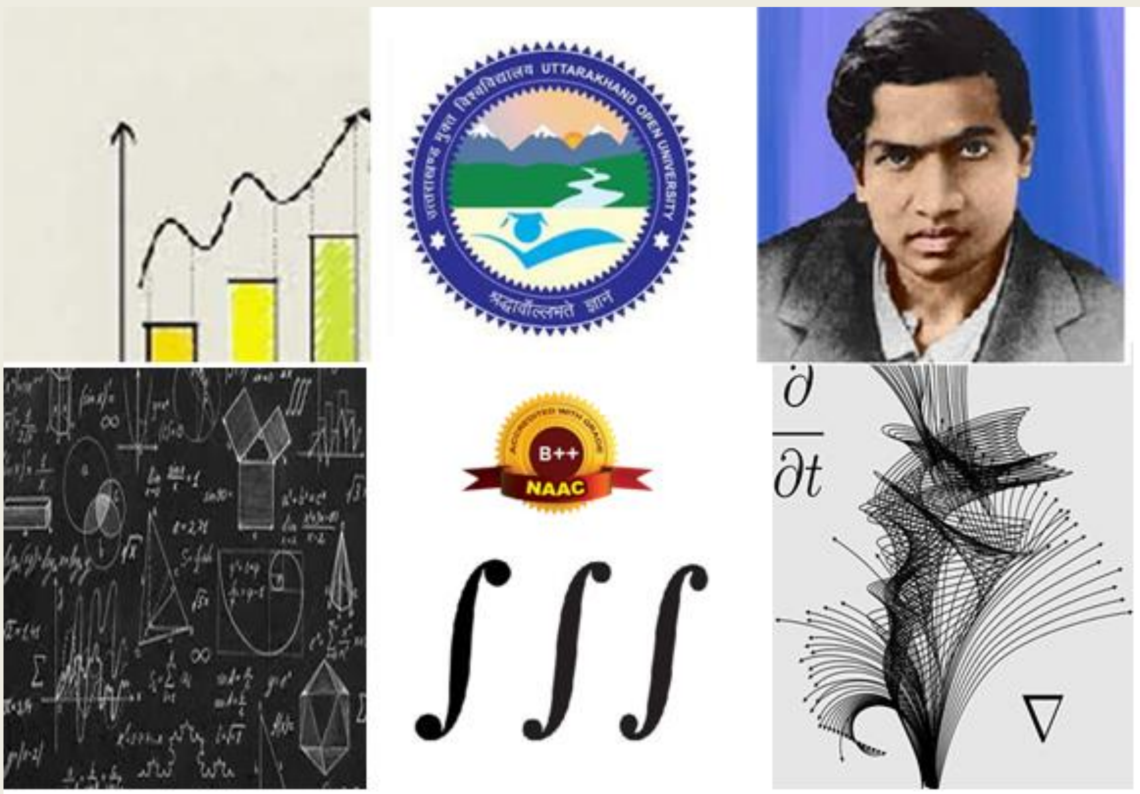
Bachelor of Science

MATHEMATICS

Second Semester

MT(N) - 102
DIFFERENTIAL EQUATION



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: DIFFERENTIAL EQUATION

COURSE CODE: MT(N) - 102

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COURSE INFORMATION

The present self-learning material **"Differential Equation"** has been designed for B.Sc. (Second Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Four Block.

First block is **FIRST ORDER DIFFERENTIAL EQUATIONS,** in this block Formation of Differential Equation, Differential Equation. Ordinary Differential Equation. Partial Differential Equation. Order of Differential Equation. Degree of a Differential Equation. Linear and non- linear differential Equation defined clearly.

Second block is **LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT,** in this block Linear Differential Equations with Constant Coefficients, Complementary Function,
Particular integral-I ($e^{ax}, Sin(ax+b), Cos(ax+b), x^n$),
Particular integral-II ($e^{ax}V(x), any\ other\ function$) defined clearly.

Third block is **HOMOGENEOUS AND SIMULTANEOUS LINEAR DIFFERENTIAL EQUATION,** in this block Simultaneous Linear Differential Equations, Linear Differential Equation of Second order, are defined.

Fourth block is **Differential Equations of first order and Higher Degree,** in this block concept of Differential Equations of first order and Higher Degree defined.

Fifth block is **Partial Differential Equations,** in this block concept of Differential Equations of first order and Higher Degree defined.
Partial Differential Equation, Order of Partial Differential Equation, Degree of Partial Differential Equation, Linear and Non-linear Partial Differential Equation, Classification of First Order Partial Differential Equations, Formation of PDEs, Cauchy's Problem for First Order PDEs, Complete Integral are defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

CONTENTS
MT(N)-102

Course Name: DIFFERENTIAL EQUATION

Course Code: MT(N) 102

BLOCK-I

UNIT 1: FORMATION OF DIFFERENTIAL EQUATION

CONTENTS:

1.1 *INTRODUCTION*

Differential equation came into existence with the invention of calculus by Isaac Newton and Gottfried Leibniz.

Isaac Newton (1642-1727)

Gottfried Wilhelm Leibniz (1646-1716)

Fig1.1

Ref:https://en.wikipedia.org

Differential equation play an important role in engineering and science. Many physical problems can be formulated as a differential equation such as the current I in an LCR circuit is described by the differential equation $LI'' + RI' + \frac{1}{c}I = E,$ which is derived from Kirchhoff's laws.

An introduction to differential equation involves understanding the fundamental concept of how these equations describe relationships involving rates of change. In essence, a differential equation is an equation that relates a function to its derivatives.

The study of differential equations consisting of formulation of differential equations, the solutions of differential equations and the physical interpretation of the solution in terms of the given problem.

1.2 *OBJECTIVES*

After studying this unit, learner will be able to

- **i.** To analyze and predict the behavior of these systems over time.
- **ii.** To provide solutions to problems that cannot be solved using other mathematical techniques.
- **iii.** To understand the definition of differential equation.

1.3 *DIFFERENTIAL EQUATION*

An Equation involving derivatives of differentials of one or more dependent variables with respect to one or more independent variables is called ***Differential Equation.***

For Example:

$$\frac{dy}{dx} = (x + sinx) \qquad \ldots (1)$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \qquad \ldots (2)$$

$$y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}} \qquad \ldots (3)$$

$$k(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2} \qquad \dots (4)$$

$$\partial^2 v/\partial t^2 = k(\partial^3 v/\partial x^3)^2 \qquad \dots (5)$$

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 = 0 \qquad \dots (6)$$

1.4 *ORDINARY DIFFERENTIAL EQUATION*

A differential Equation (Art.1.3) given in (1), (2), (3) and (4) involve only one independent variable is called an ***Ordinary Differential Equation.***

1.5 *PARTIAL DIFFERENTIAL EQUATION*

The equation (Art.1.3) given in (5) and (6) involve partial derivatives with respect to more than one independent variable is called a ***Partial Differential Equation***.

1.6 *ORDER OF A DIFFERENTIAL EQUATION*

The ***order of a differential equation*** is order of highest derivative differential equation.

In Art.(1.1) shown that the equation (2) is of 4^{th} order, equation (1) and (3) are of 1^{st} order, equations (4) and (6) are of the second order and equation (5) is of the third order.

1.7 *DEGREE OF A DIFFERENTIAL EQUATION*

The ***Degree of a differential equation*** is power of the height order derivative term in the differential equation.

In Art.(1.1) given the equation (1), (2) and (6) are of first degree. Making equation (3) free from fractions, we describe

$y\, dy/dx = \sqrt{x}(dy/dx)^2 + k$, which is of 2nd degree.

Now consider equation $\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0.$

It is clear that it involves radical sign. So to find the degree we shall discard the radical sign, which can be done by squaring, hence we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3.$$

Obviously its degree is 2.

1.8 *LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION*

A differential equation is said to be Linear if

(i) Every dependent variable and every derivative in involved occurs in the first degree only.

(ii) No products of dependent variable and /or derivatives occur.

A differential equation which is not a linear is called the ***non-linear differential equation.***

For Example:

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$. is linear.

2. $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$. is non-linear because in 2[nd] term is not of degree one.

1.9 *SOLUTION OF DIFFERENTIAL EQUATION*

The solution of a differential equation means any relation between the dependent and independent variables free from derivatives which satisfies the given differential equation is called the solution or integral (primitive) or simply general solution of the differential equation and it contains as many as arbitrary constants equal to the order of the differential equation. Yet there are other types of solutions such as particular and singular solution described as:

Let us consider an n^{th} order ordinary differential equation $F(x, y, y_1, y_2, ...y_n) = 0$ ....(1)

(i) A solution of equation (1) containing n independent arbitrary constants is called a **general solution** or simply solution of (1).

(ii) A solution of equation (1) obtained from a general solution of (1) by giving particular values to one or more of the n independent arbitrary constants are called **particular solution** of (1).

(iii) A solution of equation (1) which cannot be obtained from the general solution of (1) by any choice of the n independent arbitrary constants is called a singular solution of equation (1). For example:

Consider the differential equation $\frac{dy}{dx} = xy^{1/3}$, which has the general solution $y = \pm\left(\frac{x^2}{3} + C\right)^{3/2}$ obtained by one of the standard method (viz. Variable separable method). Since C is an arbitrary constant, taking C = 0, 1. We have four particular solutions:

$$y = \pm \frac{x^3}{3\sqrt{3}} \qquad \text{and} \qquad y = \pm \left(\frac{x^2}{3} + 1 \right)^{3/2}$$

However $y = 0$ is also a solution which cannot be obtained from the general solution by any choice of the value of C. Thus $y = 0$ is a **singular solution**.

1.10 *GEOMETRICAL INTERPRETATION OF A DIFFERENTIAL EQUATION*

For the geometrical interpretation of a differential equation here we consider a differential equation of first order and first degree $\frac{dy}{dx} = f(x, y) \quad ....(1)$.

It is well known from calculus that derivative $\frac{dy}{dx}(= m)$ of $y(x)$ stands for the slope of the tangent to the curve $y(x)$ at any general point $P(x, y)$.

Let the value of m at initial point $C_0(x_0, y_0)$ be m_0 and thus take a neighbouring point $C_1(x_1, y_1)$ of C_0 such that the slope of C_0C_1 is m_0. Let the corresponding value of m at C_1 be m_1.

Likewise take a neighbouring point $C_2(x_2, y_2)$ of C_1 such that the slope of C_1C_2 is m_1 and so on. Thus, if the successive points C_0, C_1, C_2, ... are taken very close to one another, then the broken curve $C_0C_1C_2$......... will approximate to a smooth curve [$y = \phi(x)$], which is a solution of differential equation (1) passing through the initial point $C_0(x_0, y_0)$.

A different choice of the initial point will give a different curve with the same property.

The equation of each such curve is thus a particular solution of differential equation (1) and the equation of the whole family of such curves is the general solution of differential equation (1).

NOTE: Such a simple geometrical interpretation of a second or higher order differential equation is not available.

1.11 *LINEARLY DEPENDENT AND INDEPENDENT SET OF SOLUTIONS*

Definition: the n function $y_1(x), y_2(x), \dots . y_n(x)$ are linearly dependent if $\exists$ constants $c_1, c_2, \dots . c_n$(not all zero), such that

$$c_1y_1 + c_2y_2 + \cdots\cdots + c_ny_n = 0 \qquad \dots(1)$$

If, however, identity (1) implies that $c_1 = c_2 = \cdots = c_n = 0$,

Then $y_1, y_2, \dots . y_n$ are said to be linearly independent.

1.12 *FUNDAMENTAL SET OF SOLUTIONS*

Definition: Any set $y_1, y_2 \dots \dots, y_n$ of n linearly independent solutions of the homogeneous linear nth order differential equation

$$(d^ny/dx^n) + p_1(x)(d^{n-1}y/dx^{n-1}) + p_2(x)(d^{n-2}y/dx^{n-2}) + \cdots + p_n(x)y(x) = 0, \quad x \in I$$

is said to be a fundamental set of solutions on the interval I.

1.13 *FORMULATION OF DIFFERENTIAL EQUATION*

An ordinary differential equation is formed from a relation in the variables and arbitrary constants by simply the elimination of certain arbitrary constants. Similarly the partial differential equation may be formed by the elimination of either arbitrary constants or arbitrary functions.

Suppose we are given an *n*-parameter family of curves. Then there will be *n* arbitrary constants in the family of curves and differentiation the family of curves *n* times so as to get *n* additional equations containing *n* arbitrary constants. Now eliminate *n* arbitrary constants from these (n+1) equations so obtained and then we can obtain a differential equation of order *n* whose solution is the given family of curves.

1.14 *WRONSKIAN*

The Wronskian was introduced by Josef Maria Hone (1776-1853), Polish mathematician who changed his name to Wronski after introducing the Wronskian determinant. The Wronskian of *n* functions $y_1(x)$, $y_2(x)$, $y_3(x)$, ...,$y_n(x)$ is denoted by *W* and is defined as

$$W = \begin{vmatrix} y_1 & y_2 & .... & .... & y_n \\ y_1^{(1)} & y_2^{(1)} & .... & .... & y_n^{(1)} \\ : & : & : & : & : \\ : & : & : & : & : \\ y_1^{(n-1)} & y_2^{(n-1)} & .... & .... & y_n^{(n-1)} \end{vmatrix}$$

Theorem 1: A solution $y(x)$ of equation (1) satisfying the initial conditions $y(x_0) = y^{(1)}(x_0) = y^{(2)}(x_0) =........= y^{(n-1)}(x_0) =0$ is identically zero.

Theorem 2: The Wronskian of two solutions of a differential equation $y''+Py'+Qy = 0,$ where P, Q are either constants or functions of x alone, is either identically zero or never identically zero. Which leads two important results: ***(a).*** If the Wronskian is identically zero then the two solutions are linearly dependent and ***(b).*** If the Wronskian does not vanish identically zero then the two solutions are linearly independent.

NOTE :(i). Identically zero means the solution is zero for each value of x in the interval (a, b).

(ii). If W=0 for each value of x in (a, b), then solutions are linearly dependent.

(iii). If W$\neq$ 0 for at least one value of x in (a, b), then solutions are linearly independent.

Theorem 3: If $y_1(x)$, $y_2(x)$ are any pair of linearly independent solution set of $y''+Py'+Qy=0$, where P, Q are constants or functions of x alone, then the general solution of $y''+Py'+Qy=0$ always can be put in the form $c_1y_1+c_2y_2$, where c_1, c_2 are arbitrary constants.

1.15 *SOLVED EXAMPLES*

EXAMPLE1: If $y=(A/x)+B$, then show that

$$(d^2y/dx^2)+(2/x)\times(dy/dx)=0.$$

SOLUTION: Given that

$$(d^2y/dx^2)+(2/x)\times(dy/dx)=0 \qquad \ldots (1)$$

$$y=(A/x)+B \qquad \ldots (2)$$

Now differentiating equation (2) with respect to x,

$$dy/dx=-A/x^2 \qquad \ldots (3)$$

Again differentiating (3) with respect to x, $d^2y/dx^2=(2A/x^3)$

Putting the value of dy/dx and d^2y/dx^2 in (1), we get

$(2A/x^3)+(2/x)\times -A/x^2=0$ or $0=0$

Hence eq. (2) is the solution of (1).

EXAMPLE 2: Find the differential equation of the family of curves $y = e^{mx}$, where m is arbitrary constant.

SOLUTION: Now given that the family of curves

$$y = e^{mx} \quad \text{... (1)}$$

Differentiating (1) w.r.t. x, we have

$$dy/dx = me^{mx} \quad \text{... (2)}$$

From (1) and (2) $dy/dx = my$

$$\Rightarrow \quad m = (1/y) \times (dy/dx) \quad \text{... (3)}$$

$$my = dy/dx$$

$$my = me^{mx}$$

$$\log y = mx$$

So

$$m = \frac{\log y}{x} \quad \text{... (4)}$$

Eliminating m from (3) and (4)

$$(1/y) \times (dy/dx) = (1/x) \times \log y.$$

EXAMPLE 3: Find the differential equation satisfied by family of circles$x^2 + y^2 = a^2$, a being an arbitrary constant.

SOLUTION: Let us consider the equation of any circle passing through the origin and whose centre is on the x-axis is given by

$x^2 + y^2 + 2gx = 0$, where g being arbitrary constant. ... (1)

Differentiating (1) with respect to x, we have

$$2x + 2y\frac{dy}{dx} + 2g = 0 \quad \text{... (2)}$$

From(1) $$2gx = -(x^2 + y^2)$$

$$2g = -\frac{(x^2 + y^2)}{x}$$

Now substituting the value of $2g$ in equation (2), we obtain

$$2x + 2y\frac{dy}{dx} - \frac{(x^2+y^2)}{x} = 0.$$

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0.$$

EXAMPLE 4: Find the differential equation of the family of the curves$y = e^x(Acosx + Bsinx)$, where A and B are arbitrary constant.

SOLUTION: Let

$$y = e^x(Acosx + Bsinx) \qquad \dots(1)$$

Differentiating (1) $y' = e^x(-Asinx + Bcosx) + e^x(Acosx + Bsinx)$

$$y' = e^x(-Asinx + Bcosx) + y, \quad \text{from (1)} \qquad \dots(2)$$

Again Differentiating (2)

$$y'' = -e^x(Acosx + Bsinx) + e^x(-Asinx + Bcosx) + y' \quad \dots(3)$$

Now From (2), we have

$$e^x(-Asinx + Bcosx) = y' - y. \qquad \dots(4)$$

Hence eliminating the value of A and B from (1), (3) and (4), we have

$y'' = -y + y' - y + y'$ or $y'' - 2y' + 2y = 0$

EXAMPLE5: Prove that the solutions e^x, e^x, e^{2x} of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ are linearly independent and hence or otherwise solve the given equation.

SOLUTION: The given equation $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ Or $y''' - 2y'' - y' + 2y = 0$ $\dots(1)$

Consider

$y_1 = e^x,\ y_2 = e^{-x}$ and $y_3 = e^{2x}$ $\dots(2)$

$y'_1 = e^x,\ y''_1 = e^x$ and $y'''_1 = e^x$ $\dots(3)$

$y_1''' - 2y_1'' - y'_1 + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0$, from (2) and (3)

Hence

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= (e^x \quad e^{-x} \quad e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$$

$$\begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix} = -6e^{2x}$$

Finally y_1, y_2 and y_3 are linearly independent.

EXAMPLE6: Show that $sin2x$ and $cos2x$ form a set of fundamental solutions of $y'' + 4y = 0$ and hence find the general solution of this equation.

SOLUTION: Let $y'' + 4y = 0$ ... (1)

and $y_1(x) = sin2x, \quad y_2(x) = cos2x$ ... (2)

Now $y'_1(x) = 2cos2x, \quad y_2(x) = -4sin2x$ ... (3)

$y''(x) + 4y(x) = -4sin2x + 4sin2x = 0$, from (2) and (3)

Hence we can prove that $y_1(x) = sin2x$ and $y_2(x) = cos2x$ is the solution of (1). So the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is obtained by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} sin2x & cos2x \\ 2cos2x & -2sin2x \end{vmatrix}$$

$$= -2(sin^2 2x + cos^2 2x) = -2 \neq 0.$$

Finally $W(x) \neq 0$, $sin2x$ and $cos2x$ are linearly independent solution of (1).

EXAMPLE 7: Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$ for different values of *A* and *B*.

Sol. Given that $y = Ae^{2x} + Be^{-2x}$ ....(1)

Differentiating equation (1) two times w.r.t. *x* in succession as it involves two arbitrary constants *A* and *B*.

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} \text{ and } \frac{dy}{dx} = 4\left(Ae^{2x} + Be^{-2x}\right) \quad \text{or}$$

$$\frac{dy}{dx} = 4y \qquad ....(2)$$

Thus two arbitrary constants A and B have been eliminated and equation (2) is the required differential equation of the family of curves given by equation (1).

EXAMPLE 8: Find the differential equation of all circles of radius a.

Sol. The equation of all circles of radius a is represented by as follows:

$$(x-h)^2 + (y-k)^2 = a^2 \qquad ....(1)$$

Where (h, k) is the coordinated of the centre of the circle, which can be different for different circles of radius a, hence h and k are taken to be as arbitrary constants. Now differentiating equation (1) w.r.t. x, we get $(x-h) + (y-k)y' = 0 \qquad ....(2)$

again differentiating equation (2) w.r.t. x, we have

$$1 + (y')^2 + (y-k)y'' = 0 \text{ or } (y-k) = -\frac{1+(y')^2}{y''} \qquad ....(3)$$

Substituting the value of (y-k) from equation (3) in equation (2), we get

$$(x-h) = \frac{\left\{1+(y')^2\right\}y'}{y''}$$

Putting the value of (x-h) and (y-k) in equation (1) which gives

$$\frac{\left\{1+(y')^2\right\}^2 (y')^2}{(y'')^2} + \frac{\left\{1+(y')^2\right\}^2}{(y'')^2} = a^2 \text{ or } \left\{1+(y')^2\right\}^3 = a^2 (y'')^2$$

Which is the desired differential equation of all circles of radius a.

Example 9: Find theWronskian of $\cos \omega x$ and $\sin \omega x$.

Sol. Let $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$. Their Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} = \omega(\cos^2\omega x + \sin^2\omega x) = \omega.$$

Example 10: Verify that $y = (c_1 + c_2 x)e^x$ is a general solution of $y'' - 2y' + y = 0$ and show that on any interval e^x and xe^x are linearly independent.

Sol. Given that $y = (c_1 + c_2 x)e^x$ is a general solution, it means the solution must satisfies the given differential equation. So, we have

$y' = (c_1 + c_2 x)e^x + c_2 e^x$ and $y'' = (c_1 + c_2 x)e^x + c_2 e^x + c_2 e^x$, now substituting these values in the given differential equation, we get

$$(c_1 + c_2 x)e^x + c_2 e^x + c_2 e^x - 2\left((c_1 + c_2 x)e^x + c_2 e^x\right) + (c_1 + c_2 x)e^x = 0$$

,

$\Rightarrow$ $y = (c_1 + c_2 x)e^x$ is a general solution of given differential equation.

Example 11: Prove that the functions e^x, xe^x, x^2e^x are linearly independent. Hence form the differential equation whose roots are e^x, xe^x, x^2e^x.

Sol. Let $y_1(x) = e^x$, $y_2(x) = xe^x$ and $y_3(x) = x^2e^x$, and their Wronskian W is evaluated by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & x^2e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2e^x + 4xe^x + 2e^x \end{vmatrix} = 2e^{3x} \neq 0, \forall\, x \in (-\infty, \infty).$$

$\therefore$ as $W = 0 \Rightarrow e^x$, xe^x, x^2e^x are linearly independent. Hence the general solution of the desired differential equation may be written as

$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^x + c_2 xe^x + c_3 x^2 e^x$ ....(1), where c_1, c_2, c_3 are constants. Now forming the differential equation, differentiate equation (1) w.r.t. x, we get

$$y' = c_1 e^x + c_2 (x+1) e^x + c_3 (x^2 + 2x) e^x \qquad ....(2)$$

From equation (1) and equation (2), we have

$$y' = y + c_2 e^x + 2c_3 x e^x \quad ....(3),$$

Differentiate equation (3) w.r.t. x, we get $y'' = y' + c_2 e^x + 2c_3 x e^x + 2c_3 e^x \quad ....(4)$

From eq. (3) and eq. (4), we get $y'' = 2y' - y + 2c_3 e^x \qquad ....(5)$

Differentiate eq. (5) w.r.t. x, we obtained $y''' = 2y'' - y' + 2c_3 e^x \qquad ....(6)$

From equation (5) and equation (6), we deduced that $y''' - 3y'' + 3y' - y = 0$ is the required differential equation.

Example 12: Determine the differential equation whose set of independent solution is $\{1, x, x^2\}$.

Sol. Let the general solution of the required differential equation be

$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 x + c_3 x^2 \qquad ....(1)$, [$\because$ Given solution set is independent] where c_1, c_2, c_3 are constants.

Now forming the differential equation, we have to differentiate equation (1) w.r.t. x three times in succession as it involves three arbitrary constants.

$y' = c_2 + 2c_3 x \Rightarrow y'' = 2c_3 \Rightarrow y''' = 0$, which is free from arbitrary constants, hence is the desired differential equation.

SELF CHECK QUESTIONS

1. What is the degree of a first-order, first-degree differential equation?
2. In a first-order, first-degree differential equation, what is typically represented by dx/dy?
3. Define an ordinary differential equation (ODE).
4. How are first-order, first-degree differential equations often denoted?

5. What is an initial condition in the context of solving differential equations?
6. What are solutions to first-order, first-degree differential equations typically sought for?
7. What are some common techniques used to solve first-order, first-degree differential equations?
8. In what areas of science and engineering do first-order, first-degree differential equations often arise?

1.16 SUMMARY

A first-order, first-degree differential equation involves the first derivative of a function and has the highest power of the derivative as one. It's typically written as $dx/dy = f(x, y)$, where y is the dependent variable, x is the independent variable, and $f(x, y)$ represents the relationship between them. Solutions to such equations are found by integrating and often require initial conditions for a unique solution.

1.17 GLOSSARY

- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **Rate of Change:** The speed at which a quantity changes with respect to time or another variable.
- **Derivative:** A measure of how a function changes as its input changes.

- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Dependent Variable:** The variable whose value depends on the value of another variable, often denoted as y.
- **Independent Variable:** The variable that is varied independently of other variables, often denoted as x.
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Derivative:** A measure of how a function changes as its input changes, often representing rates of change.
- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.

Understanding these terms is essential for working with first-order, first-degree differential equations and solving problems in various fields of science and engineering.

1.18 REFERENCES

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1.19 SUGGESTED READING

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1.20 TERMINAL QUESTIONS:-

(TQ-1) Find the differential equation of the family of the curves $y = Ae^{3x} + Be^{5x}$; for different values of A and B.

(TQ-2) Show that $Ax^2 + By^2 = 1$ is the solution of $x[y(d^2y/dx^2) + (dy/dx)^2] = y(dy/dx)$.

(TQ-3) Define linearly dependent and independent set of functions.

(TQ-4) Show that the linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x sinx$ and $e^x cosx$.

(TQ-5) Show that $v = \left(\frac{a}{r}\right) + B$ is the solution of

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$$

(TQ-6) Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.

(TQ-7) Find the differential equation of the family of curves $y = Ae^x + \left(\frac{B}{e^x}\right)$, for different values of A and B.

(TQ-8) Prove that the solutions e^x, e^x, e^{2x} of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ are linearly independent and hence or otherwise solve the given equation.

(TQ-9) Find the differential equation satisfied by family of circles $x^2 + y^2 = a^2$, a being an arbitrary constant.

1.21 ANSWERS:-

SELF CHECK ANSWERS

1. The degree is one.
2. The first derivative of the dependent variable y with respect to the independent variable x.
3. An equation involving derivatives of a function with respect to one independent variable.
4. They are often represented as $dx/dy = f(x, y)$.
5. A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
6. To find a function y(x) that satisfies the given equation and possibly some initial conditions.
7. Techniques such as separation of variables, integrating factors, or recognizing standard forms and applying appropriate methods.
8. They arise in fields such as physics, biology, chemistry, engineering, and economics, among others.

TERMINAL ANSWERS (TQ'S)

(TQ-1) $y'' - 8y' + 15y = 0$

(TQ-6) $8y^2 = 4yx + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$.

(TQ-7) $d^2y/dx^2 = y$

(TQ-9) $2xy\frac{dy}{dx} + x^2 - y^2 = 0$.

UNIT 2: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

CONTENTS:

2.1 *INTRODUCTION:-*

Before this unit, you have already studied

- About the differential equations and its type.
- About the general solutions of various differential equations with suitable examples.
- About the existence & uniqueness theorem with examples.

In this unit we will discuss about Differential Equation of First Order and First Degree, Variables separable, Homogeneous Equations, Equation Reducible to Homogeneous form, Pfaffian Differential Equation and Exact Differential Equation.

2.2 *OBJCETIVES:-*

After studying this unit you will be able to

- Learner will be able to solve first order first degree differential equations utilizing the standard techniques.
- Determine the first order and first degree depend on the specific context in which they are being used, and they are often used in different types of problems and situations.
- Student will be able to solve standard form of first order.
- Define a Pfaffian differential equation.

2.3 *DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE:-*

Gottfried Wilhelm Leibniz (1646-1716) made many contributions to the study of differential equations, discovering the method of separation of variables, reduction of homogeneous equations to separable ones, and the procedure for solving first order linear equations.

All the differential equations, even of first order and first degree cannot be solved. However if they belong to any of the standard forms which we are going to discuss in the subsequent articles and chapters, they can be easily solved. In general, the differential equation of first order and first degree is represented in two standard forms, namely

(i) $\frac{dy}{dx} = f(x, y)$ (ii) $M(x, y)dx + N(x, y)dy = 0$

Here, we will see that an equation in one of these forms may readily be written in the other form. As we know that it is not possible to solve all the differential equation of first order and first degree, only those differential equations which belong to the following types can be solved by standard methods. Here it will be assumed that the necessary conditions for the existence of solution are satisfied.

2.4 *VARIABLES SEPERABLE:-*

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, then the variables are said to be ***Separable***.

Working Rule:

Step1: Suppose $\frac{dy}{dx} = f_1(x)f_2(y)$ ... (1)

where $f_1(x)$ is the function of only x and$f_1(y)$ is the function of only y.

Step2: from (1), we get

$$\frac{dy}{f_2(y)} = f_1(x)dx \qquad \dots (2)$$

Step3: Integrating both sides of equation (2) , we obtain

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + c \qquad \dots (3)$$

Where c is arbitrary constant.

Note1. Remember to add an arbitrary constant c on one side (only).If arbitrary constant c is not added, then the solution derives will not be general solution.

Note2. The solution of differential equation must be expressed in the form as simple as possible.

Note3. Remember that

i. $\log x + \log y = \log xy$

ii. $\log x - \log y = \log \frac{x}{y}$

iii. $n \log x = \log x^n$

iv. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{(x+y)}{(1-xy)}\right]$

v. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left[\frac{(x-y)}{(1+xy)}\right]$

SOLVED EXAMPLES

EXAMPLE1.

Solve $(1 + x^2)dy = (1 + y^2)dx$

SOLUTION: The given equation $(1 + x^2)dy = (1 + y^2)dx$

Now separating variables

$$\frac{dy}{(1+y^2)} = \frac{dx}{(1+x^2)} \qquad \ldots(1)$$

Integrating both sides in (1)

$$\Rightarrow \int \frac{dy}{(1+y^2)} = \int \frac{dx}{(1+x^2)} \quad = \quad \tan^{-1} y = \tan^{-1} x + \tan^{-1} c$$

$\Rightarrow$ where c is constant.

$$\Rightarrow \qquad \tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

$\Rightarrow$ $\tan^{-1}\frac{(y-x)}{(1+yx)} = \tan^{-1} c$ Using $\left\{\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left[\frac{(x-y)}{(1+xy)}\right]\right\}$

$$\Rightarrow \qquad \frac{y-x}{1+yx} = c$$

EXAMPLE2.

Solve $\frac{dy}{dx} = sin(x+y) + cos(x+y) \qquad \ldots(1)$

SOLUTION. Suppose $x + y = u$.

Then differentiating both side

$$1 + \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

Substituting these value in equation (1)

$$\Rightarrow \qquad \frac{du}{dx} - 1 = sinu + cosu$$

$$\Rightarrow \qquad \frac{du}{(1+cosu) + sinu}$$

$$= \ dx, \ \ seperating\ variables$$

$$\Rightarrow \qquad \frac{du}{2cos^2\frac{u}{2} + 2sin\frac{u}{2}cos\frac{u}{2}} = \ dx$$

$$\Rightarrow \qquad \frac{\frac{1}{2}sec^2\frac{u}{2}}{1 + tan\frac{u}{2}}du = \ dx.$$

$\therefore$ Integrating both sides, we get

$\Rightarrow \qquad \log\left(1+tan\frac{1}{2}u\right)=x+c \qquad ...(2)$

Putting the value of u in equation (2)

$\Rightarrow \qquad \log\left(1+tan\frac{1}{2}(x+y)\right)=x+c$

EXAMPLE3.

Solve $\frac{dy}{dx}=e^{x+y}+x^2e^y$

SOLUTION. Given $\frac{dy}{dx}=e^{x+y}+x^2e^y$

$\Rightarrow \qquad \frac{dy}{dx}=e^x.e^y+x^2e^y$

$\Rightarrow \qquad \frac{dy}{dx}=e^y(e^x+x^2)$

Separating variables

$\Rightarrow \qquad \frac{dy}{e^y}=(e^x+x^2)dx$

Integrating both sides

$\Rightarrow \qquad \int\frac{dy}{e^y}=\int(e^x+x^2)dx$

$\Rightarrow \qquad \int e^{-y}dy=\int(e^x+x^2)dx$

$\Rightarrow \qquad \frac{e^{-y}}{-1}=e^x+\frac{x^3}{3}+c$

$\Rightarrow \qquad e^x+\frac{x^3}{3}+e^{-y}+c=0$ is required solution.

EXAMPLE4.

Solve the following differential equations:

a. $sec^2xtany\,dx+sec^2ytanxdy=0$

b. $\frac{dy}{dx}=\frac{sinx+xcosx}{y(2logy+1)}$

c. $y-x\left(\frac{dy}{dx}\right)=a\left(y^2+\frac{dy}{dx}\right)$

d. $(x^2-yx^2)dy+(y^2+xy^2)dx=0$

e. $\frac{dy}{dx}=xy+x+y+1$

SOLUTION.

a. Given $sec^2xtany\, dx + sec^2ytanxdy = 0$

$\Rightarrow \quad sec^2ytanxdy = -\ sec^2xtany\, dx$

Separating variables

$\Rightarrow \quad \frac{sec^2y}{tany}dy = -\frac{sec^2x}{tanx}dx$

Integrating both sides

$\Rightarrow \quad \int \frac{sec^2y}{tany}dy = -\int \frac{sec^2x}{tanx}dx$

$\Rightarrow \quad \log tany = -\log tanx + c_1 \qquad [c_1 = \log c]$

Finally

$\Rightarrow \quad \log tany + \log tanx = c_1 = \log c$

$\Rightarrow \quad \log tany\ tanx = \log c$

$\Rightarrow \quad tany\ tanx = c$

b. Let $\frac{dy}{dx} = \frac{sinx + xcosx}{(2ylogy+1)}$

$\Rightarrow \quad (sinx + xcosx)dx = (2ylogy + 1)dy$

$\Rightarrow \quad \int(sinx + xcosx)dx = \int(2ylogy + 1)dy \qquad \ldots(1)$

Now

$\Rightarrow \quad \int(xcosx)dx = x\ sinx + cosx$

Also

$\Rightarrow \quad \int(ylogy)dy = (logy) \times (y^2/2) - \int\{(1/y) \times (y^2/2)\}\,dy$

$\Rightarrow \quad = (logy)(y^2/2) - (y^2/4)$

Putting the value of $\int(xcosx)dx$ and $\int(ylogy)dy$ in (1)

$\Rightarrow \quad -cosx + xsinx + cosx$

$$= 2\left\{\left(\frac{y^2}{2}\right)logy - y^2/4\right\} + y^2/2 + c$$

$\Rightarrow \quad xsinx = y^2\ logy + c$

c. Let $y - x\left(\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

$\Rightarrow \quad xy - x^2\frac{dy}{dx} = y$

$\Rightarrow \quad -x^2 \frac{dy}{dx} = y - xy$

$\Rightarrow \quad -x^2 \frac{dy}{dx} = y(1-x)$

$\Rightarrow \quad x^2 \frac{dy}{dx} = y(x-1)$

$\Rightarrow \quad \int \frac{dy}{y} = \int (1-x)dx = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$\Rightarrow \quad logy = logx + \frac{1}{x} + c$

d. Given $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

$\Rightarrow \quad x^2(1-y)dy + y^2(1+x)dx = 0$

$\Rightarrow \quad \frac{1-y}{y^2} dy + \frac{1+x}{x^2} dx = 0$

or

$\Rightarrow \quad \left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \left(\frac{1}{x^2} + \frac{1}{x}\right) dy = 0$

Integrating both sides

$\Rightarrow \quad \int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy = \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dy$

$\Rightarrow \quad -\frac{1}{y} - logy - \frac{1}{x} + logx = c$

$\Rightarrow \quad log\frac{x}{y} - \left(\frac{1}{x} + \frac{1}{y}\right) = c$

e. $\frac{dy}{dx} = xy + x + y + 1$

$\Rightarrow \quad \frac{dy}{dx} = (x+1)(y+1)$

$\Rightarrow \quad \int \frac{dy}{(1+y)} = \int (x+1)dx$

$\Rightarrow \quad log(1+y) = \frac{x^2}{2} + x + c$

$\Rightarrow \quad \frac{x^2}{2} + x - log(1+y) + c = 0$

EXAMPLE5. Solve

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}.$$

Sol. Separating the variables in the given equation, we have

$$(\sin y + y\cos y)dy = (2x\log x + x)dx$$

Integrating both sides, we get

$$\int(\sin y + y\cos y)dy = \int(2x\log x + x)dx + C$$

$$\Rightarrow \qquad -\cos y + y\sin y + \cos y = 2\left[\frac{x^2}{2}\log x - \int\frac{1}{x}\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$\Rightarrow \qquad y\sin y = x^2\log x + C$, which is required solution of given differential equation.

Equations Reducible to Variable Separable Form

Sometimes, differential equations of first order and first degree are not solved by using variable separable method but by some substitution, it reduces to the variable separable form. A differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

can be easily solved by writing *ax+by+c=t*.

EXAMPLE 6 Solve $(x + y + 1)^2\dfrac{dy}{dx} = 1.$

Sol. Substituting *x*+*y*+1= *t* $\Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx} \qquad \Rightarrow \dfrac{dy}{dx} = \dfrac{dt}{dx} - 1,$

Then, the given differential equation becomes

$$t^2\left(\frac{dt}{dx} - 1\right) = 1 \qquad \text{or} \qquad \frac{dt}{dx} = \frac{1+t^2}{t^2}$$

Separating the variables and integrating, we have

$$\int\frac{t^2}{1+t^2}dt = \int dx + C \qquad \Rightarrow \int\frac{t^2+1-1}{1+t^2}dt = \int dx + C \qquad \Rightarrow t - \tan^{-1}t = x + C$$

Or $(x+y+1)-\tan^{-1}(x+y+1)=x+C.$ $[\because x+y+1=t]$

where C is an arbitrary constant of integration.

EXAMPLE 7.

Solve

$$\frac{dy}{dx}=\sin(x+y)+\cos(x+y).$$

Sol. Let $x+y=t \quad \Rightarrow 1+\frac{dy}{dx}=\frac{dt}{dx}$ or $\frac{dy}{dx}=\frac{dt}{dx}-1$

Putting these values in given equation, we have

$$\frac{dt}{dx}-1=\sin t+\cos t \quad \Rightarrow \frac{dt}{\sin t+\cos t+1}=dx$$

On integration, we get

$$\int\frac{dt}{\sin t+\cos t+1}+C=\int dx$$

Substituting $t=2\theta$ or $dt=2d\theta$, we get

$$x=2\int\frac{d\theta}{\sin 2\theta+\cos 2\theta+1}+C \quad \Rightarrow x=2\int\frac{d\theta}{2\sin\theta\cos\theta+2\cos^2\theta}+C$$

$$\therefore \quad x=\int\frac{\sec^2\theta}{1+\tan\theta}d\theta+C \quad \Rightarrow x=\log(1+\tan\theta)+C$$

$$\therefore \quad x=\log\left[1+\tan\frac{(x+y)}{2}\right]+C \quad [\because \theta=\frac{t}{2}\text{ and } t=x+y]$$

2.5 *HOMOGENEOUS EQUATIONS:-*

A differential equation of first order and first degree is said to be homogeneous if

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Working rule:

Suppose

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots (1)$$

Let $\frac{y}{x} = u$ i.e., $y = ux$ ... (2)

Now differentiating (2) w.r.t.x,

$$\frac{dy}{dx} = u + x\frac{du}{dx} \qquad \ldots (3)$$

Putting the value of (2) and (3) in (1)

$u + x\frac{du}{dx} = f(u)$ or $x\frac{du}{dx} = f(u) - u$

Separating variable x and u, we get

$$\frac{dx}{x} = \frac{du}{f(u) - u}$$

So

$$\int \frac{dx}{x} = \int \frac{du}{f(u) - u}$$

$$logx + c = \frac{du}{f(u) - u}$$

Where c is an arbitrary constant and after integrating, replace u by y/x.

SOLVED EXAMPLES

EXAMPLE1. Solve $(x^2 - y^2)dx + 2xydy = 0$

SOLUTION: The given equation can be defined as

$\Rightarrow$ $(x^2 - y^2)dx + 2xydy = 0$

$\Rightarrow$ $(x^2 - y^2)dx = -2xydy$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{(x^2-y^2)}{2xy}$ ... (1)

$\Rightarrow$ Putting $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$ in (1), we have

$\Rightarrow$ $u + x\frac{du}{dx} = -\frac{(x^2-(ux)^2)}{2xy} = -\frac{(x^2-u^2x^2)}{2ux^2} = -\frac{(1-u^2)}{2u}$

$\Rightarrow$ $x\frac{du}{dx} = -\frac{(1+u^2)}{2u}$

Separating variables

$\Rightarrow$ $\frac{2u}{(1+u^2)}du = -\frac{1}{x}dx$

$\Rightarrow$ integrating, we have $\log(1+u^2) = -\log x + \log c$

$\Rightarrow$ $\log(1+u^2) = \log\frac{c}{x}$

$\Rightarrow$ $(1+u^2) = \frac{c}{x}$

$\Rightarrow$ $1 + \frac{y^2}{x^2} = \frac{c}{x}$.

EXAMPLE2. Solve$x^2ydx - (x^3+y^3)dy = 0$.

SOLUTION: The given equation

$\Rightarrow$ $x^2ydx - (x^3+y^3)dy = 0$.

$\Rightarrow$ $x^2ydx = (x^3+y^3)dy$

$\Rightarrow$ $\frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)}$ $\Rightarrow$ Putting $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow$ $u + x\frac{du}{dx} = \frac{x^2ux}{(x^3+u^3x^3)} = \frac{x^3u}{x^3(1+u^3)} = \frac{u}{(1+u^3)}$

$\Rightarrow$ $x\frac{du}{dx} = \frac{u}{(1+u^3)} - u$ $\Rightarrow$ $x\frac{du}{dx} = -\frac{u^3}{(1+u^3)}$, separating variables

$\Rightarrow$ $\frac{(1+u^3)}{u^3}du = -\frac{1}{x}du$ $\Rightarrow$ $\left(\frac{1}{u^3} + \frac{1}{u}\right)du = -\frac{1}{x}du$

$\Rightarrow$ Integrating, we have $-\frac{u^{-3}}{3} + \log u = -\log x + \log c$

$\Rightarrow$ $-\frac{1}{3u^3} + \log u = -\log x + \log c$

$\Rightarrow$ $\log u + \log c + \log x = \frac{1}{3u^3}$

$\Rightarrow$ $\log uxc = \frac{1}{3u^3}$ $\Rightarrow$ $\log uxc = \frac{1}{3u^3}$

$\Rightarrow$ Putting the value of $u = \frac{y}{x}$

$\Rightarrow$ $\log\frac{y}{x}xc = \frac{1}{3\left(\frac{y}{x}\right)^3}$ $\Rightarrow$ $\log yc = \frac{x^3}{3y^3}$ is required solution.

EXAMPLE3. Solve $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$

SOLUTION: The given equation is $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ ... (1)

$\Rightarrow$ put $y = ux$, then $\frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow$ $u + x\frac{du}{dx} = \frac{ux+\sqrt{x^2+u^2x^2}}{x}$ $\Rightarrow$ $u + x\frac{du}{dx} = \frac{ux+x\sqrt{1+u^2}}{x}$

$\Rightarrow$ $x\frac{du}{dx} = \sqrt{1 + u^2}$ $\Rightarrow$ $\frac{du}{\sqrt{1+u^2}} = \frac{1}{x}dx$

$\Rightarrow$ integrating, we get

$\Rightarrow$ $\sinh^{-1} u = \log x + \log c$ $\Rightarrow$ $\log\left(u + \sqrt{u^2 + 1}\right) = \log cx$

$\therefore$ $\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$

$\Rightarrow$ $\left(u + \sqrt{u^2 + 1}\right) = cx$ $\Rightarrow$ $\left(\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1}\right) = cx$

$\Rightarrow$ $\frac{y+\sqrt{y^2+x^2}}{x} = cx$ $\Rightarrow$ $y + \sqrt{y^2 + x^2} = cx^2$ is required solution.

EXAMPLE4. Solve $xdy - ydx = \sqrt{x^2 + y^2}$

SOLUTION: The given equation is $xdy - ydx = \sqrt{x^2 + y^2}dx$

$\Rightarrow$ $xdy = \left(y + \sqrt{x^2 + y^2}\right)dx$

$\Rightarrow$ $\frac{dy}{dx} = \frac{\left(y+\sqrt{x^2+y^2}\right)}{x} = \frac{y}{x} + \{1 + (y/x)^2\}^{1/2}$

$\Rightarrow$ take $\frac{y}{x} = u,$ ie., $y = ux,$ $\frac{dy}{dx} = u + x\frac{du}{dx}$

So that

$\Rightarrow$ $u + x\frac{du}{dx} = u + \sqrt{1 + u^2}$ $\Rightarrow$ $\frac{dx}{x} = \frac{du}{\sqrt{1+u^2}}$

$\Rightarrow$ Integrating, $\log x + \log c = \log\left[u + \sqrt{1 + u^2}\right]$ $\Rightarrow$

$xc = u + \sqrt{1 + u^2}$

$\Rightarrow$ putting the value $u = \frac{y}{x}$ $\Rightarrow$ $xc = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}$

$\Rightarrow$ $x^2 c = y + \sqrt{x^2 + y^2}$ is required result.

2.5.1 *EQUATION REDUCIBLE TO HOMOGENEOUS FORM:-*

The differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}, \qquad \text{where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

Can be reduced to homogeneous form by taking variables X and Y such that

$$x = X + h \qquad y = Y + k \qquad \ldots(1)$$

Where h and k are constants, then $dx = dX, dy = dY$

Now given equation becomes, $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$

$$\Rightarrow \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1} = \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1x+b_1y+c_1)} \qquad \ldots(2)$$

Solving by cross multiplication

$$\Rightarrow \frac{h}{bc_1-b_1c} = \frac{k}{ca_1-c_1a} = \frac{1}{ab_1-a_1b}$$

$$\Rightarrow h = \frac{bc_1-b_1c}{ab_1-a_1b}, \qquad k = \frac{ca_1-c_1a}{ab_1-a_1b}$$

Now equation (2) becomes $\frac{dy}{dx} = \frac{aX+bY}{a_1X+b_1Y}$

Which is homogeneous equation and can be solve $= ux$. In solution putting $X = x - h$, $Y = y - k$, then we get the required solution.

SOLVED EXAMPLES

EXAMPLE1. Solve $\frac{dy}{dx} = \frac{y-x+1}{y+x-5}$

SOLUTION: The given equation

$$\frac{dy}{dx} = \frac{y - x + 1}{y + x - 5} \qquad \ldots(1)$$

$$\left[Here \quad a = -1, b = 1, a_1 = 1, b_1 = 1, \frac{a}{a_1} \neq \frac{b}{b_1}\right]$$

Now we put $x = X + h, y = Y + k$, then $dx = dX, dy = dY$

Form(1)

$$\frac{dY}{dX} = \frac{Y + k - X - h + 1}{Y + k + X + h - 5} = \frac{(-X + Y) + (-h + k + 1)}{(X + Y) + (h + k - 5)} \qquad \ldots.(2)$$

Choose h and k so that

$$\left.\begin{matrix} -h + k + 1 = 0 \\ h + k - 5 = 0 \end{matrix}\right\} \qquad \ldots(3)$$

solving equation (3), we obtain $h = 3, k = 2$

from(2) $\quad \frac{dY}{dX} = \frac{-X+Y}{X+Y} \qquad \ldots(4)$

Put put $Y = uX$, then $\frac{dY}{dX} = u + X\frac{du}{dX}$

From (4) $\quad u + x\frac{du}{dX} = \frac{-X+uX}{X+uX} = \frac{-1+u}{1+u}$

$\Rightarrow \quad X\frac{du}{dX} = \frac{-1+u}{1+u} - u$

$\Rightarrow \quad X\frac{du}{dX} = \frac{-1+u-u-u^2}{1+u}$

$\Rightarrow \quad X\frac{du}{dX} = -\frac{1+u^2}{1+u}$

Separating variables $\quad \int \frac{1+u}{1+u^2} du = -\int \frac{1}{X} dX$

Integrating $\quad \int \frac{1}{1+u^2} du + \frac{1}{2}\int \frac{2u}{1+u^2} du = -\log X + c$

$\Rightarrow \quad \tan^{-1} u + \frac{1}{2}\log(1 + u^2) = -\log X + c$

$\Rightarrow \quad$ Putting $= \frac{Y}{X}$, $\quad \tan^{-1}\frac{Y}{X} + \frac{1}{2}\log\left(1 + \frac{Y^2}{X^2}\right) = -\log X + c$

$\Rightarrow \quad \tan^{-1}\frac{Y}{X} + \frac{1}{2}\log\left(\frac{X^2+Y^2}{X^2}\right) = -\log X + c$

$\Rightarrow \quad \tan^{-1}\frac{Y}{X} + \frac{1}{2}[\log(X^2 + Y^2) - \log X^2] = -\log X + c$

$\Rightarrow \quad \tan^{-1}\frac{Y}{X} + \frac{1}{2}[\log(X^2 + Y^2) - 2\log X] = \log X + c$

$\Rightarrow \quad \tan^{-1}\frac{Y}{X} + \frac{1}{2}[\log(X^2 + Y^2)] = c$

$\therefore \qquad X = x - h = x - 3,\ Y = y - k = y - 3$

$\Rightarrow \qquad \tan^{-1}\frac{x-3}{y-2} + \frac{1}{2}[\log((x-3)^2 + (y-2)^2)] = c$ is required solution.

EXAMPLE2. Solve $(x - y)dy = (x + y + 1)dx$

SOLUTION: The given equation $\frac{dy}{dx} = \frac{x+y+1}{x-y}$ ... (1)

$\Rightarrow \qquad \left[Here \quad a = 1, b = 1, a_1 = 1, b_1 = -1, \frac{a}{a_1} \neq \frac{b}{b_1}\right]$

Put $x = X + h, y = Y + k$, then $dx = dX, dy = dY$

From (1) $\qquad \frac{dY}{dX} = \frac{X+h+Y+k+1}{X+h-Y-k} = \frac{(X+Y)+(h+k+1)}{(X+Y)-(h-k)}$ ... (2)

$\therefore$ choose h and k such that

$\Rightarrow \qquad (h + k + 1) = 0, \qquad (h - k) = 0$

Since $\qquad h = -\frac{1}{2} = k$

Putting the value of h and k in (2)

$\frac{dY}{dX} = \frac{(X+Y)+0}{(X+Y)-0} = \frac{(X+Y)}{(X+Y)}$ is required solution.

EXAMPLE3. Solve $(x - y - 2)\,dx + (x - 2y - 3)\,dy = 0$.

Sol. Re-writing given equation

$$\frac{dy}{dx} = -\frac{x - y - 2}{x - 2y - 3}, \quad \text{where } \frac{a}{A} \neq \frac{b}{B} \text{ i.e. } \frac{1}{1} \neq \frac{-1}{-2}$$ **(CASE I).**

Which is not in homogeneous form & it can be reduced to homogeneous form by the substitution of $x = X + h$, $y = Y + k \Rightarrow dy/dx = dY/dX$. Then the equation becomes

$$\frac{dY}{dX} = -\frac{X - Y + h - k - 2}{X - 2Y + h - 2k - 3} \quad \text{....(1)}$$

Choose h, k so that $h - k - 2 = 0$ & $h - 2k - 3 = 0$.

On solving, we get $h = 1$ & $k = -1$, so that $X = x - 1$ & $Y = y + 1$.

And equation (1) becomes $\dfrac{dY}{dX} = -\dfrac{X - Y}{X - 2Y} = -\dfrac{1 - Y/X}{1 - 2Y/X}$ ....(2)

Take $\dfrac{Y}{X} = v$ or $Y = vX \Rightarrow \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$ ....(3)

From equations (2) and (3), we have

$$v + X\frac{dv}{dX} = -\frac{1-v}{1-2v} \quad \Rightarrow X\frac{dv}{dX} = -\frac{1-v}{1-2v} - v$$

Or $X\dfrac{dv}{dX} = \dfrac{1-2v^2}{2v-1} \qquad \Rightarrow \dfrac{dX}{X} = \dfrac{2v-1}{1-2v^2}$

Or $\dfrac{dX}{X} = \left[-\dfrac{1}{2}.\dfrac{(-4v)}{1-2v^2} - \dfrac{1}{1-(v\sqrt{2})^2}\right]dv$

Integrating, we get $\log X = -\dfrac{1}{2}\log(1-2v^2) - \dfrac{1}{2\sqrt{2}}\log\left(\dfrac{1+v\sqrt{2}}{1-v\sqrt{2}}\right) - \dfrac{1}{2}\log C$

Or $\log\{CX^2(1-2v^2)\} = \log\left(\dfrac{1-v\sqrt{2}}{1+v\sqrt{2}}\right)^{1/\sqrt{2}}$

Or $CX^2\left(1-2\dfrac{Y^2}{X^2}\right) = \left(\dfrac{1-\sqrt{2}Y/X}{1+\sqrt{2}Y/X}\right)^{1/\sqrt{2}} \quad [\because v = Y/X]$

Or $C(X^2 - 2Y^2) = \left(\dfrac{X - \sqrt{2}Y}{X + \sqrt{2}Y}\right)^{1/\sqrt{2}}$

Or

$$C\left[(x-1)^2 - 2(y+1)^2\right] = \left(\frac{x-1-\sqrt{2}(y+1)}{x-1+\sqrt{2}(y+1)}\right)^{1/\sqrt{2}} \qquad [\because X = x-1 \ \& \ Y = y+1]$$

Hence $C(x^2 - 2y^2 - 2x - 4y - 1) = \left(\dfrac{x - y\sqrt{2} - \sqrt{2} - 1}{x + y\sqrt{2} + \sqrt{2} - 1}\right)^{1/\sqrt{2}}$ is the required solution.

EXAMPLE4.

Solve

$\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}$.

Sol. Here

$\frac{a}{A}=\frac{b}{B}\, i.e.\, \frac{1}{1}=\frac{2}{2}$

(CASE II). So substitute $x+2y=u \Rightarrow \frac{du}{dx}=1+2\frac{dy}{dx}$, so that given equation becomes $\frac{1}{2}\frac{du}{dx}-\frac{1}{2}=\frac{u-1}{u+1} \quad \Rightarrow \frac{du}{dx}=\frac{3u-1}{u+1}$

Or $\frac{u+1}{3u-1}du=dx \;\Rightarrow \frac{1}{3}\left(\frac{3u-1+4}{3u-1}\right)du=dx \quad \Rightarrow \left(\frac{1}{3}+\frac{4}{3}\frac{1}{3u-1}\right)du=dx$

Integrating, we get $\frac{1}{3}u+\frac{4}{9}\log(3u-1)=x+C$

Or $9x=3(x+2y)+4\log[3(x+2y)-1]+C' \quad [\because u=x+2y \;\&\; C'=9C]$

Hence $3x-3y=2\log[3x+6y-1]+C'/2$ is the general solution.

2.6 *PFAFFIAN DIFFERENTIAL EQUATION:-*

The Pfaffian differential equation is a type of first-order partial differential equation. It is an expression of the form:

$$\sum_{i=1}^{n} f_i(x_1, x_2, x_3 \ldots\ldots, x_n)dx_i = 0$$

where f_i is a function of n variables $x_1, x_2, x_3 \ldots\ldots, x_n$.

This equation is called Pfaffian because it can be expressed as the exterior derivative of a differential form, which is said to be the Pfaffian form.

$M(x,y)dx + N(x,y)dy = 0$ and $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ are examples of Pfaffian differential equations in two and three variables.

2.7 *EXACT DIFFERENTIAL EQUATION:-*

The equation $M(x,y) + N(x,y) = 0$ is said to be an exact differential equation when ∃ a function $f(x,y)$ of two variables x and y having continuous partial derivatives such that

$$d[f(x,y)] = Mdx + Ndy,$$
$$(\partial f/\partial x)dx + (\partial f/\partial y)dy = Mdx + Ndy$$

Remarks. The equation $y^2dx + 2xydy = 0$ is an exact differential equation, ∃ a functionxy^2, such that

$d(xy^2) = \frac{\partial}{\partial x}(xy^2)dx + \frac{\partial}{\partial y}(xy^2)dy$ or $d(xy^2) = y^2dx + 2xydy$

So the equation $y^2dx + 2xydy = 0$ may be written as $d(xy^2) = 0$. This on integration yields $xy^2 = 0$, where c as arbitrary constant. And the general solution of $xy^2 = c$.

The exact differential equation have the following important property: An exact differential equation can always be derived from its general solution directly by differentiating without any subsequent multiplication, elimination, etc.

THEOREM: To determine the necessary and sufficient condition for a differential equation of first order and first degree to be exact.

Proof:

Statement: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \qquad \dots (1)$$

to be exact $\partial M/\partial x = \partial N/\partial y \qquad \dots (2)$

Necessary condition:

Let us consider the equation $Mdx + Ndy = 0$ be exact. Hence by the definition, ∃ a function $f(x, y)$ of x and y, such that

$$\Rightarrow \qquad d[f(x,y)] = (\partial f/\partial x)dx + (\partial f/\partial y)dy = Mdx + Ndy$$

Comparing the equation, we get

$$\Rightarrow \qquad M = (\partial f/\partial x) \qquad \dots (4)$$

$$\Rightarrow \qquad N = (\partial f/\partial y) \qquad \dots (5)$$

Now differentiating equation (4) and (5) with respect to y and x , respectively obtaining

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Hence, if equation (1) is exact, M and N satisfy condition(2).

Sufficient condition:

Suppose that (2) holds and proof that (1) is an exact. For the function of $f(x, y)$, such that $d[f(x,y)] = Mdx + Ndy$

Let us consider $\qquad g(x, y) = \int M dx \qquad \dots (6)$

Be the partial integral of M, the integral defined by keeping y fixed. We first show that $(N - \partial g/\partial y)$ is a function of y only, so

$$\frac{\partial}{\partial y}(N - \partial g/\partial y) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x\, \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y\, \partial x} \text{ as}$$

$$\frac{\partial^2 g}{\partial x\,\partial y}=\frac{\partial^2 g}{\partial y\,\partial x}$$

$$=\frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y},\ \text{using (6)}$$

$$=0,\ \text{using(2)}$$

Now we take

$$f(x,y)=g(x,y)+\int\{N-(\partial g/\partial y)\}\,dy \qquad \ldots(7)$$

From (9)

$$df=dg+\left(N-\frac{\partial g}{\partial y}\right)dy=\left(\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy\right)+Ndy-\frac{\partial g}{\partial y}dy$$

$$=(\partial g/\partial\, x)dx+Ndy=Mdx+Ndy,\ \text{using (6)}$$

Hence if equation (2) is satisfied, (1) is an exact equation.

- If M and N are functions of x and y, the equation is called exact when there exists functions $g(x,y)$ of x and y, such that

  $$d[g(x,y)]=Mdx+Ndy$$

  i.e. $\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dx=Mdx+Ndy$

- The necessary and sufficient condition for the differential equation $Mdx+Ndy=0$ to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Working Rule-:

Compare the given equation with $Mdx+Ndy=0$ and find M and N. Then find $\frac{\partial M}{\partial y}and\frac{\partial N}{\partial x}$.

If $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the given equation is exact, then

STEP 1 Integrate M w.r.t x considering$y$ as constant

STEP 2 Integrates those terms of N w.r.t y which do not contain x.

STEP 3 Equate the sum of the two integrals which found from step 1 and step 2 to an arbitrary constant. Hence we obtain a required solution .i.e.

$$\int_{[treating\ y\ as\ constant]} M\,dx + \int (terms\ in\ N\ not\ containing\ x)\,dy = c,$$

c is arbitrary constant.

SOLVED EXAMPLES

EXAMPLE1: Solve $(ax + hy + g)dx + (hx + by + f)dy = 0$

SOLUTION: Let comparing the equation with $Mdx + Ndy = 0$, we obtain

$\Rightarrow \quad \frac{\partial M}{\partial y} = h, \frac{\partial N}{\partial x} = h$ so $\partial M/\partial x = \partial N/\partial y$ the given equation is exact. Hence

$\Rightarrow \quad \int Mdx \ + \ \int Ndx \quad = c$

treating y as constant taking only those term in N which do not contain x

$\Rightarrow \quad \int (ax + hy + g)dx \ + \ \int (hx + by + f)dy \ = \ 0$

$\Rightarrow \quad \frac{1}{2}ax^2 + hxy + gx + \frac{1}{2}bx^2 + fy = c$

$\Rightarrow \quad ax^2 + 2hxy + 2gx + bx^2 + 2fy + c = 0$

where c is constant and replaced $-2c = c$.

EXAMPLE2: Solve $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0.$

SOLUTION: The given differential equation can be defined as

$$\Rightarrow \quad \left[x - \frac{y}{x^2 + y^2}\right]dx + \left[y + \frac{x}{x^2 + y^2}\right]dy = 0$$

$\Rightarrow$ Here, $M = x - \frac{y}{x^2 + y^2}, \qquad N = y + \frac{x}{x^2 + y^2}.$

Then

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \& \qquad \frac{\partial N}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

So $\partial M/\partial x = \partial N/\partial y$ hence the given equation is exact.

Therefore

$\Rightarrow \quad \int M dx \quad + \quad \int N dx \quad = c$

(treating y as constant) (taking only those term in N which do not contain x)

$\Rightarrow \quad \frac{x^2}{2} - y.\frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c.$

$\Rightarrow \quad \frac{x^2}{2} - 2\tan^{-1}\frac{x}{y} + y^2 = 2c = k.$

EXAMPLE3: Solve $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

SOLUTION: Comparing the given equation can be written as

$\Rightarrow \quad (1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

$\Rightarrow \quad M = 1 + e^{x/y}, \qquad N = e^{x/y}(1 - x/y)$

$\Rightarrow \quad \frac{\partial M}{\partial y} = e^{x/y}(-x/y^2) \qquad \& \qquad \frac{\partial N}{\partial y} = e^{x/y}(-x/y^2)$

So its solution is $\partial M/\partial x = \partial N/\partial y$. Hence

$\Rightarrow \quad \int M dx \quad + \quad \int N dx \quad = c$

(treating y as constant) (taking only those term in N which do not contain x)

$\Rightarrow \quad \int(1 + e^{x/y})dx = c$ or $x + ye^{x/y} = c.$

EXAMPLE4: Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

SOLUTION: The given equation can be written as

$\Rightarrow \quad (x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$\Rightarrow \quad M = (x^2 - 4xy - 2y^2), \qquad N = (y^2 - 4xy - 2x^2)$

$\therefore \quad \frac{\partial M}{\partial y} = -4x - 4y \qquad \& \qquad \frac{\partial N}{\partial y} = -4y - 4x$

So that $\partial M/\partial x = \partial N/\partial y$.Hence

$\Rightarrow \quad \int M dx \quad + \quad \int N dx \ = c$

(treating y as constant) (taking only those term in N which do not contain x)

$\Rightarrow \quad \int(x^2 - 4xy - 2y^2)dx + \int y^2\, dy = c_1$

$\Rightarrow \quad x^3/3 - 4y \times (x^2/2) - 2y^2x + y^3/3 = c/3, \quad \therefore [c_1 = c/3]$

$\Rightarrow \quad x^3 + y^3 - 6xy(x + y) = c$, c being an arbitrary constant.

EXAMPLE5. Solve $(x^2 - ay)dx = (ax - y^2)dy$

SOLUTION:

Given equation is $(x^2 - ay)dx = (ax - y^2)dy$, which can be written as

$(x^2 - ay)dx + (y^2 - ax)dy = 0$……………………………….(1)

Now comparing with $Mdx + Ndy = 0$, we get

$M = x^2 - ay$ and $N = y^2 - ax$

Therefore $\frac{\partial M}{\partial y} = -a,\ \frac{\partial N}{\partial x} = -a \ \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Thus, the given equation is exact.

The solution is $\int_{[treating\ y\ as\ constant]} M\, dx + \int(terms\ in\ N\ not\ containing\ x)\ dy = c$

$$\Rightarrow \int\limits_{[treating\ y\ as\ constant]} (x^2 - ay)\ dx + \int y^2\ dy = c$$

$\Rightarrow \frac{x^3}{3} - ayx + \frac{y^3}{3} = c$ or $x^3 - 3axy + y^3 = 3c$

$\Rightarrow x^3 - 3axy + y^3 = C$ where $(C = 3c)$

EXAMPLE6. Solve $(3x^2 + 6xy^2)dx + (6x^2y^2 - y^2)dy = 0$

SOLUTION:

Given equation is

$(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$………………………….(1)

Now comparing with $Mdx + Ndy = 0$, we get

$M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$

Therefore $\frac{\partial M}{\partial y} = 12xy,\ \frac{\partial N}{\partial x} = 12xy\ \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Thus, the given equation is exact.

The solution is

$$\int_{[treating\ y\ as\ constant]} M\,dx + \int (terms\ in\ N\ not\ containing\ x)\,dy = c$$

$$\Rightarrow \int_{[treating\ y\ as\ constant]} (3x^2 + 6xy^2)\,dx + \int 4y^3\,dy = c$$

$$\Rightarrow \frac{3x^3}{3} + 6y^2\frac{x^2}{2} + 4\frac{y^4}{4} = c$$

$\Rightarrow x^3 + 3x^2y^2 + y^4 = c$ where c is any arbitrary constant.

EXAMPLE7. Find the values of constant λ such that

$(2xe^y + 3y^2)\frac{dy}{dx} + (3x^2 + \lambda e^y) = 0$ is exact.

Further, for this value of λ, solve the equation.

SOLUTION:

Given equation is $(2xe^y + 3y^2)\frac{dy}{dx} + (3x^2 + \lambda e^y) = 0$, which can be written as

$(3x^2 + \lambda e^y)dx + (2xe^y + 3y^2)dy = 0$ ……………………………….(1)

Now comparing with $Mdx + Ndy = 0$, we get

$M = 3x^2 + \lambda e^y$ and $N = 2xe^y + 3y^2$

Therefore $\frac{\partial M}{\partial y} = \lambda e^y,\ \frac{\partial N}{\partial x} = 2e^y$.

The given equation is exact which implies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Hence, $\lambda e^y = 2e^y \Rightarrow \lambda = 2$.

Thus, $M = 3x^2 + 2e^y$ and $N = 2xe^y + 3y^2$

Now, the solution is

$$\int_{[treating\ y\ as\ constant]} M\,dx + \int (terms\ in\ N\ not\ containing\ x)\,dy = c$$

$$\Rightarrow \int_{[treating\ y\ as\ constant]} (3x^2 + 2e^y)\,dx + \int 3y^2\,dy = c$$

$$\Rightarrow 3\frac{x^3}{3} + 2e^y x + 3\frac{y^3}{3} = c$$

$\Rightarrow x^3 + 2e^y x + y^3 = c$ where c is any arbitrary constant.

<u>SELF CHECK QUESTIONS</u>

1. What is the highest derivative involved in a first-order, first-degree differential equation?
2. Define a first-order, first-degree differential equation in mathematical notation.
3. Give an example of a first-order, first-degree differential equation.
4. What are the standard methods for solving first-order, first-degree differential equations?
5. Why are initial conditions important when solving first-order, first-degree differential equations?

2.8 *SUMMARY:-*

A first-order, first-degree differential equation involves the first derivative of a function and has the highest power of the derivative as one.

It's typically written in the form: $dy/dx = f(x, y)$,where y is the dependent variable, x is the independent variable, and $f(x, y)$ represents the relationship between x and y. These equations are fundamental in modeling various natural and physical phenomena. Solutions are sought to find a function y(x) that satisfies the given equation and any initial conditions provided. Common methods for solving such equations include separation of variables, integrating factors, and recognizing standard forms for specific techniques.

2.9 *GLOSSARY:-*

- **Differential Equation:** An equation involving derivatives of a function with respect to one or more independent variables.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, referring to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Partial Differential Equation (PDE):** A differential equation involving partial derivatives with respect to multiple independent variables.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that stands alone and is not affected by other variables.
- **Rate of Change:** The speed at which a quantity changes with respect to another variable.

- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Derivative:** A measure of how a function changes as its input changes.
- **Exact Differential Equation:** An ordinary differential equation (ODE) of the form $M(x,y)dx + N(x,y)dy = 0$ where M and N are functions of both x and y, such that the equation can be derived from a scalar potential function $F(x,y)$ as $\partial x/\partial F(dx) + \partial y/\partial F(dy) = 0$.
- **Exactness:** The property of a differential equation where the equation is derived from a scalar potential function, making the equation exact.
- **Homogeneous Differential Equation:** A differential equation in which all terms involving the dependent variable and its derivatives are of the same order.
- **Non-homogeneous Differential Equation:** A differential equation in which terms involving the dependent variable and its derivatives are of different orders.
- **Linear Differential Equation:** A differential equation in which the dependent variable and its derivatives appear linearly, without being raised to powers or involved in nonlinear functions.
- **Nonlinear Differential Equation:** A differential equation in which the dependent variable or its derivatives appear nonlinearly, such as being raised to powers or involved in nonlinear functions.

Understanding these terms is crucial for grasping the concepts and techniques involved in working with first-order, first-degree differential equations.

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2.12 *TERMINAL QUESTION:-*

(TQ-1) Solve the following differential equations:

a. $(1+x)ydx + (1-y)xdy = 0.$

b. $(1-x^2)(1-y) = xy(1+y)dy.$

c. $\frac{dy}{dx} = \frac{x(2\log x+1)}{.siny+ycosy}.$

d. $dy/dx = e^x + x^2e^{-y}.$

e. $x + y(dy/dx) = 2y.$

f. $(ds/dx) + x^2 = x^2e^{3s}.$

g. $y - x(dy/dx) = x + y(dy/dx).$

h. $x\frac{dy}{dx} + \frac{y^2}{x} = y.$

i. $2\frac{dy}{dx} = \frac{x}{y} - 1.$

j. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$

k. $(x-y)dy = (x+y+1)dx$

l. $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

m. $\frac{dy}{dx} = \frac{x+y+7}{2x+2y+3}$

n. $(dy/dx) + (1/x)y = x^n$

o. $(dy/dx) + y = e^{-x}$

2.13 *ANSWERS:-*

SELF CHECK ANSWERS

1. The highest derivative involved is the first derivative.
2. It can be represented as $dx/dy = f(x, y)$, where y is the dependent variable, x is the independent variable, and $f(x, y)$ is a function describing the relationship between them.
3. $dy/dx = x + y$ is an example of a first-order, first-degree differential equation.
4. Common methods include separation of variables, integrating factors, and recognizing standard forms for specific techniques.
5. Initial conditions are necessary to find a unique solution to the differential equation.

TERMINAL ANSWERS(TQ'S)

(a) $xy = ce^{y-x}$,

(b) $\log\{x(1-y)^2\} = \frac{1}{2}x^2 - \frac{1}{2}x^2 - 2y + c$,

(c) $y\,siny = x^2 \log x + c$

(d) $e^y = e^x + \frac{1}{3}x^3 + c$,

(e) $(e^{3s} - 1) = c_1 e^{(3s+x^3)}$, where $c_1 = e^{3s}$

(f) $\log(y - x) = c + x/(y - x)$,

(g) $\frac{1}{2}\log(x^2 + y^2) + \tan^{-1}(y/x) = logc$,

(h) $cx = e^{x/y}$, **(i)** $(x - 2y)(x + y)^2 = c$,

(j) $sin(y/x) = cx$

(k) $2\tan^{-1}\{(2y + 1)/(2x + 1)\} = \log\left\{c^2\left(x^2 + y^2 + x + y + \frac{1}{2}\right)\right\}$,

(i) $x - 2y + \log(x - y + 2) = c$,

(m) $(2/3)(x + y) - (11/9)\log(3x + 3y + 10) = x + c$

(n) $xy = x^{n+2}/(n + 2) + c$ **(o)** $ye^x = x + c$

UNIT 3: - INTEGATING FACTOR

CONTENTS:

3.1 *INTRODUCTION:-*

In previous unit we have discussed about Differential Equation of First Order and First Degree. In this unit we are discussing about the integrating factor.

An integrating factor is a function used to transform certain types of first-order ordinary differential equations into exact differential equations, making them easier to solve. It is multiplied by the given equation to adjust its form, typically enabling the application of straightforward integration techniques. Integrating factors are instrumental in solving differential equations involving variables that are not separable or exact in their original form.

3.2 *OBJCETIVES:-*

After completion of this unit learners will be able to

i. Define the concept of Integrating factor..

ii. Solve the Differential equation.

3.3 *INTEGRATING FACTOR:-*

The equation $Mdx + Ndy = 0$, is not exact can sometimes be made exact by multiplying by some suitable function of x and y. Such a function is said to be an ***Integrating Factor.*** A multiplying factor which will convert an inexact differential equation into an exact one is called an integrating factor.

Remark. Theoretically an integrating factor exists for every differential equation of the form

$$P(x,y)dx + Q(x,y)dy = 0$$

but no general rule is known to discover it. Methods have been devised for finding integrating factors for certain special types of differential equations, but the types are so special that the methods are of little practical value. It is evident that if a standard method of finding integrating factors were available then every first order equation of this form would be solvable by this means.

Theorem: The differential equation $Mdx + Ndy = 0$ possess an infinite number of integrating factor.

Proof: Let the given equation $Mdx + Ndy = 0$ ... (1)

Suppose $\mu(x,y)$be an I.F. of (1), then by definition\

$$\mu(Mdx + Ndy) = 0$$

Must be an exact differential equation and ∃ a function $V(x,y)$, such that

$$dV = \mu(Mdx + Ndy)$$

where $V =$ constant is a solution of (1)

Since $f(V)$ be any function of V. So

$\Rightarrow \quad f(V)dV = \mu f(V)(Mdx + Ndy)$

Since the expression on L.H.S. of (3) is an exact differential equation, it follows that the expression on R.H.S. of (3) must also be an exact differential.

3.4 *RULES FOR FINDING INTEGRATING FACTOR:-*

Rule I:

The integrating factor of given equation $Mdx + Ndy = 0$ can be explore by inspection as explained below.

(i). $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(ii) $\left(\frac{y}{x}\right) = \frac{xdy - ydx}{y^2}$

(iii) $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(iv) $d\left(\frac{x^2}{y}\right) = \frac{2yxdy - x^2dx}{y^2}$

(v) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$

(vi) $d\left(\frac{x^2}{y^2}\right) = \frac{2y^2xdy - 2yx^2dx}{y^4}$

(vii) $\mathrm{d}[\log(xy)] = \frac{xdy + ydx}{xy}$

(viii) $d(xy) = xdy + ydx$

(ix) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(x) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydy - xdx}{x^2 + y^2}$

(xi) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

(xii) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{ydx - xdy}{xy}$

(xiii) $\mathrm{d}\left[\frac{1}{2}\log(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$

(xv) $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$

(xvi) $d(\sin^{-1}xy) = \frac{xdy + ydx}{(1 - x^2y^2)^{1/2}}$

Rule II:

If the given equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then show that the integrating factor is $1/(Mx + Ny)$.

Proof: Let the given equation $Mdx + Ndy = 0$, we get

$$\Rightarrow \quad Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

$\Rightarrow \qquad \frac{Mdx+Ndy}{(Mx-Ny)} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{(Mx-Ny)}{(Mx+Ny)}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\} \qquad \ldots(1)$

$\Rightarrow$ Since $Mdx + Ndy = 0$ is a homogeneous, M and N must be same degree in variables x and y and hence

$\Rightarrow \qquad \frac{(Mx-Ny)}{(Mx+Ny)} = f\left(\frac{x}{y}\right) \qquad \ldots(2)$

Now putting the value of (2) in (1)

$\Rightarrow \qquad \frac{Mdx+Ndy}{(Mx-Ny)} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\} \qquad \ldots(3)$

$\Rightarrow \qquad \frac{1}{2}\left\{\log(xy)+f\left(e^{\log(x/y)}\right)d\left(\log\frac{x}{y}\right)\right\} = \frac{1}{2}\left\{\log(xy) + g\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right\} \quad \left[\because f\left(e^{\log(x/y)}\right) = g\log(x/y)\right]$

$\Rightarrow \qquad d[(1/2)\times\log(xy)+(1/2)\times\int g\log(x/y)\,d\log(x/y)]$

$\Rightarrow$ displaying that the $1/(Mx+Ny)$ is an integrating factor for a given equation $Mdx + Ndy = 0$.

Rule III:

If the given equation $Mdx + Ndy = 0$ is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$, then prove that $1/(Mx + Ny)$ is an integrating factor of $Mdx + Ndy = 0$ provided $(Mx - Ny) \neq 0$.

Proof: Suppose $Mdx + Ndy = 0 \qquad \ldots(1)$

is of the form

$f_1(xy)ydx + f_2(xy)xdy = 0 \qquad \ldots(2)$

Comparing both equations

$\Rightarrow \qquad \frac{M}{yf_1(xy)} = \frac{N}{xf_2(xy)} = \mu$

$\Rightarrow \qquad M = \mu y f_1(xy) \quad$ or $\quad N = \mu x f_2(xy) \qquad \ldots(3)$

$\Rightarrow$ Now

$\Rightarrow \qquad Mdx + Ndy = \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$

$\Rightarrow \qquad \frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\left\{\frac{(Mx+Ny)}{(Mx-Ny)}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$

$\Rightarrow \qquad = \frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}$, from (3

$\Rightarrow \qquad = \frac{1}{2}\left\{f(xy)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}$, where $\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)} =$ $f(xy)$

$\Rightarrow \qquad = \frac{1}{2}\left\{f\left(e^{\log xy}\right)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\} =$ $\frac{1}{2}\left\{g(\log xy)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}$

$\left[\therefore f\left(e^{\log(xy)}\right) = g\log(xy)\right]$

$\Rightarrow \qquad d[(1/2)\times\log(x/y)+(1/2)\times\int g\log(xy)\,d\log(xy)]$

$\Rightarrow$ Hence prove that $Mx-Ny$ is an integrating factor of

$$Mdx+Ndy=0.$$

Rule IV:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function x alone $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

Proof: The given equation $\quad Mdx+Ndy=0 \qquad \ldots(1)$

and $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$ so that $\quad Nf(x)=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \qquad \ldots(2)$

Multiplying both sides of equation (1) by $e^{\int f(x)dx}$, we have

$M_1dx+N_1dy=0,$

where

$\Rightarrow \quad M_1 = Me^{\int f(x)dx}$ and $N_1 = Ne^{\int f(x)dx} \qquad \ldots(3)$

From (3) $\qquad \frac{\partial M_1}{\partial y} = \frac{\partial M}{\partial y}e^{\int f(x)dx} \qquad \ldots(4)$

and $\quad \frac{\partial N_1}{\partial y} = \frac{\partial N}{\partial x}e^{\int f(x)dx}+Ne^{\int f(x)dx}f(x) = e^{\int f(x)dx}\left\{\frac{\partial N}{\partial x}+Nf(x)\right\}$

$\Rightarrow \qquad e^{\int f(x)dx}(\partial N/\partial x+\partial M/\partial y-\partial N/\partial x),$ from(2)

So that $\qquad \frac{\partial N_1}{\partial y} = \frac{\partial M}{\partial y}e^{\int f(x)dx}$

$\Rightarrow$ Now from (5) and (6), $\partial M_1/\partial y = \partial N_1/\partial x$

Hence $M_1dx + N_1dy = 0$ must be exact and $e^{\int f(x)dx}$ is integrating factor.

Rule V:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function y alone $f(y)$, then $e^{\int f(y)dy}$ is an integrating factor of $Mdx + Ndy = 0$.

Proof: Proceed exactly as for Rule IV.

Rule VI:

If the given equation $Mdx + Ndy = 0$, is of the form $x^{\alpha}y^{\beta}(mydx + nxdy) = 0$, then its integrating factor is $x^{km-1-\alpha}y^{kn-1-\beta}$, where k have any value.

Proof. By assumption, the given equation can be defined as

$\Rightarrow$ $$x^{\alpha}y^{\beta}(mydx + nxdy) = 0 \qquad \ldots (1)$$

$\Rightarrow$ Multiplying (1) by $x^{km-1-\alpha}y^{kn-1-\beta}$, we get

$$x^{km-1-\alpha}y^{kn-1}(mydx + nxdy) = 0$$

$\Rightarrow$ $km\, x^{km-1}y^{kn}dx + kny^{kn-1}x^{km}dy = 0$ or $d(x^{km}, y^{kn}) = 0$

$\Rightarrow$ so that $x^{km-1-\alpha}y^{kn-1-\beta}$ integrating factor of given equation

$$x^{\alpha}y^{\beta}(mydx + nxdy) = 0.$$

3.5 *SOLVED EXAMPLES:-*

EXAMPLE1. Solve $y(2xy + e^x)dx = e^x dy$.

SOLUTION: Given equation $y(2xy + e^x)dx = e^x dy$

$\Rightarrow \quad 2xy^2dx + ye^xdx = e^xdy$

$\Rightarrow \quad 2xdx + \frac{ye^xdx - e^xdy}{y^2} = 0$ or $2xdx + d\left(\frac{e^x}{y}\right) = 0$

$\Rightarrow$ Now integrating, $x^2 + \frac{e^x}{y} = c$ or $x^2 + e^x = cy$

EXAMPLE2. Solve $(x^3 + xy^2 + a^2y)dx + (y^3 + yx^2 - a^2x)dy = 0$.

SOLUTION: Given equation

$(x^3 + xy^2 + a^2y)dx + (y^3 + yx^2 - a^2x)dy = 0$

$\Rightarrow \quad x(x^2 + y^2)dx + y(x^2 + y^2)dy + a^2(ydx - xdy) = 0$

$\Rightarrow \quad xdx + ydy + a^2\frac{(ydx - xdy)}{(x^2 + y^2)} = 0$ or $xdx + ydy + a^2\tan^{-1}\frac{x}{y} = 0$

$\Rightarrow$ By integrating, $\frac{x^2}{2} + \frac{y^2}{2} + a^2\tan^{-1}\frac{x}{y} = \frac{c}{2}$ or $x^2 + y^2 + a^2\tan^{-1}\frac{x}{y} = c$

EXAMPLE3: Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

SOLUTION:

The given equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$\Rightarrow$ The given equation is homogeneous differential equation and comparing $Mdx + Ndy = 0$, $M = (x^2y - 2xy^2)$, $N = (x^3 - 3x^2y)$

$\Rightarrow \quad Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y) = x^2y^2 \neq 0,$

$\Rightarrow$ Then the integrating factor, $1/(Mx + Ny) = \frac{1}{x^2y^2}$. on multiplying factor by $\frac{1}{x^2y^2}$,

$\Rightarrow \quad (y/2 - 2/x)dx - (x/y^2 - 3/y)dy = 0,$

$\Rightarrow \quad \int\{(y/2 - 2/x)dx\} + \int(3/y)dy = 0$ or $x/y - 2\log x + 3\log y = \log c$

$\Rightarrow \quad \log y^2 - \log x^2 - \log c = -x/y$ or $\log(y^2/cx^2) = -x/y$

$\Rightarrow \quad y^2 = cx^2 e^{-x/y}$, where c is an arbitrary constant.

EXAMPLE4. Solve

$$(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$$

SOLUTION: Suppose

$$(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0 \qquad \dots (1)$$

The equation (1) Comparing $Mdx + Ndy = 0$, we get,

$M = (xy\sin xy + \cos xy)y$ and $N = (xy\sin xy - \cos xy)x$

The equation (1) is the form $f_1(xy)ydx + f_2(xy)xdy = 0$

Again,

$$Mx - Ny = xy(xy\sin xy + \cos xy) - xy(xy\sin xy - \cos xy)$$

$\therefore \quad Mx - Ny = 2xy\cos xy \neq 0.$

Since the integrating factor of (1)

$$= 1/(Mx + Ny) = 1/(2xy\cos xy)$$

On multiplying (1) by $1/(2xy\cos xy)$, we obtain

$\Rightarrow [(1/2) \times (y \tan xy + 1/x)dx + (1/2) \times (x \tan xy - 1/y)dy] \quad \dots (2)$

From (2)

$\Rightarrow \quad \int (1/2) \times (y \tan xy + 1/x)dx + \int (-1/2y)dy = (1/2)\log c$

$\Rightarrow \quad (1/2) \times (\log secxy + \log x)dx - (1/2) \times \log y = (1/2)\log c$

$\Rightarrow \quad (\log secxy + \log x/y) = \log c$ or $(y/x)secxy = c$.

EXAMPLE5. Solve $(x^2 + y^2 + x)dx + xydy = 0$

SOLUTION. Let $(x^2 + y^2 + x)dx + xydy = 0 \qquad \dots (1)$

Now the equation (1) comparing with $Mdx + Ndy = 0$, we have

$M = (x^2 + y^2 + x), \qquad N = xy$

$\Rightarrow \quad \partial M/\partial y = 2y, \qquad \partial N/\partial x = y.$

$$\partial M/\partial y \neq \partial N/\partial y.$$

Then we obtain

$\Rightarrow \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - 1) = \frac{1}{x},$ which is a function of x.

$\Rightarrow$ Since the integration factor is

$\Rightarrow \qquad e^{\int(1/x)dx} = e^{\log x} = x.$

$\Rightarrow \qquad$ multiplying (1) by x, we get

$\Rightarrow \qquad (x^3 + xy^2 + x^2)dx + x^2ydy = 0$ is an exact, so

$\Rightarrow \qquad \int(x^3 + xy^2 + x^2)dx = (1/6) \times c$ or $(1/4) \times x^4 + (1/2) \times x^2y^2 + (1/3) \times x^3 = c/6.$

$\Rightarrow \qquad 3x^4 + 6x^2y^2 + 4x^3 = c,$ where c is an arbitrary constant.

EXAMPLE 6 Solve

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0 \qquad \dots(1)$$

SOLUTION. From (1) compare with $Mdx + Ndy = 0$, we have

$$\Rightarrow \quad M = (2xy^4e^y + 2xy^3 + y), \qquad N = (x^2y^4e^y - x^2y^2 - 3x) \qquad \dots(2)$$

Here

$\Rightarrow \partial M/\partial y = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1,$

$\partial N/\partial x = 2xy^4e^y - x^2y^2 - 3x.$

$\Rightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$$= -4(2xy^4e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4e^y + 2xy^2 + y) = -\frac{4M}{y}$$

$\Rightarrow \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}$

⇒ Since the integrating factor of equation (1) is

$=e^{\int -4/y dy} = e^{-4\log y} = \left(\frac{1}{y^4}\right)$

⇒ Multiplying (1) by $1/y^4$, we get

⇒ $\{2xe^y + (2x/y) + (1/y^3)\}dx + \{x^2e^y - (x^2/y^2) - 3(x/y^4)\}dy = 0$

⇒ $\int\{2xe^y + (2x/y) + (1/y^3)\}dx = c$

or $x^2e^y + (x^2/y) + (x/y^3) = c.$

EXAMPLE 7. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ ... (1)

SOLUTION. The given equation (1) in standard form

⇒ $x^\alpha y^\beta(mydx + nxdy) + x^{\alpha'}y^{\beta'}(m'ydx + n'xdy) = 0$ ... (2)

we have

$$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0 \quad \text{... (3)}$$

⇒ From (2) and (3), we get

⇒ $\alpha = 0,\ \beta = 1,\ m = 1,\ n = -1,\ \alpha' = 2,\ \beta' = 0,\ m' = 2,\ n' = 2$

⇒ Since, the integrating factor for first term on L.H.S. of (3) is

⇒ $x^{k-1}y^{-k-1-1}$,

i.e., $x^{k-1}y^{-k-2}$ ... (4)

⇒ The second term on L.H.S.of (3) is

⇒ $2^{2k'-1-2}y^{2k'-1}$ ⇒ $i.e.,\quad 2^{2k'-3}y^{2k'-1}$ ... (5)

⇒ from (4) and (5), $k - 1 = 2k' - 3$ and $-k - 2 = 2k' - 1$

⇒ $k - 2k' = -2$ and $k + 2k' = -1 \Rightarrow k = -3/2$ and $k' = 1/4$

Putting the value of k in (4) or k' in (5), then the integrating factor of (3) or (1) is $x^{-5/2}y^{-1/2}$. Multiplying (1) by $x^{-5/2}y^{-1/2}$, we obtain

$$\Rightarrow \quad \left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \left(x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right)dy = 0$$

$$\Rightarrow \quad \frac{x^{-3/2}y^{3/2}}{-(3/2)} + \frac{2x^{1/2}y^{1/2}}{(1/2)} = \frac{2C}{3} \quad \text{or} \quad 6x^{1/2}y^{1/2} - x^{-3/2}y^{3/2} = C.$$

EXAMPLE 8. Solve

$$e^y dx + (xe^y + 2y)dy = 0$$

SOLUTION. Given equation is

$$e^y dx + (xe^y + 2y)dy = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Now rewriting the equation, we get

$$e^y dx + xe^y dy + 2ydy = 0$$

$$\Rightarrow D[xe^y] + 2ydy = 0$$

Now integrating both sides, we get

$$xe^y + \frac{2y^2}{2} = C$$

$\Rightarrow xe^y + y^2 = C$, where C is an arbitrary constant

EXAMPLE 9. Solve

$$(1 + xy)ydx + x(1 - xy)dy = 0$$

SOLUTION. Given equation is

$$(1 + xy)ydx + x(1 - xy)dy = 0$$

Now rewriting the equation, we get

$$ydx + xy^2dx + xdy - x^2ydy = 0$$

$$\Rightarrow ydx + xdy - x^2ydy + xy^2dx = 0$$

$$\Rightarrow ydx + xdy + x^2y^2\left(-\frac{dy}{y} + \frac{dx}{x}\right) = 0$$

$$\Rightarrow ydx + xdy + x^2y^2\left(\frac{dx}{x} - \frac{dy}{y}\right) = 0$$

$$\Rightarrow d(xy) + x^2y^2 d\left(\log\left(\frac{x}{y}\right)\right) = 0$$

By dividing both sides with x^2y^2, we get

$$\frac{1}{x^2y^2}d(xy) + d\left(\log\left(\frac{x}{y}\right)\right) = 0$$

$$\Rightarrow d\left(\log\left(\frac{x}{y}\right) - \frac{1}{xy}\right) = 0$$

Now integrating both sides, we get

$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = C$, where C is an arbitrary constant.

SELF CHECK QUESTIONS1

1. What is the purpose of an integrating factor in solving ordinary differential equations?
2. How does an integrating factor affect a given ordinary differential equation?
3. When is an integrating factor necessary in solving a differential equation?
4. What is the result of multiplying a differential equation by its integrating factor?
5. How can one find the integrating factor for a given differential equation?

SELF CHECK QUESTIONS2

Problem1.

Find the integrating factor of $(y - xy^2)dx - (x + x^2y)dy = 0$?

1. 1/2x
2. y/2x
3. 1/2xy
4. 1/2y

Problem 2.

If x^r is on integrating factor of $(x + y^3)$ dx + $6xy^2$ dy = 0, then r is _____

Problem 3.

The integrating factor for the differential equation $\frac{dP}{dt} + K_2P = K_1L_0e^{-K_1t}$

1. e^{-k_1t}
2. e^{-k_2t}
3. e^{k_1t}
4. e^{k_2t}

Problem 4.

An integrating factor of the differential equation $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0$ is:

1. x^2
2. 3 lnx
3. x^3

Problem 5.

The integrating factor of equation y log y dx + (x – log y) dy = 0 is

1. log x
2. log y
3. log (log x)
4. log (log y)

3.6 *SUMMARY:-*

The integral factor in differential equations is a function used to transform certain types of ordinary differential equations (ODEs) into exact differential equations, facilitating their solution. By multiplying both sides of the given ODE with this integrating factor, the equation's form is adjusted to become exact, simplifying the solution process. The choice of integrating factor is crucial, and it is often found by inspection or through methods like solving an auxiliary differential equation or using integrating factor formulas derived from the given equation. Integrating factors are particularly useful when dealing with ODEs that are not initially exact but can be made exact through multiplication by a suitable function. This concept plays a fundamental role in solving differential equations, especially in cases where standard techniques like separation of variables or substitution are not applicable.

3.7 *GLOSSARY:-*

- **Integrating Factor:** A function used to transform certain types of ordinary differential equations into exact differential equations, making them easier to solve.
- **Exact Differential Equation:** An ODE that can be expressed as the total derivative of a scalar potential function, enabling direct integration to find the solution.
- **Scalar Potential Function:** A function whose partial derivatives with respect to the variables in the ODE correspond to the coefficients of the differential terms, allowing the ODE to be written as exact.
- **Exactness:** The property of a differential equation where it can be expressed as the total derivative of a scalar potential function.
- **Exact Differential:** A differential form that can be derived from a scalar potential function.

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3.10 *TERMINAL QUESTION:-*

(TQ-1) Solve the following differential equations:

a. $p^2 + 2pycptx = y^2$

b. $p^2 - 5p + 6 = 0$

c. $y = px + a/p$

d. $y = px + logp$

(TQ-2) Choose the Correct Option:

1. The solution of differential equation $p^2 - 8p + 15 = 0$ is

(a) $p = 5, p = 3$ (b) $(y - 5x - c)(y - 3x - c) = 0$

(c) $(y + 5x)(y + 3x + c) = 0$ (d) None

2. Solution of the equation $y^2 logy = xyp + p^2$ is

(a) $\log y = cx + x^2$ (b) $\log y = cx^2 + e^x$

(c) $(y + 5x)(y + 3x + c) = 0$ (d) None

3. Solution of the equation $y = px + \log p$ is

(a) $y = e^x + c$ (b) $y = cx + \log c$

(c) $y = logcx$ (d) $x = e^y + c$

4. The differential equation $Mdx + Ndy = 0$, where M and N are the functions of x and y is exact if

(a) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (b) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$

(c) $M + N = 0$ (d) $M = N$

3.11 *ANSWERS:-*

SELF CHECK ANSWERS1

1. The integrating factor is used to transform certain types of first-order ordinary differential equations into exact differential equations, making them easier to solve.

2. An integrating factor is multiplied by both sides of the equation to adjust its form, typically enabling the equation to be written as the exact differential of a potential function.
3. An integrating factor is necessary when the given ordinary differential equation is not exact in its original form but can be made exact through multiplication by a suitable integrating factor.
4. Multiplying a differential equation by its integrating factor transforms it into an exact differential equation, simplifying the solution process.
5. The integrating factor can often be found by inspection, or it may be calculated using various methods, such as solving an auxiliary differential equation or using an integrating factor formula derived from the given equation.

SELF CHECK ANSWERS2

1. $\frac{1}{2xy}$
2. -0.5
3. $e^{k_2 t}$
4. x^3
5. $logy$

TERMINAL ANSWERS(TQ'S)

(TQ-1) **(p)** $\left(y-\frac{c}{1+cosx}\right)\left(y-\frac{c}{1-cosx}\right)=0$

(q) $(y-2x-c)(y-3x-c)=0$

(r) $y=cx+a/x$, **(s)** $y=cx+\log c$

(TQ-2) **1.** (b), **2.** (a), **3**. (b), **4**. (a)

UNIT 4: - LINEAR DIFFERENTIAL EQUATION

CONTENTS:

4.1 *INTRODUCTION*

In the previous units learners have already studied about basics of Ordinary Differential Equation and solution of Differential Equation of type First Order and First Degree.

In this unit, we discuss about the Linear Differential equation, Equations reducible to linear form Bernoulli's Equation, Principle of duality, trajectories, orthogonal trajectories in Cartesian coordinates, Orthogonal of trajectories in polar coordinates, Oblique trajectories in Cartesian coordinates.

4.2 *OBJECTIVES:-*

After studying this unit you will be able to

- **i.** Define the Linear Differential equation and it's type
- **ii.** Discussed the trajectories.
- **iii.** Understanding the orthogonal trajectories in Cartesian coordinates.
- **iv.** Analyzing the use of trajectories in this context is important for studying these systems.

4.3 *LINEAR DIFFERENTIAL EQUATION:-*

A differential equation is called linear if it can be obtained in the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (1)}$$

where P and Q are constants and are the function of x is called Linear differential equation of first order with y as dependent variable. So to solve the equation, multiply both sides by $e^{\int Pdx}$, then

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}Py = Qe^{\int Pdx}$$

Or $$\frac{d}{dx}\left\{ye^{\int Pdx}\right\} = Qe^{\int Pdx}$$

Integrating both sides

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$$

Which is the required solution of differential equation.

Working Rule:

1. The given equation in the form $\frac{dy}{dx} + Py = Q$ and $\frac{dy}{dx} + Px = Q$ as may be.
2. Find integrating factor $e^{\int Pdx}$ or $e^{\int Pdy}$.
3. The solution of Differential equation either

$$y.(I.F.) = \int\{Q.(I.F)\}dx + c$$

Or $$x.(I.F.) = \int\{Q.(I.F)\}dy + c\ .$$

4.4 *EQUATION REDUCIBLE TO THE LINEAR FORM:-*

An differential equation of the form

$$f'(y)\frac{dy}{dx}+Pf(y)=Q \qquad \ldots(1)$$

where P and Q are constants.

Putting $f(y)=v$ so that $f'(y)(dy/dx)=dv/dx,$ (1) becomes

$$dv/dx+Pv=Q \qquad \ldots(2)$$

Which is linear in v and x and its solution can be defined by Linear differential equation. Thus we get,

$I.F=e^{\int Pdx}$ and $v.e^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

Finally, replace v by $f(v)$ to solution in terms of x and y alone.

4.5 *BERNOULLI'S EQUATION:-*

Particular Case of Linear differential equation:-

An equation of the form $\frac{dy}{dx}+Py=Q\,y^n$ $\ldots(1)$

Where P and Q are constants or function of x and n is constant except 0 and 1, is known as *Bernoulli's Equation.*

From (1) $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ $\ldots(2)$

Suppose $y^{1-n}=v$ $\ldots(3)$

Differentiating (3) w.r.t. x $\frac{1}{(1-n)}\frac{dv}{dx}y^{-n}\frac{dy}{dx}=\frac{dv}{dx},$ or

$$y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dv}{dx} \qquad \ldots(4)$$

Putting the value of (3) and(4) in (1)

$\frac{1}{(1-n)}\frac{dv}{dx}+Pv=Q$ or $\frac{dv}{dx}+P(1-n)v=Q(1-n)$

Which is linear in v and x. Its $I.F.=\ e^{\int P(1-n)dx}=e^{(1-n)\int Pdx}$

Hence $v.e^{(1-n)\int Pdx}=\int Q.e^{(1-n)\int Pdx}dx+c,$ c being arbitrary constant.

$y^{1-n}e^{(1-n)\int Pdx}=\int Q.e^{(1-n)\int Pdx}dx+c,$ from(3)

SOLVED EXAMPLES

EXAMPLE1. Solve$\frac{dy}{dx}+2xy=e^{-x^2}$.

SOLUTION: The given equation

$$\frac{dy}{dx}+2xy=e^{-x^2} \qquad \dots(1)$$

where y is dependent variable

⇒ $P=2x$ and $Q=e^{-x^2}$, then $\int Pdx=\int 2xdx=2.\frac{1}{2}x^2=x^2$.

⇒ Therefore $I.F.=e^{\int Pdx}=e^{x^2}$.

Hence

⇒ $$y.(I.F.)=\int\{Q.(I.F)\}dx+c$$

⇒ $$y.e^{x^2}=\int\left(e^{-x^2}.e^{x^2}\right)dx+c$$

⇒ $$ye^{x^2}=\int dx+c \quad \text{or} \quad ye^{x^2}=x+c.$$

EXAMPLE2. Solve$\frac{dy}{dx}(x+2y^3)=y$.

SOLUTION. Let $\frac{dy}{dx}(x+2y^3)=y$ ... (1)

where x is dependent.

⇒ Thus, we have $\frac{dx}{dy}=\frac{x+2y^3}{y}$, or $\frac{dx}{dy}-\frac{1}{y}x=2y^2$ ... (2)

⇒ from (2)

⇒ $\int P dy = -\int (1/y) dy = -\log y$ so IF.of (2)= $e^{-\log y} = \frac{1}{y}$

⇒ Hence $x/y = \int 2y^2.(1/y)dx + c$

⇒ $x/y = y^2 + c$, where c is an arbitrary constant.

EXAMPLE3. Solve $(dy/dx) + x\, sin2y = x^3 cos^2 y$.

SOLUTION: Given equation

$$(dy/dx) + x\, sin2y = x^3 cos^2 y \quad ...(1)$$

Now dividing by $cos^2 y$ in equation (1)

$$sec^2 y(dy/dx) + 2x\, tany = x^3 \quad ...(2)$$

Putting $tany = v$ so that $sec^2 y(dy/dx) = dv/dx$.

Hence $dv/dx + 2xv = x^3$, which is linear in v and x and its solution

$$e^{\int 2xdx} = e^{x^2}.$$

⇒ $\boldsymbol{v}.e^{x^2} = \int x^3 . e^{x^2} dx + c,$ c being an arbitrary constant.

⇒ $\boldsymbol{v}.e^{x^2} = (1/2) \times \int t.e^t dt + c$, Now $x^2 = t$ and $2xdx = dt$

⇒ $(1/2) \times [t \times e^t - \int (1 \times e^t) dt] + c = (1/2) \times (te^t - e^t) + c$

⇒ $tany\ e^{x^2} = \left(\frac{1}{2}\right) \times e^{x^2}(x^2 - 1) + c$ as $v = tany$ & $t = x^2$

⇒ $tany = (1/2) \times (x^2 - 1) + ce^{-x^2}$, dividing by e^{x^2}

4.6 PRINCIPLE OD DUALITY:-

The principle of duality in differential equations refers to the fact that certain differential equations can be transformed into a dual form by interchanging certain variables or operators. In other words, the dual form of a differential equation is obtained by making a particular transformation that switches the roles of certain variables or operators in the original equation.

Formally, let us consider a linear differential equation of the form:

$$L[y] = f(x)$$

where L is a linear differential operator, y is the dependent variable, and $f(x)$ is a given function. The principle of duality states that if we apply a certain transformation to the differential equation, such as interchanging certain variables or operators, we can obtain a dual equation of the form:

$$L * [z] = g(x)$$

where $L *$ is the dual operator, z is the dual variable, and $g(x)$ is a new function related to $f(x)$ by the transformation.

The principle of duality has many applications in mathematics and physics, particularly in the study of partial differential equations and their solutions. Dual equations often provide a simpler or more intuitive way to understand the properties of a system, and can also lead to new insights or techniques for solving differential equations.

The principle of duality has numerous applications in mathematics and physics.

Here are some examples:

1. Electromagnetism: In electromagnetism, the principle of duality is used to relate electric and magnetic fields. Specifically, the electric and magnetic fields are related by a duality transformation that interchanges the electric and magnetic field vectors. This transformation is useful in understanding the symmetries of Maxwell's equations and in solving certain problems in electromagnetism.

2. Laplace transform: The Laplace transform is a mathematical tool used to solve differential equations. The principle of duality can be applied to the Laplace transform by interchanging the roles of time and frequency. This leads to a dual transform, known as the Fourier transform, which is useful in signal processing and other applications.

3. Partial differential equations: The principle of duality can be used to transform certain partial differential equations into dual equations, which can provide a simpler way to understand the properties of the system being studied. For example, the heat equation can be transformed into the wave equation by a duality transformation that interchanges the roles of time and space variables.

4. Quantum mechanics: In quantum mechanics, the principle of duality is used to relate particles and waves. Specifically, the wave-particle duality principle states that particles can exhibit wave-like behavior and waves can exhibit particle-like behavior.

This principle is essential to the understanding of the behavior of quantum systems, such as atoms and subatomic particles.

Overall, the principle of duality is a powerful tool for understanding the symmetries and properties of mathematical and physical systems, and has many applications in diverse areas of science and engineering.

4.7 *TRAJECTORIES:-*

Definition:

Trajectory:

A curve which cuts every number of a given family of curves in accordance with some given law is known as a ***Trajectory*** of the family of curves.

***Orthogonal Trajectory*:**

If a curve cuts every member of given family of curves at right angles, it is called an ***Orthogonal Trajectories*** of the family of the curve.

***Oblique Trajectory*:**

If a curve cuts every member of given family of curves at constant angles$\alpha(\neq 90^0)$, it is called an ***Oblique Trajectories*** of the family of the curve.

4.7.1 SELF ORTHOGONAL FAMILY OF CURVES:-

Definition:

If each member of a given family of curves intersects all other members orthogonally, then the given family of curves is said to be self orthogonal.

From self orthogonal family of the curves, if the differential equation of the family of the curves is identical with the differential equation of orthogonal trajectories, then the family of curves must be self orthogonal.

4.7.2 ORTHOGONAL TRAJECTORIES IN CARTESIAN COORDINATES:-

Let the equation of the given family of the curves be

$$f(x, y, c) = 0 \qquad \ldots(1)$$

Where c is parameter

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(x, y, dy/dx) = 0 \qquad \ldots(2)$$

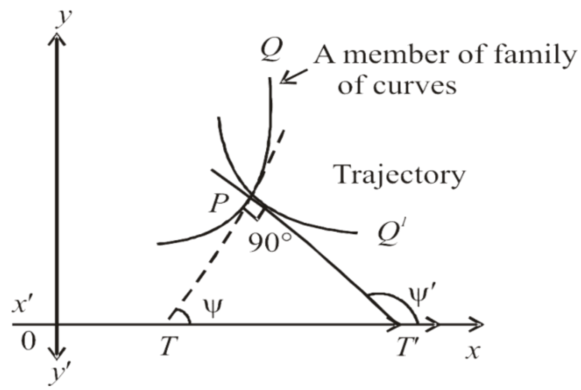

Fig.1

Let ψ be the angle between the tangents PT to the member PQ and $x-axis$ at any point $p(x,y)$, then we have

$$tan\psi = \frac{dy}{dx} \qquad ...(3)$$

Let (X,Y) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ψ' be the angle which the tangent PT' to the trajectories makes with $x-axis$.

$$tan\psi' = \frac{dY}{dX} \qquad ...(4)$$

Hence from (3) and (4), we get

$$tan\psi tan\psi' = -1 \qquad or \qquad \frac{dy}{dx}\times\frac{dY}{dX} = -1$$

$$\frac{dy}{dx}\times\frac{dY}{dX} = -1$$

$$\frac{dy}{dx} = -\frac{1}{dY/dX} = -\frac{dX}{dY}$$

Now the point of intersection of (2) with trajectory, we obtain

$$x = X, \qquad y = Y$$

Eliminating x, y and dy/dx from above equations, we have

$$F(X,Y,dX/dY) = 0$$

Hence, which is the differential equation of required family of trajectories.

Now

$$F(x, y, dy/dx) = 0$$
$$F(x, y, -dx/dy) = 0$$

Showing that it can be obtained by replacing $dr/d\theta$ by $(-dx/dy)$.

SOLVED EXAMPLES

EXAMPLE1: Find the orthogonal trajectories of family of curves $y = ax^2$, a being parameter.

SOLUTION: Given family of curves is

$$y = ax^2 \qquad \dots(1)$$

where a being parameter. Differentiating w.r.t. x, we obtain

$$dy/dx = 2ax \qquad \dots(2)$$

From (1), $a = y/x^2$

Putting the value of a in (2), we get

$$dy/dx = 2x\ y/x^2$$
$$dy/dx = 2y/x$$

Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is

$$-dx/dy = 2y/x \quad or \quad xdx + ydy = 0$$

Integrating, $x^2/2 + y^2 = b^2$ or $x^2/2b^2 + y^2/b^2 = 1$

Which is required the orthogonal trajectories, b being parameter.

EXAMPLE2: Find the orthogonal trajectories of parabolas whose equation is $y^2 = 4ax$.

SOLUTION: The equation of parabolas is

$$y^2 = 4ax \qquad \dots(1)$$

Differentiating (1) $2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a$

From (1), $a = y^2/4x$

Putting the value of a in above equation

$$y\frac{dy}{dx} = 2\,y^2/4x \Rightarrow \frac{dy}{dx} = y/2x$$

Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is

$$\frac{dx}{dy} = -y/2x \quad \Rightarrow ydy = -2xdx$$

Integrating above equation

$$\frac{y^2}{2} = -x^2 + c \Rightarrow\; y^2 = -2x^2 + c$$

EXAMPLE3: Find the orthogonal trajectories of the system of curves $(dy/dx)^2 = a/x$.

SOLUTION: The given curve is

$$(dy/dx)^2 = a/x \qquad \ldots(1)$$

Where a is constant. Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is given as below

$$-(dx/dy)^2 = a/x \qquad or \qquad dy = \pm x^{1/2}a^{1/2}dx$$

Integrating above equation

$$y + c = 1/a^{1/2} \times 2/3 \times x^{3/2}$$

$$3\sqrt{a}(y + c) = \pm 2x^{3/2}$$

Squaring both sides

$$9a(y + c)^2 = 4x^3$$

Which is required orthogonal trajectories, c being parameter.

4.7.3 ORTHOGONAL TRAJECTORIES IN POLAR

COORDINATES:-

Let the equation of the given family of the curves be

$$f(r, \theta, c) = 0 \qquad \ldots(1)$$

Where c is parameter.

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(r, \theta, dr/d\theta) = 0 \qquad \ldots(2)$$

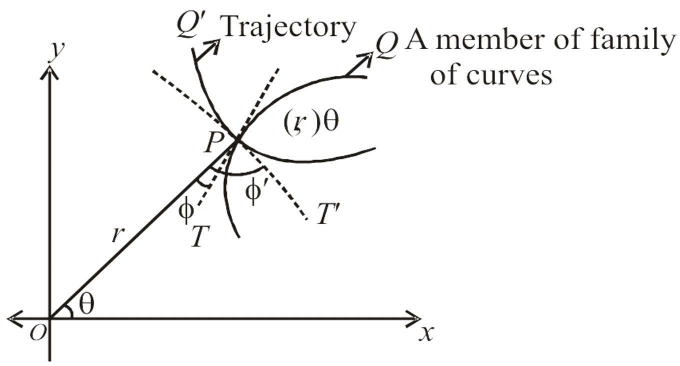

Fig.2

Let ϕ be the angle between the tangents PT to the member PQ and $x-axis$ at any point $p(r, \theta)$, then we have

$$tan\phi = \frac{dy}{dx} \qquad \ldots(3)$$

Let (R, Θ) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ϕ' be the angle which the tangent PT' to the trajectories makes with $x-axis$.

$$tan\phi' = R\frac{d\Theta}{dR} \qquad \ldots(4)$$

Hence from (3) and (4), we get

$$\phi' - \phi = 90^0 \qquad so\ much \qquad \phi' = 90^0 + \phi$$

$\therefore$ $$tan\phi' = tan(90^0 + \phi\ ) =$$

$-cot\phi \quad or \quad tan\phi tan\phi' = -1$

Putting the value of (3) and (4) in above equation

$$\left(r\frac{d\theta}{dr}\right)\left(R\frac{d\Theta}{dR}\right)=1 \quad or \quad \frac{dr}{d\theta}=-rR\frac{d\Theta}{dR}$$

Now the point of intersection of (2) with trajectory, we obtain

$$r=R, \qquad \theta=\Theta$$

Eliminating r,θ and $dr/d\theta$ from above equations, we have

$$F(R,\Theta,-R^2 d\Theta/dR)=0$$

Hence, which is the differential equation of required family of trajectories.

Now

$$F(r,\theta,dr/d\theta)=0$$

$$F(r,\theta,-r^2 d\theta/dr)=0$$

Showing that it can be obtained by replacing $dr/d\theta$ by $-r^2 d\theta/dr$.

SOLVED EXAMPLES

EXAMPLE1: Find the orthogonal trajectories of cardioids$r=a(1+cos\theta)$.

SOLUTION: The given curve is $r=a(1+cos\theta)$

Take both sides logarithm

$$\log r=\log a+\log(1+cos\theta)$$

Differentiating both sides w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta}=-\frac{sin\theta}{(1+cos\theta)}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, the differential equation of orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=-\frac{sin\theta}{(1+cos\theta)}$$

$$=-\frac{2sin\,\theta/2\,cos\,\theta/2}{(1+2cos^2\,\theta/2-1)}=-tan\,\theta/2$$

$$r\frac{d\theta}{dr}=tan\,\theta/2 \quad or \quad \frac{1}{r}\frac{dr}{d\theta}=cot\frac{\theta}{2}$$

Now integrating factor $\int \frac{1}{r}\, dr = \int cot \frac{\theta}{2}\, d\theta$

$$\log r = 2 \log sin\left(\frac{\theta}{2}\right) + \log c$$

$$\log r = \log sin^2\left(\frac{\theta}{2}\right) + \log c$$

$$r = csin^2\left(\frac{\theta}{2}\right)$$

$$r = c\frac{(1 - cos\theta)}{2} = b(1 - cos\theta)\ taking\ b = c/2$$

EXAMPLE2: Find the orthogonal trajectories of the series logarithmic spirals$r = a^{\theta}$.

SOLUTION: The given curve is

$$r = a^{\theta} \Rightarrow \log r = \theta \log a \qquad \ldots(1)$$

Differentiating both sides w.r.t.θ

$$\frac{dr}{d\theta} = a^{\theta} \log a = r \log a = r\frac{\log r}{\theta} \qquad from(1)$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, the differential equation of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = r\frac{\log r}{\theta} \Rightarrow -\theta d\theta = \frac{1}{r}\log r\, dr$$

Integrating both sides

$$\int \frac{1}{r}\log r\, dr = \int -\theta d\theta + c_2$$

$$\frac{(\log r)^2}{2} = -\frac{\theta^2}{2} + c_2, \qquad \therefore\ c_2 = \frac{c^2}{2}$$

$$\frac{(\log r)^2}{2} = -\frac{\theta^2}{2} + \frac{c^2}{2}$$

$$(\log r)^2 = c^2 - \theta^2$$

$$\log r = \sqrt{c^2 - \theta^2}$$

$$r = e^{\sqrt{c^2-\theta^2}}$$

Which is required equation.

EXAMPLE3: Find the orthogonal trajectories of $r^n cosn\theta = a^\theta$ is $r^n sinn\theta = c^n$.

SOLUTION: Given $r^n cosn\theta = a^n$, where a is a parameter.

Since taking both sides logarithm

$$n\log r + \log cosn\theta = n\log a$$

Differentiating both sides w.r..t. θ

$$\frac{n}{r}\frac{dr}{d\theta} - tann\theta = 0 \qquad or \qquad \left(\frac{1}{r}\right)\frac{dr}{d\theta} - tann\theta = 0$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, the differential equation of orthogonal trajectories is

$$\left(\frac{1}{r}\right)(-r^2)\frac{d\theta}{dr} - tann\theta = 0$$

$$\left(\frac{1}{r}\right)dr + cotn\theta d\theta = 0$$

Now integrating factor

$$\log r + \frac{1}{n}\log sinn\theta = \log c$$

Where c being constant.

$$n\log r + \log sinn\theta = n\log c$$

$$r^n sinn\theta = c^n$$

Which is the required equation of orthogonal trajectories.

4.7.4 OBLIQUE TRAJECTORIES IN CARTESIAN COORDINATES:-

Let the equation of the given family of the curv

$$f(x, y, c) = 0 \qquad \dots(1)$$

Where c is parameter.

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(x, y, dy/dx) = 0 \qquad \dots(2)$$

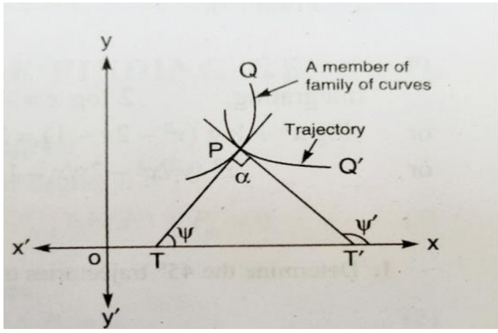

Fig.3

Let ψ be the angle between the tangents PT to the member PQ and $x-axis$ at any point $P(x, y)$, then we have

$$tan\psi = \frac{dy}{dx} \qquad \dots(3)$$

Let (X, Y) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ψ' be the angle which the tangent PT' to the trajectories makes with $x - axis$.

$$tan\psi' = \frac{dY}{dX} \qquad \dots(4)$$

Suppose PT and PT' intersect at angle α, then we get

$\therefore \qquad tan\alpha = \frac{(dy/dx)-(dY/dX)}{1+(dy/dx)(dY/dX)}$

so that $\qquad \frac{dy}{dx} = \frac{(dy/dx)+(dY/dX)}{1-(dy/dx)(dY/dX)}$

Now from (2) with trajectory, we get

$$x = X, \qquad y = Y$$

Eliminating x, y and dy/dx from above equations, we have

$$F\left(X, Y, \frac{(dy/dx) + tan\alpha}{1 - (dy/dx)\ tan\alpha}\right) = 0$$

Hence, which is the differential equation of required family of trajectories.

Now

$$F(x, y, dy/dx) = 0$$

$$F\left(x, y, \frac{(dy/dx)+tan\alpha}{1-(dy/dx)\ tan\alpha}\right) = 0$$

Showing that it can be obtained by replacing dy/dx by $\left[\frac{(dy/dx)+tan\alpha}{1-(dy/dx)\ tan\alpha}\right]$,

$i.e.,\ \ (p + tan\alpha)/(1 - ptan\alpha)$where $p = dy/dx$.

EXAMPLE: Find the family of the curves whose tangents form the angle of $\frac{\pi}{4}$ with the hyperbola $xy = c$.

SOLUTION: Let the given curve

$$xy = c \qquad \text{... (1)}$$

where c is parameter

Differentiating (1),

$$y + x(dy/dx) = 0 \quad or \quad y + px = 0, \quad where\ p = dy/dx$$

Replacing p by $\frac{p+tan\left(\frac{\pi}{4}\right)}{1-\left(\frac{\pi}{4}\right)\ tan\left(\frac{\pi}{4}\right)}$ $\quad i.e., \frac{p+1}{1-p}$ the differential equation of desired family of curves is

$$\boldsymbol{y} + \frac{p+1}{1-p}x = 0 \quad or \quad p = \frac{y+x}{y-x} \quad or \quad \frac{dy}{dx} = \frac{(y/x)+1}{(y/x)-1}$$

Suppose $\frac{y}{x} = v,\ \ i.e.,\ y = vx$ so that $\frac{dy}{dx} = v + (dv/dx)$

From above equations $v + \frac{dv}{dx} = \frac{v+1}{v-1}$ $\quad or \quad$ $x\frac{dv}{dx} = -\frac{v^2-2v-1}{v-1}$

$$\left(\frac{2}{x}\right)dx = -\left\{\frac{2(v-1)}{(v^2-2v-1)}\right\}dv$$

Integrating, $\quad 2\log x = -\log(v^2 - 2v - 1) + \log c$,

c being an arbitrary constant.

$\log x^2 + \log(v^2 - 2v - 1) = \log c \qquad or \qquad x^2\ (v^2 - 2v - 1) = c$

Putting the value of $\frac{y}{x} = v$ in above equation

$$x^2\ \left(\left(\frac{y}{x}\right)^2 - 2\frac{y}{x} - 1\right) = c$$

$$x^2 - 2xy - y^2 = c$$

SELF CHECK QUESTIONS 1

1. The solution of differential equation $p^2 - 8p + 15 = 0$ is

(a) $p = 5, p = 3$ (b) $(y - 5x - c)(y - 3x - c) = 0$

(c) $(y + 5x)(y + 3x + c) = 0$ (d) None

2. Solution of the equation $y^2 logy = xyp + p^2$ is

(a) $\log y = cx + x^2$ (b) $\log y = cx^2 + e^x$

(c) $(y + 5x)(y + 3x + c) = 0$ (d) None

3. Solution of the equation $y = px + \log p$ is

(a) $y = e^x + c$ (b) $y = cx + \log c$

(c) $y = logcx$ (d) $x = e^y + c$

4. The differential equation $Mdx + Ndy = 0,$ where M and N are the functions of x and y is exact if

(a) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (b) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$

(c) $M + N = 0$ (d) $M = N$

SELF CHECK QUESTIONS2

1. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.
2. Find the orthogonal trajectories of the system of the curve $\left(\frac{dy}{dx}\right)^2 = \frac{a}{x}$.
3. Which among the following is true for the curve $r^n = a\ sinn\theta$
   a. Given family of a curve is self orthogonal.
   b. Orthogonal trajectories is $r^n = kcosn\theta$. Where k is constant.
   c. Orthogonal trajectories is $r^n = kcosecn\theta$. Where k is constant.
   d. Orthogonal trajectories is $r^n = ksinn\theta$. Where k is constant.
4. What is oblique trajectories?

4.7 *SUMMARY:-*

In this unit, we have learned about the Linear Differential equation: Equations reducible to linear form Bernoulli's Equation. In this unit we also studied the trajectories of the family of the curve, orthogonal trajectories and oblique trajectories with example.

4.8 *GLOSSARY:-*

- **Integrating Factor:** A function used to transform certain types of ordinary differential equations into exact differential equations, making them easier to solve.

- **Exact Differential Equation:** An ODE that can be expressed as the total derivative of a scalar potential function, enabling direct integration to find the solution.
- **Scalar Potential Function:** A function whose partial derivatives with respect to the variables in the ODE correspond to the coefficients of the differential terms, allowing the ODE to be written as exact.
- **Exactness:** The property of a differential equation where it can be expressed as the total derivative of a scalar potential function.
- **Exact Differential:** A differential form that can be derived from a scalar potential function.
- **Closed Form Solution:** A solution to an exact differential equation obtained by integrating the exact equation directly, often involving finding a scalar potential function and then using it to find the solution.
- **Auxiliary Differential Equation:** An equation used to find the integrating factor for a given differential equation, often solved by inspection or through specific methods.
- **Trajectory:** The path followed by an object as it moves through space, often influenced by forces such as gravity, friction, or other interactions.
- Cartesian Coordinates
- Oblique trajectories
- Polar Coordinates

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4.10 SUGGESTED READING

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4.12 *TERMINAL QUESTIONS:-*

Terminal Question1:

(TQ-1) Solve the following differential equations:

a. $p^2 + 2pycptx = y^2$
b. $p^2 - 5p + 6 = 0$
c. $y = px + a/p$
d. $y = px + logp$

Terminal Questions 2:

(TQ-1) Find the orthogonal trajectories of the family of curves$y = ax^2$, a being a parameter.

(TQ-2) Find the orthogonal trajectories of the family of curves$3xy = x^3 - a^3$, a being a parameter.

(TQ-3) Find the orthogonal trajectories of $x^2 + y^2 = 2ax$.

(TQ-4) Find the orthogonal trajectories of the family of curves:

a. $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1, \lambda$ being the parameter.
b. $\frac{x^2}{a^2} + \frac{y^2}{(a^2+\lambda)} = 1, \lambda$ being the parameter.

(TQ-5) Find the orthogonal trajectories of the family of parabolas $y^2 = 4a(x+a)$, where a being a parameter.

(TQ-6) Find the orthogonal trajectories of the family of cardioids $r = a(1 - cos\theta)$, where a being a parameter.

(TQ-7) Find the orthogonal trajectories of the family of cardioids $r = a(1 + cos\theta)$, where a being a parameter.

(TQ-8) Find the orthogonal trajectories of $r = a(1 + cosn\theta)$.

(TQ-9) Find the orthogonal trajectories of $r^n sinn\theta = a^n$.

(TQ-10) Find the orthogonal trajectories of the family of parabolas $r = \frac{2a}{(1+cos\theta)}$, where a being a parameter.

(TQ-11) Find the orthogonal trajectories of the family of curves:

i. $y = ax^n$.
ii. $y = ax^3$.
iii. $y = 4ax$.
iv. $x^2 + y^2 = a^2$.

4.13 *ANSWERS:-*

<u>SELF CHECK ANSWERS1</u>

(TQ-2) **1.** (b), **2.** (a), **3**. (b), **4**. (a)

SELF CHECK ANSWERS2

1. $2x^2 + y^2 = k$,

2. $9a(y+c)^2 = 4x^3$**,**

3. b,

4. A curve which intersects the curves of the given family at a constant angle α is called an oblique trajectory of the given family.

TERMINAL ANSWERS 1

(TQ-1) **(p)** $\left(y - \frac{c}{1+cosx}\right)\left(y - \frac{c}{1-cosx}\right) = 0,$

(q) $(y - 2x - c)(y - 3x - c) = 0$

(r) $y = cx + a/x,$ **(s)** $y = cx + \log c$

(TQ-2) **1.** (b), **2.** (a), **3**. (b), **4**. (a)

TERMINAL ANSWERS 2

(TQ-1) $\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$

(TQ-2) $x^2 = y - (1/2) + ce^{-2y}$

(TQ-3) $x^2 + y^2 = cy$

(TQ-4)

a. $x^2 + y^2 - 2a^2 \log x = c$

b. $x^2 + y^2 - 2a^2 \log x = c$

(TQ-5) $y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$

(TQ-6) $r = b(1 + cos\theta)$

(TQ-7) $r = b(1 - cos\theta)$

(TQ-8) $r^{n^2} = b(1 - cosn\theta)$

(TQ-9) $r^n cosn\theta = c^n$

(TQ-10) $r = \frac{2c}{(1-cos\theta)}$

(TQ-11)

i. $x^2 + ny^2 = c$

ii. $x^2 + 3y^2 = c$

iii. $2x^2 + y^2 = c^2$

iv. $y = cx$

BLOCK-II

UNIT 5: LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5.1 *INTRODUCTION:-*

Linear constant coefficient ordinary differential equations are useful for modeling a wide variety of continuous time systems. The approach to solving them is to find the general form of all possible solutions to the equation and then apply a number of conditions to find the appropriate solution.

The study of these differential equations with constant coefficients dates back to Leonhard Euler, who introduced the exponential function e^x, which is the unique solution of the equation $f' = f$ such that $f(0) = 1$.

Linear Differential Equations are those Differential Equations in which the dependent variable and its derivative occur only in first degree and are not multiplied together. Hence the standard Linear Differential Equations of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = X \qquad .....(1)$$

Where a_1, a_2 ,….., a_n and X are functions of x only. If a_1, a_2,….., a_n are all constants and X is a function of x alone, then the differential equation (1) is called the general or standard form of the Linear Differential Equations with Constant Coefficients. These types of equations are most important in the study of electro-mechanical vibration and other engineering and real life problems.

In actual practice, equations of the type (1), where the coefficients are functions of x with no restrictions placed on their simplicity or complexity, do not usually have solutions expressible in terms of elementary functions and even when they do, it is in general extremely difficult to find them. If, however, each coefficient in (1) is a constant, then solutions in terms of elementary functions can be readily obtained. For the next few units, therefore, we shall concentrate on solving the differential equation $D^n y + a_1 D^{n-1} y + \ldots.. + a_{n-1} Dy + a_n y = X$.

5.2 *OBJECTIVES:-*

After studying this unit you will be able to

- **i.** Define the Linear Differential equation with constant coefficients.
- **ii.** Described the solution of Linear Differential equation with constant coefficients.
- **iii.** Evaluate the complementary function.

5.3 THE OPERATOR D AND D^{-1}:-

5.3.1 POSITIVE INDEX OF *D*:-

Let D stand for $\frac{d}{dx}$; D^2 for $\frac{d^2}{dx^2}$; and so on. The symbols D, D^2 and so on are called operators. Index of D denotes the number of times the operation of differentiation must be carried out. Thus $D^2x^3 = 6x$, which indicates that we differentiated x^3 two times. The following results are valid for such type of operators:

1. $D^m + D^n = D^n + D^m$. (Commutative over addition)
2. $D^m D^n = D^n D^m = D^{m+n}$. ( )
3. $D(u+v) = Du + Dv$, where u and v are functions of x only.
4. $(D-\alpha)(D-\beta) = (D-\beta)(D-\alpha)$, where α & β are constants.

5.3.2 NEGATIVE INDEX OF *D*:-

It is clear that *D* stands for differentiation, thus D^{-1} is equivalent to an integration. For example

$$D^{-2}x = \int\left[\int x\,dx\right]dx = \int \frac{x^2}{2}dx = \frac{x^3}{6}.$$

The negative index of *D* indicates the number of times the operation of integration is to be carried out. The negative index of *D* also satisfies the above mentioned four results.

We write, $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, ..., D^n = \frac{d^n}{dx^n}$

Now (1) may be written as

$$D^n y + a_1 D^{n-1} y + ... + a_{n-1} Dy + a_n y = X$$

$$\Rightarrow (D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n) y = X$$

$$\text{or} \quad f(D)y = X \qquad \text{..... (2)}$$

where $f(D) = D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n$

5.4 LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A differential equation of the form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-1} y}{dx^{n-2}} + a_{n-1} \frac{dy}{dx} + a_n = X \qquad \text{…….(3)}$$

where $a_1, a_2, ..., a_n$ are all constants and X is a function of x alone, is called a linear differential equation of n^{th} order with constant coefficients.

The general solution of a differential equation of n^{th} order has n arbitrary constants.

We write,

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, ..., D^n = \frac{d^n}{dx^n}$$

Now (3) may be written as

$$D^n y + a_1 D^{n-1} y + ... + a_{n-1} Dy + a_n y = X$$

$$\Rightarrow (D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n) y = X$$

or $f(D) y = X$ ….. (4)

where

$$f(D) = D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n$$

So, a linear differential equation of degree 2 with constant coefficients looks like

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X ....(4)$$

When a_0, a_1 and a_2 are constants and X is a function of x only. If we write D for the symbol d/dx and D^2 for d^2/dx^2 then dy/dx can be written as Dy and d^2y/dx^2 as D^2y and the differential equation (4) can be rewritten as

$$(a_0 D^2 + a_1 D + a_2) y = X \dots\dots\dots\dots(5)$$

Or

$f(D) y = x$ where $f(D) = a_0 D^2 + a_1 D + a_2$

We first learn how to solve differentiation equation (4) when X = 0.

This is called a homogeneous equation.

Solution to the differential equation $f(D)y = 0$

To solve the differential equation

$$f(D)y = 0 \ldots\ldots\ldots (6)$$

we first solve the equation

$$f(m) = 0 \ldots\ldots\ldots\ldots (7)$$

which is obtained by replacing D by m is the expression $f(D)$. The solution of (7) can then be written directly, depending on the nature of the roots of the equation $f(m) = 0$. We consider various cases by one. The equation $f(m) = 0$ is called the auxiliary equation (A.E.) of the differential equation (7).

For the sake of convenience, let we consider a second order linear differential equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \qquad \ldots.. \quad (6)$$

Let $y = e^{mx}$ be a solution of (6), which means that it has to satisfy (6). Therefore

$$\frac{d^2}{dx^2}\{e^{mx}\} + a_1\frac{d}{dx}\{e^{mx}\} + a_2\{e^{mx}\} = 0$$

$$\Rightarrow m^2e^{mx} + a_1me^{mx} + a_2e^{mx} = 0 \Rightarrow e^{mx}(m^2 + a_1m + a_2) = 0$$

Thus, $y = e^{mx}$ is a solution of (6) if and only if

$$m^2 + a_1m + a_2 = 0 \qquad \ldots.. (4)$$

The equation (4) is called the auxiliary equation of (3).

Theorem 1: If $y = y_1, y = y_2, \ldots\ldots\ldots\ldots. y = y_n$ are n linearly independent solutions of the differential equation:

$$(D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n) y = 0$$

then $u = c_1 y_1 + c_2 y_2 +$ ……………………………$c_n y_n$ is also its solution, where c_1, c_2…………..c_n are arbitrary constants.

Theorem 2: If $y = u$ is the complete solution of the equation If $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) $f(D)y = Q$ then the complete solution of the equation $f(D)y = Q$ is $y = u + v$.

The part $y = u$ is called the complementary function (C.F.) and the part $y = v$ is called the particular integral (P.I.) of the equation $f(D)y = Q$.

The complete solution is y = C.F. + P.I.

It is important that to solve the equation $f(D)y = Q$, we first find the C.F. It means the complete solution $f(D)y = 0$ and the particular solution of the equation $f(D)y = Q$.

- The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation or A.E.

Consider the differential equation

$f(D)y = Q$………………….(5)

Complementary function is actually the solution of the given differential equation (5) when its right hand side member it means the Q is replaced by zero. To find complementary function we first find auxiliary equation.

5.5 CASES FOR FINDING COMPLEMENTARY FUNCTION (C.F.):-

Consider the equation $(D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n) y = 0$ ....(6)

It's auxiliary equation is $m^n + a_1 m^{n-1} + .a_2 m^{n-2} .... + a_n = 0.$ …(7)

Where all a_i's are constant.

CASE I. When the roots of auxiliary equations are real and district:

Let m_1 and m_2 be the distinct roots of (7). Then $e^{m_1 x}, e^{m_2 x}$ are the solutions of (7). Hence, the general solution of (6) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \qquad ….. (8)$$

CASE II. When the roots of auxiliary equations are real and equal:

Let m and m be the roots of equation (7). From (6),

$$y = (c_1 + c_2) e^{mx}$$

$$y = a e^{mx}, a = c_1 + c_2$$

which cannot be the general solution as it contains only one arbitrary constant. To obtain the general solution, we proceed as follows:

From (4), sum of the two roots = $-a_1 \Rightarrow m + m = -a_1$

$$\Rightarrow 2m + a_1 = 0 \qquad …..(9)$$

We now show that $y = x e^{mx}$, is also a solution of (3)

We have $$\frac{dy}{dx} = 1.e^{mx} + mxe^{mx} = (1+mx)e^{mx} \qquad \ldots\ldots(10)$$

$$\frac{d^2y}{dx^2} = me^{mx} + m(1+mx)e^{mx}$$

$$= 2me^{mx} + m^2xe^{mx} \qquad \ldots\ldots..(11)$$

Substituting (7) and (8) in (3), we obtain

$$e^{mx}[(2m+m^2x)+a_1(1+mx)+a_2x]=0$$

or $(m^2+a_1m+a_2)x+(2m+a_1)=0$

This equation is identically true in view of (4) and (6)

$\therefore \; y = e^{mx}, xe^{mx},$ are two solutions of (3)

Hence, the general solution of (3) is

$y = ae^{mx} + bxe^{mx},$ where a and b are constants.

$\therefore \; y = (a+bx)e^{mx}$

CASE III. When the roots of (4) are imaginary

Let *a*+*ib* and *a*-*ib* be the roots of (3).

Using (5), the general solution of (3) is

$$y = c_1e^{(a+ib)x} + c_2e^{(a-ib)x} = e^{ax}[c_1e^{ibx}+c_2e^{-ibx}]$$

$$= e^{ax}\left[c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)\right]$$

$$= e^{ax}\left[c_1+c_2)\cos bx + (ic_1 - ic_2)\right]$$

Hence, $y = e^{ax}(A\cos bx + b\sin bx)$ is the general solution of (3) where, $A = c_1 + c_2, B = ic_1 - ic_2$.

5.6 RULES FOR FINDING COMPLEMENTARY FUNCTION (C.F.):-

Based on the above discussion, we give the following steps are adopted to find the solution of $f(D)y = 0,$ where $f(D) = D^n + D^{n-1} + ... + a_{n-1}D + a_n$.

Step 1. Obtain the auxiliary equation (A.E.) by replacing D by m in $f(D) = 0$ i.e.,

$$m^n + a_1 m^{n-1} + ... + a_{n-1}m + a_n = 0$$

Let its roots be $m_1, m_2, ..., m_n$.

Step 2. If all the roots are distinct i.e., $m_1 \neq m_2 \neq ... \neq m_n$, then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$ is the solution of $f(D)y = 0.$ Here $c_1, c_{2,} ....., c_n$ are any n arbitrary constants.

Step 3. If the roots are of repeated nature.

a) If $m_1 = m_2 = m,$ then the solution of $f(D)y = 0$ is $y = (c_1 + c_2 x)e^{mx} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$.

b) If $m_1 = m_2 = m_3 = m$, then the solution of $f(D)y = 0$ $y = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + .. + c_n e^{m_n x}$ and so on.

c) If $m_1 = m_2 = ... = m_n = m$, then the solution of $f(D)y = 0$ is $y = (c_1 + c_2 x + ... + c_n x^2)e^{mx}$.

Step 4. If the roots are complex i.e., $m_1 = a + ib, m_2 = a - ib,$ then the solution of $f(D)y = 0$ is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

S. No.	Nature of roots of auxiliary equation	Complementary Function
1.	If all the roots of auxiliary equation are real and distinct say, $m_1, m_2, m_3, \ldots$	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots$
2.	If all the roots of auxiliary equation are real and two equal roots say, $m_1, m_2, m_3, \ldots$ where $m_1 = m_2 = m$	$(c_1x + c_2)e^{mx} + c_3e^{m_3x} + \cdots$
3.	If all the roots of auxiliary equation are real and three equal roots say, $m_1, m_2, m_3, m_4 \ldots$ where $m_1 = m_2 = m_3 = m$	$(c_1x^2 + c_2x + c_3)e^{mx} + c_4e^{m_4x} + \cdots$
4.	If auxiliary equation has one pair of imaginary roots say $\alpha + i\beta, \alpha - i\beta,\ m_3, \ldots$	$e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x] + c_3e^{m_3x} + \cdots$
5.	If auxiliary equation have two pair of imaginary roots say, $\alpha \pm i\beta, \alpha \pm i\beta,\ m_5, \ldots$	$e^{\alpha x}[(C_1x + C_2)\cos\beta x + (C_3x + C_4)\sin\beta x] + c_5e^{m_5x} + \cdots$
6.	If auxiliary equation has one pair of surd roots say, $m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta},\ m_3, \ldots$	$e^{\alpha x}\left[C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}\right] + c_3e^{m_3x} + \cdots$

5.7 SOLVED EXAMPLES:-

Example1. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0.$

Solution. Given equation in the symbolic form is

$$(D^2 - 3D - 4)y = 0$$

Its auxiliary equation is $m^2 - 3m - 4 = 0$

$$(m-4)(m+1) = 0$$

$$\Rightarrow \quad m = 4 - 1$$

Hence , the solution is $y = c_1e^{-x} + c_2e^{4x}$

Example 2. solve $(D^3 + D^2 - D - 1)y = 0$

Solution . the auxiliary equation is $m^3 + m^2 - m - 1 = 0$

$\Rightarrow (m-1)(m^2 + 2m + 1) = 0)$

$((m-1)(m+1)^2 = 0$

$m = 1, -1, -1$

Hence , the Solution is $y = (c_1 + xc_2)e^{-x} + c_3 e^x$

Example 3. Solve

$(D^2 + D + 1)y = 0$

Solution . the auxiliary equation is $m^2 + m + 1 = 0$

$$\therefore m = \frac{-1 + \sqrt{1-4}}{2} = -\frac{1}{2} + \sqrt{\frac{3}{2}} i,$$

where $i = \sqrt{-1}$,

$$m = \frac{-1 - \sqrt{1-4}}{2} = -\frac{1}{2} - \sqrt{\frac{3}{2}} i,$$

where $i = \sqrt{-1}$

Hence ,the solution is $y = e^{-1/2x} \left[c_1 cos \sqrt{\frac{3}{2}} x + c_1 sin \sqrt{\frac{3}{2}} x \right]$

Example 4. Solve

$(D^2 + D + 1)y = 0.$

Solution. The auxiliary equation is $m^4 - a^4 = 0$

$$\Rightarrow (m^2 - a^2)(m^2 + a^2) = 0$$

$m = \pm a, \pm ai$

Hence, the solution $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

Example 5. Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

Subject to the condition

$y(0) = (0)$

and

y(0) =1.

Solution.

Given equation in symbolic form is

$(D^2 - 3D + 2)y = 0$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\Rightarrow$ $(m-1)(m-2) = 0$ …(i)

$\therefore$ $m = 1,2$

$\therefore$ $y = c_1e^x + c_2e^{2x}$

It is given that

$y(0) = 0.$ i.e. if $x = 0,$ then

$y = 0$ …..(ii)

Putting these values in (i),we have

$0 = c_1 + c_2$

On differentiating (i) w. r. t , we get

$$\frac{dy}{dx} = c_1e^x + 2c_2e^{2x}$$ …(iii)

It is given that

$y(0) = 1$ i.e. if $.x = 0$, then $\frac{dy}{dx} = 0$

Putting these values in (iii),we get

$1 = c_1 + 2c_2$

Solving (ii) (iv),we get

$c_1 = -1, c_2 = 1.$

putting the values of c_1 and c_2 in (i), we get …..(iv)

$$y = -e^x + e^{2x}$$

EXAMPLE6: Solve$(D^3 + D^2 + 4D + 4)y = 0$

SOLUTION: Here the given differential equation is

$$(D^3 + D^2 + 4D + 4)y = 0$$

Its corresponding auxiliary equation is$D^3 + D^2 + 4D + 4 = 0$

$$i.e., (D^2 + 4)(D + 1) = 0$$

$\Rightarrow D = -1, \ \pm 2i$

Hence the complete solution is

$$y = c_1 e^{-x} + e^{0x}(c_2 cos2x + c_3 sin2x)$$

$$\Rightarrow \qquad y = c_1 e^{-x} + c_2 cos2x + c_3 sin2x$$

EXAMPLE7: Solve $\frac{d^4x}{dt^4} + 4x = 0$

SOLUTION: Given equation in symbolic form is $(D^4 + 4)x = 0$

Therefore, Auxiliary equation is $D^4 + 4 = 0$

Or,$(D^4 + 4D^2 + 4) - 4D^2 = 0$

$$\Rightarrow (D^2 + 2)^2 - (2D)^2 = 0$$

$$\Rightarrow (D^2 + 2D + 2)(D^2 - 2D + 2) = 0$$

Therefore, either $D^2+2D+2=0$ or $D^2-2D+2=0$

$\Longrightarrow D=\frac{-2\pm\sqrt{(-4)}}{2}$ and $D=\frac{2\pm\sqrt{(-4)}}{2}$

$\Longrightarrow\ D=-1\pm iandD=1\pm i$

Hence the required solution is $x=e^{-t}(c_1cost+c_2sint)+e^{t}(c_3cost+c_4sint)$

EXAMPLE8: Solve $\frac{d^2y}{dx^2}+(a+b)\frac{dy}{dx}+aby=0$

SOLUTION: Here the given differential equation is

$$(D^2+(a+b)D+ab)y=0$$

The corresponding auxiliary equation is$D^2+(a+b)D+ab=0$

$$\Longrightarrow (D+a)(D+b)=0$$

$$\Longrightarrow D=-a,\qquad -b$$

Hence the required solution is $y=c_1e^{-ax}+c_2e^{-bx}$

EXAMPLE 9: Solve $(D^2-4D+1)y=0$

SOLUTION: Here the given differential equation is

$$(D^2-4D+1)y=0$$

The corresponding auxiliary equation is$D^2-4D+1=0$

$$\Longrightarrow D=\frac{4\pm\sqrt{(16-4)}}{2}$$

$$\Longrightarrow D=2\pm\sqrt{3}$$

Hence the required solution is $y=2c_1e^{(2+\sqrt{3})x}+c_2e^{(2-\sqrt{3})x}$

$$\Longrightarrow y=e^{2x}\{c_1e^{x\sqrt{3}}+c_2e^{-x\sqrt{3}}\}$$

EXAMPLE 10: Solve $(D^3-2D^2-4D+8)y=0$

SOLUTION: Here the given differential equation is

$$(D^3 - 2D^2 - 4D + 8)y = 0$$

The corresponding auxiliary equation is $D^3 - 2D^2 - 4D + 8 = 0$

$$\Rightarrow D^2(D-2) - 4(D-2) = 0$$

$$\Rightarrow (D-2)(D^2-4) = 0$$

$$\Rightarrow (D-2)(D-2)(D+2) = 0$$

$$\Rightarrow D = 2, \quad 2, \quad -2$$

Therefore, the required solution is $y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}$

EXAMPLE 11: Solve $(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$

SOLUTION: Here given differential equation is

$$(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$$

The corresponding auxiliary equation is $D^4 - 7D^3 + 18D^2 - 20D + 8 = 0$

$$\Rightarrow D^3(D-1) - 6D^2(D-1) + 12D(D-1) - 8(D-1) = 0$$

$$\Rightarrow (D-1)(D^3 - 6D^2 + 12D - 8) = 0$$

$$\Rightarrow (D-1)[D^2(D-2) - 4D(D-2) + 4(D-2)] = 0$$

$$\Rightarrow (D-1)(D-2)(D^2 - 4D + 4) = 0$$

$$\Rightarrow (D-1)(D-2)(D-2)^2 = 0$$

$$\Rightarrow D = 1, \quad 2 \text{ (Thrice)}$$

Therefore, the required solution is $y = c_1 e^x + (c_2 x^2 + c_3 x + c_4)e^{2x}$

EXAMPLE 12: Solve $(D^4 + 4)y = 0$

SOLUTION: Here the given differential equation is $(D^4 + 4)y = 0$

The corresponding auxiliary equation is $D^4 + 4 = 0$

$$\Rightarrow D^4 = -4$$

$$\Rightarrow D^2 = \pm 2i$$

$\Rightarrow D^2 = 2i$and$-2i$

(1)

Or,$D = \pm\sqrt{(2i)}\ and\ \pm\sqrt{(-2i)}$

Let $\sqrt{(2i)} = a + ib$

Squaring both sides, we get

$$2i = (a^2 - b^2) + (2ab)i$$

Equating real and imaginary parts on both sides, we get

$a^2 - b^2 = 0$and$2ab = 2\ or\ ab = 1$

Therefore $a^2 - \left(\frac{1}{a^2}\right) = 0$ since $b = \frac{1}{a}$

Or $a^4 - 1 = 0$ or $a^4 = 1$

$$\Rightarrow a = \pm 1, \qquad \pm i$$

If $a = 1,$ we have from $ab = 1,\ b = 1$

Hence $\sqrt{(2i)} = 1 + i$

Similarly, we can prove that$\sqrt{(-2i)} = 1 - i$

Therefore from (1), the roots of the auxiliary equation are

$\pm(1 + i)$and$\pm(1 - i)$

$$i.e., 1 \pm i\ and\ -1 \pm i$$

Therefore, the required solution is

$$y = e^x[c_1cosx + c_2sinx] + e^{-x}[c_3cosx + c_4sinx]$$

EXAMPLE 13: Solve $(D^4 + D^2 + 1)y = 0$

SOLUTION: Here the given differential equation is

$$(D^4 + D^2 + 1)y = 0$$

The corresponding auxiliary equation is

$$D^4 + D^2 + 1 = 0 \qquad \ldots(1)$$

$$\Rightarrow (D^4 + D^2 + 1) - D^2 = 0$$

$$\Rightarrow (D^2 + 1)^2 - D^2 = 0$$

$$\Rightarrow (D^2 + 1 + D)(D^2 + 1 - D) = 0$$

Now $D^2 + D + 1 = 0$ gives $D = \frac{1}{2}\left[-1 \pm \sqrt{(1-4)}\right]$

$$\Rightarrow D = \frac{1}{2}\left[-1 \pm i\sqrt{3}\right]$$

Similarly, $D^2 - D + 1 = 0$ gives $D = \frac{1}{2}\left[1 \pm i\sqrt{3}\right]$

Therefore, the solution of auxiliary equation (1) is $\frac{1}{2}\left[-1 \pm i\sqrt{3}\right], \frac{1}{2}\left[1 \pm i\sqrt{3}\right]$

Therefore, the required solution is

$$y = e^{-x/2}\left[c_1 cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 sin\left(\frac{x\sqrt{3}}{2}\right)\right] + e^{x/2}\left[c_3 cos\left(\frac{x\sqrt{3}}{2}\right) + c_4 sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

EXAMPLE 14: Solve $(D^6 - 1)y = 0$

SOLUTION: Here the given differential equation is $(D^6 - 1)y = 0$

The corresponding auxiliary equation is $D^6 - 1 = 0$

$$\Rightarrow (D^6 - 1)(D^4 + D^2 + 1) = 0$$

$$\Rightarrow (D-1)(D+1)(D^2 - D + 1)(D^2 + D + 1) = 0$$

Its roots are $1, \ -1, \ \frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$ and $-\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$

Therefore, the required solution is

$$y = c_1 e^x + c_2 e^{-x} + e^{x/2}\left[c_3 \cos\left(\frac{1}{2}x\sqrt{3}\right) + c_4 \sin\left(\frac{1}{2}x\sqrt{3}\right)\right] + e^{-x/2}\left[c_5 \cos\left(\frac{1}{2}x\sqrt{3}\right) + c_6 \sin\left(\frac{1}{2}x\sqrt{3}\right)\right]$$

EXAMPLE 15: Solve the differential equation:

$(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$, where $D = \frac{d}{dx}$,

SOLUTION: Auxiliary equation is

$(m^2+1)^3(m^2+m+1)^2 = 0.$

It implies $(m^2+1)^3 = 0$ gives $m = \pm i, \pm i, \pm i$

and $(m^2+m+1)^2 = 0$ gives $m = \frac{-1\pm\sqrt{3}i}{2}, \frac{-1\pm\sqrt{3}i}{2}$.

Hence, complimentary function (C.F.) =

$$e^{0x}[(c_1+c_2x+c_3x^2)cosx+(c_4+c_5x+c_6x^2)sinx]+e^{-x/2}\left[(c_7+c_8x)\cos\left(\frac{1}{2}x\sqrt{3}\right)+(c_9+c_{10}x)\sin\left(\frac{1}{2}x\sqrt{3}\right)\right].$$

P.I. = 0

Therefore the complete solution is

$y =$ C.F. + P.I.

$$= [(c_1+c_2x+c_3x^2)cosx+(c_4+c_5x+c_6x^2)sinx]+e^{-x/2}\left[(c_7+c_8x)\cos\left(\frac{1}{2}x\sqrt{3}\right)+(c_9+c_{10}x)\sin\left(\frac{1}{2}x\sqrt{3}\right)\right].$$

Where $c_1, c_{2,}c_3, c_{4,}c_{5,}c_6, c_{7,}c_{8,}c_8$ and c_{10} are arbitrary constants of integration.

EXAMPLE 16: Solve the differential equation:

$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC} = 0$, where $R^2C = 4L$ and R, C, L are constants.

SOLUTION: The given equation is

$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)i = 0$ where $D = \frac{d}{dt}$.

Auxiliary equation is

$$m^2+\frac{R}{L}m+\frac{1}{LC} = 0$$

This implies

$m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$

Since $R^2 = \frac{4L}{C}$.

$$m = -\frac{R}{2L}, -\frac{R}{2L}$$

Hence, C.F. = $(c_1 + c_2 t)e^{-\frac{Rt}{2L}}$, where c_1 and c_2 are arbitrary constants of integration.

5.8 SUMMARY

This unit is a combination of definition of Linear Differential equation with constant coefficients. In this unit we also described the examples of Linear Differential equation with constant coefficients and we also evaluate the complementary function. The rules for finding the complementary function is also defined in this unit.

5.9 GLOSSARY

- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **First Order:** The highest derivative involved in the equation is the first derivative.

- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.

CHECK YOUR PROGRESS

Solve the following differential equations:

1. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

3. $\frac{d^2y}{dx^2} + 7y = 0$

4. Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0\text{: given that } y(0) = 2 \text{ and } y\left(\frac{\pi}{2}\right) = -2.$$

5. Determine the general solution to the differential equation. The general solution is the sum of the complementary function and the particular integral. **True/False**

6. For the differential equation

$$(D^2 + 1)^3(D^2 + D + 1)^2y = 0, \text{where } D = \frac{d}{dx}.$$

The Auxiliary equation is $(m^2 + 1)^3(m^2 + m + 1)^2 = 1$. **True/False**

7. In solution of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ three constants are presents. **True/False**

8. The solution $y = e^x[c_1cosx + c_2sinx] + e^{-x}[c_3cosx + c_4sinx]$ is the solution of fifth order differential equation. **True/False**

5.10 REFERENCES

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5.11 SUGGESTED READING

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5.12 TERMINAL QUESTIONS:-

TQ1: $(D^3 - 13D + 12)y = 0$

TQ2: $(D^2 + 7D + 10)y = 0$

TQ3: $(D^3 - 4D^2 + 5D - 2)y = 0$

TQ4: $\frac{d^4y}{dx^4} + m^4y = 0$

5.13 ANSWERS

SELF CHECK ANSWERS

1. $y = c_1e^{-3x} + c_2e^{-2x}$

2. $y = c_1e^{-3x} + c_2xe^{-2x}$

3. $y = c_1e^{i\sqrt{7x}} + c_2e^{-i\sqrt{7x}}$ or $y = c_1cos(\sqrt{7x}) + c_2sin(\sqrt{7x})$

4. $y = 2(cosx - sinx)$

5. True

6. False

7. False

8. False

TERMINAL ANSWERS (TQ'S)

TQ1: $y = c_1 e^{x} + c_2 e^{2\sqrt{3}x} + c_2 e^{-2\sqrt{3}x}$

TQ2: $y = c_1 e^{-2x} + c_2 e^{-5x}$,
TQ3: $y = (c_1 + c_2 x) + c_3 e^{2x}$

TQ4:

$$y = e^{\frac{m}{\sqrt{2}}x}\left(c_1 cos\frac{m}{\sqrt{2}}x + c_2 sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3 cos\frac{m}{\sqrt{2}} + c_4 sin\frac{m}{\sqrt{2}}x\right)$$

UNIT 6: PARTICULAR INTEGRAL TYPE -I

6.1 *INTRODUCTION*

In this unit we are finding the particular integral of type e^{ax} or e^{ax+b} and $\sin(ax+b)\, or \cos(ax+b)$. The linear differential equation with constant coefficient of order n is

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=Q(x)$$

or, $[a_0D^n+a_1D^{n-1}+a_2D^{n-2}+\cdots+a_{n-1}D+a_n]y=Q(x)$ …(1)

where $a_0,\neq 0, a_0, a_1, a_2, \dots, a_n$ all are constant.

The general form of (1) can be written as

$[D^n+k_1D^{n-1}+\cdots+k_n]y=Q(x)$

Where $f(D)=D^n+k_1D^{n-1}+\cdots k_n$ is a polynomial in D.

$\frac{1}{f(D)}Q(x)$ is that function of x, not containing any arbitrary constant which when operated upon by $f(D)$gives $Q(x)$.

$i.e., f(D)\left[\frac{1}{f(D)}Q(x)\right]=Q(x)$.

Hence $\frac{1}{f(D)}Q(x)$ satisfies the equation $f(D)y=Q(x)$ and is therefore its particular integral. The particular integral to a relation between the variables such that satisfies the equation and that does not contain any new constant quantity. $f(D)$ and $\frac{1}{f(D)}$ are inverse operators. The solution of equation (1) is y = C.F+P.I

6.2 *OBJECTIVES*

After studying this unit you will be able to

i. Described the concept of Particular Integral.

ii. Evaluate the Particular integral of type e^{ax} or e^{ax+b}

iii. Defined the particular integral of $\sin(ax+b)$ or $\cos(ax+b)$

6.3 *CASE I. When* $Q = e^{ax}$ *or* e^{ax+b}

- $\boldsymbol{\frac{1}{D}Q(x) = \int Q(x)\,dx}$

<u>PROOF:</u> Let $\frac{1}{D}Q(x) = y$ …(2)

Operating both sides by D $D.\frac{1}{D}Q(x) = D \cdot y$

$$\Rightarrow Q(x) = D \cdot y$$

$$\Rightarrow Q(x) = \frac{dy}{dx}$$

Integrating both side with respect to x, we get

$y = \int Q(x)\,dx,$ Since equation (1) does not contain any arbitrary constant. So, no constant of integration be added.

Hence, $\frac{1}{D}Q(x) = \int Q(x)\,dx$

- $\boldsymbol{\frac{1}{(D-a)}Q(x) = e^{ax}\int Q(x)\cdot\ e^{-ax}dx}$

<u>PROOF:</u> Let $\frac{1}{(D-a)}Q(x) = y$ …(3)

Operating both sides by $(D-a)$

$$(D-a)\cdot\frac{1}{(D-a)}Q(x)=(D-a)\cdot y$$

$$\Rightarrow Q(x)=(D-a)\cdot y$$

$$\Rightarrow Q(x)=\frac{dy}{dx}-ay$$

Which is first order, first degree linear differential equation. Its integrating factor is e^{-ax}.

So, its solution is

$ye^{-ax}=\int Q(x)\ \cdot e^{-ax}dx,$ Since equation (1) does not contain any arbitrary constant. So, no constant of integration be added.

$$y=e^{ax}\int Q(x)\ \cdot e^{-ax}dx$$

Hence, $\frac{1}{(D-a)}Q(x)=e^{ax}\int Q(x)\cdot e^{-ax}dx$

Consider the non-homogenous linear differential equation of order n.

$$\frac{d^ny}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+k_ny=Q(x) \qquad \ldots(4)$$

In terms of operator D equation (1) can be rewritten as

$$(D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n)y=Q(x)$$

Therefore, particular integral is

$$\frac{1}{D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n}Q(x)$$

or, $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)}Q(x)$

Where $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$be n roots of $D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n$

Therefore $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)}Q(x)=$ $\frac{1}{(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)}\left[\frac{1}{(D-\alpha_1)}.Q(x)\right]$

$$=\frac{1}{(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)}\left\{\frac{e^{\alpha_1x}}{Q(x)e^{-\alpha_1x}}\ dx\right\}$$

Repeat this process for each factor in same manner, we get the required particular integral.

Some Particular Cases:

CASE I:

Consider $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = Q(x)$

When R.H.S. of equation (1) is of the form e^{ax}

$i.e., Q(x) = e^{ax}$ provided $f(a) \neq 0$

Since $De^{ax} = ae^{ax}$

$$D^2 e^{ax} = a^2 e^{ax}$$

In general, $D^n e^{ax} = a^n e^{ax}$

$$\Longrightarrow (D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)e^{ax}$$

$$= (a^n + k_1 a^{n-1} + k_2 a^{n-2} + \cdots + k_n)e^{ax}$$

$$i.e., [f(D)e^{ax}] = [f(a)e^{ax}]$$

Now, operating on both sides by $\frac{1}{f(D)}$,we get

$$\frac{1}{f(D)}[f(D)]e^{ax} = \frac{1}{f(D)}[f(a)]e^{ax}$$

$$\Longrightarrow e^{ax} = f(a).\frac{1}{f(D)}e^{ax}$$

$\Longrightarrow \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ Provided $f(a) \neq 0$ …(5)

Note that $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ provided $f(a) \neq 0$.

CASE OF FAILURE:

In the above case if a is simple root of auxiliary equation.

$i.e., a$ is root of the auxiliary equation $f(D) = 0$

$\Longrightarrow (D - a)$is factor of $f(D)$

$$\Longrightarrow f(D) = (D - a)\emptyset(D)$$

Where$\emptyset(a) \neq 0$

Now, P.I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)} \frac{1}{\emptyset(D)} e^{ax}$

$$= \frac{1}{(D-a)} \frac{1}{\emptyset(a)} e^{ax}$$

{by (1)}

$$= \frac{1}{\emptyset(a)} \frac{1}{(D-a)} e^{ax}$$

$$= \frac{1}{\emptyset(a)} e^{ax} \int e^{ax} e^{-ax} dx$$

$$= \frac{1}{\emptyset(a)} e^{ax} \int dx$$

$$= x \frac{1}{\emptyset(a)} e^{ax}$$

Therefore $\frac{1}{f(D)} e^{ax} = x \frac{1}{\emptyset(a)} e^{ax}$

Or,$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ provided $f'(a) \neq 0$

Similarly, if a is root of auxiliary equation of order two, then

$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$ provided $f''(a) \neq 0$

and so on.

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{\emptyset(a)} e^{ax}$$

Where$\emptyset(a) \neq 0$

6.4 *SOLVED EXAMPLES*

Example1:

Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Sol. Given equation can be written as $(D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. = $e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^{2x} = \frac{1}{f(2)} e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2) + 10} e^{2x} = \frac{1}{10} e^{2x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10} e^{2x}$

Example2:

Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Sol. Given equation can be written as $(D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1 e^{-2x} + c_2 e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} e^x$, putting $D = 1, f(1) = 0$

$\therefore$P.I $= x \frac{1}{f'(D)} e^x$ $\quad\because$ P.I. $= x \frac{1}{f'(a)} e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. $= x \frac{1}{2D+1} e^x = \frac{1}{f'(1)} e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. $= \frac{xe^x}{3}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^x + \frac{xe^x}{3}$

Example3:

Solve $D^2(D+1)^2(D^2+D+1)^2 y = e^x$

Sol: Here the given differential equation is

$$D^2(D+1)^2(D^2+D+1)^2 y = e^x$$

Its auxiliary equation is

$$m^2(m+1)^2(m^2+m+1)^2 y = 0$$

The roots are $0,\ 0,\ -1,\ -1,\ \frac{1}{2}[-1 \pm i\sqrt{3}],\ \frac{1}{2}[-1 \pm i\sqrt{3}]$

$i.e., 0,\ -1,\ \frac{1}{2}[-1 \pm i\sqrt{3}]$ twice each.

Therefore,

$$C.F. = (c_1 x + c_2)e^{0x} + (c_3 x + c_4)e^{-x}$$
$$+ e^{-x/2}\left[(c_5 x + c_6)cos\left(\frac{1}{2}\sqrt{3}x\right)\right.$$
$$\left. + (c_7 x + c_8)sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

And, $P.I. = \frac{1}{D^2(D+1)^2(D^2+D+1)^2} e^x$

$$P.I. = \frac{1}{1^2(1+1)^2(1^2+1+1)^2} e^x = \frac{1}{36} e^x$$

Therefore, required solution is

$$y = \text{C.F.} + \text{P.I.}$$

$y = (c_1x + c_2)e^{0x} + (c_3x + c_4)e^{-x} + e^{-x/2}\left[(c_5x + c_6)cos\left(\frac{1}{2}\sqrt{3x}\right) + (c_7x + c_8)sin\left(\frac{1}{2}\sqrt{3x}\right)\right] + \frac{1}{36}e^x.$

Example4: Solve

$$(D^3 + 1)y = (e^x + 1)^2$$

Sol: The auxiliary equation is

$$m^3 + 1 = 0 \quad or \ (m+1)(m^2 - m + 1) = 0$$

$$m = -1, \frac{1}{2} + \sqrt{\frac{3}{2}i}$$

$$CF = c_1 e^{-x} + e^{x2}\left[c_2 \cos\sqrt{\frac{3}{2}x + c_3 \sin\sqrt{\frac{3}{2}x}}\right]$$

$$\text{P.I} = \frac{1}{D^3+1}(e^x+1)^2 = \frac{1}{D^3+1}(e^{2x} + 2e^x + 1)$$

$$= \frac{1}{D^3+1}e^{2x} + 2.\frac{1}{D^3+1}e^x + \frac{1}{D^3+1}e^{ox}$$

$$= \frac{1}{2^3+1}e^{2x} + 2.\frac{1}{1^3+1}e^x + \frac{1}{0^3+1}e^{ox} = \frac{1}{9}e^{2x} + e^x + 1$$

Hence, the complete solution is

$$y = c\ e^{-x} + e^{x2}\left[c_2 \cos\sqrt{\frac{3}{2}}x + c_3 \sin\sqrt{\frac{3}{2}}\right] + \frac{1}{9}e^{2x} + e^x + 1$$

Example5:

Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Sol.

$$(D^2 - 8D + 15)y = 0$$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

C.F. = $c_1e^{3x} + c_2e^{5x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{3x} + c_2e^{5x}$

Example 6:

Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Sol.

$(D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ …….①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. = $c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

6.5 ***CASE I***. $When\ Q = Sin(ax + b)\ 0r\ Cos(ax + b)$

Note:

When $\boldsymbol{F(x) = Sin\ (ax + b) or\ Cos\ (ax + b)}$

If $F(x) = Sin\ (ax + b)\ or\ Cos\ (ax + b)$, put $D^2 = -a^2$,

$D^3 = D^2 D = -a^2 D, D^4 = (D^2)^2 = a^4, \ldots\ldots$

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e. if $f(-a^2) = 0$

P.I. $= x \frac{1}{f'(-a^2)} Sin\ (ax + b)\ or\ Cos\ (ax + b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.$= x^2 \frac{1}{f''(-a^2)} Sin\ (ax + b)\ or\ Cos\ (ax + b), f''(-a^2) \neq 0$

Example 7:

Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Sol.

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin x = \frac{1}{D^2+D-2} \sin x$

putting $D^2 = -1^2 = -1$

P.I $= \frac{1}{D-3} \sin x = \frac{D+3}{D^2-9} \sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{-1}{10}(D\, sin\, x + 3 \sin x)$

$= \frac{-1}{10}(\cos x + 3 \sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{-2x} + c_2e^x - \frac{1}{10}(\cos x + 3 \sin x)$

Example 8:

Solve the differential equation: $(D^2 + 2D + 1)y = cos^2x$

Sol.

Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. = $e^{-x}(c_1 + c_2x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} cos^2x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1}e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

$$\text{P.I} = \frac{1}{2} + \frac{1}{2}\frac{1}{2D-3}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{2}\frac{2D+3}{4D^2-3^2}\cos 2x, \text{ Rationalizing the denominator}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}, \text{ Putting } D^2 = -4$$

$$\therefore \text{P. I.} = \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$$

Now $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$$

Example 9:

Solve the differential equation: $(D^2 + 9)y = \sin 2x \cos x$

Sol.

Auxiliary equation is: $m^2 + 9 = 0$

$$\Rightarrow m = \pm 3i$$

$$\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{f(D)}\sin 2x \cos x = \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$$

$$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9) = 0$ for the 1st term

$$\therefore \text{P.I.} = \frac{1}{2}x\frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$$

$$\because \text{P.I.} = x\frac{1}{f'(-a^2)}\, Sin\,(ax+b),\ f'(-a^2) \neq 0$$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

SELF CHECK QUESTIONS

1. $\frac{1}{D^2+a^2} sinax = -\frac{x}{2a} cosax$. True/False

2. $\frac{1}{D^2+a^2} cosax = \frac{x}{2a} sinax$. True/False

3. P.I. of the Differential equation $D^2y = 0$ is?

4. P.I. of the Differential equation $D^2y = e^x$ is?

5. P.I. of the Differential equation $D^3y = e^{2x}$ is?

6.6 SUMMARY

This unit is a composition of different methods of fining the Particular Integral. In this unit we are evaluating the particular integral of type e^{ax} or e^{ax+b} and type of $\sin(ax + b)$ *or* $\cos(ax + b)$.

6.7 GLOSSARY

- **Differential Equation** an Equation involving derivatives of differentials of one or more dependent variables with respect to one or more independent variables is called **Differential Equation.**
- **Function:** A mathematical relation that assigns a unique output value to each input value.

- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.
- **Auxiliary Equation:** The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation or A.E.
- **Complementary function:** Complementary function is actually the solution of the given differential equation $f(D)y = Q$ when its right hand side member it means the Q is replaced by zero. To find complementary function we first find auxiliary equation.

6.8 REFERENCES

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6.9 SUGGESTED READING

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6.10 TERMINAL QUESTIONS

Q1. Solve the following Differential equation

$$(4D^2 + 12D + 9)y = 144e^{-3x}$$

Q2. Solve the following Differential equation

$$(D^2 + 4D + 4)y = e^{2x} - e^{-2x}$$

Q3. Solve the following Differential equation

$(D^4 - 1)y = e^x cosx$

Q4. Solve the following Differential equation

$D^2 y = e^x cosx$

Q5. Solve the following Differential equation

$D^2 y = 0$

6.11 *ANSWERS*

SELF CHECK ANSWERS

1. True

2. True

3. 0

4. e^x

5. $\frac{1}{8} e^{2x}$

TERMINAL ANSWERS (TQ'S)

1. y = $(c_1 + c_2 x)e^{-3x/2} + 16e^{-3x}$

2. y = $(c_1 + c_2 x)e^{-2x} + \frac{1}{16} e^{2x} - \frac{x^2}{2} e^{-2x}$

3. y = $c_1 e^x + c_2 e^{-x}$ + $c_3 cosx$ + $c_4 sinx$ – (1/5) $e^x cosx$

4. y = $c_1 + c_2 x$ + (1/2) $e^x sinx$

5. y = $c_1 + c_2 x$

Unit 7: PARTICULAR INTEGRAL -II

CONTENTS:

7.1 *INTRODUCTION*

There are different methods of finding the particular integral. The method to be applied depends on the type of function involved. There is a general method which can be applied to any kind of question but it is comparatively lengthy. The short methods are also there which are specific according to the type of function involved.

General Method

If both m_1 and m_2 are constants, the expressions $(D - m_1)(D - m_2)y$ and $(D - m_2)(D - m_1)y$ are equivalent i.e. the expression is independent of the order of operational factors.

$$\frac{1}{D-\alpha}X = e^{ax}\int Xe^{-ax}dx$$

We will explain the method with the help of following

Example: Solve $d^2y/dx^2 - 5\,dy/dx + 6y = e^{3x}$.

Solution:

The equation can be written as

$(D^2 - 5D + 6)y = e^{3x}$
$(D - 3)(D - 2)y = e^{3x}$
C.F. $= c_1 e^{3x} + c_2 e^{2x}$
And P.I. $= 1/(D-3).\ 1/(D-2)\ e^{3x}$
$= 1/(D-3)\ e^{2x}\int e^{3x} e^{-2x}dx$
$= 1/(D-3)\ e^{2x} e^{x}$
$= e^{3x}\int e^{3x} e^{-3x}\,dx = x.e^{3x}$
$y = c_1e^{3x} + c_2e^{2x} + xe^{3x}$

P.I. can be found by resolving

$1/f(D) = 1/(D-3).\ 1/(D-2)$

Now using partial fractions,

$1/f(D) = 1/(D-3).\ 1/(D-2)$

$= 1/(D-3) - 1/(D-2)$

Hence, the required P.I. is $[1(D-3) - 1/(D-2)]\ e^{3x}$
$= 1/(D-3)\ e^{3x} - 1/(D-2)\ e^{3x}$
$= e^{3x}\int e^{3x} e^{-3x}\,dx - e^{2x}\int e^{3x} e^{-2x}\,dx$
$= xe^{3x} - e^{3x}$

Second term can be neglected as it is included in the first term of the C.F.

Short Method of Finding P.I.

In certain cases, the P.I. can be obtained by methods shorter than the general method.

To find P.I. when X = e^{ax} in f(D) y = X, where a is constant

y = 1/f(D)

1/f(D) e^{ax} = 1/f(a) e^{ax}, if f(a) ≠ 0.
1/f(D) e^{ax} = $x^r/f^r(a)$ e^{ax}, if f(a) = 0, where f(D) = $(D-a)^r$f(D).

Note: The complete solution of the linear differential equation $f(D)y = F(x)$ is given by $y = C.F + P.I.$ where $C.F.$ denotes complimentary function and $P.I.$ is particular integral. $F(x) = 0$, the solution of equation $f(D)y = 0$ is given by $y = C.F.$

7.2 *OBJECTIVES*

At the end of this topic learner will be able to understand:

(i) Method for Finding Particular Function when X = $e^{ax}V(x)$
(ii) Method for Finding Particular Function when X = $x^m cosax$ or $x^m sinax$

7.3 *METHOD OF FINDING P.I. WHEN X = $e^{ax}V(x)$*

Theorem: $\frac{1}{f(D)} = e^{ax}V = \frac{1}{f(D+a)}V$ **, V being a function of x.**

Proof: By successive differentiation, we have

$$D(e^{ax}V) = e^{ax}DV + a\,e^{ax}DV = e^{ax}(D+a)V,$$

$$D^2(e^{ax}V) = e^{ax}D^2V + a\,e^{ax}DV + a\,e^{ax}DV + a^2e^{ax}V$$

$$= e^{ax}(D^2 + 2aD + a^2)V = e^{ax}(D+a)^2V,$$

Similarly, $D^3(e^{ax}V) = e^{ax}(D+a)^3V$,

………………………………………………………………

$$D^n(e^{ax}V) = e^{ax}(D+a)^nV$$

Therefore f(D) $e^{ax}V = e^{ax}f(D+a)V$. …….. (1)

The above result (1) is true for any function of x. Taking {1/f(D + a)} V in place of V in (1), we have

$$F(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\} = e^{ax}f(D+a)\left\{\frac{1}{f(D+a)}V\right\}$$

Or $e^{ax}V = f(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\}$ …….. (2)

Operating by 1/f(D) on both sides of (2), we have

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V \quad \text{…….. (3)}$$

Solved Examples

Example 1. Solve (D^2 - 2D + 1) y = x^2e^{3x} **.**

Sol. The auxiliary equation of the given equation is

D^2 - 2D + 1 = 0 then D = 1, 1

∴ C.F. = $(c_1 + c_2x)e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}x^2e^{3x} = \frac{1}{(D-1)^2}x^2e^{3x}$$

$$= e^{3x}\frac{1}{(D+3-1)^2}x^2 = e^{3x}\frac{1}{(D+2)^2}x^2 = e^{3x}\frac{1}{4(1+D/2)^2}x^2$$

$$= \frac{1}{4}e^{3x}\left(1+\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}e^{3x}\left[1-\frac{D}{2}+\frac{(-2)(-3)}{2!}\frac{D^2}{4}+\cdots\right]x^2$$

$$= \frac{1}{4}e^{3x}\left(1-\frac{D}{2}+\frac{3}{4}D^2+\cdots\right)x^2$$

$$= \frac{1}{4}e^{3x}\left\{x^2-\frac{1}{2}(2x)+\frac{3}{4}(2)\right\} = \frac{1}{8}e^{3x}(2x^2 - 4x + 3).$$

Therefore required solution is $y = (c_1 + c_2x)e^x + \frac{1}{8}e^{3x}(2x^2 - 4x + 3)$.

Example 2. Solve (D^2 - 2D + 1) y = x^2e^x.

Sol. The auxiliary equation of the given equation is

D^2 - 2D + 1 = 0 then D = 1, 1

∴ C.F. = $(c_1 + c_2x)e^x$, c_1, c_2 being arbitrary constants.

P.I. = $\frac{1}{D^2 - 2D + 1}x^2e^x = \frac{1}{(D-1)^2}x^2e^x$

$= e^x \frac{1}{(D+1-1)^2}x^2 = e^x \frac{1}{D^2}x^2 = e^x \frac{1}{D}\int x^2dx$

$= e^x \frac{1}{D}\frac{x^3}{3} = e^x \int \frac{x^3}{3}dx = \frac{e^x}{3}.\frac{x^4}{4} = \frac{1}{12}x^4e^x$

Hence the required solution is

y = $(c_1 + c_2x)e^x + \frac{1}{12}x^4e^x$.

Example 3. Solve $(D-a)^2$ y = $e^{ax}f'(x)$.

Sol. The auxiliary equation of the given equation is

$(D-a)^2 = 0$ so that D = a, a

Therefore C.F. = $(c_1 + c_2x)e^{ax}$, c_1, c_2 being arbitrary constants.

P.I. = $\frac{1}{(D-a)^2}e^{ax}f'(x) = e^{ax}\frac{1}{(D+a-a)^2}f'(x)$

$= e^{ax}\frac{1}{D}.\frac{1}{D}f'(x) = e^{ax}\frac{1}{D}\ f(x) = e^{ax}\int f(x)dx$

Hence the required solution is

y = $(c_1 + c_2x)e^{ax} + e^{ax}\int f(x)dx$.

Example 4. Solve $(D^3 - 3D - 2)$ y = $540x^3e^{-x}$.

Sol. The auxiliary equation of the given equation is

$D^3 - 3D - 2 = 0$ so that D = 2, -1, -1.

Therefore

C.F. = $c_1e^{2x} + (c_2 + c_3x)e^{-x}$, c_1, c_2 , c_3 being arbitrary constants.

P.I. = $\frac{1}{D^3-3D-2}540x^3e^{-x} = 540e^{-x}\frac{1}{(D-1)^3-3(D-1)-2}x^3$

$= 540e^{-x}\frac{1}{D^3-3D^2}x^3 = 540e^{-x}\frac{1}{-3D^2(1-\frac{D}{3})}x^3$

$= -180e^{-x}\frac{1}{D^2}\left(1+\frac{D}{3}+\frac{D^2}{9}+\frac{D^3}{27}+\cdots\right)x^3$

$= -180e^{-x}\frac{1}{D^2}\left(x^3+x^2+\frac{2}{3}x+\frac{2}{9}\right)$

$= -180e^{-x}\frac{1}{D}\left(\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{3}+\frac{2x}{9}\right)$

$= -180e^{-x}\left(\frac{x^5}{20}+\frac{x^4}{12}+\frac{x^3}{9}+\frac{x^2}{9}\right)$

$= -e^{-x}(9x^5+15x^4+20x^3+20x^2)$

Hence the required solution is

y = c_1e^{2x} +$(c_2 + c_3x)e^{-x} - e^{-x}(9x^5+15x^4+20x^3+20x^2)$.

Example 5. Solve (D^2+3D+2) y = $e^{2x}sinx$.

Sol. The auxiliary equation of the given equation is

$D^2+3D+2=0$ so that D = -2, -1.

Therefore

C.F. = $c_1e^{-2x}+c_2e^{-x}$, c_1, c_2 being arbitrary constants.

P.I. = $\frac{1}{D^2+3D+2}e^{2x}sinx$ $= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}sinx$

$= e^{2x}\frac{1}{D^2+7D+12}sinx$ $= e^{2x}\frac{1}{-1^2+7D+12}sinx$

$= e^{2x}\frac{1}{11+7D}sinx$ $= e^{2x}(11-7D)\frac{1}{(11+7D)(11-7D)}sinx$

$= e^{2x}(11-7D)\frac{1}{121-49D^2}sinx$

$= e^{2x}(11-7D)\frac{1}{121-49(-1)^2}sinx$

$= \frac{e^{2x}}{170}(11-7D)sinx$

$= \frac{e^{2x}}{170}(11sinx-7cosx)$

Therefore, required solution is

y = $c_1e^{-2x} + c_2e^{-x} + \frac{e^{2x}}{170}(11sinx - 7cosx)$.

Example 6. Solve the differential equation$(D^2+2)y = x^2e^{3x}$.
Sol. Let the given differential equation
$(D^2+2)y = x^2e^{3x}$.
Now the auxiliary equation is

$$m^2 + 2 = 0$$

$\Rightarrow \quad m^2 = -2$

$\Rightarrow \quad m^2 = \pm\sqrt{2}i$

$$C.F. = (C_1cos\sqrt{2}x + C_2sin\sqrt{2}x)$$

$$P.I. = \frac{1}{f(D)}F(x) = \frac{1}{(D^2+2)}x^2e^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$= e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$= \frac{e^{3x}}{11}\frac{1}{\left(\frac{D^2}{11}+\frac{6D}{11}+1\right)}x^2$$

$$= \frac{e^{3x}}{11}\left(1+\left\{\frac{D^2}{11}+\frac{6D}{11}\right\}\right)^{-1}x^2$$

$$= \frac{e^{3x}}{11}\left[1-\left(\frac{D^2}{11}+\frac{6D}{11}\right)+\left(\frac{D^2}{11}+\frac{6D}{11}\right)^2+\cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+\cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left(x^2-\frac{12D}{11}+\frac{50}{121}\right)$$

$\therefore \qquad P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12D}{11}+\frac{50}{121}\right)$

Complete solution is: $\qquad y = C.F. + P.I.$

$\Rightarrow \qquad y = \left(C_1cos\sqrt{2}x + C_2sin\sqrt{2}x\right) + \frac{e^{3x}}{11}\left(x^2-\frac{12D}{11}+\frac{50}{121}\right)$.

Example 7. Solve the differential equation$(D^3+1)y = e^{2x}sinx$.
Sol. Let the given differential equation
$(D^3+1)y = x^2e^{3x}$.
Now the auxiliary equation is

$$m^3 + 1 = 0$$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m^2 = -1, \frac{1 \pm \sqrt{3}i}{2}$

$$C.F. = C_1 e^{-x} + e^{\frac{x}{2}} \left(C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) x + C_3 \cos\left(\frac{\sqrt{3}}{2}x\right) \right)$$

$$P.I. = \frac{1}{f(D)} F(x) = \frac{1}{(D^3 + 1)} e^{2x} sinx$$

$$= e^{2x} \frac{1}{(D+2)^3 + 1} sinx$$

$$= e^{2x} \frac{1}{(D)^3 + 6D^2 + 12D + 9} sinx$$

$$= e^{2x} \frac{1}{-D - 6 + 12D + 9} sinx \qquad \text{Substituting } D^2 = -1$$

$$= e^{2x} \frac{1}{11D + 3} sinx$$

$$= \frac{e^{2x}(11D - 3)}{121D^2 - 9} sinx, \qquad \text{Rationalizing the denominator}$$

$$= \frac{e^{2x}}{130} (11D - 3) sinx, \qquad \text{Substituting } D^2 = -1$$

$\therefore \qquad P.I. = \frac{e^{3x}}{130} (11cosx - 3sinx)$

Complete solution is : $\quad y = C.F. + P.I.$

$\Rightarrow \quad y = C_1 e^{-x} + e^{\frac{x}{2}} \left(C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) x + C_3 \cos\left(\frac{\sqrt{3}}{2}x\right) \right)$

$$- \frac{e^{3x}}{130} (11cosx - 3sinx).$$

NOTE: When $F(x) = e^{ax} g(x)$, where $g(x)$ is any function of x.

Use the rule: $\frac{1}{f(D)}\left(xg(x)\right) = \frac{1}{f(D)} g(x) + \left(\frac{d}{dD}\frac{1}{f(D)}\right) g(x)$

Example 8. Solve the differential equation $(D^2 + 9)y = xcosx.$

Sol. Let the given differential equation

$(D^2 + 9)y = xcosx.$

Now the auxiliary equation is

$$m^2 + 9 = 0$$

$\Rightarrow \quad m^2 = -9$

$\Rightarrow \quad m^2 = \pm 3i$

$$C.F. = (C_1 cos3x + C_2 sin3x)$$

$$P.I. = \frac{1}{f(D)} F(x) = \frac{1}{(D^2 + 9)} xcosx$$

$$= x\frac{1}{(D^2+9)}cosx += \frac{-2D}{(-1^2+9)^2}cosx \quad \text{Substituting } D^2 = -1$$

$$= \frac{xcosx}{8} - \frac{2Dcosx}{64}$$

Example 9. Solve the differential equation$(D^2 + 3D + 2)y = e^{e^x}$.

Sol. Let the given differential equation

$$(D^2 + 3D + 2)y = e^{e^x}$$

Now the auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0$$

$$\Rightarrow \quad m^2 = -1, -2$$

$$C.F. = (C_1e^{-x} + C_2e^{-2x})$$

$$P.I. = \frac{1}{f(D)}F(x) = \frac{1}{(D^2+3D+2)}e^{e^x}$$

$$= \frac{1}{(D+1)(D+2)}e^{e^x}$$

$$= \left[\frac{1}{(D+1)} - \frac{1}{(D+2)}\right]e^{e^x}$$

$$= e^{-x}\int e^x\, e^{e^x}dx - e^{-2x}\int e^{2x}\, e^{e^x}dx$$

$$= e^{-x}\int D\, e^{e^x}dx - e^{-2x}\int e^xD\, e^{e^x}dx$$

$= e^{-x}e^{e^x} - e^{-2x}\left[e^xe^{e^x} - \int e^xe^{e^x}dx\right]$, Integrating second term by parts

$$= e^{-x}e^{e^x} - e^{-2x}\left[e^xe^{e^x} - \int De^{e^x}dx\right]$$

$\therefore \quad P.I. = e^{-2x}e^{e^x}$

Complete solution is : $\quad y = C.F. + P.I.$

$\Rightarrow \quad y = C_1e^{-x} + C_2e^{-2x} + e^{-2x}e^{e^x}$.

7.4 METHOD OF FINDING P.I. WHEN $X = x^m \sin ax$ or $x^m \cos ax$

Working rule for finding P.I.

(i) P.I. = $\frac{1}{f(D)}$ $x^m \cos ax$ = Real part of $\frac{1}{f(D)}$ x^m $(\cos ax + i\sin ax)$

= R.P. of $\frac{1}{f(D)} x^m e^{aix}$, by Euler's Theorem,

where R.P. stands for real part

(ii) P.I. = $\frac{1}{f(D)}$ $x^m \sin ax$ = Imaginary part of $\frac{1}{f(D)}$ x^m $(\cos ax + i\sin ax)$

= I.P. of $\frac{1}{f(D)} x^m e^{aix}$, by Euler's Theorem,

where R.P. stands for Imaginary part.

Example 1. Solve $(D^2 + 9)\, y = x \sin x$.

Sol. The auxiliary equation of the given equation is

$D^2 + 9 = 0$ then $D = \pm 3i$

∴ C.F. = $c_1 \cos 3x + c_2 \sin 3x$, c_1, c_2 being arbitrary constants.

P.I. = $\frac{1}{D^2 + 9}\, x \sin x$

= I.P. of $\frac{1}{D^2 + 9}\, x e^{ix}$,where I.P. stands for imaginary part

= I.P. of $e^{ix} \frac{1}{(D+i)^2 + 9}\, x$

= I.P. of $e^{ix} \frac{1}{D^2 + 2iD + 8}\, x$

= I.P. of $e^{ix} \frac{1}{8[1 + \left(\frac{1}{8}\right)(D^3 + 2iD\,)]}\, x$

= I.P. of $\frac{e^{ix}}{8} \left[1 + \left(\frac{iD}{4} + \frac{D^2}{8}\right)\right]^{-1} x$

= I.P. of $\frac{e^{ix}}{8} \left[1 - \frac{iD}{4} + \cdots\right] x$

= I.P. of $\frac{1}{8}$ (cosx + i sinx)(x – i/4)

$= \frac{1}{8}\{x \sin x - \left(\frac{1}{4}\right) \cos x\}$.

Required solution is y = $c_1\cos 3x + c_2\sin 3x + \frac{1}{8}\{x\, sinx - \left(\frac{1}{4}\right) cosx\}$.

Example 2. Solve $(D^2 + 2D + 1)\, y = x \cos x$.

Sol. The auxiliary equation of the given equation is

$D^2 + 2D + 1 = 0$ then D = - 1, - 1

$\therefore$ C.F. = $(c_1 + c_2 x)e^{-x}$, c_1, c_2 being arbitrary constants.

P.I. = $\frac{1}{(D+1)^2}\, x \cos x$

= R.P. of $\frac{1}{(D+1)^2}\, xe^{ix}$, where R.P. stands for real part.

= R.P. of $e^{ix}\, \frac{1}{[(D+i)+1)]^2}\, x$, where R.P. stands for real part.

= R.P. of $\frac{e^{ix}}{(1+i)^2}\left(1 + \frac{D}{1+i}\right)^{-2}$

= R.P. of $\frac{e^{ix}}{2i}\left(1 - \frac{2D}{1+i} + \cdots\right) x$

= R.P. of $\frac{e^{ix}}{2i}\left(x - \frac{2}{1+i}\right)$

= R.P. of $i\,\frac{e^{ix}}{2i^2}\left[x - \frac{2(1-i)}{(1-i)(1+i)}\right]$

= R.P. of $\frac{e^{ix}}{(-2)}\left[x - \frac{2(1-i)}{1+1}\right]$

= R.P. of $\left(-\frac{i}{2}\right)(cosx + isinx)\{(x-1) + i\}$

= R.P. of $\left(-\frac{1}{2}\right)(icosx - sinx)\{(x-1) + i\}$ ……. (1)

Therefore P.I. = $\left(-\frac{1}{2}\right)[-sinx.(x-1) - cosx]$

= $\left(\frac{1}{2}\right)[(x-1)sinx + cosx]$

Therefore, required solution is

y = $(c_1 + c_2 x)e^{-x} + \frac{1}{2}[(x-1)sinx + cosx]$.

7.5 *METHOD OF FINDING P.I. WHEN X IS THE SUM OF TWO OR MORE FUNCTIONS*

We now consider examples in which X is sum of two or more special functions of x considered separately. in such case we obtain P.I. corresponding each function separately and then add these to get the required P.I. Sof a differential equation.

Example 1. Solve (D^2 - 4D + 4) y = $x^2 + e^x + sin2x$.

Sol. The auxiliary equation of the given equation is

D^2 - 4D + 4 = 0 then D = 2, 2

∴ C.F. = $(c_1 + c_2x)e^{2x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to x^2

$$=\frac{1}{D^2-4D+4}x^2 \quad =\frac{1}{(2-D)^2}x^2$$

$$=\frac{1}{4(1-D/2)^2}x^2 \quad =\frac{1}{4}\left[1-\frac{D}{2}\right]^{-2}x^2$$

$$=\frac{1}{4}\left[1+D+\frac{(-2)(-3)}{1.2}\frac{D^2}{4}+\cdots\right]x^2$$

$$=\frac{1}{4}\left(1+D+\frac{3}{4}D^2+\cdots\right)x^2$$

$$=\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

P.I. corresponding to e^x

$$=\frac{1}{D^2-4D+4}e^x \quad =\frac{1}{1-4+4}e^x=e^x$$

P.I. corresponding to sin 2x

$$=\frac{1}{D^2-4D+4}\sin 2x \quad =\frac{1}{-2^2-4D+4}\sin 2x$$

$$=-\frac{1}{4}.\frac{1}{D}\sin 2x=\frac{1}{8}\cos 2x$$

Therefore, required solution is

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) + \frac{1}{8}cos2x.$$

Example 2. Solve $(D^2$ -1) y = $xe^x + cos^2x$.

Sol. The auxiliary equation of the given equation is

D^2 - 1= 0 then D = 1, -1

∴ C.F. = $c_1e^x + c_2e^{-x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to xe^x

$$= \frac{1}{D^2-1}xe^x = e^x\frac{1}{(D+1)^2-1}x$$

$$= e^x\frac{1}{D^2+2D}x = \frac{e^x}{2}\frac{1}{D\left(1+\frac{D}{2}\right)}x$$

$$= \frac{e^x}{2}\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}x$$

$$= \frac{e^x}{2}\frac{1}{D}\left(1-\frac{D}{2}+\cdots\right)x$$

$$= \frac{e^x}{2}\frac{1}{D}\left(x-\frac{1}{2}\right)$$

$$= \frac{e^x}{2}\left(\frac{x^2}{2}-\frac{x}{2}\right) = \frac{1}{4}e^x(x^2-x)$$

P.I. corresponding to $\cos^2$x

P.I. corresponding to $\frac{1}{2}$

$$= \frac{1}{2}\frac{1}{D^2-1}e^{0.x}$$

$$= \frac{1}{2}\frac{1}{0^2-1}e^{0.x} = -\frac{1}{2}.$$

P.I. corresponding to $\frac{1}{2}\cos 2x$

$$= \frac{1}{2}\frac{1}{D^2-1}\cos 2x$$

$$= \frac{1}{2}\frac{1}{-2^2-1}\cos 2x = -\frac{1}{10}\cos 2x.$$

Therefore, the required solution

$$y = c_1e^x + c_2e^{-x} + \frac{1}{4}e^x(x^2-x) - \frac{1}{2} - \frac{1}{10}\cos 2x.$$

Example 3. Solve $(D^2 + a^2)$ y = sin ax + xe^{2x} **.**

Sol. The auxiliary equation of the given equation is

$$D^2 + a^2 = 0 \text{ then } D = \pm ia$$

∴ C.F. = $c_1 cosax + c_2 sinax$, c_1, c_2 being arbitrary constants.

P.I. corresponding to sin ax

$$= \frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax$$

P.I. corresponding to xe^{2x}

$$= \frac{1}{D^2 + a^2} xe^{2x} = e^{2x} \frac{1}{(D+2a)^2 + a^2} x$$

$$= e^{2x} \frac{1}{D^2 + 4Da + 4 + a^2} x$$

$$= \frac{e^{2x}}{4 + a^2} \left[1 + \frac{D^2 + 4D}{4 + a^2}\right]^{-1} x$$

$$= \frac{e^{2x}}{4 + a^2} \left[1 - \frac{D^2 + 4D}{4 + a^2} + \cdots\right] x$$

$$= \frac{e^{2x}}{4 + a^2} \left(x - \frac{4}{4 + a^2}\right)$$

$$= \frac{e^{2x}}{(4 + a^2)^2} \{x(4 + a^2) - 4\}$$

Therefore, solution is

$$y = c_1 cosax + c_2 sinax - \frac{x}{2a} \cos ax + \frac{e^{2x}}{(4 + a^2)^2} \{x(4 + a^2) - 4\}.$$

7.6 *SUMMARY*

1. $\frac{1}{f(D)} = e^{ax} V = \frac{1}{f(D+a)} V$, V being a function of x.

2. $\frac{1}{f(D)} \left(xg(x)\right) = \frac{1}{f(D)} g(x) + \left(\frac{d}{dD} \frac{1}{f(D)}\right) g(x).$

3. P.I. = $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}\cos \mathrm{ax}$ = Real part of $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}\,(\cos \mathrm{ax} + \mathrm{i}\sin \mathrm{ax})$

= R.P. of $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}e^{aix}$, by Euler's Theorem,

where R.P. stands for real part

4. P.I. = $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}\sin \mathrm{ax}$ = Imaginary part of $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}\,(\cos \mathrm{ax} + \mathrm{i}\sin \mathrm{ax})$

= I.P. of $\frac{1}{f(D)}\mathrm{x}^{\mathrm{m}}e^{aix}$, by Euler's Theorem,

where R.P. stands for Imaginary part.

7.7 *GLOSSARY*

- **Particular Integral:** A specific solution to a non-homogeneous linear differential equation, obtained by guessing a solution that matches the form of the non-homogeneous term.
- **Non-homogeneous Differential Equation:** A differential equation that contains a function on the right-hand side (RHS), termed the non-homogeneous or forcing term.
- **Homogeneous Differential Equation:** A differential equation where the right-hand side (RHS) is zero, often easier to solve due to its linear and additive properties.
- **Method of Undetermined Coefficients:** A technique for finding the particular integral of a non-homogeneous differential equation by guessing a particular form for the solution and determining the coefficients by substitution.
- **Variation of Parameters:** A method for finding the particular integral by assuming a solution in the form of the homogeneous solution multiplied by unknown functions, and then solving for these functions using the original differential equation.
- **General Solution:** The complete solution to a differential equation, comprising the sum of the particular integral and the general solution of the associated homogeneous equation.
- **Linear Differential Equation:** An equation involving derivatives that is linear in the dependent variable and its derivatives.
- **Coefficient:** A constant multiplier in front of a term in an equation, often representing the strength or scale of that term.

- **Exponential Function:** A mathematical function in the form $f(x) = a^x$, where a is a constant and x is the variable, commonly appearing in particular integrals for exponential forcing terms.
- **Trigonometric Function:** A mathematical function involving angles, such as sine, cosine, or tangent, frequently encountered in particular integrals for sinusoidal forcing terms.

Overall, terms collectively form the vocabulary used to solve differential equations, particularly when dealing with particular integrals.

CHECK YOUR PROGRESS

1. What is the particular integral of $y'' + 3y' + 2y = 5e^{2x}$?
2. Determine the particular integral of $y'' - 4y = 3\sin(2x)$.
3. For the differential equation $y'' - 6y' + 9y = 4xe^{3x}$ what is the particular integral?
4. What is the particular integral of $y'' + 2y' + y = 3xe^{-x}$?
5. Determine the particular integral of $y'' - y = 2x^2$.

7.8 *REFERENCES*

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7.9 *SUGGESTED READING*

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7.10 *TERMINAL QUESTIONS*

(TQ-1): Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$

(TQ-2): Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

(TQ-3): Solve the following differential equation:

a. $(D^3 + D^2 - 5D + 3)y = 0.$
b. $(D-1)^2(D^2+1)^2y = e^x.$
c. $(D^2 - 6D + 9)y = x^2 + 2e^{2x}.$
d. $(D^2 + D - 2)y = x + sinx.$
e. $(D^2 + 1)y = (1 + e^x)^{-1}$
f. $(D^2 + 5D + 6)y = e^{-2x}sec^2x(1 + 2tanx).$

7.11 *ANSWERS*

<u>CHECK YOUR PROGRESS</u>

1. $y_p = Ae^{2x}$.
2. $y_p = Asin(2x) + Bcos(2x)$
3. $y_p = (Ax^2 + Bx + C)e^{3x}$
4. $y_p = (Ax^2 + Bx)e^{-x}$
5. $y_p = (Ax^2 + Bx + C)$

TERMINAL ANSWERS (TQ'S)

(TQ-1): $y = C_1e^{3x} + C_2e^{5x}$

(TQ-2): $y = C_1e^{x} + C_2e^{2x} + C_3e^{3x}$

(TQ-3):

a. $y = (C_1x + C_2)e^{x} + C_3e^{-3x}$

b. $y = (C_1x + C_2)e^{x} + (C_3x + C_4)cosx + (C_5x + C_6)sinx + \frac{x^2}{8}e^{x}$

c. $y = (C_1x + C_2)e^{3x} + \frac{1}{9}\left(x^2 + \frac{4x}{8} + \frac{2}{3}\right) + 2e^{2x}$

d. $y = C_1e^{-2x} + C_2e^{x} - \frac{1}{4}(2x + 1) - \frac{1}{10}(cosx + 3sinx)$

e. $y = C_1 + C_2e^{-x} + x - (1 + e^{-x})\log(1 + e^{x})$

f. $y = C_1e^{-2x} + C_2e^{-3x} + C_3e^{-2x}(tanx - 1)$

BLOCK-III

UNIT 8: HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

8.1 *INTRODUCTION*

Definition: Given functions c_0, c_1; b: R→R the differential equation in the unknown function y be given by

$$\frac{d^2y}{dt^2} + c_1(t)\frac{dy}{dt} + c_0(t)y = b(t) \qquad \text{……(1)}$$

is called a second order linear differential equation with variable coefficients.

The equation in (1) is called homogeneous iff for all $t \in R$. We have b(t) = 0.

Then eq(1) reduces to

$$\frac{d^2y}{dx^2} + c_1(t)\frac{dy}{dx} + c_0(t)y = 0$$

This is called homogeneous linear differential equation with variable coefficients.

Examples:

1. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 3y = 0$
2. $(x+2)\frac{d^2y}{dx^2} + (7x+2)\frac{dy}{dx} + 5xy = 0$

SUPERPOSITION PROPERTY**:**

Theorem: If the functions y1 and y2 are solutions of homogeneous linear equation

$y'' + c_1(t)y' + c_0(t)y = 0$

then the linear combination $a_1y_1(t) + a_2y_2(t)$ is also a solution for a_1 , $a_2 \in R$.

Proof: Given: Since y1, y2 are the solutions of the given equation

i.e. $y_1'' + c_1(t)y_1' + c_0(t)y_1 = 0$

also $y_2'' + c_1(t)y_2' + c_0(t)y_2 = 0$ ……(1)

To prove: $a_1y_1(t) + a_2y_2(t)$ is also a solution of given equation is

i.e. $(a_1y_1 + a_2y_2)'' + c_1(t)(a_1y_1 + a_2y_2)' + c_0(t)(a_1y_1 + a_2y_2) = 0$

Proof : Now, $(a_1y_1'' + a_2y_2'') + c_1(a_1y_1' + a_2y_2') + c_0(a_1y_1 + a_2y_2)$

$= a_1(y_1'' + c_1y_1' + c_0y_1) + a_2(y_2'' + c_1y_2' + c_0y_2)$ [from eq (1)]

$= a_1.0 + a_2.0$

$= 0 + 0$

$= 0$

Hence proved.

8.2 OBJECTIVES

After studying this unit you will be able to

i. Described the concept of Particular Integral.

ii. Evaluate the Particular integral of type e^{ax} or e^{ax+b}

iii. Defined the particular integral of $\sin(ax+b)\ or\ \cos(ax+b)$

8.3 METHOD OF SOLUTION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF THE TYPE

$(c_0x^nD^n + c^1x^{n-1}D^{n-1} + \ldots\ldots\ldots c^{n-1}D + a_n)y = 0$ ....(1)

In order to solve (1) introduce a new variable Z such that

$x = e^Z$

or $\log x = Z$

so that $\frac{1}{x}\frac{dx}{dZ} = 1$

Now, $\frac{dy}{dx} = \frac{dy}{dZ}.\frac{dZ}{dx}$

$= \frac{1}{x}.\frac{dy}{dZ}$

or $x\frac{dy}{dx} = \frac{dy}{dZ}$

or $xDy = D_1y$

or $xD = D_1$

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{1}{x}\frac{dy}{dZ})$

$$= \frac{d}{dx}(\frac{1}{x})\frac{dy}{dZ} + \frac{1}{x}\frac{d}{dx}(\frac{dy}{dZ})$$

$$= \frac{-1}{x^2}.\frac{dy}{dZ} + \frac{1}{x}\frac{d}{dZ}(\frac{dy}{dZ}).\frac{dZ}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}.\frac{dy}{dZ} + \frac{1}{x^2}\frac{d^2y}{dZ^2}$$

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dZ} + \frac{d^2y}{dZ^2}$$

$x^2D^2y = (D^2 – D) y$ or $x^2D^2 = D(D-1)$ and so on..

Proceeding inthe same way,

$xnDn = D_1(D_1-1)(D_1- 2)…….\{D_1 – (n-1)\}$

Substituting these values in (1)

$\{c_0D_1(D_1-1)……….(D_1-(n-1)) + …..c_{n-2}D_1(D_1-1) + c_{n-1}D_1 + c_n\}y = 0$

$f(D_1)y = 0$ which is of the form discussed in previous chapter.

Examples:

Example 1: $x^2\frac{d^2y}{dx^2} + xy' – 4y = 0$

Solution: Putting $\frac{d}{dx} = D$

The given equation reduces to

$(x^2D^2 + xD – 4)y = 0$

Now, let $x = e^z$ and $D_1 = \frac{d}{dZ}$

The above equation reduces to

$\{ D_1(D_1 – 1) + D_1 – 4 \}y = 0$

$(D_1^2 – D_1 + D_1 – 4)y = 0$

$(D_1^2 – 4)y = 0$

which is now of the form $f(D)y = 0$

The complete solution is y = C.F. + P.I

To find complementary function now

Auxilliary equation is $f(m) = 0$

$m^2 – 4 = 0$

$m = \pm 2$

C.F. = $a_1e^{2z} + a_2e^{-2z}$

C.F. = $a_1(e^z)^2 + a_2(e^z)^{-2} = a_1x^2 + a_2x^{-2}$ (since $x=e^z$)

P.I = 0 Hence $y = a_1x^2 + a_2x^{-2}$

Example 2: $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = 0$

Solution: By putting $\frac{d}{dx} = D$ the given equation becomes

$x^3D^3y + 2x^2D^2y + 3xDy – 3y = 0$

or $(x^3D^3 + 2x^2D^2 + 3xD – 3)y = 0$

Substituting $x = e^z$ and $\frac{d}{dZ} = D$

The given equation reduces to

$[D_1(D_1-1)(D_1- 2) + 2D_1(D_1-1) + 3D_1 - 3]y = 0$

$[D_1(D1^2 – 3D_1 + 2) + 2D_1(D_1-1) + 3D_1 - 3]y = 0$

$[D_1^3 – 3D_1^2 + 2D_1 + 2D_1^2 – 2D_1 + 3D_1 - 3]y = 0$

$(D_1^3 – D_1^2 + 3D_1 – 3)y = 0$

$f(D)y = 0$

Thecomplete solution is y = C.F. + P.I.

To find C.F.: The Auxilliary equation is $f(m) = 0$

$m^3 - m^2 + 3m - 3 = 0$

$m^2(m - 1) + 3(m-1) = 0$

$(m - 1)(m^2 + 3) = 0$

Or $m - 1 = 0$, $m^2 + 3 = 0$

$m = 1$ $m = \pm\sqrt{3}i$

$y = a_1e^z + [\ a_2\cos\sqrt{3}z + a_3\sin\sqrt{3}z]$

since $x = e^z$ or $z = \log x$

$y = \ y = a_1e^z + (\ a_2\cos\sqrt{3}z + a_3\sin\sqrt{3}z)$

Particular integral of y is 0.

The complete integral is

$y = C.F = \ y = a_1e^z + (\ a_2\cos\sqrt{3}z + a_3\sin\sqrt{3}z)$

Example 3 : $(x^3D^3 + 3x^2D^2 - 2xD + 2)\ y = 0$ where $D = \frac{d}{dz}$

Solution : Substituting $x = e^z$ and $D = \frac{d}{dz}$, the given equation reduces to

$[D_1(D_1 -1)(D_1 - 2) + 3D_1(D_1 - 1) - 2D_1 + 2]y = 0$

$[D_1(D_1^2 - 3D_1 + 2) + 3D_1(D_1 - 1) - 2D_1 + 2]y = 0$

$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1- 1) - 2(D_1 - 1)]y = 0$

$(D_1 - 1)[D_1(D_1 - 2) + 3D_1 - 2]y = 0$

$(D_1 - 1)(D_1^2 + D_1 - 2)y = 0$

$(D_1 - 1)\ (D_1^2 + 2D_1 - D_1 - 2)y = 0$

$(D_1 - 1)[\ (D_1(D_1 + 2) -1(D_1 + 2)]y = 0$

$(D_1 - 1)[(D_1 - 1)(D_1 + 2)]y = 0$

$(D_1 - 1)(D_1 - 1)(D_1+2)y = 0$

$(D_1- 1)^2(D_1+2)y = 0$

i.e. $f(D_1)y = 0$

The complete solution is y = C.F. + P.I.

To find C.F.: The Auxilliary equation is f(m) = 0

$(m - 1)^2(m + 2) = 0$

$m = 1, 1, -2$

C.F. = $(a_1 + a_2 z)e^z + a_3 e^{-2z}$

P.I. = 0

y = C.F. + P.I.

$y = (a_1 + a_2 z)e^z + a_3 e^{-2z}$

Put $x = e^z$ or $z = \log x$

$y = (a_1 + a_2 \log x)x + a_3 x^{-2}$

Example 4: $x^2D^2 - 2y = 0$

Solution: Substitute $x = e^z$ or $\frac{d}{dz} = D_1$

The given equation reduces to

$[D_1(D_1 - 1) - 2]y = 0$

$(D_1^2 - D_1 - 2)y = 0$

$f(D_1)y = 0$

The complete solution is y= C.F. + P.I.

To find complementary solution f(m) = 0

or $m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$(m+1)(m-2) = 0$

$m+1 = 0$, $m-2 = 0$

$m = -1$, $m = 2$

C.F. = $c_1 e^{-z} + c_2 e^{2z}$

C.F. = $c_1(e^z)^{-1} + c_2(e^z)^2$

Since $x = e^z$ or $z = \log x$

C.F. = $c_1x^{-1} + c_2x^2$

Also P.I. = 0

Complete solution y = C.F. + P.I.

$y = \frac{c1}{x} + c_2x^2$

Exercise:

1. $(x^3D^3 - 3x^2D^2 + xD - 1)y = 0$
2. $(x^2D^2 - 2xD + 1)y = 0$
3. $(x^2D^2 - xD + 1)y = 0$
4. $(x^3D^3 - 2x^2D^2 - xD + 1)y = 0$

Answers:

1. $c_1x + c_2\sqrt{x}$
2. $x^{3/2}\,[c_1\cosh\frac{\sqrt{5}}{2}(\log x) + c_2\sinh\frac{\sqrt{5}}{2}(\log x)]$
3. $(c_1 + c_2\log x)x$
4. $c_1x + c_2x^{-1}$

An Equation Of The Form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

where P, Q are functions of x only is called homogeneous linear equation of second order.

8.4 *THE COMPLETE SOLUTION IN TERMS OF A KNOWN INTEGRAL*

If an integral included in the complement any function of such equations are known then the general equation can be found in terms of known integral.

Let y = u be the known integral of the complementary function of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \text{.......(1)}$$

i.e. it is a solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \text{......(2)}$$

Let y = uv be the solution of (1)

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + \frac{du}{dx}.\frac{dv}{dx} + \frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Substituting these value in (1)

$$(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}) + P(v\frac{du}{dx} + u\frac{dv}{dx}) + Quv = 0$$

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}(2\frac{du}{dx} + Pu) + (v\frac{d^2u}{dx^2} + Pv\frac{du}{dx} + Quv) = 0$$

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}(2\frac{du}{dx} + Pu) + v(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu) = 0$$

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}(2\frac{du}{dx} + Pu) + v.0 = 0$$

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}(2\frac{du}{dx} + Pu) = 0$$

$$u[\frac{d^2v}{dx^2} + (\frac{2}{u}\frac{du}{dx} + P)\frac{dv}{dx}] = 0$$

$\frac{d^2v}{dx^2} + (\frac{2}{u}\frac{du}{dx} + P)\frac{dv}{dx} = 0$

Now let $\frac{dv}{dx} = z$

$\frac{d}{dx}(\frac{dv}{dx}) = \frac{dz}{dx}$

$\frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\frac{dz}{dx} + (P + \frac{2}{u}\frac{du}{dx})z = 0$

This is a linear equation of the form

$\frac{dz}{dx} + Pz = Q$, where Q = 0

I.F. = $e^{\int(P + 2/udu/dx)dx}$

$= e^{2\int Pdx}.e^{\int 2/udu}$

$= e^{2\int du/u}.e^{\int P.dx}$

$= e^{2logu}.e^{\int P.dx}$

$= e^{logu^2}.e^{\int P.dx}$

I.F. = $u^2.e^{\int P.dx}$

The solution is

z.IF = $\int$Q.IFdx + C

$z.\ u^2.e^{\int P.dx} = 0 + C$

$z.\ u^2.e^{\int P.dx} = 0$

$z = \frac{c}{u^2}\ e^{\int -Pdx}$

Since, $z = \frac{dv}{dx}$

$\frac{dv}{dx} = \frac{c}{u^2}\ e^{\int -Pdx}$

or $dv = \frac{c}{u^2}\ e^{\int -Pdx}.\ dx$

Integrating this we get

$v = c\int \frac{e^{\int -Pdx}}{u^2} dx + c_1$

y = uv be the complete solution of (1)

or $y = u[\, c\int \frac{e^{\int -Pdx}}{u^2} dx + c_1 \,]$

is the complete solution.

To find One Integral in C.F. by inspection :

$y = e^{mx}$ is a part of C.F. of the equation

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ ……(1)

Then we have $y = e^{mx}$

$\frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2e^{mx}$

equation(1) becomes

$m^2e^{mx} + Pe^{mx} + Qe^{mx} = 0$

$(m^2 + Pm + Q)e^{mx} = 0$

Since $e^{mx} \neq 0$

Therefore $m^2 + Pm + Q = 0$ i.e. y = emx is a part of C.F if $m^2 + Pm + Q = 0$

Deductions: 1.$y = e^x$ is a solution of equation (1) $1 + P + Q = 0$

$y = e^{-x}$ is a solution of equation (1) $1 - P + Q = 0$

$y = e^{2x}$ is a solution of equation (1) $2^2 + 2P + Q = 0$

2. $y = x^m$ is a solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

Since $y = x^m$

$\frac{dy}{dx} = mx^{m-1}$

$\frac{d^2y}{dx^2} = m(m - 1)x^{m-2}$

The equation reduces to

$m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$

$[m(m-1) + Pmx + Qx^2]x^{m-2} = 0$

Since $x^{m-2} \neq 0$

$m(m-1) + Pmx + Qx^2 = 0$

Hence $y = x^m$ is a solution of (1) if

$m(m-1) + Pmx + Qx^2 = 0$

Hence $y = x$ is a solution of (1) if $P + Qx = 0$

Also, $y = x^2$ is a solution of (1) if $2.1 + 2Px + Qx^2$

$$2 + 2Px + Qx^2 = 0$$

Example 1: $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = 0$

Solution: Here $\frac{d^2y}{dx^2} - \frac{(2x+5)}{(x+2)}\frac{dy}{dx} + \frac{2}{(x+2)}y = 0$ …….(1)

Comparing equation with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

We have $P = \frac{-(2x+5)}{(x+2)}$, $Q = \frac{2}{(x+2)}$

Now try to find solution of (1) by inspection

$2^2 + 2P + Q = 4 - 2.\frac{(2x+5)}{(x+2)} + \frac{2}{(x+2)}$

$= \frac{4(x+2) - 2.(2x+5) + 2}{(x+2)}$

$= \frac{4x+8-4x-10+2}{(x+2)}$

$= 0$

Therefore $u = e^{2x}$ is a solution of equation (1)

Let $y = uv$ be the complete solution of (1)

i.e. $y = e^{2x}.v$ be the complete solution of (1)

Therefore the given equation reduces to

$\frac{d^2v}{dx^2} + (P + \frac{2}{u}\frac{du}{dx})\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} +[\frac{-(2x+5)}{(x+2)} + \frac{2}{e^{2x}}.2e^{2x}]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} +[\frac{-(2x+5)}{(x+2)} + 4]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-(2x+5)+4(x+2)}{(x+2)}]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-2x-5+4(x+2)}{(x+2)}]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + (\frac{2x+3}{x+2})\frac{dv}{dx} = 0$ ……(2)

Let $\frac{dv}{dx} = z$

$\frac{d}{dx}(\frac{dv}{dx}) = \frac{dz}{dx}$

Then equation (2) becomes

$\frac{dz}{dx} + (\frac{2x+3}{x+2})z = 0$

$\frac{dz}{dx} = -(\frac{2x+3}{x+2})z$

$\frac{dz}{z} = -(\frac{2x+3}{x+2})dx$

Integrating both sides

$\int\frac{dz}{z} = \int -(\frac{2x+3}{x+2})dx$

$\int\frac{dz}{z} = \int [2 -(\frac{1}{x+2})]dx$

$\int\frac{dz}{z} = -2x + \log(x+2) + \log C$

$\log z = -2x + \log(x+2) + \log C$

$\log z = \log[e^{-2x}(x+2).C]$

$z = ce^{-2x}(x+2)$

Since we have $z = \frac{dv}{dx}$

Therefore $\frac{dv}{dx} = ce^{-2x}(x+2)$

$dv = ce^{-2x}(x+2)dx$

Integrating both side, we have

$v = c\int e^{-2x}(x+2)dx$

$v = c[(x+2)\int e^{-2x}dx - \int \frac{d}{dx}(x+2).\int e^{-2x}dx]$

$v = c[\frac{-1}{2}(x+2)e^{-2x} - \int(\frac{-1}{2})e^{-2x}dx]$

$v = c[\frac{-(x+2)e^{-2x}}{2} + \frac{1}{2}\int e^{-2x}dx + c_1$

$v = c[\frac{-(x+2)e^{-2x}}{2} - \frac{1}{4} e^{-2x}]+ c_1$

$y = uv$ be the complete solution

$y = e^{2x} [c(\frac{-(x+2)e^{-2x}}{2} - \frac{1}{4} e^{-2x})+ c_1]$

$y = \frac{-c}{2} (x+2) - \frac{c}{4} + c_1e^{2x}$

$y = -c[\frac{(x+2)}{2} +\frac{1}{4}]+ c_1e^{2x}$

$y = \frac{-c}{4}(2x+5) +c_1e^{2x}$

Example 2: $(x+1)\frac{d^2y}{dx^2} - 2(x+3)\frac{dy}{dx} + (x+5)y = 0$

Solution : $\frac{d^2y}{dx^2} - \frac{(2x+3)}{x+1}\frac{dy}{dx} + \frac{(x+5)}{(x+1)}y = 0$

Comparing the equation with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

We have $P= - \frac{(2x+3)}{x+1}$, $Q = \frac{(x+5)}{(x+1)}$

Now we have to find a solution of (1) by inspection

$1 + P+Q = 1- \frac{(2x+3)}{x+1} + \frac{(x+5)}{(x+1)}$

$= \frac{(x+1)- 2(x+3)+(x+5)}{(x+1)}$

$= \frac{x+1-2x-6+x+5}{(x+1)}$

$= 0$

Therefore u = e^x is a solution of equation(1)

Let y = uv be the complete solution of (1)

i.e. y = e^x.v be the complete solution of (1)

Therefore the given equation reduces to

$\frac{d^2v}{dx^2} + (P + \frac{2}{u})\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-2(x+3)}{(x+1)} + \frac{2}{e^x}.e^x]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-2(x+3)}{(x+1)} + 2]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-2x-6+2(x+1)}{(x+1)}]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + [\frac{-2x-6+2x+2}{(x+1)}]\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + \frac{-4}{(x+1)}\frac{dv}{dx} = 0$

Let $\frac{dv}{dx} = z$

Then $\frac{d}{dx}(\frac{dv}{dx}) = \frac{dz}{dx}$

Equation (2) becomes

$\frac{dz}{dx} - \frac{4}{(x+1)} = 0$ ……(3)

This is a linear differential equation of the form

$\frac{dz}{dx} + Pz = Q$

Where P = $\frac{-4}{(x+1)}$, Q = 0

Now I.F. = $e^{\int P.dx}$

$= e^{\int -4/x+1 dx}$

$= e^{-4\int dx/x+1}$

$= e^{-4\log(x+1)}$

$= e^{\log(x+1)^{-4}}$

$= (x+1)^{-4}$

$= \frac{1}{(x+1)^4}$

The solution of (3) is

z.IF = ∫Q.IFdx + C

$z.\frac{1}{(x+1)^4} = \int 0dx + C$

$z.\frac{1}{(x+1)^4} = C$

or $z = C(x+1)^4$

Since we have $z = \frac{dv}{dx}$

Therefore $\frac{dv}{dx} = C(x+1)^4$

$dv = C(x+1)^4dx$

Integrating both sides, we have

$\int dv = \int C(x+1)^4dx$

$v = C\frac{(x+1)^5}{5} + C_1$

Since y = uv is the complete solution

Therefore, $y = e^x[\ C\frac{(x+1)^5}{5} + C_1\ ]$

$y = Ce^x\,\frac{(x+1)^5}{5} + C_1e^x$

Removal of first Derivative (Reduction to normal form):

If we are not able to find a part of the C.F. of the solution of the differential equation

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ …… (1)

where P and Q are functions of x

Then we have y = uv

By this substitution change the dependent variable from y to v.

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

Change the dependent variable from y to 'v'

Equation (1) becomes

$(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}) + P(u\frac{dv}{dx} + v\frac{du}{dx}) + Quv = 0$

$u\frac{d^2v}{dx^2} + u(P + \frac{2}{u})\frac{dv}{dx} + v(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu) = 0$ …..(2)

Now to remove the first derivative

$P + \frac{2}{u}\frac{du}{dx} = 0$

$\frac{2}{u}\frac{du}{dx} = - P$

$\frac{du}{u} = \frac{-P}{2}\, dx$

$\int\frac{du}{u} = \frac{-1}{2}\int Pdx$

$\log u = \frac{-1}{2}\int Pdx$

$u = e^{-1/2 \int Pdx}$

Now the equation (2) reduces to

$u\frac{d^2v}{dx^2} + v(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu) = 0$ ……(4)

Now since, $u = e^{-1/2 \int P dx}$

We have $\frac{du}{dx} = \frac{-1}{2} Pu.$ (using 3)

$\frac{d^2u}{dx^2} = \frac{-1}{2} (P\frac{du}{dx} + u\frac{dP}{dx})$

$\frac{d^2u}{dx^2} = \frac{-1}{2} [P(\frac{-1}{2}Pu) + u\frac{dP}{dx}]$

$= \frac{-1}{2} (\frac{-1}{2}P^2u + u\frac{dP}{dx})$

$\frac{d^2u}{dx^2} = u = e^{-1/2 \int P dx}$

Substituting $\frac{d^2u}{dx^2}$, $\frac{du}{dx}$, u in equation (4)

$u\frac{d^2v}{dx^2} + v[\frac{1}{4} P^2u - \frac{1}{2} u\frac{dP}{dx} + P(\frac{-1}{2}Pu) + Qu] = 0$

$u\frac{d^2v}{dx^2} + uv[\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} + P(\frac{-1}{2}P) + Q] = 0$

$u\frac{d^2v}{dx^2} + uv(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}) = 0$

$u[\frac{d^2v}{dx^2} + v(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx})] = 0$

since $u \neq 0$

Therefore $\frac{d^2v}{dx^2} + v(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}) = 0$

NOTE: This equation is applicable if $Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ = constant

Example 1: $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = 0$

Solution: Comparing this equation

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ ……(1)

We have $P = -4x$, $Q = (4x^2 – 1)$

We choose $u = e^{-1/2\int P.dx}$

$u = e^{-1/2\int -4x dx}$

$u = e^{\int 2x dx}$

$u = e^{x^2}$

Putting $y = uv = e^{x^2}.v$ the given equation reduces to

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right) v = 0 \qquad \text{.....(2)}$$

Now $Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

$= (4x^2 - 1) - \frac{1}{2}.\frac{d}{dx}(-4x) - \frac{1}{4}(-4x)^2$

$= (4x^2 - 1) - \frac{1}{2}(-4) - \frac{1}{4}(16x^2)$

$= 4x^2 - 1 + 2 - 4x^2$

$= 1$

Then equation (2) becomes

$$\frac{d^2v}{dx^2} + v = 0$$

i.e. $(D^2 + 1)v = 0$

$f(D)v = 0$

The complete solution of v = C.F + P.I

Auxilliary equation $f(m) = 0$

$m^2 + 1 = 0$

$m = \pm 1$

$v = c_1\cos x + c_2\sin x$

P.I. = 0

So $v = c_1\cos x + c_2\sin x$

Since y = u.v is the solution of the given equation

Therefore, $y = e^{x^2}(c_1\cos x + c_2\sin x)$

Example 2: $\frac{d^2y}{dx^2}$ - 2tanx$\frac{dy}{dx}$ +5y = 0

Solution : Comparing this equation with $\frac{d^2y}{dx^2}$ + P$\frac{dy}{dx}$+ Qy = 0

We have P = -2tanx, Q = 5

Choose u = $e^{-1/2\int P.dx}$

u = $e^{-1/2\int -2tanx}$

u = $e^{\int tanx}$

u = $e^{logsecx}$

u = secx

Put y= uv , the given equation reduces to

$\frac{d^2v}{dx^2}$ + (Q - $\frac{1}{2}\frac{dP}{dx}$ - $\frac{1}{4}$ P^2) v = 0

$\frac{d^2v}{dx^2}$ + [5 - $\frac{1}{2}.\frac{d}{dx}$(-2tanx) - $\frac{1}{4}$(4tan^2x)]v = 0

$\frac{d^2v}{dx^2}$ + [5 + sec^2x – tan^2x]v = 0

$\frac{d^2v}{dx^2}$ + 6v = 0

(D^2 + 6)v = 0

f(D)v = 0

v = C.F. + P.I

C.F. : Auxilliary equation f(m) = 0

$m^2 + 6 = 0$

$m = \pm\sqrt{6}i$

C.F. = $c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x$

P.I. = 0, therefore v = $c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x$

y = uv is the complete solution of given equation

$y = \sec x(c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x)$

Exercises :

1. $(D^2 – 4xD + 4x^2 – 1) y = 0$
2. $D^2y - \frac{2}{x}Dy + (1 +\frac{2}{x^2})y = 0$
3. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} - (a^2 + 1)y = 0$
4. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2y = 0$

Answers:

1. $y = e^{x2(}\ a_1\cos x + a_2\sin x)$

2. $y = x(a_1\cos x + a_2\sin x)$

3. $y = \sec x(c_1e^{ax} + c_2e^{-ax})$

4. $y = ex^2 (c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$

8.5 *TRANSFORMATION OF THE EQUATION BY CHANGING THE INDEPENDENT VARIABLE*

Consider the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \text{……(1)}$$

where P, Q are functions of x.

Let the independent variable changing from **x** to **t**

Let f(x) = t

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$$

$= \frac{d}{dx}(\frac{dy}{dt} . \frac{dt}{dx})$

$= \frac{d}{dt}(\frac{dy}{dt}).\frac{dt}{dx}.\frac{dt}{dx} + \frac{dy}{dt}.\frac{d^2t}{dx^2}$

$\frac{d^2y}{dx^2} \quad = \frac{d^2y}{dx^2}(\frac{dt}{dx})^2 + \frac{dy}{dt}.\frac{d^2t}{dx^2}$

Substituting these values in equation(1)

$\frac{d^2y}{dt^2}.(\frac{dt}{dx})^2 + \frac{dy}{dt}.\frac{d^2t}{dx^2} + P(\frac{dy}{dt}\frac{dt}{dx}) + Qy = 0$

$\frac{d^2y}{dt^2}.(\frac{dt}{dx})^2 + \frac{dy}{dt}(\frac{d^2t}{dx^2} + P\frac{dt}{dx}) + Qy = 0$

$\frac{d^2y}{dt^2} + \frac{\frac{d^2t}{dx^2} + P\frac{dt}{dx}}{\left(\frac{dt}{dx}\right)^2}.\frac{dy}{dt} + \frac{Q}{\left(\frac{dt}{dx}\right)^2}y = 0$

Therefore the given equation reduces to

$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + Q_1y = 0$

where $P_1 = \frac{\frac{d^2t}{dx^2} + P\frac{dt}{dx}}{\left(\frac{dt}{dx}\right)^2}$, $\quad Q_1 = \frac{Q}{\left(\frac{dt}{dx}\right)^2}$

where P_1, Q_1 are functions of **x** and may be expressed as function of **t** with the help of relation

t = f(x)

Solved Examples:

Example 1: $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 0$

Solution: The given equation can be converted to

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

The given equation can be written as

$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 0$

where $P = -1/x$, $Q = -4x^2$

Changing the independent variable from **x** to **t** by the relation of the form **t = f(x)**

The given equation reduces to

$$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + Q_1y = 0 \qquad \text{....(2)}$$

where $P_1 = \dfrac{\frac{d^2t}{dx^2} + P\frac{dt}{dx}}{\left(\frac{dt}{dx}\right)^2}$, $\quad Q_1 = \dfrac{Q}{\left(\frac{dt}{dx}\right)^2}$

$Q_1 = -4x^2/(dt/dx)^2 = \text{constant}$

or $\dfrac{-4x^2}{\left(\frac{dt}{dx}\right)^2} = -4$

$(\frac{dt}{dx})^2 = x^2$

$\frac{dt}{dx} = x$

$dt = xdx$

Integrating both sides

$t = \frac{x^2}{2}$

$P_1 = \dfrac{1+\left(\frac{-1}{x}\right).x}{x^2} = \dfrac{1-1}{x^2} = 0$

Therefore the equation (2) reduces to

$\frac{d^2y}{dt^2} + (-4)y = 0$

$(D^2 - 4)y = 0 \qquad \text{....(3)}$

i.e. $f(D)y = 0$

The complete solution of y = C.F. + P.I.

The Auxilliary equation is given for complementary function is

$f(m) = 0$

$m^2 - 4 = 0$

$m = \pm 2$

C.F. = $c_1e^{2t} + c_2e^{-2t}$

P.I. = 0

y = C.F. + P.I.

$y = c_1e^{2t} + c_2e^{-2t}$

$y = c_1e^{2.x2/2} + c2e^{-2x2/2}$ (Since $t = x^2/2$)

$y = c_1e^{x2} + c2e^{-x2}$

Example 2 : $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2\cos^3 xy = 0$

Or $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} - 2\cos^2 xy = 0$

Solution: $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2\cos^3 xy = 0$

Or $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} - 2\cos^2 xy = 0$ …….(1)

Comparing the given equation with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

we have P = tanx , $Q = -2\cos^2 x$

Substituting t = f(x) the given equation reduces to

$\frac{d^2y}{dt^2} + P_1\frac{dy}{dt} + Q_1y = 0$ ….(2)

where $P_1 = \frac{\frac{d^2t}{dx^2} + P\frac{dt}{dx}}{\left(\frac{dt}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dt}{dx}\right)^2}$

$Q1 = \frac{-2\cos^2 x}{\left(\frac{dt}{dx}\right)^2} = -2$ (say)

$\left(\frac{dt}{dx}\right)^2 = \cos^2 x$

$\frac{dt}{dx} = \cos x$

t = sinx

$P_1 = \frac{\frac{d^2t}{dx^2} + P\frac{dt}{dx}}{\left(\frac{dt}{dx}\right)^2}$

$P_1 = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$

Therefore the equation (2) reduces to

$\frac{d^2y}{dt^2} - 2y = 0$

i.e. $(D^2 - 2)y = 0$

f(D) y = 0

y = C.F. + P.I.

The Auxilliary equation for complementary function is given by

f(m) = 0

or $m^2 - 2 = 0$

or $m = \pm\sqrt{2}$

$y = c_1\cosh\sqrt{2}t + c_2\sinh\sqrt{2}t$ (Since t = sinx)

$y = c_1\cosh\sqrt{2}\sin x + c_2\sinh\sqrt{2}\sin x$

SELF CHECK QUESTIONS

1. $\frac{1}{D^2+a^2} sinax = -\frac{x}{2a} cosax$. True/False

2. $\frac{1}{D^2+a^2} cosax = \frac{x}{2a} sinax$. True/False

3. P.I. of the Differential equation $D^2y = 0$ is?

4. P.I. of the Differential equation $D^2y = e^x$ is?

5. P.I. of the Differential equation $D^3y = e^{2x}$ is?

8.6 *SUMMARY*

This unit is a composition of different methods of fining the Particular Integral. In this unit we are evaluating the particular integral of type e^{ax} or e^{ax+b} and type of $\sin(ax + b)\ or \cos(ax + b)$.

8.7 *GLOSSARY*

- **Differential Equation** an Equation involving derivatives of differentials of one or more dependent variables with respect to one or more independent variables is called **Differential Equation.**
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.
- **Auxiliary Equation:** The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation or A.E.
- **Complementary function:** Complementary function is actually the solution of the given differential equation $f(D)y = Q$ when its right hand side member it means the Q is replaced by zero. To find complementary function we first find auxiliary equation.

8.8 *REFERENCES*

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8.9 *SUGGESTED READING*

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8.10 *TERMINAL QUESTIONS*

Q1. $x^6\frac{d^2y}{dx^2} + 3x^5\frac{dy}{dx} + 9y = 0$

Q2. $\frac{d^2y}{dx^2} + (\tan x - 1)^2\frac{dy}{dx} - 6y\sec^4 x = 0$

Q3. $(x^3 - x)\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4x^3y = 0$

***Q*4.** $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = 0$

Q5. $(D^2 - 4xD + 4x^2 - 1)\, y = 0$

Q6. $D^2y - \frac{2}{x}Dy + \left(1 + \frac{2}{x^2}\right)y = 0$

Q7. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} - (a^2 + 1)y = 0$

Q8. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2y = 0$

8.11 *ANSWERS*

TERMINAL ANSWERS (TQ'S)

1. $c_1\cos\frac{3}{2x^2} + c_2\frac{3}{2x^2}$

2. $c_1e^{-3\tan x} + c_2^{e2\tan x}$

3. $y = c_1 \sin (2\sqrt{x}^2 - 1 + C)$

4. $y = c_1\cos (x^2 + c_2)$

5. $y = e^{x2(}a_1\cos x + a_2\sin x)$

6. $y = x (a_1\cos x + a_2\sin x)$

7. $y = \sec x (c_1e^{ax} + c_2e^{-ax})$

8. $y = ex^2 (c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$

Unit 9: SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

CONTENTS:

9.1 *INTRODUCTION*

In this unit we shall discuss differential equations is which there is one independent variable and two or more than two variable and two or more than two dependent variables. To solve such equations completely, there must be as many equations as there are dependent variables. Such equations are called its ordinary simultaneous differential equations.

9.2 *OBJECTIVES*

At the end of this topic learner will be able to understand:

(i) Method for Finding Particular Function when X = $e^{ax}V(x)$

(ii) Method for Finding Particular Function when X = $x^m cosax$ or $x^m sinax$

9.3 *METHODS FOR SOLVING SIMULTANEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS*

Let x and y be the dependent variables and t be the independent variable. Thus, in such equations there occur differential coefficients of x, y with respect to t. let D = d/dt then such equations can be put in the form

$$f_1(D)x + f_2(D)y = T_1 \qquad \text{………. (1)}$$

And

$$g_1(D)x + g_2(D)y = T_2 \qquad \text{………. (1)}$$

Where T_1 and T_2 are functions of the independent variable t and $f_1(D)$, $f_2(D)$, $g_1(D)$ and $g_2(D)$ all are rational integral functions of D with constant coefficients. Such equations can be solved by the following two methods.

First method. Methods of elimination (use of operator D)

In order to eliminate y between (1) and (2). Operating on the both sides of (1) by $g_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting, we get

$$\{f_1(D)g_2(D) - g_1(D)f_2(D)\}x = g_2(D)T_1 - f_2(D)T_2 \qquad \text{……. (3)}$$

Which is a linear differential equation with constants coefficients in x and t and can be solved to give the value of x in term of t. substituting this value of x in either (1) or (2), we get the value of y in terms of t.

Note 1. The above equations (1) and (2) can be solved by first eliminating x between them and solving the resulting equation to get y in term of t. substituting this value of y in either (1) or (2), we get the value of x in terms of t.

Note 2. Since $f_2(D)$ and $g_2(D)$ are functions of D with constant coefficients, so

$$f_2(D)g_2(D) - g_2(D)f_2(D)$$

Note 3. In the general solutions of (1) and (2) the number of arbitrary constants is equal to the degree of D in the determinant $\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ g_1(D) & g_2(D) \end{vmatrix}$, provided $\Delta \neq 0$.

If $\Delta = 0$, then the system of equations (1) and (2) is dependent and such cases will not be considered.

Second method. Methods of differentiation.

Sometimes, x or y can be eliminated easily if we differentiate (1) or (2). For example, assume that the given equations (1) or (2) connect four quantities x, y, dx/dt and dy/dt. Differentiating (1) and (2) with respect to t, we obtain four equations containing x, dx/dt, d^2x/dt^2 , y, dy/dt and d^2y/dt^2. Eliminating three quantities y, dy/dt. d^2y/dt^2 from these four equations, y is eliminated and we get an equation of second order with x as the dependent and t as the independent variable. Solving this equation, we get value of x in term of t. substituting this value of x in either (1) or (2), we get value of y in term of t.

Example 1. Solve the simultaneous equations $\frac{dx}{dt}$ – 7x + y = 0 and $\frac{dy}{dt}$ – 2x -5y = 0.

Sol. We shall solve the given system by two methods.

First method. Methods of elimination (use of operator D)

Step 1. Writing D for d/dt, given equation can be rewrite in the symbolic form as follows:

$(D - 7)x + y = 0$ …… (1)

And $-2x + (D - 5)y = 0$ ……. (2)

Step 2. We now eliminate x (say) as follows. Multiplying (1) by 2 and operating (2) by

(D – 7), we get $2(D - 7)x + 2y = 0$ …… (3)

$-2(D - 7)x + (D - 7)(D -5)y = 0$ ….. (4)

Adding (3) and (4) we get, $[(D - 7)(D - 5) + 2]y = 0$ or $(D^2 - 12D + 37)y = 0$

Which is linear equation with constant coefficients,

Its auxiliary equation is $D^2 - 12D + 37 = 0$ so that $D = 6 \pm i$

Therefore $y = e^{6t}(c_1 cost + c_2 sint)$, c_1, c_2 being arbitrary Constants. …… (5)

Step 3. We now try to get x by using (5). In this connection remember that we must avoid integration to get x. thus if we use (1) to get x, then after putting value of y we have to integrate for getting x. Now from (5), differentiating w.r.t. ‘t’, we get

$$Dy = 6e^{6t}(c_1 cost + c_2 sint) + e^{6t}(-c_1 sint + c_2 cost)$$

Or $$Dy = e^{6t}[6c_1 + c_2)cost + (6c_2 - c_1)sint] \quad \text{........ (6)}$$

Substituting the value of y and Dy given by (5) and (6) in (2) , we have

$$2x = Dy - 5y = e^{6t}[(6c_1 + c_2)cost + (6c_2 - c_1)sint] - 5(c_1 cost + c_2 sint)]$$

$$x = \frac{1}{2}e^{6t}[(c_1 + c_2)cost + (c_2 - c_1)sint] \quad \text{........ (7)}$$

thus (5) and (7) together give the required solution.

Remark. We can also eliminate y first (as we did to eliminate x) and then obtain x. thus value of x can be put in (1) to get the desired value of y.

Second Method. Method of differentiation.

Given that $\frac{dx}{dt} - 7x + y = 0$ .... (1)

and $\frac{dy}{dt} - 2x - 5y = 0$ .... (2)

to eliminate x, we differentiate (2) w.r.t. ‘t’ and obtain

$$(d^2y/dt^2) - 2\,(dx/dt) - 5(dy/dt) = 0 \quad \text{........ (3)}$$

Now, from (2), we have $x = \frac{1}{2}\left(\frac{dy}{dt} - 5y\right)$ .... (4)

Then from (1), we get $\frac{dx}{dt} = 7x - y = \frac{7}{2}\left(\frac{dy}{dt} - 5y\right) - y$ using (4)

Therefore $\frac{dx}{dt} = = \frac{7}{2}\left(\frac{dy}{dt} - \frac{37y}{2}\right)$

Substituting this value of dx/dt in (3), we get

$(d^2y/dt^2) - 2\,(dy/dt) + 37y - 5(dy/dt) = 0$ or $(D^2 - 12D + 37)y = 0$.

Now get y as done in first method. in fact repeat the whole method after this step. Thus get the same values of x and y as in the first method.

Note 1. Second method will be used when found very necessary, in almost all problems we we shall use the first method.

Note 2. Generally, t will be the independent variable and x and y will be dependent variables.

In some problems any other variable, x say, will be given as the independent variable and y and z as the dependent variables. This point should be noted carefully while doing any problem.

9.4 *SOLVED EXAMPLES*

Example 1. Solve $\frac{dx}{dt}$ – y = t, $\frac{dy}{dt}$ + x = 1.

Sol. Writing D for $\frac{d}{dt}$, the given equation become

Dx – y = t ……. (1)

x + Dy = 1 ……. (2)

Differentiating (1) w.r.t. 't', D^2x – Dy = 1 ……. (3)

To eliminate y between (2) and (3), we add them and get

D^2x + x = 2 or (D^2 + 1)x = 2 …….(4)

Now the auxiliary equation of (4) is D^2 + 1 = 0

D = $\pm i$.

∴ C.F. = $c_1 \cos t$ + $c_2 \sin t$,

c_1 and c_2 being arbitrary constants.

P.I. = $\frac{1}{1+D^2}$ 2 = $(1 + D^2)^{-1}$ 2 = $(1 - D^2 + \cdots)$ 2 = 2

Hence the general solution of (4) is

x = $c_1 \cos t$ + $c_2 \sin t$ + 2 ….. (5)

From (5), Dx = $\frac{dx}{dt}$ = - $c_1 \sin t$ + $c_2 \cos t$ ….. (6)

∴ From (1), y = Dx – t = - $c_1 \sin t$ + $c_2 \cos t$ – t. ...... (7)

The required solution is given by (5) and (7).

Example 2. Solve the simultaneous differential equation

(D – 17)y + (2D – 8)z = 0,

(13D – 53)y – 2z = 0, where D ≡ $\frac{d}{dt}$

Sol. Given (D – 17)y + 2(D – 4)z = 0 …..(1)

(13D – 53)y – 2z = 0 …..(2)

Operating on both side of (2) by (D – 4) and then adding to (1) , we have

{(D – 17) + (D – 4) (13D – 53)}y = 0

Or (D^2 – 8D – 15)Y = 0 ….(3)

Here auxiliary equation is

$$D^2 - 8D - 15 = 0$$

So that D = 3, 5.

$\therefore$ y = C.F. = $c_1e^{3x} + c_2e^{5x}$,

c_1 and c_2 being arbitrary constants ….. (4)

From (4), Dy = $\frac{dy}{dx}$ = $3c_1e^{3x} + c_2e^{5x}$, ….. (5)

From (2), 2z = 13Dy – 53y

Or 2z = $13(c_1e^{3x} + 5c_2e^{5x}) - 53(c_1e^{3x} + c_2e^{5x})$, by (4) and (5)

$\therefore$ z = $6C_2e^{5x} - 7C_1e^{3x}$ …….. (6)

The required general solution is given by (4) and (6).

Example 3. Solve $\frac{dx}{dt} + 5x + y = e^t$, $\frac{dy}{dt} - x + 3y = e^{2t}$.

Sol. Given (D + 5)x +y = e^t ……(1)

and -x +(D + 3)y = e^{2t} ……(2)

Operating on both sides of (2) by (D + 5), we get

- (D + 5)x + (D + 5)(D + 3)y = (D + 5)e^{2t} = $2e^{2t} + 5\ e^{2t}$, …..(3)

Adding (1) and (3), {1 + (D + 5)(D + 3)}y = $e^t + 7e^{2t}$

Or $(D + 4)^2$ y = $e^t + 7e^{2t}$

Its auxiliary equation is $(D + 4)^2 = 0$

So that D = - 4 , - 4.

$\therefore$ C.F. = $(c_1 + c_2t)\ e^{-4t}$ c_1 and c_2, being arbitrary constants.

P.I.= $\frac{1}{(D+4)^2}(e^t + 7e^{2t}) = \frac{1}{(D+4)^2}e^t + 7\frac{1}{(D+4)^2}e^{2t}$

$$= \frac{1}{(1+4)^2}e^t + 7\frac{1}{(2+4)^2}e^{2t}$$

$$= \frac{1}{25}e^t + \frac{7}{36}e^{2t}\ .$$

$\therefore$ Solution of (4) is y = C.F. + P.I.

y = $(c_1 + c_2t)\ e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}$ …… (5)

from (5), Dy = $\frac{dy}{dt}$ = $-4(c_1 + c_2t)\ e^{-4t} + c_2e^{-4t} + \frac{1}{25}e^t + \frac{7}{18}e^{2t}$ …(6)

$\therefore$ from (2),

x = Dy + 3y - e^{2t} , using (5) and (6) , this gives

$$x = -4(c_1 + c_2 t)e^{-4t} + c_2 e^{-4t} + \frac{1}{25}e^t + \frac{7}{18}e^{2t}$$

$$+ 3[(c_1 + c_2 t)\, e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}] - e^{2t}$$

or $x = -(c_1 + c_2 t)e^{-4t} + c_2 e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t}$ …..(7)

The required general solution is given by (5) and (7) .

Example 4. Solve $\frac{dy}{dt}$ = y , $\frac{dx}{dt}$ = 2y + x .

Sol. Given that $\frac{dy}{dt}$ = y, ….(1)

$\frac{dx}{dt}$ = 2y + x …..(2)

From (1), $(\frac{1}{y})$ dy = dt .

Integrating, log y – log c_1 = t

y = $c_1\, e^t$ …..(3)

substituting this value of y in (2), we have

$$\frac{dx}{dt} = 2c_1\, e^t + x$$

or $\frac{dx}{dt}$ - x = $2c_1\, e^t$, which is a linear equation .

Its I.F. = $e^{\int(-1)dt}$ = e^{-t} and solution is

$$x.\, e^{-t} = \int (2\, c_1\, e^t).\, e^t dt + c_2 = 2c_1 t + c_2$$

or x = $(2c_1 t + c_2)\, e^t$, c_1 and c_2 are arbitrary constants.

Hence the required solution is given by

$$x = (2c_1 t + c_2)\, e^t , \quad y = c_1\, e^t .$$

9.5 *SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS INVOLVING OPERATORS x(d/dt) OR t(d/dt)*

Example 1. Solve t (dx/dt) + y = 0, t (dy/dt) + x = 0.

Sol. Let t = e^z . let $D_1 = d/dz$ = t (d/dt). Then given equations becomes

D_1x + y = 0 ……. (1)

And x + D_1x = 0 ……. (2)

Eliminating y from (1) and (2), ${D_1}^2x$ – x = 0 or (${D_1}^2$ – 1) x = 0 ……. (3)

Its auxiliary equation is ${D_1}^2$ – 1 =0 so that D_1 = 1, -1.

Therefore Solution of (3) is x = $C_1e^z + C_2e^{-z}$ and so $D_1x = C_1e^z - C_2e^{-z}$

Therefore from (1) y = $-D_1x$ = $C_2e^{-z} - C_1e^z$

Since t = e^z, the required solution is x = $C_1t + C_2t^{-1}$, y = $C_2t^{-1} - C_1t$.

Example 2. Solve t (dx/dt) + 2(x – y) = t, t (dy/dt) + x + 5y = t^2

Sol. Let t = e^z . let $D_1 = d/dz$ = t (d/dt). Then given equations becomes

(D_1 + 2)x - 2y = e^z ……. (1)

And x + $(D_1 + 5)y$ = e^{2z} ……. (2)

Eliminating y from (1) and (2), $(D_1 + 5)$ (D_1 + 2)x + 2x = $(D_1 + 5)$ e^z+ 2e^{2z}

Or $({D_1}^2 + 7D_1 + 12)x = 6e^z$+ $2e^{2z}$ …….. (3)

Its auxiliary equation is ${D_1}^2 + 7D_1 + 12$ = 0 giving D_1 = - 3, -4

Therefore, C.F. = $C_1e^{-3z} + C_2e^{-4z}$, where C_1 and C_2 are arbitrary constants.

P.I. corresponding to $6e^z$ = 6 $\frac{1}{{D_1}^2+7D_1+12}e^z = \frac{3}{10}e^z$

P.I. corresponding to $2e^{2z}$ = 2 $\frac{1}{{D_1}^2+7D_1+12}e^{2z} = \frac{1}{15}e^{2z}$

Therefore, solution of (3) is x = $C_1e^{-3z} + C_2e^{-4z}$ + $\frac{3}{10}e^z + \frac{1}{15}e^{2z}$ …… (4)

Therefore, D_1x = - $3C_1e^{-3z} - 4C_2e^{-4z}$ + $\frac{3}{10}e^z + \frac{2}{15}e^{2z}$ ……. (5)

From (1) and (5), y = - (1/5) $C_1e^{-3z} - C_2e^{-4z}$ – (1/20) e^z + (2/15) e^{2z} ...... (6)

Putting t = e^z in (4) and (6), the required general solution is

x = $C_1t^{-3} + C_2t^{-4} + \frac{3t}{10} + \frac{t^2}{15}$, y = - (1/2) C_1t^{-3} - C_2t^{-4} +$\frac{2t^2}{15}$ - $\frac{t}{20}$.

SELF CHECK QUESTIONS

1. $\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ g_1(D) & g_2(D) \end{vmatrix}$, If $\Delta = 0$, then the system of equations is dependent and such cases will not be considered.
2. Since $f_2(D)$ and $g_2(D)$ are functions of D with constant coefficients, so $f_2(D)g_2(D) - g_2(D)f_2(D)$
3. The equations $\frac{dx}{dt}$ – 7x + y = 0 and $\frac{dy}{dt}$ – 2x -5y = 0 are simultaneous equation.
4. The equations $\frac{dx}{dt}$ – 7x + y = 0 and $\frac{dy}{dt}$ – 2x -5y = 0 are not simultaneous equation.
5. The Solution of $\frac{dx}{dt}$ – y = t, $\frac{dy}{dt}$ + x = 1 is x = $c_1 \cos t + c_2 \sin t + 2$ for x.

9.6 *SUMMARY*

It solves simultaneous ordinary differential equations describing the transmutation, the growth and the decay of the nuclide densities, and performs an accurate depletion calculation with fine description of the irradiation history and the isotopic chain. A simultaneous differential equation is one of the mathematical equations for an infinite function of one or more than one variable that relate the values of the function. Differentiation of an equation in various orders. An Ordinary differential equation (ODE) contains only ordinary derivatives.

9.7 *GLOSSARY*

- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **Rate of Change:** The speed at which a quantity changes with respect to time or another variable.
- **Derivative:** A measure of how a function changes as its input changes.
- **First Order:** The highest derivative involved in the equation is the first derivative.

- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Dependent Variable:** The variable whose value depends on the value of another variable, often denoted as y.
- **Independent Variable:** The variable that is varied independently of other variables, often denoted as x.
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Derivative:** A measure of how a function changes as its input changes, often representing rates of change.
- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.

Understanding these terms is essential for working with first-order, first-degree differential equations and solving problems in various fields of science and engineering.

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9.10 TERMINAL QUESTIONS

(TQ-1) Solve dx/dt = ny – mz, dy/dt = lz – nx, dz/dt = mx – ly .

(TQ-2) Solve foe x dx/dt = 2y, dy/dt = 2z and dz/dt = 2x.

(TQ-3) Solve for y dx/dt = x – 2y , dy/dt = 5x + 3y

(TQ-4) Solve dx/dt + x – y = e^t, dy/dt + y – x = 0

(TQ-5) Solve dx/dt – y = e^{-t}, dy/dt + x = e^t

9.11 ANSWERS

SELF CHECK ANSWERS

1. True
2. True
3. True

4. False

5. True

TERMINAL ANSWERS (TQ'S)

(TQ-1) $x^2 + y^2 + z^2 = c_1$, $lx^2 + my^2 + nz^2 = c_2$, $lx + my + nz = c_3$

(TQ-2) $x = c_1e^{2t} + c_2e^{-t}\cos(\sqrt{3t} + c_3)$

(TQ-3) $y = \{(3c_1 - c_2)sin3t - (c_1 + 3c_2)cos3t\}/2$

(TQ-4) $x = c_1 + c_2e^{-2t} + (\frac{2t}{3})$, $y = c_1 - c_2e^{-2t} + (\frac{1}{3})e^t$

(TQ-5) $x = c_1cost + c_2sint + (e^t - e^{-t})/2$

UNIT–10: LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

CONTENTS:

10.1 *INTRODUCTION:-*

A linear differential equation of second order with variable coefficients has a central role in the physical sciences. We know that the general solution or the complete solution of a linear differential equation contains two parts, first part is complementary function (C.F.) and second part is particular integral (P.I.) that is, Complete solution = C.F. + P.I. In the present unit,we shall study thelinear differential equation of 2^{nd} order with variable coefficients and will discuss some standard methods to find the general solution ofsuch equations, when one part of C.F. is known.

10.2 *OBJECTIVES :-*

After completing this unit, students will be able to –

- understand the standard form of linear differential equation of second order
- find the complete solution of a linear differential equation of second order
- reduce the a linear differential equation into a normal form
- use the method of variation of parameters to find the complete solution of a linear differential equation of second order

10.3 *STANDARD FORM OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH VARIABLE COFFICENTS*

A differential equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

where P, Q and X are functions of x only, are known as a linear differential equation of second order with variable coefficients. There is no general rule to solve such type of differential equations. In this chapter, we shall discuss some important methods by which we will able to solve such type of differential equations.

10.4 *COMPLETE SOLUTION IN TERMS OF A KNOWN INTEGRAL*

We begin with the following theorem which has great importance in linear differential equation of second order:

Theorem: If an integral (solution), which is the part of the complementary function of linear differential equation of second order be known then the complete solution or the general solution can be derive in terms of the known integral.

Proof: Let $y = u$ be a known integral in the complementary function of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \qquad (1)$$

That is, it is a solution of(1), therefore $y = u$ will satisfy the equation (1),

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0. \qquad (2)$$

On substituting $y = uv$, we get

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}.$$

Putting these values in equation(1),we obtain

$$\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}.\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = X.$$

We can rewrite the above equation such as

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{du}{dx} + Pu\right) + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = X$$

From the above equation and in view of equation (2), we get

$$u\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{du}{dx} + Pu\right) = X,$$

or

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}.\frac{du}{dx}\right)\frac{dv}{dx} = \frac{X}{u} \qquad (3)$$

Letting $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx} = \frac{dp}{dx}$ in equation(3), we get

$$\frac{dp}{dx} + \left(P + \frac{2}{u}.\frac{du}{dx}\right)p = \frac{X}{u},$$

which is a linear differential equation with p as dependent variable.Therefore

$$I.F. = e^{\int\left(P + \frac{2}{u}.\frac{du}{dx}\right)dx}$$

$$= e^{(2\log u + \int P dx)} = u^2 e^{\int P dx},$$

and the solution of equation (3) is

$$pu^2e^{\int pdx} = \int \left[\frac{X}{u}.u^2e^{\int Pdx}\right]dx + C_1$$

$$p = \frac{dv}{dx} = \frac{C_1e^{-\int Pdx}}{u^2} + \frac{e^{-\int Pdx}}{u^2}\int uXe^{Pdx}\,dx. \qquad (4)$$

Integrating both side of the equation (4), we get

$$v = C_2 + C_1\int\frac{\int e^{-\int pdx}}{u^2}\,dx + \int\left[\frac{e^{-\int Pdx}}{u^2}\int uXe^{\int Pdx}dx\right]dx.$$

Thus, the solution of equation (1) is

$$y = uv = C_2\,u + C_1\,u\int\frac{e^{-\int pdx}}{u^2}dx + u\int\left[\frac{e^{-\int pdx}}{u^2}\int uXe^{\int\, pdx}\;dx\right]dx \qquad (5)$$

Since equation (5) contains two arbitrary constants therefore it is the complete solution of the given equation in terms of the known integral.

10.5 *TO FIND AN INTEGRAL IN COMPLIMENTARY FUNCTIONBY INSPECTION*

Consider a linear differential equation of the second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X. \qquad (1)$$

(i). If $y = e^{mx}$ is a solution of (1) then $m^2 + Pm + Q = 0$.

Let$y = e^{mx}$. Then $\frac{dy}{dx} = me^{mx}$and$\frac{d^2y}{dx^2} = m^2e^{mx}$. If$y = e^{mx}$is a solution of equation (1) then$(m^2 + Pm + Q)e^{mx} = 0$ implies that$m^2 + Pm + Q = 0$.

(ii). If $y = x^m$ is a solution of (1) then $m(m-1) + Pmx + Qx^2 = 0$.

We have $y = x^m$ then $\frac{dy}{dx} = mx^{m-1}$ and $\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$. If $y = x^m$ is a solution of equation (1) then $m(m-1)x^{m-2} + P\,mx^{m-1} + Qx^m = 0$ implies that $m(m-1) + Pmx + Qx^2 = 0$

From the above results, we can conclude the following:

(iii). If$y = x$is a solution of (1) then $P + Qx = 0$.

(iv). If$y = x^2$ is a solution of (1) then $2 + 2Px + Qx^2 = 0$.

(v). If$y = e^x$is a solution of (1)then$1 + P + Q = 0$.

(vi). If $y = e^{-x}$ s a solution of (1) then$1 - P + Q = 0$.

(vii). If$y = e^{ax}$ s a solution of (1) then$\frac{P}{a} + \frac{Q}{a^2} = 0$.

ILLUSTRATIVEEXAMPLES:

Example 1: Find thecomplete solution of the following differential equation:

$$x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3.e^x$$

Solution: Writing the above equation in the standard form as

$$\frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x}+\frac{2}{x^2}\right)y = xe^x \qquad (1)$$

Comparing equation (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

we get $P = -\left(1+\frac{2}{x}\right)$, $Q = \left(\frac{1}{x}+\frac{2}{x^2}\right)$ and $X = xe^x$. Here

$$P + Q\,x = -\left(1+\frac{2}{x}\right) + x\left(\frac{1}{x}+\frac{2}{x^2}\right) = 0.$$

Hence $y = x$ is a part of the C.F. of the solution of equation (1).

Letting $y = vx$, we get

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \text{ and } \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.$$

Putting these values in equation (1), we get

$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x.$$

Let $\frac{dv}{dx} = p$, then

$$\frac{dp}{dx} - p = e^x.$$

This is a linear differential equation in p, therefore its integrating factor will be

$$\text{I.F.} = e^{\int -dx} = e^{-x}$$

and solution will be

$pe^{-x} = \int e^x.e^{-x}\,dx + C_1 = x + C_1$ (or)

$$p = \frac{dv}{dx} = xe^x + C_1e^x.$$

Integrating the above equation with respect to x, we get

$$v = xe^x - e^x + C_1e^x + C_2.$$

Hence,the complete solution of equation (1) is

$$y = vx = x^2e^x - xe^x + C_1xe^x + C_2x.$$

Example 2: Find a complete solution of the following

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0.$$

Solution: The given equation can be written as

$$\frac{d^2y}{dx^2} - \left(2-\frac{1}{x}\right)\frac{dy}{dx} + \left(1-\frac{1}{x}\right)y = 0. \qquad (1)$$

Comparing equation (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

we have $P = -\left(2-\frac{1}{x}\right)$, $Q = \left(1-\frac{1}{x}\right)$ and $X = 0$. Here, since $1 + P + Q = 1 - 2 + \frac{1}{x} + 1 - \frac{1}{x} = 0$, therefore $y = e^x$ is a part of the C.F. of the solution of (1).

Putting $y = ve^x$, we get

$$\frac{dy}{dx} = \frac{dv}{dx}e^x + ve^x$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^x + 2\frac{dv}{dx}e^x + ve^x.$$

Using the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation(1), we get

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0.$$

Let $\frac{dv}{dx} = p$, then

$$\frac{dp}{dx} + \frac{p}{x} = 0 \quad or \frac{dp}{p} = -\frac{dx}{x}$$

or $log\, p = -\, log\, x + log\, C_1$

or $\quad p = \frac{dv}{dx} = \frac{C_1}{x} or v = C_1 \log x + C_2$.

Hence, the complete solution of equation (1) will be

$$y = ve^x = (C_1 \log x + C_2)e^x.$$

Example3: Solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$.

Solution: The above equation can be written in the form

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x} + 1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2} + \frac{1}{x}\right) y = x. \qquad (1)$$

Comparing equation (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

we have $P = -2\left(\frac{1}{x} + 1\right)$, $Q = 2\left(\frac{1}{x^2} + \frac{1}{x}\right)$ and $X = x$. Here, since $P + Q\,x = 0$, therefore $y = x$ is part of the C.F. of (1).

Putting $y = vx$, $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $\frac{d^2y}{dx^2}$

$$= x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} \text{ in equation (1), we get } \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1.$$

Let $p = \frac{dv}{dx}$. Then we get

$$\frac{dp}{dx} - 2p = 1,$$

which is linear differential equation in p. Therefore its integrating factor will be

$$I.F. = e^{\int -2dx} = e^{-2x}$$

and its solution is

$$pe^{-2x} = \int 1\,.\,e^{-2x}dx + C_1 = -\frac{1}{2}e^{-2x} + C_1 \text{(or)}$$

$$p = \frac{dv}{dx} = -\frac{1}{2} + C_1e^{2x}.$$

Integrating both side with respect to x, we get

$$v = -\frac{1}{2}x + \frac{1}{2}C_1 e^{2x} + C_2.$$

Hence the complete solution of equation (1) is

$$y = vx = -\frac{1}{2}x^2 + \frac{1}{2}C_1 x e^{2x} + C_2 x.$$

Example4: Solve$(x+1)\frac{d^2y}{dx^2} - 2(x+3)\frac{dy}{dx} + (x+6)y = e^x$.

Solution: The above equation can be written as

$$\frac{d^2y}{dx^2} - \frac{2(x+3)}{x+1}\frac{dy}{dx} + \frac{x+5}{x+1}\, y = \frac{e^x}{x+1}\,. \qquad (1)$$

Comparing equation (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

we have$P = -\frac{2(x+3)}{x+1}$, $Q = \frac{x+5}{x+1}$ and $X = \frac{e^x}{x+1}$. Here, since $1 + P + Q = 0$.

Therefore, $y = e^x$ is a part of the C.F. of equation (1).

By putting$y = ve^x, \frac{dy}{dx} = e^x\frac{dy}{dx} + ve^x$ and $\frac{d^2y}{dx^2} = e^x\frac{d^2v}{dx^2} + 2e^x\frac{dv}{dx} + ve^x$ in (1), we get

$$e^x\left(\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + v\right) - \frac{2(x+3)}{x+1}e^x\left(\frac{dv}{dx} + v\right) + \frac{x+5}{x+1}ve^x = \frac{e^x}{x+1},$$

$$\frac{d^2y}{dx^2} + \left\{2 - \frac{2(x+3)}{x+1}\right\}\frac{dy}{dx} = \frac{1}{x+1},$$

$$\frac{d^2v}{dx^2} - \frac{4}{x+1}\frac{dv}{dx} = \frac{1}{x+1}.$$

Putting $\frac{dv}{dx} = p$, weget

$$\frac{dp}{dx} - \frac{4}{x+1}\, p = \frac{1}{x+1}.$$

This is a linear differential equation in p and its integrating factor is

$$\text{I. F} = e^{\int\left(-\frac{4}{x+1}\right)dx} = \frac{1}{(x+1)^4}\,.$$

Hence$p.\frac{1}{(x+1)^4} = \int \frac{1}{(x+1)^5}dx + C_1$(or)

$$p.\frac{1}{(x+1)^4} = -\frac{1}{4(x+1)^4} + C_1 \ (or)$$

$$p = \frac{dv}{dx} = -\frac{1}{4} + C_1(x+1)^4,$$

which gives $v = -\frac{1}{4}x + \frac{C_1}{5}(x+1)^5 + C_2$.

Hence the complete solution of equation (1) is

$$y = ve^x = \left[-\frac{1}{4}x + \frac{C_1}{5}(x+1)^2 + C_2\right]e^x.$$

Example 5: Solve $x\frac{d^2y}{dx^2}-(2+x)\frac{dy}{dx}+2y=x^3$.

Solution: Given equation can be written as

$$\frac{d^2y}{dx^2}+\left(-\frac{2}{x}-1\right)\frac{dy}{dx}+\frac{2}{x}y=x^2 . \qquad (1)$$

Here, $P=\left(-\frac{2}{x}-1\right)$, $Q=\frac{2}{x}$ and $R=x^2$, which shows that $1+P+Q=0$.

It means $u=e^x$ is C.F. or known function. Let $y=uv$ i.e., $y=e^x v$.Then

$$\frac{dy}{dx}=e^x v'+e^x v \text{ and } \frac{d^2y}{dx^2}=e^x v''+2e^x v'+e^x v.$$

Using these values in equation (1) we get

$$\frac{d^2v}{dx^2}+\left(1-\frac{2}{x}\right)\frac{dv}{dx}=x^2e^{-x} \quad . \qquad (2)$$

Let $\frac{dv}{dx}=p$. Then $\frac{d^2v}{dx^2}=\frac{dp}{dx}$ and equation (2) becomes

$$\frac{dp}{dx}+\left(1-\frac{2}{x}\right)p=x^2e^{-x}, \qquad (3)$$

which is linear differential equation of first order in p therefore its integration factor is

$$\text{I.F}=e^{\int(1-\frac{2}{x})dx}=e^x\frac{1}{x^2} .$$

Then the solution of equation (3) will be

$$p.e^x.\frac{1}{x^2}=\int x^2e^{-x}.e^x.\frac{1}{x^2}dx+C_1 \text{(or)}$$

$$\frac{dv}{dx}.p=x^3e^{-x}+C_1x^2e^{-x}.$$

Integration both side of the above equation we have

$$v=-x^3e^{-x}-(3+C_1)x^2e^{-x}-2(3+C_1)(x+1)e^{-x}+C_2.$$

Hence the complete solution of equation is

$$y=uv=-x^3-(3+C_1)x^2-2(3+C_1)(x+1)+C_2e^x.$$

EXERCISE

1. $x\frac{(d^2y)}{dx^2}-2(1+x)\frac{dy}{dx}+(x+2)y=(x-2)e^x$
2. $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$
3. $\frac{d^2y}{dx^2}+(1-cotx)\frac{dy}{dx}-ycotx=sin^2x.$
4. $\sin^2 x\frac{d^2y}{dx^2}=2y$, given that $y=cotx$ is a solution.
5. $x\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}+(x+2)y=(x-2)e^{2x}$

ANSWERS

1. $y=e^{2x}+\left(\frac{C_1}{3}x^3+C_2\right)e^x$
2. $y=1+C_1x\int x^{-2}e^{\frac{x^3}{3}}\,dx+C_2x.$
3. $y=-\frac{1}{10}(sin2x-2cos2x)+\frac{C_1}{2}(sinx-cosx)+C_2e^{-x}.$

4. $y = C_1 - C_1 x cot x + C_2 \cot x$.
5. $y = \frac{1}{3} C_1 x^3 e^x + C_2 e^x + e^{2x}$.

10.6 *METHOD TO REMOVE OF THE FIRST DERIVARIVE (REDEUCTION TO THE NORMAL FORM)*

Sometimes it is difficult to find a part of the C.F. of solution of the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X. \qquad (1)$$

by inspection. Hence the method learned will not be useful. Here, we use the second method that is, reducing the differential equation to the normal form, in which the term containing the first derivative will be absent. For this we will change the dependant variable y to v in the equation (1) by letting$y = uv$, where$u$ is some function of x. Now,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}.\frac{du}{dx} + u\frac{d^2v}{dx^2}.$$

Substituting these values in equation (1), we get

$$\left\{v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}.\frac{du}{dx} + u\frac{d^2v}{dx^2}\right\} + P\left\{v\frac{du}{dx} + u\frac{dv}{dx}\right\} + Quv = X,$$

$$\text{or}\, u\frac{d^2v}{dx^2} + u\frac{dv}{dx}\left[P + \frac{2}{u}.\frac{du}{dx}\right] + v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = X. \qquad (2)$$

To remove the first derivative term, we will substitute the value of u such that

$$P + \frac{2}{u}\frac{du}{dx} = 0 \quad \text{or}\, \frac{du}{u} = -\frac{P}{2}dx.$$

Integrating this, we will have

$$\log u = -\int\left(\frac{P}{2}\right)dx \quad \text{or} \quad u = e^{\left\{-\int\left(\frac{P}{2}\right)dx\right\}}. \qquad (3)$$

Now, equation (2) reduces to

$$\frac{d^2v}{dx^2} + \frac{v}{u}\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] = \frac{X}{u}. \qquad (4)$$

But from equation (3), we get

$$\frac{du}{dx} = -\frac{P}{2}u,$$

and

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right] = -\frac{1}{2}\left[P\left(-\frac{1}{2}Pu\right) + u\frac{dP}{dx}\right] = \frac{1}{4}P^2u - \frac{u}{2}\frac{dP}{dx}$$

Again substituting these values in (4), we get

$$\frac{d^2v}{dx^2}+v\left[\frac{1}{4}P^2-\frac{P^2}{2}+Q-\frac{1}{2}\frac{dP}{dx}\right]=X.e^{\left\{\int\left(\frac{1}{2}Pdx\right)\right\}}$$

or $\frac{d^2v}{dx^2}+v\left[Q-\frac{1}{4}p^2-\frac{1}{2}\frac{dP}{dx}\right]=Xe^{\left\{\int\frac{1}{2}pdx\right\}}$. (5)

This reduced equation can easily be integrated if $Q-\frac{P^2}{4}-\frac{1}{2}\frac{dP}{dx}=\text{constant}$ or $\frac{1}{x^2}$. The equation (5) is called the normal form of the equation(1). One should remember the values of u, X and Y so that they can write the reduced equation directly.

ILLUSTRATIVE EXAMPLES

Example 1: Solve$x^2\frac{d^{2y}}{dx^2}-2(x^2+x)\frac{dy}{dx}+(x^2+2x+2)y=0.$

Solution: The given equation in the standard will be

$$\frac{d^2y}{dx^2}-\frac{2(x+1)}{x}\frac{dy}{dx}+(x^2+2x+2)y=0.$$

Here, $P=-\frac{2(x+1)}{x}, Q=\frac{x^2+2x+2}{x^2}$ and $X=0.$

Now, $u=e^{\left\{-\frac{1}{2}\int Pdx\right\}}=e^{\left\{-\frac{1}{2}\int -2\left(1+\frac{1}{x}\right)dx\right\}}=xe^x$

By substituting$y=\ uv$, the normal form of the given equation becomes

$$\frac{d^2v}{dx^2}+v\left(Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right)=X.e^{\left\{\frac{1}{2}\int pdx\right\}}$$

or $\frac{d^2v}{dx^2}+v\left\{\frac{x^2+2x+2}{x^2}-\frac{4}{x^2}.\frac{(1+x)^2}{4}-\frac{1}{2}.\frac{2}{x^2}\right\}=0$

or$\frac{d^2v}{dx^2}+\frac{v}{x^2}\{(x^2+2x+2)-(1+x^2+2x)-1\}=0$

or $\frac{d^2v}{dx^2}=0.$

Integrating it again, we get

$$\frac{dv}{dx}=C_1\text{and } v=C_1x+C_2.$$

So, the general solution of the given equation is

$$y=uv=(C_1x+C_2)xe^x.$$

Example2: Solve $\frac{d^2y}{dx^2}+\frac{1}{x^{\frac{1}{3}}}\frac{dy}{dx}+\left(\frac{1}{4x^{\frac{2}{3}}}-\frac{1}{6x^{\frac{5}{3}}}-\frac{6}{x^2}\right)y=0.$

Solution: From the given equation, we deduce

$$P=\frac{1}{x^{\frac{1}{3}}}, Q=\frac{1}{4x^{\frac{2}{3}}}-\frac{1}{6x^{\frac{4}{3}}}-\frac{6}{x^2}\text{and } X=0.$$

Now,

$$u=e^{-\int(p/2)dx}=e^{-\int\left(\frac{1}{2}x^{-1/3}\right)dx}=e^{-\left(\frac{3}{4x^{2/3}}\right)}$$

By substituting the value of u, the normal form of the given equation is

$$\frac{d^2y}{dx^2}+v\left[Q-\frac{1}{4}p^2-\frac{1}{2}\frac{dP}{dx}\right]=Xe^{\frac{\int p}{2}dx}$$

or $\frac{d^2v}{dx^2} + v\left[\left(\frac{1}{4x^{\frac{2}{3}}} - \frac{1}{6x^{\frac{5}{3}}} - \frac{6}{x^2}\right) - \frac{1}{4x^{\frac{2}{3}}} - \frac{1}{2}(-\frac{1}{3x^{\frac{4}{3}}})\right] = 0$

or $\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0$

or $x^2\frac{d^2v}{dx^2} - 6v^2 = 0.$

This is a homogeneous equation. To solve this, we put $x = e^z$ and let $\frac{d}{dz} = D$, then

$$D(D-1)v - 6v = 0,$$
$$(D+2)(D-3)v = 0.$$

Its auxiliary equation is$(m+2)(m-3) = 0$,gives$m = -2, 3.$

$$\therefore v = C_1e^{-2z} + C_2e^{3z} = C_1\frac{1}{x^2} + C_2x^3.$$

Hence the complete solution is

$$y = uv = \left(\frac{C_1}{x^2} + C_2x^3\right)e^{\left(-\frac{3}{4}x^{\frac{2}{3}}\right)}.$$

Example 3: Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}sin2x.$

Solution: Here, $P = -4x,\quad Q = 4x^2 - 1$ and $X = -3e^{x^2}sin2x$

We will choose$u = e^{\left\{-\frac{1}{2}\int pdx\right\}} = e^{\left\{-\frac{1}{2}\int(-4xdx)\right\}} = e^{x^2}.$

On substituting $y = uv$ in the original equation, it reduces to

$$\frac{d^2v}{dx^2} + v\left[Q - \frac{1}{4}p^2 - \frac{1}{2}\frac{dP}{dx}\right] = Xe^{\left\{\frac{1}{2}\int pdx\right\}}$$

or $\frac{d^2v}{dx^2} + v\left[(4x^2-1) - 4x^2 - \frac{1}{2}(-4)\right]$

$$= -3e^{x^2}sin2x\,.\,e^{\frac{1}{2}\int -4xdx}$$

$$\frac{d^2v}{dx^2} + v = -3\,sin2x.$$

Auxilliary equation is$m^2 + 1 = 0$ $i.e.$ $m = \pm i.$

Hence, $\quad$ C.F.$= C_1cosx + C_2sinx.$

$$\text{P.I.} = \frac{1}{D^2+1}(-3sin2x) = \frac{1}{-4+1}(-3sin2x) = sin2x$$

and$v = C_1cosx + C_2sinx + sin2x.$

Hence, the complete solution is given by

$$y = uv = (C_1cosx + C_2sinx + sin2x)e^{x^2}.$$

Example 4: Solve $\frac{d^2y}{dx^2} + 2\,tanx\frac{dy}{dx} + y(1 + 2\,tan^2 x) = secx.tanx.$

Solution: Here, we have

$$P = 2tanx, \quad Q = (1 + \tan^2 x)\text{and } X = secx\,tanx$$

Let $\quad u = e^{\left\{-\frac{1}{2}\int pdx\right\}} = e^{\left\{-\frac{1}{2}\int tanxdx\right\}} = cosx.$

By putting $y = uv$, we get the normal form of the equation as

$$\frac{d^2v}{dx^2}+v\left[Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right]=Xe^{\left\{\frac{1}{2}\int pdx\right\}}$$

or $\frac{d^2v}{dx^2}+v\left[1+2\,tan^2\,x-\frac{1}{4}.4\,tan^2\,x-\frac{1}{2}.2\,sec^2\,x\right]=sec^2\,x\,tanx$

or $\frac{d^2v}{dx^2}=sec^2\,x\,tanx$

Integrating ittwice, we get

$$v=\frac{1}{2}(tanx-x)+C_1x+C_2$$

Hence, complete solution will be

$$y=vu=cosx\left\{\frac{1}{2}tanx+C_1'x+C_2\right\}.$$

Example 5: Solve$y''-4xy'+(4x^2-3)y=e^{x^2}$.

Solution: Here, $P=\ -4x, Q=4x^2-3,\ X=e^{x^2}$

Now, $Q-\frac{1}{2}.\frac{dP}{dx}-\frac{P^2}{4}=4x^2-3-\frac{1}{2}(-4)-\frac{16x^2}{4}=-1=$ constant.

Normal form of equation is

$$\frac{d^2v}{dx^2}-v=Y;\quad Y=\frac{X}{u};\quad u=e^{-\int\left(\frac{p}{2}\right)dx}$$

$i.e.$ $\quad v''-v=1;\quad u=e^{x^2}$

Auxiliary equation$m^2-1=0\quad\Rightarrow\quad m=\pm 1$.

C.F.=$C_1e^x+C_2e^{-x}$and $\text{P.I.}=\frac{1}{D^2-1}=-1$

Hence, general solution is

$$v=C_1e^x+C_2e^{-x}-1$$

and complete solution of given equation is

$$y=uv=e^{x^2}(C_1e^x+C_2e^{-x}-1)$$

Example 6: Solve$y''-2tanxy'+5y=secx\,e^x$

Solution: Here, $P=-2\,tanx,\ \ Q=5$ and $X=e^xsecx$

Now, $\quad Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}=5+sec^2\,x-tanx=6=$ constant

Normal form of equation is

$$v''+6v=Y,\quad Y=\frac{X}{u}\text{and }u=e^{\int-(\frac{p}{2})dx}\ =secx.$$

$i.e.$ $\quad v''+6v=e^x,\quad u=secx$

which is linear with constant coefficients. Therefore

$$\text{A. E.}=m^2+6=0\ i.e,\ \ m=\pm\sqrt{6}i$$

Hence $C.F.=\left(C_1cos\sqrt{6x}+C_2sin\sqrt{6x}\right),\qquad P.I.=\frac{e^x}{D^2+6}=\frac{e^x}{7}$ and

$$v=\left(C_1cos\sqrt{6x}+C_2sin\sqrt{6x}\right)+\frac{e^x}{7}.$$

Hence, the complete solution of given equation is

$$y=uv=secx[\left(C_1cos\sqrt{6x}+C_2sin\sqrt{6x}\right)+\frac{e^x}{7}.$$

10.7 *TRANSFORMATION OF THE EQUATION BY CHANGING THE INDEPENDENT VARIABLE*

Sometimes it is suitable to find the solution of second order differential equation by changing the independent variable by some suitable substitution.

Let the differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X. \qquad (1)$$

By changing the independent variable from x to z where z is a function of x we get

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

and

$$= \frac{d}{dx}\left(\frac{dy}{dz}.\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}.$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$\left\{\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}.\frac{d^2z}{dx^2}\right\} + P\left\{\frac{dy}{dz}.\frac{dz}{dx}\right\} + Qy = X.$$

or $$\frac{d^2y}{dz^2} + \left\{\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right\}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}\ y = \frac{X}{\left(\frac{dz}{dx}\right)^2}$$

or $$\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + Q_1y = X_1, \qquad (2)$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } X_1 = \frac{X}{\left(\frac{dz}{dx}\right)^2} \qquad (3)$$

P_1, Q_1, X_1 are the functions of x but can be expressed in form of z by the help of given relation between z and x.

1. Now , we will choose z in such a way so that that P_1 vanishes ,that is

$$\frac{d^2z}{dx^2} + p\frac{dz}{dx} = 0 \quad \text{(or)}$$

$$z = \int e^{\{-\int p dx\}}dx.$$

Then the equation (1) will convert into

$$\frac{d^2y}{dx^2} + Q_1y = X_1.$$

This equation is solvable if Q comes out to be a constant or of the form $\frac{\text{constant}}{z^2}$

2. We may choose a suitable z in such a way that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ is constant then

$$\frac{Q}{\left(\frac{dz}{dx}\right)^2} = C^2 \text{or} \qquad C\frac{dz}{dx} = \sqrt{Q}$$

$$\therefore \qquad Cz = \int \sqrt{Qdx}.$$

With this substitution of z in equation (1), we have

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + C^2y = X_1.$$

This differential equation can be solved if P_1 also comes out to be a constant.

ILLUSTRATIVE EXAMPLES

Example 1: Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$.

Solution: The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2$$

Here $P = -\frac{1}{x}, Q = -4x^2$ and $X = 8x^2 \sin x^2$. Changing the independent variable x to z, the given equation is transformed into.

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = X_1, \qquad (1)$$

where, $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $X_1 = \frac{X}{\left(\frac{dz}{dx}\right)^2}$.

Now, choose z in such a way that $Q_1 = -1$.

$$-1 = Q = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -\frac{4x^2}{\left(\frac{dz}{dx}\right)^2} \text{or} \quad \frac{dz}{dx} = 2x$$

implies that $z = x^2$. Substituting these values in equation (1), we get

$$\frac{d^2y}{dz^2} + \frac{2 - \frac{1}{x}.2x}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} - y = \frac{8x^2 \sin x^2}{4x^2} = 2\sin z$$

(or) $\frac{d^2y}{dz^2} - y = 2\sin z$.

Hence the auxiliary equation is $m^2 - 1 = 0, m = \pm 1$ and C. F. is $C_1e^z + C_2e^{-z}$.

$$\text{P.I.} = \frac{1}{D^2-1}2\sin z = -\sin z = -\sin x^2.$$

Therefore, complete solution is given by

$$y = C_1e^{x^2} - C_2e^{-x^2} \sin x^2.$$

Example 2: Solve $\frac{d^2y}{dx^2} + cotx\frac{dy}{dx} + 4\, cosec^2x\, y = 0$

Solution: On comparing with the standard form we have

$$P = cotx, \;\; Q = 4cosec^2x, \qquad X = 0.$$

Changing the independent variable from x to z by a relation of the form $z = f(x)$ the given equation is transferred into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = X_1\,, \qquad (1)$$

where$P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad X_1 = \dfrac{X}{\left(\frac{dz}{dx}\right)^2}.$

Substitute z such that $Q = \dfrac{4cosec^2x}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 4$ (say), we have

$$\left(\frac{dz}{dx}\right)^2 = cosec^2x, \text{ (or)}$$

$$\frac{dz}{dx} = cosecx \quad \text{(or)} \;\; z = log\, tan\frac{x}{2}.$$

Now, $$P_1 = \frac{-cosecx\, cotx + cotx\, cosecx}{cosec^2x} = 0, X_1 = 0.$$

The transformed equation (1) is

$$\frac{d^2y}{dz^2} + 4y = 0. \qquad (2)$$

$A.E.$ is$m^2 + 4 = 0 \;\; \Rightarrow m = \pm 2i$and the solution of the equation (2) is

$$y = C_1cos2z + C_2sin2z.$$

Hence, the complete solution of the given equation is

$$y = C_1\cos\left(2\log\tan\frac{x}{2}\right) + C_2\sin\left(2\log\tan\frac{x}{2}\right).$$

10.8 *METHOD OF VARIATION OF PARAMETERS*

Consider a second order linear differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \qquad (1)$$

where P, Q and X are the function of x or constant. Suppose that the general of the equation homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad (2)$$

be given by

$$y = Au + Bv,$$

where A and B are arbitrary constant and u and v are the function of x only. Since u and v will be solution of homogenous differential equation (2), therefore we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Q\,u = 0 \text{ and} \frac{d^2v}{dx^2} + P\frac{dv}{dx} + Q\,v = 0. \qquad (3)$$

Now suppose that

$$y = Au + Bv \qquad (4)$$

is the complete solution of equation (1), where A and B are function of x and chosen so that (1) is satisfied. Differentiating (4), we get

$$\frac{dy}{dx} = Au' + uA' + Bv' + B'v.$$

In order to find A and B , we take $A'u + B'v = 0.$ (5)

Therefore

$$\frac{dy}{dx} = Au' + Bv'. \qquad (6)$$

Differentiation (6), we get

$$\frac{d^2y}{dx^2} = Au'' + A'u' + Bv'' + B'v'. \qquad (7)$$

Putting the value of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$Au'' + Bv'' + A'u' + B'v' + P(Au' + Bv') + Q(Au + Bv) = X$$

or $$A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = X.$$

Using (3), we get

$$A'u' + B'v' = X. \qquad (8)$$

Solving equation (5) and (8) for A' and B', we get

$$A' = \frac{dA}{dx} = -\frac{vX}{uv' - vu'} \text{ and } B' = \frac{dB}{dx} = \frac{uX}{uv' - vu'}$$

or $A = \int \frac{-vX}{uv' - vu'} dx + C_1$ and $B = \int \frac{uX}{uv' - vu'} dx + C_2$.

By putting the value of A and B from the above equation in (4), the solution will be obtained. Since In this method, the arbitrary constants of the complementary function varies in order to obtain the general solution. Therefore this method is called as method of variation parameters. This method is very effective method to find a particular integral and it can be applied where the earlier methods are not applicable.

ILLUSTRATIVE EXAMPLES

Example 1: Apply method of variation of parameter:

$$(1 - x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1 + x^2)y = x .$$

Solution: We can convert the given equation in the form of

$$\frac{d^2y}{dx^2} - \frac{4x}{(1 - x^2)}\frac{dy}{dx} - \frac{(1 + x^2)}{(1 - x^2)} y = \frac{x}{(1 - x^2)} . \qquad (1)$$

Comparing the equation (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X,$$

We get $P = -\frac{4x}{(1 - x^2)}, Q = -\frac{(1 + x^2)}{(1 - x^2)}$ and $X = \frac{x}{(1 - x^2)}$.

Putting $y = ve^{\left(-\frac{1}{2}\int P du\right)} = ve^{\int 2x/(1 - x^2)dx} = \frac{v}{(1 - x^2)}$.

Then the given equation reduces to

$$\frac{d^2v}{dx^2} + v = x. \qquad (2)$$

The solution of the equation $\frac{d^2v}{dx^2} + v = 0$ is

$$v = A\cos x + B\sin x . \qquad (3)$$

Suppose A and B are function of x then

$$\frac{dv}{dx} = -A\sin x + B\cos x + \cos x\, A' + \sin x\, B'.$$

We choose A and B such that

$$\cos x\, A' + \sin x\, B' = 0. \qquad (4)$$

Then $\frac{dv}{dx} = -A\sin x + B\cos x$ and $\frac{d^2v}{dx^2} = -A\cos x - B\sin x - \sin x\frac{dA}{dx} + \cos xB'.$

Suppose that v in (3) satisfies the equation (2) then by putting the value of v and $\frac{d^2v}{dx^2}$ in (2), we get

$$-\sin x\, A' + \cos x\, B' = x. \qquad (5)$$

Solving equation (4) and (5), we obtain

$$A' = -x\sin x \quad \text{that is} \quad A = x\cos x - \sin x + C_1$$

and

$$B' = x\sin x + \cos x + C_1, \qquad \text{that is } B = x\sin x + \cos x + C_1.$$

Example 2 Find the solution of the following differential equation by using the method of variation of parameter.

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x .$$

Solution: The solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

is $y = A\cos 2x + B\sin 2x$, where A and B are constants. Let

$$y = A\cos 2x + B\sin 2x \qquad (1)$$

be the complete primitive of the given equation where A and B are function of x and chosen so that (1) satisfy the given equation. Then

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + A'\cos 2x + B'\sin 2x .$$

We choose A and B such that

$$A'\cos 2x + B'\sin 2x = 0. \qquad (2)$$

Then

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x \quad \text{and}$$

$$\frac{d^2y}{dx^2} = -2A'\sin 2x + 2B'\cos 2x - 4A\cos 2x - 4B\sin 2x .$$

Using these values in the given equation, we obtain

$$-2\sin 2x\, A' + 2\cos 2x\, B' = 4\tan 2x \quad \text{or}$$

$$-\sin 2x\, A' + \cos 2x\, B' = 2\tan 2x. \qquad (3)$$

Form (2) and (3), we obtain

$$A' = \frac{-2\sin^2 2x}{\cos 2x} \text{ and } B' = 2\sin 2x\ .$$

Integration theses, we get

$$A = -2\int \frac{(1-\cos^2 2x}{\cos 2x} dx + c_1$$

$$= -2\int\ \sec 2x - \cos 2x)\ dx + c_1$$

$$= -\log(\sec 2x + \tan 2x\ ) + \sin 2x + c_1$$

and $$B = -\cos 2x + c_2.$$

The complete solution of give equation is obtained by putting the values of A and B in equation (1),

$$\mathrm{y = c_1 \cos 2x + c_2 \sin 2x - [\log(\sec 2x + \tan 2x\ )]. \cos 2x.}$$

ANSWERS

1. $y = c_1e^x +\ c_2e^{-x} +\ e^x \log x \frac{1+e^x}{e^x} - 1 - e^{-x}\log(1+e^x)$
2. $y = c_1x + c_2x\ e^{2x} - \frac{1}{2}x^2 - \frac{1}{4}x$
3. $y = c_1(\sin x - \cos x\ ) + c_2e^{-x} - \frac{1}{10}(\sin 2x - 2\cos 2x\ )$
4. $y = c_1 \cos nx + c_2 \sin nx + \frac{1}{n^2}\cos nx \log cosnx + \frac{x}{n}\sin nx$
5. $y = c_1(2x+5) +\ c_2e^{2x} - e^x$

10.9 *METHOD OF OPERATIONAL FACTORS*

Let us consider a linear differential equation of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

where P, Q, R are the functions of x. Writing D for d/dx , then equation (1) can be rewrite in the form of

$$(D^2 + PD + Q)y = X \quad \text{or} \quad f(D)y = X.$$

Sometime it will be possible to factorise the $f(D)$ into two linear operators $f_1(D)$ and $f_2(D)$ such that if $f_2(D)$ operates upon y then $f_1(D)$ operates on the result of this operation and the same result is obtained as if $f(D)$ act on y. Since the factors of $f(D)$are not usually commutative hence a great care is to be taken in writing them in right order. W will illustrate this method with the help of following examples.

ILLUSTRATIVE EXAMPLES

Example 1. Factorise the operator on the left hand side of $xy'' - (x+2)y' + 2y = x^3$ and hence solve it.

Solution: We can rewrite the give equation as

$$[xD^2 - (x+2)D + 2 = x^3]$$

$$or \qquad (xD-2)(D-1)y = x^3. \qquad (1)$$

Suppose $(D-1)y = v$. Then (1) becomes

$$(xD-2)v = x^3$$
$$or\ x\frac{dv}{dx} - 2v = x^3$$
$$or\ \frac{dv}{dx} - \frac{2}{x} = x^2.$$

This is linear differential equation in y , therefore its integrating factor will be

$$I.F. = e^{\int(-2/x)dx} = e^{-2\log x} = \frac{1}{x^2},$$

and its solution will be obtained by

$$v\frac{1}{x^2} = \int\left(x^2\frac{1}{x^2}\right)dx + c_1 = x + c_1$$
$$or\ v = x^3 + c_1x^2.$$

Putting the value of v in $(D-1)y = v$, we get

$$(D-1)y = x^3 + c_1x^2$$
$$or\ \frac{dy}{dx} - y = x^3 + c_1x^2,$$

which is linear differential equation in y and therefore its integrating factor will be

$$I.F. = e^{\int -dx} = e^{-x}$$

and its solution will be obtained by

$$ye^{-x} = \int e^{-x}(x^3 + c_1x^2)dx + c_2$$
$$= -(x^3 + c_1x^2)e^{-x} - (3x^2 + 2c_1x)e^{-x} - (6x + 2c_1)e^{-x} - 6e^{-x} + c_2$$
$$y = -(x^3 + c_1x^2)e^{-x} - (3x^2 + 2c_1x) - (6x + 2c_1) - 6 + c_2e^x$$
$$= -x^3 - (c_1+3)x^2 - 2(c_1+3)x - 2(c_1+3) + c_2e^x$$
$$y = x^3 - (c_1 + 3)(x^2 + 2x + 2) + c_2e^x,$$

which is the general solution of the given differential equation.

Example 2 Solve $y'' + (1-x)y' - y = e^x$.

Solution : The given equation can be written as

$$[xD^2 + (1-x)D - 1]y = e^x$$
$$or \qquad (xD+1)(D-1)y = e^x. \qquad (1)$$

Let $(D-1)y = v$. Then (1) gives

$$(Dx+1)v = e^x. \qquad (2)$$

We first solve (2) that is,

$$x\frac{dv}{dx} + v = e^x,$$

which is linear differential equation y and its integrating factor will be

$$I.F. = e^{\int 1/x\, dx} = e^{logx} = x$$

And its solution will be obtain by

$$vx = \int x\left(\frac{1}{x}\right)e^x dx + c_1;$$

$c_1$being an arbitrary constant.

$$v = \frac{1}{x}e^x + \left(\frac{1}{x}\right)c_1.$$

Using this value of v in $(D-1)y = v$, we get

$$(D-1)y = \left(\frac{1}{x}\right)e^x + \left(\frac{1}{x}\right)c_1 \text{ or } \frac{dy}{dx} - y = \left(\frac{1}{x}\right)e^x + \left(\frac{1}{x}\right)c_1$$

which is again linear equation in y , so its integrating factor will be

$$I.F. e^{-\int dx} = e^{-x}.$$

Thus, its solution is given by

$$ye^{-x} = c_2 + \int\left(\frac{1}{x}e^x + \frac{1}{x}c_1\right)e^{-x}dx = c_2 + \log x + c_1\int\frac{e^{-x}}{x}dx$$

and the solution of the given equation is

$$y = c_2e^x + e^x \log x + c_1e^x\int\frac{e^{-x}}{x}dx.$$

SELF CHECK QUESTIONS

TRUE/FALSE

1. Solution of $3x^2y'' + (2+6x-6x^2)y' - 4y = 0$ is
$y = c_2e^{2x/3} + c_1e^{2x/3}\int(1/x^2)e^{2x-(2/3x)}dx.$

2. Solution of $xy'' + (x-1)y' - y = x^4$ is
$y = c_1e^{-x} + c_2(x-1) + \left(\frac{1}{3}\right)x^4 - \left(\frac{4}{3}\right)x^3 + 4x^2$

3. Solution of $(x+1)y'' + (x-1)y' - 2y = 0$ is
$y = c_1(x^2+1) + c_2e^{-x}$

4. Solution of $xy'' + (x-2)y' - 2y = x^3$ is
$y = x^3 + c_1(x^2-2x+2) + c_2e^{-x}$

5. Solution of $x^2y'' + y' - (1+x^2)y = e^{-x}$ is
$y = c_1e^x\int e^{-2x+\frac{1}{x}}dx + c_2e^x - \frac{1}{2}e^{-x}.$

10.10 *SUMMARY*

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations.

10.11 *GLOSSARY*

- **Differential Equation:** An equation involving derivatives of a function with respect to one or more independent variables.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, referring to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Partial Differential Equation (PDE):** A differential equation involving partial derivatives with respect to multiple independent variables.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that stands alone and is not affected by other variables.
- **Rate of Change:** The speed at which a quantity changes with respect to another variable.
- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Derivative:** A measure of how a function changes as its input changes.
- **Exact Differential Equation:** An ordinary differential equation (ODE) of the form $M(x,y)dx + N(x,y)dy = 0$ where M and N are functions of both x and y, such that the equation can be derived from a scalar potential function $F(x,y)$ as $\partial x/\partial F(dx) + \partial y/\partial F(dy) = 0$.
- **Exactness:** The property of a differential equation where the equation is derived from a scalar potential function, making the equation exact.
- **Homogeneous Differential Equation:** A differential equation in which all terms involving the dependent variable and its derivatives are of the same order.
- **Non-homogeneous Differential Equation:** A differential equation in which terms involving the dependent variable and its derivatives are of different orders.

- **Linear Differential Equation:** A differential equation in which the dependent variable and its derivatives appear linearly, without being raised to powers or involved in nonlinear functions.
- **Nonlinear Differential Equation:** A differential equation in which the dependent variable or its derivatives appear nonlinearly, such as being raised to powers or involved in nonlinear functions.

Understanding these terms is crucial for grasping the concepts and techniques involved in working with first-order, first-degree differential equations.

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10.14 *TERMINAL QUESTION*

(TQ-1) Solve the following differential equations

1. $\frac{d^2y}{dx^2} - \boldsymbol{y} = \frac{\mathbf{2}}{\mathbf{1} + \boldsymbol{e^x}}$

2. $x^2\frac{d^2y}{dx^2} - 2x\,(1+x)\frac{dy}{dx} + 2(x+1)y = x^3$

3. $\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = \sin^2 x$

4. $\frac{d^2y}{dx^2} + n^2y = \sec nx$

5. $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

10.15 *ANSWERS*

SELF CHECK ANSWERS

1. True
2. True
3. True
4. True
5. True

TERMINAL ANSWERS(TQ'S)

1. $y = c_1e^x + c_2e^{-x} + e^x\log x\frac{1+e^x}{e^x} - 1 - e^{-x}\log(1+e^x)$

2. $y = c_1x + c_2x\,e^{2x} - \frac{1}{2}x^2 - \frac{1}{4}x$

3. $y = c_1(\sin x - \cos x) + c_2e^{-x} - \frac{1}{10}(\sin 2x - 2\cos 2x)$

4. $y = c_1 \cos nx + c_2 \sin nx + \frac{1}{n^2} \cos nx \log cosnx + \frac{x}{n} \sin nx$

5. $y = c_1(2x + 5) + c_2 e^{2x} - e^x$

BLOCK-IV

UNIT 11: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE - I

CONTENTS:

11.1 INTRODUCTION

The present chapter is devoted to the differential equations of first order and higher degree, which arises in many field of physical and biological sciences and all are non-linear in nature. From analysis point of view it introduces an interesting type of solution called **"singular"** solution. Before going into details, we shall introduce the definition of direction field and geometrical interpretation of differential equation of first order and higher degree.

11.2 OBJECTIVES

After studying this block, you should be able to understands :

i. Geometrical interpretation of differential equation of first order and first/higher degree

ii. Methods for solving differential equations of first degree and higher degree.

iii. Clairaut's differential Equations.

11.3 DIRECTION FIELD

A directions field (slope field) is a mathematical object used to graphically represented solutions to the first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

For example, we consider a differential equation

$$y' = \frac{dy}{dx} = y - x \qquad \textbf{... (1)}$$

and a space [–2, 2] × [–2, 2] ≤ $\mathbf{R}^2$. Suppose x = {–2, –1, 0, 1, 2} and y = {–2, –1, 0, 1, 2} i.e., 25 coordinate points. Plugging each of these 25 points into differential equation (1) to find an associated value for y' i.e.,

$x = -2$

y	–2	–1	0	1	2
y'	0	1	2	3	4

$x = -1$

y	–2	–1	0	1	2
y'	–1	0	1	2	3

$x = 0$

y	–2	–1	0	1	2
y'	–2	–1	0	1	2

$x = 1$

y	–2	–1	0	1	2
y'	–3	–2	–1	0	1

$x = 2$

y	–2	–1	0	1	2
y'	–4	–3	–2	–1	0

In order to sketch this information into the direction field, we navigate to the point (x, y) and then sketch the tiny line that has slope equal to the corresponding value y'. For y' = 0, the small line, will be horizontal, for $y' < 0$, the small line will be tilting to the right and for y' > 0, the small line will be tilting to the left. For larger the value y' either positive or negative, the steeper the line will be.

In order to get a clearer picture of the direction field, we should complete more values except for {– 2, – 1, 0, 1, 2}× {–2, –1, 0, 1, 2} points.

The solution of differential equation passing through (1, 1) is

$$y = x - e^{x-1} + 1 \qquad \textbf{... (2)}$$

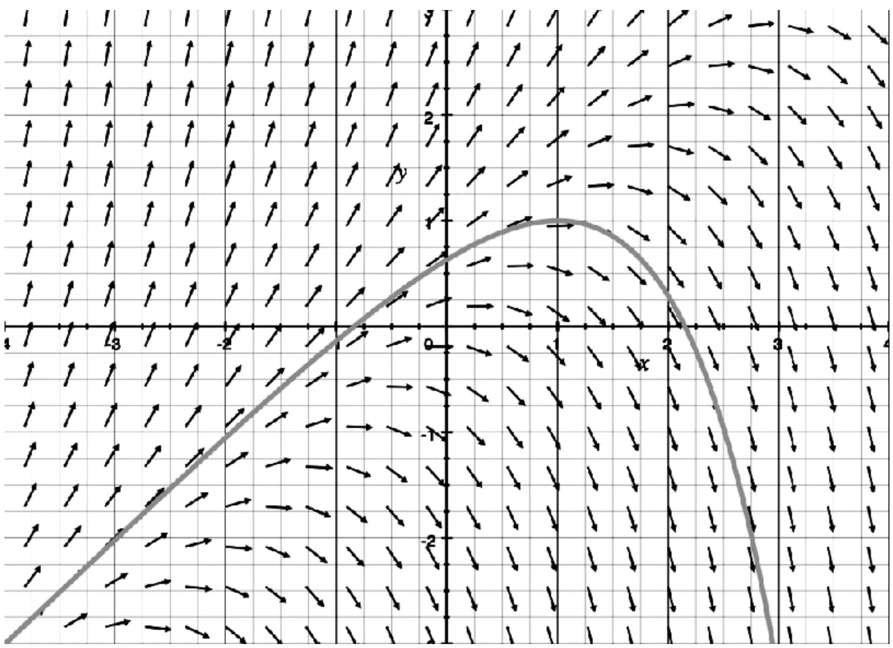

Fig. 11.1 Direction Field of $f\,(x,\,y) = y - x$

Similarly, a direction field for the function f (x, y) = - xy; y >:0 and solution curve for the differential equation.

$$y' = -\,x\,y;\; x > 0$$

Passing through (0, 1) is given below

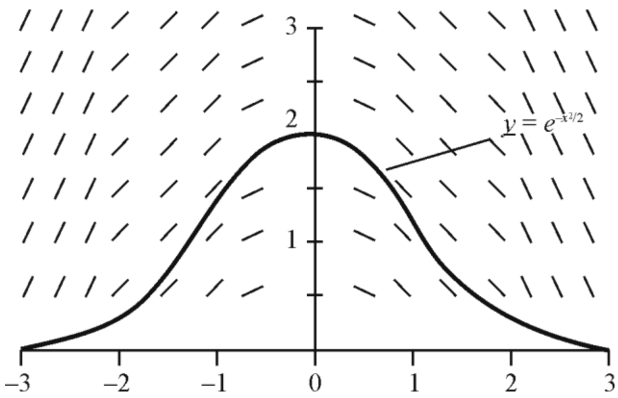

Fig.11.2. The direction Field of the function $f(x, y) = -xy\,(y > 0)$

11.4 GEOMETRICAL INTERPRETATION OF A DIFFERENTIAL EQUATION OF FIRST ORDER AND HIGHER DEGREE

In figure of previous section, it has been observed that at any point (x, y), y' have only one value (one direction) because y' appears in the one degree.

Thus, in case of second degree equation y' appears in the second degree and at any point (x, y), y' will have two direction (two values). Hence the moving point can pass through it in either of these direction i.e., there arise two curves of the problem which will be the locii of a general solution passing through the point (x, y).

If $f(x, y, c) = 0$ is the general solution of a differential equation of order one and degree two, then arbitrary constant c will appear in the second degree at any point (x, y).

In case of n degree differential equation, there are n values of y' (directions) at any point (x, y) and therefore there are n curves of the problem which will be the locii of a general solution passing through the point (x, y) and degree of arbitrary constant c in general solution will be n.

11.5 MATHEMATICAL REPRESENTATION OF DIFFERENTIAL EQUATION OF ORDER ONE AND HIGHER DEGREE

If n is any positive integer greater than one and if $p = y' = \frac{dy}{dx}$ then a differential equation of order one and degree n may be expressed as

$$a_0 p^n + a_1 p^{n+1} + \ldots + p_{n-1} p + a_n = 0 \quad \textbf{... (3)}$$

Where $a_i = a_i(x);\, i = 0, 1, 2, \ldots n$ are real valued continuous functions defined on some interval $I = [a, b] \leq \mathbf{R}$. The equation (3) represented in form $F = (x, y, y') = 0$ or $F = (x, y, p) = 0$.

11.6 COMPLETE SOLUTION OF DIFFERENTIAL EQUATIONS OF ORDER ONE AND HIGHER DEGREE

It is well-established fact that a general solution of differential equation containing as many arbitrary constants as the order of the differential equation is called a **complete solution**. Thus, complete solution of differential equation of order one and higher degree contains only one arbitary constant.

The aim of this section is to study the solvability of equation $F(x, y, y') = 0$ and we observe that equation may be classified into one or more of the following calegories.

- **i.** Solvable for p i.e., $p = F(x, y)$
- **ii.** Solvable for y i.e., $y = F(x, p)$
- **iii.** Solvable for x i.e., $x = F(y, p)$
- **iv.** Not containing x i.e., $f(y, p) = 0$
- **v.** Not containing y i.e., $f(x, p) = 0$
- **vi.** Homogeneous is x and y
- **vii.** x and y appear in first degree only
- **viii.** Clairaut's form
- **ix.** Reducible to Clairaut's form.

Case I. Solvable for *p* :If differential equation $f(x, y, p) = 0$ can be expressed as a product of linear factors as

$$(p - f_1)(p - f_2)\dots(p - f_n) = 0$$

Where $f_i, i = 1, 2, \dots n$ are continuous functions define on 1, then a function defined on I is a solution of $f(x, y, p) = 0$ iff it is a solution of one of the factors $(p - f_i) = 0$

Therefore, in order to find solution of $f(x, y, p) = 0$ equate each factors $(p - f_i)$ to zero and we obtain n differential equation of first order and first degree as

$$p - f_i = 0 \quad \text{or} \quad \frac{dy}{dx} = f_i\,(x, y), i = 1\dots n \quad \textbf{... (4)}$$

Since f_1's are continuous functions on 1, then equation given in equation (4) have solutions. On solving these equations, we obtain n solutions of equation (1) which may be expressed as

$$F_1(x, y, c_1) = 0, \dots, F_n\,(x, y, c_n) = 0 \quad \textbf{... (5)}$$

Where $c_1, \ldots, c_n$ are arbitrary constants. Since the constants $c_1 \ldots, n$ are arbitrary, we can choose $c_1 = \ldots = c_n = c$ and the solutions given in equation (5) may be expressed as :

$$F_1(x, y, c) = 0, \ldots, F_n\ (x, y, c) = 0 \ldots \textbf{(6)}$$

The n solutions so obtained may be left distinct as in equation (6) or the may be combined into one solution by taking their product as :

$$F_1(x, y, c) \ldots F_n(x, y, c) = 0 \ldots \textbf{(7)}$$

When solution of equations (1) are combined into one solutions as in equations (7), there is only one arbitrary constant c which in n th degree (as required according to theory of differential equations).

EXAMPLES

Example 1. Solve : $p^2 - 5p + 6 = 0$

Solution : Factorizing, we have $(p-2)(p-3) = 0$, we

$$p = 2 \text{ and } p = 3$$

i.e.,
$$\frac{dy}{dx} = 2 \text{ and } \frac{dy}{dx} = 3$$

$\therefore$ Corresponding sulution are

$$y = 2x + c, \quad y = 3x + c$$

Hence required solutions

$$(y - 2x - c)(y - 3x - c) = 0$$

Example 2. Solve : $p^2 + 2py \cot x - y^2 - 0$

Solution : From given equation, we have

$$p = \frac{-2y\cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$

$$= -y \cot x \pm y \operatorname{cosec} x \; ... \text{(A)}$$

$\therefore$ $$p = y(\operatorname{cosec} x - \cot x)$$

and $$p = y(\operatorname{cosec} x + \cot x)$$

From (A) $$\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx \qquad \because \frac{dy}{dx} = p$$

On integrating, we get

$$\log y = \log \tan \frac{x}{2} - \log \sin x + \log c$$

$$= -\log(1 + \cos x) + \log c$$

Hence, $$y(1 + \cos x) = c$$

Similarly, the solution is $y(1 - \cos x) = c$ ... (B)

Hence the required solution of the given equation is

$y(1 \pm \cos x) = c$ ... (C)

Example 3. Solve : $p^3 + (2x - y^2)\, p^2 - 2x^2\, p = 0$

Solution : The given equation can be expressed as a product of linear factors as follows :

$$p(p + 2x)(p - y^2) = 0$$

The component equation are :

$$p = 0;\ p + 2x = 0;\ p - y^2 = 0$$

whose solutions are

$$y - c = 0;\ y + x^2 - c = 0;\ xy + cy + 1 = 0$$

Example 4. Solve : $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0$

Solution : Given equation can be put as

$$9p^2y^2 - 6xyp + 12y^2 - 3x^2 = 0$$

or $$(3py - x)^2 - 4(x^2 - 3y^2) = 0 \quad \text{... (D)}$$

Put $$x^2 - 3y^2 = v^2$$

$\therefore$ $$2x - 6y\frac{dy}{dx} = 2v\frac{dv}{dx}$$

or $$(x - 3py) = v\frac{dy}{dx}$$

Putting in (D) we get

$$v^2\left(\frac{dv}{dx}\right)^2 - 4v^2 = 0$$

$\therefore$ $$\frac{dv}{dx} = \pm 2 \quad \text{... (E)}$$

or $$v = c \pm 2x \quad \text{or} \quad v^2 = (c \pm 2x)^2$$

or $$x^2 - y^2 = (c \pm 2x)^2$$

Example 5. Solve $\left(1 - y^2 + \dfrac{y^4}{x^2}\right)p^2 - 2\dfrac{y}{x}p + \dfrac{y^2}{x^2} = 0$

Solution : The given equation can be written as

$$p^2 - \frac{2y}{x}p + \frac{y^2}{x^2} = p^2y^2\left(1 - \frac{y^2}{x^2}\right)$$

or $$(px - y)^2 = p^2y^2\,(x^2 - y^2)$$

or $$px - y = \pm\, py\sqrt{x^2 - y^2}$$

$\therefore$ $$p[x \mp \sqrt{(x^2 - y^2)}] = y$$

or $$\frac{dx}{dy} = \frac{x \mp y\sqrt{x^2 - y^2}}{y} \quad \therefore \quad p = \frac{dy}{dx}$$

Put $$x = vy, \frac{dx}{dy} = x + y.\frac{dv}{dx}$$

$\therefore$ $$v + \frac{dv}{dy} = \frac{vy \mp y^2\sqrt{(v^2-1)}}{y}$$

$$= v \mp y\sqrt{(x^2-1)}$$

or $$\frac{dv}{dy} = \mp\sqrt{(v^2-1)}$$

or $$\frac{dv}{\sqrt{(v^2-1)}} = \mp y\,dy$$

$\therefore$ $$\cos h^{-1} v = \left(c \mp \frac{y^2}{2}\right)$$

or $$\log[v + \sqrt{(v^2-1)}] = c \mp \frac{y^2}{2}$$

$\therefore$ $$\log\frac{x + \sqrt{(x^2 - y^2)}}{y} = c \mp \frac{y^2}{2}$$

Case II. Equations Solvable for *y*

The different equations

$$a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} + a_3 p^{n-3} + \ldots + a_{n-1} p + a_0 = 0$$

can be expressed as

$$y = F(x, p) \ \ldots \text{(F)}$$

Differentiate (F) w.r.t. x to obtain the equation of the form

$$p = F\left(x,\ p,\ \frac{dp}{dx}\right)$$

which is a differential equation of first order and first degree in variables x and p. Solving this equation we obtain a relation of the form

$$\psi\ (x, p, c) = 0 \ldots \text{(G)}$$

where c is an arbitrary constant. Eliminating p between equation (F) and (G) we have a relation of the form

$$\phi\ (x, y, c) = 0$$

which is a solution of equation (F). If elimination of p is not easy, then a solution of equation (F) may be given by expressing x and y in terms of p

EXAMPLES

Example 1. Solve : $y + px = x^4 p^2$

Solution : We have $y + px = x^4 p^2$

or $$y = -px + x^4 p^2 \qquad \ldots \text{(H)}$$

Differentiating with respect to x and denoting $\frac{dy}{dx}$ by p, we have

$$p = -p - x\frac{dp}{dx} + x^4, 2p.\frac{dp}{dx} + p^2.4x^2$$

or $$2p + x\frac{dp}{dx} = 2px^3\left(2p + x\frac{dp}{dx}\right)$$

or $$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

$\therefore$ $$2p + x\frac{dp}{dx} = 0 \text{ or } 1 - 2px^2 = 0$$

Let $$\frac{2}{x}dx + \frac{dp}{p} = 0$$

Then $$\log x^2 + \log p = \log c$$

or $$px^2 = c, \quad \text{i.e.,} \quad p = \frac{c}{x^2}$$

Eliminating p from (1) and (2), we get

$$y = -x, \frac{c}{x^2} + x^4, \frac{c^2}{x^4}$$

$$xy = -c + c^2 x$$

Example 2. Solve : $y = 2x + f(xp^2)$

Solution : The given equation is

$$y = 2px + f(xp^2)$$

Differentiating with respect to *x*, we get

$$p = 2p + 2x\frac{dp}{dx} + f'(xp^2)\,.\left\{p^2 + 2px\frac{dp}{dx}\right\}$$

or $$0 = \left(p + 2x\frac{dp}{dx}\right) + pf'(xp^2)\left\{p + 2x\frac{dp}{dx}\right\}$$

$\therefore$ $$p + 2x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dx}{x} + \frac{2}{p}\,dp = 0$$

or $$\log x + \log p^2 = \log c$$

or $$p^2 x = c \quad \therefore \quad p = \frac{c}{\sqrt{x}} \quad \text{... (I)}$$

Putting the value of *p* in (I), we get

$$y = 2x\frac{c}{\sqrt{x}} + f\left(x \cdot \frac{c^2}{x}\right)$$

or $$y = 2c\sqrt{x} + f(c^2)$$

Example 3. Solve : $xp^2 - 2yp + x = 0$

Solution : The given equation is

$$2yp = x(p^2 + 1)$$

or $$2y = x\left(p + \frac{1}{p}\right)$$

Differentiating wight respect to *x*, we get

$$2p = \left(p + \frac{1}{p}\right) + x\left(1 - \frac{1}{p^2}\right)\frac{dp}{dx}$$

or $$\left(p-\frac{1}{p}\right)=x\left(1-\frac{1}{p^2}\right)\frac{dp}{dx}$$

or $$\left(p-x\frac{dp}{dx}\right)(p^2-1)=0$$

$\therefore$ $$\left(p-x\frac{dp}{dx}\right)=0$$

Let $\frac{dp}{p}=\frac{dx}{x}$ i.e., $p=\log x+\log c$

Then $$p=cx$$

Putting for p in the given equation, (1) we get

$$x(cx)^2=2y(cx)+x=0$$

or $$2cy=c^2x^2+1$$

Case III. Equations Solvable for *x* :

The differential equation

$$a_0p^n+a_0p^{n-1}+a_2p^{n-2}+\ldots+a_{n-1}p+a_0=0 \text{ .. (J)}$$

where $a_1, a_2, \ldots a_n$ have their usual meaning. Let us suppose that it can be expressed in the form.

$$x=F(y,\ p)$$

Differentiating (J) we respect to y to obtain

$$\frac{1}{p}=\frac{dx}{dy}=f\left(x,\ p,\frac{dp}{dy}\right) \text{ ... (K)}$$

Now, (K) is a new differential equation with variable y and p. Solving this we obtain a relation of the form

$$\psi(x,\ p,c)=0 \qquad \ldots \text{ (L)}$$

Eliminating p between (J) and (L), we get either $\phi(x,\ p,c)=0$ as the required solution or in parametric form. If elimination of p is not easy, then a solution of given equation may be expressed as x, y are functions of p.

EXAMPLES

Example 1. Solve : $x = y + p^2$

Solution : Differentiating given equation with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = 1 + 2p\,\frac{dp}{dy}$$

or $$dy = \frac{2p^2}{1-p}\,dp = -\left(p + 1 + \frac{1}{p-1}\right) dp$$

Integrating gives $$y = -2\left[\frac{1}{2}\,p^2 + p\log(p-1)\right] + c$$

or $$y = -[p^2 + 2p + 2\log(p-1)] + c \quad \text{.....(M)}$$

Substituting this the given equation, we get

$$x = c - [2p + 2\log(p-1)] \;\text{... (N)}$$

Relations (1) and (2) together give the required solution.

Note : This problem may be solved regarding solvable for y and p

(I) Suppose $y = x - p^2$

Differentiating w.r.t. x

$$\frac{dy}{dx} = 1 - 2p\,\frac{dp}{dx}$$

$$p = 1 - 2p\,\frac{dp}{dx}$$

$$\frac{dp}{dx} = \frac{1-p}{2p} \text{ (separable equation)}$$

$$\frac{2p}{1-p}\,dp = dx$$

Integration gives

$$x = c - 2p - 2\ln(p-1) \ \dots \text{(O)}$$

Using in given equation we have

$$y = c - 2p - p^2 - 2\ln(p-1) \dots \text{(P)}$$

Eqs. (O) and (P) represent complete solution of differential equation in parametric from

(II) Suppose $p = \sqrt{x-y}$

$$\frac{dy}{dx} = \sqrt{x-y}$$

let $x - y = v^2$. Then

$$1 - \frac{dy}{dx} = 2v\frac{dv}{dx}$$

Using it we get

$$1 - 2v\frac{dv}{dx} = v$$

$$\frac{dv}{dx} = \frac{1-v}{2v}$$

$$\frac{2v}{1-v}\,dv = dx$$

$$-2v - 2\ln(v-1) = x + c$$

$$-2\sqrt{x-y} - 2\ln(\sqrt{x-y-1}) = x + c$$

Example 2. Solve : $y^2 \log y = xyp + p^2$

Solution : The given equation is

$$x = \frac{y \log y}{p} - \frac{p}{y}$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{p\,(1+\log y) - y\log y\,.\dfrac{dp}{dy}}{p^2} - \frac{y\,.\dfrac{dp}{dy} - p}{y^2}$$

or $$\frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log y - \frac{y}{p^2}\log y\,\frac{dp}{dy} - \frac{1}{y}\,.\frac{dp}{dy} + \frac{p}{y^2}$$

or $$\frac{1}{y}\frac{dp}{dy}\left(1 + \frac{y^2}{p^2}\log y\right) = \frac{p}{y^2}\left(1 + \frac{y^2}{p^2}\log y\right)$$

On cancelling the common factor on either side

$\therefore$ $$\frac{dp}{p} = \frac{dy}{y}; \quad \therefore \ \log p - \log y = \log c$$

or $$\frac{p}{y} = c \qquad \text{i.e.,} \quad p = cy$$

Substituting the value of p in the given equation, we get

$$\log y = cx + x^2$$

Example 3. Solve : $ayp^2 + (2x - b)\,p - y = 0$

Solution : The given equation can be written

$$2x = \frac{y}{p} + b - ayp$$

Differentiating with respect to y, we get

$$2\cdot\frac{1}{p} = \frac{1}{p} - \frac{1}{p^2}\,y\,\frac{dp}{dy} - a\left(p + y\,\frac{dp}{dy}\right)$$

or $$\frac{1}{p^2}\left(p + y\,\frac{dp}{dy}\right) + a\left(p + y\,\frac{dp}{dy}\right) = 0$$

$$\left(\frac{1}{p^2}+a\right)\left(p+y\frac{dp}{dy}\right)=0$$

$\therefore$ $\frac{y}{p}\frac{dp}{dy}+1=0$ i.e., $\frac{dp}{p}+\frac{dy}{y}=0$

or $\log p+\log y=\log c$ i.e., $py=c$

Eliminating p form given equation, we get

$$ac^2+(2x-b)\,c-y^2=0$$

Example 4. Solve : $p^3-4xyp+8y^2=0$

Solution : We equation can be written as

$$x=\frac{p^2}{4y}+\frac{2y}{p}$$

Differentiating w.r.t. gives

$$\frac{dx}{dy}=\frac{1}{p}=\frac{p}{2y}\frac{dp}{dy}-\frac{p^2}{4y^2}+\frac{2}{p}-\frac{2y}{p^2}\frac{dp}{dy}$$

$$\frac{-1}{p}+\frac{p^2}{4y^2}=\left(\frac{p}{2y}-\frac{2y}{p^2}\right)\frac{dp}{dy}$$

$$\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)=\frac{2y}{p}\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)\frac{dp}{dy}$$

let $1=\frac{2y}{p}\frac{dp}{dy}$

Then $p^2=cy$

Using is given equation we get

$$(cy)^{3/2}-4xy\sqrt{cy}+8y^2=0$$

$$y^{3/2}\,[c^{3/2}-4x\sqrt{c}+8y^{1/2}]=0$$

$$8y^{1/2}=4x\sqrt{c}-c^{3/2}$$

$$y=A(x-A^2)^2$$

Case IV. Equation that do not contain *x* :

Suppose equation can be expressed in the form

$$F(y, p) = 0$$

there arise the following subcases :

Subcase (i) It may be solvable for p and can be expressed as

$$p = f(y)$$

which can be solved as in Case (I)

Subcase (ii) It may be solvable for y and can be expressed as

$$y = g(p)$$

which can be solve as in Case (II)

Case V. Equations that do not contain *y* :

Suppose equations can be expressed in the form

$$F(x, p) = 0$$

there arise the following subcases :

Subcase (i) It may be solvable for p and can be expressed as

$$p = f(x)$$

which can be solve as in Case (I).

Subcase (ii) It may be solvable for x and can be expressed as

$$x = g(p)$$

which can be solved as in Case (III)

Case VI. Equations that are homogenous in *x* and *y :*

Suppose equation is homogenous in x and y, then it can be expressed as

$$F\left(p, \frac{y}{x}\right) = 0 \qquad \text{.........(Q)}$$

In such case, there arise the following subcases :

(i) It may be solvable for p and can be expressed as

$$p = f\left(\frac{y}{x}\right)$$

Which is a homogenous differential equation of first order and first degree.

Hence, we can substitute $v = \frac{y}{x}$, separate the variables, and obtain a solution of equation (Q)

(ii) It may be solvable for $\frac{y}{x}$ and can be expressed as

$$y = yg(p)$$

Which can be solvable in Case (II).

EXAMPLES

Example 1. Solve equation $y^2 + xyp - x^2 p^2 = 0$

Solution : The given equation is homogenous in *x* and *y*. Dividing by x^2 it becomes

$$\left(\frac{y}{x}\right)^2 + \frac{y}{x} p - p^2 = 0$$

This can be expressed as

$$\left(\frac{y}{x} + \frac{p}{2}\right)^2 = \frac{5}{4} p^2$$

From which, we get

$$\frac{y}{x} = \pm \frac{\sqrt{5} - 1}{2} p = \pm \frac{\sqrt{5} - 1}{2} \frac{dy}{dx}$$

Separating the variable and integrating, we find that the solution of the given equation are

$$y = cx^{\frac{2}{\sqrt{5}-1}} \text{ and } y = cx^{\frac{-2}{1-\sqrt{5}}}$$

where c is an arbitrary constant.

Example 2. Solve the equation $y = yp^2 + 2xp$

Solution : The given equation can be expressed in the form.

$$y = x\left(\frac{2p}{1-p^2}\right)$$

which is case (II). Differentiating the equation with respect to x and rearranging the terms, we obtain

$$\frac{dx}{x} = \left\{\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right\} dp$$

Integrating this equation, we have

$$x = c\left(1 - \frac{1}{p^2}\right)$$

where c is an arbitrary constant. Substituting this value of x in the equation for y, we obtain

$$y = \frac{-2c}{p}$$

Substituting this value, in $x = c\left(1 - \frac{1}{p^2}\right)$, we have

$$x = \frac{1}{4c}(4c^2 - y^2)$$

or $$y^2 = 4c^2 - 4cx$$

which is a solution of the given equation.

Case (VII) Equations in which x and y appear in the first degree only :

If x and y appear in the first degree, then equation can be solved either for y or for x and proceed to obtain solutions of equation either as in Case (II) or as in Case (III).

Case (VIII) Clairaut's Equations :

The equation $y = px + f(p)$ known as **Clairaut's equation,** where p has its usual meaning.

Differentiating with respect to x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{i.e.,} \quad \frac{dp}{dx}[x + f'(p)] = 0$$

Now, either $$\frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0$$

But $$\frac{dp}{dx} = 0 \text{ gives } p = c$$

Hence solution of Clairaut's equations is obtained by putting $p = c$ in equation.

$$y = cx + f(c)$$

If we eliminate p between the following equation.

$$y = px + f(p)$$

and $$x + f'(p) = 0$$

Then we shall obtain another solution which is not general solution of Clairaut's equation. This solution does not contain any arbitrary constant, not can it be derived from the general solution by giving particular values to the arbitrary constant. It is known as **singular solution**.

EXAMPLES

Example 1. Solve : $y = px + \frac{a}{p}$

Solution : Differentiate with respect to *x*, we get

$$p = p + x\frac{dp}{dx} - \frac{a}{p^2}.\frac{dp}{dx}$$

$$\therefore \qquad \frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0$$

Taking $\frac{dp}{dx} = 0$, we get $p = c$ and cutting in the given equation, we get

$$y = cx + a/c$$

as the general solution.

Example 2. Solve : $p^2 x\,(x-2) + p\,(2y - 2xy - x + 2) + y^2 + y = 0$

Solution : The given can be written as

$$(y^2 - 2pxy + p^2x^2) + 2p(y - px) + (y - px + 2p) = 0$$

or $$(y - px + 1)\,(y - px + 2p) = 0$$

or $$(y - px + 1)\,(y - px + 2p) = 0$$

or $$p = x - 1 \quad \text{and} \quad y = px - 2p$$

Each of them is of Clairaut's form and hence the solution is obtained by putting $p = c$ and is

$$(\,y\text{-}\ cx + 2c)\ (\,y\text{-}\ cx + 1) = 0$$

Example 3. Solve $(x - a)\ p^2 + (x - y)\ p - y = 0$

Solution : The given equation can be written as

$$y(p+1) = xp\,(p+1) - ap^2 \quad \text{or} \quad y = px - \frac{ap^2}{p+1}$$

This is in Clairaut's form. Hence its solution will be

$$y = cx - \frac{ac^2}{c+1}$$

Example 4. Solve $p = \log\,(px - y)$

Solution : Given equation can be written as

$$px - y = e^p$$

$$y = px - e^p$$

It is Clairaut's form. Hence its solutions be

$$y = cx - e^c$$

Case (IX) Equations Reducible to Clairant's form

By proper substitutions equations may be transformed into Clairaut's form.

EXAMPLES

Example 1. Solve : $x^2(y - px) = p^2 y$

Solution : We shall reduce the above equation to Clairaut's form by change of variables.

Suppose $\quad x^2 = u \quad \text{and} \quad y^2 = v$

Therefore, $\quad 2x\,dx = du \quad \text{and} \quad 2y\,dy = dv$

or $\quad \dfrac{y\,dy}{x\,dx} = \dfrac{dv}{du} \quad \text{or} \quad \dfrac{y}{x}p = \dfrac{dv}{du}$

Putting for p is the given equation, we get

$$x^2\left(y - x.\frac{x}{y}\frac{dv}{du}\right) = \frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 y$$

or $$\left(y^2 - x^2.\frac{dv}{du}\right) = \left(\frac{dv}{du}\right)^2$$

[Put $x^2 = u$ and $y^2 = v$]

or $$9v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

If we take $\frac{dv}{du} = P$, then $v = uP + P^2$ is

Clairaut's form, and its is $v = cu + c^2$ or $y^2 = cx^2 + c^2$

Example2.

Solve $(x^2 + y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$

Solution : The given equations can be written as

$$x^2 + y^2 - 2(x+y)\frac{(x+yp)}{(1+p)} + \left(\frac{x+yp}{1+p}\right)^2 = 0$$

Put $$x^2 + y^2 = v; \qquad \therefore \quad 2x + 2yp = \frac{dv}{dx}$$

and $$x + y = u, \qquad (1+p) = \frac{du}{dx}$$

$\therefore$ $$\frac{x+yp}{1+p} = \frac{1}{2}\frac{dv}{dx}$$

Making the above substitutions, we get

$$v - 2u.\frac{1}{2}\frac{dv}{du} + \frac{1}{4}\left(\frac{dv}{du}\right)^2 = 0 \quad \text{or} \quad v = u\frac{dv}{du} - \frac{1}{4}.\left(\frac{dv}{du}\right)^2$$

Putting $\frac{dv}{du} = P$, we get

$$v = uP - \frac{1}{4}P^2$$

which is Clairaut's form and hence its solution is

$$v = u.c - \frac{1}{4}c^2$$

or $$x^2 + y^2 = c(x+y) - \frac{1}{4}c^2$$

Example 3. Use the transformation $u = x^2$ and $v = y^2$ to solve $(px - y)(py + x) = h^2 p$.

Solution : Just as in Example 1, we have on putting $x^2 = u$ and $y^2 = v$, we have

$$\frac{y}{x}p = \frac{dv}{du} \quad \text{or} \quad p = \frac{x}{y}\frac{dv}{du}$$

Putting for p in the given equation, we get

$$p^2 xy + p(x^2 - y^2 - h^2) - xy = 0$$

or $$xy.\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 + \frac{x}{y}\left(\frac{dv}{du}\right)(x^2 - y^2 - h^2) - xy = 0$$

or $$x^2\left(\frac{dv}{du}\right)^2 + \left(\frac{dv}{du}\right)(x^2 - y^2 - h^2) - y^2 = 0$$

Put $\frac{dv}{du} = P$, then

$$uP^2 + P(u - v - h^2) - v = 0$$

or $$uP(P+1) - v(P+1) - Ph^2 = 0 \quad \text{or} \quad v = uP - \frac{P}{(P+1)}h^2$$

which is of the form $v = Pu + f(P)$, i.e., Clairaut's form.

Hence, the solution is $v = uc - \frac{c}{c+1}h^2$ or $y^2 = cx^2 - \frac{c}{c+1}h^2$

Example 4. Reduce $x^2\left(\frac{dy}{dx}\right)^2 + y\,(2x+y)\,\frac{dy}{dx} + y^2 = 0$ to Clairaut's form by using the substitution $y = u$ and $xy = v$.

Solution : We have $y = u$ and $xy = v$. Then $x = \frac{v}{u}$

$$\therefore \qquad dx = \frac{u\,dv - u\,du}{u^2} \quad \text{and} \quad dy = du$$

$$\therefore \qquad \frac{dy}{dx} = \frac{u^2 du}{u\,dv - v\,du} = \frac{u}{\frac{dv}{du} - \frac{v}{u}}$$

Putting the values of x, y and $\frac{dy}{dx}$ in the given equation, we get

$$\frac{v^2}{u^2}\cdot\frac{u^2}{\left(\frac{dv}{du}-\frac{v}{u}\right)^2} + u\left(2\frac{v}{u}+u\right)\frac{u}{\left(\frac{dv}{du}-\frac{v}{u}\right)} + u^2 = 0$$

or
$$\frac{v^2}{u^2} + \left(\frac{2v}{u}+u\right)\left(\frac{dv}{du}-\frac{v}{u}\right) + \left(\frac{dv}{du}-\frac{v}{u}\right)^2 = 0$$

or

$$\frac{v^2}{u^2} + \frac{2v}{u}\frac{dv}{du} + u\frac{dv}{du} - \frac{2v^2}{u^2} - v + \left(\frac{dv}{du}\right)^2 - 2\frac{v}{u}\frac{dv}{du} + \frac{v^2}{u^2} = 0$$

or
$$v = u\,\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

If we put $\frac{dv}{du} = P$, we get $u = uP + P^2$

This is clearly a Clairaut's form. Hence the solution is

$$v = uc + c^2 \quad \text{or} \quad xy = cy + c^2$$

11.7 SUMMARY

1. Differential equations of higher degree (or greater than one) are non-linear.
2. The concept of singular solutions is associated with the uniqueness of solutions. If condition of uniqueness is violated then solution is singular.
3. Clairaut's equation may have more than one singular solution.

11.8 GLOSSARY

- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **Derivative:** A measure of how a function changes as its input changes.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.

- **Dependent Variable:** The variable whose value depends on the value of another variable, often denoted as y.
- **Independent Variable:** The variable that is varied independently of other variables, often denoted as x.
- **Derivative:** A measure of how a function changes as its input changes, often representing rates of change.
- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.
- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.

CHECK YOUR PROGRESS

Solving following differential equations :

1. $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 4\left(\frac{dx}{dy}\right)^2$

2. $(p-1)\,e^{4x} + p^{2y}\;p^2 = 0 \quad \because\; u = c^{2x}, v = e^{2y}$

3. $(y - e^x + c)\,(y + e^{-x} + c) = 0$

4. $y = \sin\,p - p\cos\,p$

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11.11TERMINAL QUESTIONS

Exercise (A)

Solve the following differential equations :

1. **(a)** $p^2 - 7p + 12 = 0$ **(b)** $p^2 - 7p + 18 = 0$
2. $p^2x^2 - xyp - y^2 = 0$
3. $x^2p^2 + xyp - 6y^2 = 0$
4. $xp^2 + (y - x)\ p - y = 0$
5. $xy\,(p^2 + 1) - (x^2 + y^2)\ p = 0$
6. $4y^2p^2 + 2\,pxy\,(3x+1) + 3x^2 = 0$
7. $p^3 + 2p^2x - p^2y^2 - 2\,pxy^2 = 0$
8. $x^2\left(\frac{dy}{dx}\right)^2 + 3xy\frac{dy}{dx} + 2\,y^2 = 0$
9. $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + 2\,y^2 - x^2 = 0$
10. $p^2 - 2\,p\cosh x + 1 = 0$

Exercise (B)

Solve following differential Equations :

1. $x^2p^2 + x^2py + a^2 = 0$
2. $y = 2px + p^4x^2$
3. $y = 2px - p^2$
4. $x - yp = ap^2$
5. $x + yp = ap^2$
6. $y = 3x + \log\ p$
7. $p^3 + mp^2 = a(y + mx)$
8. $4p^3 + 3px = y$
9. $y = \frac{1}{\sqrt{(1 + p^2)}} + b$
10. $y = p^2x + p^4$

11. $y = p\tan p + \log\cos p$

Exercise (C)

Solving following differential equations :

1. $y = 2px + y^2 p^3$

2. $yp^2 - 2px + y = 0$

3. $y = 2px + p^2 y$

4. $y = 3px + 6p^2 y^2$

5. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$

6. $x + \dfrac{p}{\sqrt{(1+p^2)}} = a$

7. $x = y + a\log p$

Exercise (D)

1. Solve the following equation

i. $y = 2p + 3p^2$

ii. $y^2 = a^2(1+p^2)$

Exercise (E)

1. Solve the following equations :

a) $x^2 = a^2(1+p^2)$

b) $x(1+p^2) = 1$

Exercise (F)

1. Solve : $y = px + p - p^2$ and obtain the singular solution as well.

2. Solve : $y = px + ap(1-p)$

3. Solve : $y = x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$

4. Solve : $y = px - 1/p^2$ and obtain the singular solution as well.

5. Solve : $xp^2 - yp + 2 = 0$

6. Solve : $p^2 x = py - 1$ and obtain the singular solution as well.

7. Solve : $(y - px)(p - 1) = p$

8. Solve : $xp^3 - (y + 3)\ p^2 + 4 = 0$

9. Solve : $\dfrac{(y - px)^2}{(1 + p^2) - a^2} = 0$

10. Solve : $\sin px \cos y = \cos px \sin y + p$

Exercise (G)

Change the following equations to Clairaut's form and solve

1. $xy\,(y - px) = x + py \because u = x^2, y = v^2$

2. $xyp^2 - (x^2 + y^2 + 1)\ p + xy = 0 \because u = x^2, v = y^2$

3. $y = px + \dfrac{p}{x} \because\ u = x^2, v = y^2$

4. $(px - y)(x - py) = 2p \because u = x^2, v = y^2$

5. $y = 2px + y^2 p^3 \because u = x^2, v = y^2$

6. $(y + px)^2 = x^2 p \because\ xy = v$

7. $e^{3x}\,(p - 1) + p^3 e^{2y} = 0 \because u = e^x, v = e^y$

8. Using substitutions $x^2 = u$ and $y^2 = v$, solve the equation

   $axy\ p^2 + (x^2 - ay^2 - b)\ p - xy = 0$

9. Using the substitutions $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$ solve the equations

$y^2\,(y - px) = x^4 p^2$

10. Reduce the equations $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by using the substitution $y - x = v$ and $x^2 = u$

11.12ANSWERS

CHECK YOUR PROGRESS

CHQ1.

$(y - cx)^2 = 4/c^2$

CHQ2.

$e^{2y} = ce^{2x} + c^2$

CHQ3.

$p^2 - 2p\cosh x + 1 = 0$

CHQ4.

$x = c - \cos p,\ y = \sin p - p\cos p$

TERMINAL ANSWERS (TQ'S)

Exercise (A)

Answers :

1. **(a)** $(x-4x+c)\,(y-3x+c)=0$

(b) $(y-3x+c)\,(y-6x+c)=0$

2. $y^2 = ce^{\left(\frac{1\pm\sqrt{3}}{2}\right)x^2}$

3. $\left(\frac{y}{x^3}-c\right)(yx^2-c)=0$

4. $(y-x-c)\,(xy-c)=0$

5. $(y^2-x^2-c)\,(y-cx)=0$

6. $x^2+2y^2=c,\ x^3+y^2=c$

7. $(y-c)\,(x+x^2-c)\,(xy+cy+1)=0$

8. $(xy-c)\,(x^2y-c)=0$

9. $\sin^{-1}\left(\frac{y}{x}\right)=\pm\log cx$

10. $(y-e^x+c)\,(y+e^{-x}+c)=0$

Exercise (B)

Answers :

1. $c^2+cxy+a^2x=0$

2. $(y-c^2)^2=4cx$

3. $x=cp^{-2}+\frac{2}{3}\,p,\ y=2cp^{-1}+\frac{1}{3}\,p$

4. $y=\frac{1}{\sqrt{(1-p^2)}}(c+a\sin^{-1}p)-ap,\ x=\frac{p}{\sqrt{(1-p^2)}}(c+a\sin^{-1}p)$

5. $x\frac{\sqrt{(p^2+1)}}{p}=(a\sin h^{-1}\ p+c);\ y=-\frac{x}{p}+ap$

6. $y = 3x + \log \dfrac{3}{1 - ce^{3x}}$

7. $ax = c + \dfrac{3p^2}{2} - mp + m^2 \log (p + m);$

$ay = -m\left[c + \dfrac{3}{2}p^2 + mp + m^2 \log (p + m)\right] + mp^2 + p^3$

8. $y = -\dfrac{8}{7}p^3 + cp^{-1/2}; \; x = -\dfrac{12}{7}p^2 + cp^{-3/2}$

9. $(x + c)^2 + (y - b)^2 = 1$

10. $x = \dfrac{1}{(1-p)^2}\left(\dfrac{4}{3}p^3 - p^4 + c\right), \; y = p^2 x + p^4$

11. $x = \tan p + c, \; y = p \tan p + \log \cos p$

Exercise (C)

Answers :

1. $y^2 = 2cx + c^3$

2. $y^2 = 2x - c^2$

3. $y^2 = 2cy + c^2$

4. $y^2 = 3cx + 6c^2$

5. $y = c\dfrac{1}{1+p^2}; \; x = \tan^{-1} p + \dfrac{p}{1+p^2}$

6. $(y + c)^2 + (x - a)^2 = 1$

7. $x = c + a \log \dfrac{p}{p-1}$

Exercise (D)

Answers

(i) $2 \ln p + 6p = x + c;\ y = 2p + 3p^2$

(ii) $y = a \cos h\left(\frac{x}{a} + c\right)$

Exercises (E)

Answers :

(a) $y = \frac{1}{2}\left[\frac{x}{a}\sqrt{x^2 + a^2} - a \log\left(x + \sqrt{x^2 - a^2}\right)\right] + c$

(b) $x^2 + (y + c)^2 = x$

Exercise (F)

Answers :

1. $ycx + c - c^2, 4y = (x + 1)^2$

2. $y = cx + ac\,(1 - c)$

3. $y = cx + c^2$

4. $y = cx - \frac{1}{c^2};\ y^2 = 2x^2$

5. $y = cx + \frac{2}{c}$

6. $y = cx + \frac{1}{c};\ y^2 = 4x$

7. $y + cx + \frac{c}{c - 1}$

8. $y = cx + \frac{4}{c^2} - 3$

9. $(y - cx)^2/(1 + c^2) = a^2$

10. $y = cx - \tan^{-1} c$

Exercise (G)

Answers :

1. $y^2 = cx^2 + (1 + c)$

2. $y^2 = cx^2 - \dfrac{c}{c - 1}$

3. $y^2 = cx^2 + c$

4. $y^2 = cx^2 - \dfrac{2c}{1 - c}$

5. $y^2 = cx + \dfrac{c^3}{8}$

6. $xy = cx - c^2$

7. $e^y = ce^x + c^3$

8. $y^2 = cx^2 - \dfrac{bc}{ac + 1}$

9. $c^2xy + cy - x = 0$

10. $2c^2x^2 - 2c(y - x) + 1 = 0$

UNIT 12:

DIFFERENTIAL EQUATIONS OF FIRST ORDER

AND HIGHER DEGREE -II

CONTENTS:

12.1 INTRODUCTION

In continuation of previous unit we discuss here briefly basic properties of differential equations of first-order and higher degree. In this unit we are explaining singular solution, envelope P and C Discriminats, Determination of Singular Solution and Riccati Equation.

12.2 OBJECTIVES

After studying this unit the learners will be able to

i. Describe the singular solution, envelope and P and C Discriminats.

ii. Determination of Singular Solution

iii. Understand the Riccati Equation

12.3 SINGULAR SOLUTIONS

The concept of **"singular solution"** is associated with the notion of existence and uniqueness of solution of initial value problems (IVP) and it is evident that solution of linear differential equation is always unique. However differential equations of first order and higher degree are non-linear and solution of non-linear differential equations may or may not be unique if solution exists.

A function $y = \phi(x)$ is called the singular solution of the differential equation $F(x, y, y') = 0$ if uniqueness of solution is violated at each point of the domain of the equation. Geometrically, if more than one integral curve with the common tangent line passes through each point (x, y) then singular solution exists. It should be noted that if uniqueness of solution is violated only at some points then it may be used as weaker definition of singular solution.

Thus a solution is called singular solution of differential equation if it cannot be obtained from general solution for any particular value of arbitrary constant 'c'. In otherwards a solution of differential equation which does not belong to the family of curves (integral curves/solution) or in the neighbourhood of singular solution there is no other solution of the equation. Thus a singular solution is an **isolated solution** of the equation.

For example, $y=(x+c)^2$ is general/complete solution differential equation $(y')^2-4y=0$ and is graphically represented by the family parabolas as

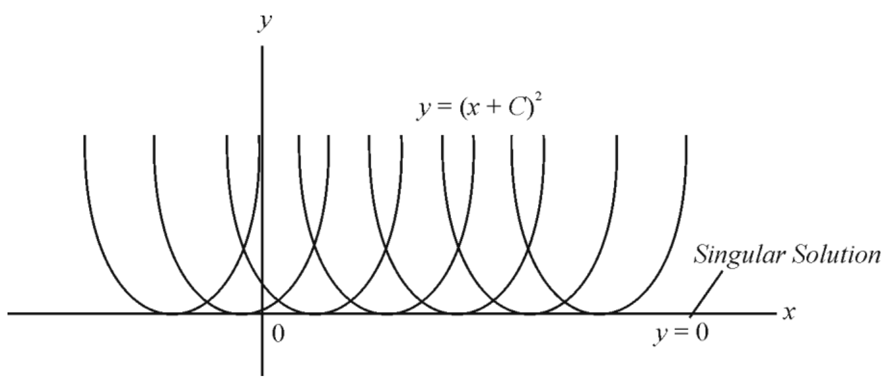

Fig. 12.3.1. General and Singular Solutions

But the function $y=0$ also satisfies the differential equation and this function is not contained in the general solution. It means there are more than one integral curves passes through each point of the straight line $y=0$ i.e., condition of uniqueness of solution is violated at each point of the line $y=0$. Hence $y=0$ is **singular solution** of the differential equation.

In case of differential equation $y=xy'+a\sqrt{1+(y')^2}$ the general solution is $y=cx+a\sqrt{1+c^2}$ is family of straight lines but $x^2+y^2=a^2$

(circle) is also a solution of this equation which cannot be obtained from general/complete solution for particular value of arbitrary constant.

Note : The singular solution usually corresponds to the envelope of the family of the integral curves of the general solution of the differential equation.

12.4 ENVELOPE

In mathematics, a curve which is tangential to each one of the family of curves in a plane and in three dimensions, a surface which is tangent to each one of the family of surfaces are called **"envelope"**.

For example, $y=0$ (singular solution) is tangent to each one of the family of curves $y=(x+c)^2$ (general solution) i.e., $y=0$ singular solution is envelope of the general solution.

Let $f\left(x, y, c\right)=0$; c is arbitrary constant, be a family of curves. Then parametric equations of the envelope are defined as

$$f\left(x, y\right)=0\ ;\ \ f_c'(x, y, c)=0$$

and eliminating c from these equations we can obtain the equations of the envelope in explicit or implicit form.

The system of equations $f\ (x,\ y,\ c) = 0$, $f_c'\ (x,\ y,\ c) = 0$ is necessary condition for the existence of an envelope and to find the equation of envelope uniquely, the sufficient conditions are

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f'}{\partial x} & \dfrac{\partial f'}{\partial y} \end{vmatrix} \neq 0, \frac{\partial^2 y}{\partial c^2} \neq 0$$

For example, consider a family of circles

$$(x-c)^2+(y-c)^2=1$$

$$f(x, y, c) = (x - c)^2 + (y - c)^2 - 1 = 0$$
$$f'(x, y, c) = -2(x - c) - 2(y - c) = 0$$

On solving we get $c = \frac{x + y}{2}$

and using in $f(x, y, c) = 0$ we get

$(y - x)^2 = 2 \quad i.e., \quad y = x \pm \sqrt{2}$

These two straight lines are envelope lines for given family of circles and are shown as

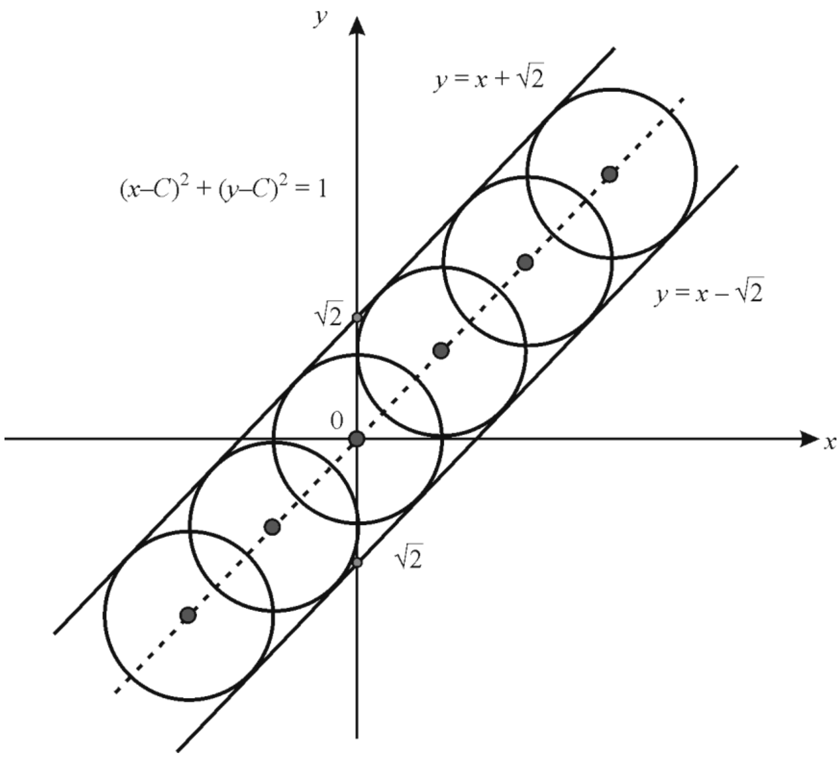

Fig. 12.3.2. General and Singular Solutions

12.5 p AND c-DISCRIMINANT

As we know that the discriminant of the quadratic equation

$ax^2 + bx + c = 0; a \neq 0$ is $b^2 - 4ac$

which is elimination of x form $f = ax^2 + bx + c = 0$ and $\frac{\partial f}{\partial x} = 0$

Similarly, if $\phi(x, y, c) = 0$ is the solution of the differential equation $f(x, y, p) = 0$ or $f(x, y, y') = 0$ then $p-$discriminant is obtained by eliminating p from $f(x, y, p) = 0$ and $\frac{\partial f}{\partial p} = 0$ and c-discriminant is obtained by eliminating c-discriminant is obtained by eliminating c from $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$.

12.6 DETERMINATION OF SINGULAR SOLUTIONS

If $f(x, y, y')$ and its partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y},$ are continuous in the domain of the differential equation then singular solution can be obtained from the system of equations

$$f(x, y, y') = 0; \frac{\partial f}{\partial y'}(x, y, y') = 0 \quad (p\text{ -discriminant})$$

and the corresponding curves obtained by p-discriminant is called p-discriminant curve. If it is solution of the differential equation then it is called singular solution. The another way to find the singular solution of the differential equation is c-discriminant method which is same as p-discriminant method.

Example 1. For the quadratic equation $Ap^2 + Bp + C = 0$ with A, B, C are functions of x and y, show that the p-discriminant is $B^2 - 4AC = 0$

Solution : We have

$$f = Ap^2 + Bp + c = 0 \quad \ldots (1)$$

$$\text{Then} \quad f' = \frac{\partial f}{\partial p} = 2Ap + B \quad \ldots (2)$$

and p-discriminant is obtained by eliminating p from $f = 0$ and $f' = 0$

i.e., $A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{24}\right) + C = 0 \; \frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$

i.e., $B^2 - 4AC = 0 \quad \ldots (3)$

Example 2. For the cubic $p^3 + Ap + B = 0$, show that p-discriminant equation is $4A^2 + 27B^2 = 0$

Solution : We have $f = p^3 + Ap + B = 0 \quad \ldots (1)$

with *A, B, C*are functions of *x, y*

Then,

$f' = \frac{\partial f}{\partial p} = 3p^2 + A \quad \ldots (2)$

Eliminating p from $f = 0$ and $f' = 0$ we have

$$p = \sqrt{\frac{-A}{3}}$$

and $$\left(\sqrt{\frac{-A}{3}}\right)^3 + A\left(\sqrt{\frac{-A}{3}}\right) + B = 0$$

$$\left(\frac{-A}{3}\right)^{\frac{3}{2}} - 3\left(\frac{-A}{3}\right)^{\frac{3}{2}} + B = 0$$

$$-2\left(\frac{-A}{3}\right)^{\frac{3}{2}} + B = 0$$

$$4\left(\frac{-A}{3}\right)^3 = B^2 \quad \text{or} \quad 27B^2 + 4A^3 = 0 \;\ldots (3)$$

Example 3. Find the singular solutions of the equation

$$1 + (y')^2 = \frac{1}{y^2}$$

Solution : We have

$$f = 1 + (y')^2 - \frac{1}{y^2} = 0$$

i.e., $f = 1 + p^2 - \dfrac{1}{y^2} = 0 \quad \ldots (1)$

Then $f' = \dfrac{\partial f}{\partial p} = 2p \quad \ldots (2)$

(I) *p*-discriminant Methods :

Eliminating p from $f = 0$ and $f' = 0$ we obtain

$y^2 = 1$ i.e., $y = \pm 1 \;\ldots (3)$

(II) *c*-discriminant Methods : First we determine general solution of given differential equation. We can write it in the following form

$$y' = p = \pm \frac{\sqrt{1-y^2}}{y}, \qquad \frac{y\,dy}{\sqrt{1-y^2}} = \pm\, dx$$

Integration gives

$$\sqrt{1-y^2} = \pm (x+c)$$

or $(x+c)^2 + y^2 = 1$; c is arbitrary constant………… (4)

Let $\phi(x, y, c) = (x+c)^2 + y^2 - 1 = 0$

Then $$\frac{\partial \phi}{\partial c} = 2\,(x + c)$$

Eliminating c from $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$ we get

$$y = \pm 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (5)$$

Thus $y = \pm 1$ are singular solution of given differential equation or envelope of family of circles $(x + c)^2 + y^2 = 1$ and it can be represented as

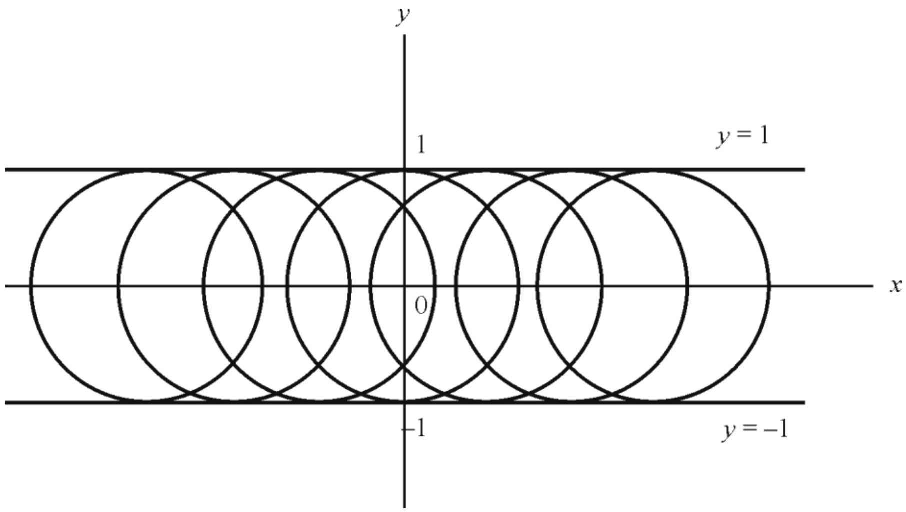

Fig. 12.6.1

Example 4. Find the singular solution of the differential equation $y = (y')^2 - 3xy' + 3x^2$

Solution : We have

$$f = y - p^2 + 3xp - 3x^2 = 0 \quad \ldots (1)$$

***p* - discriminant Methods :**

$$\frac{\partial f}{\partial p} = -2p + 3x$$

Eliminating p from $f = 0$ and $\frac{\partial f}{\partial p} = 0$ we have

$$y-\left(\frac{3x}{2}\right)^2+3x\left(\frac{3x}{2}\right)-3x^2=0$$

i.e., $4y=3x^2$ … (3)

c-discriminant Method :

First we determine the general solution of given differential equation we write it as

$$y=p^2-3xp+3x^2;\ p=\frac{dy}{dx} \quad \ldots (4)$$

Differentiating w.r.t. x using $p=\frac{dy}{dx}$ we have

$$p=2p\frac{dp}{dx}-3p-3x\frac{dp}{dx}+6x$$

$$2p\frac{dp}{dx}-4p-3x\frac{dp}{dx}+6x=0$$

$$(2p-3x)\frac{dp}{dx}-2(2p-3x)=0$$

$$(2p-3x)\left(\frac{dp}{dx}-2\right)=0 \quad \ldots (5)$$

Taking $\frac{dp}{dx}-2=0$, we have

$$p=2x+c$$

Using in (4) we get

$$y=(2x+c)^2-3x(2x+c)+3x^2$$

$$y=x^2+cx+c^2$$

i.e., $\phi=y-x^2-cx-c^2=0$ … (6)

It is general solution of given differential equation. Then $\frac{\partial\phi}{\partial c}=-x-2c$ and eliminating c from $\phi=0$ and $\frac{\partial\phi}{\partial c}=0$ we have

$$y = \frac{3}{4}x^2 \quad \text{i.e.,} \quad 4y = 3x^2 \; \dots (7)$$

Note : For singular solution, consider $2p - 3x = 0$ (From (6))

Example 5. Investigate the singular solution of the differential equation $(y')^2 (1-y)^2 = 2 - y$

Solution : We have

$$f = (y')^2 (1-y)^2 - 2 + y = 0$$

i.e., $f = p^2 (1-y)^2 - 2 + y = 0 \quad \dots (1)$

Then $\frac{\partial f}{\partial p} = 2p(1-y)^2 \quad \dots(2)$

Eliminating p from $f = 0$ and $\frac{\partial f}{\partial p} = 0$ we have

$$(1-y)^2 (2-y) = 0 \; \dots (3)$$

For general solution, we rewrite the given equation as

$$p = \pm \frac{\sqrt{2y}}{1-y}; \; p = \frac{dy}{dx}$$

Integration gives

$$4(2-y)(1+y)^2 = 9(x+c)^2 \; \dots (4)$$

i.e., $\phi = 4(x-y)(1+y)^2 - 9(x+c)^2 = 0$

Then $\frac{\partial \phi}{\partial c} = 18(x+c)$

Eliminating c from $\phi = 0$ and $\frac{\partial \phi}{\partial c} = 0$ we get

$$(y+1)^2 (2-y) = 0 \; \dots (5)$$

From two discriminant, only the envelope $y = 2$ is the singular solution of the differential equation (because it is common in both and satisfies the given differential equation).

Example 6. Find the singular solutions of differential equation $y^4 p^3 - 6xp + 2y = 0$

Solution : We have

$$f = y^4 p^3 - 6xp + 2y = 0 \quad \ldots (1)$$

Then $$f' = \frac{\partial f}{\partial p} = 3y^4 p^2 - 6x \quad \ldots (2)$$

Eliminating p from $f = 0$ and $f' = 0$ we get

$$p^2 = \frac{2x}{y^4} \text{ and } 2y = p(6x - y^4 p^2)$$

$$p^2 = \frac{2x}{y^4} \text{ and } 4y^2 = p^2 (6x - p^2 y^4)^2$$

i.e., $$4y^2 = \frac{2x}{y^4} (6x - 2x)^2$$

$$y^4 = 8x^3 \text{ i.e., } y^2 = 2x \quad \ldots (3)$$

Which satisfies the given differential equation. Hence, $y^2 = 2x$ singular solution.

Example 7. Find singular solution of Clairaut's equation $y = px + g(p)$.

Solution : We have

$f = y - px - g(p) = 0 \ldots (1)$

Then,

$f' = -x - g'(p)$ ... (2)

Eliminating p from $f = 0$ and $f'(p) = 0$ we obtain singular solution of the differential equation ($\because f$ ¢ (p) = 0 gives $p = (g')^{-1}(x)$)

Example 8. Find singular solution of the different equation

$$(p-1)\,e^{4x} + e^{2y}\,p^2 = 0$$

Solution : In order to transform into Clairaut's form we use $u = e^{2x}$.

Then, $du = 2e^{2x}\,dx = 2udx,\ dv = 2e^{2y}\,dy = 2vdy$

i.e., $\frac{dv}{du} = \frac{v}{u}\frac{dy}{dx}$

$$P = \frac{v}{u}p; \quad p = \frac{dy}{dx}, \quad P = \frac{dv}{du}$$

Using in given equation, we obtain

$$\left(P\frac{u}{v} - 1\right)u^2 + v\frac{u^2}{v^2}P^2 = 0$$

$$P\frac{u^3}{v} + \frac{u^2}{v}P^2 - u^2 = 0 \quad \text{or} \quad \frac{Pu}{v} + \frac{P^2}{v} - 1 = 0$$

$$v = Pu + P^2 \qquad \ldots (1)$$

It is Clairaut's form and its general solution is

$$v = cu + c^2$$

$$\text{or} \quad e^{2y} = ce^{2x} + c^2 \quad \ldots (2)$$

In order to find singular solution, we eliminate p from

$$f = v - Pu + P^2 = 0 \quad \text{and} \quad \frac{\partial f}{\partial P} = 0 \ \text{i.e.,}$$

$$v - Pu + P^2 = 0 \quad \text{and} \quad u + 2P = 0$$

Hence $$v - u\left(\frac{-u}{2}\right) - \left(\frac{-u}{2}\right)^2 = 0$$

$$v + \frac{u^2}{4} = 0$$

Thus $\qquad 4e^{2y} + e^{4x} = 0 \quad \ldots (3)$

It can easily be proved that it satisfies the given equation.

Example 9. Find singular solution of differential equation

$$y = x^4 p^2 - px$$

Solution : In order to transform into Clairaut's form

we use $\qquad x = \frac{1}{t}$ with $p = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = P\left(\frac{-1}{x^2}\right) = -Pt^2$

Using is given equation we obtain

$$y = \frac{1}{t^4}(P^2 t^4) - (-Pt^2)\frac{1}{t}$$

$$y = P^2 + Pt \quad \text{or} \quad y = Pt + P^2 \quad \ldots (1)$$

It is Clairaut's form and its general solution is

$$y = ct + c^2 \quad \text{or} \quad y = \frac{c}{x} + c^2 \quad \ldots (2)$$

In order to find singular solution, eliminate P from $f = y - Pt - P^2$

and $\qquad \frac{\partial f}{\partial P} = -t - 2P = 0$

Hence $\qquad y = Pt + P^2 \quad \text{and} \quad P = -\frac{t}{2}$

i.e., $\qquad y = -\frac{t^2}{2} + \frac{t^2}{4} \quad \text{or} \quad 4y = -t^2 \quad \text{or} \quad 4y = -\frac{1}{x^2}$

i.e., $\qquad 4x^2 y + 1 = 0$

It can easily be verified that it is solution of given equation.

Example 10. Solve and find singular solution of the differential equation.

$$4x(x-1)(x-2)\,p^2-(3x^2-6x+2)^2=0$$

Solution : Given equation is solvable for p i.e.,

$$\frac{dy}{dx}=p=\pm\frac{1}{2}\frac{(3x^2-6x+2)}{\sqrt{x(x-1)(x-2)}}$$

It variable separable differential equation and integration gives

$$dy=\pm\left(\frac{3x^2-6x+2}{2\sqrt{x^3-3x^2+2x}}\right)$$

$$y=\pm\sqrt{x^3-3x^2+2x}+c$$

$$(y-c)^2=x(x-1)(x-2)$$

$$c^2-2yc+y^2-x(x-1)(x-2)=0 \quad \ldots (2)$$

It is quadratic c. Therefore C-discriminant is

$$4y^2-4(y^2-x(x-1)(x-2))=0$$

$$\text{i.e., } x(x-1)(x-2)=0 \ldots (3)$$

It is singular solution of the differential equation.

Example 11. Find the singular solution of differential equation.

$$p^3-4xyp+8y^2=0$$

Solution : In order to transform the given differential equation into Clairaut's form, we Substitute $y=v^2$ with $\frac{dy}{dx}=2v\frac{dv}{dx}$

i.e., $P=2vP;\ P=\frac{dv}{dx}$

Using is given equation to obtain

$$v=xP-P^3 \qquad \ldots (1)$$

It is Clairaut's form and its general solution

$$v = xc - c^3 \quad \text{or} \quad y = c^2\,(x - c^2)^2 \qquad \ldots (2)$$

For singular solution, eliminate p from $v - xP + P^3 = 0$ and $x - 3P^2 = 0$ and obtain.

$$v = \frac{2x}{3}\sqrt{\frac{x}{3}}$$

$$\boxed{27y = 4x^3} \qquad \ldots (3)$$

12.7 RICCATI EQUATION

A differential equation $y' + q(x)\,y^2 + p(x) = 0 \;\ldots (1)$

is called a Ricaati equation, where $q(x) \neq 0$, p are continuous functions on an interval $I = [a, b] \subseteq \mathbf{R}$. A more general form of Riccati equation is given by

$$y' + a(x)\,y + b(x)y^2 + c(x) = 0 \quad \ldots (2)$$

Where a, b, c are continuous functions on an interval $I = [a, b] \subseteq \mathbf{R}$. If $b(x)$ $=0$ on I then Eq. (2) becomes a linear differential equation of order one and therefore it is solvable. For $a(x) = 0$ on I, Eq. (2) and Eq. (1) are same. If $a(x) \neq 0$ on I, then Eq. (2) can easily be transformed into Eq. (1).

Theorem (1) : Every Riccati equation $y'' + a(x)y + b(x)y^2 + c(x) = 0$

can be transformed into $z' + q(x)z^2 + p(x) = 0$

Proof : Multiplying given equation by $e^{\int a(x)\,dx}$ We have

$$y' e^{\int a(x)\,dx} + a(x)\,y\,e^{\int a(x)\,dx} + b(x)\,y^2\,e^{\int a(x)\,dx} + c(x)\,e^{\int a(x)\,dx} = 0$$

i.e.,
$$\frac{d}{dx}\left(y\,e^{\int a(x)\,dx}\right) + y^2\,e^{2\int a(x)\,dx}.\,b(x)\,e^{-\int a(x)\,dx} + c(x)\,e^{\int a(x)\,dx} = 0$$

Substitute $y\,e^{\int a(x)\,dx} = z$ in above equation to obtain

$$\frac{dz}{dx} + b(x)\,e^{-\int a(x)\,dx}\;z^2 + c(x)\,e^{\int a(x)\,dx} = 0$$

or $$z' + q(x)\,z^2 + p(x) = 0$$

where $q(x) = b(x)\,e^{-\int a(x)\,dx}$, $p(x) = c(x)\,e^{\int a(x)\,dx}$

Theorem (2) : Every Riccati equation can be reduced to a self-adjoint equation of order two and vice-versa.

Proof : Suppose $z' + q(x)\,z^2 + p(x) = 0$ be a Riccati equation and substitute $z = \dfrac{y'}{qy}$ to obtain

$$\frac{qyy'' - (qy' + q'y)\,y'}{(qy)^2} + q\left(\frac{y'}{qy}\right)^2 + p\left(\frac{qy}{qy}\right)^2 = 0$$

$$qyy'' - q(y')^2 - q'yy' + q(y')^2 + p(qy)^2 = 0$$

$$\frac{y''}{q} - \frac{q'y'}{q^2} + py = 0$$

$$\frac{d}{dx}\left(\frac{y'}{q}\right) + py = 0$$

or $$\frac{d}{dx}(r(x)y') + p(x)\,y = 0;\ r(x) = \frac{1}{q(x)}$$

It is a **self-adjoint equation** of order two. Again let $\frac{d}{dx}\left(\frac{y'}{q}\right) + py = 0$ be a self-adjoint different equation of order two let

$$z = \frac{y'}{qy} \quad \text{or} \quad yz = \frac{y'}{q}$$

Using in above equation we obtain

$$\frac{d}{dx}(yz) + py = 0$$

$$yz' + y'z + py = 0$$

$$z' + \frac{y'}{y}z + p = 0$$

$$\text{or} \qquad z' + qz^2 + p = 0$$

Example 1. Find all the solutions of the Riccati equation

$$y' - y^2 - 1 = 0$$

Solution : We have

$$y' - y^2 - 1 = 0$$

On comparing with $y' + q(x)\, y^2 + p(x) = 0$ we have

$$q(x) = -1,\ p(x) = -1$$

Substitute $y = \frac{z'}{qz}$ i.e., $y = -\frac{z'}{z}$ in given equation we obtain

$$\frac{d}{dx}\left(\frac{-z'}{q}\right) + p(x)\, z = 0 \quad \text{i.e.,} \quad z'' + z = 0 \ \ldots (2)$$

and the general are given by

$$z = c_1 \cos x + c_2 \sin x \ \ldots (3)$$

Hence, all the solution of the given Riccati equation are given by

$$y = -\frac{z'}{z} = \frac{c_1 \sin x - c_2 \cos x}{c_1 \cos x + c_2 \sin x} \quad \dots (4)$$

If $c_1 = 0$ then $y = -\cot x$ and if $c_2 = 0$ then $y = \tan x$

Example 2. Find all the solutions of the Riccati equation $y' + y - e^x y^2 - e^{-x} = 0$

Solution : Given equation is transformed into

$$z' + q(x)\, z^2 + p(x) = 0 \quad \dots (1)$$

where $q(x) = b(x)\, e^{-\int a(x)\,dx},\ p(x) = c(x)\, e^{\int a(x)\,dx}$

and $z = ye^{\int a(x)\,dx}$

i.e., $q(x) = -e^x\, e^{-\int dx} = -1,\ p(x) = -e^{-x}\, e^{\int dx} = -1$

Substitute $z = \frac{t'}{qt} = -\frac{t'}{t}$ in Eq. (1) to obtain

$$\frac{d}{dx}\left(\frac{t'}{q}\right) + pt = 0 \quad \text{i.e.,} \quad \frac{-d}{dx}(t') - t = 0$$

and its general solution are given $t = c_1 \cos x + c_2 \sin x$

Hence, all solutions of Eq. (1) are

$$z = -\frac{t'}{t} = \frac{c_1 \sin x - c_2 \cos x}{c_1 \cos x + c_2 \sin x}$$

Thus, all the solutions of given equations are

$$y = ze^{-\int a(x)\,dx} = e^{-x}\left[\frac{c_1 \sin x - c_2 \cos x}{c_1 \cos x + c_2 \sin x}\right]$$

Note : Since Riccati equation is a differential equation of order one therefore except that its solutions should contain only one arbitrary constant. But solutions of Riccati equation two arbitrary constants. This is contrary to our expectation.

This confusion can easily be removed as :

Let u and v be two linearly independent solutions self-adjoint equation.

$$\frac{d}{dx}\left(\frac{z'}{q}\right)+p(x)\,z=0$$

Then general solution of Riccati equation,

$$y'+q(x)\,y^2+p(x)=0$$

is given by

$$y=\frac{1}{q(x)}\,\frac{c_1u'(x)+c_2v'(x)}{c_2u(x)+c_2v(x)}$$

Assume that $c_1 \neq 0$ then

$$y=\frac{1}{q(x)}\,\frac{u'(x)+kv'(x)}{u(x)+kv(x)}$$

where $k=\frac{c_2}{c_1}$ is an arbitrary constant.

Thus shows that the solution of Riccati equation involve only one arbitrary constant.

Example 3 : If y_1 is a particular solutions of Raccati equation $y'+a(x)y+b(x)\,y^2+c(x)=0$ then general solutions containing one arbitrary constant can be obtained by substituting $y=y_1(x)+\frac{1}{v(x)}$ where $v(x)$ is solutions of linear differential equation

$$\frac{dv}{dx}-(a(x)+2b(x)y_1)\,v=b(x)$$

Solution : Since $y=y_1(x)$ is a particular solutions of given Raccati equation therefore,

$$y_1'+a(x)\,y_1+b(x)\,y_1^2+c(x)=0$$

Substituting, $y=y_1+\dfrac{1}{v(x)}$ in given Riccati equation we obtain

$$\left(y_1'-\frac{v'}{v^2}\right)+a(x)\left(y_1+\frac{1}{v(x)}\right)+b(x)\left(y_1+\frac{1}{v(x)}\right)^2+c(x)=0$$

$$y_1'+a(x)\,y_1+b(x)\,y_1^2+c(x)+\left(\frac{-v'}{v^2}+\frac{a(x)}{v}+\frac{b(x)}{v^2}+\frac{2y_1b(x)}{v}\right)=0$$

Using (1) we get

$$-v'+(a(x)+2y_1b(x))\,v+b(x)=0$$

$$v'-(a(x)+2y_1b(x))\,v=b(x) \qquad \dots (2)$$

Which is a linear differential equation of order one and its solution will contain only one arbitrary constant.

Example4:

Solve the Riccati equation $y'+2xy-y^2-(1+x^2)=0;\ y_1\,(x)=x$

Solution : On comparing given equation with

$$y'+2xy-y^2-(1+x^2)=0$$

we have $a(x)=2x,\ b(x)=-1,\ c\,(x)=-\,(1+x^2)$

Since y_1 is a particular solution of given differential equation therefore general solution of given Riccatti equation

$$y=y_1+\frac{1}{v(x)}=x+\frac{1}{v(x)}$$

Where v is solution of differential equation

$$\frac{dv}{dx} - (a(x) + 2y_1\, b(x))\, v = b(x)$$

Hence, $$\frac{dv}{dx} - (2x + 2x\,(-1))\, v = -1$$

$$\frac{dv}{dx} = -1 \quad \text{i.e.,} \quad v = c - x$$

Thus, general solution of given Riccati equation

$$y = x + \frac{1}{c - x}$$

12.8SUMMARY

This unit is a Description of the singular solution, envelope and P and C Discriminats, Determination of Singular Solution and solution of the Riccati Equation. In briefly a Riccati equation in the narrowest sense is any first-order ordinary differential equation that is quadratic in the unknown function. In geometry, an envelope of a planar family of curves is a curve that is tangent to each member of the family at some point, and these points of tangency together form the whole envelope. In mathematics, the discriminant of a polynomial is a quantity that depends on the coefficients and allows deducing some properties of the roots without computing them. One of the common ways to determine existence of a singular solution is to eliminate constant from the system of equations: A singular solution in a stronger sense is such function that is tangent to every solution from a family of solutions, forming the envelope of this family of solutions.

12.9GLOSSARY

- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Dependent Variable:** The variable whose value depends on the value of another variable.
- **Independent Variable:** The variable that is varied independently of other variables.
- **Rate of Change:** The speed at which a quantity changes with respect to time or another variable.
- **Derivative:** A measure of how a function changes as its input changes.
- **First Order:** The highest derivative involved in the equation is the first derivative.
- **First Degree:** The degree of the equation, which refers to the highest power of the highest-order derivative, is one.
- **Ordinary Differential Equation (ODE):** A differential equation involving only ordinary derivatives with respect to one independent variable.
- **Dependent Variable:** The variable whose value depends on the value of another variable, often denoted as y.
- **Independent Variable:** The variable that is varied independently of other variables, often denoted as x.
- **Function:** A mathematical relation that assigns a unique output value to each input value.
- **Derivative:** A measure of how a function changes as its input changes, often representing rates of change.
- **Initial Condition:** A condition that specifies the value of the dependent variable at a particular point in the independent variable's domain.

- **Solution:** A function or set of functions that satisfy the given differential equation and any accompanying initial conditions.

CHECK YOUR PROGRESS

1. Find differential equation if the complete primitive i.e., $c^2 + 2cy + x^2 + 1 = 0$ and also find singular solution.
.

2. Reduce, the equation $3xp^2 - 6yp + x + 2y = 0$ to Clairaut's form using $u = x^2, v = x - 3y$ and also find its singular solution.

3. Solve $xp^2 - 2yp + x + 2y = 0$ using $u = x^2, v = y - x$ and also find its singular solution.

4. Find all the solutions of the following Riccati equation

$$y' - 2y + y^2 + 2 = 0$$

12.10 REFERENCES

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12.11 SUGGESTED READING

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12.12 TERMINAL QUESTIONS:-

EXERCISE (A):

Find singular solution of following differential equations :

1. $y + px = x^4 p^2$

2. $xp^2 - 2yp + x = 0$

3. $x^3 p^2 + x^2 py + a^3 = 0$

4. $y = 2px + p^4 x^2$

5. $y = \frac{1}{\sqrt{1+p^2}} + b$

6. $y = \sin p - p \cos p$

7. $p^3 - 4xyp + 2y^2 = 0$

8. $y = px + \frac{a}{p}$

9. $p = \ln(px - y)$

10. $y = px - \frac{1}{p^2}$

EXERCISE (B):

1. Find all the solutions of the following Riccati equation

   (a) $y' + y^2 - 1 = 0$

   (b) $y' - y + y^2 - 2 = 0$

2. Solve the following Riccati equations

   (a) $y' + \frac{y}{x} - y^2 + \frac{1}{x^2} = 0;\ y_1(x) = \frac{1}{x}$

   (b) $y' - \frac{y^2}{2\cos x} + \frac{\sin^2 x - 2\cos^2 x}{2\cos x};\ y_1(x) = \sin x$

12.13 ANSWERS:-

CHECK YOUR PROGRESS

CHQ 1:

$p^2(1-x^2) + xyp + x^2 = 0$ and $x^2 + y^2 = 1$

CHQ 2:

$c^2xy + cy - x = 0;\ y = 0,\ y + 4x^2 = 0$

CHQ 3:

$2c^2x^2 - 2c(y-x) + 1 = 0\ 3x^2 - 2xy + y^2 = 0$

CHQ 4:

$$y = e^x \left(\frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} \right)$$

TERMINAL ANSWERS (TQ'S)

Answers (A) :

1. $4x^2 y + 1 = 0$

2. $y = \pm\, x$

3. $xy^2 = 4a^3$

4. $16y^3 + 27x^2 = 0$

5. $y = 1 + b$

6. $y = 0$

7. $27y = 4x^3$

8. $y^2 = 4ax$

9. $y = x\,(\ln x - 1)$

10. $4y = (x+1)^2$

Answers (B) :

1. (a) $y = \dfrac{c_1 e^x - c_2 e^{-x}}{c_1 e^x + c_2 e^{-x}}$

(b) $y = \dfrac{-c_1 e^{-x} + 2c_2 e^{2x}}{c_1 e^{-x} + c_2 e^{2x}}$

2. (a) $y = \dfrac{1}{x} + \dfrac{2x}{(c - x^2)}$

(b) $y = \sin x + \left(c \cos x - \dfrac{1}{2} \sin x \right)^{-1}$

BLOCK-V

Unit 13: Solution of Partial Differential Equations

CONTENTS:

13.1 *INTRODUCTION*

Partial Differential Equations (PDEs) play a crucial role in describing and understanding a wide range of physical phenomena and mathematical concepts. They are fundamental tools in fields such as physics, engineering, biology, finance, and more. PDEs describe how functions and variables change with respect to multiple independent variables, including time and space. This introduction provides an overview of the formation and solution of PDEs, highlighting their significance and the approaches used to tackle them.

PDEs are essential mathematical tools for modeling dynamic processes across various fields. Their formation involves translating physical systems into mathematical equations, while their solution requires a combination of analytical and numerical techniques. PDEs provide a bridge between theory and real-world applications, enabling us to make informed decisions and advancements in science and technology.

In this unit, we propose to study various methods to solve partial differential equations.

13.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To develop a fundamental understanding of what partial differential equations are and how they differ from ordinary differential equations.
- To Understand the Basics of PDEs.
- To Analyzing the Physical Phenomena.

The objectives of studying the formation of solutions to PDEs are designed to equip learners with a solid foundation in the theory and application of partial differential equations, preparing them to tackle diverse challenges in mathematics, physics, engineering, and other scientific disciplines.

13.3 *PARTIAL DIFFERENTIAL EQUATION:-*

A Partial Differential Equation (PDE) is a type of differential equation that involves multiple independent variables and their partial derivatives with respect to those variables. Unlike ordinary differential equations (ODEs), which involve a single independent variable, PDEs deal with functions of two or more independent variables.

Or

"*An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as partial differential equation.*"

Mathematically, a partial differential equation typically takes the form:

$$F\left(x_1, x_2, \cdots x_n z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \cdots \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1^2}, \frac{\partial^2 z}{\partial x_2^2}, \cdots \frac{\partial^2 z}{\partial x_n^2}\right) = 0 \quad \cdots (1)$$

where

- $x_1, x_2, \cdots x_n$ are the independent variables,
- z is the unknown function of these variables,
- $\frac{\partial z}{\partial x_i}, (i = 1,2 \cdots n)$ represents the partial derivative of u with respect to x_i **(the first-order partial derivative)**.
- $\frac{\partial^2 z}{\partial x_i^2}$ represents the **second-order partial derivative** of u with respect to x_i, and F is some mathematical expression that relates u and its partial derivatives.

13.4 *ORDER OF PARTIAL DIFFERENTIAL EQUATION:-*

The order of a partial differential equation (PDE) refers to the highest order of partial derivatives present in equation (1).

For Example:

- The equations $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy,\quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz,\quad z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$ are of the **first order.**
- The equations $\frac{\partial^2 z}{\partial y^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2},\ \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ are of the **second and third order.**

13.5 *DEGREE OF PARTIAL DIFFERENTIAL EQUATION:-*

The degree of a partial differential equation (PDE) is the highest power to which the highest-order partial derivative term is raised in the equation.

For Example: $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + du = 0$

In this PDE, the highest-order partial derivative term is$\frac{\partial^2 u}{\partial x^2}$, and its degree is 2 because it is raised to the power of 2.

13.6 *LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATION:-*

The partial differential equation is called ***LINEAR*** if the dependent variable and its partial derivatives occur only in the first degree and not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation.

Example: The equation $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy, \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$ are Linear.

A partial differential equation which is not Linear is known as **NON-LINEAR** partial differential equation.

Example: The equation $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x, \frac{\partial^2 z}{\partial y^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$ are non-linear.

Notation: Now we adopt the following notations throughout the study of PDEs

$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y}$ and $r = \frac{\partial^2 z}{\partial y^2}$

Let we take $x_1, x_2 \dots \dots . x_n$($n$ independent variable) and z is then regarded as the dependent variable. Hence we use the following notation.

$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}$ and $p_n = \frac{\partial z}{\partial x_n}$

$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$ and so on.

13.7*CLASSIFICATION OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS:-*

First-order partial differential equations (PDEs) can be classified into four categories: linear, semi-linear, quasi-linear, and non-linear. These classifications are based on the degree of linearity in the PDEs. Here's an explanation of each category with examples:

A first order partial differential equation in two variables in its most general form can be expressed as

$$F(x, y, z, p, q) = 0 \qquad \dots (1)$$

Where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$, z is dependent variable and x, y is independent variables.

a. **Linear PDEs:** Linear PDEs are those in which all terms involving the dependent variable and its partial derivatives are of the first degree. OR

A first order partial differential equation $F(x, y, z, p, q) = 0$ is said to be ***LINEAR*** if it is linear p, q and z *i.e.*, If given equation is of the form

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y).$$

For example:

$$yx^2p + xy^2q = xyz + x^2y^3$$
$$pcos(x + y) + qsin(x + y) = z + e^y sinx$$
$$p + 3q = 5z + tan(y - 3x)$$
$$p + q = z + xy \text{ etc.,}$$

are all linear partial differential equations.

b. **Semi-Linear PDEs:** A first order partial differential equation $F(x, y, z, p, q) = 0$ is said to be ***SEMI-LINEAR*** if it is linear p, q and the coefficient of p and q are the functions of x and y *i.e.*, If it is of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

For example:

$$(x + y^2)p + xlogy\, q = 2z^2x + xy + e^x$$
$$pcos(x + y) + qsin(x + y) = z^3 + e^y + sinx$$
$$xyp + x^2yq = x^2y^2z^2$$
$$yp + xq = (x^2z^2/y^2) \text{ etc.,}$$

are all Semi-linear partial differential equations.

c. **Quasi-Linear PDEs:** A first order partial differential equation $F(x, y, z, p, q) = 0$ is said to be ***QUASI-LINEAR*** if it is linear in p, q *i.e.*, If it is of the form

$$P(x, y, z) + Q(x, y, z)q = R(x, y, z)$$

For example:

$$(x + y + z)p + xyz + xz = 3x^2 + 5y^2 + 6z^2$$
$$(x^2 + y^2)p + 4xyzq = 3z + e^{x+y}$$
$$(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy) \text{ etc.,}$$

are all Quasi-linear partial differential equations.

d. Non-Linear PDEs: A first order partial differential equation $F(x, y, z, p, q) = 0$ which does not come under the above three types, called Non-Linear equation.

For example:

$$p^2 + q^2 = 1$$
$$pq = z$$
$$x^2p^2 + y^2q^2 = z^2 \text{ etc.,}$$

are all Non-linear partial differential equations.

13.8 *FORMATION OF PDEs:-*

Partial differential equation can be formed either by elimination of arbitrary constants or by elimination of arbitrary functions.

<u>RULE1:</u> **Derivation of a partial differential equation by elimination of arbitrary constants.**

Let us consider $F(x, y, z, p, q) = 0$ ... (1)

where a, b are arbitrary constants. Let z be the function of two independent variables x and y.

Now differentiating (1) w.r.t x and y, we obtain

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0$$
$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0$$

Solving these two equations we can formulate partial differential equation (1).

Situation I: *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

Example: Let $z = ax + y$

Differentiating above equation w.r.t. x and y , we obtain

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = 1$, then

$$z = x\left(\frac{\partial z}{\partial x}\right) + y$$

Situation II: *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants usually gives rise to unique partial differential equation of order one.*

Example: Let $az + b = a^2 x + y$

Differentiating above equation w.r.t. x and y , we get

$a\frac{\partial z}{\partial x} = a^2$ and $a\left(\frac{\partial z}{\partial y}\right) = 1$, then

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

Situation III: *When the number of arbitrary constants is grater then the number of independent variables, then the elimination of arbitrary constants leads to unique partial differential equation usually greater than one.*

Example: Let $z = ax + by + cxy$ ... (1)

Differentiating above equation w.r.t. x and y , we obtain

$$\frac{\partial z}{\partial x} = a + cy, \frac{\partial z}{\partial y} = b + cx \quad \text{... (2)}$$

and

$$\frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y^2} = 0, \ \frac{\partial^2 z}{\partial x \partial y} = c \quad \text{... (3)}$$

then

$\left(\frac{\partial z}{\partial x}\right)x = ax + cxy$ and $\left(\frac{\partial z}{\partial y}\right)y = by + cxy$

Now from (1)

$$z = \left(\frac{\partial z}{\partial x}\right)x - cxy + \left(\frac{\partial z}{\partial y}\right)y - cxy + cxy$$

$z + cxy = \left(\frac{\partial z}{\partial x}\right)x + \left(\frac{\partial z}{\partial y}\right)y \Rightarrow \ z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \left(\frac{\partial z}{\partial x}\right)x + \left(\frac{\partial z}{\partial y}\right)y.$

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Solve the partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.

SOLUTION: The given equation is

$$z = ax + by + a^2 + b^2 \quad \text{... (1)}$$

Differentiating (1) equation w.r.t. x and y , we obtain

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$.

Putting the value of a and b in (1), we have

$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$. Which is required solution (PDEs).

EXAMPLE2: Solve the partial differential equation by eliminating h and k from $(x-h)^2 + (y-k)^2 + z^2 = \lambda^2$.

SOLUTION: The given equation is

$$(x-h)^2 + (y-k)^2 + z^2 = \lambda^2 \quad \text{... (1)}$$

Differentiating (1) equation w.r.t. x and y , we obtain

$2(x-h)+2z\left(\frac{\partial z}{\partial x}\right)=0$ and $(x-h)=-z\left(\frac{\partial z}{\partial x}\right)$

and

$2(y-k)+2z\left(\frac{\partial z}{\partial y}\right)=0$ and $(x-k)=-z\left(\frac{\partial z}{\partial y}\right)$

Putting the value of $(x-h)$ and $(x-k)$in (1), we obtain

$z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$ and $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right]=\lambda^2$

Which is required solution (PDEs).

EXAMPLE3: Solve the partial differential equation by eliminating a and b from the following relations:

a. $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
b. $2z=(ax+y)^2+b$

SOLUTION:

a. Let the equation

$$2z=\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad \dots(1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$$2\left(\frac{\partial z}{\partial x}\right)=\frac{2x}{a^2} \Rightarrow p=\left(\frac{\partial z}{\partial x}\right)=\frac{x}{a^2} \Rightarrow a^2=\frac{x}{p}$$

$$2\left(\frac{\partial z}{\partial y}\right)=\frac{2y}{b^2} \Rightarrow q=\left(\frac{\partial z}{\partial y}\right)=\frac{y}{b^2} \Rightarrow b^2=\frac{y}{p}$$

Putting the value of a^2 and b^2in (1), we get

$2z=\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2}{\left(\frac{x}{p}\right)^2}+\frac{y^2}{\left(\frac{y}{p}\right)^2}=px+qy$. is required solution (PDEs).

b. Let the equation

$$2z=(ax+y)^2+b \qquad \dots(1)$$

Differentiating (1) equation w.r.t. x and y , we have

$$2p=2a(ax+y) \Rightarrow p=a(ax+y)$$

$$2q=2(ax+y) \Rightarrow q=(ax+y)$$

where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$.

Dividing both above equation $\frac{p}{q}=a$.

Putting the value of a in (1), we obtain

$q=\left(\frac{p}{q}\right)x+y$ or $px+qy=q^2$.

EXAMPLE4: Solve the partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

SOLUTION: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ... (1)

Differentiating (1) equation w.r.t. x and y , we can write

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \Rightarrow c^2x+a^2z\frac{\partial z}{\partial x}=0 \quad ...(2)$$

$$\frac{2x}{b^2}+\frac{2z}{c^2}\frac{dz}{dx}=0 \Rightarrow c^2x+a^2z\frac{dz}{dx}=0 \quad ...(3)$$

Differentiating (2) w.r.t. x and(3) w.r.t. y , we obtain

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \quad ...(4)$$

$$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \quad ...(5)$$

Now again from (2),

$$c^2x=-a^2z\frac{\partial z}{\partial x}$$

$$c^2=-\frac{za^2}{x}\frac{\partial z}{\partial x}$$

Substituting the value of c^2 in (4) and dividing by a^2, we obtain

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0$$

$$-\frac{za^2}{xa^2}\frac{\partial z}{\partial x}+\frac{a^2}{a^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{a^2}{a^2}z\frac{\partial^2 z}{\partial x^2}=0$$

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0 \text{ or } zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \quad ...(7)$$

Similarly from (3) and (5), we get

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \quad ...(8)$$

Again differentiating (2) partially w.r.t. y,

$$0+a^2\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0$$

Or

$$a^2\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)+z\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right\}=0 \quad ...(9)$$

Hence (7), (8) and (9) are three possible forms of the required PDEs.

EXAMPLE5: Solve the partial differential equation by eliminating a, b, c from $ax^2+by^2+cz^2=1$.

SOLUTION: Given the equation

$$ax^2 + by^2 + cz^2 = 1 \qquad \dots(1)$$

Differentiating (1) equation w.r.t. x and y , we obtain

$$2ax + 2cz\left(\frac{\partial z}{\partial x}\right) = 0 \qquad \dots (2)$$

$$2by + 2cz\left(\frac{\partial z}{\partial y}\right) = 0 \qquad \dots (3)$$

Differentiating (2) equation w.r.t. y , we have

$$0 + 2c\left\{\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) + z\left(\frac{\partial^2 z}{\partial x \partial y}\right)\right\} = 0$$

or

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) + z\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0 \quad \dots(4)$$

Since c is arbitrary constant. The equation (4) is the desired PDEs.

Again, differentiating (2) equation w.r.t. x and (3) w.r.t.y , we get

$$2a + 2c\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\left(\frac{\partial^2 z}{\partial x^2}\right)\right\} = 0$$

and

$$2b + 2c\left\{\left(\frac{\partial z}{\partial y}\right)^2 + z\left(\frac{\partial^2 z}{\partial y^2}\right)\right\} = 0$$

Again from (2), $a = -\frac{cz}{x}\frac{\partial z}{\partial x}$. Substituting this in above equation, we obtain

$$-\left(\frac{cz}{x}\right) \times \left(\frac{\partial z}{\partial x}\right) + c\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\left(\frac{\partial^2 z}{\partial x^2}\right)\right\} = 0$$

or $$zx\left(\frac{\partial^2 z}{\partial x^2}\right) + x\left(\frac{\partial z}{\partial x}\right)^2 - z\left(\frac{\partial z}{\partial x}\right) = 0 \quad \dots(5)$$

Similarly from (3), we get

$$zy\left(\frac{\partial^2 z}{\partial y^2}\right) + y\left(\frac{\partial z}{\partial y}\right)^2 - z\left(\frac{\partial z}{\partial y}\right) = 0 \quad \dots(6)$$

Hence required the PDEs.

<u>RULE2:</u> Derivation of a partial differential equation by elimination of Arbitrary function $\boldsymbol{\phi}$ from the equation $\boldsymbol{\phi(u, v) = 0}$,where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the functions of $\boldsymbol{x, y}$ and $\boldsymbol{z}$.

Let $$\phi(u, v) = 0 \qquad \dots(1)$$

be the given equation and let

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial y}{\partial x} = 0 \text{ and } \frac{\partial x}{\partial y} = 0 \qquad \dots(2)$$

Now differentiating (1) w.r.t.x, we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

Now from (2)

$$\frac{\left(\frac{\partial \phi}{\partial u}\right)}{\left(\frac{\partial \phi}{\partial v}\right)}=-\frac{\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)} \qquad \dots (3)$$

Similarly Differentiating (1) w.r.t. y, we have

$$\frac{\left(\frac{\partial \phi}{\partial u}\right)}{\left(\frac{\partial \phi}{\partial v}\right)}=-\frac{\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)} \qquad \dots (4)$$

Now eliminating ϕ with the help of (3) and (4), we obtain

$$\frac{\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)}=\frac{\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)}$$

$$\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+p\frac{\partial v}{\partial z}\right)$$

or $$Pp+Qq=R$$

where $P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$, $R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

SOLVED EXAMPLE

EXAMPLE1: Solve the partial differential equation by eliminating the arbitrary function f from the equation $x+y+z=f(x^2+y^2+z^2)$.

SOLUTION: The given equation is

$$x+y+z=f(x^2+y^2+z^2) \qquad \dots (1)$$

Differentiating (1) with w.r.t. x and y

$$1+p=f'(x^2+y^2+z^2)\,(2x+2zp) \dots (2)$$

$$1+q=f'(x^2+y^2+z^2)\,(2y+2zq) \dots (3)$$

From (2) and (3), we obtain

$$\frac{1+p}{(2x+2zp)}=\frac{1+q}{(2y+2zq)}$$

$$(1+p)(y+zq)=(1+q)(x+zp)$$

$$(y+zq)+p(y+zq)=(x+zp)+q(x+zp)$$

$$(y+zq)+py+zqp=(x+zp)+qx+zpq$$

$$(y+zq)+py=(x+zp)+qx$$

$$y+zq+py=x+zp+qx$$

$$zq + py - zp - qx = x - y$$

$p(y - z) + q(z - x) = x - y$ is required the PDEs.

EXAMPLE2: Eliminate the arbitrary functions f and F from $y = f(x - at) + F(x + at)$.

SOLUTION: The given equation is

$$y = f(x - at) + F(x + at) \quad \dots (1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{\partial y}{\partial x} = f'(x - at) + F'(x + at)$$

Again, differentiating,

$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) + F''(x + at) \quad \dots (2)$$

Also, differentiating (1) w.r.t. t, we obtain

$$\frac{\partial y}{\partial t} = f'(x - at)(-a) + F'(x + at)(a)$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x - at)(-a)^2 + F''(x + at)(a)^2$$

$$\frac{\partial^2 y}{\partial t^2} = a^2[f''(x - at) + F''(x + at)] \quad \dots (3)$$

From (1) and (2), we obtain

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

EXAMPLE3: From the partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$.

SOLUTION: Let the given equation

$$z = f(x^2 - y) + g(x^2 + y) \quad \dots (1)$$

Now differentiating (1) w.r.t. x and y , we have

$$\frac{\partial z}{\partial x} = 2xf'(x^2 - y) + 2gx(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}$$

$$\frac{\partial z}{\partial y} = -f' \ (x^2 - y) + g'(x^2 + y)$$

Again differentiating above equation w.r.t. . x and y , we obtain

$$\frac{\partial^2 z}{\partial x^2} = 2\{f' \ (x^2 - y) + g' \ (x^2 + y)\}$$

$$+ 4x^2\{(x^2 - y) + g'' \ (x^2 - y)\} \dots (2)$$

$$\frac{\partial^2 z}{\partial y^2} = f'' \ (x^2 - y) + g'' \ (x^2 + y)$$

Again (2),

$$f'\ (x^2-y)+g'(x^2+y)=\left(\frac{1}{2x}\right)\times\left(\frac{\partial z}{\partial x}\right)$$

Putting the value of $f''\ (x^2-y)+g''\ (x^2+y)$ and $f'\ (x^2-y)+g'(x^2+y)$ in (2), we get

$$\frac{\partial^2 z}{\partial x^2}=2\times\left(\frac{1}{2x}\right)\times\left(\frac{\partial z}{\partial x}\right)+4x^2\frac{\partial^2 z}{\partial y^2}$$

$x\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial z}{\partial x}\right)+4x^3\frac{\partial^2 z}{\partial y^2}$ is required the solution.

13.9 *CAUCHY'S PROBLEM FOR FIRST ORDER PDEs:-*

If

a. $x_0(\mu), y_0(\mu)$ and$z_0(\mu)$ are functions which together with their first derivatives, are continuous in interval I defined by $\mu_1<\mu<\mu_2$.

b. And if $f(x,y,z,p,q)$ is continuous function of x,y,z,p and q in certain region U of the $xyzpq$ space, then it is required to establish the existence of function $\phi(x,y)$ with the following properties:

   i. $\phi(x,y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

   ii. For all value of x and y lying in R, the point $\{x,y,\phi(x,y),\phi_x(x,y),\phi_y(x,y)\}$ lies in U and $f\{x,y,\phi(x,y),\phi_x(x,y),\phi_y(x,y)\}=0$.

   iii. For all μ belonging to interval I, the point $\{x_0(\mu),y_0(\mu)\}$belongs to the region R and $\phi\{x_0(\mu),y_0(\mu)\}=z_0$.

   Stated geometrically, what we wish to prove is that there exists a surface $z=\phi(x,y)$ which passes through the curve C whose parametric equations are given by $x=x_0(\mu), y=y_0(\mu), z=z_0(\mu)$ and every point of which the direction $(p,q,-1)$ of the normal is such that $f(x,y,z,p,q)=0$.

EXAMPLE: Solve the Cauchy's problem for $zp+q=1$, when the initial data curve in $x_0=\mu, y_0=\mu, z_0=\frac{\mu}{2}, 0\le\mu\le 1$.

SOLUTION: The given equation

$$f(x, y, z, p, q) = zp + q - 1 = 0 \quad \dots (1)$$

And the given initial data curve

$$x_0 = \mu, y_0 = \mu, z_0 = \frac{\mu}{2}, 0 \le \mu \le 1 \quad \dots (2)$$

Now from (1), we have

$\frac{\partial f}{\partial p} = z,$ $\frac{\partial f}{\partial q} = 1$ and $\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu,$ for $0 \le \mu \le 1$.

Now we have $\frac{dx}{dt} = \frac{\partial f}{\partial p}$ $\frac{dy}{dt} = \frac{\partial f}{\partial q}$ and $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

And $\frac{dx}{dt} = z$, $\frac{dy}{dt} = 1 \quad \dots (3)$

$$\frac{dz}{dt} = p\left(\frac{\partial f}{\partial p}\right) + q\left(\frac{\partial f}{\partial q}\right) = pz + q = 1, by\ (1) \quad \dots (4)$$

Now integrating (3) and (4) , we have

$$y = t + C_1, \qquad z = t + C_2 \quad \dots (5)$$

Again from (2) at $t = 0, x(\mu, 0) = \mu, y(\mu, 0) = \mu$ and $z(\mu, 0) = \frac{\mu}{2}$ ... (6)

Using (6), (5) reduce to $y = t + \mu$ and $z = t + \frac{\mu}{2} \dots (7)$

Again from (3) and (7), we get

$$\frac{dx}{dt} = t + \frac{\mu}{2} \text{ so that } x = \frac{1}{2} \times t^2 + \frac{1}{2} \times \mu t + C_3$$

Now Using (6),in above equation, we get

$$x = \frac{1}{2} \times t^2 + \frac{1}{2} \times \mu t + \mu$$

And then solving $y = t + \mu$ with above equation for μ and t in terms of x and y, we obtain

$t = \frac{y - x}{1 - \left(\frac{y}{2}\right)}$ and $\mu = \frac{x - (y^2/2)}{1 - \left(\frac{y}{2}\right)}$

Substituting these values in $z = t + \frac{\mu}{2}$, the required surface passing through the initial data curve is

$$z = \frac{\left\{2(y - x) + x - \frac{y^2}{2}\right\}}{2 - y}.$$

SELF CHECK QUESTIONS

Choose the Correct Option:

1. The equation $ptany + qtanx = sec^2 z$ is of order

a. 1

b. 2

c. 3

d. 4

2. The equation $\frac{\partial^2 z}{\partial x^2} - 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial y}\right)^2 = 0$ is of order

a. 1
b. 2
c. 3
d. None

3. The equation $(2x + 3y)p + 4xq - 8pq = x + y$ is

a. Linear
b. Non-linear
c. Quasi- linear
d. Semi-linear

4. The equation $(x + y - z)\left(\frac{\partial z}{\partial x}\right) + (3x + 2y)\left(\frac{\partial z}{\partial y}\right) + 2z = x + y$ is

a. Linear
b. Quasi-linear
c. Semi-linear
d. Non-linear

5. If the coefficient of highest derivative does not contain either the dependent variable or its derivatives such partial differential equation is

a. Linear
b. Non-linear
c. Quasi-linear
d. Semi-linear

6. Choose the correct option:

a. Every semi-linear partial differential equation is quasi-linear.
b. Every quasi-linear partial differential equation is semi-linear.
c. Every semi-linear partial differential equation is linear
d. Every quasi-linear partial differential equation is linear.

7. A semi-linear partial differential equation which is linear is dependent variable and its derivative, then it is

a. Linear
b. Non-linear
c. Quasi-linear
d. Semi-linear

8. Consider the surfaces $z = F(x, y, a, b)$ then corresponding partial differential equation is of the form

a. $f(x, y, z, p, q) = 0$
b. $f(x, y, p, q) = 0$
c. $f(x, y, z) = 0$

d. $f(p,q) = 0$

9. If we eliminate arbitrary constants from the surface Consider the surfaces $z = F(x,y,a,b)$ then corresponding partial differential equation is of the form $F(x,y,z,p,q) = 0,\ \ a,b$ are constants, then the obtained partial differential equation is

a. Quasi-linear

b. Non-linear

c. Both a and b

d. None

10. Consider the surface $F(u,v) = 0$ where u and v are known functions of x,y,z. After eliminating the arbitrary functions from given surface, we obtain

a. A quasi-linear partial differential equation

b. A semi-linear partial differential equation

c. A non-linear partial differential equation

d. A linear partial differential equation

11. A partial differential equation $z = pq$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ is formed by eliminating arbitrary constants a and b from the equation

a. $z = (a+x) + (a+y)$

b. $z = (a+x)(a+y)$

c. $z = ax + by$

d. $2z = (ax+y)^2 + b$

13.10 *SUMMARY :-*

In this unit we have studied the PDEs, order and degree of PDEs, linear and non-linear PDEs, classification of first order PDEs, origin of PDEs, Cauchy problem for first order PDEs. The partial differential equations continue to be fundamental in various scientific and engineering disciplines. They play a crucial role in fields such as physics, engineering, economics, and biology, providing a powerful mathematical framework for understanding and predicting complex phenomena with multiple variables.

13.11 *GLOSSARY :-*

- **Differential Equation:** An equation that relates one or more functions and their derivatives. In the context of partial differential equations, these equations involve partial derivatives with respect to multiple independent variables.

- **Partial Differential Equation (PDE):** A type of differential equation that involves partial derivatives. It describes a relation between a function and its partial derivatives with respect to two or more independent variables.
- **Cauchy problem for a first-order partial differential equation :** The Cauchy problem for a first-order partial differential equation (PDE) involves specifying initial conditions for the unknown function and its partial derivatives.

13.12 *REFERENCES:-*

- Walter A. Strauss(2008), Partial Differential Equations: An Introduction.
- Stanley J. Farlow (1993), Partial Differential Equations for Scientists and Engineers.
- Lawrence C. Evans(1998), Partial Differential Equations. J.
- David Logan (2015),Applied Partial Differential Equations.

13.13 *SUGGESTED READING:-*

- Sandro Salsa(2008), Partial Differential Equations in Action: From Modelling to Theory.
- Robert C. McOwen (2009), Partial Differential Equations: Methods and Applications.
- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations

13.14 *TERMINAL QUESTIONS:-*

(TQ-1): Form partial differential equations by eliminating arbitrary constants a and b from the following relations:

a. $z = a(x + y) + b$
b. $z = ax + by + ab$
c. $z = ax + a^2y^2 + b$
d. $z = (x + a)(x + b)$

(TQ-2): Find the partial differential equation of planes having equal x and y intercepts.

(TQ-3): Find the partial differential equation of all spheres whose centres lie on z −axis.

(TQ-4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding PDEs

a. $z = Ae^{pt} sinpx, (p\ and\ A)$
b. $z = Ae^{-p^2t} cospx, (p\ and\ A)$
c. $z = ax^3 + by^3; (a, b)$
d. $4z = \left[ax + \frac{y}{a} + b\right]^2; (a, b)$
e. $z = ax^2 + bxy + cy^2; (a, b, c)$
f. $z^2 = ax^3 + ab + by^3; (a, b, c)$
g. $ax^2 + z^2 + cy^2 = 1$

(TQ-5) Eliminate arbitrary function f from

a. $z = f(x^2 - y^2)$
b. $z = f(x^2 + y^2)$

13.15 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1.a	2.b	3.b	4.b
5.a	6.b	7. a	8.a
9.b	10.c	11. c	

TERMINAL ANSWERS (TQ'S)

(TQ-1):

a. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$
b. $z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$
c. $\left(\frac{\partial z}{\partial y}\right) = 2y\left(\frac{\partial z}{\partial x}\right)^2$
d. $z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

(TQ-2): $p - q = 0$

(TQ-3): $xq - yp = 0$

(TQ-4):

a. $\frac{\partial^2 z}{\partial x^2} = \frac{dz}{dt}$

b. $\frac{\partial^2 z}{\partial x^2} = \frac{dz}{dt}$

c. $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 3z$

d. $z = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

e. $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 2z$

f. $9x^2y^2z = 6x^3y^2\left(\frac{\partial z}{\partial x}\right) + 6x^2y^3\left(\frac{\partial z}{\partial y}\right) + 4z\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

g. $z\left[z - x\left(\frac{\partial z}{\partial x}\right) - y\left(\frac{\partial z}{\partial y}\right)\right] = 1$

Unit 14: Linear PDEs of Order One

CONTENTS:

14.1 *INTRODUCTION:-*

In this unit, we will study appears to cover a broad range of topics related to first-order linear partial differential equations, Lagrange's method, integral surfaces, orthogonal surfaces, and extensions to multiple independent variables. Each of these topics contributes to a deeper understanding of the geometric and analytical aspects of partial differential equations.

14.2 *OBJECTIVES:-*

After studying this unit learner's will be able to

- Study Lagrange's equation and method for solving specific linear first-order PDEs. Understand the steps involved and the conditions under which this method is applicable.
- Learn about integral surfaces associated with solutions to linear first-order PDEs. Understand their geometric interpretation and significance in the context of differential equations.
- Explore the concept of surfaces orthogonal to integral surfaces. Understand the relationship between these surfaces and the geometric interpretation of solutions.
- Develop the ability to provide a geometrical description of solutions to linear first-order PDEs.

The main objectives of this unit, learners gain a comprehensive understanding of linear first-order PDEs and their applications, preparing them for more advanced studies in differential equations and mathematical modeling.

14.3 *LAGRANGE EQUATION:-*

Lagrange equations in the context of partial differential equations (PDEs) typically refer to a specific type of quasi-linear first-order PDE. The Lagrange equation of order one is given by:

$$Pp + Qq = R$$

where P, Q and R are the functions of x, y, z.

Example: $xyz + yzp = zx$ is Lagrange equation.

14.4 *GENERAL SOLUTION OF LAGRANGE EQUATION:-*

Theorem: The general solution of Lagrange equation

$$Pp + Qq = R \qquad \dots(1)$$

is

$$\phi(u, v) = 0 \qquad \dots(2)$$

where ϕ is an arbitrary function and

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ $\quad \dots(3)$

are two independent solutions of

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots(4)$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if $\frac{u}{v}$ is not merely constant.

Proof: Now differentiating (2) with respect to 'x' and 'y', we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \dots(5)$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \dots(6)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (5) and (6), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial x} + p\left(\frac{\partial v}{\partial z}\right) \\ \frac{\partial u}{\partial y} + q\left(\frac{\partial u}{\partial z}\right) & \frac{\partial v}{\partial y} + q\left(\frac{\partial v}{\partial z}\right) \end{vmatrix} = 0$$

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} = 0 \quad \dots(7)$$

Similarly

Taking the differentials of $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$, we have

$$\left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz = 0 \qquad \dots(8)$$

$$\left(\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy + \left(\frac{\partial v}{\partial z}\right)dz = 0 \qquad \dots(9)$$

where u and v are independent functions.

From (8) and (9) for $dx: dy: dz$, gives

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \quad \dots(10)$$

Now from (4) and (10), we have

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P} = \frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q} = \frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R} = k,$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = kQ, \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = kR$$

Putting these values in (7), we obtain

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} = 0$$

$$(kP)p + (kQ)q = kR$$

$$k(Pp + Qq) = kR$$

$$Pp + Qq = R \text{ which is given equation (1).}$$

Therefore if $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ (where c_1, c_2 are constants) are two independent solutions of the system of differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, then $\phi(u,v) = 0$ is the solution of $Pp + Qq = R$, ϕ being arbitrary function.

14.5 *WORKING RULE: -*

Working rule for solving $Pp + Qq = R$ by Lagrange's method:-

Step1: Substitute the given equation in the standard form of a linear first-order partial differential equation.

$$Pp + Qq = R \qquad ...(1)$$

Step2: Write down the Lagrange's auxiliary equations for (1) namely,

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$$

Step3: Solve (2) by using the well known methods. Let $u(x, y, z) = c_1$and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step4: The general solution of (1) is obtained in one of the following three equivalent forms:

$\phi(u, v) = 0$, $u = \phi(v)$ or $v = \phi(u)$, ϕ being arbitrary function.

TYPE1 based on rule I for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$:

Given the partial differential equation $Pp + Qq = R$, the Lagrange's auxiliary equation is given by:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Now, let's consider two fractions, say, $\frac{dx}{P}$ and $\frac{dy}{Q}$. If one of the variables (x or y) is either absent or cancels out, then we can set up a differential equation and integrate.

For example, if $\frac{dx}{P}$ and $\frac{dy}{Q}$, are given, and let's say y is absent or cancels out, then we have:

$$\frac{dx}{P} = \frac{dz}{R}$$

Now, you can integrate this equation with respect to x and z separately:

$$\frac{1}{P}dx = \frac{1}{R}dz$$

Similarly, you can repeat the procedure with another set of two fractions. For example, if $\frac{dy}{Q}$ and $\frac{dz}{R}$ are given, and z is absent or cancels out, then we have:

$$\frac{dy}{Q} = \frac{dx}{P}$$

Integrate this equation with respect to y and x separately:

$$\frac{1}{Q}dy = \frac{1}{P}dx$$

These integrations will give you solutions involving the variables x, y and z. The constants of integration can be determined by any initial or boundary conditions provided.

SOLVED EXAMPLES

EXAMPLE1: Solve the partial differential equation $2p + 3q = 1$ by Lagrange's methods.

SOLUTION: Let the given Differential equation is

$$Pp + Qq = R \qquad \text{... (1)}$$

where $P = 2, Q = 3, R = 1$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \qquad \text{..(2)}$$

Taking two fraction of two, we obtain

$$\frac{dx}{2} = \frac{dy}{3} \quad \text{or} \quad 3dx - 2dy = 0 \quad \text{... (3)}$$

Now integrating (3), we have

$$3x - 2y = c_1, \quad c_1 \text{ being an arbitrary constant}$$

$\therefore u(x, y, z) = 3x - 2y = c_1$ is one solution of the given partial differential equation

Similarly, taking last two fraction of two, we get

$\frac{dy}{3} = \frac{dz}{1}$ or $dy - 3dz = 0$ ... (4)

Now integrating (4), we get

$$y - 3z = c_2, \quad c_2 \text{ being an arbitrary constant}$$

$\therefore v(x, y, z) = y - 3z = c_2$ is another solution of the given partial differential equation.

Hence the general solution is given below

$$\phi = (3x - 2y, y - 3z) = 0$$

Where ϕ is an arbitrary constant.

EXAMPLE2: Find the general solution of $zp + x = 0$.

SOLUTION: Let the given Differential equation is $zp + x = 0$

$$Pp + Qq = R \quad \text{... (1)}$$

where $P = z, Q = 0, R = -x$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \quad \text{.. (2)}$$

Taking first and last fraction of (2), we obtain

$\frac{dx}{z} = \frac{dz}{-x}$ or $xdx + zdz = 0$ ... (3)

Now integrating (3), we have

$\frac{x^2}{2} + \frac{z^2}{2} = k$, or $x^2 + z^2 = c_1$ being an arbitrary constant

$\therefore u(x, y, z) = x^2 + z^2 = c_1$ is one solution of the given partial differential equation.

Also the second fraction of (2), we get

$$dy = 0$$

Integrating, $y = c_2$

$\therefore v(x, y, z) = \; y = c_2$ is another solution of the given partial differential equation.

Hence the desired solution is

$$\phi = (x^2 + z^2, y) = 0$$

where ϕ is an arbitrary constant.

EXAMPLE3: Solve $y^2 p - xyp = -x(z - 2y)$.

SOLUTION: Let the given equation

$$y^2 p - xyp = -x(z - 2y) \qquad \dots (1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \qquad \dots (2)$$

Taking the first two fractions of (1) , we get

$2xdx + 2ydy = 0$ so $x^2 + y^2 = c_1$

Now again taking the last two fractions of (1), we have

$\frac{dz}{dy} = \frac{z-2y}{y}$ or $\frac{dz}{dy} + \frac{z}{y} = 2$

So which is linear in z and y. Its integrating factor = $e^{\int (1/y) dy} = e^{\log y} = y$.

Hence $z.y = \int 2ydy + c_2$ or $zy - y^2 = c_2$

From above equations, the required general integral is $\phi(x^2 + y^2, zy - y^2)$ being an arbitrary function.

EXAMPLE4: Solve $p\, tanx + qtany = tanz$.

SOLUTION: The given equation is

$$p\, tanx + qtany = tanz \quad \dots (1)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$$

Taking first two fraction of above equation, we obtain

$$\frac{dx}{tanx} = \frac{dy}{tany} = cotxdx - cotydy = 0$$

Now integrating, $logsinx - logsiny = logc_1$ or $\frac{sinx}{siny} = c_1$

Again last two fraction of above equation, we get

$$\frac{dy}{tany} = \frac{dz}{tanz} = cotydx - cotzdz = 0$$

Now integrating, $logsiny - logsinz = logc_1$ or $\frac{siny}{sinz} = c_2$

Hence the required general solution is $\frac{sinx}{siny} = \phi\left(\frac{siny}{sinz}\right), \phi$ being an arbitrary function.

TYPE2 based on rule II for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

Let the Lagrange's auxiliary equations for the partial differential

$$Pp + Qq = R \qquad ...(1)$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad ...(2)$$

Suppose that one integral of (2) is known by using rule 1 derived in previous article and suppose also that another integral cannot be derived by using the rule I of previous article. Then, one (the first) integral known to us is used to find another (the second) integral as shown in the following solved examples. Note that in the second integral, the constant of integration of the first integral should be removed later on equation.

SOLVED EXAMPLE

EXAMPLE 1: Solve $xzp + yzq = xy$

SOLUTION: Given $xzp + yzq = xy \qquad ...(1)$

From (1), we have

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots(2)$$

Taking first two fraction, we get

$$\frac{dx}{x} - \frac{dy}{y} = 0 \qquad \ldots(3)$$

Integrating (3), we obtain

$\log x - \log y = \log c_1$ or $\frac{x}{y} = c_1$ or $x = yc_1$ ... (4)

From second and third fraction of (2), we have

$\frac{dy}{yz} = \frac{dz}{c_1 y^2}$ or $c_1 y dy - z dz = 0$ ... (5)

Integrating (3), we have

$$\frac{1}{2}c_1 y^2 - \frac{1}{2}z^2 = c_2$$

$$xy - z^2 = c_2$$

From (4) and (5), the required general solution is $\phi\left(xy - z^2, \frac{x}{y}\right)$, ϕ being an arbitrary function.

EXAMPLE 2: Solve $p + 3q = 5z + \tan(y - 3x)$

SOLUTION: Given $p + 3q = 5z + \tan(y - 3x)$ ... (1)

From (1), we get

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)} \qquad \ldots(2)$$

Taking first two fraction

$$dy - 3dx = 0$$

Now integrating above equation $y - 3x = c_1$, c_1 being an arbitrary constant.

Again from (2), we obtain

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y - 3x)}$$

Integrating, $x - \frac{1}{5}\log(5z + tanc_1) = \frac{1}{5}\ c_2$

where c_2 being arbitrary constant.

$$5x - \log[5z + \tan(y - 3x)] = \ c_2$$

Hence the required general integral is

$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$, where ϕ is an arbitrary function.

TYPE3 based on rule III for solving $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$**:**

The Lagrange's auxiliary equations for the partial differential

$$Pp + Qq = R \qquad \text{... (1)}$$

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{... (2)}$$

If P_1, Q_1 and R_1 be the function of x, y and z,then by a well- known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}. \qquad \text{... (3)}$$

If denominator is zero ($P_1 P + Q_1 Q + R_1 R$), then $P_1 dx + Q_1 dy + R_1 dz$ is also zero which is integrated to obtain $u_1(x, y, z) = c_1$. This method may be repeated to another integral$u_2(x, y, z) = c_2$. Here, $P1$, $Q1$, and $R1$ are called as **Lagrange's multipliers**. As special case, these can be constants also. In such cases second integral should be obtained by using rule I and rule II as the case may be.

SOLVED EXAMPLE

EXAMPLE1: Solve$(mz - ny)p + (nx - lz)q = ly - mx.$

SOLUTION: Given $(mz - ny)p + (nx - lz)q = ly - mx$ ... (1)

The Lagrange's auxiliary equation of (1) is

$$\frac{dx}{(mz-ny)}=\frac{dy}{(nx-lz)}=\frac{dz}{ly-mx}$$

Changing x, y, z multipliers, each fraction of (1), we get

$$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+n(ly-mx)}=\frac{xdx+ydy+zdz}{0}$$

$\therefore\ xdx+ydy+zdz=0$ so that $2xdx+2ydy+2zdz=0$

Now integrating, we have

$$x^2+y^2+z^2=c_1 \quad \dots (2)$$

where c_1 being an arbitrary constant.

Again, choose l, m, n multipliers, each fraction of (1), we obtain

$$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}$$

$\therefore\ ldx+mdy+ndz=0$ so that $lx+my+nz=c_2$ ... (3)

From (2) and (3), the required general solution is given by

$\phi(x^2+y^2+z^2, lx+my+nz)=0$, ϕ being an arbitrary function.

EXAMPLE2: Solve $x(y^2-z^2)q-y(z^2+x^2)q+z(x^2+y^2)$.

SOLUTION: Given $x(y^2-z^2)q-y(z^2+x^2)q+z(x^2+y^2)$ ... (1)

The Lagrange's auxiliary equation of (1) is

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{-y(z^2+x^2)}=\frac{dz}{z(x^2+y^2)} \quad \dots (2)$$

Changing x, y, z multipliers, each fraction of (2), we have

$$\frac{xdx+ydy+zdz}{x(y^2-z^2)q-y(z^2+x^2)q+z(x^2+y^2)}=\frac{xdx+ydy+zdz}{0}$$

$xdx+ydy+zdz=0$ so that $x^2+y^2+z^2=c_1$ ... (3)

Again, choose $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ multipliers, each fraction of (2), we obtain

$$\frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy + \left(\frac{1}{z}\right)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{ldx + mdy + ndz}{0}$$

$\left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\log\{x/(yz)\} = \log c_2$ or $\frac{x}{yz} = c_2$ ... (4)

∴ The required solution is $\phi(x^2 + y^2 + z^2, x/(yz)) = 0$ ϕ being an arbitrary function.

EXAMPLE3: Solve the general solution of the equation $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$.

SOLUTION: Given $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$ ... (1)

The Lagrange's auxiliary equations are

$$\frac{dx}{(y+zx)} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \qquad \text{... (2)}$$

Changing x, y, z multipliers, each fraction of (2), we get

$$\frac{xdx + ydy - zdz}{x(y + zx) - y(x + yz) - z(x^2 - y^2)} = \frac{xdx + ydy - zdz}{0}$$

$xdx + ydy - zdz = 0$ so that $2xdx + 2ydy - 2zdz = 0$

Integrating, $x^2 + y^2 - z^2 = c_1$ ... (3)

where c_1 being an arbitrary constant.

Choose $x, y, 1$ multipliers, each fraction of (2), we obtain

$$\frac{xdx + ydy + dz}{y(y + zx) - x(x + yz) + x^2 - y^2} = \frac{xdx + ydy + dz}{0}$$

∴ $xdx + ydy + dz = 0$ or $d(xy) + dz = 0$

Integrating, $xy + z = c_2$ ... (4)

where c_2 being arbitrary constant.

∴ Hence the required solution is $\phi\ (x^2 + y^2 - z^2, xy + z) = 0$, ϕ being an arbitrary function.

EXAMPLE4: Solve $(y-z)p+(z-x)q=x-y$ **.**

SOLUTION: The Lagrange's auxiliary equations are

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y} \qquad \dots(1)$$

Changing 1,1,1 multipliers, each fraction of (1), we obtain

$$\frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)}=\frac{dx+dy+dz}{0}$$

$\therefore \quad dx+dy+dz=0 \quad \Rightarrow \quad x+y+z=c_1$

Choosing x,y,z multipliers, each fraction of (1), we obtain

$$=\frac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)}=\frac{xdx+ydy+zdz}{0}$$

$\therefore \quad 2xdx+2ydy+2zdz=0 \quad \Rightarrow \quad x^2+y^2+z^2=c_2$

Hence the required solution is $\phi\,(x+y+z, x^2+y^2+z^2)=0,\ \phi$ being an arbitrary function.

TYPE4 based on rule IV for solving $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$**:**

The Lagrange's auxiliary equations for the partial differential

$$Pp+Qq=R \qquad \dots(1)$$

$$\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R} \qquad \dots(2)$$

If P_1, Q_1 and R_1 be the function of x, y and z ,then by a well- known principle of algebra, each fraction in (1) will be equal to

$$\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}. \qquad \dots(3)$$

Let us consider that the numerator of (3) is an exact differential of the denominator of (3), then (3) can be combined with a suitable fraction in (2) to obtain an integral. But, in some problems, another set of multipliers P_2, Q_2 and R_2 are so obtain that the fraction

$$\frac{P_2dx+Q_2dy+R_2dz}{P_2P+Q_2Q+R_2R}. \qquad \dots(3)$$

is such that its numerator is an exact differential of denominator. Fractions (3) and (4) are then combined to give an integral. This method may be repeated in some problems to get another integral. Sometimes, only one integral is possible by using the rule IV. In such cases, the second integral should be derived by using rule 1 or rule 2 or rule 3 of previous articles. The following solved examples will illustrate the rule:

SOLVED EXAMPLE

EXAMPLE1: Solve$(y+z)p+(z+x)q=x+y$.

SOLUTION: Given $(y+z)p+(z+x)q=x+y \quad \dots(1)$

The Lagrange's auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad \dots(2)$$

Changing $1,-1,0$ multipliers, each fraction of (2), we obtain

$$\frac{dx-dy}{(y+z)-(x+y)}=\frac{d(x-y)}{-(x-y)} \quad \dots(3)$$

Again choose $0,1,-1$ multipliers, each fraction of (2), we get

$$\frac{dy-dz}{(z+x)-(x+y)}=\frac{d(y-z)}{-(y-z)} \quad \dots(4)$$

Again finally choose 1,1,1 multipliers, each fraction of (2), we have

$$\frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)}=\frac{dx+dy+dz}{2(x+y+z)} \quad \dots(5)$$

Now from (3),(4) and (5), we have

$$\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)}=\frac{dx+dy+dz}{2(x+y+z)} \quad \dots(6)$$

Taking first two fraction of (6), we get

$$\frac{d(x-y)}{-(x-y)}=\frac{d(y-z)}{-(y-z)}$$

$\therefore$ Integrating it, we obtain $\log(x-y)=\log(y-z)+\log c_1$

$$\frac{x-y}{y-z}=c_1 \Rightarrow x-y=c_1(y-z) \quad \dots(7)$$

Taking first and third fraction of (6), we have

$$\frac{2d(x-y)}{(x-y)}+\frac{dx+dy+dz}{2(x+y+z)}=0$$

∴ Integrating it, we get

$$2\log(x-y)+\log(x+y+z)=\log c_2 \Rightarrow (x-y)^2(x+y+z)=c_2$$

$$\Rightarrow (x-y)^2(x+y+z)=c_2 \quad ...(8)$$

From (8) and (9), we get

$$\phi\left[\left(\frac{x-y}{y-z},\right)(x-y)^2(x+y+z)\right]=0$$

Where ϕ is an arbitrary function.

EXAMPLE2: Solve $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$.

SOLUTION: Given $y^2(x-y)p+x^2(y-x)q=z(x^2+y^2)$ ...(1)

The Lagrange's auxiliary equations are

$$\frac{dx}{y^2(x-y)}=\frac{dy}{-x^2(x-y)}=\frac{dz}{z(x^2+y^2)} \quad ...(2)$$

Taking first two fraction of (2), we have

$$\frac{dx}{y^2}=\frac{dy}{-x^2}$$

$$x^2dx+y^2dy=0$$

∴ Integrating it, we obtain $x^2+y^2=c_1$ ...(3)

choose $1,-1,0$ multipliers, each fraction of (2), we have

$$=\frac{dx-dy}{y^2(x-y)+x^2(x-y)}=\frac{dx-dy}{(y^2+x^2)(x-y)} \quad ...(4)$$

Combining the third fraction of (2) with (4), we obtain

$$\frac{dx-dy}{(y^2+x^2)(x-y)}=\frac{dz}{z(x^2+y^2)}\Rightarrow\frac{d(x-y)}{(x-y)}-\frac{dz}{z}=0$$

Integrating it, we get $\log(x-y)-\log(z)=\log c_2$

$$\frac{\log(x-y)}{z} = \log c_2 \Rightarrow \frac{x-y}{z} = c_2$$

Hence, the required solution is

$$\phi\left[x^2 + y^2, \frac{x-y}{z}\right] = 0$$

Where ϕ is an arbitrary function.

14.6 *GEOMETRICAL DESCRIPTION OF SOLUTIONS OF LAGRANGE'S EQUATION $Pp + Qq = R$ AND LANGRAGE'S AUXILIARY EQUATIONS* $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$ **:-**

Let

$$Pp + Qq = R \qquad \dots(1)$$

and

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots(2)$$

where P, Q, R are the function of x, y, z.

Let $$z = \phi(x, y) \qquad \dots(3)$$

Represents the solution of the Lagrange's partial differential equation (1). Then (3) expresses a surface whose normal at any point (x, y, z) has direction ratios$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ $i.e., p, q, -1$. Also, we know that the system of simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we obtain

$$Pp + Qq + R(-1) = 0 \qquad \dots(4)$$

which expresses that the normal to the surface (3) at any point is

perpendicular to the member of family of curves (2) through that point. Hence, the member must touch the surface at that point. Since this holds for each point on (3), therefore, we consider that the curves (2) lies completely on the surface (3) whose differential equation is obtain by (1).

14.7 *GEOMETRICAL INTERPRETATION OF* $\boldsymbol{Pp + Qq = R}$:-

To show that the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solution of

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots (1)$$

Are orthogonal to the system of surfaces whose satisfied the equation

$$Pdx + Qdy + Rdz = 0 \qquad \ldots (2)$$

Again, from (1) lie on the surface represented by

$$Pp + Qq = R \qquad \ldots (3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

14.8 *LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE WITH n INDEPENDENT VARIABLES:-*

Let $x_1, x_2, x_3, \ldots \ldots \ldots \ldots x_n$ be the n independent variables and z be dependent function depending on $x_1, x_2, x_3, \ldots \ldots \ldots \ldots x_n$. Also, let $p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, p_3 = \frac{\partial z}{\partial x_3}, \ldots \ldots \ldots \ldots . p_n = \frac{\partial z}{\partial x_n}$

Then, the general linear partial differential equation of order one with n independent variables is obtained by

$$P_1p_1 + P_2p_2 + P_3p_3 + \cdots \cdots + P_np_n = R \qquad (1)$$

where $P_1P_2P_3 \ldots \ldots \ldots P_n$ are the functions of $x_1, x_2, x_3, \ldots \ldots \ldots \ldots x_n$ and R is the function of $x_1, x_2, x_3, \ldots \ldots \ldots \ldots x_n$ and z.

Therefore, the system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \cdots \cdots = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad \ldots (2)$$

Let $u_1(x_1, x_2, x_3, \ldots \ldots \ldots . . x_n, z) = c_1, u_2(x_1, x_2, x_3, \ldots \ldots \ldots . . x_n, z) = c_2, u_3(x_1, x_2, x_3, \ldots \ldots \ldots . . x_n, z) = c_3, \ldots \ldots \ldots \ldots n(x_1, x_2, x_3, \ldots \ldots \ldots . . x_n, z) = c_n,$) be any independent integral of (2), then the general solution of (2) is written by

$$\phi(u_1, u_{2,} u_3 \ldots \ldots . . u_n) = 0$$

SOLVED EXAMPLE

EXAMPLE1: Solve $(y+z)p + (z+x)q = x + y$

SOLUTION: Here the Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \ldots (1)$$

Changing $1, -1, 0$ as multipliers of (1), we have

$$\frac{dx - dy}{(y+z) - (z+x)} = \frac{d(x-y)}{-(x-y)} \qquad \ldots (2)$$

Again, choosing $0, 1, -1$ as multipliers of (1), we obtain

$$\frac{dy - dz}{(z+x) - (x+y)} = \frac{d(x-y)}{-(y-z)} \qquad \ldots (3)$$

Finally, choosing $1,1,1$ as multipliers of (1), we get

$$= \frac{dx + dy + dz}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \qquad \ldots (4)$$

From (2), (3) and (4), we get

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x-y)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \qquad \ldots (5)$$

Taking the two fractions of (5), we have

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x-y)}{-(y-z)}$$

Integrating, $\log(x-y) = \log(y-z) + \log c_1$, where c_1 being an arbitrary constant.

$\log\left(\frac{x-y}{y-z}\right) = \log c_1$ or $\left(\frac{x-y}{y-z}\right) = c_1$

Again taking first and third fraction of (5),

$$2\frac{d(x-y)}{(x-y)}+\frac{d(x+y+z)}{(x+y+z)}=0$$

Integrating, $2\log(x-y)+\log(x+y+z)=\log c_2$ or

$$(x-y)^2(x+y+z)=c_2$$

Hence the required general solution is $\phi\left[(x-y)^2(x+y+z),\frac{x-y}{y-z}\right]=0,\phi$ being arbitrary function.

EXAMPLE2: Solve $(1+y)p+(1+x)q=z$

SOLUTION: Here the Lagrange's auxiliary equations are

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z} \quad \ldots(1)$$

Taking the first two fractions of (1), we get

$(1+x)dx=(1+y)dy$ or $2(1+x)dx-2(1+y)dy=0$ ... (2)

Integrating, $(1+x)^2-(1+y)^2=c_1,\ c_1$ being an arbitrary constant.

Taking 1,1,0 as multipliers of each fraction of (1)

$$=\frac{dx+dy}{1+y+1+x}=\frac{d(2+x+y)}{2+x+y} \quad \ldots(3)$$

Combining the last fraction of (1) with (3), we have

$\frac{d(2+x+y)}{2+x+y}=\frac{dz}{z}$ or $\frac{d(2+x+y)}{2+x+y}-\frac{dz}{z}=0$

Integrating, $\log(2+y+x)-\log z=\log c_2$ or $\frac{(2+x+y)}{z}=c_2$ ... (4)

From (2) and (4), the required general solution is obtained by

$\phi\left[(1+x)^2-(1+y)^2,\frac{(2+x+y)}{z}\right]=0,\phi$ being an arbitrary function.

EXAMPLE3: Find the tangent vector at the point $\left(0,1,\frac{\pi}{2}\right)$ to the helix described by the parametric equations $x=cost, y=sint, z=t.$

SOLUTION: The tangent vector to the helix at (x,y,z) is obtained by

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=(-sint,cost,1)$$

We state that the given point $\left(0,1,\frac{\pi}{2}\right)$ corresponds $t=\frac{\pi}{2}$. Therefore, the required tangent to vector to helix is obtain by

$$\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=(-sint,cost,1)=(-1,0,1)$$

EXAMPLE4: Find the equation of the tangent line to the space circle $x^2+y^2+z^2=1, x+y+z=0$ at the point $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},-\frac{3}{\sqrt{14}}\right)$.

SOLUTION: Let

$F(x,y,z)=x^2+y^2+z^2-1=0$ and

$$F(x,y,z)=x+y+z=0 \qquad \dots(1)$$

The equation of the tangent line at the point (x_0,y_0,z_0) is

$$\frac{(x-x_0)}{\frac{\partial(F,G)}{\partial(y,z)}}=\frac{(y-y_0)}{\frac{\partial(F,G)}{\partial(z,x)}}=\frac{(z-z_{0)}}{\frac{\partial(F,G)}{\partial(x,y)}} \qquad \dots(2)$$

Where

$$\frac{\partial(F,G)}{\partial(y,z)}=\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}=2y-2z=\frac{4}{\sqrt{14}}+\frac{6}{\sqrt{14}}=\frac{10}{\sqrt{14}}$$

$$\frac{\partial(F,G)}{\partial(z,x)}=\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}=2z-2x=-\frac{6}{\sqrt{14}}-\frac{2}{\sqrt{14}}=\frac{8}{\sqrt{14}}$$

$$\frac{\partial(F,G)}{\partial(x,y)}=\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}=2z-2y=\frac{2}{\sqrt{14}}-\frac{4}{\sqrt{14}}=\frac{-2}{\sqrt{14}}$$

The required solution of the point $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},-\frac{3}{\sqrt{14}}\right)$ is obtained by

$$\frac{\left(x-\frac{1}{\sqrt{14}}\right)}{\frac{10}{\sqrt{14}}}=\frac{\left(y-\frac{2}{\sqrt{14}}\right)}{\frac{8}{\sqrt{14}}}=\frac{\left(z-\frac{3}{\sqrt{14}}\right)}{\frac{-2}{\sqrt{14}}}$$

EXAMPLE5: Find the integral surface of the linear partial differential equation$x(x^2+z)p-y(x^2+z)q=(x^2-y^2)z$.

SOLUTION: Given $x(x^2+z)p - y(x^2+z)q = (x^2-y^2)z$ ... (1)

The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{x(x^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \qquad \text{... (2)}$$

The two independent solutions of (2) may be given as

$$u(x,y,z) = xyz = c_1 \qquad \text{... (3)}$$

$$v(x,y,z) = x^2+y^2-2z = c_2 \qquad \text{... (4)}$$

Taking t as parameter, the obtained equation of the straight line $x+y=0, z=1$ can be put in parametric form

$$x=t, y=-t, \qquad z=1 \qquad \text{... (5)}$$

Putting the value of (5) in (3) and (4), we have $-t^2=c_1$ and $2t^2-2=c_2 \Rightarrow -2c_1-2=c_2 \;\Rightarrow\; 2c_1+2+c_2=0$ ... (6)

Now, substituting the values of c_1 and c_2 from (3) and (4) in (6), we obtain

$$2xyz+x^2+y^2-2z=0$$

which is the desired integral surface of the given PDE.

EXAMPLE6: Find the equation of integral surface satisfying $4yzp+q+2y=0$ and passing through $y^2+z^2=1, x+z=2$.

SOLUTION: Given $4yzp+q+2y=0$ ... (1)

The equation of the obtained curve is

$$y^2+z^2=1, \;\; x+z=2 \qquad \text{... (2)}$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y} \qquad \text{... (3)}$$

Taking the first and third fractions of (3), we obtain

$dz+2zdz=0$ so that $x+z^2=2$ ... (4)

Taking the last two fractions of (3), we get

$dz+2ydy=0$ so that $x+y^2=2$ ... (5)

Adding (4) and (5), we have

$$(z^2+y^2)+(x+z)=c_1+c_2$$

From (2), $1+2=c_1+c_2$

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required integral surface is written by

$3=x+z^2+y^2+z$ or $x+z^2+y^2+z-3=0$

EXAMPLE7: Find the surface which intersects the surfaces of the system $z(x+y)=c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1, z=1$.

SOLUTION: The equation of the given system of surfaces is

$$f(x,y,z)\equiv\frac{z(x+y)}{3z+1}=C \qquad \dots(1)$$

$$\therefore\ \frac{\partial f}{\partial x}=\frac{z}{3z+1},\frac{\partial f}{\partial y}=\frac{z}{3z+1},\frac{\partial f}{\partial z}=\left[\frac{3z+1-3z}{(3z+1)^2}\right](x+y)=\frac{(x+y)}{(3z+1)^2}$$

$$z(3z+1)q+z(3z+1)q=x+y \quad \dots(2)$$

The Lagrange's auxiliary equations is

$$\frac{dx}{z(3z+1)}=\frac{dy}{z(3z+1)}=\frac{dz}{(x+y)} \quad \dots(3)$$

Taking first two fraction of (3), we have $dx-dy=0$

Integrating it, $x-y=c_1$

Taking $x, y, -z(3z+1)$ as multipliers, each fraction of (3) is

$$xdx+ydy-z(3z+1)dz=0$$

$$xdx+ydy-3z^2dz-zdz=0$$

Or

$$2xdx+2ydy-6z^2dz-2zdz=0$$

Integrating it, we obtain $x^2+y^2-2z^3-z^2=c_2$ ... (4)

Hence, the equation (1) is given by

$$x^2+y^2-2z^3-z^2=\phi(x-y)$$

where ϕ is an arbitrary function.

EXAMPLE8: Find the family orthogonal to $\phi[z(x+y)^2, x^2-y^2]=0$.

SOLUTION: Given

$$\phi[z(x+y)^2, x^2-y^2]=0 \qquad \dots(1)$$

Let $u=z(x+y)^2$ $\qquad v=x^2-y^2$

From (1) becomes,

$$\phi(u,v)=0 \qquad ..(2)$$

Differentiating two w.r.t. x and y, we have

$$\left.\begin{aligned}\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0\\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+q\frac{\partial v}{\partial z}\right)=0\end{aligned}\right\}\dots(3)$$

From (2),

$\frac{\partial u}{\partial x} = 2z(x+y),\ \frac{\partial u}{\partial y} = 2z(x+y), \frac{\partial u}{\partial z} = (x+y)^2, \frac{\partial v}{\partial x} = 2x,\ \frac{\partial v}{\partial y} = -2y, \frac{\partial v}{\partial z} = 0$ ... (4)

Substituting the values of (4) in (3), we get

$$\left(\frac{\partial\phi}{\partial u}\right)[2z(x+y)+p(x+y)^2] + \left(\frac{\partial\phi}{\partial v}\right)[2x+0] = 0$$
$$\left(\frac{\partial\phi}{\partial u}\right)[2z(x+y)+q(x+y)^2] + \left(\frac{\partial\phi}{\partial v}\right)[-2y+0] = 0$$

Now

$$\frac{\frac{\partial\phi}{\partial u}}{\frac{\partial\phi}{\partial v}} = \frac{2x}{2z(x+y)+p(x+y)^2} = \frac{-2y}{2z(x+y)+p(x+y)^2}$$
$$x(x+y)[2z+p(x+y)] = -y(x+y)[2z+q(x+y)]$$
$$2xz + xq(x+y) + 2zy + yp(x+y) = 0$$
$$xq(x+y) + py(x+y) = -2z(x+y)$$
$$(qx+py)(x+y) = -2z(x+y)$$

$$qx + py = -2z \qquad \text{... (4)}$$

which is a partial differential equation of the family of surfaces given by (1).

The differential equation of the family of surfaces orthogonal to (4) is obtain by

$ydx + xdy - 2zdz = 0$ or $d(xy) - 2zdz = 0$

Integrating it,

$$xy - z^2 = c$$

which is the represented family of orthogonal surfaces.

EXAMPLE9: Solve $x_2x_3p_2 + x_3x_1p_2 = x_1x_2x_3$

SOLUTION: The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \cdots + P_np_n = R$, we obtain

$P_1 = x_2x_3, P_2 = x_3x_1, P_3 = x_1x_2$ and $R = x_1x_2x_3$

$\therefore$ The system of Lagrange's auxiliary equations is given by

$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \frac{dx_3}{p_3}$ or $\frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{x_1x_2x_3}$ ... (1)

Taking first and second fraction of (1), we have

$$x_1dx_1 = x_2dx_2$$

so

$$\frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2}$$

which give

$$u_1 = x_1^2 - x_2^2 = C_1 \qquad \dots (2)$$

Taking second and third fraction of (1), we have

$$x_2 dx_2 = x_3 dx_3$$

so

$$\frac{x_2^2}{2} = \frac{x_3^2}{2} + \frac{C_2}{2}$$

which give

$$u_2 = x_2^2 - x_3^2 = C_2 \qquad \dots (3)$$

Taking third and fourth fraction of (1), we have

$$dz = x_3 dx_3$$

so

$$z = \frac{x_3^2}{2} + \frac{C_3}{2}$$

which give

$$u_3 = 2z - x_3^2 = C_3 \qquad \dots (4)$$

Finally, from (2), (3) and (4), the general solution of the obtained partial differential equation is

$$\phi = (x_1^2 - x_2^2, x_2^2 - x_3^2, 2z - x_3^2) = 0.$$

EXAMPLE9: Solve $P_1p_1 + P_2p_2 + P_3p_3 = az + (x_1x_2)/x_3$

SOLUTION: The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \cdots + P_np_n = R$, we obtain

$P_1 = x_1, P_2 = x_2, P_3 = x_3$ and $R = az + (x_1x_2)/x_3$

$\therefore$ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} \quad \text{or} \quad \frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{az + (x_1x_2)/x_3} \qquad \dots (1)$$

Taking first and second fraction of (1), we have

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2}$$

so

$$\log x_1 = \log x_2 + \log c_1$$

which give

$$u_1 = \frac{x_1}{x_2} = C_1 \qquad \dots (2)$$

Taking second and third fraction of (1), we have

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3}$$

so

$$\log x_2 = \log x_3 + \log c_2$$

which give

$$u_1 = \frac{x_2}{x_3} = C_2 \qquad \dots (3)$$

Taking first and fourth fraction of (1), we have

$$\frac{dx_1}{x_1} = \frac{dz}{az+(x_1x_2)/x_3} = \frac{dz}{az+C_2x_1} \qquad \text{since } \frac{x_2}{x_3} = C_2$$

$$\frac{dz}{dx_1} = \frac{az + C_2x_1}{x_1}$$

i.e.,

$$\frac{dz}{dx_1} - \frac{a}{x_1}z = C_2 \quad \dots (4)$$

which is a linear differential equation whose integrating function (I.F.) is given as follows :

so

$I.F\ of\ (4)$, we have

$$= e^{-a\int\frac{dx_1}{x_1}} = e^{-a\log x_1} = x_1^{-a}$$

The solution of the linear differential equation (4) is given by

$$zx_1^{-a} = C_2 \int x_1^{-a} dx + C_3$$

$$zx_1^{-a} = C_2 \frac{x_1^{-a+1}}{1-a} + C_3$$

$$zx_1^{-a} = \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) + C_3 \text{ since } \frac{x_2}{x_3} = C_2$$

$$\frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) = C_3$$

i.e.,

$$u_3 = \frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right) = C_3 \quad \dots (5)$$

Finally, from (2), (3) and (5), the general solution of the given partial differential equation is

$$\phi\left(\frac{x_1}{x_2}, \frac{x_2}{x_3}, \left\{\frac{z}{x_1^a} - \left(\frac{x_2}{x_3}\right)\left(\frac{x_1^{-a+1}}{1-a}\right)\right\}\right) = 0.$$

SELF CHECK QUESTIONS

Choose the Correct Option:

1. The PDE $Pp + Qq = R$ is popularly known as
   a. Lagrange's equation
   b. Euler's equation
   c. Monge's equation
   d. Leibnitz equation
2. Lagrange's auxiliary equations for $xzp + yzq = xy$ are
   a. $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$
   b. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$
   c. $\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{1}$
   d. $\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{z}$
3. The solution of the PDE $xzp + yzq = xy$ is
   a. $\phi\left(\frac{x}{y}, xy - z^2\right) = 0$
   b. $\phi(x^2, y) = 0$
   c. $\phi(x^2z, y) = 0$
   d. $\phi(x^2zx, yx) = 0$

14.9 *SUMMARY*

In this unit we have studied Lagrange's equation, General solution of Lagrange Equation, working rule (Example based), Geometrical description of solutions of $Pp + Qq = R$ and of the system of equations $\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$ and to establish relationship and Linear Partial Differential Equations of order one with n independent variables and understanding and solving linear first-order PDEs are fundamental in the study of more complex partial differential equations and their applications in physics, engineering, and other scientific disciplines.

14.10 GLOSSARY:-

- **Partial Differential Equation (PDE):** An equation involving partial derivatives of an unknown function with respect to two or more independent variables.
- **Linear PDE:** A PDE where the unknown function and its derivatives appear linearly (i.e., without products or powers) in the equation.
- **Order One:** Refers to the highest order of derivatives present in the PDE. For linear PDEs of order one, the highest derivative is first-order.
- **Dependent Variable:** The variable that depends on other variables. In PDEs, this is typically the function being solved for.
- **Independent Variables:** Variables with respect to which partial derivatives are taken. In PDEs, these represent dimensions or parameters that influence the behavior of the dependent variable.
- **Coefficient Functions:** Functions that multiply the derivatives of the dependent variable in the PDE. These coefficients may depend on the independent variables.
- **Characteristics:** Curves or surfaces along which the PDE simplifies to an ordinary differential equation (ODE). They help in transforming the PDE into a simpler form for solution.
- **Initial Conditions:** Specified values or conditions given at a particular point in the domain of the PDE, often used to determine a unique solution.
- **Boundary Conditions:** Conditions specified on the boundary of the domain, essential for determining a unique solution to the PDE.
- **Method of Characteristics:** A technique used to solve linear first-order PDEs. It involves finding characteristic curves along which the PDE reduces to an ODE.
- **Compatibility Conditions:** Conditions that ensure the existence and uniqueness of solutions to the PDE, often related to the coefficients and boundary/initial conditions.
- **Transport Equation:** A specific type of linear first-order PDE that describes the advection or transport of a quantity along characteristic curves.
- **General Solution:** The set of all possible solutions to the PDE, often involving arbitrary functions or constants determined by initial/boundary conditions.

This glossary provides a foundational understanding of terms related to linear first-order PDEs. Each term plays a crucial role in formulating, understanding, and solving these equations in various applications across science and engineering.

14.11 REFERENCES:-

- Sandro Salsa(2008), Partial Differential Equations in Action: From Modelling to Theory.
- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations

14.12 *SUGGESTED READING:-*

- M.D.Raisinghania 20th eddition (2020), Ordinary and Partial Differential Equations.
- David Logan (2015), Applied Partial Differential Equations.

14.13 *TERMINAL QUESTIONS:-*

(TQ-1): Solve $p + 3q = 5z + \tan(y - 3x)$.

(TQ-2): Solve $z(z^2 + xy)(px - qy) = x^4$.

(TQ-3): Solve $xyp + y^2q = zxy - 2x^2$

(TQ-4): Solve $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

(TQ-5): Solve $xzp + yzq = xy$

(TQ-6): Solve $(y - zx)p + (x + yz)q = x^2 + y^2$

(TQ-7): Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

(TQ-8): Solve $(y - z)p + (z - x)q = x - y$

(TQ-9): Solve $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$

(TQ-10): Solve$2y(z - 3)p + (2x - z)q = y(2x - 3)$

(TQ-11): Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$

(TQ-12): Solve $ptanx + qtany = tanz$

(TQ-13): Solve $zp = -x$

(TQ-14): Find the general solution of differential equation

$$x^2\left(\frac{\partial z}{\partial x}\right)+y^2\left(\frac{\partial z}{\partial y}\right)=(x+y)z$$

(TQ-15): Find the general solution of differential equation

$$px(x+y)-qy(x+y)+(x-y)(2x=2y+z)=0$$

(TQ-16): Solve $p+q=x+y+z$

14.14 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1. (a) 2.(a) 3.(a) 4.(a)

TERMINAL ANSWERS (TQ'S)

(TQ-1): $5x-\log[5z+tan(y-3x)]=\phi(y-3x)$
(TQ-2): $\phi(xy,x^4-z^4-2xyz^2)=0$
(TQ-3): $x-\log[z-2(x^2/y^2)]=\phi(x/y)$
(TQ-4): $(y^2-ax-x^3)/x=\phi\left(\frac{z}{y}\right)$
(TQ-5): $\phi\left(xy-z^2,\frac{x}{y}\right)=0$
(TQ-6): $\phi(x^2-y^2+z^2,xy-z)=0$
(TQ-7): $\phi(x^2+y^2-2z,xyz)=0$
(TQ-8): $\phi(x+y+z,x^2+y^2+z^2)=0$
(TQ-9): $\phi(x^2+y^2-z^2,xy+z)=0$
(TQ-10): $\phi\left(\frac{1}{y}-\frac{1}{x},\frac{xy}{z}\right)=0$
(TQ-11): $\phi(x^3-y^3,x^2-z^2)=0$
(TQ-12): $\frac{sinx}{siny}=\phi\left(\frac{siny}{sinz}\right)$
(TQ-13): $\phi(x^2+y^2,zy-y^2)=0$
(TQ-14): $\phi\left(\frac{x,y}{z},\frac{x-y}{z}\right)=0$
(TQ-15): $\phi[xy,(x+y)(x+y+z)]=0$
(TQ-16): $\phi[x-y,e^{-x}(2+x+y+z)]=0$

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