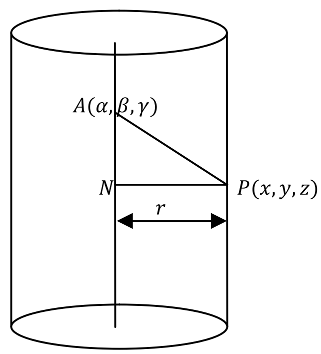


Fig 10.6.1

From the figure, we have

$$PN^2 = AP^2 - AN^2$$

$$AP^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$

and

AN =Projection of AP on the axis= $\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{(l^2+m^2+n^2)}}$

Hence

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\frac{[(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2}{l^2+m^2+n^2}=r^2$$

$$(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]-[(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2 = r^2(l^2+m^2+n^2).$$

By Lagrange's identity, this becomes

$$[n(y-\beta)-m(z-\gamma)]^2+[l(z-\gamma)-n(x-\alpha)]^2+[m(x-\alpha)-l(y-\beta)]^2=r^2(l^2+m^2+n^2).$$

This is the required equation of the right circular cylinder.

Equation of a cylinder whose base curve is a circle $x^2+y^2=a^2,\ z=0$ and axis is z axis

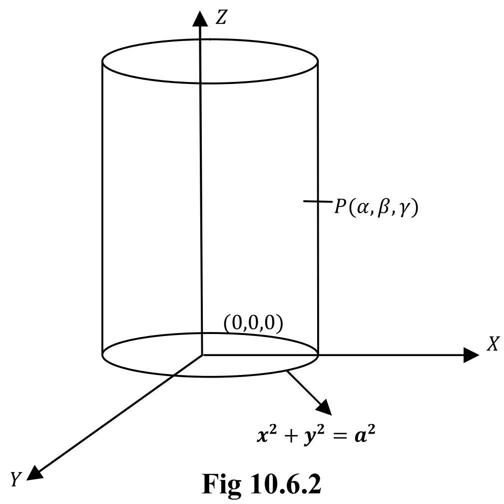


Fig 10.6.2

Let P be any point on the surface of cylinder then the equation of generator line is given by $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ and equation of axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$.

Any point on the surface is given by $(\alpha,\beta,\gamma+r)$ satisfies given point on the base curve, we get

$\alpha^2+\beta^2=a^2$ and $\gamma+r=0$

Replacing α and β by x and y, we get equation of cylinder $x^2+y^2=a^2$

Alternate method

Let P be any point on the surface of cylinder $PQ=a$ then

$$OP^2=PQ^2+OQ^2$$

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}=a^2+z^2$$

$$\Rightarrow x^2+y^2+z^2=a^2+z^2$$

$$\Rightarrow x^2+y^2=a^2$$

Ex.3. Find the equation of the right circular cylinder of radius 2 and having as axis the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

Sol. We have,

the given axis of cylinder whose equation is $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$…………………(1)

Now, a point on the axis of cylinder is $P(1,2,3)$ and its direction ratios are 2,1,2.

Therefore, its direction cosines are $\frac{2}{\sqrt{2^2+1^2+2^2}},\frac{1}{\sqrt{2^2+1^2+2^2}},\frac{2}{\sqrt{2^2+1^2+2^2}}$ i.e., $\frac{2}{3},\frac{1}{3},\frac{2}{3}$

Let $Q(x,y,z)$ be a point on the cylinder.

Now, the length of the perpendicular from the point $Q(x,y,z)$ to the given axis (1)

$=$ radius of cylinder $=2$.

Draw QR perpendicular to the axis PX.

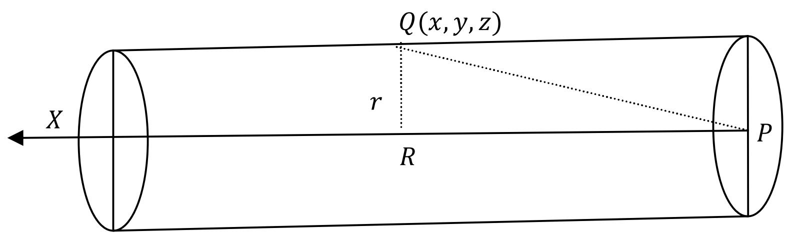


Fig. 10.6.3

Therefore, $QR = 2$.

Now, $PR =$ Projection of PQ on PR

$$= (x-1)\frac{2}{3} + (y-2)\frac{1}{3} + (z-3)\frac{2}{3}$$

$$= \frac{2x + y + 2z - 2 - 2 - 6}{3} = \frac{2x + y + 2z - 10}{3}$$

Also, $PQ = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$

Therefore, in right angled triangle PRQ

$$PQ^2 = PR^2 + QR^2$$

$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = \frac{(2x + y + 2z - 10)^2}{9} + 4$$

$$\Rightarrow 9(x^2 + 1 - 2x + y^2 + 4 - 4y + z^2 + 9 - 6z)$$
$$= 4x^2 + y^2 + (2z-10)^2 + 4xy + 2y(2z-10) + 4x(2z-10) + 36$$

$$\Rightarrow 9(x^2 + y^2 + z^2 - 2x - 4y - 6z + 14)$$
$$= 4x^2 + y^2 + 4z^2 + 100 - 40z + 4xy$$
$$+ \quad 4yz - 20y + 8zx - 40x + 36$$

$$\Rightarrow 9x^2 + 9y^2 + 9z^2 - 18x - 36y - 54z + 126$$
$$= 4x^2 + y^2 + 4z^2 + 4xy + 4yz + 8zx - 40x - 20y - 40z + 136$$

$$\Rightarrow 9x^2 - 4x^2 + 9y^2 - y^2 + 9z^2 - 4z^2 - 4xy - 4yz - 8zx - 18x + 40x - 36y + 20y - 54z + 40z + 126 - 136 = 0$$

$$\Rightarrow 5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z - 10 = 0$$

SELF CHECK QUESTIONS

(SCQ-1) Any line on the surface of cylinder is called its

(a) Generator (b) radius (c) axis (d) none of these

(SCQ-2) Equation of enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whose generator are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

(a) $(x^2 + y^2 + z^2 - a^2)(l^2 + m^2 + n^2) = (lx + my + nz)^2$

(b) $(x^2 + y^2 + z^2 + a^2)(l^2 + m^2 + n^2) = (lx + my + nz)^2$

(c) $(x^2 + y^2 + z^2 + a^2)(l^2 + m^2 + n^2) = (lx - my - nz)^2$

(d) $(x^2 + y^2 + z^2 - a^2)(l^2 + m^2 + n^2) = (lx - my - nz)^2$

(SCQ-3) The equation of right circular cylinder whose axis is z-axis and radius a is

(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 + a^2 = 0$

(c) $x^2 + z^2 = a^2$ (d) $x^2 + z^2 + a^2 = 0$

(SCQ-4) In three dimensional geometry, equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

(a) Circle (b) Cylinder

(b) conics (d) None of these

10.7 PARABOLOIDS

The elliptic paraboloid

The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c} \qquad (1)$$

is called an elliptic paraboloid we have the following points about the nature and shape of this surface.

(1) No point bisects all chords through it and therefore there is no centre of the surface.
(2) The coordinate planes x = 0, and y = 0 bisect all chords perpendicular to them and are, therefore, its two planes of symmetry or two principal planes.
(3) Z cannot be negative (c being positive), and hence there is no part of the surface below the XOY planes i.e. the surface is above the XOY plane.
(4) The section by the plane z = k (k > 0) is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}, z=k$$

which represents an ellipse whose semi-axes are

$$a\sqrt{\frac{2k}{c}}, \qquad b\sqrt{\frac{2k}{c}}$$

And whose centre lies on the z-axis. It increases in size as k increases; there being no limit to the increases; for k = 0 it is a point ellipse. The surface is therefore generated by a variable ellipse parallel to the XOY plane and is consequently called the elliptic paraboloid. Hence the surface is entirely on the positive side of the XOY plane, and extended to infinity.

The section of the surface by plane parallel to ZOX plane is the parabola given by the equations

$$y=k, x^2=\frac{2a^2}{c}\left(z-\frac{k^2c}{2b^2}\right)$$

Similarly, the section of the surface by the plane x = k is the parabola whose equation are

$$x=k, y^2=\frac{2b^2}{c}\left(z-\frac{k^2c}{2b^2}\right)$$

Thus, the paraboloid is also generated by a variable parabola in two different ways.

10.8 STANDARD EQUATION OF A PARABOLOID

To begin with, let us go back to the theorem which state that any second degree equation reduced an equation of form

$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ ....(1)

Now, let us assume that (1) represent a non central conicoid. Since the conicoid has no centre, we find that either

i) Exactly two of the a, b and c are zero, or
ii) Only one of a, b and c is zero

Let us look at the cases separately. First we consider case (i) i.e. let us assume that $a = 0, b = 0$ and $c \neq 0$ (we can similarly deal with $a = 0, c = 0$ and $b \neq 0$ or $b = 0, c = 0$ and $a \neq 0$)

In this condition equation (1) become

$$cz^2 + 2ux + 2vy + 2wz + d = 0$$

$$\Rightarrow c\left(z^2 + \frac{2w}{c}z + \frac{w^2}{c^2}\right) = -2ux - 2vy - d + \frac{w^2}{c}$$

$$\Rightarrow c\left(z + \frac{w}{c}\right)^2 = -2ux - 2vy - d + \frac{w^2}{c}$$

by shifting origin to $\left(0,0,-\frac{w}{c}\right)$, we get

$cz^2 + 2ux + 2vy + d_0 = 0$ where $d_0 = -d + \frac{w^2}{c}$ what does this equation represent? Let's see if both u, v are zero then surface represents a pair of lines.

If one of the coefficients u and v is non zero say $v \neq 0$ and $u = 0$ then you can see that the surface is built up of a series of parabolas along a line parallel to the x −axis. Thus it is a parabolic cylinder. In fact even if both u,v are non zero, the surface is parabolic cylinder.

Let us go to case (ii).

Here we assume that $a \neq 0$ and $b, c = 0$ (similarly we can deal with other two cases i.e. $b \neq 0$ and $a, c = 0$ and $c \neq 0$ and $a, b = 0$).

In this case equation (1) reduces to the form

$$by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

$$\Rightarrow b\left(y + \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = -2ux - d + \frac{v^2}{b} + \frac{w^2}{c}$$

$\Rightarrow b\left(y + \frac{v}{b}\right)^2 + c\left(z + \frac{w}{c}\right)^2 = -2ux + d_1$ where $d_1 = -d + \frac{v^2}{b} + \frac{w^2}{c}$

Shifting centre at the point $\left(0, \frac{-v}{b}, \frac{-w}{c}\right)$, we get

$$bY^2 + cZ^2 = -2uX + d_1 = -2u\left(X - \frac{d_1}{2u}\right)$$

Again shifting new origin to $\left(-\frac{d_1}{2u}, 0, 0\right)$, we get

$$bY^2 + cZ^2 = -2uX$$

Do you agree that this equation is a three dimensional version of standard equation of parabola?

We call this surface a paraboloid.

For example $2y^2 + z^2 = 12x$ represents a paraboloid.

10.9 THE HYPERBOLIC PARABOLOID

The locus of the equation

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$ is called a hyperbolic paraboloid.

(1) The surface is symmetric with respect to the YOZ and ZOX planes. Thus, the coordinates planes x= 0, y = 0 are the two principal planes.
(2) The section of the surface by the plane z = k is given by the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}, z=k$$

which is a hyperbola with its centre on z-axis.

If k is a positive, the transverse axis of the hyperbola is parallel to x- axis, and if k is negative the transverse axis is parallel to y- axis. the section of the plane z =0 is the pair of lines

$\frac{x}{a}+\frac{y}{b}=0,\ z=0\ and\ \frac{x}{a}-\frac{y}{b}=0,\ z=0,$

Which are parallel to the asymptotes of all hyperbolic section. The surface is therefore generated by a variable hyperbola parallel to the XOY plane and is consequently called the hyperbolic paraboloid.

The sections by the planes parallel to YOZ and ZOX planes are parabolas whose equations are given by

$x=k, y^2=-\frac{2b^2}{c}(z-\frac{k^2c}{2a^2})$ and $y=k, x^2=-\frac{2a^2}{c}(z+\frac{k^2c}{2b^2})$ respectively

10.10 INTERSECTION OF A LINE WITH PARABOLOID

To find the points of intersection of the line

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ say (1)

With the paraboloid

$ax^2+by^2=2cz$ (2)

The coordinates of any point on (1) are

$$(\alpha+lr,\beta+mr,\gamma+nr)$$

If this point lies on (2), then

$$a(\alpha + lr)^2 + b(\beta + mr)^2 = 2c(\gamma + nr)$$

$$\Rightarrow r^2(al^2 + bm^2) + 2r(a\alpha l + b\beta m - cn) + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0 \qquad (3)$$

This is the quadratic equation in *r* and therefore gives two values of *r,* i.e. every line meets a paraboloid in two points. It follows from this that the plane sections of paraboloid are conics.

If $l = m = 0$, then one root of this equation is infinite. Therefore, one point of intersection is at infinity, and the others point is at a finite distance.

Thus, a line parallel to z-axis meets the paraboloid in one point at an infinite distance from (α, β, γ) and so meets it in one finite points only its distance is given by

$$r = \frac{a\alpha^2 + b\beta^2 - 2c\gamma}{2cn}$$

Such a line is called a diameter of the paraboloid; and the point at finite distance is called the extremity of the diameter.

10.11 TANGENT LINES AND TANGENT PLANE

If the point (α, β, γ) lies on the surface $(ax^2 + by^2 = 2cz)$, we have

$$a\alpha^2 + b\beta^2 - 2c\gamma = 0$$

The coordinates of any point are$(\alpha + lr, \beta + mr, \gamma + nr)$

If this point lies on paraboloid, then

$$a(\alpha + lr)^2 + b(\beta + mr)^2 = 2c(\gamma + nr)$$

$$\Rightarrow r^2(al^2 + bm^2) + 2r(a\alpha l + b\beta m - cn) + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0$$

W hich shows the root of above equation is zero. The other roots is also zero, if in addition to it$a\alpha l + b\beta m - cn = 0$ touch the paraboloid at (α, β, γ).

If we eliminate $l, m, n$with the help of equation of line. We get the locus of all the tangent lines through (α, β, γ) which is given by

$$a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)=0$$

$$\Rightarrow a\alpha x+b\beta y-cz=a\alpha^2+b\beta^2-c\gamma$$

$$\Rightarrow a\alpha x+b\beta y=c(z+\gamma)$$

This is the equation of the tangent plane at (α, β, γ) to the paraboloid.

In particular, $z = 0$ is the tangent plane at the origin and the z −axis is the normal there at.

10.12 CONDITION OF TANGENCY

The condition that the plane $lx+my+nz=p$may touch the paraboloid$ax^2+by^2=2cz$

is$\frac{l^2}{a}+\frac{m^2}{b}+\frac{2np}{c}=0$

and the point of contact is$(\frac{-cl}{an}, \frac{-cm}{bn}, \frac{np}{n})$

Corollary. The plane $2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$is a tangent plane to the paraboloid for all values of l, m, n.

Ex. 4. Prove that the plane $2x-4y-z+3=0$ touches the paraboloid$x^2-2y^2=3z$. Find the point of contact

Sol. The equation of given paraboloid is $x^2-2y^2=3z$............................(1)

We have the given equation of plane

$2x-4y-z+3$ ....(2)

First comparing (1) with the standard equation $ax^2+by^2=2cz$ of the paraboloid, we get $a=1$, $b=-2$, $2c=3 \Rightarrow c=\frac{3}{2}$.

Now, comparing (2) with the standard equation of plane $lx + my + nz = p$, we get

$$l = 2, m = -4, n = -1, p = -3.$$

Now, we will check the condition of tangency i.e., $\frac{l^2}{a} + \frac{m^2}{b} + \frac{2n}{c} = 0$

Therefore, $\frac{2^2}{1} + \frac{(-4)^2}{-2} + \frac{2(-1)(-3)}{\frac{3}{2}} = 4 - 8 + 4 = 0.$

which implies that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid$x^2 - 2y^2 = 3z$.

Now we will find the coordinates of the point of contact.

Let the coordinates of the point of contact $'P'$ be (x_1, y_1, z_1).

We know that the equation of the tangent plane at the point $(\alpha, \beta, \gamma)$to the paraboloid $ax^2 + by^2 = 2cz$ is $a\alpha x + b\beta y = c(\gamma + z)$.

Therefore, the equation of the tangent plane at the point(x_1, y_1, z_1) to the paraboloid $x^2 - 2y^2 = 3z$ is

$$1. x_1 x + (-2) y_1 y = \frac{3}{2}(z_1 + z)$$

$$\Rightarrow 2x_1 x - 4y_1 y = 3z_1 + 3z$$

$\Rightarrow 2x_1 x - 4y_1 y - 3z - 3z_1 = 0$ ....(3)

If the plane (3) touches the given paraboloid at the point $P(x_1, y_1, z_1)$ then the plane (2) must be the same.

Therefore comparing the equations (2) and (3), we get

$$\frac{2x_1}{2} = \frac{-4y_1}{-4} = \frac{-3}{-1} = \frac{3z_1}{3} \Rightarrow \frac{x_1}{1} = \frac{y_1}{1} = \frac{3}{1} = \frac{z_1}{1}$$

Therefore, $x_1 = 3, y_1 = 3, z_1 = 3$ which implies that the required point of contact is (3,3,3).

10.13 EQUATIONS OF NORMAL

The tangent plane to the paraboloid

$$ax^2 + by^2 = 2cz$$

At any given point (α, β, γ) is given by

$$a\alpha x + b\beta y = c(z + \gamma)$$

Therefore, the equations of normal at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha} + \frac{y-\beta}{b\beta} + \frac{z-\gamma}{-c}$$

10.14 NORMALS FROM A GIVEN POINT

The equations normal at any point (α, β, γ) of the paraboloid

$$ax^2 + by^2 = 2cz$$

are

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c}$$

If it passes through the given point (x', y', z'), say, then

$$\frac{x'-\alpha}{a\alpha} = \frac{y'-\beta}{b\beta} = \frac{z'-\gamma}{-c} = \lambda$$

Therefore, $\alpha = \frac{x'}{1+a\lambda}, \beta = \frac{y'}{1+b\lambda}, \gamma = z' + c\lambda$

But (α, β, γ) lies on the paraboloid, therefore,

$$a\frac{x'^2}{(1+c\lambda)^2}+b\frac{y'^2}{(1+b\lambda)^2}=2c(z'+c\lambda)$$

This is the equation of fifth degree in λ. From which it follows that there are five points on the paraboloid, the normal at which pass through the given point (x', y', z').

Thus, in general, five normal can be drawn through a given point to a paraboloid.

Ex. 5. Show that the normal from (x', y', z') to the paraboloid $ax^2+by^2=2cz$ lie on the cone $\frac{x\prime}{x-x\prime}-\frac{y\prime}{y-y\prime}+c.\frac{\frac{1}{a}-\frac{1}{b}}{z-z\prime}=0$

Sol. The equation of paraboloid **is** $ax^2+by^2=2cz$……………………………….(1)

Let (x_1, y_1, z_1) be a point on (1) then $a{x_1}^2+b{y_1}^2=2cz_1$……………………..(2)

Now, the equations of the normal at (x_1, y_1, z_1) to given equation of paraboloid

$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{-c}=k$ (say)…………………………………………………….(3)

if the normal passes through (x', y', z'), we get

$$\frac{x'-x_1}{ax_1}=\frac{y'-y_1}{by_1}=\frac{z'-z_1}{-c}=k$$

Therefore, $x'-x_1=kax_1 \Rightarrow x'=kax_1+x_1 \Rightarrow (ka+1)x_1=x' \Rightarrow x_1=\frac{x'}{ka+1}$

$$y'-y_1=kby_1 \Rightarrow y'=kby_1+y_1 \Rightarrow (kb+1)y_1=y' \Rightarrow y_1=\frac{y'}{ka+1}$$

and $z'-z_1=-ck \Rightarrow z_1=z'+ck$

i.e., $x_1=\frac{x'}{ka+1}, y_1=\frac{y'}{ka+1}, z_1=z'+ck$……………………………………….(4)

Hence, the equation of the cubic curve which lie on the feet of the normals from (x', y', z') are given by

$x = \frac{x'}{ka+1}, y = \frac{y'}{ka+1}, z = z' + ck$ ....(5)

Now we will prove that the feet of the normals will lie on the given cone.

If the values of x, y, z from (5)satisfythe equation of the given cone then the result is true

Now, L.H.S of given equation of cone at (x, y, z)

$= \frac{x'}{\frac{x'}{ka+1} - x'} - \frac{y'}{\frac{y'}{kb+1} - y'} + c.\frac{\frac{1}{a} - \frac{1}{b}}{z' + ck - z'} = \frac{ka+1}{-ka} - \frac{kb+1}{-kb} + \frac{(b-a)}{kab} = \frac{-kab-b+kab+a}{kab} - \frac{(a-b)}{kab}$

$$= \frac{a-b}{kab} - \frac{(a-b)}{kab} = 0 = RHS$$

It implies that the normal from (x', y', z') to the paraboloid $ax^2 + by^2 = 2cz$ lie on the cone $\frac{x\prime}{x - x\prime} - \frac{y\prime}{y - y\prime} + c.\frac{\frac{1}{a} - \frac{1}{b}}{z - z\prime} = 0$

SELF CHECK QUESTIONS

(SCQ-5) Equation of tangentto the paraboloid $ax^2 + by^2 = 2cz$at any point (x_1, y_1, z_1) on it are

(a) $ax_1x - by_1y = c(z_1 + z)$

(b) $ax_1x + by_1y = c(z_1 + z)$

(c) $ax_1x + by_1y = -c(z_1 + z)$

(d) None of these

(SCQ-6) The equation of normalto the paraboloid $ax^2 + by^2 = 2cz$at any point (x_1, y_1, z_1) on it are

(a) $\frac{x - x_1}{ax_1} = \frac{y - y_1}{-by_1} = \frac{z - z_1}{-c}$

(b) $\frac{x - x_1}{-ax_1} = \frac{y - y_1}{-by_1} = \frac{z - z_1}{-c}$

(c) $\frac{x - x_1}{ax_1} = \frac{y - y_1}{by_1} = \frac{z - z_1}{-c}$

(d) $\frac{x - x_1}{-ax_1} = \frac{y - y_1}{by_1} = \frac{z - z_1}{-c}$

(SCQ-7) The equation of normalto the paraboloid $4x^2 + 5y^2 = 2z$at any point (2,3,4) on it are

(a) $\frac{x-2}{8} = \frac{y-3}{-15} = \frac{z-4}{-1}$

(b) $\frac{x-2}{8} = \frac{y-3}{15} = \frac{z-4}{-1}$

(c) $\frac{x-2}{-8} = \frac{y-3}{15} = \frac{z-4}{-1}$

(d) $\frac{x-2}{8} = \frac{y-3}{15} = \frac{z-4}{1}$

(SCQ-8) The equation of equation of paraboloid $ax^2 + by^2 = 2cz$ represents a hyperbolic paraboloid if

(a) a and b both are positive
(b) a and b both are negative
(c) a and b both are opposite signs
(d) none of these

FILL IN THE BLANKS

(SCQ-9) Any straight line lying on the surface of a cylinder is called its ………………….

(SCQ-10) The generator of the cylinder $y^2 + z^2 = a^2$ are straight line parallel to ………..

(SCQ-11) If all the generator of a cylinder intersect a curve then the curve is called……….

(SCQ-12) The surface generated by the tangent lines to the sphere $x^2 + y^2 + z^2 = 25$ which are parallel to the given line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is an enveloping ………..of the given sphere.

(SCQ-13) The condition that the plane $lx + my + nz = p$ touches the paraboloid $ax^2 + by^2 = 2cz$ is…………….

(SCQ-14) The plane ……………….. is a tangent plane to the paraboloid for all values of l, m, n.

10.15 SUMMARY

In this unit, we have learned about cylinder. Also we have learned that surface generated by straight line which is parallel to fixed line. We find the general equation of a cylinder whose generators are parallel to the line and intersect the conic. We also find the equation of right circular cylinder, enveloping cylinder and equation of tangent plane to cylinder. We have also learned about paraboloid and find the equation of tangent plane. We have find the condition of tangency and equations of nbormal at any point of the paraboloid. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics. To check your progress, self check questions (SCQ's) and (TQ's) are given place to place.

10.16 GLOSSARY

1. Cylinder- surface generated by straight line which is parallel to fixed line
2. Generator of cylinder- line on surface of cylinder
3. Enveloping cylinder – locus of tangents to a surface which parallel to given line
4. Paraboloid- solid generated by the rotation of a parabola about its axis of symmetry
5. Tangent- straight line that touches curved surface at a point
6. Normal- a straight line which is perpendicular to the tangent.

10.17 REFERENCES

1. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
2. 3 D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
3. Leonard E. Dickson :The First course in the Geometry.
4. Burnside,William Show, Panton &Arthur William :Volumetric Theory.
5. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series ,McGraw Hill.
6. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

10.18 SUGGESTED READINGS

1. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
2. Analytical geometry –Nazrul Islam , Tata McGraw Hill .
3. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
4. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

10.19 TERMINAL QUESTIONS

(TQ-1) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1,\ z=3.$

(TQ-2) Find the equation of a right circular cylinderof radius 2 whose axis passes through $(1,2,3)$ and has direction cosine proportional to $(2,-3,6)$

(TQ-3) Find the equation of the right circular cylinder of radius 3 and whose axis is $\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$

(TQ-4) Find the equation of the right circular cylinder described on the circle through the point $(1,0,0), (0,1,0)$ and $(0,0,1)$ as guiding curve.

(TQ-5) Prove that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ and coordinate of the point of contact is $(8,9,5)$

(TQ-6) By completing the square find the vertex of the paraboloid

$$3x^2+2y^2-6x+8y-12z-13=0$$

(TQ-7) Prove that the normals from (x',y',z') to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ lie on the cone

$$\frac{x'}{x-x'}-\frac{y'}{y-y'}+\frac{a^2-b^2}{z-z'}=0$$

10.20 ANSWERS

SELF CHECK QUESTIONS (SCQ'S)

(SCQ-1) a (SCQ-2) a (SCQ-3) a (SCQ-4) b

(SCQ-5) a (SCQ-6) b (SCQ-7) c (SCQ-8) c

(SCQ-9) envelope

(SCQ-10) x −axis (SCQ-11) enveloping curve

(SCQ-12) cylinder (SCQ-13) $\frac{2np}{c}$

(SCQ-14) $2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$

TERMINAL QUESTIONS (TQ'S)

(TQ-1) $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$
(TQ-2) $45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0$
(TQ-3) $5x^2 + 5y^2 + 8z^2 + 4yz + 4zx - 8xy - 6x - 42y - 96z + 225 = 0$
(TQ-4) $x^2 + y^2 + z^2 - yz - zx - xy - 1 = 0$
(TQ – 6) $(1, -2, -2)$

UNIT-11: THE CENTRAL CONICOIDS

CONTENTS

11.1 INTRODUCTION

The locus of s general equation of second degree, viz
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ ....(1)
is called a quadratic surface or a conicoid.

Apparently, the general equation of the second degree contains ten constants, but only nine are effective; for the whole equation may be divided by any one of the constants, leaving nine ratios of the ten constants a,b,c etc. to one another .thus a conicoid can be determined to satisfy nine conditions each of which gives rise to one relation between the constants, e.g. a conicoid can be determined so as to pass through nine given points no four of which are coplanar.

In reality, it is not easy to determine the characteristic to the general equation (1). Therefore, a device of suitable transformations of the coordinate axes is adopted to reduce the general equation of the second degree to a number of standard forms. Some of the standard forms are given below:

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of one sheet

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of two sheet

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c^2}$ Elliptic paraboloid

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c^2}$ Hyperbolic paraboloid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ Imaginary cone

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Cone

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Elliptic cylinder

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Hyperbolic cylinder

The equation representing cones and cylinders have already been described in the preceding chapters.

In this chapter we shall discuss the nature and some of the important geometrical properties of the surfaces represents by the equations(A) to (I).

11.2 OBJECTIVES

Since for the learners of any branch of mathematics knowledge of central conicoid is essential. The object of this unit is to be help the student to develop the basic concept on geometry. The main objectives of this unit is to give student a good foundation of central conicoid of a unit student might be taking in depth later on either in central conicoid of mathematics.

11.3 THE ELLIPSOID

The locus of the equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ……………………………………………………………(1)

is called an ellipsoid.

Now let us discuss the nature and shape of the surface

(i) **The center**. If (α, β, γ) be the coordinates of any point on (1),then

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1$$

or

$$\frac{(-\alpha)^2}{a^2} + \frac{(-\beta)^2}{b^2} + \frac{(-\gamma)^2}{c^2} = 1$$

This shows that the point (-α,-β,-γ) also lies on (1). But these points are on a straight line through the origin and are equidistant from the origin (as origin is the middle point of the line joining these points). Hence the origin bisects every chord which passes through it and is, on this account, called the center of the surface. Thus, the origin is the center of ellipsoid (1).

(ii) **Symmetry.**If thepointwith coordinates (α,β,γ) lies on (1), then so does also the point (α,β,-γ). The middle point of the line joining these point (α,β,0). This point lies on the XOY plane (i.e z=0 plane) and also the line is perpendicular to this plane. sects Hence the XOY plane bisects every chord perpendicular to it and therefore the surface is symmetrical with respect to this plane.

Similarly, it can be shown that the surface is symmetrical with respect to the YOZ and the ZOX planes.

These three planes are called the principal planes of the ellipsoid. The three lines of intersection of the three principal planes, taken in pairs, are called the principal axes of the ellipsoid. In this case, the coordinate axes are the principal axes.

(iii) Surface is closed. Leta, b, c in(1) be in the descending order of magnitude $(a > b > c)$, so that(α, β, γ) being any point on the surface, we see that

$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \leq 1$ **and** $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \geq 1$

This shows that no point on the surface is at a distance greater than a, or less than c from the origin. The surface is therefore limited in every direction. Also the x-coordinate of any point on the surface cannot numerically exceed a,for otherwise the first term alone in (1) would be greater than 1 and thus either y or z will be imaginary. Similarly, the y- and z-coordinates of any point cannot numerically exceed b and c respectively. Therefore, the surface is bounded by the planes $x = \pm a, y = \pm b, z = \pm c$i.e. the surface is closed.

(iv) Section of the surface. The section of the surface (1) by the plane $z = k$ (parallel to XOY plane) is given by the equations

$$z = k, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$$

This is an ellipse with its semi-axes

$$\sqrt[a]{1 - \frac{k^2}{c^2}}, \sqrt[b]{1 - \frac{k^2}{c^2}}$$

and center on the z-axis. This is a real ellipse if $k^2 < c^2$, a point ellipse if $k^2 = c^2$ and an imaginary ellipse if $k^2 > c^2$. The surface is therefore generated by a variable ellipse whose plane is parallel to the plane $z = 0$ and whose center lies on z-axis. This ellipse diminishes in size as k varies from 0 to c or from 0 to -c.

Similarly, it can be shown that the sections by the planes parallel to other coordinate planes are also ellipses and the ellipsoid is generated by them.

(v) Length of the axes. The$x$ −axis ($y = 0, z = 0$) meets the surface in the two points $(a, 0, 0)$ and $(-a, 0, 0)$. Thus, the surface intercepts a length $2a$ on the x −axis. Similarly, the length intercepted on $y-$ and $z-$axis are $2b$ and $2c$ respectively. The lengths, $2a, 2b, 2c$ intercepted on the principal axes are called the lengths of axes of the ellipsoid.

11.4 THE HYPERBOLOID OF ONE SHEET

The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (1)$$

is called a hyperboloid of one sheet.

For the nature and shape of the surface, similar to the case of ellipsoid, it can be easily shown that

(i) Origin is the centre of the surface represented by (2).

(ii) The surface is symmetrical with respect to each of the coordinate planes are the principal planes of the surface. The coordinate axes are its principal axes.

(iii) The surface is obviously unbounded, for each of the coordinates x, y, and z can taken any value provided only that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

is not less than unity.

(iv) The section of the surface by the plane $z = k$ (parallel to XOY plane) is given by the equations

$$z=k,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$$

The section is, therefore, an ellipse whose semi-axis are$a\sqrt{1+\frac{k^2}{c^2}},\ b\sqrt{1+\frac{k^2}{c^2}}$

and whose centre lies on z −axis and which increase in size as k increases. There is no limit to the increase of k. The surface may, therefore, be generated by the variable ellipse parallel to the

XOY plane which increases in size as it moves rather away from the coordinate planes i.e. as k varies from 0 to ∞ or from 0 to -∞.

But the section of the surface parallel to the other coordinate planes are not ellipses. Thus, the section y the plane $x = k$ (parallel to YOZ plane) are hyperbolas

$$x = k, \ \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}$$

and

$y = h, \ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{h^2}{b^2}$respectively

(v) The x-axis meets the surface in the two points $(a, 0,0)$ and $(-a, 0,0)$ and thus the length intercepted on x-axis is 2a. similarly, the length intercepted on $y -$ axis is $2b$, whereas the $z -$ axis does not meet the surface in real points.

11.5 THE HYPERBOLOID OF TWO SHEETS

The surface represented by

$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (3) is

called a hyperboloid of two sheets.

For the nature and shape of the surface, similar to the case of ellipsoid, it is easy to prove that

(i) Origin is the centre of the surface represented by (3)

(ii) The surface is symmetrical with respect to the coordinate planes. The coordinate planes are the principal plane; and coordinate axes the principal axes of the surface.

(iii) The surface is unbounded in all the coordinates x, y, z but the intersection with x- axis are real while those with y-axis and z-axis are imaginary.

(iv) The section of the surface by the plane x = k is given by the equation

$$x = k, \ \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1$$

The section is, therefore, an ellipse with semi-axes

$$b\sqrt{\frac{k^2}{a^2}-1}, \qquad c\sqrt{\frac{k^2}{a^2}-1}$$

It is real ellipse if $k^2 > a^2$; a point ellipse if $k^2 = a^2$ and an imaginary ellipse if $k^2 < a^2$. Thus, there is no portion of the surface included between the plane x = -a and x = a; for $k^2 > a^2$, the section is a real ellipse and it increase in size as k increase.

The section by the planes y = k and z = k are the hyperbolas

$$y = k, \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}$$

and

$$z = k, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

respectively. These hyperbolas extend to infinity as k varies from 0 to ∞ of from 0 to -∞. Thus, the surface may also be considered as generated by variable hyperbolas parallel to ZOX or XOY planes and is consequently called a hyperboloid of two sheets.

(v) The x- axis meets the surface in the two points (a,0,0) and (-a,0,0) and therefore the length intercepted on x- axis is 2a, whereas the y-and z- axis meet the surface in imaginary points.

11.6 TANGENT LINES AND TANGENT PLANES

A straight line which intersects a central conicoid in two coincident points is called a tangent line to the conicoid at that point.

The locus of tangent lines at a point on the conicoid is called the tangent plane to the conicoid at that point.

To find the equation of the tangent plane at the point (α, β, γ) of the central conicoid

$$ax^2 + by^2 + cz^2 = 1 \qquad (1)$$

Equation of any line through $(\alpha, \beta, \gamma)$is

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r\ (say)$ (2)

Any point on the line is

$$(\alpha + lr, \beta + mr, \gamma + nr)$$

If it lies on (1), then

$$a\ (\alpha + lr)^2 + b\ (\beta + mr)^2 + c\ (\gamma + nr)^2 = 1$$

or

$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$ (3)

But $(\alpha, \beta, \gamma)$is a point on the conicoid (1)

$\therefore a\alpha^2 + b\beta^2 + c\gamma^2 = 1$ (4)

$\therefore$ Equation (3) becomes

$$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) = 0$$

Clearly one value of r is zero so that one of the points of intersection of (1) and (2) coincides with (α, β, γ). The line (2) will be a tangent line to (1) if the other value of r is also zero, for which the condition is

$(al\alpha + bm\beta + cn\gamma) = 0$ (5)

Thus (2) will be a tangent line to the conicoid (1) at (α, β, γ) obtained by eliminating l, m, n between (2) and (5), is

$$a\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma) = 0$$

or

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

or

$$a\alpha x + b\beta y + c\gamma z = 1$$

which is a plane

Hence the tangent lines at (α, β, γ) lies in the plane

$$a\alpha x + b\beta y + c\gamma z = 1$$

which is therefore, the tangent plane at (α, β, γ) to the conicoid (1).

Ex. 1. Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x', y', z').

Sol. The equation of given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$……………………….(1)

As we know that, the equation of tangent plane to the central conicoid $ax^2 + by^2 + cz^2 = 1$ at the point (x_1, y_1, z_1) is $ax_1x + by_1y + cz_1z = 1$

Now campairing given equation (1) with $ax^2 + by^2 + cz^2 = 1$, we get

$$a = \frac{1}{a^2}, b = \frac{1}{b^2}, c = \frac{1}{c^2}$$

Hence, equation of the tangent plane to the ellipsoid (1) at the point (x', y', z').

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} + \frac{z'z}{c^2} = 1$$

11.7 CONDITION OF TANGENCY

To find the condition that the plane $lx + my + nz = p$ should touch the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

Let $lx + my + nz = p$ ……………………………………………………………..(1)

touch the conicoid

$ax^2 + by^2 + cz^2 = 1$……………………………………………………………….(2)

at the point (α, β, γ).

But the equation of the tangent plane to the conicoid at (α, β, γ) is

$a\alpha x + b\beta y + c\gamma z = 1$ ……………………………………………………………...(3)

Consequently, (1) and (3) must represent the same plane comparing the coefficient of like power, we have

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p}$$

$\alpha = \frac{l}{ap},\ \beta = \frac{m}{bp},\ \gamma = \frac{n}{cp}$ ……………………………………………………………….(4)

But (α, β, γ) lies on the conicoid (2)

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

$\Rightarrow a\left(\frac{l^2}{a^2p^2}\right) + b\left(\frac{m^2}{b^2p^2}\right) + c\left(\frac{n^2}{c^2p^2}\right) = 1$

$\Rightarrow \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

This is the required condition of tangency.

Also, from (4) the coordinates of the point of contact are $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

Ex. 2. Find the condition that the plane $lx + my + nz = p$ may touch the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol. The equation of given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$……………………….(1)

and the equation of plane is $lx + my + nz = p$................................(2)

As we know thatthe plane $lx + my + nz = p$ should touch the conicoid

$ax^2 + by^2 + cz^2 = 1$ if $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

Now campairing given equation (1) with $ax^2 + by^2 + cz^2 = 1$, we get

$$a = \frac{1}{a^2}, b = \frac{1}{b^2}, c = \frac{1}{c^2}$$

Hence, the condition that the plane (2) may touch the ellipsoid (3) is

$$\frac{l^2}{\frac{1}{a^2}} + \frac{m^2}{\frac{1}{b^2}} + \frac{n^2}{\frac{1}{c^2}} = p^2 \Rightarrow a^2l^2 + b^2m^2 + c^2n^2 = 1$$

Ex. 3. A tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the coordinate axes in A, B and C. Prove that the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 16$$

Sol. The equation of given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$...........................(1)

The equation of any tangent plane to the ellipsoid (1) is

$lx + my + nz = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$................................(2)

the plane (2)meets the co-ordinate axes in the points given by

$$A\left(\frac{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{l}, 0,0\right);\ B\left(0, \frac{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{m}, 0\right); C\left(0,0, \frac{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}{n}\right)$$

Let (x_1, y_1, z_1) be the co-ordinates of the centroid of the tetrahedron $OABC$, then

$$x_1 = \frac{\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{l}+0+0+0}{4} = \frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{4l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$y_1 = \frac{\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{m}+0+0+0}{4} = \frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{4m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$z_1 = \frac{\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{n}+0+0+0}{4} = \frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{4n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$4lx_1 = \sqrt{a^2l^2+b^2m^2+c^2n^2}$$

$$\Rightarrow 16l^2{x_1}^2 = a^2l^2+b^2m^2+c^2n^2$$

Therefore, $\frac{a^2}{{x_1}^2} = \frac{16l^2a^2}{16l^2{x_1}^2} = \frac{16l^2a^2}{a^2l^2+b^2m^2+c^2n^2}$ ……………………………………(6)

Similarly,

$$\frac{b^2}{{x_2}^2} = \frac{16m^2b^2}{16\ {}^2{x_2}^2} = \frac{16m^2b^2}{a^2l^2+b^2m^2+c^2n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$\frac{c^2}{{x_3}^2} = \frac{16n^2c^2}{16n^2{x_3}^2} = \frac{16n^2c^2}{a^2l^2+b^2m^2+c^2n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Adding (6), (7) and (8), we get

$$\frac{a^2}{{x_1}^2}+\frac{b^2}{{x_2}^2}+\frac{c^2}{{x_3}^2} = \frac{16l^2a^2+16m^2b^2+16n^2c^2}{a^2l^2+b^2m^2+c^2n^2}$$

$$\Rightarrow \frac{a^2}{{x_1}^2}+\frac{b^2}{{x_2}^2}+\frac{c^2}{{x_3}^2} = \frac{16(l^2a^2+m^2b^2+n^2c^2)}{a^2l^2+b^2m^2+c^2n^2}$$

$$\Rightarrow \frac{a^2}{{x_1}^2}+\frac{b^2}{{x_2}^2}+\frac{c^2}{{x_3}^2} = 16$$

11.8 DIRECTOR SPHERE

The locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid is a sphere, concentric with the conicoid, called the director sphere of the conicoid.

To find the locus of the point of the intersection of three mutually perpendicular tangent planes to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad (1)$$

Let the three mutually perpendicular tangent planes to (1) be

$$l_1x + m_1y + n_1z = \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}} \quad (2)$$

$$l_2x + m_2y + n_2z = \sqrt{\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}} \quad (3)$$

$$l_1x + m_1y + n_1z = \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}} \quad (4)$$

so that

$$\left.\begin{array}{c} l_1^2 + l_2^2 + l_3^2 = 1 \\ l_1l_2 + m_1m_2 + n_1n_2 = 0 \\ l_1^2 + m_1^2 + n_1^2 = 1 \\ l_1m_1 + l_2m_2 + l_3m_3 = 0 \end{array}\right\} \quad (5)$$

$l_1, m_1, n_1; l_2, m_2, n_2;\ l_3, m_3, n_3$ being the actual direction cosines of the normal to the three tangent planes (2), (3), (4) respectively.

The locus of the point of intersection of (2), (3) and (4) is obtained by eliminating $l_1, m_1, n_1; l_2, m_2, n_2;\ l_3, m_3, n_3$between them. This is easily done by squaring and adding (2), (3) and using (5), i.e.

$x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)+2yz(m_1n_1+m_2n_2+m_3n_3)+2zx(n_1l_1+n_2l_2+n_3l_3)+2xy(l_1m_1+l_2m_2+l_3m_3)=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^3)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$.

or

$$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

This is the required equation of locus, i.e., the direct sphere.

Ex. 4. Find the equation of director sphere of the ellipsoid$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol. The equation of given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$……………………….(1)

As we know thatthe equation of director sphere of conicoid $ax^2+by^2+cz^2=1$ is $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Now campairing given equation (1) with $ax^2+by^2+cz^2=1$, we get

$$a=\frac{1}{a^2},b=\frac{1}{b^2},c=\frac{1}{c^2}$$

Hence, the equation of director sphere of the ellipsoid(1) is

$$x^2+y^2+z^2=\frac{1}{\frac{1}{a^2}}+\frac{1}{\frac{1}{b^2}}+\frac{1}{\frac{1}{c^2}}\Rightarrow x^2+y^2+z^2=a^2+b^2+c^2$$

SELF CHECK QUESTIONS

(SCQ-1) The surface represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

(a) Hyperboloid of one sheet (b) sphere

(c) Hyperboloid of two sheet (d)Ellipsoid

(SCQ-2) The surface represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is

(a) Hyperboloid of one sheet (b) sphere

(c) Hyperboloid of two sheet (d)Ellipsoid

(SCQ-3) The equation of director sphere of the central conicoid is $ax^2+by^2+cz^2=1$

(a) $x^2+y^2-z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

(b) $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$

(c) $x^2+y^2-z^2=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$

(d) $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

(SCQ-4) The centre of the central conicoid $ax^2+by^2+cz^2=1$ is at the

(a) origin (b) $(a,b,0)$

(c) (a,b,c) (d) none of these

11.9 POLAR PLANE OF A POINT

If any secant APQ through a given point A meets a conicoid in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q (i.e. AP, AR and AQ are in harmonic progression) is called the polar plane of A with respect to the conicoid.

$$AR=\frac{2AP.AQ}{AP+AQ}$$

11.10 EQUATION OF A POLAR PLANE

To find the equation of the polar plane of a given point $A\ (\alpha,\beta,\gamma)$ with respect to the conicoid $ax^2+by^2+cz^2=1$

Let the equation of any secant APQ through the point $A\ (\alpha,\beta,\gamma)$ be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,(say) \qquad (1)$$

It meets the given conicoid in the points P and Q, where the measures of AP and AQ are r_1 and r_2 respectively.

Therefore, from section 5.4, r_1 and r_2 are the roots of the quadratic equation

$$r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \quad (2)$$

Let δ be the measure of AR. Then, since AP, AR, AQ are in harmonic progression

$$AR=\frac{2AP.AQ}{AP+AQ}$$

i.e. $\delta=\frac{2r_1r_2}{r_1+r_2}=\frac{2(a\alpha^2+b\beta^2+c\gamma^2-1)}{-2(al\alpha+bm\beta+cn\ \ )}$

$$al\alpha\delta+bm\beta\delta+cn\gamma\delta=-(a\alpha^2+b\beta^2+c\gamma^2-1) \quad (3)$$

Let the coordinates of R be (x', y', z'), then from (1)

$$x'-\alpha=\delta l, y'-\beta=\delta m, z'-\gamma=\delta n$$

Therefore, from (3), we have

$$a\alpha(x'-\alpha)+b\beta(y'-\beta)+c\gamma(z'-\gamma)=-(a\alpha^2+b\beta^2+c\gamma^2-1)$$

$$\Rightarrow a\alpha x'+b\beta y'+c\gamma z'=1$$

Hence the locus of (x', y', z') is the plane given by

$$a\alpha x+b\beta y+c\gamma z=1$$

which is the required equation of polar plane of (α,β,γ) with respect to the given conicoid $ax^2+by^2+cz^2=1$.The point A (α,β,γ) is called the pole of the polar plane.

11.11 COORDINATES OF THE POLE

The pole of the plane

$$lx + my + nz = p$$

With respect to the same conicoid is obtained on comparing it with

$$a\alpha x + b\beta y + c\gamma z = 1$$

i.e. this gives

$$\alpha = \frac{l}{ap}, \beta = \frac{m}{bp}, \gamma = \frac{n}{cp}$$

Hence the coordinates of the pole are

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

Ex. 5. Prove that the pole of the plane $4x + 8y - 3z = 15$ with respect to conicoid $3x^2 + 7y^2 + 2z^2 = 12$ is $\left(\frac{16}{15}, \frac{32}{35}, \frac{-6}{5}\right)$.

Sol. The equation of given conicoid is $3x^2 + 7y^2 + 2z^2 = 12$ ....………(1)

The equation of plane is $4x + 8y - 3z = 15$ ....(2)

Let $P(x', y', z')$ be the pole of the plane (2) w.r.t conicoid (1).

Polar plane of $P(x', y', z')$ w.r.t (1) is $\frac{3}{12}x'x + \frac{7}{12}y'y + \frac{2}{12}z'z = 1$……………(3)

Therefore, the equations (2) and (3) represent same plane and so compairing their coefficients, we get

$$\frac{\frac{3}{12}x'}{4} = \frac{\frac{7}{12}y'}{8} = \frac{\frac{2}{12}z'}{-3} = \frac{1}{15}$$

$$\Rightarrow \frac{x'}{16} = \frac{7y'}{96} = \frac{z'}{-18} = \frac{1}{15}$$

$$\Rightarrow x' = \frac{16}{15}, y' = \frac{96}{7 \times 15}, z' = \frac{-18}{15} \Rightarrow x' = \frac{16}{15}, y' = \frac{32}{35}, z' = \frac{-6}{5}$$

Therefore, the pole of the plane $4x + 8y - 3z = 15$ with respect to conicoid $3x^2 + 7y^2 + 2z^2 = 12$ is $\left(\frac{16}{15}, \frac{32}{35}, \frac{-6}{5}\right)$

11.12 CONJUGATE POINTS AND CONJUGATE PLANES

If we interchange α, β, γ with x', y', z' respectively, the equation of a polar plane is unaltered, which shows that if the polar plane of (α, β, γ) passes through (x', y', z') then the polar plane of (x', y', z') passes through (α, β, γ). Two such points are called **conjugate points**.

Similarly, it will be seen that if the pole of a plane P_1 lies on another plane P_2, then the pole of P_2 lies on P_1. Two such planes are called **conjugate planes**.

11.13 POLAR LINES

Two lines which are such that the polar plane of any point on either line with respect to a conicoid passes through the other line are called the **polar lines** with respect to the conicoid.

11.14 EQUATION OF POLARLINE

To find the equation of the polar line of the given line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (1)$$

with respect to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \qquad (2)$$

The polar plane of any point $(\alpha + lr, \beta + mr, \gamma + nr)$ on the given line is

$$a(\alpha + lr)x + b(\beta + mr)y + c(\gamma + nr)z = 1$$

or

$$(a\alpha x + b\beta y + c\gamma z - 1) + r(alx + bmy + cnz) = 0$$

This plane for all values of r, passes through the line given by the intersection of the planes

$$a\alpha x + b\beta y + c\gamma z - 1 = 0$$

$$alx + bmy + cnz = 0$$

These equations determine the polar line of the given line.

Hence the equations of the polar lines of (1) with respect to the conicoids (2) are

$$a\alpha x + b\beta y + c\gamma z - 1 = 0$$

$alx + bmy + cnz = 0.$

SELF CHECK QUESTIONS

(SCQ-5) Theequation of the polar plane of a given point $A\,(\alpha, \beta, \gamma)$ with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$

(a) $a\alpha x + b\beta y + c\gamma z = -1$

(b) $a\alpha x + b\beta y - c\gamma z = 1$

(c) $a\alpha x + b\beta y + c\gamma z = 1$

(d) $a\alpha x - b\beta y - c\gamma z = 1$

(SCQ-6) Two lines, which are such that the polar plane of any point on either line with respect to a conicoid passes through the other line, are called

(a) Parallel line (b) Tangent line

(c) Line (d) Polar line

(SCQ-7) The pole of the plane $lx + my + nz = p$ with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

(a) $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$ (b) $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{-n}{cp}\right)$

(c) $\left(\frac{l}{ap}, \frac{-m}{bp}, \frac{-n}{cp}\right)$ (d) $\left(\frac{-l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

FILL IN THE BLANKS

(SCQ-8) The central conicoid $ax^2 + by^2 + cz^2 = 1$ is an ellipsoid if the constants a, b, c are all...........................

(SCQ-9) The centre of the director sphere of the central conicoid $3x^2 + 4y^2 - 5z^2 = 1$ is the point..............................

(SCQ-10) If the pole of a plane P_1 lies on another plane P_2, then the pole of P_2 lies on P_1. Two such planes are called.............................

(SCQ-11) The plane $lx + my + nz = p$ touches the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if............

11.15 SUMMARY

In this unit, we have learned about central conicoid.We find the general equation of a tangent plane, director sphere. We also find the equation of polar plain and conjugate planes.. We have find the condition of tangency. To make the concepts more clear, many solved examples are given in the unit after clearing the selected articles or topics. To check your progress, self check questions (SCQ's) and (TQ's) are given place to place.

11.16 GLOSSARY

1. Conicoid-a surface of second degree
2. Ellipsoid- a surface that may be obtained from a sphere by deforming it
3. Tangent- straight line that touches curved surface at a point
4. Normal- a straight line which is perpendicular to the tangent.

11.17 REFERENCES

1. Volumetric Analysis- Kumar Mittal, Nageen Prakashan Pvt. Ltd., Meerut.
2. 3 D Geometry,-R.K. Pandey, Krishna Prakashan media (P) Ltd. , Meerut.
3. Leonard E. Dickson :The First course in the Geometry.
4. Burnside,William Show, Panton &Arthur William :Volumetric Theory.
5. Murray R. Spiege l:Vector Analysis, Schaum's Outline Series ,McGraw Hill.
6. P.K. Jain and Khalil Ahmed :A textbook of Analytical Geometry, Wiley Eastern publication, New Age.

11.18 SUGGESTED READINGS

5. Fundamentals of Solid geometry - Jearl walker, John wiley , Hardy Robert and Sons.
6. Analytical geometry –Nazrul Islam , Tata McGraw Hill .z
7. Engineering Physics- S.K. Gupta, Krishna Prakashan Media (P) Ltd., Meerut
8. Volumetric Analysis- M.D. Rai Singhania, S. Chand Publication, New Delhi.

11.19 TERMINAL QUESTIONS

(TQ-1) Find the equation of the tangent plane to the ellipsoid at the points (5,-2,3)

(TQ-2) Find the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoids.

(TQ-3) Show that $(-\alpha\,, -\beta, -\gamma)$ is the only centre of the sphere$x^2+y^2+z^2+2\alpha x+2\beta y+2\gamma z+d=0$.

(TQ-4) Prove that the pole of the plane $lx+my+nz=p$ with respect to conicoid $ax^2+by^2+cz^2=1$ is $\left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$.

11.20 ANSWERS

SELF CHECK QUESTIONS (SCQ'S)

(SCQ-1) d (SCQ-2) c (SCQ-3) d (SCQ-4) a

(SCQ-5) c (SCQ-6) d (SCQ-7) a

(SCQ-8) positive

(SCQ-9) (0,0,0)

(SCQ-10) conjugate planes

(SCQ-11) $a^2l^2 + b^2m^2 + c^2n^2 = p^2$

TERMINAL QUESTIONS (TQ'S)

(TQ-1)

(TQ-2)

(TQ-3) … … … … … … … … …

(TQ-4) … … … … … … … … … … ….

(TQ – 5) ------------

(TQ – 6) $(1, -2, -2)$

(TQ – 7) ……………

BLOCK V

POLAR FORM OF CONICS

UNIT 12 TRACING OF CONICS I

CONTENTS

12.1 INTRODUCTION

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic sectionthrough the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture.

In previous unit we analyzed equation of a polar plane, conjugate points and conjugate planes, equation of polar line. In this unit we discuss about conic section, centre of Conic, nature of conic, eccentricity, foci and equation of directrix.

12.2 OBJECTIVES

After reading this unit learners will be able to

1. analyze conics
2. analyze and find centre of Conic
3. constructand find asymptote
4. find nature of Conic
5. find Length and equations of the axes of central conic
6. findeccentricity, foci and Equation of directrix

12.3 CONIC SECTION

Conic sections are one of the important topics in Geometry. A conic section (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. A cone has two identically shaped parts called nappes. One nappe is what most people mean by "cone," and has the shape of a party hat.

A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown in Figure 12.1.

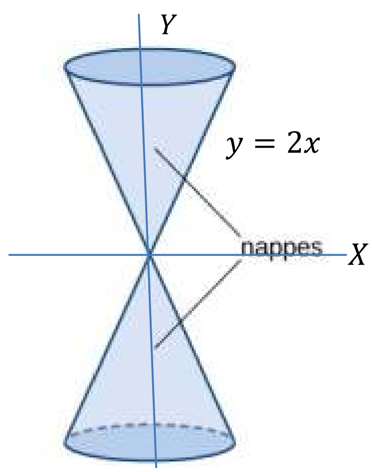


Fig 12.3.1. A cone generated by revolving the line $y = 3x$ around the y −axis.

Conic sections can be generated by intersecting a plane with a cone. The three types of conic sections are

1. **Hyperbola:** If the plane is parallel to the axis of revolution (the y −axis) then the conic section is a hyperbola. In simple way, to generate a hyperbola the plane intersects both pieces of the cone. For this, the slope of the intersecting plane should be greater than that of the cone.

Fig 12.3.2 Hyperbola

2. **Parabola:** If the plane is parallel to the generating line, the conic section is a parabola. We can say that to generate a parabola, the intersecting plane must be parallel to one side of the cone and it should intersect one piece of the double cone.

Fig 12.3.3 Parabola

3. **Ellipse:** If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse. In other words, if the plane intersects one of the pieces of the cone and its axis but is not perpendicular to the axis, the intersection will be an ellipse.

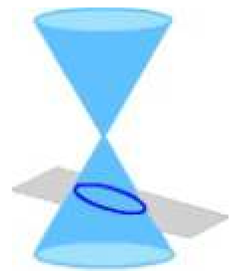


Fig 12.3.4 Ellipse

NOTE: 1. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.If the plane is perpendicular to the axis of revolution, the conic section is a circle.
2. A central conic is **a conic section which has a center of symmetry**. Hence a central conic is a conic section which is a circle, an ellipse or a hyperbola.

Analytically, a conic section or conic is defined as follows:

A conic section is the locus of a point which moves so that its distance from a fixed point (called **focus**) is in a constant ratio to its perpendicular distance from a fixed straight line (called the **directrix**).

The constant ratio is called eccentricity and is denoted by e.

NOTE: If the focus does not lie upon the directrix, the conic is ellipse if $e < 1$, parabola if $e = 1$ and hyperbola if $e > 1$. The circle is a special case of an ellipse when $e = 0$.

Theorem 1. Every conic section is represented by an equation of the second degree in x and y.

Proof.

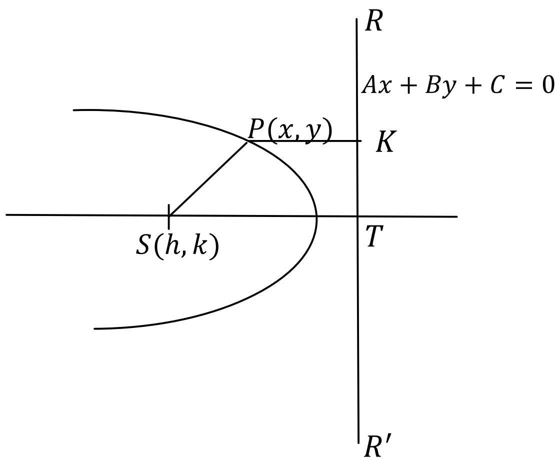


Fig 12.5

In Fig 12.5,let e be the eccentricity, (h, k) be the coordinates of focus S and $Ax + By + C = 0$ be the equation of directrix of conic.

Now, Let $P(x, y)$ be any point on the conic, then by the definition of eccentricity, we have

$$e = \frac{PS}{PK} \Rightarrow PS = e\,PK \Rightarrow \sqrt{(x-h)^2 + (y-k)^2} = e\frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

Now by squaring both side, we get

$$(x-h)^2 + (y-k)^2 = e\frac{(Ax + By + C)^2}{A^2 + B^2}$$

Clearly we can see that the given equation is an equation of second degree of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Theorem 2 The second degree in x and y always represents a general equation of conic.

Proof. Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$…………………(1)

represents any general equation of second degree in x and y.

For removing the term of xy, we turn the coordinate axes through an angle θ, the origin remaining the same.

Thus replacing x by $x\cos\theta - y\sin\theta$ and y by $x\sin\theta + y\cos\theta$ in equation (1), we get

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0$$

$\Rightarrow \quad a(x^2\cos^2\theta + y^2\sin^2\theta - 2xy\cos\theta\sin\theta) + 2h(x^2\cos\theta\sin\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\cos\theta\sin\theta) + b(x^2\sin^2\theta + y^2\cos^2\theta + 2xy\cos\theta\sin\theta) + 2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0$

$\Rightarrow \quad x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2xy(-a\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta) + b\cos\theta\sin\theta) + y^2(a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta) + 2x(g\cos\theta + f\sin\theta) + 2y(-g\sin\theta + f\cos\theta) + c = 0$

$\Rightarrow x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2xy\big((b-a)\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta)\big) + y^2(a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta) + 2x(g\cos\theta + f\sin\theta) + 2y(f\cos\theta - -g\sin\theta) + c = 0$……………(2)

Now from equation (2), we choose θ so that the coefficient of xy becomes 0. i.e.

$$(b-a)\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \frac{(b-a)}{2}\sin 2\theta + h\cos 2\theta = 0$$

$$\Rightarrow \frac{(b-a)}{2}\sin 2\theta + h\cos 2\theta = 0 \Rightarrow (a-b)\sin 2\theta = 2h\cos 2\theta$$

$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$ ……..(3)

Here we can observe that the relation (3) always gives real values of $\theta a, b$ and h.

If we substitute the values of $\cos\theta$ and $\sin\theta$ from equation (3) in (2), then the equation (2) becomes

$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$....................(4)

Now the following cases arise

Case I: If $A \neq 0$ and $B \neq 0$

Then equation (4) can be written as

$$A\left(x^2 + \frac{2Gx}{A} + \frac{G^2}{A^2}\right) + B\left(y^2 + \frac{2Fy}{B} + \frac{F^2}{B^2}\right) - \frac{G^2}{A} - \frac{F^2}{B} + C = 0$$

$$\Rightarrow A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = \lambda$$

Now shifting the origin to $(-\frac{G}{A}, -\frac{F}{B})$, the given equation becomes

$Ax^2 + By^2 = \lambda$ ………………………………………………………(5)

Now the following subcases arise

(i) If $\lambda = 0$, the equation (5) becomes $Ax^2 + By^2 = 0$ which represents a pair of straight lines.

(ii) If $\lambda \neq 0$, the equation (5) can be written as

$\frac{x^2}{K/A} + \frac{y^2}{K/B} = 1$ ....……………………….(6)

a) If K/A and K/B are both positive then the equation (6) represents an ellipse.

b) If K/A and K/B are of opposite signs then the equation (6) represents a hyperbola.

c) If K/A and K/B are both negative then the equation (6) represents an imaginary ellipse.

Case II: If $A = 0$ or $B = 0$

Let $A = 0$ and $B \neq 0$, then the equation (4) becomes

$By^2 + 2Gx + 2Fy + C = 0 \Rightarrow y^2 + \frac{2F}{B}y = -\frac{2G}{B}x - \frac{C}{B}$

$\Rightarrow y^2 + \frac{2F}{B}y + \frac{F^2}{B^2} = -\frac{2G}{B}x - \frac{C}{B} + \frac{F^2}{B^2}$

$\Rightarrow \left(y + \frac{F}{B}\right)^2 = -\frac{2G}{B}x - \frac{C}{B} + \frac{F^2}{B^2}$ ……………………………………(7)

(i) If $G = 0$, the equation (7) represent two parallel straight lines, which are coincident if $F^2 - BC = 0$

(ii) If $G \neq 0$, the equation (7) can be written as

$$\Rightarrow \left(y + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x - \left(-\frac{C}{2G} + \frac{F^2}{2GB}\right)\right)$$

Shifting the origin to $\left(\frac{F^2}{2G} - \frac{C}{2G}, -\frac{F}{B}\right)$, then given equation can be written as

$\Rightarrow y^2 = -\frac{2G}{B}x$, which represents a parabola

If $A \neq 0$ and $B = 0$, the procedure and the result remain same.

Thus in every case the general equation of second degree represents a conic section.

CHECK YOUR PROGRESS

(CQ 1) Every conic section is represented by an equation of the second degree in x and y. (T/F)

(CQ 2) The second degree in x and y need not represents a general equation of conic. (T/F)

(CQ 3) If the focus lie upon the directrix, the conic is ellipse if $e > 1$ (T/F)

(CQ 4) The circle is a special case of an ellipse when________________.

12.4 CENTRE OF A CONIC

Centre: The centre of a conic section is a point such that all chords of conic which pass through it are bisected there

When the equation to the conic in the form $ax^2 + 2hxy + by^2 + c = 0$the origin at the centre.

Proof. Given equation of conic is

$ax^2 + 2hxy + by^2 + c = 0$…………………………(1)

Let$P(x_0, y_0)$ be any point on the conic (1). Then

$ax_0^2 + 2hx_0y_0 + by_0^2 + c = 0$ …………………………..(2)

Let$Q(-x_0, -y_0)$ be any point and equation (1) can be written as

$a(-x_0)^2 + 2h(-x_0)(-y_0) + b(-y_0)^2 + c = 0$, which implies that $Q(-x_0, -y_0)$ also lies on (1).

But the points $P(x_0, y_0)$ and $Q(-x_0, -y_0)$ lies on the same plane through the origin $O(0,0)$ and P and Q are at equal distances from the origin O.

Hence, the chord of the conic which passes through the origin O and any point P of the curve is bisected at the origin.

Therefore, the origin is the centre.

When the equation to the conic in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$the origin is centre iff $f = g = 0$ (first degree terms are absent from the equation of the conic.

Proof. Given equation of conic is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$………………..(1)

If the origin O be the centre, then corresponding to each point (x_0, y_0) on (1), there must be also a point $(-x_0, -y_0)$ lying on the curve.

Hence we must have

$ax_0^2 + 2hx_0y_0 + by_0^2 + 2gx_0 + 2fy_0 + c = 0$ ………..…..(2)

and $a(-x_0)^2 + 2h(-x_0)(-y_0) + b(-y_0)^2 + +2g(-x_0) + 2f(-y_0) + c = 0$

$ax_0^2 + 2hx_0y_0 + by_0^2 - 2gx_0 - 2fy_0 + c = 0$ ……..…..(3)

Subtracting (3) from (2), we get

$$4gx_0 + 4fy_0 = 0 \Rightarrow gx_0 + fy_0 = 0$$

Now given relation is to be true for all points (x_0, y_0) which lie on the curve (1).

However, it can only be the case, when $f = g = 0$.

Ex.1. Find the standard form of equation of a conic with centre at origin.

Sol. Let equation of conic with centre at the origin O be

$ax^2 + 2hxy + by^2 + c = 0$…………..(1)

Now, $ax^2 + 2hxy + by^2 = -c \Rightarrow \left(\frac{-a}{c}\right)x^2 + 2\left(-\frac{h}{c}\right)xy + \left(-\frac{b}{c}\right)y^2 = 1$

Let $\left(\frac{-a}{c}\right) = A, \left(-\frac{h}{c}\right) = H$ and $\left(-\frac{b}{c}\right) = B$

Hence, $Ax^2 + 2Hxy + By^2 = 1$ this is the standard form of all conics with centre at origin.

To find the coordinates of the centre of the conic given by the general equation and to find the equation to the curve referred to axes through the centre to the original axes.

Let equation of conic is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$……………………….…..(1)

Let (x_0, y_0) be the centre of given conic.

Now we transform the origin to the point (x_0, y_0) on replacing x by $x + x_0$ and y by $y + y_0$,.

The equation we get

$$a(x + x_0)^2 + 2h(x + x_0)(y + y_0) + b(y + y_0)^2 + 2g(x + x_0) + 2f(y + y_0) + c = 0$$

$$\Rightarrow ax^2 + 2axx_0 + ax_0^2 + 2hxy + 2hxy_0 + 2hx_0y + 2hx_0y_0 + by^2 + 2byy_0 + by_0^2 + 2gx + 2gx_0 + 2fy + 2fy_0 + c = 0$$

$\Rightarrow ax^2 + 2hxy + by^2 + 2(ax_0 + hy_0 + g)x + 2(hx_0 + by_0 + f)y + ax_0^2 + 2hx_0y_0 + by_0^2 + 2gx_0 + 2fy_0 + c = 0$ ………..(2)

If the point (x_0, y_0) be the centre of the conic section, the coefficients of x and y in the equation (2) must be zero, thus we have

$ax_0 + hy_0 + g = 0$……………………………………………...(3)

and

$hx_0 + by_0 + f = 0$ ……………………………………………...(4)

Now solving (3) and (4) for x_0 and y_0 , we get

$\frac{x_0}{hf-bg} = \frac{y_0}{hg-af} = \frac{1}{ab-h^2}$ ……………………………………….….(5)

Hence, centre (x_0, y_0) of the conic (1) is $\left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right)$

Therefore, value of the constant term (c') in (2)

$$c' = ax_0^2 + 2hx_0y_0 + by_0^2 + 2gx_0 + 2fy_0 + c$$

$$= x_0(ax_0 + hy_0 + g) + y_0(hx_0 + by_0 + f) + gx_0 + fy_0 + c$$

Using (3) and (4), we get

$c' = x_0.0 + y_0.0 + gx_0 + fy_0 + c = gx_0 + fy_0 + c$ ……………(6)

Now substituting the value of $x_0$andy_0, we get

$$c' = g\left(\frac{hf - bg}{ab - h^2}\right) + f\left(\frac{hg - af}{ab - h^2}\right) + c$$

$$= \frac{fgh - bg^2 + fgh - af^2 + abc - ch^2}{ab - h^2}$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{ab - h^2}$$

Here Δ is the discriminant of the given equation (1).

Therefore equation (2) can be written in the form

$x^2 + 2hxy + by^2 + \frac{\Delta}{ab-h^2} = 0$ ……………………………..(7)

This is the required equation to the curve referred to axes through the centre to the original axes.

Some important cases:

1. If$\Delta \neq 0$ and $ab - h^2 \neq 0$, i.e. $h^2 \neq ab$, then conic (1) has a unique centre. Hence conic (1) is either a pair of intersecting straight lines or circle or ellipse or hyperbola.
2. If$\Delta = 0$ and $ab - h^2 \neq 0$, the equation (7) becomes$x^2 + 2hxy + by^2 = 0$, which represents a pair of straight lines intersecting at the point (x_0, y_0).
3. If $ab - h^2 = 0$ and $hf - bg = 0$ or $gh - af = 0$, $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$, thus we conclude that after solving equation (3) and (4), we get infinite solutions. Therefore equation (1) represents pair of parallel straight line. Also both lines are parallel to $ax + hy + g = 0$.

4. If $ab - h^2 = 0$ and $hf - bg \neq 0$ and $gh - af = 0, \frac{a}{h} = \frac{h}{b} \neq \frac{g}{f}$, thus we conclude that after solving equation (3) and (4), we get no solutions. Therefore equation (1) represents a parabola which has no centre.

Note: The value of Δ can also be written as $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Example 2. Find the coordinate of the centre of the conic

$$\mathbf{2x^2 + 3xy + 4y^2 + 8x - 40y + 45 = 0}$$

Proof. Let

$F(x, y) \equiv 2x^2 + 3xy + 4y^2 + 8x - 40y + 45 = 0$............(1)

Let the centre be (x_0, y_0).

Then the centre is given by equations

$2x_0 + \frac{3}{2}y_0 + 4 = 0$ *i.e.* $4x_0 + 3y_0 + 8 = 0$ ....(2)

and

$\frac{3}{2}x_0 + 4y_0 - 20 = 0$ *i.e.* $3x_0 + 8y_0 - 40 = 0$ ....(3)

Solving (2) and (3), we get

$$\frac{x_0}{-120 - 64} = \frac{y_0}{24 + 160} = \frac{1}{32 - 9}$$

$$\Rightarrow x_0 = \frac{-184}{23} = -8, y_0 = \frac{184}{23} = 8$$

Hence, the centre of given conic is $(-8,8)$.

Example 3. Find the coordinate of the centre of the conic

$$\boldsymbol{x^2 + xy + 8y^2 - 4x - 7y + 15 = 0}$$

and hence reduced it to the standard form.

Proof. Let

$F(x, y) \equiv x^2 + xy + 2y^2 - 4x - 9y + 15 = 0$ ……………………..(1)

Let the centre be (x_0, y_0).

Then the centre is given by equations

$x_0 + \frac{1}{2}y - 2 = 0$ $i.e.$ $2x_0 + y_0 - 4 = 0$ ………………………..(2)

$\frac{1}{2}x_0 + 2y - \frac{9}{2} = 0$ $i.e.$ $x_0 + 4y_0 - 9 = 0$ ……………………....(3)

Solving (2) and (3), we get

$$\frac{x_0}{-9+16} = \frac{y_0}{-4+18} = \frac{1}{8-1}$$

$$\Rightarrow x_0 = \frac{7}{7} = 1, y_0 = \frac{14}{7} = 2$$

Hence, the centre(x_0, y_0) of given conic is$(1,2)$.

Now, $g = \frac{1}{2}$ of coefficient of x in (1)$= -2$ and

$f = \frac{1}{2}$ of coefficient of y in (1)$= -\frac{9}{2}$

and $c = 15$

Therefore the new constant term is

$$c' = gx_0 + fy_0 + c = -2(1) - \frac{9}{2}(2) + 15 = -2 - 9 + 15 = 4$$

Hence the equation of the given conic referred to the centre as origin is

$$x^2 + xy + 2y^2 + c_1 = 0$$

$$\Rightarrow x^2 + xy + 2y^2 + 4 = 0$$

$$\Rightarrow -\frac{1}{4}x^2 - \frac{1}{4}xy - \frac{1}{2}y^2 = 1$$

This is the standard form of conic.

CHECK YOUR PROGRESS

(CQ 5) If$\Delta = 0$ and $ab - h^2 \neq 0$ represents a pair of straight lines. (T/F)

(CQ 6) The __________ of a conic section is a point such that all chords of conic which pass through it are bisected there.

12.5 ASYMPTOTES

Asymptote: An asymptote is a straight line, which meets the conic in two points both of which are situated at an infinite distance, but which is itself not altogether at infinity. Hyperbolas are the only conic sections with asymptotes.

To find the equation of the asymptotes of the central conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ………………………..(1)

Proof. As we know that the equation of conic and its asymptote differ by constant term.

Hence, consider the equation of asymptote of conic (1) be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + K = 0$ ………(2)

where K is to be chosen in such a way so that equation (2) represent a pair of straight lines.

Now the condition for (2) represent a pair of straight lines be

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+K \end{vmatrix} = 0$$

$$\Rightarrow a(b(c+k) - f^2) - h(h(c+K) - fg) + g(fh - bg) = 0$$

$$\Rightarrow ab(c+K) - af^2 - h^2(c+K) + fgh + fgh - bg^2 = 0$$

$$\Rightarrow abc + abK - af^2 - ch^2 - Kh^2 + 2fgh - bg^2 = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 + K(ab - h^2) = 0$$

$$\Rightarrow \Delta + K(ab - h^2) = 0$$

$\Rightarrow K(ab - h^2) = -\Delta$ or $K = -\frac{\Delta}{(ab-h^2)}$

Now putting this value of K in (2), we get the equation of asymptote

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{(ab - h^2)} = 0$$

To find the equation of the hyperbola conjugate to the equation of the central conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ …………..(1)

Proof. As we know that the equation of conjugate hyperbola differs from the equation of asymptote by same constant term as the equation of asymptote differs from the equation of hyperbola.

As we earlier studied that the equation of asymptote is formed by adding a constant term $-\frac{\Delta}{(ab-h^2)}$ in equation of central conic (1).

Now by adding the same constant term in the equation of asymptote, we get the conjugate hyperbola

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{2\Delta}{(ab - h^2)} = 0$$

Ex.5. If (x_0, y_0) are the coordinates of the centre of hyperbola

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Then prove that the equation of asymptote is $F(x, y) = F(x_0, y_0)$ and the equation of the conjugate hyperbola is $F(x, y) = 2F(x_0, y_0)$.

Proof. As we know thatthe equation of conjugate hyperbola differs from the equation of asymptote by same constant term.

Now let the equation of hyperbola be $F(x, y) = 0$

Thenthe equation of asymptote of the hyperbola be

$F(x, y) + K = 0$ (K = constant)… ……………….(1)

Since the asymptote of a hyperbola passes through its centre, therefore the point (x_0, y_0) can satisfy the equation of hyperbola (1), so we have

$$F(x_0, y_0) + K = 0 \Rightarrow K = -F(x_0, y_0)$$

Now using this K in (1), we get

$F(x, y) - F(x_0, y_0) = 0 \Rightarrow F(x, y) = F(x_0, y_0)$ …………(2)

This is the equation ofasymptotes of given hyperbola.

Now the equation of conjugate hyperbola differs from equation of asymptotes by same constant as the equation of the asymptotes differs from the equation of the hyperbola.

So adding the constant term $-F(x_0, y_0)$ to the equation (2), we get the equation of hyperbola conjugate to the given hyperbola as

$$F(x,y) - 2F(x_0, y_0) = 0 \Rightarrow F(x,y) = 2F(x_0, y_0)$$

Ex, 6. Find the equation of the asymptotes of the conic

$$\boldsymbol{x^2 + xy + 8y^2 - 4x - 7y + 15 = 0}$$

Sol. As we know that the equation of the asymptotes differs from the equation of the conic only by a constant term.

So let the equation of the asymptotes be

$x^2 + xy + 8y^2 - 4x - 7y + 15 + K = 0$ ……………...(1)

Where K is a constant to be determined by the fact that (1) should represent a pair of straight lines.

Now we can compare equation (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Then we get $a = 1, h = \frac{1}{2}, b = 8, g = -\frac{7}{2}, f = -\frac{7}{2}, c = 15 + K$

The equation (1) will represent a pair of straight lines if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Therefore

$$1.8.(15 + K) + 2. -\frac{7}{2}.(-2).\frac{1}{2} - 1.\left(-\frac{7}{2}\right)^2 - 8.(-2)^2 - (15 + K)(\frac{1}{2})^2 = 0$$

$\Rightarrow 120 + 8K + 7 - \frac{49}{4} - 32 - \frac{15+K}{4} = 0$

$\Rightarrow 120 + 8K + 7 - \frac{49}{4} - 32 - \frac{15+K}{4} = 0$

$$\Rightarrow 480 + 32K + 28 - 49 - 128 - 15 - K = 0$$

$$\Rightarrow 316 + 31K = 0$$
$$\Rightarrow K = -\frac{31}{316}$$

Therefore equation of asymptote is

$$x^2 + xy + 8y^2 - 4x - 7y + 15 - \frac{31}{316} = 0$$
$$\Rightarrow 316x^2 + 316xy + 2528y^2 - 1264x - 2212y + 4709 = 0$$

12.6 NATURE OF CONIC

We can find the nature of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ from its second terms as shown below

(i) Let $\Delta \neq 0$, so that the equation of conic does not simply represent a pair of straight lines. The equation of straight lines passing through the origin and parallel to the asymptotes of the conic (1) is

$$ax^2 + 2hxy + by^2 = 0$$

(a) If $h^2 - ab > 0$, the straight lines (2) are real and so the asymptotes are real. Hence in this case the equation of conic is a hyperbola

(b) If $h^2 - ab = 0$, the second degree terms in equation of conic are in a perfect square and so the equation of conic is a parabola.

(c) If $h^2 - ab < 0$, the straight lines (2) are imaginary and so the asymptotes are imagianary. Hence in this case the equation of conic is an ellipse.

Result for the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Conditions	**Nature of Conics**
$h^2 - ab < 0$	Ellipse
$h^2 - ab = 0$	Parabola
$h^2 - ab > 0$	Hyperbola
$a = b$ and $h = 0$	Circle
$a + b = 0$	Rectangular hyperbola
$\Delta = 0$	Two straight line (real or imaginary)
$h^2 - ab = 0$ and $\Delta = 0$	Two parallel straight line

12.7 LENGTH AND EQUATIONS OF THE AXES OF A CENTRAL CONIC

Let the standard equation of central conic with centre at origin is given by

$Ax^2 + 2Hxy + By^2 = 1$ ………………….(1)

Let the given conic (1) be cut by a concentric circle of radius r whose equation is

$x^2 + y^2 = r^2$ ……………………………..….(2)

Let the circle intersect the conic at four points say $P_1, P_2,$ P_3 and P_4 as shown in fig 12.7.1(A) and 12.7.1(B).

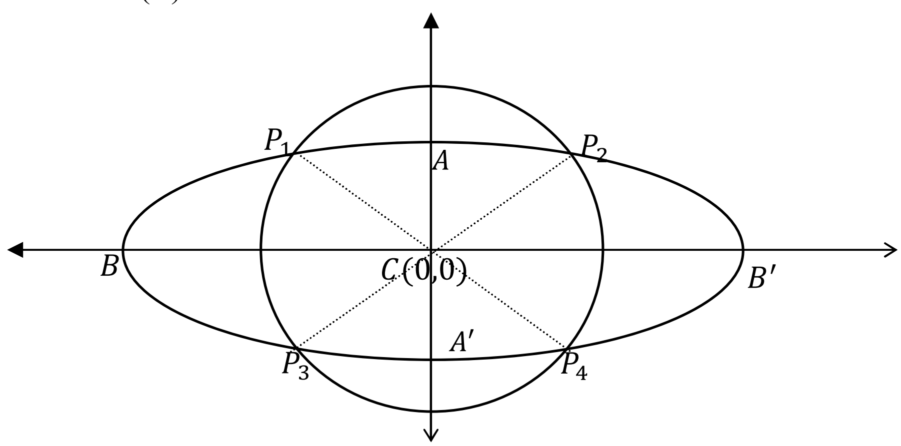


Fig. 12.7.1 (A)

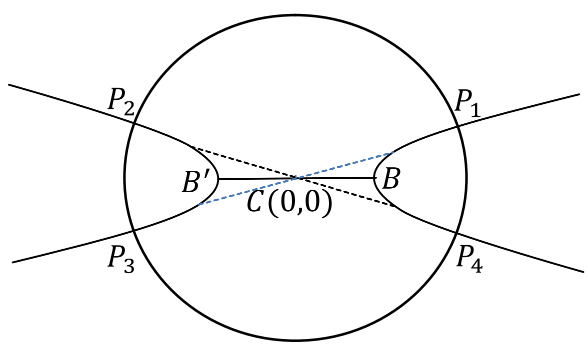


Fig. 12.7.1 (B)

Let P_1P_3 and P_2P_4 are the lines joining the point of intersection.

Now the combined equation of the lines P_1P_3 and P_2P_4 is

$Ax^2 + 2Hxy + By^2 = \frac{x^2+y^2}{r^2}$ (making equation (1) homogeneous with the help of equation (2))

$\Rightarrow (A - \frac{1}{r^2})x^2 + 2Hxy + (B - \frac{1}{r^2})y^2 = 0$.....................(3)

Now the two lines represented by equation (3) will coincide iff circle touches the conic at the extremities of either axis of conic as shown in fig 12.7.2 (A) and 12.7.2 (B)

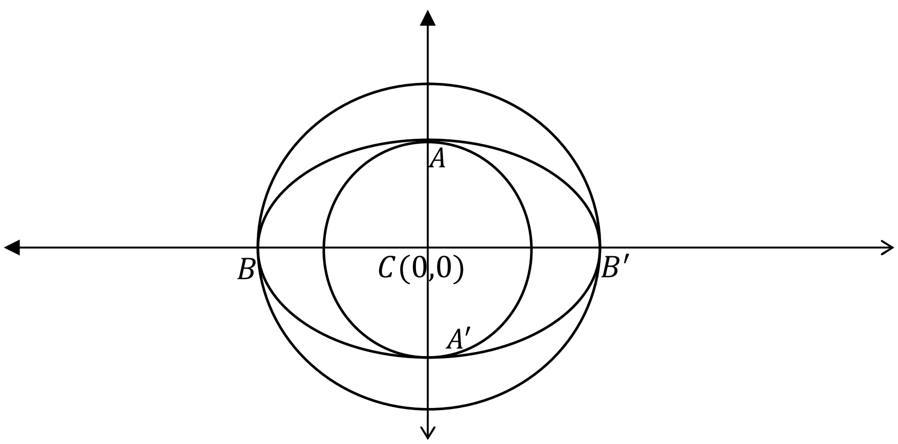


Fig. 12.7.2 (A)

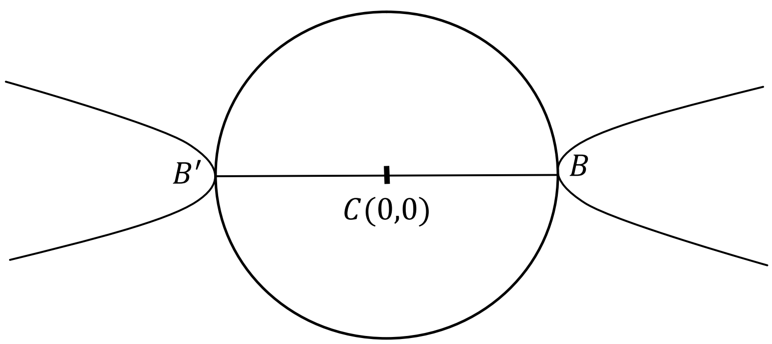


Fig. 12.7.2 (B)

Therefore, in such case the radius of the circle is equal to the length of either of the semi axes of the conic.

Now equation (3) will represent a pair of coincident straight lines if

$H^2 = \left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right)$ (Using the condition $h^2 = ab$)........(4)

$$\Rightarrow H^2 = AB - \frac{1}{r^2}A - \frac{1}{r^2}B + \frac{1}{r^4}$$

$\Rightarrow \frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$

$\Rightarrow (AB - H^2)r^4 - (A + B)r^2 + 1 = 0$........................,,,,,,,,,,.(5)

As we can see the given equation is quadratic in r^2.

Let r_1^2 and r_2^2 be the roots of givenequation (5), then

$$r_1^2 + r_2^2 = \frac{A + B}{AB - H^2}, \quad r_1^2 r_2^2 = \frac{1}{AB - H^2}$$

Some Important Cases

1. If $AB - H^2 > 0$, the conic is an **ellipse**. As we know in such case rots are positive. If r_1^2 and $r_2^2 > 0$. If $r_1^2 > r_2^2$, then length of semi major and semi minor axes are $2r_1$ and $2r_2$ respectively.
2. If $AB - H^2 < 0$, the conic is a **hyperbola**. As we know in such case one root is positive and another root is negative. If $r_1^2 > 0$ and $r_2^2 < 0$, then length of semi transverse axis and semi conjugate axes are $2r_1$ and $2\sqrt{|r_2^2|}$ respectively.

THE EQUATION OF AXES

As we know the axes of conic coincide with the coincident lines given by the equation (3).

Now multiplying equation (3) with $\left(A-\frac{1}{r^2}\right)$, we get

$\left(A-\frac{1}{r^2}\right)^2 x^2 + 2H\left(A-\frac{1}{r^2}\right)xy + \left(A-\frac{1}{r^2}\right)(B-\frac{1}{r^2})y^2 = 0$ ……….(5)

From equation (4), we have

$$H^2 = \left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)$$

Therefore equation (5) become

$$\left(A-\frac{1}{r^2}\right)^2 x^2 + 2H\left(A-\frac{1}{r^2}\right)xy + H^2y^2 = 0$$

$$\Rightarrow \left(\left(A-\frac{1}{r^2}\right)x\right)^2 + 2\left(\left(A-\frac{1}{r^2}\right)x\right)Hy + (Hy)^2 = 0$$

$$\Rightarrow \left(\left(A-\frac{1}{r^2}\right)x + Hy\right)^2 = 0$$

$\Rightarrow \left(A-\frac{1}{r^2}\right)x + Hy = 0$ …………………………..(6)

Let r_1^2 be the algebraically greater value of r^2. Now putting r_1^2 in place of r^2, we get

$\left(A-\frac{1}{r_1^2}\right)x + Hy = 0$ …………………………(7)

This is the equation of the major axis in the case of the conic being an ellipse or that of the transverse axis in the case of conic being a hyperbola.

Let r_2^2 be the algebraically smaller value of r^2. Now putting r_2^2 in place of r^2, we get

$\left(A-\frac{1}{r_2^2}\right)x + Hy = 0$ ………………………(8)

This is the equation of the minor axis in the case of the conic being an ellipse or that of the conjugate axis in the case of conic being a hyperbola.

Equation (7) and (8) are the equation of axes of the conic whose centre is at origin.

However, if the equation of given conic is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ………………(9)

Then shifting the origin to its centre (x_0, y_0), the equation takes standard form of conics and equation of the axes are

$\left(A - \frac{1}{r_1^2}\right)(x - x_1) + H(y - y_1) = 0$ and

$$\left(A - \frac{1}{r_2^2}\right)(x - x_1) + H(y - y_1) = 0$$

Ex. 8. Find the length and the equations of the axes of the conic

$$\mathbf{4x^2 - 3xy + 4y^2 - 19x + 14y - 4 = 0}$$

Proof. The equation of given conic is

Let

$F(x, y) \equiv 4x^2 - 3xy + 4y^2 - 19x + 14y - 4 = 0$ …….(1)

Let the centre be (x_0, y_0).

Then the centre is given by equations

$4x_0 - \frac{3}{2}y_0 - \frac{19}{2} = 0$ $i.e.$ $8x_0 - 3y_0 - 19 = 0$ ……………..(2)

$-\frac{3}{2}x_0 + 4y_0 + 7 = 0$ $i.e.$ $-3x_0 + 8y_0 + 14 = 0$…………..(3)

Solving (2) and (3), we get

$$\frac{x_0}{-42+152} = \frac{y_0}{57-112} = \frac{1}{64-9}$$

$$\Rightarrow x_0 = \frac{110}{55} = 2, y_0 = \frac{-55}{55} = -1$$

Hence, the centre(x_0, y_0) of given conic is$(2, -1)$.

Now the new constant term is

$$c' = gx_1 + fy_1 + c = -\frac{19}{2}(2) + 7.(-1) - 4 = -19 - 11 = -30$$

Therefore, the equation of the conic referred to the centre as origin is

$$second\ degree\ terms\ of\ the\ given\ conic + new\ constant\ c' = 0$$

$$4x^2 - 3xy + 4y^2 - 30 = 0$$

$$\Rightarrow \frac{4}{30}x^2 - \frac{4}{30}xy + \frac{4}{30}y^2 = 1$$

This is the standard form of required conics.

As we know that the quadratic in r^2 giving the squares of the length of the semi axes.

Hence

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

$$\Rightarrow \left(\frac{4}{30} - \frac{1}{r^2}\right)\left(\frac{4}{30} - \frac{1}{r^2}\right) = \left(\frac{-3}{30}\right)^2$$

$$\Rightarrow \left(\frac{4}{30} - \frac{1}{r^2}\right)^2 - \left(\frac{3}{30}\right)^2 = 0$$

$$\Rightarrow \left(\frac{4}{30} - \frac{1}{r^2} - \frac{3}{30}\right)\left(\frac{4}{30} - \frac{1}{r^2} + \frac{3}{30}\right) = 0$$

$$\Rightarrow \left(\frac{1}{30}-\frac{1}{r^2}\right)\left(\frac{7}{30}-\frac{1}{r^2}\right)=0$$

Therefore$r_1^2 = 30$ and$r_2^2 = \frac{30}{7}$

Since both r_1^2 and r_2^2 are positive. Hence the given is an eliipse.

The length of the axes of the ellipse $= 2r_1$and $2r_2$

$= 2\sqrt{30}$and $2\sqrt{\frac{30}{7}}$

The equation of major axis referred to the original coordinate axes is

$\left(A-\frac{1}{r_1^2}\right)(x-x_0)+H(y-y_0)=0$

$$\Rightarrow \left(\frac{4}{30}-\frac{1}{30}\right)(x-2)+\left(-\frac{3}{30}\right)(y+1)=0$$

$$\Rightarrow (x-2)-(y+1)=0$$

$\Rightarrow x-y-3=0 \Rightarrow y=x-3$ ....(3)

The equation of of the minor axis which is line perpendicular to the major axis and passingthrough the centre$(2,-1)$ is

$(y+1)=-(x-2) \Rightarrow x+y-1=0$ ....(4)

Thus the equations of the axes of the given conic are given by equation (3) and (4).

12.8 ECCENTRICITY, FOCI AND EQUATION OF DIRECTRIX OF CENTRAL CONICS

The equation of central conic is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ……………………(1)

Let (x_0, y_0) be the center of conic and after shifting the origin to the centre (x_0, y_0), we get the standard form

$Ax^2 + 2Hxy + By^2 = 1$ ………………………………………(2)

Suppose α_1^2 and α_2^2 be the roots of quadratic equation $\left(A - \frac{1}{\alpha^2}\right)\left(B - \frac{1}{\alpha^2}\right) = H^2$ in α^2.

Assume $\alpha_1^2 > \alpha^2 > \alpha_2^2$

Now in case of ellipse, the eccentricity can be defined as

$e^2 = 1 - \frac{b^2}{a^2}$………………………………………(3)

And in case of hyperbola, the eccentricity can be defined as

$e^2 = 1 + \frac{b^2}{a^2}$…………………………………………(4)

If the equation of central conic (1) is an ellipse then both the roots α_1^2 and $\alpha_2^2$are positive and

$\alpha_1^2 = a^2$ and $\alpha_2^2 = b^2$.

Also eccentricity e is given as $e^2 = 1 - \frac{\alpha_1^2}{\alpha_2^2}$

If the equation of central conic (1) is an hyperbola then both the roots α_1^2 is positive and $\alpha_2^2$is negative and $\alpha_1^2 = a^2$ and $\alpha_2^2 = -b^2$.

Also eccentricity e is given as $e^2 = 1 - \frac{\alpha_1^2}{\alpha_2^2}$.

To find the coordinates of the foci, let φ be the inclination of the major or transverse axis to the x-axis.

Then $\tan\varphi =$ the gradient of the line represented by the equation of major axis or transverse axis

$$= \frac{-\left(A - \frac{1}{r_1^2}\right)}{H}$$

When the conic is an ellipse: Let $C(x_0, y_0)$ be the centre of ellipse and foci be the points on the major axis at a distance er_1 from the centre $C(x_0, y_0)$.

Then the coordinates of foci of the ellipse are

$(x_0 + er_1\cos\varphi\,, y_1 + er_1\sin\varphi)$ and $(x_0 - er_1\cos\varphi\,, y_1 - er_1\sin\varphi)$

Where $er_1 = \sqrt{(r_1^2 - r_2^2)}$

The directrices are the lines perpendicular to the major axis at a distance $\frac{r_1}{e}$ $i.e. \frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$ from the centre C.

There equations are

$$(x - x_0)\cos\varphi + (y - y_0)\sin\varphi = \pm\frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$$

In case the conic (1) is a hyperbola the same formulae hold but in this case the value of r_2^2 is negative.

Ex. 9. Find the eccentricity, equations of its axes, the coordinates of foci and equation directrices of the conic

$$\mathbf{17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0}$$

Proof. The equation of given conic is

$F(x,y) \equiv 17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ ....(1)

Let the centre be (x_0, y_0).

Then the centre is given by equations

$17x_0 - 6y_0 + 23 = 0$ ....(2)

$-6x_0 + 8y_0 - 14 = 0$ $i.e. -3\,x_0 + 4y_0 - 7 = 0$ ....(3)

Solving (2) and (3), we get

$$\frac{x_0}{42-92} = \frac{y_0}{-69+119} = \frac{1}{68-18}$$

$$\Rightarrow x_0 = \frac{-50}{50} = -1, y_0 = \frac{50}{50} = 1$$

Hence, the centre(x_0, y_0) of given conic is$(-1,1)$.

Now the new constant term is

$$c' = gx_1 + fy_1 + c = 23(-1) + (-14).(1) + 17 = -23 + 3 = -20$$

Therefore, the equation of the conic referred to the centre as origin is

second degree terms of the given conic+new constant c'=0

$$17x^2 - 12xy + 8y^2 - 20 = 0$$

$$\Rightarrow \frac{17}{20}x^2 - \frac{12}{20}xy + \frac{8}{20}y^2 = 1$$

This is the standard form of required conics.

As we know that the quadratic in r^2 giving the squares of the length of the semi axes.

Hence

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2$$

$$\Rightarrow\left(\frac{17}{20}-\frac{1}{r^2}\right)\left(\frac{8}{20}-\frac{1}{r^2}\right)=\left(\frac{6}{20}\right)^2$$

$$\Rightarrow(17r^2-20)(8r^2-20)=36r^4$$

$$\Rightarrow 136r^4-500r^2+400=36r^4$$

$$\Rightarrow\ 100r^4-500r^2+400=0$$

$$\Rightarrow\ r^4-5r^2+4=0$$

$$\Rightarrow\ (r^2-4)(r^2-1)=0$$

Therefore$r_1^2=4$ and$r_2^2=1$

Since both r_1^2 and r_2^2 are positive. Hence the given conic is an eliipse.

The length of the major axis of the ellipse $=2r_1=2.2=4$

The length of the minor axis of the ellipse$=2r_2=2.1=2$

The equation of major axis referred to the original coordinate axes is

$\left(A-\frac{1}{r_1^2}\right)(x-x_0)+H(y-y_0)=0$

$$\Rightarrow\left(\frac{17}{20}-\frac{1}{4}\right)(x-(-1))+\left(-\frac{6}{20}\right)(y-1)=0$$

$$\Rightarrow\frac{12}{20}(x+1)+\left(-\frac{6}{20}\right)(y-1)=0\Rightarrow 2(x+1)-(y-1)=0$$

$\Rightarrow 2x-y+3=0\Rightarrow y=2x+3$ ....(3)

The equation of of the minor axis which is line perpendicular to the major axis and passingthrough the centre$(2,-1)$ is

$2(y-1) = -(x-(-1)) \Rightarrow x + 2y - 3 = 0$ ......(4)

Thus the equations of the axes of the given conic are given by equation (3) and (4).

The eccentricity is

$$e = \sqrt{\left(1 - \frac{r_1^2}{r_2^2}\right)} = \sqrt{\left(1 - \frac{1}{4}\right)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

If φ is the angle which the major axis $y = 2x + 3$makes with the x-axis, then

$\tan\varphi = 2 \Rightarrow \sin\varphi = \frac{2}{\sqrt{5}}$ and $\cos\varphi = \frac{1}{\sqrt{5}}$.

Now, the distance of a focus from the centre $C = er_1$

Coordinate of the foci =

$(x_0 + er_1\cos\varphi,\ y_1 + er_1\sin\varphi)$ and $(x_0 - er_1\cos\varphi,\ y_1 - er_1\sin\varphi)$

$\Rightarrow \left(-1 + \frac{\sqrt{3}}{2}.2.\frac{1}{\sqrt{5}},\ 1 + \frac{\sqrt{3}}{2}.2.\frac{2}{\sqrt{5}}\right)$ and $\left(-1 + \frac{\sqrt{3}}{2}.2.\frac{1}{\sqrt{5}},\ 1 + \frac{\sqrt{3}}{2}.2.\frac{2}{\sqrt{5}}\right)$

$\Rightarrow \left(-1 + \sqrt{\frac{3}{5}},\ 1 + 2\sqrt{\frac{3}{5}}\right)$ and $\left(-1 - \sqrt{\frac{3}{5}},\ 1 - 2\sqrt{\frac{3}{5}}\right)$

Therefore, coordinate of the foci = $\left(-1 + \sqrt{\frac{3}{5}},\ 1 + 2\sqrt{\frac{3}{5}}\right)$ and $\left(-1 - \sqrt{\frac{3}{5}},\ 1 - 2\sqrt{\frac{3}{5}}\right)$

The distance of a directrix from the centre $C = \frac{r_1}{e} = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}}$

The directrices are the straight lines perpendicular to the major axis and at a distance $\frac{r_1}{e}$ from the centre on either side of it.

So the equation of the directrice is

$$(x - x_1)\cos\varphi + (y - y_1)\sin\varphi = \pm\frac{r_1}{e}$$

$$\Rightarrow (x+1)\frac{1}{\sqrt{5}} + (y-1)\frac{2}{\sqrt{5}} = \pm\frac{4}{\sqrt{3}}$$

$$\Rightarrow x + 2y = 1 \pm 4\sqrt{\frac{5}{3}}$$

The equation of the directrice is $x + 2y = 1 \pm 4\sqrt{\frac{5}{3}}$

CHECK YOUR PROGRESS

(CQ 7) The eccentricity of the conics $x^2 + 2x - y^2 + 5 = 0$ is $\sqrt{2}$. (T/F)

(CQ 8) The foci of the conics $x^2 + 2x - y^2 + 5 = 0$ is $(0, \pm 2\sqrt{2})$ (T/F)

(CQ 9) The length of the axes of the conic $9x^2 - 6x + 4y^2 + 4y + 1 = 0$ are 1 and 2/3. (T/F)

(CQ 10) The standard equation of conic is $ax^2 + 2hxy + ay^2 = d$ (T/F)

(CQ 11) The equation of central conic is $-ax^2 + 2hxy + by^2 - 2gx + 2fy + c = 0$ (T/F)

(CQ 12) If $h^2 - ab < 0$, the straight lines are real and so the asymptotes are real. (T/F

(CQ 13) ________ are the only conic sections with asymptotes.

12.9 SUMMARY

In this unit we discussed about the geometric definition of ellipse, parabola and hyperbola and standard equation of conics. We derived the equation of axes and directrix and nature of conics. We also learnt to find the centre of conic and its eccentricity.

12.10 GLOSSARY

1. **Curve:** a continuous and smooth flowing line without any sharp turns
2. **Eccentricity:** the ratio of the distance from any point on the conic section to the focus to the perpendicular distance from that point to the nearest directrix
3. **Hyperbola:** set of all the points, the difference of whose distances from the two fixed points in the plane (foci) is a constant

12.11 REFERENCES

1. Jain, P. K. A Textbook of Analytical Geometry of Three Dimensions. New Age International, 2005.
2. Khan, Ratan Mohan. Analytical Geometry of Two and Three Dimensions and Vector Analysis. New Central Book Agency, 2012.

12.12 SUGGESTED READINGS

1. Robert J. T. Bell, An Elementary Treatise on Coordinate Geometry of Three Dimensions. Macmillan India Ltd, 1994.
2. D. Chatterjee, Analytical Geometry: Two and Three Dimensions. Narosa Publishing House, 2009.

12.13 TERMINAL QUESTION

MULTIPLE CHOICE QUESTION

(TQ-1) A cone has two identically shaped parts called

a)Nappes b) directrix c) loci d) None of these

(TQ-2) The standard form of all conics with centre at origin is

a) $Ax^2 + 2Hxy - By^2 = 1$
b) $Ax^2 + 2Hxy + By^2 = -1$
c) $Ax^2 + 2Hxy + By^2 = 1$
d) $Ax^2 - 2Hxy - By^2 = 1$

(TQ-3) If the plane is perpendicular to the axis of revolution, the conic section is

a) ellipse b)circle c) Hyperbola d) parabola

(TQ-4) The nature of the conic represented by the equation $x^2 + 2xy + y^2 - 2x - 1 = 0$ is

a) Ellipse b) Hyperbola c) Parabola d) Circle

(TQ-5) Focus of the conic represented by the equation $x^2 - 4xy + 4y^2 - 12x - 6y + 5 = 0$ is

a) (2,1) b) (-2,1) c) (2,-1) d) (-2,-1)

LONG ANSWER QUESTIONS

(TQ-6) Find the coordinate of the centre of the conic

$$2x^2 + 3xy + 4y^2 + 8x - 40y + 45 = 0$$

(TQ-7) What conics do the following equations represents? Possible, find their centre and also the standard form of conics.

(i) $12x^2 - 23xy + 10y^2 - 25x + 26y - 14 = 0$

(ii) $3x^2 - 8xy - 3y^2 + 10x - 13y = -8$

(TQ-8) Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ for its asymptotes and passes through the origin

(TQ-9) Find the equation of the hyperbola whose asymptotes are parallel to $2x + 3y = 0$ and $3x + 2y = 0$, whose centre is at $(1,2)$ and which passes through $(5,3)$.

(TQ-10) Find the length and the equations of the axes of the conic

$$4x^2 - 3xy + 4y^2 - 19x + 14y - 4 = 0$$

12.14 ANSWERS

(CQ 1) T	(CQ 2) F	(CQ 3) F
(CQ 4) e=0	(CQ 5) T	(CQ 6) centre
(CQ 7) T	(CQ 8) F	(CQ 9) T
(CQ 10) T	(CQ 11) F	(CQ 12) F
(CQ 13) Hyperbola		
(TQ-1) a)	(TQ-2) c)	(TQ-3) b)
(TQ-4) c)	(TQ-5) d)	(TQ-6) a)

UNIT 13 TRACING OF CONICS II

CONTENTS

13.1 INTRODUCTION

In previous unit we we discussed about conic section, centre of Conic, nature of conic, eccentricity, foci and equation of directrix.
In this unit, we will try to trace an ellipse, hyperbola and parabola.

13.2 OBJECTIVES

After reading this unit learners will be able to

1. constructto trace ellipse.
2. analyzeto trace hyperbola.
3. understand to trace parabola.

13.3 WORKING RULE TO TRACE AN ELLIPSE OR A HYPERBOLA

Let the equation of the given conic be

$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ....(1)

Now we will see the steps for tracing the hyperbola or ellipse

Step 1. Finding the Centre of Conic

First, we will try to find the centre of the conic (1) which we already discussed in previous unit i.e. First we replace x by $x + x_0$ and y by $y + y_0$ and after that taking the coefficients of x and y of new equation equal to 0 and solving them to get centre (x_0, y_0)

Step 2. Finding the new constant term when the equation of conic shifted the centre to origin.

After shifting the origin to centre i.e(x_0, y_0), the given equation of conic (1) becomes

$ax^2 + 2hxy + by^2 + c' = 0$ where $c' = gx_1 + fy_1 + c$

Step 3. Rewriting the given equation of conic in standard form

If $c' \neq 0$, the equation of conic (1) can be written as

$Ax^2 + 2Hxy + By^2 = 1$ where $A = -\frac{a}{c'}, H == -\frac{h}{c'}, B = -\frac{b}{c'}$,

This the standard form of conic

Step 4. Finding the length of the conics

As we discussed in previous unit that the squares of the lengths of the semi-axes of the conic are the roots of the equation

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

If r_1^2 and r_2^2 are the roots of given equation

(i) If both the roots r_1^2 and r_2^2 are positive then the conic will be an ellipse and if $r_1^2 > r_2^2$ then $2r_1$ and $2r_2$ are the length of major and minor axes respectively.

(ii) If both the roots r_1^2 and r_2^2 are of opposite sign (one of the root is negative) then the conic will be a hyperbola and if $r_1^2 > 0\ i.e. r_1^2$ is positive and $r_2^2 < 0\ i.e. r_2^2$ is negative then $2r_1$ and $2\sqrt{|r_2|}$ are the length of transverse and conjugate axes respectively.

Step 5. Finding the equation of axes

In ellipse, if$2r_1$ and $2r_2$ are the length of major and minor axes respectively. Then the equation of major axis and minor axis referred to the center as origin is given by equation (2) and (3) respectively.

$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$ ……………………………….(2)

$\left(A - \frac{1}{r_2^2}\right)x + Hy = 0$ ……………………………………….(3)

Similarly, $2r_1$ and $2\sqrt{|r_2|}$ are the length of transverse and conjugate axes respectively. Then equation (2) and (3) represent the equation of transverse axis and conjugate axis respectively referred to the center as origin.

Now the equation of major axis or transverse axis referred to the original coordinate axes be

$$\left(A - \frac{1}{r_1^2}\right)(x - x_0) + H(y - y_0) = 0$$

And the equation of minor axis or conjugate axis referred to the original coordinate axes be

$$\left(A - \frac{1}{r_2^2}\right)(x - x_0) + H(y - y_0) = 0$$

Step 6. Finding eccentricity

The eccentricity is given by $e = \sqrt{\left(1 - \frac{r_1^2}{r_2^2}\right)}$

Step 7. Finding coordinates of Foci

Ifφ is the inclination of the major or transverse axis to the x- axis, then the coordinate of the foci with respect to the original coordinate axes are

$(x_0 + er_1 \cos\varphi, y_0 + er_1 \sin\varphi)$ and $(x_0 - er_1 \cos\varphi, y_0 - er_1 \sin\varphi)$

Step 8. Finding length of latus rectum

Length of latus rectum= $\frac{2|r_2^2|}{r_1}$

Step 9. Finding special points

We will try to find that point where the given conic (1) meets original x and y axes. It help us to trace the given conic more accurately

Step 10. The sketching of curve

(i) First mark the centre $C(x_0, y_0)$ in XY plane and draw line CX' and CY'through C parallel to the X and Y axes respectively.

(ii) Draw the axes of conics through the point C (major and minor axes or transverse and conjugate axes)

(iii) Mark length CA and $CA^{'}$ each equal to r_1 on the major axis (ellipse) or on the transverse axis (Hyperbola) and mark length CB and CB^'each equal to $\sqrt{|r_2|}$ on the minor axis (ellipse) or on the conjugate axis (Hyperbola).

(iv) Draw line through A and $A^{'}$ perpendicular to AA' and line through B and B' parallel to AA', so that the rectangular $PQRS$ is formed.

(v) The ellipse lie within this rectangle touching it at the points $A, B, A^{'}$ and B'. Hence it can be easily trace.

(vi) The hyperbola is outside of the given rectangle touching it at the points A and $A^{'}$. The asymptotes of the hyperbola are the diagonals PR and QS of this rectangle. Hence it can be easily trace.

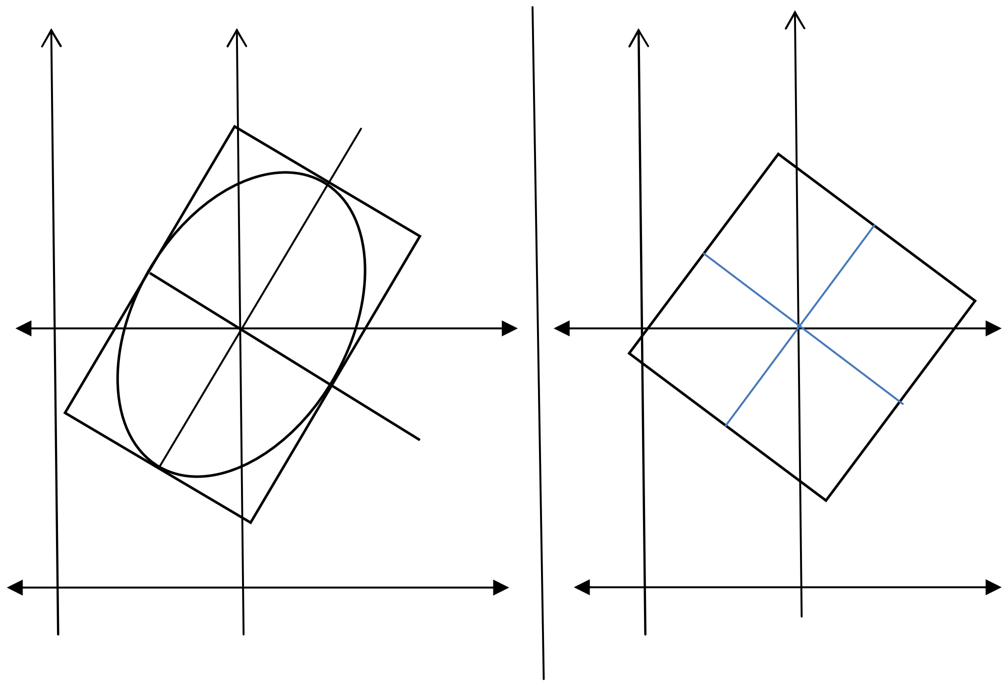


Fig. 13.1.The sketching of curve

Ex 1. Trace the conic $\mathbf{5x^2+4xy+8y^2-12x-12y=0}$

Sol. The given equation of conic is

$F(x,y)\equiv 5x^2+4xy+8y^2-12x-12y=0$ ....(1)

With compare with general equation of conic i.e.

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

We get, $a=5, h=2,\ b=8, g=-6\ ,\ f=-6\ and\ c=0$

Now $h^2=4\ and\ ab=40\Rightarrow h^2\neq ab$

i.e. The second degree terms are not in a perfect square. Hence the given equation of conic represent central conic.

Now we will see the steps for tracing the central conic

Step 1. Centre of Conic

Replace x by $x + x_0$ and y by $y + y_0$ in equation (1), we get

$$5(x + x_0)^2 + 4(x + x_0)(y + y_0) + 8(y + y_0)^2 - 12(x + x_0) - 12(y + y_0) = 0$$

$$5(x^2 + x_0^2 + 2xx_0) + 4(xy + xy_0 + x_0y + x_0y_0) + 8(y^2 + y_0^2 + 2yy_0) - 12x - 12x_0 - 12y - 12y_0 = 0$$

$\Rightarrow \quad 5x^2 + 4xy + 8y^2 + (10x_0 + 4y_0 - 12)x + (4x_0 + 16y_0 - 12)y + 5x_0^2 + 4x_0y_0 + 8y_0^2 - 12x_0 - 12y_0 = 0$

Taking the coefficients of x and y equal to 0, we get

$10x_0 + 4y_0 - 12 = 0 \Rightarrow 5x_0 + 2y_0 - 6 = 0$ ……….(2)

and

$4x_0 + 16y_0 - 12 = 0 \Rightarrow 2x_0 + 8y_0 - 6 = 0$ …………….(3)

$$\frac{x_0}{-12 + 48} = \frac{y_0}{-12 + 30} = \frac{1}{40 - 4}$$

$\Rightarrow x_0 = \frac{36}{36} = 1$ and $y_0 = \frac{18}{36} = \frac{1}{2}$

Therefore the centre of conic is $\left(1, \frac{1}{2}\right)$.

Step 2. New constant term when the equation of conic shifted the centre to origin.

$$c' = gx_1 + fy_1 + c = -6 \times 1 + (-6) \times \frac{1}{2} + 0 = -9$$

Step 3. Rewriting the given equation of conic in standard form

Now the equation of conic (1) refereed to centre as origin can be written as

$$5x^2 + 4xy + 8y^2 - 9 = 0$$

Now standard form of above equation be

$$\frac{5}{9}x^2 + \frac{4}{9}xy + \frac{8}{9}y^2 = 1$$

4. Finding the length of the conics

Now compairing with general standard form of conic i.e. $Ax^2 + 2Hxy + By^2 = 1$, we get

$$A = \frac{5}{9}, H = \frac{2}{9} \; and \; B = \frac{8}{9}$$

As we know the squares of the lengths of the semi-axes of the conic are the roots of the equation

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

$\Rightarrow \left(\frac{5}{9} - \frac{1}{r^2}\right)\left(\frac{8}{9} - \frac{1}{r^2}\right) = \left(\frac{2}{9}\right)^2$

$$\Rightarrow \frac{40}{81} + \left(-\frac{5}{9} - \frac{8}{9}\right)\frac{1}{r^2} + \frac{1}{r^4} = \frac{4}{81}$$

$$\Rightarrow \frac{40}{81} - \frac{13}{9}\frac{1}{r^2} + \frac{1}{r^4} = \frac{4}{81}$$

$$\Rightarrow 40 - 117\frac{1}{r^2} + 81\frac{1}{r^4} = 4$$

$$\Rightarrow 36r^4 - 117r^2 + 81 = 0$$

$$\Rightarrow (36r^2 - 81)(r^2 - 1) = 0$$

Hence, $r^2 = \frac{81}{36}, 1, 81$ i.e $r_1^2 = \frac{81}{36} = \frac{9}{4}$ and $r_2^2 = 1$

Here we can see that both the roots r_1^2 and r_2^2 are positive then the conic will be an ellipse.

Therefore, the length of major axis$=2r_1=2.\frac{3}{2}=2$ and the length of minor axis$=2r_2=2.1=2$

Step 5. Equation of axes

As we know that the equation of major axis referred to center as origin be

$$\left(A-\frac{1}{r_1^2}\right)x+Hy=0$$

$\Rightarrow\left(\frac{5}{9}-\frac{4}{9}\right)x+\frac{2}{9}y=0$

$\Rightarrow x+2y=0$

Shifting the origin back, we get

$$(x-1)+2\left(y-\frac{1}{2}\right)=0\Rightarrow x+2y-2=0$$

which is the required the equation of major axis of conic (1)

Similarly,

The equation of minor axis referred to center as origin be

$$\left(A-\frac{1}{r_2^2}\right)x+Hy=0$$

$\Rightarrow\left(\frac{5}{9}-1\right)x+\frac{2}{9}y=0$

$\Rightarrow -4x+2y=0\Rightarrow 2x-y=0$

Shifting the origin back, we get

$$2(x-1)-\left(y-\frac{1}{2}\right)=0\Rightarrow 2x-y-\frac{3}{2}=0\Rightarrow 4x-2y-3=0$$

Step 6. Finding eccentricity

The eccentricity is given by $e = \sqrt{\left(1 - \frac{r_2^2}{r_1^2}\right)} = \sqrt{\left(1 - \frac{1}{\frac{9}{4}}\right)} = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

Step 7. The points of intersection of conic with coordinate axes

Put $y = 0$ in equation (1), we get

$$5x^2 - 12x = 0$$

$\Rightarrow x(5x - 12) = 0 \Rightarrow x = 0, \frac{12}{5}$

Hence the given conic cut the x-axis at $x = 0$ and $x = \frac{12}{5}$

Similarly, put $y = 0$ in equation (1), we get

$$8y^2 - 12y = 0$$

$\Rightarrow 4y(2y - 3) = 0 \Rightarrow y = 0,\ \frac{3}{2}$

Hence the given conic cut the y-axis at $y = 0$ and $y = \frac{3}{2}$.

Hence the shape of the given conic is as shown in figure.

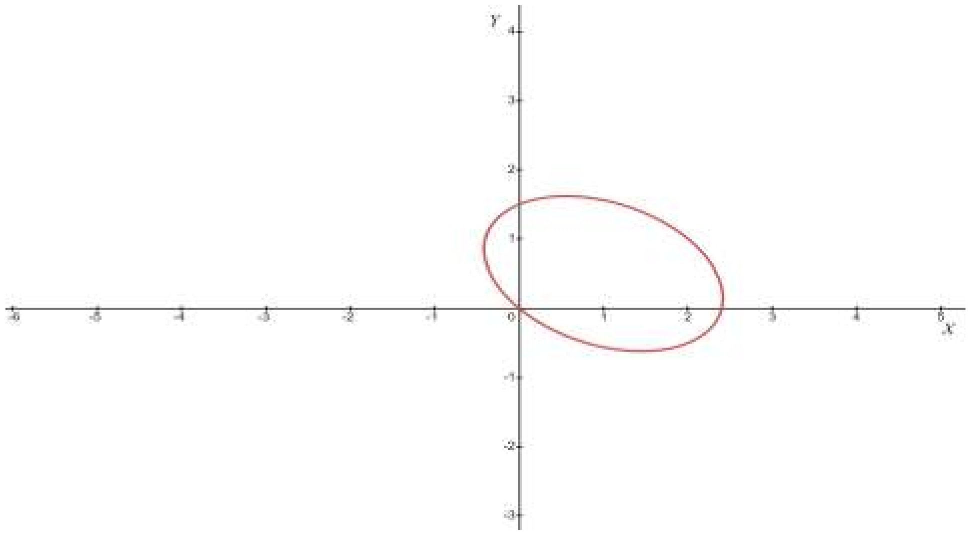


Fig. 13.2. $\mathbf{5x^2 + 4xy + 8y^2 - 12x - 12y = 0.}$

Ex 2. Trace the curve $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$. Find the coordinates of its foci and the length of latus rectum.

Sol. The given equation of conic is

$F(x,y) \equiv 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ ……………(1)

With compare with general equation of conic i.e.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

We get, $a = 14, h = -2, \ b = 11, g = -22\ , \ f = -29$ and $c = 71$

Now $h^2 = 4 \ and \ ab = 154 \Rightarrow h^2 \neq ab$

i.e. The second degree terms are not in a perfect square. Hence the given equation of conic represent central conic.

Now we will see the steps for tracing the central conic

Step 1. Centre of Conic

Replace x by $x + x_0$ and y by $y + y_0$ in equation (1), we get

$$14(x + x_0)^2 - 4(x + x_0)(y + y_0) + 11(y + y_0)^2 - 44(x + x_0) - 58(y + y_0) + 71 = 0$$

$$14(x^2 + x_0^2 + 2xx_0) - 4(xy + xy_0 + x_0y + x_0y_0) + 11(y^2 + y_0^2 + 2yy_0) - 44x - 44x_0 - 58y - 58y_0 + 71 = 0$$

$\Rightarrow \quad 14x^2 - 4xy + 11y^2 + (28x_0 - 4y_0 - 44)x + (-4x_0 + 22y_0 - 58)y + 514 - 4x_0y_0 + 11y_0^2 - 44x_0 - 58y_0 + 71 = 0$

Taking the coefficients of x and y equal to 0, we get

$28x_0 - 4y_0 - 44 = 0 \Rightarrow 7x_0 - y_0 - 11 = 0$ …………………………(2)

and

$-4x_0 + 22y_0 - 58 = 0 \Rightarrow -2x_0 + 11y_0 - 29 = 0$ ………(3)

Solving (2) and (3), we get

$$\frac{x_0}{29+121}=\frac{y_0}{22+203}=\frac{1}{77-2}$$

$\Rightarrow x_0=\frac{150}{75}=2$ and $y_0=\frac{225}{75}=3$

Therefore the centre of conic is $(2,3)$.

Step 2. New constant term when the equation of conic shifted the centre to origin.

$$c'=gx_1+fy_1+c=-22\times 2+(-29)\times 3+71=-60$$

Step 3. Rewriting the given equation of conic in standard form

Now the equation of conic (1) refereed to centre as origin can be written as

$$14x^2-4xy+11y^2-60=0$$

Now standard form of the above equation be

$$\frac{7}{30}x^2-\frac{1}{15}xy+\frac{11}{60}y^2=1$$

4. Finding the length of the conics

Now compairing with general standard form of conic i.e. $Ax^2+2Hxy+By^2=1$, we get

$$A=\frac{7}{30},H=-\frac{1}{30}\ and\ B=\frac{11}{60}$$

As we know the squares of the lengths of the semi-axes of the conic are the roots of the equation

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2$$

$\Rightarrow \left(\frac{7}{30}-\frac{1}{r^2}\right)\left(\frac{11}{60}-\frac{1}{r^2}\right)=\left(\frac{1}{30}\right)^2$

$$\Rightarrow \frac{77}{1800} + \left(-\frac{7}{30} - \frac{11}{60}\right)\frac{1}{r^2} + \frac{1}{r^4} = \frac{1}{900}$$

$$\Rightarrow \frac{77}{1800} - \frac{25}{60}\frac{1}{r^2} + \frac{1}{r^4} = \frac{1}{900}$$

$$\Rightarrow 77 - 750\frac{1}{r^2} + 1800\frac{1}{r^4} = 2$$

$$\Rightarrow 75r^4 - 750r^2 + 1800 = 0$$

$$\Rightarrow r^4 - 10r^2 + 24 = 0$$

$$\Rightarrow (r^2 - 4)(r^2 - 6) = 0$$

Hence, $r^2 = 6,4,81$ i.e $r_1^2 = 6$ and $r_2^2 = 4$

Here we can see that both the roots r_1^2 and r_2^2 are positive then the conic will be an ellipse.

Therefore, the length of major axis= $2r_1 = 2.\sqrt{6} = 2\sqrt{6}$ and the length of minor axis= $2r_2 = 2.2 = 4$

Step 5. Equation of axes

As we know that the equation of major axis referred to center as origin be

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$$

$\Rightarrow\left(\frac{7}{30} - \frac{1}{6}\right)x - \frac{1}{30}y = 0$

$\Rightarrow 2x - y = 0$

Shifting the origin back, we get

$$2\,(x-2) - (y-3) = 0 \Rightarrow 2x - y - 1 = 0$$

which is the required the equation of major axis of conic (1)

Similarly,

The equation of minor axis referred to center as origin be

$$\left(A-\frac{1}{r_2^2}\right)x+Hy=0$$

$\Rightarrow\left(\frac{7}{30}-\frac{1}{4}\right)x-\frac{1}{30}y=0$

$\Rightarrow -x-2y=0 \Rightarrow x+2y=0$

Shifting the origin back, we get

$$(x-2)+2(y-3)=0 \Rightarrow x+2y-8=0$$

Step 6. Finding eccentricity

The eccentricity is given by $e=\sqrt{\left(1-\frac{r_2^2}{r_1^2}\right)}=\sqrt{\left(1-\frac{4}{6}\right)}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$

Step 7. The points of intersection of conic with coordinate axes

Put $y=0$ in equation (1), we get

$$14x^2-44x+71=0$$

$\Rightarrow x=\frac{44\pm\sqrt{(44)^2-4\times14\times71}}{2\times14}$

We can easily see that we get the imaginary values of x.

Hence the given conic does not cut x −axis

Again put $x=0$ in equation (1), we get

$$11y^2-58y+71=0$$

$\Rightarrow y = \frac{58 \pm \sqrt{(58)^2 - 4 \times 11 \times 71}}{2 \times 11} = \frac{58 \pm \sqrt{3364 - 312}}{2 \times 11} = \frac{58 \pm \sqrt{240}}{22} = \frac{29 \pm 2\sqrt{15}}{11}$

Hence, $y = 3.34\,,\ 1.93\ (nearly)$

Step 8: Cordinates of focii

If φ be the inclination of major axis to the x −axis, then $tan\,\varphi = 2$

Then $\sin\varphi = \frac{2}{\sqrt{5}}$ and $\cos\varphi = \frac{1}{\sqrt{5}}$.

Hence, the coordinates of focii are

$(x_0 + er_1\cos\varphi\,, y_0 + er_1\sin\varphi)$ and $(x_0 - er_1\cos\varphi\,, y_0 - er_1\sin\varphi)$

$= \ (2 + \frac{1}{\sqrt{3}}.\sqrt{6}.\frac{1}{\sqrt{5}}, 3 + \frac{1}{\sqrt{3}}.\sqrt{6}.\frac{2}{\sqrt{5}})$ and $(2 + \frac{1}{\sqrt{3}}.\sqrt{6}.\frac{1}{\sqrt{5}}, 3 + \frac{1}{\sqrt{3}}.\sqrt{6}.\frac{2}{\sqrt{5}})$

$= (2 + \sqrt{\frac{2}{5}}, 3 + 2\sqrt{\frac{2}{5}})$ and $(2 - \sqrt{\frac{2}{5}}, 3 - 2\sqrt{\frac{2}{5}})$

Step 8. Finding length of latus rectum

Length of latus rectum$= \frac{2|r_2^2|}{r_1} = \frac{2.4}{\sqrt{6}} = \frac{4\sqrt{6}}{3}$

Hence the shape of the given conic is as shown in figure.

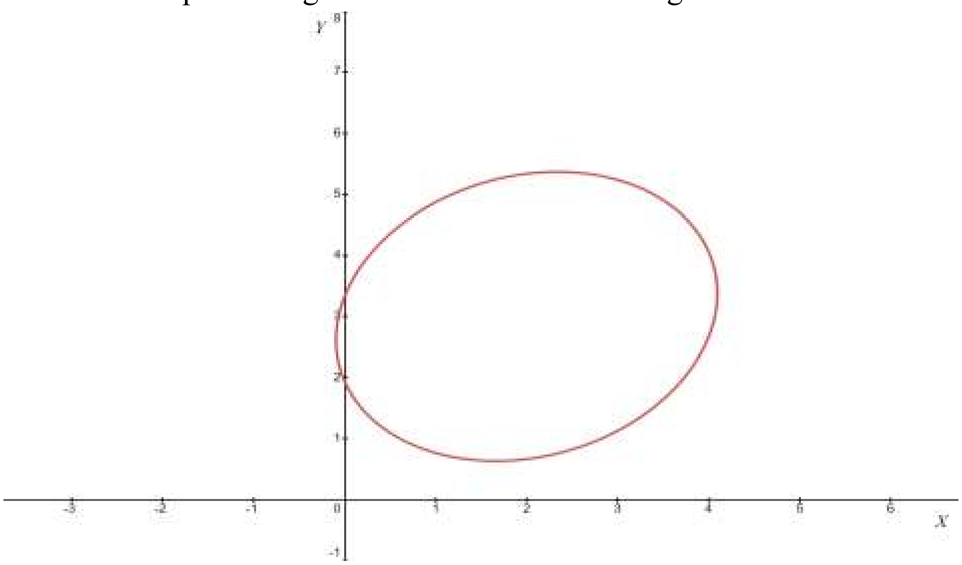


Fig. 13.3. $\mathbf{14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0.}$

Ex 3. Trace the curve $6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$.

Sol. The given equation of conic is

$F(x,y) \equiv 6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0$ …….(1)

With compare with general equation of conic i.e.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

We get, $a = 6, h = \frac{5}{2},\ b = -6, g = -2\ ,\ f = \frac{7}{2}$ and $c = 11$

Now $h^2 = \frac{25}{4}\ and\ ab = -36 \Rightarrow h^2 \neq ab$

i.e. The second degree terms are not in a perfect square. Hence the given equation of conic represent central conic.

Now we will see the steps for tracing the central conic

Step 1. Centre of Conic

Replace x by $x + x_0$ and y by $y + y_0$ in equation (1), we get

$$6(x+x_0)^2 + 5(x+x_0)(y+y_0) - 6(y+y_0)^2 - 4(x+x_0) + 7(y+y_0) + 11 = 0$$

$$6(x^2 + x_0^2 + 2xx_0) + 5(xy + xy_0 + x_0y + x_0y_0) - 6(y^2 + y_0^2 + 2yy_0) - 4x - 4x_0 + 7y + 7y_0 + 11 = 0$$

$\Rightarrow\quad 6x^2 + 5xy + 11y^2 + (12x_0 + 5y_0 - 4)x + (5x_0 - 12y_0 + 7)y + 6x_0^2 + 5x_0y_0 - 6y_0^2 - 4x_0 + 7y_0 + 11 = 0$

Taking the coefficients of x and y equal to 0, we get

$12x_0 + 5y_0 - 4 = 0$ ……………………………….(2)

and

$5x_0 - 12y_0 + 7 = 0$ ……………………….(3)

Solving (2) and (3), we get

$$\frac{x_0}{35-48} = \frac{y_0}{-20-84} = \frac{1}{-144-25}$$

$\Rightarrow x_0 = \frac{-1}{-169} = \frac{1}{13}$ and $y_0 = \frac{-104}{-169} = \frac{8}{13}$

Therefore the centre of conic is $\left(\frac{1}{13}, \frac{8}{13}\right)$.

Step 2. New constant term when the equation of conic shifted the centre to origin.

$$c' = gx_0 + fy_0 + c = -2 \times \frac{1}{13} + \frac{7}{2} \times \frac{8}{13} + 11 = \frac{-4+56+286}{26} = \frac{338}{26} = 13$$

Step 3. Rewriting the given equation of conic in standard form

Now the equation of conic (1) refereed to centre as origin can be written as

$$6x^2 + 5xy - 6y^2 + 13 = 0$$

Now standard form of the above equation be

$$-\frac{6}{13}x^2 - \frac{5}{13}xy + \frac{6}{13}y^2 = 1$$

4. Finding the length of the conics

Now compairing with general standard form of conic i.e. $Ax^2 + 2Hxy + By^2 = 1$, we get

$$A = -\frac{6}{13}, H = -\frac{5}{26} \text{ and } B = \frac{6}{13}$$

As we know the squares of the lengths of the semi-axes of the conic are the roots of the equation

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

$\Rightarrow \left(-\frac{6}{13}-\frac{1}{r^2}\right)\left(\frac{6}{13}-\frac{1}{r^2}\right)=\left(\frac{-5}{26}\right)^2$

$$\Rightarrow \frac{-36}{169}+\left(\frac{6}{13}-\frac{6}{13}\right)\frac{1}{r^2}+\frac{1}{r^4}=\frac{25}{676}$$

$$\Rightarrow \frac{-36}{169}+\frac{1}{r^4}=\frac{25}{676}$$

$$\Rightarrow \frac{-169}{676}+\frac{1}{r^4}=0$$

$$\Rightarrow \frac{-1}{4}+\frac{1}{r^4}=0$$

$$\Rightarrow r^4-4=0$$

$$\Rightarrow (r^2-2)(r^2+2)=0$$

Hence, $r^2=2,-2,81$ i.e $r_1^2=2$ and $r_2^2=-2$

Here we can see that one root r_1^2 is positive while r_2^2 is negative then the conic will be a Hyperbola.

Therefore, the length of transverse axis$=2r_1=2.\sqrt{2}=2\sqrt{2}$ and the length of minor axis$=2\sqrt{|r_2^2|}=2.\sqrt{2}=2\sqrt{2}$

Step 5. Equation of axes

As we know that the equation of major axis referred to center as origin be

$$\left(A-\frac{1}{r_1^2}\right)x+Hy=0$$

$\Rightarrow\left(\frac{-6}{13}-\frac{1}{2}\right)x-\frac{5}{26}y=0$

$\Rightarrow-\frac{25}{26}x-\frac{5}{26}y=0$

$\Rightarrow 5x + y = 0$

Shifting the origin back, we get

$$5\left(x - \frac{1}{13}\right) + \left(y - \frac{8}{13}\right) = 0 \Rightarrow 5x + y - 1 = 0$$

which is the required the equation of transverse axis of conic (1)

Similarly,

The equation of minor axis referred to center as origin be

$$\left(A - \frac{1}{r_2^2}\right)x + Hy = 0$$

$\Rightarrow \left(\frac{-6}{13} + \frac{1}{2}\right)x - \frac{5}{26}y = 0$

$\Rightarrow x - 5y = 0$

Shifting the origin back, we get

$$\left(x - \frac{1}{13}\right) - 5\left(y - \frac{8}{13}\right) = 0 \Rightarrow x - 5y + 3 = 0$$

Step 6. Finding eccentricity

The eccentricity is given by $e = \sqrt{\left(1 - \frac{r_2^2}{r_1^2}\right)} = \sqrt{\left(1 + \frac{2}{2}\right)} = \sqrt{2}$

Step 8: Cordinates of focii

If φ be the inclination of transverse axis to the x –axis, then $tan\,\varphi = -5$

Then $\sin\varphi = -\frac{5}{\sqrt{26}}$ and $\cos\varphi = \frac{1}{\sqrt{26}}$.

Hence, the coordinates of focii are

$(x_0 + er_1 \cos\varphi, y_0 + er_1 \sin\varphi)$ and $(x_0 - er_1 \cos\varphi, y_0 - er_1 \sin\varphi)$

$= (\frac{1}{13} + \sqrt{2}.\sqrt{2}.\frac{1}{\sqrt{26}}, \frac{8}{13} + \sqrt{2}.\sqrt{2}.(\frac{-5}{\sqrt{26}})$ and $(\frac{1}{13} - \sqrt{2}.\sqrt{2}.\frac{1}{\sqrt{26}}, \frac{8}{13} - \sqrt{2}.\sqrt{2}.(\frac{-5}{\sqrt{26}})$

$= (\frac{1}{13} + \frac{\sqrt{26}}{13}, \frac{8}{13} - \frac{5\sqrt{26}}{13})$ and $(\frac{1}{13} - \frac{\sqrt{26}}{13}, \frac{8}{13} + \frac{5\sqrt{26}}{13})$

Hence the shape of the given conic is as shown in figure.

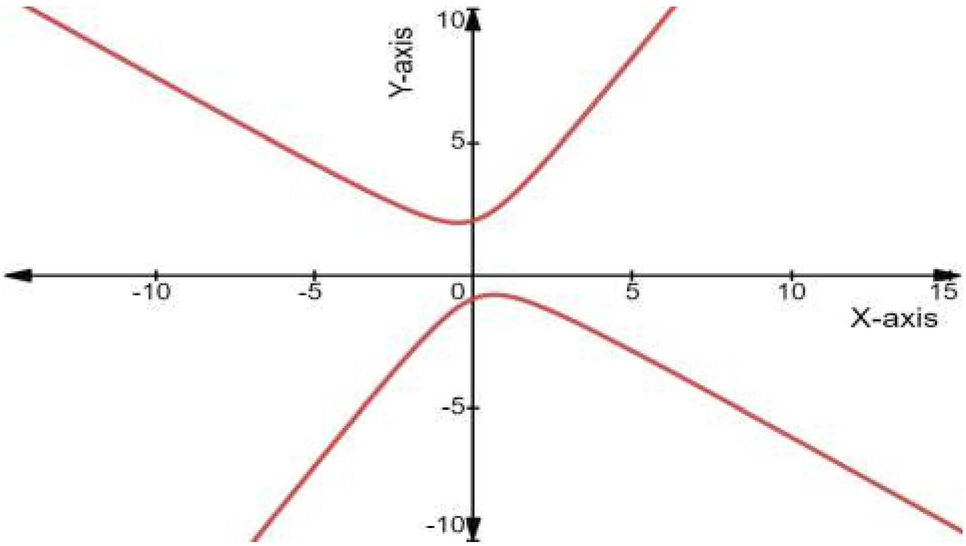


Fig. 13.3. $\boldsymbol{6x^2 + 5xy - 6y^2 - 4x + 7y + 11 = 0}$.

Ex 4. Trace the curve $\boldsymbol{7x^2 + 52xy - 32y^2 - 170x + 140y = 0}$ and find the equation of its asymptote

Sol. The given equation of conic is

$F(x, y) \equiv 7x^2 + 52xy - 32y^2 - 170x + 140y = 0$ ……………(1)

With compare with general equation of conic, i.e.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

We get, $a = 7, h = 26,\ b = -32, g = -85\ ,\ f = 70$ and $c = 0$

Now $h^2 = 676\ \ and\ ab = -182 \Rightarrow h^2 \neq ab$

i.e. The second degree terms are not in a perfect square. Hence the given equation of conic represent central conic.

Now we will see the steps for tracing the central conic.

Step 1. Centre of Conic

Replace x by $x + x_0$ and y by $y + y_0$ in equation (1), we get

$$7(x + x_0)^2 + 52(x + x_0)(y + y_0) - 32(y + y_0)^2 - 170(x + x_0) + 140(y + y_0) + 11 = 0$$

$$7(x^2 + x_0^2 + 2xx_0) + 52(xy + xy_0 + x_0y + x_0y_0) - 32(y^2 + y_0^2 + 2yy_0) - 170x - 170x_0 + 140y + 140y_0 = 0$$

$\Rightarrow 7x^2 + 52xy - 32y^2 + (14x_0 + 52y_0 - 170)x + (52x_0 - 64y_0 + 140)y + 7x_0^2 + 52x_0y_0 - 32y_0^2 - 170x_0 + 140y_0 = 0$

Taking the coefficients of x and y equal to 0, we get

$14x_0 + 52y_0 - 170 = 0 \Rightarrow 7x_0 + 26y_0 - 85 = 0$ ………………….(2)

and

$52x_0 - 64y_0 + 140 = 0 \Rightarrow 13x_0 - 16y_0 + 35 = 0$ ……..……….(3)

Solving (2) and (3), we get

$$\frac{x_0}{910 - 1360} = \frac{y_0}{-1105 - 245} = \frac{1}{-112 - 338}$$

$\Rightarrow x_0 = \frac{-450}{-450} = 1$ and $y_0 = \frac{-1350}{-450} = 3$

Therefore the centre of conic is $(1,3)$.

Step 2. New constant term when the equation of conic shifted the centre to origin.

$$c' = -85 \times 1 + 70 \times 3 + 0 = -85 + 210 = 125$$

Step 3. Rewriting the given equation of conic in standard form

Now the equation of conic (1) refereed to centre as origin can be written as

$$7x^2 + 52xy - 32y^2 + 125 = 0$$

Now standard form of the above equation be

$$-\frac{7}{125}x^2 - \frac{52}{125}xy + \frac{32}{125}y^2 = 1$$

4. Finding the length of the conics

Now comparing with general standard form of conic i.e. $Ax^2 + 2Hxy + By^2 = 1$, we get

$$A = -\frac{7}{125}, H = -\frac{26}{125} \; and \; B = \frac{32}{125}$$

As we know the squares of the lengths of the semi-axes of the conic are the roots of the equation

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

$$\Rightarrow \left(-\frac{7}{125} - \frac{1}{r^2}\right)\left(\frac{32}{125} - \frac{1}{r^2}\right) = \left(-\frac{26}{125}\right)^2$$

$$\Rightarrow \frac{-224}{15625} + \left(\frac{7}{125} - \frac{32}{125}\right)\frac{1}{r^2} + \frac{1}{r^4} = \frac{676}{15625}$$

$$\Rightarrow \frac{-224}{15625} - \frac{25}{125}\frac{1}{r^2} + \frac{1}{r^4} = \frac{676}{15625}$$

$$\Rightarrow -\frac{36}{625} - \frac{1}{5}\frac{1}{r^2} + \frac{1}{r^4} = 0$$

$$\Rightarrow 36\, r^4 + 125r^2 - 625 = 0$$

$\Rightarrow r^2 = \frac{-12\ \pm\sqrt{(125)^2-4\times36\times(-625)}}{2\times36} = \frac{-125\pm\sqrt{(125)^2-4\times36\times(-625)}}{72} = \frac{-125\pm\sqrt{105625}}{72}$

$\Rightarrow r^2 = \frac{-125\pm325}{72}$

Hence $r^2 = \frac{-125+325}{72} = \frac{200}{72} = \frac{25}{9}$ or $r^2 = \frac{-125-325}{72} = \frac{-450}{72} = \frac{-2}{4}$

Hence, $r^2 = \frac{25}{9}, -\frac{225}{36}$, i.e. $r_1^2 = \frac{25}{9}$ and $r_2^2 = -\frac{25}{2}$

Here we can see that one root r_1^2 is positive while r_2^2 is negative then the conic will be a Hyperbola.

Therefore, the length of transverse axis$= 2r_1 = 2.\frac{5}{3} = \frac{10}{3}$ and the length of conjugate axis$=$ $2\sqrt{|r_2^2|} = 2.\frac{5}{2} \ = 5$

Step 5. Equation of axes

As we know that the equation of major axis referred to center as origin be

$$\left(A - \frac{1}{r_1^2}\right)x + Hy = 0$$

$\Rightarrow\left(-\frac{7}{125} - \frac{9}{25}\right)x - \frac{26}{125}y = 0$

$\Rightarrow -\frac{52}{125}x - \frac{26}{125}y = 0$

$\Rightarrow 2x + y = 0$

Shifting the origin back, we get

$$2(x-1) + (y-3) = 0 \Rightarrow 2x + y - 5 = 0$$

which is the required the equation of transverse axis of conic (1)

Now as we know transverse axis and conjugate axis are perpendicular to each other.

Hence, the equation of conjugate axis can be written as

$$(x-1)-2(y-3)=0 \Rightarrow x-2y+5=0$$

Step 6. Finding eccentricity

The eccentricity is given by $e=\sqrt{\left(1-\frac{r_2^2}{r_1^2}\right)}=\sqrt{\left(1+\frac{25/4}{25/9}\right)}=\sqrt{1+\frac{9}{4}}=\frac{\sqrt{13}}{2}$

Step 7: Cordinates of focii

If φ be the inclination of transverse axis to the x −axis, then $tan\,\varphi=-2$

Then $\sin\varphi=-\frac{2}{\sqrt{5}}$ and $\cos\varphi=\frac{1}{\sqrt{5}}$.

Hence, the coordinates of focii are

$(x_0+er_1\cos\varphi\,,y_0+er_1\sin\varphi)$ and $(x_0-er_1\cos\varphi\,,y_0-er_1\sin\varphi)$

$=\ (1+\frac{\sqrt{13}}{2}.\frac{5}{3}.\frac{1}{\sqrt{5}},3+\frac{\sqrt{13}}{2}.\frac{5}{3}.\left(\frac{-2}{\sqrt{5}}\right)$ and $(1-\frac{\sqrt{13}}{2}.\frac{5}{3}.\frac{1}{\sqrt{5}},3-\frac{\sqrt{13}}{2}.\frac{5}{3}.\left(\frac{-2}{\sqrt{5}}\right)$

$=\left(1+\frac{\sqrt{65}}{6},3-\frac{\sqrt{65}}{3}\right)$ and $\left(1-\frac{\sqrt{65}}{6},3+\frac{\sqrt{65}}{3}\right)$

Step 8. Asymptote

As we know that the equation of asymptote differ from equation of the hyperbola only by a constant term.

Let K be that constant and equation of asymptote can be written as

$7x^2+52xy-32y^2-170x+140y+K=0$.

Now it will represent the straight line when $\Delta=0$

Now,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 7 & 26 & -85 \\ 26 & -32 & 70 \\ -85 & 70 & K \end{vmatrix}$$

$$= 7(-32K - 4900) - 26(26K + 5950) - 85(1820 - 2720)$$

$$= (-224 - 676)K - 34300 - 154700 + 76500 = 900K - 81630$$

Hence

$$\Delta = 0 \Rightarrow 900K - 112{,}500 = 0 \Rightarrow K = \frac{112500}{900} = 125$$

Therefore, equation of asymptote is

$$7x^2 + 52xy - 32y^2 - 170x + 140y + 125 = 0$$

Hence the shape of the given conic is as shown in figure.

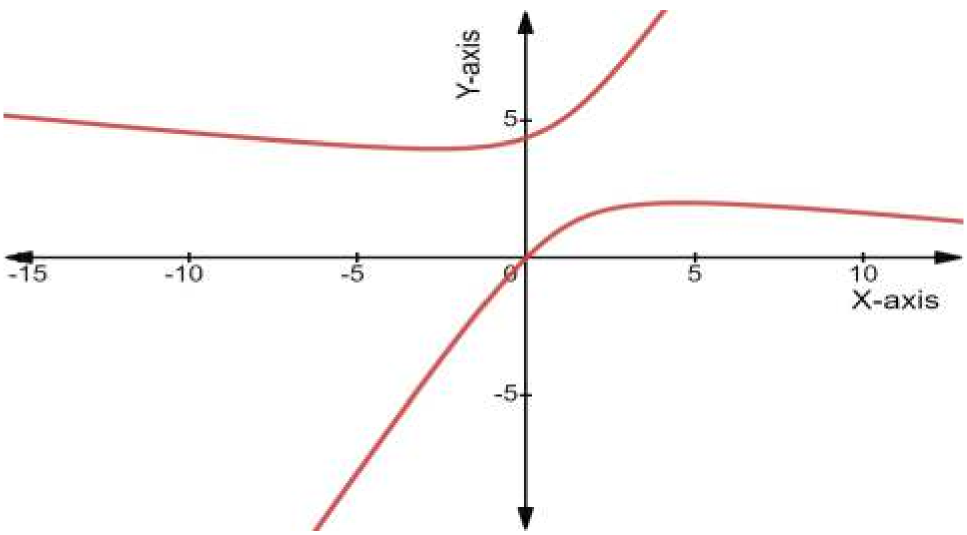


Fig. 13.3. $\mathbf{7x^2 + 52xy - 32y^2 - 170x + 140y = 0}$.

CHECK YOUR PROGRESS

(CQ1) $F(x,y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a equation of the conic. (T/F)

(CQ 2) Eccentricity $e =$ ______________

13.2 STANDARD FORM OF PARABOLA

Let the general equation of the second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ …………….(1)

If given equation of conic (1) is parabola, then the second degree terms form a perfect square.

So we can take $ax^2 + 2hxy + by^2 = (ux + vy)^2$

where $a = u^2,\ h = uv$ and $b = v^2$.

Now we can rewrite equation (1) as

$$(ux + vy)^2 + 2gx + 2fy + c = 0$$

$\Rightarrow (ux + vy)^2 = -(2gx + 2fy + c)$ ……………………………….(2)

Let K be an arbitrary constant then

$$(ux + vy + K)^2 = (ux + vy)^2 + 2K(ux + vy) + K^2$$

$$= -(2gx + 2fy + c) + 2Kux + 2Kvy + K^2 \quad \text{(from (2))}$$

Therefore

$(ux + vy + K)^2 = 2x(Ku - g) + 2y(Kv - f) + K^2 - c$ ………..(3)

Now, we choose K such that the lines $ux + vy + K = 0$ and $2x(Ku - g) + 2y(Kv - f) + K^2 - c = 0$ are at right angles.

Therefore, the product of the slopes of two given line is equal to -1.

$$\Rightarrow \left(-\frac{v}{u}\right)\left(-\frac{2(kv-f)}{2(Ku-g)}\right) = -1 \Rightarrow v(Kv-f) = -u(Ku-g)$$

$$\Rightarrow Ku^2 + Kv^2 = ug + vf$$

$\Rightarrow K = \frac{ug+v}{u^2+v^2}$ ……………………………………………………(4)

For this value of K, the coefficient of $2x$ on the right hand side of equation (3) is equal to

$$\frac{ug+vf}{u^2+v^2}.u - g = \frac{(u^2g+uvf)-(u^2g+v^2g)}{u^2+v^2} = \frac{uvf-v^2g}{u^2+v^2} = \frac{v(uf-vg)}{u^2+v^2}$$

Similarly, the coefficient of $2y$ on the right hand side of equation (3) is equal to

$$\frac{ug+vf}{u^2+v^2}.v - f = \frac{(uvg+v^2f)-(u^2f+v^2f)}{u^2+v^2} = \frac{uvg-u^2f}{u^2+v^2} = \frac{-u(uf-vg)}{u^2+v^2}$$

Therefore the equation (3) become

$$(ux+vy+K)^2 = 2x.\frac{v(uf-vg)}{u^2+v^2} + 2y.\frac{-u(uf-vg)}{u^2+v^2} + K^2 - c$$

$\Rightarrow (ux+vy+K)^2 = 2\frac{(uf-vg)}{u^2+v^2}\left(vx-uy+\frac{(K^2-c)(u^2+v^2)}{2(uf-vg)}\right)$…………………………………..(5)

Let $c' = \frac{(K^2-c)(u^2+v^2)}{2(uf-vg)}$ then equation (5) become

$$(ux+vy+K)^2 = 2\frac{(uf-vg)}{u^2+v^2}(vx-uy+c')$$

$\Rightarrow \left(\frac{ux+v\quad +K}{(u^2+v^2)^{\frac{1}{2}}}\right)^2 = 2\frac{(uf-v\quad)}{(u^2+v^2)^{\frac{3}{2}}}\frac{(vx-u\quad +c')}{(u^2+v^2)^{\frac{1}{2}}}$……………………..(6)

In equation (6), we can see that

$\frac{ux+v\ \ +K}{(u^2+v^2)^{\frac{1}{2}}}$ and $\frac{(vx-u\ \ +c')}{(u^2+v^2)^{\frac{1}{2}}}$ are the perpendicular distances of the point (x, y) on the curve from the mutually perpendicular straight lines $ux + vy + K = 0$ and $vx - uy + c' = 0$ respectively.

Now we can transform the coordinate axes so that the straight line

$ux + vy + K = 0$ ………………………………………(7)

and

$vx - uy + c' = 0$ ……………………………………………(8)

Become the new axes of x and y respectively.

Let x' and y' are the new coordinates of the point (x, y) with respect to these new coordinate axes, then

x' = perpendicular distance of the point (x, y) from the new y −axis(equation (7))

$$= \frac{(vx - uy + c')}{(u^2 + v^2)^{\frac{1}{2}}}$$

and

y' = perpendicular distance of the point (x, y) from the new x −axis(equation (8))

$$= \frac{ux + vy + K}{(u^2 + v^2)^{\frac{1}{2}}}$$

Hence using x' and y' in equation (6), we get

$y'^2 = 2\frac{(uf-vg)}{(u^2+v^2)^{\frac{3}{2}}}x'$ ………………………………………(9)

Let assume $a = \frac{(uf-v\ \ )}{2(u^2+v^2)^{\frac{3}{2}}}$

Hence equation (9) become

$$y'^2 = 4ax'$$

which is the standard form of equation of parabola.

13.3 LENGTH OF LATUS RECTUM, DIRECTRIX AND FOCUS OF PARABOLA

As we earlier discussed that the standard form of parabola is $y'^2 = 4ax'$ where

$x' == \frac{(vx-u \ +c')}{(u^2+v^2)^{\frac{1}{2}}}, y' \ = \frac{ux+vy+K}{(u^2+v^2)^{\frac{1}{2}}}$ and $a = \frac{(uf-vg)}{2(u^2+v^2)^{\frac{3}{2}}}$.

Now,

The equation of the axis of parabola is$y' = 0 \Rightarrow \frac{ux+vy+K}{(u^2+v^2)^{\frac{1}{2}}} = 0 \Rightarrow ux + vy + K = 0 \ldots\ldots\ldots..(1)$

The equation of the tangent at vertex is $x' = 0 \Rightarrow \frac{vx-uy+c'}{(u^2+v^2)^{\frac{1}{2}}} = 0 \Rightarrow vx - uy + c' = 0 \ldots\ldots\ldots(2)$

The length of latus rectum $= 4a = \frac{2(uf-vg)}{(u^2+v^2)^{\frac{3}{2}}}$

The vertex of parabola is the point of intersection of axis of parabola and tangent at vertex of parabola. So after solving equation (1) and (2), we get the vertex of parabola.

Now, equation of latus rectum of parabola is

$$x' = a \Rightarrow \frac{vx - uy + c'}{(u^2 + v^2)^{\frac{1}{2}}} = \frac{(uf - vg)}{2(u^2 + v^2)^{\frac{3}{2}}}$$

$\Rightarrow vx - uy + c' = \frac{(uf-vg)}{2(u^2+v^2)}$……………………………………………………..(3)

The focus of parabola is the point of intersection of axis of parabola and latus rectum of parabola.

Therefore, by solving equation (1) and equation (3), we get the focus of parabola.

The equation of directrix of parabola is $x' = a \Rightarrow \frac{vx-uy+c'}{(u^2+v^2)^{\frac{1}{2}}} = \frac{-(uf-v\ \ )}{2(u^2+v^2)^{\frac{3}{2}}}$

$\Rightarrow vx - uy + c' = -\frac{(uf-v\ \ )}{2(u^2+v^2)}$..(4)

Foot of directrix is the point of intersection of axis of parabola and directrix of parabola. So we can find foot of directrix by solving equation (1) and equation (4).

Ex 5. Find the axis, the vertex, the latus rectum, the focus and the equation of the directrix of the parabola $x^2 + 2xy + y^2 - 2x - 1 = 0$.

Sol. The given equation of conic is

$x^2 + 2xy + y^2 - 2x - 1 = 0$ ....**(1)**

We can easily see that the second degree terms i.e. $x^2 + 2xy + y^2$ form a perfect square.

Therefore the given equation represents a parabola.

So we can write the equation (1) as

$$(x + y)^2 - 2x - 1 = 0$$

$\Rightarrow (x + y)^2 = 2x + 1$..**(2)**

Here we can observe that $x + y = 0$ and $2x + 1 = 0$ does not perpendicular to each other (because product of the slopes of given lines is not equal to -1).

Let K be an arbitrary constant then

$$(x + y + K)^2 = (x + y)^2 + K^2 + 2K(x + y)$$

Using equation (2), we get

$$(x+y+K)^2 = 2x+1+K^2+2K(x+y)$$

$\Rightarrow (x+y+K)^2 = 2(K+1)x+2Ky+K^2+1$……………….(3)

Now, we choose K such that the lines $x+y+K=0$ and $2(K+1)x+2Ky+K^2+1=0$are perpendicular to each other.

Therefore, the product of the slopes of two given line is equal to -1.

$$\Rightarrow (-1)\left(-\frac{2(k+1)}{2k}\right) = -1 \Rightarrow K+1 = -K$$

$$\Rightarrow -2K = 1 \Rightarrow K = -\frac{1}{2}$$

Putting the value of K in equation (3), we get

$$\left(x+y-\frac{1}{2}\right)^2 = 2(-\frac{1}{2}+1)x+2\left(-\frac{1}{2}\right)y+\left(-\frac{1}{2}\right)^2+1$$

$$\Rightarrow (2x+2y-1)^2 = 4\left(x-y+\frac{5}{4}\right)$$

$$\Rightarrow (2x+2y-1)^2 = 4x-4y+5$$

$$\Rightarrow (2x+2y-1)^2 = 4x-4y+5$$

$$\Rightarrow \left(\sqrt{2^2+2^2}\,.\frac{2x+2y-1}{\sqrt{2^2+2^2}}\right)^2 = \sqrt{4^2+4^2}\left(\frac{4x-4y+5}{\sqrt{4^2+4^2}}\right)$$

$$\Rightarrow 8\left(\frac{2x+2y-1}{2\sqrt{2}}\right)^2 = 4\sqrt{2}\left(\frac{4x-4y+5}{4\sqrt{2}}\right)$$

$\Rightarrow \left(\frac{2x+2y-1}{2\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}}\left(\frac{4x-4y+5}{4\sqrt{2}}\right)$ ………………………………………..(4)

Let $x' = \frac{4x-4y+5}{4\sqrt{2}} y' = \frac{2x+2y-1}{2\sqrt{2}}$ and $a = \frac{1}{4\sqrt{2}}$.

Then equation (4) becomes

$y'^2 = 4ax'$ which is the standard form of parabola.

Now,

The equation of the axis of parabola is $y' = 0 \Rightarrow \frac{2x+2y-1}{2\sqrt{2}} = 0 \Rightarrow 2x + 2y - 1 = 0 \ldots\ldots\ldots..(5)$

The equation of the tangent at vertex is $x' = 0 \Rightarrow \frac{4x-4y+5}{4\sqrt{2}} = 0 \Rightarrow 4x - 4y + 5 = 0 \ldots\ldots\ldots(6)$

The vertex of parabola is the point of intersection of axis of parabola and tangent at vertex of parabola. So after solving equation (1) and (2), we get

$\frac{x}{10-4} = \frac{y}{-4-10} = \frac{1}{-8-8} \Rightarrow x = \frac{6}{-16} = -\frac{3}{8}$ and $y = \frac{-14}{-1} = \frac{7}{8}$

Hence, vertex of parabola = $\left(-\frac{3}{8}, \frac{7}{8}\right)$

The length of latus rectum = $4a = 4.\frac{1}{4\sqrt{2}} = \sqrt{2}$

Now, equation of latus rectum of parabola is

$x' = a \Rightarrow \frac{4x-4y+5}{4\sqrt{2}} = \frac{1}{4\sqrt{2}} \Rightarrow 4x - 4y + 4 = 0 \Rightarrow x - y + 1 = 0$ …….(7)

The focus of parabola is the point of intersection of axis of parabola and latus rectum of parabola.

Therefore, by solving equation (1) and equation (3), we get

$\frac{x}{2-1} = \frac{y}{-1-2} = \frac{1}{-2-2} \Rightarrow x = \frac{1}{-4} = -\frac{1}{4}$ and $y = \frac{-3}{-4} = \frac{3}{4}$

the focus of parabola= $\left(-\frac{1}{4}, \frac{3}{4}\right)$.

The equation of directrix of parabola is $x' = -a \Rightarrow \frac{vx-uy+c'}{(u^2+v^2)^{\frac{1}{2}}} = \frac{-(uf-v\ \ )}{2(u^2+v^2)^{\frac{3}{2}}}$

$x' = -a \Rightarrow \frac{4x-4y+5}{4\sqrt{2}} = \frac{-1}{4\sqrt{2}} \Rightarrow 4x - 4y - 6 = 0$…………..….(8)

CHECK YOUR PROGRESS

(CQ 3) $y^2 = 4ax$ is a standard form of hyperbola.. (T/F)

(CQ 4) $y^2 = 4ax$ is a standard form of parabola.. (T/F)

(CQ 5) The focus of parabola is the point of intersection of axis of parabola and ____________ of parabola.

13.4 WORKING RULE TO TRACE A PARABOLA

Step 1.Draw the rectangular axis OX and OY and plot the vertex A.

Step 2. Draw the axis and the line perpendicular to it through the vertex A, i.e. tangent at vertex

Step 3. Find the points where the given parabola meets the coordinate axes. These points will enable us to know that on what side of the tangent at vertex the curve lies.

Step 4. Now on the axis of the parabola, towards the side of the curve lies, take a point S (focus) such that $AS = a = \frac{1}{4}\ of\ latus\ rectum.$

Step 5. Draw $LSL^{'}$ perpendicular to the axis of parabola and mark off $SL = SL' = 2a$.

Step 6. Draw the figure of curve touching the tangent at A (vertex) , symmetrical about the axis of parabola and passing through the points L and L ^' and also the points where the curve meets the coordinate axes.

Ex.6. Trace the parabola $16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$. Also find the coordinate of its focus.

Sol. The given equation of parabola is

$16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$…………………(1)

The given equation can be written as

$(4x - 3y)^2 = -77x + 64y - 95$…………………(2)

Here we can observe that $4x - 3y = 0$ and $-77x + 64y - 95 = 0$ does not perpendicular to each other (because product of the slopes of given lines is not equal to -1).

Let K be an arbitrary constant then

$$(4x - 3y + K)^2 = (4x - 3y)^2 + K^2 + 2K(4x - 3y)$$

Using equation (2), we get

$$(4x - 3y + K)^2 = -77x + 64y - 95 + K^2 + 2K(4x - 3y)$$

$\Rightarrow (4x - 3y + K)^2 = (8K - 77)x + (64 - 6K)y + K^2 - 95$……….(3)

Now, we choose K such that the lines $4x - 3y + K = 0$ and $(8K - 77)x + (64 - 6K)y + K^2 - 95 = 0$are perpendicular to each other.

Therefore, the product of the slopes of two given line is equal to -1.

$$\Rightarrow \left(\frac{4}{3}\right)\left(-\frac{8K - 77}{64 - 6K}\right) = -1 \Rightarrow 4(8K - 77) = 3(64 - 6K) \Rightarrow 32K - 308 = 192 - 18K$$

$$\Rightarrow 32K + 18K = 192 + 308 \Rightarrow 50K = 500 \Rightarrow K = 10$$

Putting the value of K in equation (3), we get

$$(4x - 3y + 10)^2 = (8.10 - 77)x + (64 - 6.10)y + (10)^2 - 95$$

$$\Rightarrow (4x - 3y + 10)^2 = 3x + 4y + 5$$

$$\Rightarrow \left(\sqrt{4^2 + (-3)^2}\,.\frac{(4x - 3y + 10)^2}{\sqrt{4^2 + (-3)^2}}\right)^2 = \sqrt{3^2 + 4^2}\left(\frac{3x + 4y + 5}{\sqrt{3^2 + 4^2}}\right)$$

$\Rightarrow \left(5 \cdot \frac{(4x-3y+10)^2}{5}\right)^2 = 5\left(\frac{3x+4y+5}{5}\right)$

$\Rightarrow 25\left(\frac{(4x-3y+10)^2}{5}\right)^2 = 5\left(\frac{3x+4y+5}{5}\right)$

$\Rightarrow \left(\frac{4x-3y+10}{5}\right)^2 = \frac{1}{5}\left(\frac{3x+4y+5}{5}\right)$ ……………………………………….(4)

Let $x' = \frac{3x+4y+5}{5} y' = \frac{4x-3y+10}{5}$ and $4a = \frac{1}{5}$.

Then equation (4) becomes

${y'}^2 = 4ax'$ which is the standard form of parabola.

Now,

The equation of the axis of parabola is$y' = 0 \Rightarrow \frac{4x-3y+10}{5} = 0 \Rightarrow 4x - 3y + 10 = 0 \ldots..(5)$

The equation of the tangent at vertex is $x' = 0 \Rightarrow \frac{3x+4y+5}{5} = 0 \Rightarrow 3x + 4y + 5 = 0 \ldots \ldots \ldots (6)$

The vertex of parabola is the point of intersection of axis of parabola and tangent at vertex of parabola. So after solving equation (5) and (6), we get

$\frac{x}{-15-40} = \frac{y}{30-20} = \frac{1}{16+9} \Rightarrow x = \frac{-55}{25} = -\frac{11}{5}$and $y = \frac{10}{25} = \frac{2}{5}$

Hence, vertex of parabola = $\left(-\frac{11}{5}, \frac{2}{5}\right)$

The length of latus rectum $= 4a = \frac{1}{5}$

Now, equation of latus rectum of parabola is

$\frac{3x+4y+5}{5} = \frac{1}{20} \Rightarrow 4(3x + 4y + 5) = 1 \Rightarrow 12x + 16y + 19 = 0$ …….(7)

The focus of parabola is the point of intersection of axis of parabola and latus rectum of parabola.

Therefore, by solving equation (5) and equation (7), we get

$\frac{x}{-57-160} = \frac{y}{120-76} = \frac{1}{64+3} \Rightarrow x = \frac{-217}{100}$and $y = \frac{44}{100} = \frac{11}{25}$

the focus of parabola= $\left(\frac{-217}{100}, \frac{11}{25}\right)$.

The given curve cuts the x-axis where y=0, equation (1) become

$$16x^2 + 77x + 95 = 0$$

We can see that it gives imaginary value of x (because $(77)^2 - 4.16.95 < 0$)

From this we conclude that the curve does not cut the x-axis.

The given curve cuts the y −axis where $x = 0$, equation (1) become

$$9y^2 - 64y + 95 = 0$$

$$\Rightarrow y = \frac{64 \pm \sqrt{(-64)^2 - 4.\ 9.\ 95}}{2.9} = \frac{64 \pm \sqrt{4096 - 3420}}{18} = \frac{64 \pm 26}{18}$$

$$\Rightarrow y = \frac{64+26}{18} = \frac{90}{18} = 5\ or\ y = \frac{64-26}{18} = \frac{38}{18} = \frac{19}{9} = 2.1\ (nearly)$$

Tracing:

Draw the rectangular axes OX and OY and plot the vertex A whose coordinates are$\left(-\frac{11}{5}, \frac{2}{5}\right)$ and y −cut the y-axis at the point.

Draw the axis of parabola $4x - 3y + 10 = 0$. The line passes through the vertex A and cuts the y −axis at the point $(0,10/3)$.

Draw the tangent at vertex $A\left(-\frac{11}{5}, \frac{2}{5}\right)$.

Now the parabola does not cut the x −axis and it cuts the y-axis at the points whose y-coordinates are 5 and 2.1 (nearly). Therefore the curve lies on the origin side of the tangent at vertex. Now mark focus S on the axis of parabola such that $AS = a = \frac{1}{5}$. Draw a line through S and perpendicular to the x −axis $4x - 3y + 10 = 0$ and take two points L and L' on it such that $SL = SL' = 2a = \frac{2}{5}$. Also plot the points where the curve cut the y −axis.

Now draw a parabola touching the tangent at the vertex at the point A, symmetrical about the axis $4x - 3y + 10 = 0$ and passing through L, L' and the points on the y-axis whose y −coordinates are 5 and 2.1 (nearly).

The shape of the curve is as shown in figure.

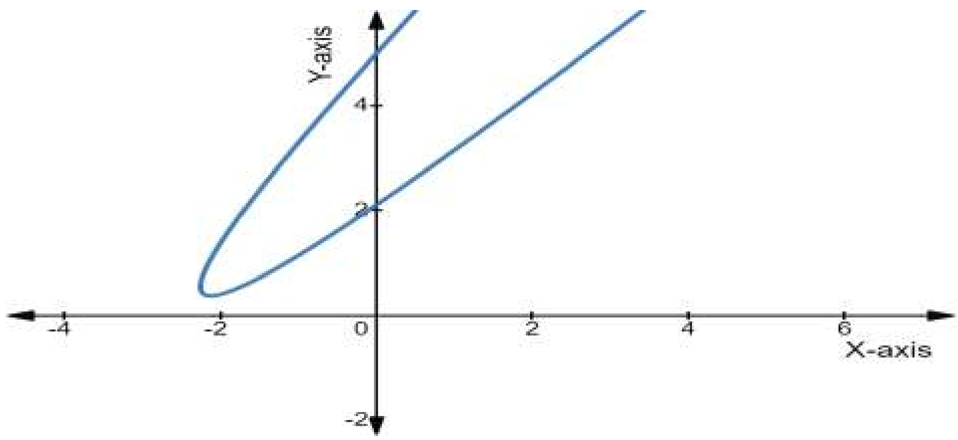


Fig. 13.5. $\mathbf{16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0}$

13.5 SUMMARY

In this unit we learned the working rule to trace an ellipse or a hyperbola. We also discussed about standard form of Parabola and length of latus rectum, directrix and focus of parabola. We try to trace a parabola.

13.6 GLOSSARY

1. **Curve:** a continuous and smooth flowing line without any sharp turns
2. **Eccentricity:** the ratio of the distance from any point on the conic section to the focus to the perpendicular distance from that point to the nearest directrix
3. **Hyperbola:** set of all the points, the difference of whose distances from the two fixed points in the plane (foci) is a constant

13.7 REFERENCES

1. Robert J. T. Bell, An Elementary Treatise on Coordinate Geometry of Three Dimensions. Macmillan India Ltd, 1994.
2. D. Chatterjee, Analytical Geometry: Two and Three Dimensions. Narosa Publishing House, 2009.

13.8 SUGGESTED READINGS

1. Jain, P. K. A Textbook of Analytical Geometry of Three Dimensions. New Age International, 2005.
2. Khan, Ratan Mohan. Analytical Geometry of Two and Three Dimensions and Vector Analysis. New Central Book Agency, 2012.

13.9 TERMINAL QUESTION

MULTIPLE CHOICE QUESTION

(TQ-1) The point of intersection of axis of parabola and tangent at vertex of parabola.

a) Vertex b) focus c) origin d) None of these

(TQ-2) The point of intersection of axis of parabola and latus rectum of parabola.

a) Vertex b) focus c) origin d) None of these

(TQ-3) The nature of the conic represented by the equation $5x^2 - 6xy + 5y^2 + 26x - 22y + 5 = 0$is

a) Ellipse b) Hyperbola c) Parabola d) Circle

(TQ-4) Focus of the conic represented by the equation $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$ is

a) (2,3) b) (-2,3) c) (2,-3) d) (-2,-3)

FILL IN THE BLANKS

(TQ-5) The conic $32x^2 + 52xy - 7y^2 - 64x - 52y - 248 = 0$ is a ________ ______________.

(TQ-6) The equation of transverse axis of the hyperbola $x^2 + 4xy + y^2 - 4 = 0$ is __ _______

(TQ-7) The equation of major axis of the ellipse $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$ is ___________.

LONG ANSWER QUESTIONS

(TQ-8) Trace the curve $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$

(TQ-9) Trace the curve $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$

(TQ-10) Trace the conic $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$

(TQ-11) Trace $7x^2 + 52xy - 32y^2 - 170x + 140y = 0$ and find the equation of its asymptote.

(TQ-12) Find the axis, the vertex, the latus rectum, the focus and the equation of the directrix of the parabola $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$.

(TQ-13) Trace the parabola $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$ and the find the coordinate of focus.

(TQ-14) Show that $\frac{1}{x+y-a} + \frac{1}{x-y+a} + \frac{1}{-x+y-a} = 0$ represents a parabola. Find the coordinates of the focus and the equation of directrix.

13.10 ANSWERS

(CQ 1) T

(CQ 2) $\sqrt{\left(1-\frac{r_2^2}{r_1^2}\right)}$

(CQ 3) F

(CQ 4) T

(CQ 5) latus rectum

(TQ-1) a)

(TQ-2) b)

(TQ-3) a)

(TQ-4) c)

(TQ-5) Hyperbola

(TQ-6) $x - y = 0$

(TQ-7) 4

UNIT 14: POLAR EQUATION OF A CONIC

CONTENTS

14.1 INTRODUCTION

The working rule to trace an ellipse or a hyperbola was explained to learners in the previous unit. Additionally, we learned about the standard form of a parabola, the length of the latus rectus, the directrix, and the parabola's focus.

In this unit we will examine about Polar coordinates, Chord, tangent and asymptotes of a conic. We will also try to find Auxiliary circle and director circle and also Pair of tangents and chord of contact.

14.2 OBJECTIVES

After reading this unit learners will be able to

- familiar with polar coordinates
- analyze and find chord, tangent and asymptotes of a conic

- determine auxiliary circle and director circle
- establish pair of tangents and chord of contact

14.3 POLAR COORDINATES

Polar equation of a conic with its latus rectum of length $2l$, eccentricity e and the focus S being the pole

Proof. Consider S be the focus which can be taken as the pole. Let ON be the directrix of the conic. Draw SO perpendicular to the directrix and take SO as the initial line SX.

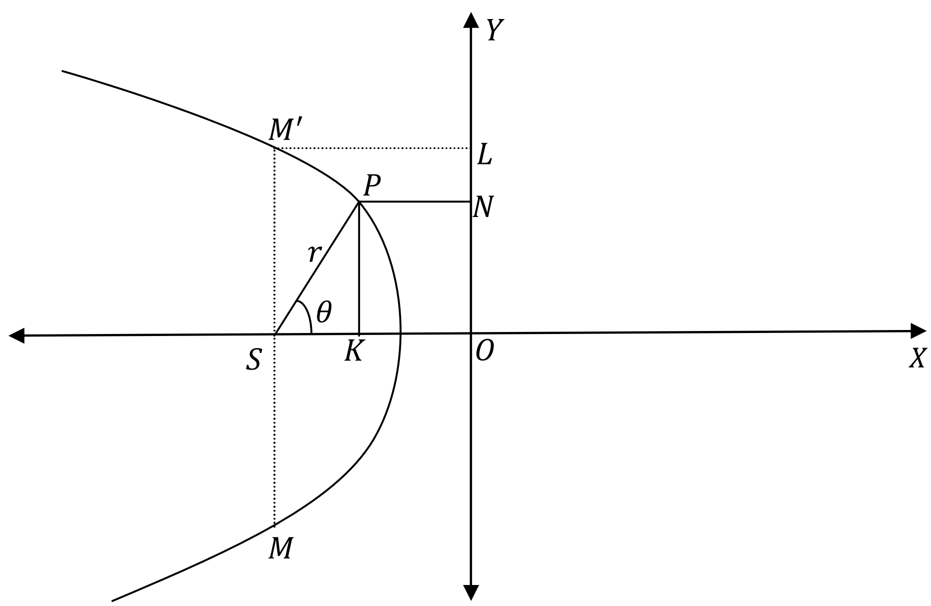


Fig. 14.3.1

Let a point P on the conic and let its polar coordinates be (r, θ)so that $SP = r, \angle XSP = \theta$.

Let MSM' be tha latus rectum of length $2l$ so that the semi rectum $SM = l$.

Since the point P is on point is on the conic, therefore by the definition of a conic, we have

$SP = e.PN = e.NZ = e(SZ - SK)$

Therefore, $r = e.SO - e.SK = e.SZ - e.SP\cos\theta$ (because $= SP\cos\theta$)
$\Rightarrow r = e.SO - e.r\cos\theta$ ………………………………(1)
But the point M is also on the conic. Therefore
$l = SM = e.ML = e.SO$
Using the value of $e.SO$ in eq (1), we get
$\Rightarrow r = l - e.r\cos\theta \Rightarrow r(1 + e\cos\theta) = l$
$\Rightarrow \frac{l}{r} = 1 + e\cos\theta$ ………………………(2)
which is the required polar equation of conic.
NOTE: If we take the positive direction of the initial line opposite the direction directed from the focus towards the directrix,

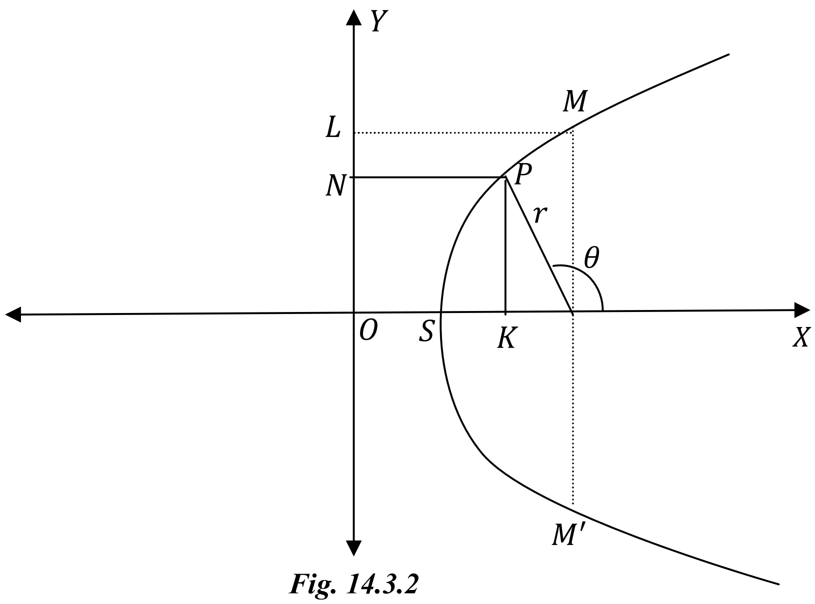


Fig. 14.3.2

the equation of the conic will come out to be
$\frac{l}{r} = 1 - e\cos\theta$ ……………………………………(3)
We clearly see that if we rotate the initial line through an angle π in the equation (2) . i.e. if we replace θ by $\pi + \theta$ in the equation (2), we get the equation (3). Hence any result for the conic (3)

can be obtained the corresponding result for the conic (2) by increasing each vectorial angle by π i.e. writing $\pi + \theta$ for $\theta, \pi + \alpha$ for α $\pi + \beta$ for β where α and β are vectorial angles.

Important Results

1. If the conic is parabola then $e = 1$, the polar form of conic becomes

   $\frac{l}{r} = 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \Rightarrow r = \frac{l}{2}\sec^2\frac{\theta}{2}$

2. As we observe in equation (2) , $r = \frac{l}{1+e\cos}$. Hence the coordinates of a point $P(r,\theta)$ on the conic (2) can be written as $\left(\frac{1}{(1+e\,co\quad)},\theta\right)$.

Equation of the directrix of conic $\frac{l}{r} = 1 + e\cos\theta$

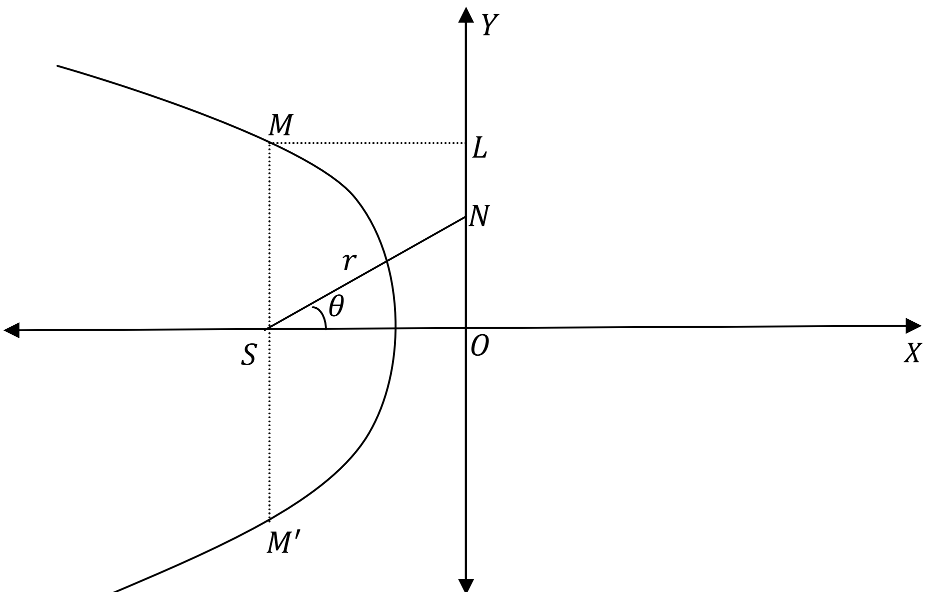


Fig. 14.3.3

Let S be the focus. Let N be any point on the directrix OY. Let the polar coordinates of N be (r,θ).

Hence $\angle OSN = \theta,\ SN = r$

Now, $\frac{SO}{SN} = \cos\theta \Rightarrow SO = r\cos\theta$

Also $SM = e.ML = e.SO = er\cos\theta$

$$\Rightarrow l = ercos\,\theta \Rightarrow \frac{l}{r} = \cos\theta$$

This is the required equation of directrix.

Polar equation of a conic with its focus as the pole and its axis inclined at an angle α to the initial line

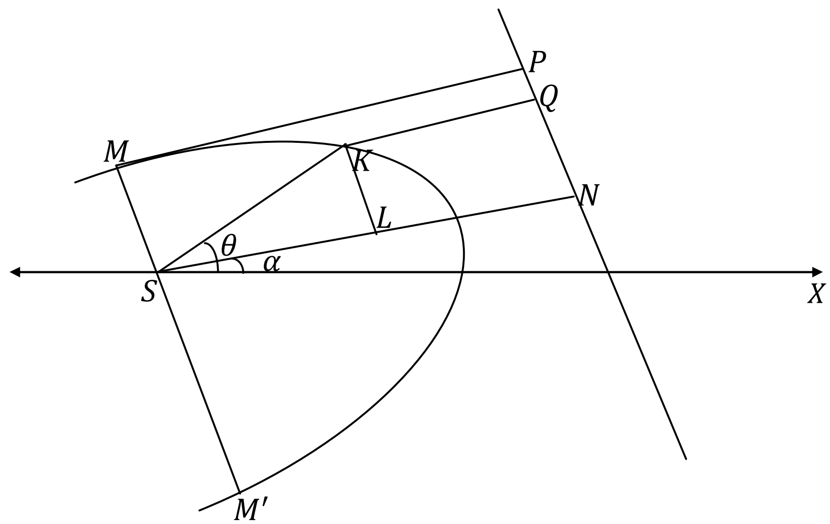


Fig. 14.3.4

Assume SN be the axis of conic to be inclined to to the initial line at an angleα. Let K be a point with coordinates (r, θ) on the conic.

Now, $r = SN = e.KQ = e.LN = e(SN - SL) = e(MP - SL)$

$$\Rightarrow r = e\left\{\frac{MS}{e} - SK\cos(\theta - \alpha)\right\} = e\left\{\frac{l}{e} - r\cos(\theta - \alpha)\right\} = l - er\cos(\theta - \alpha)$$

$$\Rightarrow l = r(1 + er\cos(\theta - \alpha))$$

$$\Rightarrow \frac{l}{r} = 1 + er\cos(\theta - \alpha)$$

This is the required polar equation.

Ex.1. Show that the equations $\frac{l}{r} = 1 + e\ \cos\theta$ and $\frac{l}{r} = 1 - e\ \cos\theta$ represent the same conic.

Proof. Consider the given equations

$\frac{l}{r} = 1 + e\ \cos\theta$……………………………………………….(1)

$\frac{l}{r} = 1 - e\ \cos\theta$ …………………...(2)

First we try to prove that the every pointon the equation of conic (1) lies on the equation of conic (2).

Let $K(r', \theta')$ be any point on the equation of curve (1), then

$\frac{l}{r'} = 1 + e\cos\theta'$ ……………………..(3)

As we know that the coordinates of K can be expressed $(-r', \theta' + \pi)$ instead of (r', θ').

Now, $(-r', \theta' + \pi)$ satisfy the equation (2), therefore

$\frac{l}{-r'} = -1 + e\cos(\theta' + \pi)$

$\Rightarrow \frac{l}{-r'} = -1 - e\cos\theta' \Rightarrow \frac{l}{r'} = 1 + e\cos\theta'$

Therefore given equation is true by virtue of equation (3).

Thus every point K on the conic (1) also lies on the conic (2).

Similarly we can show that every point on the conic (2) is also a point on the curve (1).

Hence the equation (1) and equation (2) represent the same conic.

Ex.2. If MSM' and NSN' be two perpendicular focal chords of a conic, prove that

$$\frac{1}{MS.M'S}+\frac{1}{NS.N'S}$$ **is constant**

Proof. Consider the equation of conic

$$\frac{l}{r}=1+e\ \cos\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\Rightarrow \frac{1}{r}=\frac{1+e\ \text{co}}{l}$$

Let the vectorical angle of M be $\emptyset$ then vectorial angle of M' be $(\pi+\emptyset)$

From equation (1), we get

$$\frac{1}{MS}=\frac{1+e\ \cos\emptyset}{l} \text{ and } \frac{1}{M'S}=\frac{1+e\ \cos(\pi+\emptyset)}{l}\Rightarrow\frac{1}{M'S}=\frac{1-e\ \cos\emptyset}{l}$$

Therefore,

$$\frac{1}{MS}\cdot\frac{1}{M'S}=\frac{1+e\ \cos\emptyset}{l}\cdot\frac{1-e\ \cos\emptyset}{l}=\frac{1-e^2\cos\emptyset}{l^2}$$

$$\Rightarrow\frac{1}{MS}\cdot\frac{1}{M'S}=\frac{1-e^2\cos^2\emptyset}{l^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

MSM' and NSN' be two perpendicular focal chords of a conic.

Therefore, the vectorical angle of M be $\left(\emptyset+\frac{\pi}{2}\right)$ and vectorical angle of M' be $\left(\emptyset+\frac{\pi}{2}+\pi\right)$

Now Using equation (1), we get

$$\frac{1}{NS}=\frac{1+e\ \cos\left(\emptyset+\frac{\pi}{2}\right)}{l}=\frac{1-e\ \sin\emptyset}{l} \text{ and } \frac{1}{N'S}=\frac{1+e\ \cos\left(\emptyset+\frac{\pi}{2}+\pi\right)}{l}\Rightarrow\frac{1}{N'S}=\frac{1+e\ \sin\emptyset}{l}$$

Therefore,

$\frac{1}{NS} \cdot \frac{1}{N'S} = \frac{1-e\ \sin\emptyset}{l} \cdot \frac{1+e\ \sin\emptyset}{l} = \frac{1-e^2 \sin^2\emptyset}{l^2}$

$\Rightarrow \frac{1}{NS} \cdot \frac{1}{N'S} = \frac{1-e^2 \cos\emptyset}{l^2}$ ……………………………………(4)

Adding (3)and (4), we get

$\frac{1}{MS} \cdot \frac{1}{M'S} + \frac{1}{NS} \cdot \frac{1}{N'S} = \frac{1-e^2 \cos^2\emptyset}{l^2} + \frac{1-e^2 \sin^2\emptyset}{l^2} = \frac{2-e^2(\cos^2\emptyset+\sin^2\emptyset)}{l^2}$

Therefore,

$\frac{1}{MS} \cdot \frac{1}{M'S} + \frac{1}{NS} \cdot \frac{1}{N'S} = \frac{2-e^2}{l^2} = constant$

Ex.3. If MSN and $MS'L$ be two chord of an ellipse through the foci S and S' , prove that $\frac{MS}{SN} + \frac{MS'}{S'L}$ is independent of the position of M.

Proof.

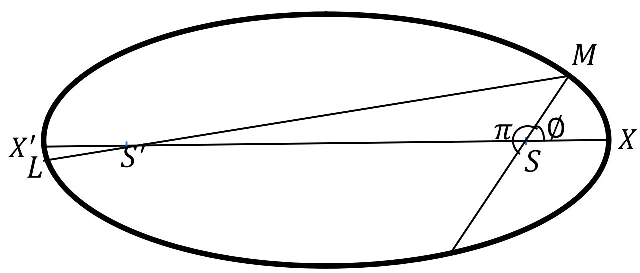


Fig. 14.3.4

Let the polar equation of ellipse be

$\frac{l}{r} = 1 + e\cos\theta$……………………………………….(1)

Let the vectorical angle of M be $\emptyset$ then vectorial angle of N be $(\pi + \emptyset)$

From equation (1), we get

$\frac{1}{MS} = \frac{1+e\ \cos\emptyset}{l}$ ……………………………………………….(2)

and

$\frac{1}{SN} = \frac{1+e\ \cos(\pi+\emptyset)}{l}$

$\Rightarrow \frac{1}{SN} = \frac{1-e\ \cos\emptyset}{l}$ …………………………….……….(3)

Adding equation (2)and equation (3), we get

$\frac{1}{MS} + \frac{1}{SN} = \frac{1+e\ \cos\emptyset}{l} + \frac{1-e\ \cos\emptyset}{l}$

$\Rightarrow \frac{1}{MS} + \frac{1}{SN} = \frac{2}{l}$ ………………………………………………….(4)

Now by multiplying both side with MS in equation (4), we get

$\frac{MS}{MS} + \frac{MS}{SN} = \frac{2MS}{l} \Rightarrow 1 + \frac{MS}{SN} = \frac{2\,MS}{l} \Rightarrow \frac{1}{SN} = \frac{2MS}{l} - 1$

Similarly for the focal chord $MS'L$, we have

$\frac{1}{S'L} = \frac{2MS'}{l} - 1$ ………………………………………….……….(5)

Adding equation (3) and equation (4), we get

$\frac{1}{SN} + \frac{1}{S'L} = \frac{2MS}{l} - 1 + \frac{2MS'}{l} - 1 \Rightarrow \frac{1}{SN} + \frac{1}{S'L} = \frac{2(MS+MS')}{l} - 2$

$\Rightarrow \frac{1}{SN} + \frac{1}{S'L} = \frac{2(MS+MS')}{l} - 2$ ……………………………….……..(6)

As we know that the sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.

Let the length of the major axis be $2a$.

Then, $MS + MS' = 2a$

Using above condition in equation (6), we get

$\frac{1}{SN}+\frac{1}{S{\cdot}L}=\frac{2.2a}{l}-2=\frac{4a}{l}-2=constant$

which implies that the $\frac{MS}{SN}+\frac{MS'}{S'L}$ is independent of the position of M.

Ex.4. A chord MN of a conic whose eccentricity e and semi-latus rectum l subtends a right angle at the focus S, show that

$$\left(\frac{1}{MS}-\frac{1}{l}\right)^2+\left(\frac{1}{SN}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}$$

Proof. Let the polar equation of given conic be

$\frac{l}{r}=1+e\cos\theta$……………………………….(1)

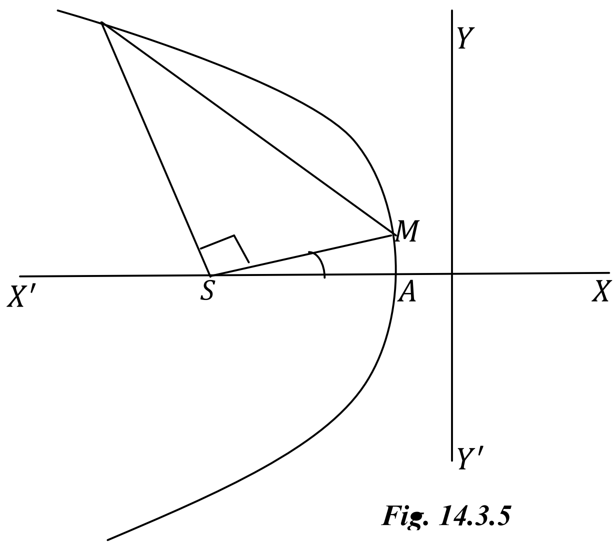


Fig. 14.3.5

Let MNbe chord of given conic which subtends right angle at focus S and the vectorial angle of M be $\emptyset$ then vectorial angle of N be $\left(\frac{\pi}{2}+\emptyset\right)$

From equation (1), we get

$\frac{l}{MS}=1+e\ \cos\emptyset\Rightarrow\frac{1}{MS}=\frac{1+e\ \cos\emptyset}{l}$

$\Rightarrow\frac{1}{MS}-\frac{1}{l}=e\cos\emptyset$ ……………………………(2)

and

$$\frac{l}{SN} = 1 + e\ \cos\left(\frac{\pi}{2} + \emptyset\right) \Rightarrow \frac{1}{SN} = \frac{1 - e \sin\emptyset}{l}$$

$$\Rightarrow \frac{1}{SN} - \frac{1}{l} = \frac{-e\ \sin\emptyset}{l}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Squaring and adding equation (2) and equation (3)

$$\left(\frac{1}{MS} - \frac{1}{l}\right)^2 + \left(\frac{1}{SN} - \frac{1}{l}\right)^2 = \frac{e^2 \cos^2\emptyset}{l^2} + \frac{e^2 \sin^2\emptyset}{l^2} = \frac{e^2(\sin^2\emptyset + \cos^2\emptyset)}{l^2}$$

Hence

$$\left(\frac{1}{MS} - \frac{1}{l}\right)^2 + \left(\frac{1}{SN} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}$$

Ex. 5. Prove that in a conic, the sum of the reciprocals of two perpendicular focal chords is constant.

Proof. Let MSM' and NSN' be two focal chords of a conic as shown in figure

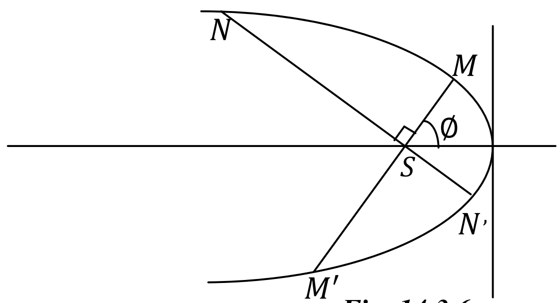


Fig. 14.3.6

Now we prove that $\frac{1}{MSM'} + \frac{1}{NSN'} = constant$

As we know the equation of conic in polar form is defined as

$$\frac{l}{r} = 1 + e\cos\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let the vectorical angle of M be $\emptyset$ then vectorial angle of M' be $(\pi + \emptyset)$

From equation (1), we get

$\frac{l}{MS} = 1 + e\cos\emptyset$

$\Rightarrow MS = \frac{l}{1+e\,co\ \ \emptyset}$……………………………(2)

and

$\frac{1}{SM'} = \frac{1+e\ \cos\ (\pi+\emptyset)}{l}$

$\Rightarrow SM' = \frac{l}{1-e\,co\ \ \emptyset}$…………………………………………(3)

Now $MM' = MS + SM' = \frac{l}{1+e\,co\ \ \emptyset} + \frac{l}{1-e\,co\ \ \emptyset} = \frac{2l}{1-e^2\cos^2\emptyset}$

$\Rightarrow \frac{1}{MM'} = \frac{1-e^2\cos^2\emptyset}{2l}$……………………………………………(4)

Now MM' and $NN'$are perpendicular to each other.

Then the vectorial angle of N be $(\frac{\pi}{2} + \emptyset)$ and vectorial angle of N be $(\pi + \frac{\pi}{2} + \emptyset)$

Similarly, using equation (1), we get

$\frac{l}{NS} = 1 + e\cos(\frac{\pi}{2} + \emptyset)$

$\Rightarrow NS = \frac{l}{1-e\sin\emptyset}$………………………………………(5)

and

$\frac{1}{SN'} = \frac{1+e\ \cos\ (\pi+\frac{\pi}{2}+\emptyset)}{l}$

$\Rightarrow SN' = \frac{l}{1+e\sin\emptyset}$……………………………………(6)

Now $NN' = NS + SN' = \frac{l}{1-e\,\mathrm{si}\quad\emptyset} + \frac{l}{1-e\sin\emptyset} = \frac{2l}{1-e^2\sin^2\emptyset}$

$\Rightarrow \frac{1}{NN'} = \frac{1-e^2\sin^2\emptyset}{2l}$...............................(7)

Adding equation (4) and equation (7), we get

$\frac{1}{MM'} + \frac{1}{NN'} = \frac{1}{MM'} = \frac{1-e^2\cos^2\emptyset}{2l} + \frac{1-e^2\sin^2\emptyset}{2l} = \frac{2-e^2(\cos^2\emptyset+\sin^2\emptyset)}{2l} = \frac{2-e^2}{l^2}$

Therefore $\frac{1}{MM'} + \frac{1}{NN'} = constant$

CHECK YOUR PROGRESS

(CQ 1) $\frac{l}{r} = 1 + e\cos\theta$ is polar equation of conic. (T/F)

(CQ 2) The sum of the reciprocals of two perpendicular focal chords is 0. (T/F)

(CQ 3) Equation of the directrix of conicin polar form is_____________.

14.4 CHORD, TANGENT AND ASYMPTOTES OF A CONIC

Equation of the chord of a conic $\frac{l}{r} = 1 + e\cos\theta$, whose extremeties are $(\rho, \emptyset)$ and $(\rho', \emptyset')$

Consider the given equation of conic is

$\frac{l}{r} = 1 + e\cos\theta$..(1)

Let $M(\rho, \emptyset)$ and $N(\rho', \emptyset')$ lie on given conic (1).

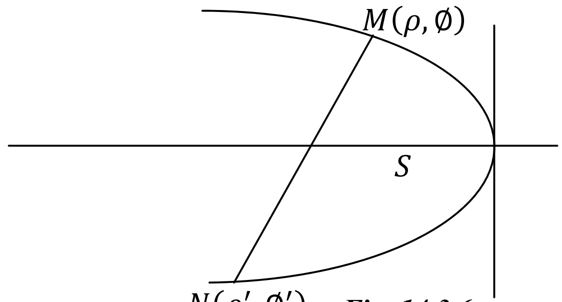


Fig. 14.3.6

Using equation (1), we get

$\frac{l}{\rho} = 1 + e\cos\ \emptyset$……………………………………(2)

and

$\frac{l}{\rho'} = 1 + e\cos\ \emptyset'$……………………………………(3)

Let the polar equation of any line be

$A\cos\theta + \mathrm{B}\sin\theta = \frac{\mathrm{l}}{\mathrm{r}}$ ………………………………………………(4)

If the straight line passes through $(\rho, \emptyset)$, then

$A\cos\emptyset + \mathrm{B}\sin\emptyset = \frac{\mathrm{l}}{\rho}$…………………………………………(5)

Using the value of $\frac{l}{\rho}$ from equation (2) in equation (4), we get

$A\cos\emptyset + \mathrm{B}\sin\emptyset = 1 + e\cos\ \emptyset$

$\Rightarrow (A - e)\cos\emptyset + \mathrm{B}\sin\emptyset - 1 = 0$……………………………(6)

Similarly if the straight line passes through $(\rho', \emptyset')$, we get

$(A - e)\cos\emptyset' + \mathrm{B}\sin\emptyset' - 1 = 0$……………………………………(7)

Solving equation (5) and equation (6) for $(A-e)$ and B, we get

$$\frac{A-e}{-\sin\emptyset+\sin\emptyset'}=\frac{B}{-\cos\emptyset'+\cos\emptyset}=\frac{1}{\cos\emptyset\sin\emptyset'-\cos\emptyset'\sin\emptyset}$$

$$\Rightarrow\frac{A-e}{\sin\emptyset'-\sin\emptyset}=\frac{B}{\cos\emptyset-c\quad\emptyset'}=\frac{1}{\sin(\emptyset'-\emptyset)}$$

$$\Rightarrow\frac{A-e}{\frac{1}{2}\cos\frac{(\emptyset'+\emptyset)}{2}\sin\frac{(\emptyset'-\emptyset)}{2}}=\frac{B}{\frac{1}{2}\sin\frac{(\emptyset'+\emptyset)}{2}\sin\frac{\emptyset'-\emptyset}{2}}=\frac{1}{\frac{1}{2}\cos\frac{(\emptyset'-\emptyset)}{2}\sin\frac{(\emptyset'-\emptyset)}{2}}$$

Therefore,

$$A-e=\frac{\frac{1}{2}\cos\frac{(\emptyset'+\emptyset)}{2}\sin\frac{(\emptyset'-\emptyset)}{2}}{\frac{1}{2}\cos\frac{(\emptyset'-\emptyset)}{2}\sin\frac{(\emptyset'-\emptyset)}{2}}=\cos\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\Rightarrow A=e+\cos\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}$$

and

$$B=\frac{\frac{1}{2}\sin\frac{(\emptyset'+\emptyset)}{2}\sin\frac{\emptyset'-\emptyset}{2}}{\frac{1}{2}\cos\frac{(\emptyset'-\emptyset)}{2}\sin\frac{(\emptyset'-\emptyset)}{2}}=\sin\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}$$

Using the given value of A and B in equation (4), we get

$$\left(e+\cos\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\right)\cos\theta+\left(\sin\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\right)\sin\theta=\frac{l}{r}$$

$$\Rightarrow e\ \cos\theta+\cos\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\cos\theta+\sin\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\sin\theta=\frac{l}{r}$$

$$\Rightarrow e\ \cos\theta+\cos\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\cos\theta+\sin\frac{(\emptyset'+\emptyset)}{2}\sec\frac{(\emptyset'-\emptyset)}{2}\sin\theta=\frac{l}{r}$$

$$\Rightarrow e\ \cos\theta+\left(\cos\frac{(\emptyset'+\emptyset)}{2}\cos\theta+\sin\frac{(\emptyset'+\emptyset)}{2}\sin\theta\right)\sec\frac{(\emptyset'-\emptyset)}{2}=\frac{l}{r}$$

$$\Rightarrow e\ \cos\theta+\left(\cos\frac{(\emptyset'+\emptyset)}{2}\cos\theta+\sin\frac{(\emptyset'+\emptyset)}{2}\sin\theta\right)\sec\frac{(\emptyset'-\emptyset)}{2}=\frac{l}{r}$$

$$\Rightarrow\frac{l}{r}=e\ \cos\theta+\cos\left(\theta-\frac{(\emptyset'+\emptyset)}{2}\right)\sec\frac{(\emptyset'-\emptyset)}{2}\ldots\ldots\ldots\ldots(5)$$

This is the required equation of chordMN in polar form.

Equation of the tangent at the point $(\rho, \emptyset)$ of the conic $\frac{l}{r} = 1 + e\cos\ \theta$

Let $M(\rho, \emptyset)$ be a given point on the given conic

$\frac{l}{r} = 1 + e\cos\ \theta$ ……………………………………………..(1)

Let $N(\rho', \emptyset')$ be another point on the given conic.

As discussed above, the equation of the chord joining the points M and N as

$\frac{l}{r} = e\ \cos\theta + \cos\left(\theta - \frac{(\emptyset' + \emptyset)}{2}\right)\sec\frac{(\emptyset' - \emptyset)}{2}$………………..(2)

By the definition of tangent,

the tangent at M is the limiting position of the chord MN as $\emptyset' \to \emptyset$.

Therefore by taking the limit in equation (2) as $\emptyset' \to \emptyset$, we get

$\lim_{\emptyset' \to \emptyset}\frac{l}{r} = \lim_{\emptyset' \to \emptyset}\left(e\ \cos\theta + \cos\left(\theta - \frac{(\emptyset' + \emptyset)}{2}\right)\sec\frac{(\emptyset' - \emptyset)}{2}\right)$

$\Rightarrow \frac{l}{r} = e\ \cos\theta + \cos(\theta - \emptyset)$ ....(3)

This is the required equation of tangent at M.

Some important results

1. If the conic is $\frac{l}{r} = 1 + e\cos(\theta - \theta_1)$, the tangent at the point $M(\rho, \emptyset)$ be
   $\frac{l}{r} = e\ \cos(\theta - \theta_1) + \cos(\theta - \emptyset)$
2. Equation (3) can be written as
   $\frac{l}{r} = e\ \cos\theta + \cos\theta\cos\emptyset + \sin\theta\sin\emptyset$
   $\Rightarrow l = re\ \cos\theta + r\cos\theta\cos\emptyset + r\sin\theta\sin\emptyset$..(4)

As we know that $x = r\cos\theta, y = r\sin\theta$, equation (4) become

$l = xe + x\cos\emptyset + y\sin\emptyset \Rightarrow y = -\frac{e+\cos\emptyset}{\sin\emptyset}x + \frac{l}{\sin\emptyset}$

Hence the slope of tangent is $-\frac{e+\cos\emptyset}{\sin\emptyset}$

Ex. 6. Two equal ellipses of eccentricity e, are placed with their axes at right angles and they have one focus S in common. If MN be a common tangent, show that the angle MSN is equal to $\sin^{-1}\left(\frac{e}{\sqrt{2}}\right)$.

Proof. Consider two equal ellipses with eccentricity e and having common focus S.

Let axis of one ellipse as the initial line such that the axis of other ellipse makes an angle $\frac{\pi}{2}$ with initial line,

Assume the equation of one of the ellipse is

$\frac{l}{r} = 1 + e\cos\theta$ ……………………………………………..(1)

and the equation of another ellipse is

$\frac{l}{r} = 1 + e\cos\left(\theta - \frac{\pi}{2}\right)$

$\Rightarrow \frac{l}{r} = 1 + e\sin\theta$ ……………………………….……..(2)

Let MN be a common tangent and the vectorial angles of M on conic (1) be φ_1,

Hence equation of tangent at M is

$\frac{l}{r} = e\cos\theta + \cos(\theta - \varphi_1)$

$\Rightarrow \frac{l}{r} = e\cos\theta + \cos\theta\cos\varphi_1 + \sin\theta\sin\varphi_1$

$\Rightarrow \frac{l}{r} = (e + \cos\varphi_1)\cos\theta + \sin\varphi_1\sin\theta$................................(3)

Similarly if the vectorial angles of N on conic (2) be φ_2, equation of tangent at N is

$\frac{l}{r} = e \sin\theta + \cos(\theta - \varphi_2)$

$\Rightarrow \frac{l}{r} = e \sin\theta + \cos\theta\cos\varphi_2 + \sin\theta\sin\varphi_2$

$\Rightarrow \frac{l}{r} = \cos\varphi_2\cos\theta + (e + \sin\varphi_2)\sin\theta$...............(4)

Now equation (3) and (4) should be identical,

Comparing equation (3) and (4), we get

$$1 = \frac{(e + \cos\varphi_1)}{\cos\varphi_2} = \frac{\sin\varphi_1}{(e + \sin\varphi_2)}$$

Hence,

$$\cos\varphi_2 = e + \cos\varphi_1$$

$\Rightarrow \cos\varphi_2 - \cos\varphi_1 = e$ ....(5)

and

$$e + \sin\varphi_2 = \sin\varphi_1 \Rightarrow \sin\varphi_1 - \sin\varphi_2 = e$$

$\Rightarrow \sin\varphi_1 - \sin\varphi_2 = e$ ....(6)

Squaring and adding both side of equation (5)and (6), we get

$(\cos\varphi_2 - \cos\varphi_1)^2 + (\sin\varphi_1 - \sin\varphi_2)^2 = 2e^2$

$\Rightarrow \cos^2\varphi_2 + \cos^2\varphi_1 - 2\cos\varphi_2\cos\varphi_1 + \sin^2\varphi_1 + \sin^2\varphi_2 - 2\sin\varphi_1\sin\varphi_2 = 2e^2$

$\Rightarrow \cos^2\varphi_2 + \sin^2\varphi_2 + \cos^2\varphi_1 + \sin^2\varphi_1 - 2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) = 2e^2$

$\Rightarrow 2 - 2\cos(\varphi_1 - \varphi_2) = 2e^2 \Rightarrow 2 - 2\cos(\varphi_1 - \varphi_2) = 2e^2$

$\Rightarrow 1 - \cos(\varphi_1 - \varphi_2) = e^2$

$\Rightarrow 2\sin^2\frac{(\varphi_1-\varphi_2)}{2} = e^2 \Rightarrow \sin^2\frac{(\varphi_1-\varphi_2)}{2} = \frac{e^2}{2} \Rightarrow \sin\frac{(\varphi_1-\varphi_2)}{2} = \sqrt{\frac{e^2}{2}}$

$\Rightarrow \varphi_1 - \varphi_2 = 2\sin^{-1}\sqrt{\frac{e^2}{2}} = 2\sin^{-1}(\frac{e}{\sqrt{2}})$

Hence, $\angle MSN = \varphi_1 - \varphi_2 = 2\sin^{-1}(\frac{e}{\sqrt{2}})$

Ex.7. Show that the two conics $\frac{l}{r} = 1 + e\cos\theta$ and $\frac{l'}{r} = 1 + e'\cos(\theta - \varphi)$ will touch one another if

$$l^2(1 - e'^2) + l'^2(1 - e^2) = 2ll'(1 - ee'\cos\varphi)$$

Proof. Assume the equation of one of the conic is

$\frac{l}{r} = 1 + e\cos\theta$ ……………………………………………..(1)

and the equation of another conic is

$\frac{l'}{r} = 1 + e'\cos(\theta - \varphi)$ ………………………….……..(2)

Let M be a common point of contact and $\emptyset$ be the vectorial angle of M, then tangent at this point with respect to the conic (1) is

$\frac{l}{r} = e\cos\theta + \cos(\theta - \emptyset)$

$\Rightarrow \frac{l}{r} = (e + \cos\emptyset)\cos\theta + \sin\emptyset\sin\theta$ ……………..…..(3)

Similarly, tangent at this point with respect to the conic (2) is

$\frac{l'}{r} = e'\cos(\theta - \varphi) + \cos(\theta - \emptyset)$

$\Rightarrow \frac{l'}{r} = e'(\cos\theta\cos\varphi + \sin\theta\sin\varphi) + \cos\theta\cos\emptyset + \sin\theta\sin\emptyset$

$\Rightarrow \frac{l'}{r} = (e'\cos\varphi + \cos\emptyset)\cos\theta + (e'\sin\varphi + \sin\emptyset)\sin\theta$...(4)

Now we compare both equation (3) and equation (4) as both equations represent same straight line, we get

$\frac{l}{l'} = \frac{e + \cos\emptyset}{(e'\cos\varphi + \cos\emptyset)} = \frac{\sin\emptyset}{e'\sin\varphi + \sin\emptyset}$

$\Rightarrow l(e' \cos\ \varphi + \cos\ \emptyset) = l'(e + \cos\emptyset)$

$\Rightarrow le' \cos\ \varphi + l \cos\ \emptyset = l'e + l' \cos\emptyset \Rightarrow (l - l')\cos\emptyset = l'e - le' \cos\ \varphi$

$\Rightarrow \cos\emptyset = \frac{l'e - le' \cos\ \varphi}{l - l'}$ ……………………………………………….(5)

and

$\Rightarrow l(e' \sin\ \varphi + \sin\ \emptyset) = l' \sin\emptyset \Rightarrow le' \sin\ \varphi + l \sin\ \emptyset = l' \sin\emptyset$

$\Rightarrow (l - l') \sin\emptyset = -\ le' \sin\ \varphi$

$\Rightarrow \sin\emptyset = -\frac{le' \sin\ \varphi}{l - l'}$……………………………(6)

Squaring and adding on both side of equation (5) and (6), we get

$\cos^2\emptyset + \sin^2\emptyset = \left(\frac{l'e - le' \cos\ \varphi}{l - l'}\right)^2 + \left(-\frac{le' \sin\ \varphi}{l - l'}\right)^2$

$\Rightarrow (l - l')^2 = l'^2 e^2 + l^2 e'^2 \cos^2\varphi - 2ll'ee' \cos\ \varphi + l^2 e'^2 \sin^2\varphi$

$\Rightarrow (l - l')^2 = l'^2 e^2 + l^2 e'^2 \cos^2\varphi - 2ll'ee' \cos\ \varphi + l^2 e'^2 \sin^2\varphi$

$\Rightarrow l^2 + l'^2 - 2ll' = l'^2 e^2 + l^2 e'^2 - 2ll'ee' \cos\ \varphi$

$\Rightarrow l^2\left(1 - e'^2\right) + l'^2(1 - e^2) = 2ll'(1 - ee' \cos\ \varphi)$

Ex. 8. A chord of a conic subtends a constant angle at a focus of the conic. Show that the chord touches another conic.

Proof. Let the equation of given conic be

$\frac{l}{r} = 1 + e\ cos\ \theta$ ………………………………………………… (1)

Let MN be a chord and the vectorial angle at M and N be $\gamma - \varphi$ and $\gamma + \varphi$ respectively such that chord MN of the given conic (1) subtends a constant angle 2φ at the focus S.

Now the equation of the chord MN be

$\frac{l}{r} = e\ \cos\theta + \cos\left(\theta - \frac{(\gamma+\varphi+\gamma-\varphi)}{2}\right)\sec\left(\frac{\gamma+\varphi-(\gamma-\varphi)}{2}\right)$

$\Rightarrow \frac{l}{r} = e\ \cos\theta + \cos(\theta-\gamma)\sec\varphi$

$\Rightarrow \frac{(l\cos\varphi)}{r} = (e\ \cos\varphi)\ \cos\theta + \cos(\theta-\gamma)$ ....(2)

Here we can see that the equation (2) is an equation of tangent of line $\frac{l\,\text{co}}{r} = 1 + (e\ \cos\varphi)\ cos\ \theta$ at the point M.

Hence chord touches another curve $\frac{l\cos\varphi}{r} = 1 + (e\ \cos\varphi)\ cos\ \theta$ at point M.

Ex. 9. If a chord MN of a conic $\frac{l}{r} = 1 + e\cos\theta$ subtends a constant angle 2φ at a focus of the conic. Prove that the locus of the point L where it meets the internal bisector SL of the angle $\angle MSN$ is conic

$$\frac{l\cos\varphi}{r} = 1 + e\cos\varphi\cos\theta$$

Proof. Consider the equation of conic is

$\frac{l}{r} = 1 + e\cos\theta$ ....(1)

Let MN be a chord subtends a constant angle 2φ at a focus such that $\angle MSN = 2\varphi$ and SL be the bisector of angle $\angle MSN$ meeting curve at L.

Let the vectorial angle at L be γ.

Therefore, the vectorial angle at M and N will be $\gamma-\varphi$ and $\gamma+\varphi$ respectively.

Hence the equation of chord MN will be

$\frac{l}{r} = e\ \cos\theta + \cos\left(\theta - \frac{(\gamma+\varphi+\gamma-\varphi)}{2}\right)\sec\left(\frac{\gamma+\varphi-(\gamma-\varphi)}{2}\right)$

$\frac{l}{r} = e\ \cos\theta + \cos\left(\theta - \frac{(\gamma+\varphi+\gamma-\varphi)}{2}\right)\sec\left(\frac{\gamma+\varphi-(\gamma-\varphi)}{2}\right)$

$\Rightarrow \frac{l}{r} = e\ \cos\theta + \cos(\theta-\gamma)\sec\varphi$ ....(2)

Solving equation (2) with vectorial angle of P i.e. γ, we get

$\frac{l}{r} = e\ \cos\theta + \cos(\gamma-\gamma)\sec\varphi \Rightarrow \frac{l}{r} = e\ \cos\theta + \sec\varphi$

$\Rightarrow \frac{l\cos\varphi}{r} = 1 + e\ \cos\varphi\cos\theta$

Equation of asymptotes of conic $\frac{l}{r} = 1 + e\cos\ \theta$

Consider the given equation of conic is

$\frac{l}{r} = 1 + e\cos\ \theta$..(1)

Let $M(\rho, \emptyset)$ be any point on given conic (1), then

$\frac{l}{\rho} = 1 + e\cos\ \emptyset$..(2)

And the equation of tangent at $M(\rho, \emptyset)$ be

$\frac{l}{r} = e\ \cos\theta + \cos(\theta - \emptyset)$ ....(3)

$\Rightarrow l = re\ \cos\theta + r\cos\theta\cos\emptyset + r\sin\theta\sin\emptyset$.......................(4)

By the definition of asymptote, the limiting position of the tangent as the point of contact tends to infinity.

Hence equation (2) will tends to an asymptote if $\rho \to \infty$.

Taking limit $\rho \to \infty$ on both sides in the equation (1), we get

$0 = 1 + e\cos\emptyset \Rightarrow \cos\emptyset = -\frac{1}{e}$ and $\sin\emptyset = \pm\sqrt{1-\frac{1}{e^2}}$

Using given value of $\cos\emptyset$ and $\sin\emptyset$ in equation (4), we get

$$l = re\cos\theta + r\cos\theta\left(-\frac{1}{e}\right) + r\sin\theta\left(\pm\sqrt{1-\frac{1}{e^2}}\right)$$

$$\Rightarrow le = re^2\cos\theta - r\cos\theta \pm r\sin\theta\left(\sqrt{e^2-1}\right)$$

$$\Rightarrow \frac{le}{r} = (e^2-1)\cos\theta + \left(\sqrt{e^2-1}\right)\sin\theta \ldots\ldots\ldots\ldots\ldots(5)$$

and

$$\Rightarrow \frac{le}{r} = (e^2-1)\cos\theta - \left(\sqrt{e^2-1}\right)\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Hence equation (5) and equation (6) are required equations of asymptotes.

Clearly we can observe that asymptotes are real only when $e > 1$ or we can say the conic is hyperbola.

CHECK YOUR PROGRESS

(CQ 4) Equation of asymptotes of conic $\frac{l}{r} = 1 + e\cos\theta$. (T/F)

(CQ 5) The slope of tangent is $-\frac{e+\cos\emptyset}{\cos\emptyset}$. (T/F)

14.5 AUXILIARY CIRCLE AND DIRECTOR CIRCLE

Auxiliary Circle: The locus of the foot of the perpendicular from the focus on any tangent to a central conic is a circle called the auxiliary circle of the conic.

Polar Equation of the auxiliary circle of the conic $\frac{l}{r} = 1 + e\cos\theta$

Let the equation of conic be

$\frac{l}{r} = 1 + e\cos\theta$……………………………(1)

Let $M(\rho, \emptyset)$be a point on the given curve.

Hence the equation of tangent at M will be

$\frac{l}{r} = e\cos\theta + \cos(\theta - \emptyset)$ ……………………………………………(2)

$\Rightarrow l = r(e\cos\theta + \cos\theta\cos\emptyset + \cos\theta\cos\emptyset)$

$\Rightarrow l = r(e\cos\theta + \cos\theta\cos\emptyset + \sin\theta\sin\emptyset)$

$\Rightarrow l = r\cos\theta(e + \cos\emptyset) + r\sin\theta\sin\emptyset$

Now changing above equation into Cartesian form by taking$x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow l = (e + \cos\emptyset)x + \sin\emptyset\ y$…………………………………………………(3)

Now the equation of line passing through the focus S (origin)and perpendicular to the equation (2) can be written as

$0 = \sin\emptyset\ x - (e + \cos\emptyset)y$

Now changing above equation in polar form, we get

$0 = r\sin\emptyset\cos\theta - r(e + \cos\emptyset)\sin\theta$

$\Rightarrow 0 = r\sin\emptyset\cos\theta - er - r\cos\emptyset\sin\theta \Rightarrow 0 = -\sin(\theta - \emptyset) - e\sin\theta$

$\Rightarrow -e\sin\theta = \sin(\theta - \emptyset)$ …………………………..(4)

Now the foot of the perpendicular from the focus S to the tangent (2) is given by the intersection of (2) and (4).

Now equation (2) can be written as

$\frac{l}{r} - e\cos\theta = \cos(\theta - \emptyset)$ ……………………………………(5)

Squaring and adding equation (4) and equation (5), we get

$(-e\sin\theta)^2 + \left(\frac{l}{r} - e\cos\theta\right)^2 = \sin^2(\theta - \emptyset) + \cos^2(\theta - \emptyset)$

$\Rightarrow e^2\sin^2\theta + \frac{l^2}{r^2} + e^2\cos^2\theta - 2\frac{l}{r}.e\cos\theta = 1$

$\Rightarrow e^2 + \frac{l^2}{r^2} - \frac{2l}{r}\cos\theta = 1$

$\Rightarrow (e^2 - 1)r^2 - 2ler\cos\theta + l^2 = 0$

This is the required equation of auxiliary circle.

Special Case: When conic is parabola i.e. $e = 1$

The equation of auxiliary circle becomes

$$-2lr\cos\theta + l^2 = 0 \Rightarrow 2r\cos\theta - l = 0 \Rightarrow \frac{l}{r} = 2\cos\theta$$

$$\Rightarrow \frac{l}{r} = \cos(\theta - 0) + 1.\cos\theta$$

which is the equation of tangent to the parabola $\frac{l}{r} = 1 + \cos\theta$ at the point $\theta = 0$.

Ex. 10. Find the point of intersection of the two tangent at the points $M(\rho_1, \varphi_1)$ and $N(\rho_2, \varphi_2)$

Proof. Let $M(\rho_1, \varphi_1)$ and $N(\rho_2, \varphi_2)$ be two points on a given curve

$\frac{l}{r} = 1 + e\cos\theta$ ……………………………………(1)

Now, the equation of tangent at $M(\rho_1, \varphi_1)$ can be written as

$\frac{l}{r} = \cos(\theta - \varphi_1) + e\cos\theta$ ....(2)

and the equation of tangent at $N(\rho_2, \varphi_2)$ can be written as

$\frac{l}{r} = \cos(\theta - \varphi_2) + e\cos\theta$ ....(3)

Now subtracting equation (2)from equation (3), we get

$0 = \cos(\theta - \varphi_2) - \cos(\theta - \varphi_1) \Rightarrow \cos(\theta - \varphi_1) = \cos(\theta - \varphi_2)$

$\Rightarrow (\theta - \varphi_1) = \pm(\theta - \varphi_2)$

If$(\theta - \varphi_1) = (\theta - \varphi_2)$, then$\varphi_1 = \varphi_2$, which cannot be possible.

Therefore, $(\theta - \varphi_1) = -(\theta - \varphi_2) \Rightarrow 2\theta = \varphi_1 + \varphi_2 \Rightarrow \theta = \frac{\varphi_1+\varphi_2}{2}$

Now using this value of θ in equation (2), we get

$$\frac{l}{r} = \cos(\frac{\varphi_1 + \varphi_2}{2} - \varphi_1) + e\cos\left(\frac{\varphi_1 + \varphi_2}{2}\right) \Rightarrow \frac{l}{r} = \cos\frac{\varphi_2 - \varphi_1}{2} + e\cos\left(\frac{\varphi_1 + \varphi_2}{2}\right)$$

If the point of intersection is (r', θ'), then

$\theta' = \frac{\varphi_1+\varphi_2}{2}$ and $\frac{l}{r'} = \cos\frac{\varphi_2-\varphi_1}{2} + e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$

Ex. 11. **L , M and $N$are three points on the conic $\frac{l}{r} = 1 + \cos\theta$the focus S being the pole. The tangents at M meets SL and SN in A and B so that $SA = SB = l$. Prove that the chord $\frac{l}{r} = 1 + 2e\cos\theta$**

Proof. The equation of conic is

$\frac{l}{r} = 1 + e\cos\theta$...(1)

Letφ_1, φ_2 and φ_3 be the vectorial angles at L , M and N respectively.

Now tangent at M meets SL and SN in A and B respectively such that $SA = SB = l$.

Therefore, the polar coordinates of A and B are (l, φ_1) and (l, φ_3) respectively.

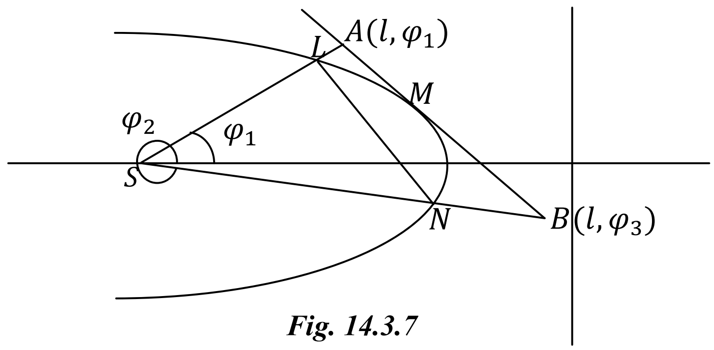


Fig. 14.3.7

Now the tangent at M is

$1 = e\cos\theta + \cos(\theta - \varphi_2)$ …………………………….(2)

and the points $A(l, \varphi_1)$ lies on given tangent. Hence

$1 = e\cos\varphi_1 + \cos(\varphi_1 - \varphi_2)$

$\Rightarrow 1 = e\cos\varphi_1 + \cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2$

$\Rightarrow (e + \cos\varphi_2)\cos\varphi_1 + \sin\varphi_1\sin\varphi_2 - 1 = 0$……(3)

Similarly the point $B(l, \varphi_3)$ lies on given tangent. Therefore

$(e + \cos\varphi_2)\cos\varphi_3 + \sin\varphi_3\sin\varphi_2 - 1 = 0$……………..(4)

Solving equation (3) and (4) for $(e + \cos\varphi_2)$ and $\sin\varphi_2$, we have

$$\frac{e+\cos\ _2}{\sin\varphi_3 - \sin\varphi_1} = \frac{\sin\varphi_2}{\cos\varphi_1 - \cos\varphi_3} = \frac{1}{\cos\varphi_1\sin\varphi_3 - \cos\varphi_3\sin\varphi_1}$$

$$\Rightarrow \frac{e+\cos\ \ {}_{2}}{2\cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)} = \frac{\sin\varphi_2}{2\,s\ \ \left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)} = \frac{1}{\sin(\varphi_3-\varphi_1)}$$

$$\Rightarrow \frac{e+\cos\ \ {}_{2}}{2\cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)} = \frac{\sin\varphi_2}{2\sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)} = \frac{1}{2\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)\cos\left(\frac{\varphi_3-\varphi_1}{2}\right)}$$

Therefore,

$$e+\cos\varphi_2 = \frac{2\cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)}{2\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)\cos\left(\frac{\varphi_3-\varphi_1}{2}\right)}$$

$$= \cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right)$$

$$\Rightarrow e+\cos\varphi_2 = \cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and $\sin\varphi_2 = \frac{2\sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)}{2\sin\left(\frac{\varphi_3-\varphi_1}{2}\right)\cos\left(\frac{\varphi_3-\varphi_1}{2}\right)}$

$$\sin\varphi_2 = \sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The equation of chord LN can be written as

$$\frac{l}{r} = e\ \cos\theta + \cos\left(\theta - \left(\frac{\varphi_3+\varphi_1}{2}\right)\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right)$$

$$\Rightarrow \frac{l}{r} = e\ \cos\theta + \left(\cos\theta\cos\left(\frac{\varphi_3+\varphi_1}{2}\right) + \sin\theta\sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right)$$

$$\Rightarrow \frac{l}{r} = e\ \cos\theta + \left(\cos\theta\cos\left(\frac{\varphi_3+\varphi_1}{2}\right) + \sin\theta\sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right)$$

$$\Rightarrow \frac{l}{r} = e\ \cos\theta + \cos\theta\cos\left(\frac{\varphi_3+\varphi_1}{2}\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right) + \sin\theta\sin\left(\frac{\varphi_3+\varphi_1}{2}\right)\sec\left(\frac{\varphi_3-\varphi_1}{2}\right)$$

Now using the equation (5) and equation (6), we get

$$\frac{l}{r} = e\ \cos\theta + (e+\cos\varphi_2)\cos\theta + \sin\varphi_2\sin\theta$$

$\Rightarrow \frac{l}{r} = 2e \ \cos\theta + \cos\varphi_2 \cos\theta + \sin\varphi_2 \sin\theta$

$\Rightarrow \frac{l}{r} = 2e \ \cos\theta + \cos(\theta - \varphi_2)$

Now we can see that the above equation is the equation of tangent at the point M to the conic $\frac{l}{r} = 1 + 2e \ \cos\theta$.

It implies that the chord LN touches the conic $1 + 2e \ \cos\theta$.

Director circle: The locus of the point of intersection of two perpendicular tangents to a conic is called the director circle of the conic.

Equation of the director circle

Let the equation of conic be

$\frac{l}{r} = 1 + e\cos\theta$ ……………………………………….(1)

By the definition, director circle of conic is the locus of the point of perpendicular tangents to the conic.

Let the equation of tangents to the given conic at the points $M(\rho_1, \varphi_1)$ and $N(\rho_2, \varphi_2)$ be

$\frac{l}{r} = e\cos\theta + \cos(\theta - \varphi_1)$ ………………………….(2)

and

$\frac{l}{r} = e\cos\theta + \cos(\theta - \varphi_2)$ …………………………….(3)

Now we will find the point of intersection of (2) and (3). For this subtracting equation (3) from equation (2), we have

$0 = \cos(\theta - \varphi_2) - \cos(\theta - \varphi_1) \Rightarrow \cos(\theta - \varphi_2) = \cos(\theta - \varphi_1) \Rightarrow (\theta - \varphi_2) = \pm(\theta - \varphi_1)$

If $(\theta - \varphi_2) = (\theta - \varphi_1) \Rightarrow \varphi_2 = \varphi_1$, which cannot be possible.

Therefore, $(\theta - \varphi_2) = -(\theta - \varphi_1) \Rightarrow 2\,\theta = \varphi_1 + \varphi_2$

$\Rightarrow \theta = \frac{\varphi_1+\varphi_2}{2}$ ……………………………….(4)

Using the value of θ in equation (2), we get

$\frac{l}{r} = e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos(\frac{\varphi_1+\varphi_2}{2} - \varphi_1)$

$\Rightarrow \frac{l}{r} = e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos\left(\frac{\varphi_2-\varphi_1}{2}\right)$……………………………(5)

Let (r', θ') be the point of intersection of the tangents (2) and (3), then it satisfies equation (4) and (5), therefore

$\theta' = \frac{\varphi_1+\varphi_2}{2}$ and $\frac{l}{r'} = e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos\left(\frac{\varphi_2-\varphi_1}{2}\right)$

Now we will change equation (2) in Cartesian form, equation (2) implies

$\frac{l}{r} = e\cos\theta + \cos(\theta - \varphi_1) \Rightarrow \frac{l}{r} = e\cos\theta + \cos\theta\cos\varphi_1 + \sin\theta\sin\varphi_1$

$\Rightarrow \frac{l}{r} = (e + \cos\varphi_1)\cos\theta\ + \sin\varphi_1\sin\theta$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$\frac{l}{r} = (e + \cos\varphi_1)x\ + \sin\varphi_1\,y$ ………………………………………….(6)

Let m_1 be the slope of equation (6), therefore

$m_1 = -\frac{e+\cos\ \ _1}{\sin\varphi_1}$

Similarly if m_2 be the slope of the Cartesian form of equation (3), then

$m_2 = -\frac{e+\cos\varphi_2}{\sin\varphi_2}$

Now it is given that the given two tangents are perpendicular to each other, therefore

$$m_1.m_2 = -1 \Rightarrow \left(-\frac{e+\cos\ \ _1}{\text{si}\ \ _1}\right).\left(-\frac{e+\cos\ \ _2}{\sin\varphi_2}\right) = -1$$

$$\Rightarrow e^2 + (\cos\varphi_1 + \cos\varphi_2) + \cos\varphi_1\cos\varphi_2 = \sin\varphi_1\sin\varphi_2$$

$$\Rightarrow e^2 + e(\cos\varphi_1 + \cos\varphi_2) + \cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 = 0$$

$$\Rightarrow e^2 + 2e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_2-\varphi_1}{2}\right) + 2\cos^2\frac{\varphi_2-\varphi_1}{2} - 1 = 0 \text{ ....(7)}$$

As we know $\theta' = \frac{\varphi_1+\varphi_2}{2}$ and $\frac{l}{r'} = e\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) + \cos\left(\frac{\varphi_2-\varphi_1}{2}\right)$

$$\Rightarrow \cos\left(\frac{\varphi_2-\varphi_1}{2}\right) = \frac{l}{r'} - e\cos\theta'$$

Therefore equation (7) becomes

$$e^2 + 2e\cos\theta'\left(\frac{l}{r'} - e\cos\theta'\right) + 2\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1 = 0$$

$$\Rightarrow e^2 + \frac{2l}{r'}e\cos\theta' - 2e^2\cos^2\theta' + \frac{2l^2}{r'^2} - \frac{4l}{r'}e\cos\theta' + 2e^2\cos^2\theta' - 1 = 0$$

$$\Rightarrow (1-e^2){r'}^2 + 2ler'\cos\theta' - 2l^2 = 0$$

Therefore the locus of (r', θ') is

$$(1-e^2)r^2 + 2ler\cos\theta - 2l^2 = 0$$

This is the required equation of director circle.

CHECK YOUR PROGRESS

(CQ 6)The locus of the foot of the perpendicular from the focus on any tangent to a central conic is a circle called the auxiliary circle of the conic.(T/F)
(CQ 7) The locus of the point of intersection of two perpendicular tangents to a conic is called _______________________.

14.6 PAIR OF TANGENTS AND CHORD OF CONTACT

The equation of pair of tangent drawn the conic $\frac{l}{r} = e\cos\theta$ from the point $(\rho, \emptyset)$ is

$(S_1^2 - 1)(S_2^2 - 1) = M^2$, where $S_1 \equiv \frac{l}{r} - e\cos\theta, S_2 \equiv \frac{l}{\rho} - e\cos\emptyset$ and $M \equiv \left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\emptyset\right) - \cos(\theta - \emptyset)$

Proof. Consider a point M with vectorial angle φ on the conic

$\frac{l}{r} = 1 + e\cos\theta$……………………………………..(1)

The equation of tangent to the given conic (1) at the point φ is

$\frac{l}{r} = \cos(\theta - \varphi) + e\cos\theta$………………………………….(2)

If the above tangent (2) passes through the point $(\rho, \emptyset)$, then

$\frac{l}{\rho} = \cos(\emptyset - \varphi) + e\cos\emptyset$………………….(3)

Now we find the required equation of pair of tangents by eliminating φ between equation (2) and equation (3).

Now it is given that $S_1 \equiv \frac{l}{r} - e\cos\theta$ and$S_2 \equiv \frac{l}{\rho} - e\cos\emptyset$. Hence

$$(S_1^2-1)(S_2^2-1) = \left[\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right]\left[\left(\frac{l}{\rho} - e\cos\emptyset\right)^2 - 1\right]$$

Now using equation (2) and equation (3), we get

$$(S_1^2-1)(S_2^2-1) = [\cos^2(\theta-\varphi)-1][\cos^2(\emptyset-\varphi)-1]$$

$$= [-\sin^2(\theta-\varphi)][-\sin^2(\emptyset-\varphi)]$$

$\Rightarrow (S_1^2-1)(S_2^2-1) = \sin^2(\theta-\varphi)\sin^2(\emptyset-\varphi)$ ....(4)

Now $M = \left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\emptyset\right) - \cos(\theta-\emptyset)$

Using equation (2) and equation (3), we get

$$M = \cos(\theta-\varphi)\cos(\emptyset-\varphi) - \cos(\theta-\emptyset)$$

$$= \frac{1}{2}[2\cos(\theta-\varphi)\cos(\emptyset-\varphi)] - \cos(\theta-\emptyset)$$

$= \frac{1}{2}[\cos(\theta+\emptyset-2\varphi) + \cos(\theta-\emptyset)] - \cos(\theta-\emptyset)$ ($\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)$

Therefore,

$$M = \frac{1}{2}[\cos(\theta+\emptyset-2\varphi) - \cos(\theta-\emptyset)] = -\sin(\theta-\varphi)\sin(\emptyset-\varphi)$$

Hence,

$M^2 = \sin^2(\theta-\varphi)\sin^2(\emptyset-\varphi)$ ....(4)

From equation (4)and equation (5), we get

$(S_1^2 - 1)(S_2^2 - 1) = P^2$

Here we can see that the above equation does not containφ, therefore it is the required equation of the pair of tangents drawn from the point $(\rho, \emptyset)$ to the conic $\frac{l}{r} = 1 + e\cos\theta$.

14.7 SUMMARY

In this unit we discussed about chord, tangent and asymptotes of a conic, auxiliary circle and director circle.

14.8 GLOSSARY

1. **Curve:** a continuous and smooth flowing line without any sharp turns
2. **Eccentricity:** the ratio of the distance from any point on the conic section to the focus to the perpendicular distance from that point to the nearest directrix
3. **Hyperbola:** set of all the points, the difference of whose distances from the two fixed points in the plane (foci) is a constant
4. **Tangent:**straight line that "just touches" the curve at that point.
5. **Auxillary circle:** circle of an ellipse mainly determined by the diameter of the major axis.

14.9 REFERENCES

1. Jain, P. K. A Textbook of Analytical Geometry of Three Dimensions. New Age International, 2005.
2. Khan, Ratan Mohan. Analytical Geometry of Two and Three Dimensions and Vector Analysis. New Central Book Agency, 2012

14.10 SUGGESTED READINGS

1. Robert J. T. Bell, An Elementary Treatise on Coordinate Geometry of Three Dimensions. Macmillan India Ltd, 1994.

2. D. Chatterjee, Analytical Geometry: Two and Three Dimensions. Narosa Publishing House, 2009.

14.11 TERMINAL QUESTION

MULTIPLE CHOICE QUESTION

(TQ-1) The point of intersection of axis of parabola and tangent at vertex of parabola.

a) Vertex b) focus c) origin d) None of these

(TQ-2) The point of intersection of axis of parabola and latus rectum of parabola.

a) Vertex b) focus c) origin d) None of these

(TQ-3) The nature of the conic represented by the equation $5x^2 - 6xy + 5y^2 + 26x - 22y + 5 = 0$ is

a) Ellipse b) Hyperbola c) Parabola d) Circle

(TQ-4) Focus of the conic represented by the equation $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$ is

a) (2,3) b) (-2,3) c) (2,-3) d) (-2,-3)

FILL IN THE BLANKS

(TQ-5) The conic $32x^2 + 52xy - 7y^2 - 64x - 52y - 248 = 0$ is a ________ ___________

(TQ-6) The equation of transverse axis of the hyperbola $x^2 + 4xy + y^2 - 4 = 0$ is __ _____

(TQ-7) The equation of major axis of the ellipse $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$ is ______________.

LONG ANSWER QUESTIONS

(TQ-8) Trace the curve $32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0$

(TQ-9) Trace the curve $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$

(TQ-10) Trace the conic $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$

(TQ-11) Trace $7x^2 + 52xy - 32y^2 - 170x + 140y = 0$ and find the equation of its asymptote.

(TQ-12) Find the axis, the vertex, the latus rectum, the focus and the equation of the directrix of the parabola $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$.

(TQ-13) Trace the parabola $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$ and the find the coordinate of focus.

(TQ-14) Show that $\frac{1}{x+y-a} + \frac{1}{x-y+a} + \frac{1}{-x+y-a} = 0$ represents a parabola. (TQ-15) Find the coordinates of the focus and the equation of directrix.

14.12 ANSWERS

(CQ 1) T	(CQ 2) F	(CQ 3) $\frac{l}{r} = \cos\theta$
(CQ 4) T	(CQ 5) F	(CQ 6) T
(CQ 7) director circle of the conic		
(TQ-1) a)	(TQ-2) b)	(TQ-3) a)
(TQ-3) c)	(TQ-5) Hyperbola	(TQ-6) $x - y = 0$
(TQ-7) 4		

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University, Haldwani, Nainital-263139
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,
E. mail info@uou.ac.in
Website:htpp://uou.ac.in

MT(N)-120-1(003882)