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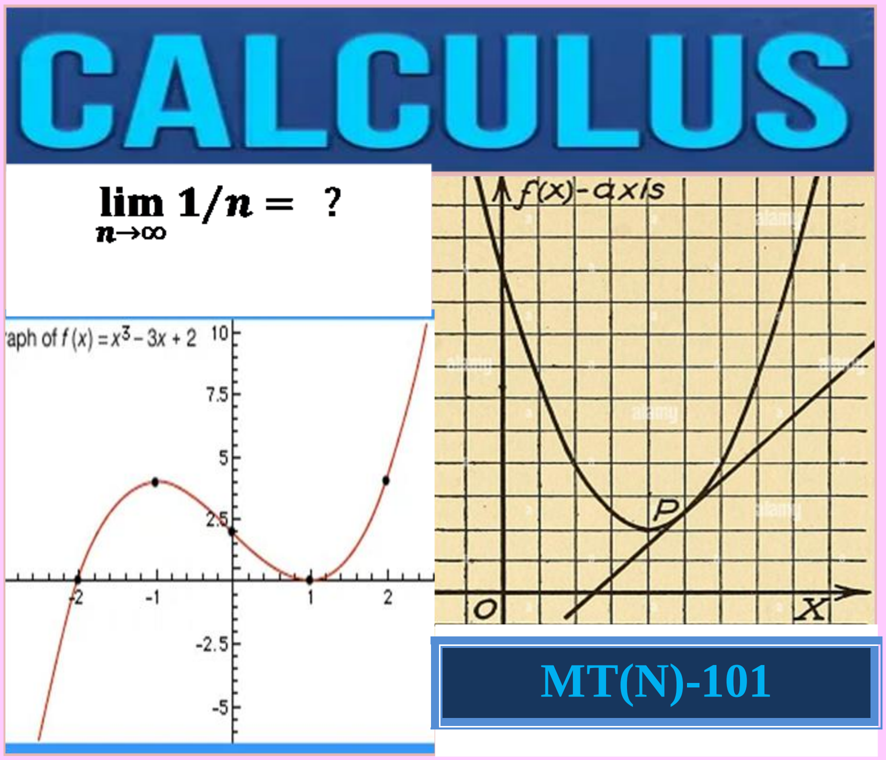

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: CALCULUS

COURSE CODE: MT(N) 101

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India,
263139

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   Haldwani, Uttarakhand

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1. **Dr. Arvind Bhatt** Department of Mathematics Uttarakhand Open University Haldwani, Nanital, Uttarakhand	I	1
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5. **Dr. Shivangi Upadhyay** Department of Mathematics Uttarakhand Open University Haldwani, Nanital, Uttarakhand	II	9
6. **Deepak kr Sharma** Department of Mathematics Uttarakhand Open University Haldwani, Nanital, Uttarakhand	IV	14,15

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COURSE CONTENTS

COURSE INFORMATION

The present self learning material “Calculus” has been designed for B.Sc. (First Semester) learners of Uttarkhand Open University, Haldwani. This self learning material is writing for increase learner access to high-quality learning materials. This course is divided into 15 units of study. The whole course is divided into four blocks. The first block is related basics of calculus, limit, continuity, differentiability and mean value theorems. In second block explain about Successive Differentiation, Expansion of a function, Indeterminate forms, Maxima and Minima for one variable and Integrals. In third block contained the Asymptotes, Double and Triple Integrals. The last block is related to function of several variables. In this block the topics partial differentiation, Expansion of function in two variables and Jacobian are explained in simple manner. This material also used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the leaner’s to grasp the subject easily.

BLOCK-I
SEQUENCES, CONTINUITY AND DIFFERENTIABILITY

UNIT–1:-REAL NUMBERS AND SEQUENCE OF REAL NUMBERS

CONTENTS:-

1.1 INTRODUCTION:

This is a course on Calculus. Present unit will be help the learners to learn the topic. We will be talking about the real line on which we have the functions. We will be doing something with the functions in this unit. Calculus is a branch of mathematics that studies change. It focuses on limits, functions, derivatives, integrals and infinite series. In this unit we are discussing mostly about basics of sequence and series. The sequence and series are depends on Set. In this unit we are also defined the set, relation, function.

1.2 OBJECTIVES:

The objective of this topic is to at the end of this topic learner will be able to:

- **i.** Explain the Sets, interval, relation and function
- **ii.** Describe the real number system and its properties.
- **iii.** Memorize the basic concepts of sequence and properties

1.3 BASICS OF CALCULUS:

The concept of set theory and function theory is an important part of calculus. After that the topic of differentiation and integration defined by the mathematicians. Some concepts, like continuity, exponents, are the foundation of advanced calculus.

1.3.1 SETS:

Any well-defined collection of objects or numbers are referred to as a set. The number, letter or any other object contained in a set are called elements of the set. The sets are denoted by capital letters e.g. X, Y, Zor . The elements are denoted by lower case letters a, b, c, …., x, y, z. To indicate that 'a' is an element of the set X we use the notation $a \in X$. This read as "a is in X" or "a belongs to X". For example $A = \{1, 3, 5, 7, 11, 13, 17, 20\}$.

1.3.2 INTERVAL:

An open interval does not contain its endpoints, and is indicated with parentheses. $(a, b) =]a, b[= \{x\epsilon\mathbb{R}: a < x < b\}$. A closed interval is an interval which contain all its limit points, and is expressed with square brackets. $[a, b] = [a, b] = \{x\epsilon\mathbb{R}: a \le x \le b\}$. A half-open interval includes only one of its endpoints, and is expressed by mixing the notations for open and closed intervals. $(a, b] =]a, b] = \{x\epsilon\mathbb{R}: a < x \le b\}$. $[a, b) = [a, b[= \{x\epsilon\mathbb{R}: a \le x < b\}$.

1.3.3 ORDERED PAIRS:

An ordered pair (a, b) is a set of two elements for which the order of the elements is of significance. Thus $(a, b) \ne (b, a)$ unless $a = b$. In this respect (a, b) differs from the set $\{a, b\}$.Again $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$ .If X and Y are two sets, then the set of all ordered pairs (*x*, *y*), such that $x \in X$ and $y \in Y$ is called Cartesian product of *X* and *Y*. It is denoted by $X \times Y$.

1.3.4 RELATION:

A subset *R* of $X \times Y$ is called relation of *X* on *Y*. It gives a correspondence between the elements of *X* and *Y*. If (*x*, *y*) be an element of *R*, then *y* is called image of *x*. A relation in which each element of *X* has a single image is called a function.

If $X = \{1,2,3,4\}$ and $Y = \{a, b, c\}$ then,

$$X \times Y = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c),(4,a),(4,b),(4,c)\}$$

$$R_1 = \{(1,a),(2,b),(3,c),(4,b)\}$$

R_1 is a relation as well as a function while $R_2 = \{(1,a),(2,b),(2,c),\ (3,c)\}$ is a relation but not a function (since 2 has two images).

1.3.5 FUNCTION(MAPPING):

The equation $y = x^2$ gives a rule which determines for each number *x*, a corresponding number *y*. The set of all such pairs of numbers (*x*, *y*) determines a function.

Definition. Let *X* and *Y* are two sets and suppose that to each element *x* of *X* corresponds, by some rule, a single element *y* of *Y*. Then the set of all ordered pairs (*x*, *y*) is called function. The set *X* is called the domain of the function. The element *y*, which corresponds to the element *x* is called the value of function at *x*. It is denoted by $f(x)$, read as "*f* of *x*". The set of all the values of the function is called the range of the function. The term mapping is also used for a function and we say that the set *X* maps into the set *Y* under the mapping *f*. We write as $f : X \to Y$ and read as " the function *f* which maps *X* into *Y* ". We shall also use the notation $f : y = f(x)$ to denote "the function *f* defined by the rule $y = f(x)$". Basically function is a rule which binds one set *X* to another set *Y*.The rule is that for all elements of *X* their should be unique image in *Y*.

1.3.6 VARIABLE:

A symbol such as *x* or *y*, used to represent an arbitrary element of a set is called a variable. For example $y = f(x)$.The symbol *x* which represents an element in the domain is called the independent variable, and the symbol *y* which represent the element corresponding to *x* is called the dependent variable. This is based on the fact that value of x can be arbitrary chosen, then y has a value which depends upon the chosen value of x.

1.4 *REAL NUMBERS:*

Numbers initiate with Natural Numbers. The natural numbers are the standard numbers, $1, 2, 3, ..$. with which humans count. Natural numbers were discovered by Pythagoras $(582-500\ BC)$ and Archimedes $(287-212\ BC)$ (both are Greek philosophers and mathematicians). After Natural Number the integer was introduced in the year 1563 when Arbermouth Holst was busy with his bunnies and elephants experiment. He stored count of the amount of bunnies in the cage and after 6 months he saw that the amount of bunnies increased. Then he concludes the addition and multiplication of a number system then rational number is defined. In arithmetic, a number that can be considered as the quotient p/q of two integers such that $q \neq 0$. In addition to all the fractions, the set of rational numbers added all the integers, each of which can be written as a quotient with the integer as the numerator and 1 as the denominator. Rational numbers were discovered in the sixth century BCE by Pythagoras. Later this Irrational numbers are the numbers that cannot be considered as a simple fraction. It cannot be considered in the form of a ratio, such as p/q, where p and q are integers, q≠0. It is a contradiction of rational numbers. The Greek mathematician Hippasus of Metapontum is the person who invented irrational numbers in the 5th century B.C., according to an article from the University of Cambridge. Subsequently real number introduced in the 16th century, Simon Stevin designed the basis for modern decimal notation, and asserted that there is no difference between rational and irrational numbers in this regard.

Richard Dedekind
(6 October 1831 – 12 February 1916)
Fig 1.4.1
Ref:https://en.wikipedia.org/wiki/Richard_Dedekind#/media/File:Richard_Dedekind_1900s.jpg

In the 17th century, Descartes invented the term "real" to describe roots of a polynomial, distinguishing them from "imaginary" ones. Mathematician Richard Dedekind quarried these problems 159 years ago at ETH Zurich, and became the first person to characterize the real numbers. Bob sinclar defined the whole numbers in 1968. Whole Numbers is the subset of the number system that includes of all positive integers contained zero. In mathematics, a real number is a value of a continuous amount that can act for a distance along a line (or alternatively, a number that can be summarised as an infinite decimal expansion. The set of real numbers is expressed using the symbol R or $\mathbb{R}$. Real numbers can be consider of as points on an infinitely long line called the number line or real line, where the points interrelated to integers are equally spaced. Any real number can be resolved by a possibly infinite decimal representation. We can write the set of real numbers in the form of rational and irrational number as, $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$.

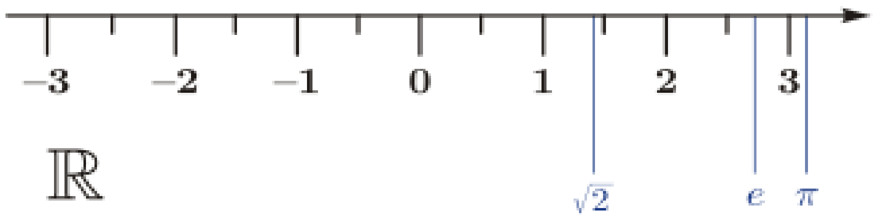


Fig 1.4.2
Ref: https://en.wikipedia.org/wiki/Real_number

The above points show that real numbers incorporate natural numbers, whole numbers, integers, rational numbers, and irrational numbers. $\sqrt{2},\ e, \pi$ are irrational numbers.

1.4 1 PROPERTIES OF REAL NUMBERS:

The main properties of real numbers are as follows:

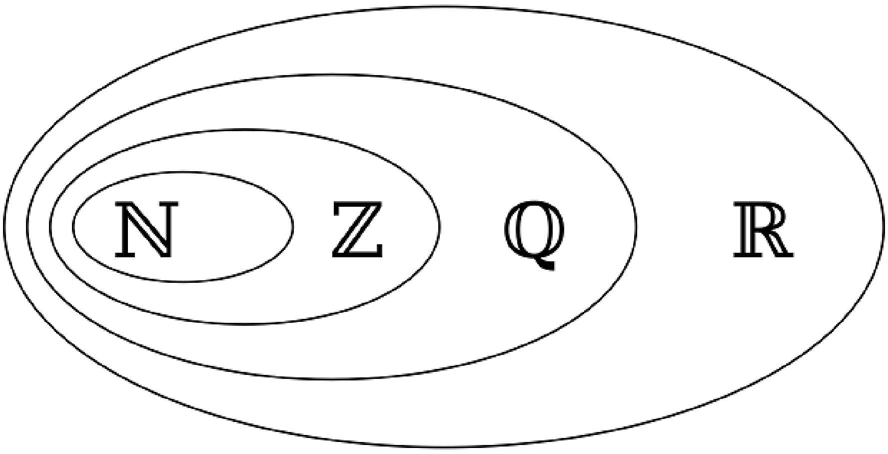
i. **Closure Property:** If $a, b \epsilon \mathbb{R}, a + b \epsilon \mathbb{R}$ and $ab \epsilon \mathbb{R}$. It shows that sum and product of two real numbers is always a real number.

ii. **Associative Property:** If $a, b, c \epsilon \mathbb{R}$, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$. It follows that sum or product of any three real numbers remains the same even when the grouping of numbers is changed.

iii. **Commutative Property:** If $a, b \epsilon \mathbb{R}, a + b = b + a$ and $a \times b = b \times a$. It means that the sum and the product of two real numbers remain the same even after interchanging the order of the numbers

iv. **Distributive Property:** Real numbers satisfy the distributive property. If $a, b, c \in \mathbb{R}$.

- $a \times (b + c) = (a \times b) + (a \times c)$ is the distributive property of multiplication over addition.
- $a \times (b - c) = (a \times b) - (a \times c)$ is the distributive property of multiplication over subtraction.



https://en.wikipedia.org/wiki/Real_number#/media/File:Number-systems.svg

1.4.2 DEFINITION:

If a and b are real numbers, we say that

(i) $a > b$ if $a - b$ is a positive number,
(ii) $a < b$ if $a - b$ is a negative number.

A relation involving $>$ or $<$ is known as an inequality. The following useful laws of inequalities can be easily obtained from the definition.

(i) If $a > b$, then $b < a$.
(ii) If $a > b$ and $b > c$, then $a > c$.
(iii) If $a > b$ and $c > d$, then $a + c > b + d$. (addition of inequalities).
(iv) If $a > b$, then $a + c > b + c$.
(v) If $a > b$ and c is a positive number, $ac > bc$.
(vi) If $a > b$ and c is a negative number, $ac < bc$.
(vii) If $a + c > b$, then $a > b$-c (transposition of a term). A particular case of transposition is:
If $a > b$, then $-b > -a$.

1.4.3 DISTANCE BETWEEN TWO POINTS:

The distance between two points x and a on the real line is denoted by $|x-a|$, and define as follows :

$$|x-a| = x-a \text{if} x \geq a,$$
$$|x-a| = x-a \text{if} x < a.$$

It is the numerical difference between the numbers x and a.

1.4.4 ABSOLUTE VALUE:

The absolute value$|x|$ of a real number x is defined by

i. $|x| = x$ if $x \geq 0$.

ii. $|x| = -x$ if $x < 0$.

- In particular, $(-\infty, +\infty)$ denotes the set of all ordinary real numbers.
- $|x| \geq 0$.
- $|-x| = |x|$.
- $|x|$ =max$(x, -x)$.
- $-|x|$ =min$(x, -x)$.
- If $x, y \in \mathbb{R}$, then (i) $|x|^2 = x^2 = |-x|^2$.(ii) $|xy| = |x|.|y|$ (iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ provided $y \neq 0$.

Let A be a nonempty subset of $\mathbb{R}$.

(i) The set is said to be bounded above if three exists a number $u \epsilon \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number u is called an upper bound of S.

(ii) The set is said to be bounded below if three exists a number $w \epsilon \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such number w is called an lower bound of S.

(iii) A set is said to be bounded if it is both bounded above and bounded below. A set is said to be unbounded if it is not bounded.

(iv) If A is bounded above, then a number u is said to be supremum (or a least upper bound) of A if it satisfies the conditions:

(a) u is an upper bound of A, and

(b) If v is any upper bound of A, then $u \leq v$.

(v) If A is bounded below, then a number w is said to be infimum (or a greatest lower bound) of A if it satisfies the conditions:

(c) w is an upper bound of A, and

(d) If t is any upper bound of A, then $t \leq w$.

- The least upper bound or the greatest lower bound may not belong to the set A.1 is least upper bound of the sets $\{x: 0 < x < 1\}, \{x: 0 \leq x \leq 1\}$ and $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$.

1.4.5 COMPLETENESS PROPERTY OF REAL NUMBER SYSTEM:

- Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

1.4.6 ARCHIMEDEAN PROPERTY:

If $x, y \in \mathbb{R}$and $x > 0$, then there is a positive integer n such that $nx > y$.

Proof : Let us assume that $nx \leq y$ for every positive integer n . Then y is an upper bound of the set $S = \{nx: n \in \mathbb{N}\}$. By the least upper bound property, let $u$be a l.u.b. of A. Since $n \in \mathbb{N}$ it implies $+1 \in \mathbb{N}$. So $(n+1)x \in S$. Then $(n+1)x \leq u$ for all n and so $nx \leq u - x < u$ for all n i.e. $u - x$ is also an upper bound of set S which is smaller than u. Since u be a l.u.b. of S. Then it is impossible $u - x$ is also an upper bound of set S . It contradicts the assumption $nx \leq y$. It means that $nx > y$.

Another form

If $x \in \mathbb{R}$ then there exists $n \in \mathbb{N}$ such that $x < n$.

Proof: Consider that $n \leq x$ for all $n \in \mathbb{N}$; therefore, x is an upper bound of $\mathbb{N}$. By the Completeness Property, the nonempty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Subtracting 1 from u gives a number $u - 1$ which is smaller than the supremum$u$ of $\mathbb{N}$. Therefore $u - 1$ is not an upper bound of $\mathbb{N}$, so there exists $m \in \mathbb{N}$ with $u - 1 < m$.Adding1 gives $u < m + 1$, and since $m + 1 \in \mathbb{N}$, this inequality Contradicts the fact that u is an upper bound of $\mathbb{N}$.

- If $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then $\inf S = 0$.
- If $t > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < t$.
- If $y > 0$, there exists $n \in \mathbb{N}$ such that $n - 1 \leq y < n$.
- Let $A = \{r \epsilon Q: r > 0, r^2 < 2\}$ this is a non-empty and bounded subset of$\mathbb{Q}$. The set A does not have l.u.b. in$\mathbb{Q}$. This shows that $\mathbb{Q}$ does not have the least upper bound property.
- If $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \epsilon \mathbb{Q}$ such that $x < p < y$.

Proof: By Archimedean property $x, y \in \mathbb{R},\ x > 0$ then there exists $\in \mathbb{N}, nx > y,\ y - x > 0$. As $x < y \Rightarrow y - x > 0$. For $y - x, 1 \in \mathbb{R}$ and $y - x > 0$.Then there exists $n \in \mathbb{N}, n(y - x) > 1, ny - nx > 1, ny > 1 + nx,$

$1 + nx < ny$**……………………………..(1.4.6.1)**

Now we are searching such type of integer which is greater then nx and smaller then ny. Let $A = \{m: m\epsilon Z,\ \ nx < m\}$.Since any subset $A \subseteq \mathbb{Z}, A \neq \emptyset, A$ has lower bound then A is bounded below.For $nx, 1 \in \mathbb{R}$ and $1 > 0$ by Archimedean property there exists $n_0 \epsilon \mathbb{N}$ such that $n_0 > nx\ \Rightarrow nx < n_0$ it implies $n_0 \epsilon \mathbb{N}$ it implies $n_0 \epsilon \mathbb{Z}$. Now $n_0 \epsilon \mathbb{Z}$. and $nx < n_0$ it implies $n_0 \epsilon A$. It shows that $A \neq \emptyset$. It impliesnx is a lower bound of A.As $A \neq \emptyset$, $A \subset \mathbb{Z}$ and A has a lower bound it implies A has a minimal element.Since we are taking $A = \{m: m\epsilon Z,\ \ nx < m\}$. Say $m_1 \in A$ is its minimal element $m_1 - 1 \notin A m_1 - 1 \leq$ nx. m_1 ≤1+nx.m_1 <ny,$m_1 \in A$, m_1>nx , $nx < m_1 < ny$,$x < \frac{m_1}{n} < y$,It implies that $x < P < y$ when $P = \frac{m_1}{n} \in \mathbb{Q}$.

1.5 SEQUENCE OF REAL NUMBERS:

A set of numbers $a_1, a_2, a_3 \dots \dots \dots., a_n, \dots \dots ..$ in a definite order of occurrence is called a sequence. It is denoted briefly by $\{a_n\}$. A sequence is really a function of the natural number n, written down in the natural order.A sequence $\{a_n\}$ is said to be bounded above if $\{a_n\} \leq a$ for every n,Where a is some fixed number. Similarly A sequence $\{a_n\}$ is said to be bounded below if $\{a_n\} \geq b$for every n, where b is a fixed number.A sequence which is bounded both above and below is called a bounded sequence.For such a sequence$a \leq \{a_n\} \leq b$ for every n, where a and b are fixed numbers.We can easily rewrite this relation as $|s_n| < c$ for every n, where c is a fixed positive number.A sequence $\{a_n\}$ is said to be monotonically increasing if $a_n \leq a_{n+1}$ for every n.A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_n \geq a_{n+1}$ for every n.

- The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots \dots ..$ is a monotonically decreasing sequence, while $1, 2,3,4 \dots \dots \dots$..is a monotonically increasing sequence.

1.6 EXAMPLES:

Problem 1: Show that there is no rational number whose square is 2.
Solution. Let**,** if possible, there exist a rational number p/q, where $q \neq 0$ and p, q are integers prime to each other (i.e. having no common factor) whose square is equal to 2,

i.e., $(p/q)^2 = 2$ or $p^2 = 2q^2$ ……………………….(**1.6.1**)
Now q is an integer and so is $2q^2$. Thus, p^2 is an integer divisible by 2. As such p must be divisible by 2, for otherwise p^2 would not be divisible by 2. Let $p = 2m$, where m is an integer. Then from (1.6.1),

$$2m^2 = q^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(\mathbf{1.6.2})$$

Thus, it follows that q is also divisible by 2. Hence, p and q are both divisible by 2 which contradicts the hypothesis that p and q have no common factor. Thus, there exists no rational number whose square is 2.

Problem 2: Show that $\sqrt{8}$ is not a rational number.

Solution. Let, if possible $\sqrt{8}$ be the rational number p/q, where $q \neq 0$ and where p, q are positive integers prime to each other, so that $\sqrt{8} = p/q$. But $2 < \sqrt{8} < 3, 2 < p/q < 3 \Rightarrow 2q < p < 3q$ or $0 < p - 2q < q$. Thus, $p - 2q$ is a positive integer less than q, so that $\sqrt{8}(p - 2q)$ or $p/q\,(p - 2q)$ is not an integer. But $\sqrt{8}(p - 2q) = p/q\,(p - 2q) = \frac{p^2}{q} - 2p$ it implies that $\frac{p^2}{q^2}q - 2p = 8q - 2p$, which is an integer it shows that $\sqrt{8}(p - 2q)$ is an integer. This is a contradiction. Hence,$\sqrt{8}$ is not a rational number.

Problem 3: Prove that the greatest member of a set, if it exists, is the supremum (l.u.b) of the set.

Solution. Let G be the greatest member of the set S. Clearly $x \leq G \forall x \in S$. So that G is an upper bound of S. Again no number less than G can be an upper bound of S, for if y be any number less than G, there exists at least one member g of S which is greater than y. Thus,G is the least of all the upper bound of S, i.e., G is the supremum of S.

Problem 4: For all real numbers x, y show that

i $|x + y| \leq |x| + |y|$, and

ii $|x + y| \geq \big||x| - |y|\big|$,

Solution: i.$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$

$$\leq |x|^2 + |y|^2 + 2|x|.|y| [\because xy \leq |xy| = |x|.|y|]$$
$$= (|x| + |y|)^2.$$

Since $|x + y|$ and $|x| + |y|$ are both non-negative, therefore taking positive square roots on both sides, we have $|x + y| \leq |x| + |y|$.

ii.$|x - y|^2 = (x - y)^2 = x^2 + y^2 - 2xy$

$$\geq |x|^2 + |y|^2 - 2|x|.|y| [\because -(xy) \geq |xy| = -|x|.|y|]$$
$$= (|x| - |y|)^2 = \big||x| - |y|\big|^2.$$

Since $|x - y|$ and $\big||x| - |y|\big|$ are both non-negative, therefore taking positive square roots on both sides, we have $|x + y| \geq \big||x| - |y|\big|$.

Problem 5: For real numbers $x, a, \mathcal{E} > 0$ show that

a) $|x| < E \Leftrightarrow -E < x < E$,

b) $|x - a| < E \Leftrightarrow a - \mathcal{E} < x < a + \mathcal{E}$.

Solution: a) $|x|$ = max $(x, -x) < E \Leftrightarrow x < E \wedge -x < E$

$$\Leftrightarrow -\mathcal{E} < x < E$$

b) $|x-a| = \max\{(x-a), -(x-a)\} < E$
$$\Leftrightarrow (x-a) < E \wedge -(x-a) < E$$
$$\Leftrightarrow x < a + \mathcal{E} \wedge a - \mathcal{E} < x$$
$$\Leftrightarrow a - \mathcal{E} < x < a + \mathcal{E}.$$

Problem 6: Show that a set S of real numbers is bounded if there exists a real number $G > 0$ such that $|x| \leq G, \forall S$.

Solution. Suppose that$S$ is bounded, therefore it is bounded both above and below. Let K be an upper bound and k, a lower bound for S. On taking a real number $G = \max\,(|K|, |k| + 1)$, we have, $K \leq |K| \leq G$ and $-k \leq |k| \leq |k| + 1 \leq G, i.e., k > -G$. This implies $-G < k \leq x \leq K \leq G, \forall\, x \in S$. Hence $|x| \leq G, \forall S$.

Problem 7: If$a, b \in \mathbb{R}$ such that $a < b + \mathcal{E}$ for each $\mathcal{E} > 0$, then $a \leq b$.

Solution. Suppose $a > b$. Then $a - b > 0$, so that $a < b + (a-b)$(by taking $\mathcal{E} = a - b$) and so $a < a$.This is a contradiction. Hence our assumption $a > b$ must be false. Therefore $a \leq b$.

Problem 8: If$a, b \in \mathbb{R}$ such that $a \leq b + \frac{1}{n}$, for all $n \in \mathbb{N}$, then $a \leq b$.

Solution. Assume $a \leq b + \frac{1}{n}$, for all $n \in \mathbb{N}$, and $a > b$. Then $a - b > 0$ and by the Archimedean property, we have, $n_0(a-b) > 1$, for some $n_0 \in \mathbb{N}$. Then $a > b + \frac{1}{n_0}$, contrary to our assumption.

Problem 9: If for any $\mathcal{E} > 0, |b - a| < E$, then $b = a$.

Solution. We have for any$\mathcal{E} > 0, b < a + \mathcal{E}$ and $a - \mathcal{E} < b$.Since$b < a + \mathcal{E}$ for any $\mathcal{E} > 0$ this implies $a \leq b$. Hence $b = a$.

Problem 10: If $a, b \in \mathbb{R}$ and $a < c$ for each $c > b$, then $a \leq b$.

Solution. Assume that$a$ and b satisfy the hypothesis but not the conclusion. Then $a > b$, and so there is $a, c \in \mathbb{R}$ such that $a > c > b$. Now $c > b \Rightarrow a < c$ in contradiction to $a > c$.

CHECK YOUR PROGRESS

1: For what values of x is $\sqrt{(2x+3)}$ a real number?

2: Find the union and intersection of sets A and B:

$A = \{x|x$=rational number$\}$

$B = \{x|x$=irrational number$\}$

3: A function f is defined by $f(x) = x^2 - 3x + 4$. Find the value of the function at $x = 1,2$ and 3. Also find

$\frac{f(x+h)-f(x)}{x}$ $(h \neq 0)$.

1.7. SUMMARY:

In this unit we are explaining Sets, interval, relation and function. In this unit our main focus is properties of real number system. We are explaining Archimedean property and its proof in simple form.

1.8. GLOSSARY:

- **i.** Set.
- **ii.** Relation and Function.
- **iii.** Number System and its properties.
- **iv.** Basics of sequence.

1.9. REFERENCES:

- **i.** Tom M. Apostol (1996). Mathematical Analysis (2nd edition), Narosa Book Distributors Pvt Ltd-New Delhi.
- **ii.** Gorakh Prasad (2016). Differential Calculus (19th edition). PothishalaPvt. Ltd.
- **iii.** Walter Rudin. (2017). Principles of Mathematical Analysis (3rd edition). McGraw Hill Education .
- **iv.** R.G. Bartley and D.R. Sherbert (2000) Introduction of real analysis, John Wiley and Sons (Asia) P. Ltd., Inc.
- **v.** Gilbert Strang (1991). Calculus. Wellesley-Cambridge Press.

1.10.SUGGESTED READINGS:

- **i.** Howard Anton, I. Bivens and Stephan Davis (2016). Calculus (10th edition). Wiley India.
- **ii.** George B. Thomas Jr, Ross L.Finney (1998), Calculus and Analytical Geometry, Adison Wiley Publishing Company.
- **iii.** James Stewart (2012). Multivariable Calculus (7th edition). Brooks/Cole. Cengage.
- **iv.** S.C. Malik and SavitaArora (2021). Mathematical Analysis (6th edition). New Age International Private Limited.

1.11.TERMINAL QUESTIONS:

TQ1. If$a, b \in \mathbb{R}$ then show that max $(a, b) = \frac{a+b+|a-b|}{2}$ and min $(a, b) = \frac{a+b-|a-b|}{2}$.

TQ2. State and proof Archimedean property?.

TQ3. Prove that $|x+y| = |x| + |y|$ iff $xy \geq 0$.
TQ4. Prove that $|x+y| < |x| + |y|$ iff $xy < 0$.
TQ5. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?
TQ6. There are 35 learner in art class and 57 learners in dance class. Find the number of learners who are either in art class or in dance class?

- When two classes meet at different hours and 12 learners are enrolled in both activities?
- When two classes meet at the same hour?

1.12.ANSWERS:

ANSWER OF CHEK YOUR PROGESS:
SCQ1: $\left(-\frac{3}{2}, \infty\right)$.
SCQ2: $(-\infty, \infty), \emptyset$.
SCQ3: $2,2,4; 2x - 3 + h$

ANSWER OF TERMINAL QUESTIONS:
TQ4: 9
TQ5: (i) 80 (ii) 92.

UNIT–2:- LIMIT AND CONTINUITY

CONTENTS:-

2.1. INTRODUCTION

Grégoire de Saint-Vincent gave the first definition of limit (terminus) of a geometric series.The modern definition of a limit goes back to Bernard Bolzano who, in 1817, developed the basics of the epsilon-delta technique to define continuous functions. However, his work remained unknown to other mathematicians until thirty years after his death. Augustin-Louis Cauchy in 1821, followed by Karl Weierstrass, formalized the definition of the limit of a function which became known as the (ε, δ)-definition of limit. The modern notation of placing the arrow below the limit symbol is due to G. H. Hardy. In mathematics, a continuous function is a function such that a continuous variation (that is a change without jump) of the argument induces a continuous variation of the value of the function. This means that there are no abrupt changes in value, known as discontinuities. More precisely, a function is continuous if arbitrarily small changes in its value can be assured by restricting to sufficiently small changes of its argument. In previous unit we have discussed about basics of Calculus.Now in this unit we have explained about limit and continuity.

2.2. OBJECTIVES

The objective of this topic is to at the end of this topic learner will be able to:

- **i.** Explained the concept of limit of a function.
- **ii.** Describe the meaning of continuity and discontinuity.
- **iii.** Defined the Uniform Continuity.

2.3 LIMIT OF A FUNCTION

Definition. Let *f* be a function given by the rule *y* = *f* (*x*). Choose any set of positive numbers $h_1, h_2, h_3, ..., h_n, ...$ which continuously decreases *i.e.*,

$$h_1 > h_2 > h_3 > .... > h_n > .... > 0 ....(1)$$

and can be made as small as we want by taking n large enough. Then the values $f(a+h_1), f(a+h_2), f(a+h_3), ..., f(a+h_n), ...(2)$

of the function continuously approach a number *A* as h_n gets smaller and smaller. This number *A* is called the "**Limit of *f* (*x*) at *a***" or "the limit of *f* (*x*) as *x* → *a*". We write $\underset{x \to a}{Limit} f(x) = A$. Here *x* → *a* is read as "*x* tends to *a*".In fact *A* is the limit on the right since we have considered only the value of *x* greater than *a* *i.e.*, on . If we consider the value of the function

$$f(a-h_1), f(a-h_2), f(a-h_3), ..., f(a-h_n), ...$$

And find that they are continuously approach a number *B* as h_n gets smaller and smaller, we call *B* the limit of *f* (*x*) on the left. When *A*= *B*, we call *A* the limit of *f* (*x*) at *a*. The limits on the right and left are respectively denoted by $\underset{x \to a+}{Limit} f(x)$ and $\underset{x \to a-}{Limit} f(x)$.If the numbers $h_1, h_2, h_3, ..., h_n, ...$ considered above from a sequence, having the limit zero. Similarly the numbers $f(a+h_1), f(a+h_2), f(a+h_3), ..., f(a+h_n), ...$ form another sequence. It should be noted that for the limit to exist, $f(a+h_n)$ should approach *A* for every sequence of type (1).

ANOTHER DEFINITION OF LIMIT

A number l is said to be the limit of function $f(x)$ at x=a if for arbitrary $\varepsilon > 0, \exists \delta > 0$ (positive real number) such that, whenever

$$0 < |x-a| < \delta \text{ we have } |f(x)-l| < \varepsilon$$

or we write $\underset{x \to a}{Limit}\ f(x) = l$ if given $\varepsilon > 0, \exists \delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x)-l| < \varepsilon.$$

2.4. ALGEBRA OF LIMITS

Theorem 1. If $\underset{x \to a}{Limit}\ f(x) = l \neq 0$ then $\exists$ a number $k > 0$ and $\delta > 0$ such that $|f(x)| > k$ whenever $0 < |x-a| < \delta$. Also then $\underset{x \to a}{Limit}\ \frac{1}{f(x)} = \frac{1}{l}$.

Proof. Let $\varepsilon = \frac{1}{2}|l|$ then $\varepsilon > 0$ because $l \neq 0$. Since $\underset{x \to a}{Limit}\ f(x) = l$, therefore given $\varepsilon > 0, \exists \delta > 0$ such that

$|f(x)-l| < \varepsilon$ whenever $0 < |x-a| < \delta$............................(1)

Now $|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)|$

$< \varepsilon + |f(x)|$ whenever $0 < |x-a| < \delta$ (from (1))

This implies that whenever $0 < |x-a| < \delta$

$|f(x)| > |l| - \varepsilon = |l| - \frac{1}{2}|l| = \frac{1}{2}|l| > 0$..................(2)

Taking $\frac{1}{2}|l| = k > 0$ we get

$$|f(x)| > k \text{ whenever } 0 < |x-a| < \delta$$

Now to prove $\underset{x \to a}{Limit}\ \frac{1}{f(x)} = \frac{1}{l}$. We have

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| = \left|\frac{l - f(x)}{l.f(x)}\right| = \frac{|l - f(x)|}{|l.f(x)|} = \frac{|l - f(x)|}{|l||f(x)|} \text{............................(3)}$$

Now from the first part $\exists$ $k > 0$ and $\delta_1 > 0$ such that whenever $0 < |x-a| < \delta_1$

$|f(x)| > k$..(4)

Let $\varepsilon' > 0$ be given. Since $\underset{x \to a}{Limit}\ f(x) = l$, therefore given $\varepsilon' > 0, \exists \delta_2 > 0$ such that

$|f(x)-l|<\varepsilon'$ whenever $0<|x-a|<\delta_2$..................................(5)

Let $\delta=\min\{\delta_1,\delta_2\}$ then from (3), (4) and (5) we get

$$\left|\frac{1}{f(x)}-\frac{1}{l}\right|<\frac{1}{|l|}.\varepsilon'.\frac{1}{k}=\frac{\varepsilon'}{k.|l|} \text{ whenever } 0<|x-a|<\delta$$

If we take $\varepsilon'=k.|l|.\varepsilon>0$ then we have

$$\left|\frac{1}{f(x)}-\frac{1}{l}\right|<\frac{\varepsilon.k.|l|}{k.|l|}<\varepsilon \text{ whenever } 0<|x-a|<\delta$$

Hence $\underset{x\to a}{Limit}\,\frac{1}{f(x)}=\frac{1}{l}$.

Theorem 2. The limit of a sum is equal to the sum of limits.

Proof. Let $\underset{x\to a}{Limit}\, f(x)=l$ and $\underset{x\to a}{Limit}\, g(x)=m$.

To show that $\underset{x\to a}{Limit}\,[(f+g)(x)]=l+m$.

Let $\varepsilon>0$ be given. Since $\underset{x\to a}{Limit}\, f(x)=l$, therefore, $\exists\delta_1>0$ such that

$$|f(x)-l|<\frac{1}{2}.\varepsilon \text{ whenever } 0<|x-a|<\delta_1$$

And $\underset{x\to a}{Limit}\, g(x)=m$, therefore, $\exists\delta_2>0$ such that

$$|g(x)-m|<\frac{1}{2}.\varepsilon \text{ whenever } 0<|x-a|<\delta_2$$

If we take $\delta=\min\{\delta_1,\delta_2\}$, then for $0<|x-a|<\delta$ both $0<|x-a|<\delta_1$ and $0<|x-a|<\delta_2$ holds. And so whenever $0<|x-a|<\delta$ then both $|f(x)-l|<\frac{1}{2}.\varepsilon$ and $|g(x)-m|<\frac{1}{2}.\varepsilon$ are true.

Now if $0<|x-a|<\delta$ then

$$\begin{aligned}|(f+g)(x)-(l+m)| &= |(f(x)-l)+(g(x)-m)| \\ &\le |(f(x)-l)|+|(g(x)-m)| \\ &< \frac{1}{2}.\varepsilon+\frac{1}{2}.\varepsilon=\varepsilon\end{aligned}$$

Thus $|(f+g)(x)-(l+m)|<\varepsilon$ whenever $0<|x-a|<\delta$.

Hence $\underset{x\to a}{Limit}\,[(f+g)(x)]=l+m$.

Similarly we can show that $\underset{x\to a}{Limit}\,[(f-g)(x)]=l-m$.

Theorem 3. The limit of product is equal to the product of limits.

Proof. Let $\underset{x\to a}{Limit}\, f(x) = l$ and $\underset{x\to a}{Limit}\, g(x) = m$. To show that $\underset{x\to a}{Limit}\, [(fg)\ (x)] = l.m$.

Now

$$|(fg)(x) - (l.m)| = |f(x).g(x) - l.m)|$$
$$= |f(x).g(x) - l.g(x) + l.g(x) - l.m)|$$
$$\leq |f(x).g(x) - l.g(x)| + |l.g(x) - l.m)|$$
$$\leq |g(x)||f(x) - l| + |l||g(x) - m| \text{ ....(1)}$$

Since $\underset{x\to a}{Limit}\, g(x) = m$, therefore $g(x)$ is bounded in some deleted neighbourhood of $x = a$. Hence $\exists\ k > 0$ and $\delta_1 > 0$ such that whenever $0 < |x - a| < \delta_1$ then $|g(x)| \leq k$.

Now let $\varepsilon > 0$ be given. Since $\underset{x\to a}{Limit}\, f(x) = l$, therefore , $\exists \delta_2 > 0$ such that

$|f(x) - l| < \frac{1}{2}.\varepsilon$ whenever $0 < |x - a| < \delta_2$.

And $\underset{x\to a}{Limit}\, g(x) = m$, therefore, $\exists \delta_3 > 0$ such that

$|g(x) - m| < \frac{1}{2}.\varepsilon$ whenever $0 < |x - a| < \delta_3$

If we take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then from (1) whenever $0 < |x - a| < \delta$ we get

$$|(fg)(x) - (l.m)| < k.\frac{\varepsilon}{2k} + |l|.\frac{\varepsilon}{2(|l|+1)} < \frac{\varepsilon}{2} + |l|.\frac{\varepsilon}{2(|l|)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus for $\varepsilon > 0$ we have $\delta > 0$ such that

$|(fg)(x) - (l.m)| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Hence $\underset{x\to a}{Limit}\, [(fg)\ (x)] = \underset{x\to a}{Limit}\ f(x).g\ (x) = l.m$.

Similarly we can show that the limit of quotient is equal to the quotient of the limits provided that the limit of denominator is not zero.

2.4.1. RIGHT HAND AND LEFT HAND LIMITS

A. A function $f(x)$ is said to approach l as $x \to a$ from right if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $a < x < a + \delta$

It is written as $\underset{x\to a+0}{Limit}\ f(x) = l$ or $f(a+0) = l$.

"Put $a+h$ for x in $f(x)$, where $h > 0$ and very small and make h approach zero" *i.e.*, $f(a+0) = \underset{h\to 0}{Limit}\ f(a+h)$.

B. A function $f(x)$ is said to approach l as $x\to a$ from left if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } a - \delta < x < a$$

It is written as $\underset{x\to a-0}{Limit}\ f(x) = l$ or $f(a-0) = l$."Put $a - h$ for x in $f(x)$, where $h > 0$ and very small and make h approach zero" *i.e.*, $f(a-0) = \underset{h\to 0}{Limit}\ f(a-h)$.

Note. If both Right Hand Limit and Left Hand Limit of $f(x)$ as $x\to a$ are equal in value, their common value will be the limit of $f(x)$ as $x\to a$. If either or both of these limits do not exist, then the limit of $f(x)$ as $x\to a$ does not exist. Even if both of these limits exists but are not equal in value then also the limit of $f(x)$ as $x\to a$ does not exist.

2.4.2. LIMITS AS $x \to +\infty$ ($-\infty$)

A. A function $f(x)$ is said to approach l as $x \to +\infty$, if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } \delta \le x$$

Then we write $\underset{x\to\infty}{Limit}\ f(x) = l$ or $f(x) \to l$ as $x\to\infty$.

B. A function $f(x)$ is said to approach l as $x \to -\infty$, if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } x \le -\delta$$

Then we write $\underset{x\to-\infty}{Limit}\ f(x) = l$ or $f(x) \to l$ as $x\to-\infty$.

2.4.3. INFINITE LIMITS

A function $f(x)$ is said to approach $+\infty$ or $-\infty$ as $x \to a$, if for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x) > \varepsilon \text{ or } f(x) < -\varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Then in other words, $\underset{x\to a}{Limit}\ f(x) = \infty$ or $\underset{x\to a}{Limit}\ f(x) = -\infty$.

Illustrative examples

Example 1. Find $\underset{x\to 0}{Limit}\dfrac{\sin x}{x}$.

Solution. Let $f(x) = \dfrac{\sin x}{x}$ Here

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit}\frac{\sin h}{h}$$

$$= \underset{h \to 0}{Limit} \frac{h - h^3/3! + h^5/5! - h^7/7! + \ldots}{h}$$

$$= \underset{h \to 0}{Limit} 1 - h^2/3! + h^4/5! - h^6/7! + \ldots = 1$$

And $f(0-0) = \underset{h \to 0}{Limit} f(0-h) = \underset{h \to 0}{Limit} f(-h) = \underset{h \to 0}{Limit} \frac{\sin(-h)}{-h}$

$= \underset{h \to 0}{Limit} \frac{\sin(h)}{h} = 1$.Since f (0+0)= f(0-0)= 1 and hence $\underset{h \to 0}{Limit} \frac{\sin x}{x} = 1$.

Example 2. Find $\underset{x \to \infty}{Limit} \frac{\sin x}{x}$.

Solution. Let $f(x) = \frac{\sin x}{x}$. Put $x = 1/y$ so as $x \to \infty$, $y \to 0$. Then

$$\underset{x \to \infty}{Limit} \frac{\sin x}{x} = \underset{y \to 0}{Limit} \frac{\sin(1/y)}{1/y} = \underset{y \to 0}{Limit}\ y \sin\left(\frac{1}{y}\right)$$

Let $g(y) = y \sin\left(\frac{1}{y}\right)$. Then, right hand limit is

$g(0+0) = \underset{h \to 0}{Limit} g(0+h) = \underset{h \to 0}{Limit} g(h)$

$= \underset{h \to 0}{Limit} h \sin\left(\frac{1}{h}\right)$

$= 0 \times$ finite quantity which lies between -1 and +1

$= 0$

and the left hand limit is

$g(0-0) = \underset{h \to 0}{Limit} g(0-h) = \underset{h \to 0}{Limit} g(-h)$

$= \underset{h \to 0}{Limit} h \sin\left(\frac{1}{h}\right) = 0$

Since g (0 + 0) = g (0 - 0) = 0 therefore $\underset{y \to 0}{Limit}\ y \sin\left(\frac{1}{y}\right) = 0$ and hence

$\underset{x \to \infty}{Limit} \frac{\sin x}{x} = 0.$

Example 3. Find $\underset{x \to \infty}{Limit} \sin\left(\frac{1}{x}\right)$.

Solution. Let $f(x) = \sin\left(\frac{1}{x}\right)$. Here

$f(0+0) = \underset{h \to 0}{Limit} f(0+h) = \underset{h \to 0}{Limit} f(h) = \underset{h \to 0}{Limit} \sin\left(\frac{1}{h}\right)$

As $h \to 0$, the value of $\sin\left(\frac{1}{h}\right)$ oscillates between -1 and +1 passing through zero. Hence there is no definite number l to which $\sin\left(\frac{1}{h}\right)$ tends to as $h \to 0$. Therefore right hand limit does not exist.Similarly left hand limit $f(0-0)$ also does not exists.

Thus $\underset{x\to\infty}{Limit} \sin\left(\frac{1}{x}\right)$ does not exist.

Example 4. Find $\underset{x\to 0}{Limit} (1+x)^{1/x}$.

Solution. Let $f(x) = \underset{x\to 0}{Limit} (1+x)^{1/x}$. Now right hand limit is

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit} (1+h)^{1/h}$$

$$= \underset{h\to 0}{Limit}\left[1+\frac{1}{h}.h+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2!}.h^2+\frac{\frac{1}{h}\left(\frac{1}{h}-1\right)\left(\frac{1}{h}-2\right)}{3!}.h^3+....\right]$$

$$= \underset{h\to 0}{Limit}\left[1+\frac{1}{1!}+\frac{1.(1-h)}{2!}+\frac{1.(1-h)(1-2h)}{3!}+....\right]$$

$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+....\infty = e$$

Similarly, the left hand limit is

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h) = \underset{h\to 0}{Limit} (1-h)^{-1/h} = e$$

Thus both f (0 + 0) and f (0 – 0) exists and equal to e. Hence $\underset{x\to 0}{Limit} (1+x)^{1/x} = e$.

Example 5. Show that $\underset{x\to 2}{Limit} \frac{|x-2|}{(x-2)}$ does not exist.

Solution. Let $f(x) = \underset{x\to 2}{Limit} \frac{|x-2|}{(x-2)}$. Now right hand limit is

$$f(2+0) = \underset{h\to 0}{Limit}\ f(2+h) = \underset{h\to 0}{Limit} \frac{|2+h-2|}{(2+h-2)}$$

$$= \underset{h\to 0}{Limit} \frac{|h|}{(h)} = \underset{h\to 0}{Limit} \frac{h}{h} = 1$$

and the left hand limit is

$$f(2-0) = \underset{h\to 0}{Limit}\ f(2-h) = \underset{h\to 0}{Limit}\ \frac{|2-h-2|}{(2-h-2)}$$

$$= \underset{h\to 0}{Limit}\ \frac{|-h|}{(-h)} = \underset{h\to 0}{Limit}\ \frac{h}{-h} = -1$$

Since $f(2+0) \neq f(2-0)$. Hence $\underset{x\to 2}{Limit}\ \frac{|x-2|}{(x-2)}$ does not exist.

Example 6. Find $\underset{x\to 0}{Limit}\ \frac{1}{x} e^{1/x}$.

Solution. Let $f(x) = \underset{x\to 0}{Limit}\ \frac{1}{x} e^{1/x}$. Then

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h) = \underset{h\to 0}{Limit}\ \frac{1}{h} e^{1/h}$$

$$= \infty \text{ (since } \frac{1}{h} \to \infty \text{ and } e^{1/h} \to \infty \text{ as } h \to 0)$$

and

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h) = \underset{h\to 0}{Limit}\ -\frac{1}{h} e^{-1/h}$$

$$= \underset{h\to 0}{Limit}\ -\frac{1}{he^{1/h}}$$

$$= \underset{h\to 0}{Limit}\ \frac{-1}{h\left[1 + \frac{1}{h} + \frac{1}{2!}\frac{1}{h^2} + \frac{1}{3!}\frac{1}{h^3} + \ldots.\infty\right]} = 0$$

Since $f(0+0) \neq f(0-0)$.

Hence $\underset{x\to 0}{Limit}\ \frac{1}{x} e^{1/x}$ does not exist.

2.16 THE FOUR FUNCTIONAL LIMITS AT A POINT

Let a function $f(x)$ be defined in (a, b). let $c \in (a, b)$ and $h > 0$. We give to h a sequence of diminishing value $<h_n>$ with $\underset{n\to\infty}{Limit}\ h_n = 0$.

A. Consider the right hand neighbourhood $(c, c+h)$ of the point c. let $M(h_n)$ be supremum of $f(x)$ in $(c, c+h_n)$ and $m(h_n)$ be the infimum of $f(x)$ in $(c, c+h_n)$, then

$$M(h_1) \geq M(h_1) \geq M(h_1) \geq \ldots..$$

$$m(h_1) \leq m(h_1) \leq m(h_1) \leq \ldots..$$

This means that the sequences $<M(h_n)>$ and $<m(h_n)>$ are monotonically non-increasing and non-decreasing respectively.

Hence $\underset{n\to\infty}{Limit}\ M(h_n)$ and $\underset{n\to\infty}{Limit}\ m(h_n)$ exists. We write

$$\overline{f(c+0)} = \underset{n\to\infty}{Limit}\, M(h_n) \text{ and } \underline{f(c+0)} = \underset{n\to\infty}{Limit}\, M(h_n)$$

These limits are respectively called the upper and lower limits of $f(x)$ at $x = c$ on the right.

B. Next we consider the left hand neighbourhood $(c\text{-}h,\ c)$ of the point c. let $M'(h_n)$ be supremum of $f(x)$ in $(c,\ c+h_n)$ and $m'(h_n)$ be the infimum of $f(x)$ in $(c\text{-}h_n,\ c)$. Arguing as above, we find that $\underset{n\to\infty}{Limit}\, M'(h_n)$ and $\underset{n\to\infty}{Limit}\, m'(h_n)$ exists and we write

$$\overline{f(c-0)} = \underset{n\to\infty}{Limit}\, M(h_n) \text{ and } \underline{f(c-0)} = \underset{n\to\infty}{Limit}\, M(h_n)$$

These limits are called the upper and lower limits of $f(x)$ at $x = c$ on the left respectively.

2.6. CONTINUITY

The intuitive concept of continuity is derived from geometrical consideration. If the graph of the function $y = f(x)$ is a continuous curve, then it is to call the function continuous.

2.6.1. CAUCHY'S DEFINITION OF CONTINUITY.

A real valued function $f(x)$ defined on an interval I is said to be continuous at $x = a \in I$ if and only if for any arbitrarily chosen positive number ε, however small, we can find a corresponding number $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta.$

We say that $f(x)$ is continuous if it is continuous at every $x \in I.$

or

$f(x)$ is continuous at $x = a$ is given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$

2.6.2. GEOMETRICAL INTERPRETATION OF CONTINUITY.

The geometrical interpretation of the above definition is that, corresponding to any pre-assigned positive number ε, we can determine an interval of width 2δ about the point $x = a$ such that for any point x lying in the interval $(a-\delta, a+\delta)$, $f(x)$ is confirmed to lie between $f(a) - \varepsilon$ and $f(a) + \varepsilon$.

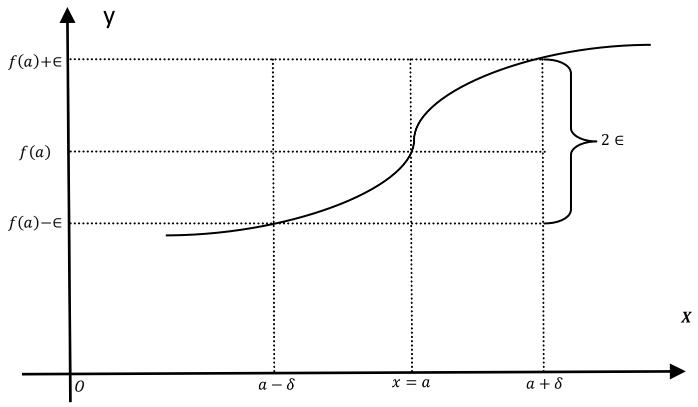


Fig 2.6.2.1

A. For a function $f(x)$ to be a continuous at $x = a$, it is necessary that $\underset{x \to a}{Limit} f(a)$ must exist.

B. $|f(x) - f(a)| < \varepsilon \Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$ i.e., $f(x)$ lies between $f(a) - \varepsilon$ and $f(a) + \varepsilon$ and $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$ *i.e.*, x lies between $a - \delta$ and $a + \delta$.

C. The function must be defined at the point of continuity.

D. The value of δ depends upon the value of ε and a.

2.6.3. HEINE'S DEFINITION OF CONTINUITY

A function $f(x)$ is said to be continuous at $x = a$, if and only if every convergent sequence $< x_n >$ of real numbers such that $\underset{n \to \infty}{Limit}\, x_n = a$, the sequence $< f(x_n) >$ converges to $f(a)$ *i.e.*, f is continuous at $x = a$ if and only if $\underset{n \to \infty}{Limit}\, x_n = a \Rightarrow \underset{n \to \infty}{Limit}\, f(x_n) = f(a)$.

2.6.4. ALTERNATIVE DEFINITION OF CONTINUITY OF A FUNCTION AT A POINT

A function $f(x)$ defined on an interval I is said to be continuous at $x = a \in I$ iff $\underset{x \to a}{Limit} f(x)$ exists, is finite and equal to $f(a)$. Otherwise, the function is discontinuous at $x = a$. Thus a function $f(x)$ is said to be continuous at $x = a$ if $f(a + 0) = f(a + 0) = f(a)$. This is also the working rule for testing the continuity of a function at a given point.

2.6.5. POLYNOMIAL FUNCTION

Theorem. A polynomial function is always a continuous function.

Proof. If $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ is a polynomial of degree *n* in *x*, then we are to show that $f(x)$ is continuous for all $x \in R$. For this, let $c \in R$, then

$$\underset{x \to c}{Limit}\ f(x) = \underset{x \to c}{Limit}\ \left[a_0 + a_1 x + a_2 x^2 + ... + a_n x^n\right]$$

$$= \underset{x \to c}{Limit}\ a_0 + \underset{x \to c}{Limit}\ a_1 x + \underset{x \to c}{Limit}\ a_2 x^2 + ... + \underset{x \to c}{Limit}\ a_n x^n$$

$$= a_0 + a_1 c + a_2 c^2 + ... + a_n c^n$$

$$= f(c)$$

Since $\underset{x \to c}{Limit}\ f(x) = f(c)$, therefore $f(x)$ is continuous at $x = c$.

Note. The polynomial function $f(x)$ is always continuous at each points of its domain.

2.6.6. CONTINUITY FROM LEFT AND CONTINUITY FROM RIGHT

A function $f(x)$ is said to be continuous from left at $x = a$ if $\underset{x \to a-0}{Limit}\ f(x)$ exists and equal to $f(a)$ *i.e.*, $\underset{h \to 0}{Limit}\ f(a-h) = f(a)$.

Similarly, $f(x)$ is said to be continuous from right at $x = a$ if $\underset{x \to a+0}{Limit}\ f(x)$ exists and equal to $f(a)$ *i.e.*, $\underset{h \to 0}{Limit}\ f(a+h) = f(a)$

and $f(x)$ is continuous at $x = a$ iff

$$\underset{x \to a-0}{Limit}\ f(x) = \underset{x \to a+0}{Limit}\ f(x) = f(a)$$

$$\underset{h \to 0}{Limit}\ f(a-h) = \underset{h \to 0}{Limit}\ f(a+h) = f(a)$$

2.7. DISCONTINUITY

If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of the function.***e. g.*** The function $f(x) = \dfrac{1}{x-a}$ does not exists at $x = a$ so $f(x)$ is not continuous at $x = a$.

2.7.1. TYPES OF DISCONTINUITY

A. Removable discontinuity

A function $f(x)$ is said to have a removable discontinuity at a point $x = a$ if $\underset{x \to a}{Limit}\ f(x)$ exist but is not equal to $f(a)$ *i.e.*, if

$f(a-0) = f(a+0) \neq f(a)$. The function can be made continuous by defining it in such a way that $\underset{x \to a}{Limit}\ f(x) = f(a)$.

B. Discontinuity of first kind (Ordinary discontinuity)

A function $f(x)$ is said to have a discontinuity of the first kind or ordinary discontinuity at $x = a$ if $f(a + 0)$ and $f(a - 0)$ both exist but not equal. The point $x = a$ is said to be a point of discontinuity from the left or right according as

$f(a-0) \neq f(a) = f(a+0)$ or $f(a-0) = f(a) \neq f(a+0)$.

C. Discontinuity of second kind

A function $f(x)$ is said to have a discontinuity of the second kind at $x = a$ if none of the limits $f(a + 0)$ and $f(a - 0)$ exist. The point $x = a$ is said to be a point of discontinuity of second kind from the left or right according as $f(a-0)$ or $f(a+0)$ does not exist.

D. Mixed discontinuity

A function $f(x)$ is said to have a mixed discontinuity at $x = a$ if it has a discontinuity of second kind on one side of a and on the other side a discontinuity of first kind or may be continuous.

E. Infinite discontinuity

A function $f(x)$ is said to have an infinite discontinuity at $x = a$ if $f(a + 0)$ or $f(a - 0)$ is $+\infty$ or $-\infty$ *i.e.*, if $f(x)$ is discontinuous at $x = a$ and $f(x)$ is unbounded in every neighbourhood of $x = a$.

2.7.2 JUMP OF A FUNCTION AT A POINT

If both $f(a + 0)$ and $f(a - 0)$ exists, then the jump in the function at $x = a$ is defined as the non-negative difference $f(a-0) \sim f(a+0)$. A function having a finite number of jumps in a given interval is called piecewise continuous.

Illustrative Examples

Example 1. Test the continuity of $f(x)$ at $x = 1$ when

$$f(x) = \begin{cases} x^2 + 2 & if \;\; x > 1 \\ 2x + 1 & if \;\; x = 1 \\ 3 & if \;\; x < 1 \end{cases}$$

Solution. Here $f(1) = 2.1 + 1 = 3$

$$f(1+0) = \underset{h\to 0}{Limit}\; f(1+h) = \underset{h\to 0}{Limit}\; (1+h)^2 + 2$$

$$= \underset{h\to 0}{Limit}\; 1 + h^2 + 2h + 2 = 3 \text{ as } 1 + h > 1.$$

$$f(1-0) = \underset{h\to 0}{Limit}\; f(1-h) = \underset{h\to 0}{Limit}\; (1-h)^2 + 2$$

$$= \underset{h\to 0}{Limit}\; 1 + h^2 - 2h + 2 = 3 \text{ as } 1 - h < 1.$$

So $f(1) = f(1+0) = f(1-0)$ .Hence $f(x)$ is continuous at $x = 1$.

Example 2. Discuss the continuity of the function $f(x) = \dfrac{1}{1 - e^{-1/x}}$ when $x \neq 0$ and $f(0) = 0$ for all values of x.

Solution. Test the continuity at $x = 0$

$$f(0+0) = \underset{h\to 0}{Limit}\; f(0+h) = \underset{h\to 0}{Limit}\; f(h)$$

$$= \underset{h\to 0}{Limit}\; \frac{1}{1 - e^{-1/h}} = 1$$

$$f(0-0) = \underset{h\to 0}{Limit}\; f(0-h) = \underset{h\to 0}{Limit}\; f(-h)$$

$$= \underset{h\to 0}{Limit}\; \frac{1}{1 - e^{1/h}} = 0$$

Thus we have $f(0 + 0) \neq f(0 - 0) = f(0)$. So $f(x)$ is not continuous at $x = 0$ and it is a discontinuity of first kind *i.e.*, $f(x)$ is continuous on the left and has a discontinuity of first kind on right at $x = 0$.

Now test the continuity at $x = a \neq 0$

$$f(a) = \frac{1}{1 - e^{-1/a}}$$

$$f(a+0) = \underset{h\to 0}{Limit}\; f(a+h) = \underset{h\to 0}{Limit}\; \frac{1}{1 - e^{-1/(a+h)}}$$

$$= \frac{1}{1 - e^{-1/a}} = f(a)$$

$$f(a-0) = \underset{h\to 0}{Limit}\; f(a-h) = \underset{h\to 0}{Limit}\; \frac{1}{1 - e^{-1/(a-h)}}$$

$$= \frac{1}{1 - e^{-1/a}} = f(a)$$

Thus we have $f(a + 0) = f(a - 0) = f(a)$. Hence $f(x)$ is continuous at every point except $x = 0$.

Example 3. Test the continuity of the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. Here

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ h \sin \frac{1}{h} = 0$$

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ (-h)\sin\left(-\frac{1}{h}\right) = \underset{h\to 0}{Limit}\ h \sin \frac{1}{h} = 0$$

Thus we have $f(0 + 0) = f(0 - 0) = f(0)$. Hence $f(x)$ is continuous at $x = 0$.

Note. 1. If we check the continuity at $x = c \neq 0$ of the above function, then we see that

$$\underset{x\to c}{Limit}\ f(x) = \underset{x\to c}{Limit}\ x \sin \frac{1}{x}$$

$$= c \sin \frac{1}{c} = f(c)$$

So $f(x)$ is continuous at $x = c$. Thus f (x) is continuous for all $x \in R$ *i.e.*, $f(x)$ is continuous on the whole real line.

Note 2. If we take $f(0) = 2$, in the above function, then $f(0 + 0) = f(0 - 0) \neq f(0)$. The function becomes discontinuities at $x = 0$ and has a removable discontinuity at $x = 0$.

Example 4. If a function $f(x)$ is defined by $f(x) = x - [x]$, where x is a positive variable and $[x]$ denotes the integral part of x. Show that it is discontinuous for integral values of x and continuous for all others. Draw the graph.

Solution. From the definition of the function f (x) we have

$$f(x) = \begin{cases} x-(n-1) & for\ n-1 < x < n \\ 0 & for\ x = n \\ x-n & for\ n < x < n+1 \end{cases} \qquad \text{where } n \text{ is an integer}$$

First we test the continuity of $f(x)$ at $x = n$. We have $f(n) = 0$.

$$f(n+0) = \underset{h\to 0}{Limit}\ f(n+h) = \underset{h\to 0}{Limit}\ (n+h) - n$$

$$= \underset{h \to 0}{Limit}\ h = 0 \quad [\text{as } n < n + h < n+1]$$

$$f\,(n-0) = \underset{h \to 0}{Limit}\ f\,(n-h) = \underset{h \to 0}{Limit}\ (n-h) - \overline{n-1}$$

$$= \underset{h \to 0}{Limit}\ 1 - h = 1 \quad [\text{ as } n\text{-}1 < n\text{-}h < n]$$

Since $f\,(n-0) \neq f\,(n+0)$, so the function $f\,(x)$ is discontinuous at $x = n$. Thus $f\,(x)$ is discontinuous for all integral values of x. it is obviously continuous for all other values of x. Since x is a positive variable putting = 1, 2, 3, 4, 5, ….,we see that graph of the function consists of the following straight lines.

$$y = f(x) = \begin{cases} x & when\ 0 < x < 1 \\ 0 & when\ x = 1 \\ x-1 & when\ 1 < x < 2 \\ 0 & when\ x = 2 \\ x-2 & when\ 2 < x < 3 \\ 0 & when\ x = 3 \\ x-3 & when\ 3 < x < 4 \\ 0 & when\ x = 5 \end{cases}$$

and so on.

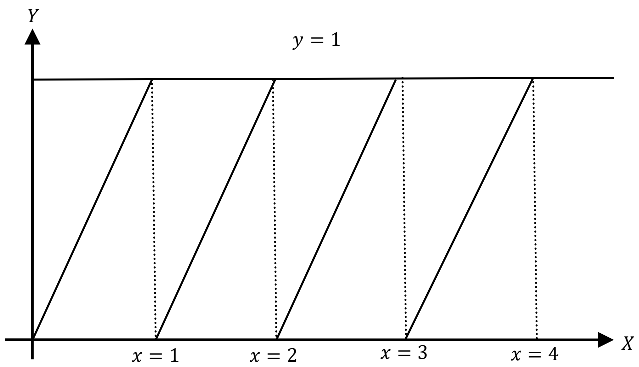


Fig. 2.7.2.1

It is clear from the graph that

(1) The function is discontinuous for all integral values of x but continuous for other values of x.
(2) The function is bounded between 0 and 1 in every domain which includes an integer.
(3) The lower bound 0 is attained but upper bound 1 is not attained since $f\,(x) \neq 1$ for any value of x.

Example 5. Show that the function $f(x) = [x] + [-x]$ has a removable discontinuity for integral values of x.

Solution. We see that $f(x) = 0$ when x is an integer and $f(x) = -1$ when x is not an integer. Hence if n is an integer then

$$f(n-0) = f(n+0) = -1 \text{ and } f(n) = 0.$$

So the function $f(x)$ has a removable discontinuity at $x = n$, where n is an integer.

Example 6. Show that the function $f(x)$ defined on R by

$$f(x) = \begin{cases} 1, & when\, x \text{ is rational} \\ -1 & when\, x \text{ is irrational} \end{cases}$$

is discontinuous at every point of R.

Solution. Let us suppose first that x is rational. Then $f(x) = 1$. For each positive integer n, let x_n be an irrational number such that $|x_n - x| < 1/n$. Then the sequence $<x_n>$ converges to x. Now by the definition $f(x_n) = 1$ for all n. So $\underset{n\to\infty}{Limit}\, f(x_n) = -1 \neq f(x)$ Hence $f(x)$ is discontinuous at each rational point. Now suppose that x is an irrational number then $f(x) = -1$. For each positive integer n, let x_n be an rational number such that $|x_n - x| < 1/n$, then the sequence $<x_n>$ converges to x. Now by the definition $f(x_n) = -1$ for all n, so that $\underset{n\to\infty}{Limit}\, f(x_n) = 1 \neq f(x)$. Hence $f(x)$ is discontinuous at each irrational point. Therefore $f(x)$ is discontinuous at every point of R.

Example 7. Prove that the function $f(x) = \dfrac{|x|}{x}$ for $x \neq 0$ and $f(0) = 0$, is continuous at all the points except $x = 0$.

Solution. If $x > 0$ then, $f(x) = \dfrac{x}{x} = 1$ and if $x < 0$ then, $f(x) = \dfrac{-x}{x} = -1$.

Therefore the given function can define as:

$$f(x) = \begin{cases} -1, & if\ x < 0 \\ 0, & if\ x = 0 \\ 1, & if\ x > 0 \end{cases}$$

If $x < 0$, then $f(x) = -1$ *i.e.*, $f(x)$ is a constant function and a constant function is always continuous at each point of its domain. This implies that $f(x)$ is continuous for all $x < 0$. Similarly, we can show that $f(x)$ is continuous for all $x > 0$. Now we see the continuity at $x = 0$.

$$f(0+0) = \underset{h\to 0}{Limit}\, f(0+h) = \underset{h\to 0}{Limit}\, f(h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\, 1 = 1$$

$$f(0-0) = \underset{h\to 0}{Limit}\, f(0-h) = \underset{h\to 0}{Limit}\, f(-h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\, -1 = -1$$

Here $f(0+0) \neq f(0-0) \neq f(0)$.
Hence $f(x)$ is not continuous at $x = 0$.

Example 8. Show that the function ϕ defined as

$$\phi(x) = \begin{cases} 0 & for\, x = 0 \\ \frac{1}{2} - x & for\, 0 < x < \frac{1}{2} \\ \frac{1}{2} & for\, x = \frac{1}{2} \\ \frac{3}{2} - x & for\, \frac{1}{2} < x < 1 \\ 1 & for\, x = 1 \end{cases}$$

has three points of discontinuity which you are required to find. Also draw the graph of the function.

Solution. Here the domain of the function $\phi(x)$ is a closed interval [0, 1]. When $0 < x < \frac{1}{2}$, $\phi(x) = \frac{1}{2} - x$, which is a polynomial of degree one in x. we know that a polynomial function is continuous at each point of its domain and so $\phi(x)$ is continuous at each point of the open interval $\left(0, \frac{1}{2}\right)$. Again when $\frac{1}{2} < x < 1$, $\phi(x) = \frac{3}{2} - x$, which is also a polynomial in x and so $\phi(x)$ is continuous at each point of the open interval $\left(\frac{1}{2}, 1\right)$. Now we will test the continuity of the function $\phi(x)$ at the points $x = 0, \frac{1}{2}, 1$,

(a) At $x = 0$ We have $\phi(0) = 0$ and

$$\phi(0+0) = \underset{h\to 0}{Limit}\, \phi(0+h) = \underset{h\to 0}{Limit}\, \phi(h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\, \frac{1}{2} - h = \frac{1}{2}$$

Since $\phi(0) \neq \phi(0+0)$, so the function $\phi(x)$ is discontinuous at $x = 0$ and the discontinuity is ordinary.

(b) At $x = \frac{1}{2}$ We have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$ and

$$\phi\left(\frac{1}{2}+0\right) = \underset{h\to 0}{Limit}\, \phi\left(\frac{1}{2}+h\right) = \underset{h\to 0}{Limit}\, \frac{3}{2} - \left(\frac{1}{2}+h\right),\ h > 0$$

$$= \underset{h\to 0}{Limit}\, 1 - h = 1 \neq \phi\left(\frac{1}{2}\right)$$

$$\phi\left(\frac{1}{2}-0\right)=\underset{h\to 0}{Limit}\,\phi\left(\frac{1}{2}-h\right)=\underset{h\to 0}{Limit}\ \frac{1}{2}-\left(\frac{1}{2}-h\right),\ h>0$$

$$=\underset{h\to 0}{Limit}\ h=0\neq\phi\left(\frac{1}{2}\right)$$

Since $\phi\left(\frac{1}{2}+0\right)\neq\phi\left(\frac{1}{2}-0\right)\neq\phi\left(\frac{1}{2}\right)$, so the function $\phi(x)$ is discontinuous at $x=\frac{1}{2}$ and $\phi(x)$ is discontinuous from left as well as from right.

(c) At $\boldsymbol{x}=1$. We have $\phi(1)=1$ and

$$\phi(1-0)=\underset{h\to 0}{Limit}\,\phi(1-h)=\underset{h\to 0}{Limit}\ \frac{3}{2}-(1-h),\ h>0$$

$$=\underset{h\to 0}{Limit}\ \frac{1}{2}+h=\frac{1}{2}\neq\phi(1)$$

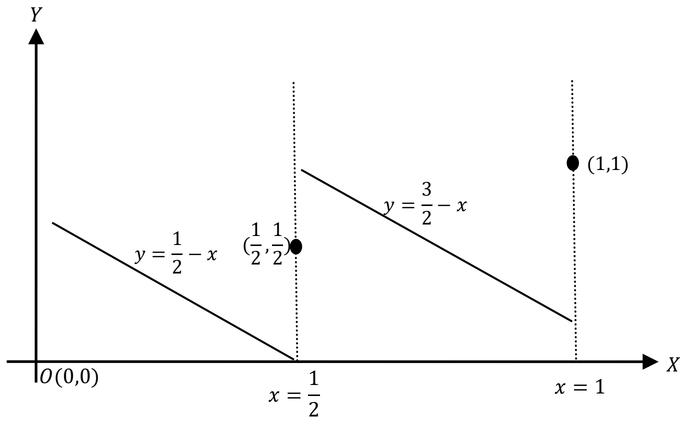


Fig. 2.7.2.1

Since $\phi(1-0)\neq\phi(1)$, so the function $\phi(x)$ is discontinuous at $x=1$ and the discontinuity is ordinary. Hence the function $\phi(x)$ has three points of discontinuity at $x=0,\ \frac{1}{2},\ 1$. The graph of the function consist of the point (0, 0); the line $y=\frac{1}{2}-x$ for $\left(0,\frac{1}{2}\right)$, the point $\left(\frac{1}{2},\frac{1}{2}\right)$; and the line segment $y=\frac{3}{2}-x$ for $\left(\frac{1}{2},1\right)$ and the point(1, 1).

Example 9. Discuss the discontinuity of the function defined by

$$f(x) = \begin{cases} x^2, & \text{if } x < -2 \\ 4, & \text{if } -2 \le x \le 2 \\ x^2, & \text{if } x > 2 \end{cases}$$

Solution. Here we shall check the continuity for $f(x)$ at $x = -2$ and $x = 2$.

<u>At $x = -2$:</u>

We have $f(-2) = 4$.

$$f(-2+0) = \underset{h\to 0}{Limit}\ f(-2+h) = \underset{h\to 0}{Limit}\ 4,\ h > 0$$
$$= 4$$
$$f(-2-0) = \underset{h\to 0}{Limit}\ f(-2-h) = \underset{h\to 0}{Limit}\ (-2-h)^2,\ h > 0$$
$$= 4$$

Hence $f(x)$ is continuous at $x = -2$.

<u>At $x = 2$:</u>

We have $f(2) = 4$.

$$f(2+0) = \underset{h\to 0}{Limit}\ f(2+h) = \underset{h\to 0}{Limit}\ (2+h)^2,\ h > 0$$
$$= 4$$
$$f(2-0) = \underset{h\to 0}{Limit}\ f(2-h) = \underset{h\to 0}{Limit}\ 4,\ h > 0$$
$$= 4$$

Hence $f(x)$ is continuous at $x = 2$.

Example 9. Let $y = E(x)$ denotes the integral parts of x. Prove that the function is discontinuous where x has an integral value. Also draw the graph.

Solution. From the definition of $E(x)$ we have

$$E(x) = \begin{cases} n-1, & \text{for } n-1 < x < n \\ n, & \text{for } n < x < n+1 \\ n+1, & \text{for } n+1 < x < n+2 \end{cases}$$

and so on where n is an integer.We consider $x = n$, $E(n) = n,\quad E(n-0) = n-1,\quad E(n+0) = n$

Since $E(n+0) \ne E(n-0)$. Hence the function $E(x)$ is discontinuous at $x = n$, where x has an integral value.

Evidently it is continuous for all other values of x. To draw the graph, we put $n = \ldots.-4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots.$

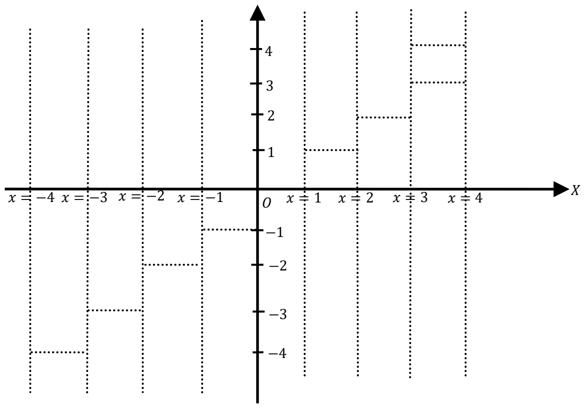


Fig. 2.7.2.2

$$y = E(x) = \begin{cases} -4 & when\ -4 \le x < -3 \\ -3 & when\ -3 \le x < -2 \\ -2 & when\ -2 \le x < -1 \\ -1 & when\ -1 \le x < 0 \\ 0 & when\ 0 \le x < 1 \\ 1 & when\ 1 \le x < 2 \\ 2 & when\ 2 \le x < 3 \\ 3 & when\ 3 \le x < 4 \\ 4 & when\ 4 \le x < 5 \text{ and so on} \end{cases}$$

Example

09. Discuss the kind of discontinuity, if any, of the function

$$f(x) = \begin{cases} \dfrac{x - |x|}{x}, & if\ x \ne 0 \\ 2, & if\ x = 0 \end{cases}$$

Solution. The function is continuous at all points except possible at the origin. Now at $x = 0$,

$$f(0+0) = \underset{h\to 0}{Limit}\ f(0+h) = \underset{h\to 0}{Limit}\ f(h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ \frac{h - |h|}{h} = 0$$

$$f(0-0) = \underset{h\to 0}{Limit}\ f(0-h) = \underset{h\to 0}{Limit}\ f(-h),\ h > 0$$

$$= \underset{h\to 0}{Limit}\ \frac{-h - |-h|}{h} = 2$$

Also $f(0) = 2$. So $f(0-0) = f(0) \ne f(0+0)$. Hence the given function $f(x)$ is discontinuous at $x = 0$ and this is discontinuity of first kind.

2.8. INTERMEDIATE VALUE THEOREM

Statement. Let $f\ (x)$ be a function, continuous on the closed and bounded interval $[a, b]$. If k be any real number between $f\ (a)$ and $f\ (b)$ *i.e.*, $(f\ (a)<k<f\ (b))$, then $\exists$ a real number c between a and b *i.e.*, $(a<c<b)$ such that $f(c) = k$.

Proof. Let us suppose that
$f\ (a)<k<f\ (b)$..................(1)
Define a function $g\ (x)$ such that
$g\ (x)= f\ (x)-k; \quad \forall\ x \in[a,b]$ ....(2)
Now since $f\ (x)$ is continuous on $[a, b]$ and k is constant, so
$g\ (x)$ is also continuous on $[a, b]$.........................(3)

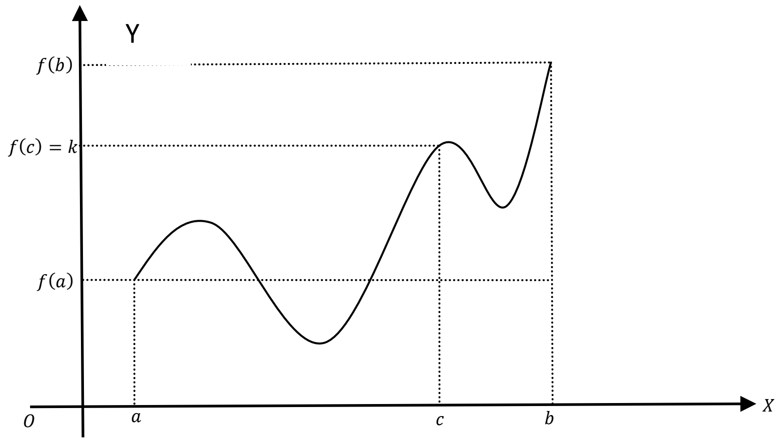


Fig 2.8.1

Now from (1) and (2), we have

$$\begin{array}{l} g\ (a)= f\ (a)-k<0 \\ g\ (b)= f\ (b)-k>0 \end{array} \Rightarrow g\ (a).g\ (b)< 0 \text{ ....(4)}$$

Now from (3) and (4) $\exists$ a real number c between a and b *i.e.*, $(a<c<b)$ such that $g\ (c) = 0$.

$$\Rightarrow g\ (c)= f\ (c)-k=0$$

$$\Rightarrow f\ (c)=k$$

Hence there exists a point $c\in(a,b)$ such that $f\ (c)=k$.

Note. The converse of the above theorem is not necessarily true. For example, Let $f\ (x)$ be a function defined by

$$f\ (x) = \begin{cases} \sin\dfrac{1}{x} & if\ \ x\neq 0 \\ 0 & if\ \ x=0 \end{cases}$$

Then, in the interval $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$ this function takes all values between $f\left(-\frac{2}{\pi}\right)$ and $f\left(\frac{2}{\pi}\right)$ i.e., between -1 and +1. But this function is not continuous in $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$ as it is discontinuous at $x = 0$.

2.9. UNIFORM CONTINUITY

Definition. A real valued function $f(x)$ defined on an interval I is said to be uniformly
continuous in I, if for each $\varepsilon > 0$ $\exists$ a positive number $\delta > 0$ (depending upon ε but independent of $x \in I$) such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta \text{ where } x_1, x_2 \in I.$$

Theorem. If a function $f(x)$ is uniformly continuous on an interval $[a, b] = I$, then it is continuous on I.

Proof. Let us suppose that $f(x)$ is uniformly continuous on I then for given $\varepsilon > 0$ $\exists$ a positive number $\delta > 0$ (depending upon ε but independent of $x \in I$) such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta \text{ where } x_1, x_2 \in I.$$

Let $x_1 \in I, x_2 = x \in I$ then we have

$$|f(x) - f(x_1)| < \varepsilon \text{ whenever } |x - x_1| < \delta$$

$$\Rightarrow f(x) \text{ is continuous at } x_1 \in I.$$

Since x_1 is arbitrary, consequently $f(x)$ is continuous on I.

Note. The converse of the above theorem is not true, can be seen in the given example.

Example 1. Let $f : R \to R$ given by $f(x) = x^2$ $\forall x \in R$ which is continuous $\forall x \in R$. Now we will show that $f(x)$ is not uniformly continuous.

Solution. Let $\varepsilon > 0$ be given. The function $f(x)$ will be uniformly continuous if we find $\delta > 0$ such that

$$x_1, x_2 \in R, \ |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon \qquad \text{................................(1)}$$

The function $f(x)$ will not be uniformly continuous on R if we find some $\varepsilon > 0$ for which no δ works. So here we shall show that for some given $\varepsilon > 0$ there exists no $\delta > 0$ which satisfy (1).

By the axioms of Archimedes for any $\delta > 0$ there exists a positive integer *n* such that

$n\delta^2 > \varepsilon$ ....(2)

If we take $x_1 = n\delta$ and $x_2 = n\delta + \frac{\delta}{2}$ then

$$|x_1 - x_2| = \frac{\delta}{2} < \delta$$

But $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2|$

$$= \frac{\delta}{2}\left(2n\delta + \frac{\delta}{2}\right) = n\delta^2 + \frac{\delta^2}{4} > \varepsilon \text{ [by (2)]}$$

Hence for those two points $x_1, x_2 \in R$ we always have

$$|f(x_1) - f(x_2)| > \varepsilon \text{ whenever } \delta > 0.$$

This contradicts (1). Hence *f* (*x*) is not uniformly continuous on *R*.

Example 2. Show that the function $f(x) = x^2 + 3x \ \forall x \in [-1,1]$ is uniformly continuous in $[-1,1]$.

Solution. Let $\varepsilon > 0$ be given. Let $x_1, x_2 \in [-1,1]$ then

$$|f(x_1) - f(x_2)| = |x_1^2 + 3x_1 - \overline{x_2^2 + 3x_2}|$$

$$= |x_1^2 + 3x_1 - x_2^2 - 3x_2|$$

$$= |x_1^2 - x_2^2 + 3(x_1 - x_2)|$$

$$= |(x_1 - x_2) + (x_1 + x_2 + 3)|$$

$$\leq |x_1 - x_2||x_1 + x_2 + 3|$$

$$< 5|x_1 - x_2|$$

[Since $x_1, x_2 \in [-1,1] \Rightarrow |x_1| < 1, |x_2| < 1$]

$$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon \text{ for } |x_1 - x_2| < \frac{\varepsilon}{5}.$$

Thus for any $\varepsilon > 0 \ \exists \ \delta = \frac{\varepsilon}{5} > 0$ such that $\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta \ \forall x \in [-1,1]$. Hence *f* (*x*) is uniformly continuous in $[-1,1]$.

Example 3. Show that the function $f(x) = x^3 \ \forall x \in [-2,2]$ is uniformly continuous in $[-2,2]$.

Solution. Let $\varepsilon > 0$ be given. Let $x_1, x_2 \in [-2,2]$ then

$$\begin{aligned}|f(x_1)-f(x_2)| &= |x_1^3 - x_2^3| \\ &= |(x_1-x_2)+(x_1^2+x_2^2+x_1x_2)| \\ &\le |x_1-x_2||x_1^2+x_2^2+x_1x_2| \\ &\le |x_1-x_2|\left[|x_1^2|+|x_2^2|+|x_1x_2|\right] \\ &< 12|x_1-x_2|\end{aligned}$$

[Since $x_1, x_2 \in [-2,2] \Rightarrow |x_1| \le 2, |x_2| \le 2$]

$$\Rightarrow |f(x_1)-f(x_2)| < \varepsilon \text{ for } |x_1-x_2| < \frac{\varepsilon}{12}.$$

Thus for any $\varepsilon > 0 \ \exists \ \delta = \dfrac{\varepsilon}{12} > 0$ such that

$\Rightarrow |f(x_1)-f(x_2)| < \varepsilon$ whenever $|x_1-x_2| < \delta \ \forall \ x \in [-2,2]$.

Hence $f(x)$ is uniformly continuous in $[-2,2]$.

2.10. SUMMARY

In this unit following definition of limit , continuity, Type of discontinuity and regarding uniform continuity. These concepts will be helpful for learner to understand the concept of calculus.

1. A number l is said to be the limit of function $f(x)$ at x= a if for arbitrary $\varepsilon > 0, \exists \delta > 0$ (positive real number) such that, whenever

$$0 < |x-a| < \delta \text{ we have } 0 < |x-a| < \delta$$

2. A function $f(x)$ is said to be continuous at $x = a$ is given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon.$$

3. A function $f(x)$ is continuous at $x = a$ iff

$$\underset{x\to a-0}{Limit} f(x) = \underset{x\to a+0}{Limit} f(x) = f(a)$$

$$\underset{h\to 0}{Limit} f(a-h) = \underset{h\to 0}{Limit} f(a+h) = f(a)$$

4. The polynomial function $f(x)$ is always continuous at each points of its domain.

5. The removable discontinuity at a point $x = a$exists if

$$f(a-0) = f(a+0) \ne f(a)$$

6. An Ordinary discontinuity at a point $x = a$exists if

$$f(a-0) \ne f(a) = f(a+0) \text{ or } f(a-0) = f(a) \ne f(a+0).$$

7. Discontinuity of the second kind at $x = a$, exists if $f(a + 0)$ or $f(a - 0)$ or both does not exist.

8. A real valued function $f(x)$ defined on an interval I is said to be uniformly continuous in I if for each $\varepsilon > 0 \; \exists$ a positive number $\delta > 0$ (depending upon ε but independent of $x \in I$) such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta \text{ where } x_1, x_2 \in I.$$

2.11. GLOSSARY

i. Sets-Well defined collection of objects

ii. Continuous-sketch its curve on a graph without lifting your pen even once.

iii. Discontinuity-lack of continuity

CHECK YOUR PROGRESS

1. A polynomial function is always ………………...

2. $\underset{x \to 0}{Limit} \dfrac{a^x - 1}{x} =$ ....

3. A function is said to have ....if $f(a + 0) = f(a - 0) \neq f(a)$.

4. The value of $f(a + 0) \sim f(a - 0)$ is known as ....

5. Every uniformly continuous function is ....

6. $\underset{x \to 0}{Limit} \dfrac{\sin 3x}{x} = 1$. True/False

7. $\underset{x \to 0}{Limit} \dfrac{|x-3|}{x-3} = 1$. True/False

8 .Every continuous function in closed interval is bounded. True\False

9. The function must be defined at the point of continuity. True\False

10. If $f(x)$ is uniformly continuous on closed interval I, then it is continuous on I. True/False

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2.14. TERMINAL QUESTIONS

1. Test the continuity at x = 0 if

$$f(x) = \begin{cases} x\log x, & x > 0 \\ 0, & x = 0 \end{cases}$$

2. Show that the function $f(x) = \dfrac{x^2-1}{x-1}$ is continuous for all values of x except x = 1.

3. Discuss the continuity of the following function at x = 0:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{\sin 5x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

4. Discuss the continuity of the following function

$$f(x) = \begin{cases} x+4, & x \leq 1 \\ x^2+3x+1, & 1 < x \leq 2 \\ 2x^2+x+1, & 2 < x \leq 3 \\ 8x-2, & x \geq 3 \end{cases}$$

5. Determine the constants a and b so that the following functions are continuous everywhere.

i. $$f(x) = \begin{cases} x+4, & x \leq 1 \\ x^2+3x+1, & 1 < x \leq 2 \\ 2x^2+x+1, & 2 < x \leq 3 \\ 8x-2, & x \geq 3 \end{cases}$$

ii. $f(x) = \begin{cases} 1, & x \le 3 \\ ax + b, & 3 < x < 5 \\ 7, & x \ge 5 \end{cases}$

6. The value of the $\underset{x \to 0}{Limit} \dfrac{\sin x}{x}$ is

(a) 1 (b) 0 (c) ∞

(d) does not exist.

7. The value of k for which $f(x) = \begin{cases} \dfrac{\sin 5x}{3kx}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ is continuous at x = 0

(a) 1/3 (b) 3/5 (c) 0 (d) 5/3

8. $\underset{x \to 0}{Limit} \sin \dfrac{1}{x}$ is

(a) 1(b) 0 (c) ∞ (d) does not exists

9.A function f (x) is said to be continuous at x = a if

(a) $\underset{x \to a}{Limit} f(x)$ exists (b) $f(a)$ exists

(c) $\underset{x \to a}{Limit} f(x) = f(a)$ (d) None of these

10. The function $f(x) = |x|$ is

(a) Continuous for all x(b)Discontinuous at x = 0 only

(c) Continuous at x = 0 only (d) None of these

2.15. ANSWERS

CHECK YOUR PROGRESS

SCQ1: Continuous

SCQ2: $\log_e a$

SCQ3: Removable discontinuity

SCQ4: Jump

SCQ5: Continuous

SCQ6:F

SCQ7:F

SCQ8:T

SCQ9:T

SCQ10:T

TERMINAL QUESTIONS

(TQ-1) Continuous at $x = 0$
(TQ-3) Removable Discontinuity at $x = 0$
(TQ-4) Continuous at $x = 1, 2, 3$
(TQ-5) **(i)** $a = 2$, $b = 1$ **(ii)** $a = 3$, $b = -8$
(TQ-6)(a)
(TQ-7)(d)
(TQ-8)(c)
(TQ-9)(c)
(TQ-10)(a)

UNIT–3:-DIFFERENTIABILITY

CONTENTS:-

3.1 INTRODUCTION

We observe several phenomena where changes are taking place. The motion of the planet around the Sun, the speed of a car and temperature at a fixed point of a place are some examples. Some question arises here:

i. The speed at which it is move at any time.
ii. Instantaneous direction.
iii. The position of planet relative to the Sun after some time.

The answer of such questions is responsible for the origin and development of the derivative**. Sir Issac Newton** and **G.W. Leibniz** get the credit of development of differentiability.

A function is differentiable at a point when there's a defined derivative at that point. The meaning is that the slope of the tangent line of the points from the left is approaching the same value as the slope of the tangent of the points from the right.

Sir Issac Newton
(25 December 1642 –20 March 1726/27)

Ref:https://www.neh.gov/humanities/2011/januaryfebruary/feature/newton-the-last-magician

G.W. Leibniz
(1 July 1646– 14 November 1716)

Ref: https://iep.utm.edu/leib-met/

Fig 3.1

3.2 OBJECTIVES

After completion of this unit the learner will be able to

- **i.** The meaning of term 'Derivative at a point'.
- **ii.** Differentiability in an interval.
- **iii.** Algebra of derivatives.
- **iv.** Darboux Theorem.

3.3. DERIVATIVE AT A POINT

A function $f:(a,b) \to R$ is said to be differentiable or derivable at $c \in (a,b)$ if and only if

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\ i.e.\ \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists (finitely).

This limit, known as the derivative of f(x) at x=c, is denoted by $f'(c)$ or $Df(c)$ or $\left[\frac{d}{dx}f(x)\right]_{x=c}$. The process of evaluating $f'(c)$ is called differentiation.

3.4. PROGRESSIVE AND REGRESSIVE DERIVATIVES:-

The progressive derivative (or the Right hand derivative) of $f(x)$ at $x = c$ is denoted by $Rf'(c)$ or $f'(c+0)$ and is defined as

$$Rf'(c) = f'(c+0) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\ ,h>0$$

The regressive derivative (or Left hand derivative) of $f(x)$ at $x = c$ is denoted by $Lf'(c)$ or $f'(c-0)$and is defined as

$$Lf'(c) = f'(c-0) = \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}\ ,h>0$$

Note:-$f'(c)$ exists if and only if

(i) $Lf'(c)$ and $Rf'(c)$ both exists and

(ii) $Lf'(c) = Rf'(c)$

Examples 1. The function f defined by

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ x, & x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$ and not differentiable at any point because

$$f'(h) = \frac{f(h)-f(0)}{h-0} = \frac{f(h)}{h}.$$

So that $f'(h) = 0$ if h is rational and $f'(h) = 1$ if h is irrational. Therefore $\lim_{h\to 0} f'(x)$ does not exists.

Examples 2. The function $f(x)$ defined by $f(x) = |x|$ is continuous for all $x \in R$ and differentiable for all $x \in R$ except $x = 0$. For differentiability at $x = 0$ we see that

$$f'(h) = \frac{f(h)-f(0)}{h-0} = \frac{|h|}{h}$$

so $f'(h) = 1$ if $h > 0$ and $f'(h) = -1$ if $h < 0$. Therefore $Rf'(0) = 1$ and $Lf'(0) = -1$ so $f'(0)$ does not exists, for $x > 0$, $f'(x) = 1$ and for $x < 0$, $f'(x) = 1$.

Note:- If $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$ is a polynomial in x of degree n, then $f(x)$ is differentiable at every point $x \in R$.

3.5. DIFFERENTIABILITY IN AN INTERVAL:-

(i) **Open interval (a,b) :-** A function $f: (a,b) \to R$ is said to be differentiable in (a,b) iff it is differentiable at every point of (a,b).

(ii) **Closed interval [a,b] :-** A function $f:[a,b] \to R$ is said to be differentiable in [a,b] iff $Rf'(a), Lf'(b)$ exists and f is differentiable in (a,b).

Alternative definition of differentiability:-

Let $f(x)$ be a function defined on a interval I and let c be an interior point of I.Then by the definition of derivative, assuming that $f'(c)$ exists we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

i.e. $f'(c)$ exists if for given $\in > 0, \exists \delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \in \text{ whenever } 0 < |x - c| < \delta$$

or $x \in (c - \delta, c + \delta) \Rightarrow f'(c) - \in < \frac{f(x)-f(c)}{x-c} < f'(c) + \in$

Geometrical Meaning of Derivative:-

Let we take two neighbouring point $P[a, f(a)]$ and $Q[a + h, f(a + h)]$ on the curve $y = f(x)$. Let the chord PQ and the tangent at P meet the x-axis at L and T respectively. Let $\angle QLX = \propto$ and $\angle PTX = \varphi$. Draw PN and $QM \perp$ to x-axis and $PH \perp QM$.

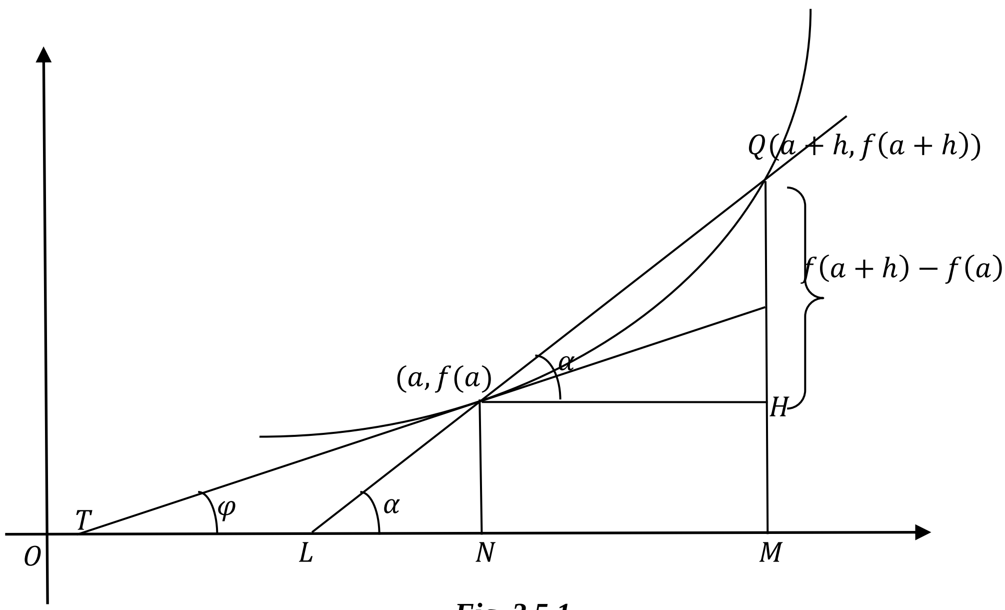

Fig. 3.5.1

Then $PH = NM = OM - ON = a + h - a = h$

$$QH = MQ - MH = MQ - PN = f(a + h) - f(a)$$

$$\tan \alpha = \frac{QH}{PH} = \frac{f(a+h)-f(a)}{h} \quad \text{.......................(i)}$$

As $h \to 0$, the point Q moving along the curve approaches to P and chord $PQ \to$ Tangent at P, i.e. TP and $\to \varphi$. Taking $h \to 0$ in (i) we get

$$\tan\varphi = f'(a)$$

Hence that derivative of $f(x)$ at a point $'a'$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $x = a$ makes with x-axis.

Meaning of the sign of derivatives :-

Let $f'(c) > 0$ where c is an interior point of the domain of the function ; then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) > 0 .$$

If $\in > 0$ be any number $< f'(c), \exists\delta > 0$ s.t.

$$|x-c| < \delta \Rightarrow \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \in \text{.....................(i)}$$

i.e. $x \in (c-\delta, c+\delta), x \neq c \Rightarrow \frac{f(x)-f(c)}{x-c} \in (f'(c) - \in, f'(c) + \in)$ Since $\in$ is chosen smaller than $f'(c)$; we conclude from (i) $\frac{f(x)-f(c)}{x-c} > 0$ when $\in (c-\delta, c+\delta)$, $x \neq c$.Then we have $f(x) - f(c) > 0$ when $c < x < c+\delta$ and $f(x) - f(c) < 0$ when $c - \delta < x < c$.Thus we conclude that if $f'(c) > 0$, $\exists$ a neighbourhood $[c-\delta, c+\delta]$ of c such that $f(x) > f(c), \forall x \in (c, c+\delta)$ and $f(x) < f(c),, \forall x \in (c-\delta, c)$. If $f'(c) < 0$, it is similarly shown that $\exists$ a neighbourhood $[c-\delta, c+\delta]$ of c such that $f(x) > f(c) \forall x \in (c-\delta, c)$ and $f(x) < f(c) \forall x \in (c, c+\delta)$.

A Necessary condition for the existence of a Finite derivative:-

Theorem 1. Continuity is a necessary but not a sufficient condition for the existence of a finite derivative at a point

Proof :Ist Part-

Let $f(x)$ have a finite derivative at $x = c$.

Then $Rf'(c) = Lf'(c)$..............................(i)

To prove that $f(x)$ is continuous at $x = c$, we have from (i)

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\, and$$

$$f'(c) = \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

Therefore $f'(c) = \frac{f(c+h)-f(c)}{h} + \in$ and $f'(c) = \frac{f(c-h)-f(c)}{-h} + \in '$

where $\in, \in ' \to 0$ as $h \to 0$. i.e. $hf'(c) = f(c+h) - f(c) + h\in$ and $-hf'(c) = f(c-h) - f(c) - h\in'$.Taking $h \to 0$ we get $0 = \lim_{h\to 0} f(c+h) - f(c)$, $0 = \lim_{h\to 0} f(c-h) - f(c)$. i.e. $\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c-h) = f(c)$.

Hence $f(x)$ is continuous at $x = c$.

2nd Part –

The converse of the first part is not true i.e. if $f(x)$ is continuous at $x = c$, then $f(x)$ may or may not be differentiable at $x = c$. We put an example in favour of this statement. Consider $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, Then $f(0+h) = h\sin\left(\frac{1}{h}\right), f(0-h) = -h\sin\left(-\frac{1}{h}\right) = h\sin\left(\frac{1}{h}\right)$ so that $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0) = 0$. Thus $f(x)$ is continuous at $x = 0$. But $Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$ which does not exist. Hence $f(x)$ is not differentiable at $x = 0$.

3.6. ALGEBRA OF DERIVATIVES

Theorem 1. If a function $f(x)$ is differentiable at a point x_0 and c is any real number, then the function cf(x) is also differentiable at x_0 and $(cf)'x_0 = cf'(x_0)$.

Proof :- Since the function $f(x)$ is differentiable at a point x_0 then by the definition

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

Now, $$(cf)'x_0 = \lim_{x\to x_0}\frac{(cf)(x)-(cf)(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0}\frac{cf(x)-cf(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0}\left\{\frac{c[f(x)-f(x_0)]}{x-x_0}\right\}$$

$$= c\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$= cf'(x_0) \text{ exists.}$$

Hence$f(x)$ is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

Theorem 2. Let the functions $f(x)$ and $g(x)$ are defined in an interval I. If f and g are differentiable at $x = x_0 \in I$, then so also $f+g$ and $(f+g)'(x_0) = f'(x_0) + g(x_0)$.

Proof :- Since $f(x)$ and $g(x)$ are differentiable at $x = x_0$, Therefore,

$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$.....................(i)

and $\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = g'(x_0)$.....................(ii)

Now, $(f+g)'(x_0) = \lim\limits_{x\to x_0} \frac{(f+g)x-(f+g)x_0}{x-x_0} = \lim\limits_{x\to x_0} \frac{[f(x)+g(x)]-[f(x_0)+g(x_0)]}{x-x_0}$

$$= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}\right]$$

$= \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim\limits_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}$

[as the limit of the sum is equal to sum of the limits]

$= f'(x_0) + g'(x_0)$ exists. [using (i) and (ii)]

Hence f+g is differentiable at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g(x_0).$$

Theorem 3. Let $f(x)$ and $g(x)$ be defined on an interval I. If f and g are differentiable at $x = x_0 \in I$, then so also is fg and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Proof :- Since the $f(x)$ and $g(x)$ are differentiable at x_0, we have

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)\text{....................(i)}$$

$$\text{and } \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0)\text{......................(ii)}$$

Now, $(fg)'(x_0) = \lim\limits_{x\to x_0} \frac{(fg)x-(fg)x_0}{x-x_0}$

$$= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}.g(x) + f(x_0).\frac{g(x)-g(x_0)}{x-x_0}\right]$$

$$= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.\lim_{x\to x_0} g(x) + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}.f(x_0)$$

$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$ exists.

[using (i) and (ii) and $\lim\limits_{x\to x_0} g(x) = g(x_0)$]

[Since $g(x)$ is differentiable at $x_0 \Rightarrow g(x)$ is continuous at x_0]

Hence fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Theorem 4. If f(x) is differentiable at $x = x_0$ and $f(x_0) \neq 0$ then the function $\frac{1}{f(x)}$ is differentiable at $x_0$and $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{[f(x_0)]^2}$

Proof :- Since $f(x)$ is differentiable at $x = x_0$, therefore

$\lim\limits_{x\to 0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$........................(i)
and $f(x)$ is also continuous at $x=x_0$
$\lim\limits_{x\to 0}f(x)=f(x_0)\neq 0$................(ii)
(every differentiable function is continuous at very point in its domain)
Also $f(x_0)\neq 0$, hence $f(x_0)\neq 0$ in some neighbourhood N of x_0.
Now we have for $x\in N$, $\left(\frac{1}{f}\right)'(x_0)=\lim\limits_{x\to x_0}\frac{\frac{1}{f(x)}-\frac{1}{f(x_0)}}{x-x_0}$

$$=\lim_{x\to x_0}\left[-\frac{f(x)-f(x_0)}{x-x_0}.\frac{1}{f(x)}.\frac{1}{f(x_0)}\right]$$

$$=-\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.\lim_{x\to x_0}\frac{1}{f(x)}.\frac{1}{f(x_0)}$$

$$=-f'(x_0).\frac{1}{f(x_0)f(x_0)} \qquad \text{[using (i) and (ii)]}$$

$$=-\frac{f'(x_0)}{[f(x_0)]^2} \text{ exists.}$$

Hence $\frac{1}{f(x)}$ is differentiable at x_0 and $\left(\frac{1}{f}\right)'(x_0)=-\frac{f'(x_0)}{[f(x_0)]^2}$

Note :- Let f and g be defined on I. If f and g are differentiable at $x=x_0\in I$ and $g(x_0)\neq 0$, then the function $\frac{f}{g}$ is differentiable at x_0 and using the theorem ③ and ④ we can prove

$$\left(\frac{f}{g}\right)'(x_0)=\frac{[g(x_0)f'(x_0)-f(x_0)g'(x_0)]}{[g(x_0)]^2}$$

3.7. THE CHAIN RULE OF DIFFERENTIABILITY

Theorem – Let $f(x)$ and $g(x)$ are two function such that the range of $f(x)$ is contained in the domain of $g(x)$. If $f(x)$ is differentiable at (x_0) and $g(x)$ is differentiable at $f(x_0)$, then gof is differentiable at x_0 and

$$(gof)'(x_0)=g'\big(f(x_0)\big)f'(x_0)$$

Proof :- Let $y=f(x)$ and $y_0=f(x_0)$
Since f(x) is differentiable at x_0, we have

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$$

or $f(x)-f(x_0)=(x-x_0)[f'(x_0)+\lambda(x)]$............(i)
where $\lambda(x)\to 0$ as $x\to x_0$
Since $g(x)$ is differentiable at y_0, we have

$$\lim_{y\to y_0}\frac{g(y)-g(y_0)}{y-y_0}=g'(y_0)$$

or $g(y)-g(y_0)=(y-y_0)[g'(y_0)+\mu(y)]$.............(ii) where $\mu(y) \to 0$ as $y \to y_0$

Now, $(gof)(x) - (gof)x_0 = g\big(f(x)\big) - g\big(f(x_0)\big) = g(y) - g(y_0)$

$= (y - y_0)[g'(y_0) + \mu(y)]$....... from (ii)

$= [f(x) - f(x_0)][g'(y_0) + \mu(y)]$

$= (x - x_0)[f'(x_0) + \lambda(x)][g'(y_0) + \mu(y)]$.......from(i)

Thus if $x \neq x_0$ then

$\frac{(gof)x-(gof)x_0}{x-x_0} = [g'(y_0) + \mu(y)][f'(x_0) + \lambda(x)]$.........(iii)

Since f is differentiable at x_0 so f is continuous at x_0

i.e. $x \to x_0 \Rightarrow f(x) \longrightarrow f(x_0)$ i.e. $y \to y_0$

and $\mu(y) \longrightarrow 0$ as $x \to x_0$ and $\lambda(x) \longrightarrow 0$ as $x \to x_0$

taking the limit $x \to x_0$ we get from (iii)

$$\lim_{x \to x_0} \frac{(gof)(x) - (gof)(x_0)}{x - x_0} = g'(y_0)f'(x_0)$$

$$= g'\big(f(x_0)\big).f'(x_0)$$

Hence the function (gof) is differentiable at x_0 and

$$(gof)'(x_0) = g'\big(f(x_0)\big)f'(x_0)$$

3.8. DERIVATIVE OF THE INVERSE FUNCTION

Theorem – If $f(x)$ be a continuous one-one onto function defined on a interval. Let $f(x)$ is differentiable at $x = x_0$, with $f'(x_0) \neq 0$, then inverse of the function $f(x)$ is differentiable at $f(x_0)$ and its derivative at $f(x_0)$ is $\frac{1}{f'(x_0)}$.

Proof :- We know that if the domain of f be X and its range be Y, then inverse function g of f is denoted by f^{-1}. f^{-1} is a function with domain Y and range X such that

$f(x) = y \Leftrightarrow g(y) = x$, also g exists if f is one-one onto.

Let $y = f(x)$ and $y_0 = f(x_0)$

Since f is differentiable at x_0, then we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$\Rightarrow f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)]$...............(i)

where $\lambda(x) \longrightarrow 0$ as $x \to x_0$.

Further $g(y) - g(y_0) = x - x_0$.........(by definition of g)

$$\Rightarrow \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}$$

$= \frac{1}{[f'(x_0)+\lambda(x)]}$...............by (i)

Since f is continuous at $x = x_0 \Rightarrow g = f^{-1}$ is continuous at $f(x_0) = y_0$ and so $g(y) \to g(y_0)$ as $y \to y_0$

Hence

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(x_0) + \lambda(x)} = \frac{1}{f'(x_0)}$$

$\Rightarrow g'(y_0) = \frac{1}{f'(x_0)}$ or $g'(f(x_0)) = \frac{1}{f'(x_0)}$

3.9. DARBOUX THEOREM

Theorem. If $f(x)$ is differentiable in $[a,b]$ and $f'(a)$,$f'(b)$ have opposite signs, then $\exists$ at least one point $c \in (a,b)$ such that $f'(c) = 0$.

Proof :- For definiteness, let $f'(a) > 0$ and $f'(b) < 0$ then there are intervals $(a, a+h]$ and $[b-h, b), h > 0$ such that

$$f(x) > f(a), \forall x \in (a, a+h]$$

$$\text{and } f(x) > f(b), \forall x \in [b-h, b)$$

By the assumption, f is differentiable in $[a,b]$ so f is continuous in $[a,b]$. Consequently f attains its Supremum and Infimum in $[a,b]$.

Let $Supf = M$ then $\exists c \in [a,b]$ such that $f(c) = M$.

Then by the definition of Supremum

$f(x) \le M$ i.e. $f(x) \le f(c), , \forall x \in [a,b]$ ....(i)

To prove : $f'(c) = 0$. Suppose it is not true. Then either $f'(c) > 0$ or $f'(c) < 0$.If $f'(c) > 0, \exists . (c, c+\in], \in > 0$ such that $f(x) > f(c), \forall x \in (c, c+\in]$.This is contrary to (i).If $f'(c) < 0, \exists [c-\in, c)$ such that $f(x) > f(c), , \forall x \in [c-\in, c)$ .Again we get contradiction to (i).Hence $f'(c) = 0$.But $f'(a) > 0$ and $f'(b) < 0$, therefore $c \ne a$ and $c \ne b$ i.e. $c \in (a,b)$.

Corollary 1. If f is differentiable in $[a,b]$ and $f'(a) \ne f'(b)$, then $f'(x)$ takes all the values between $f'(a)$ and $f'(b)$ at least once in (a,b).

Proof :- Assume that $f'(a) < f'(b)$, let $f'(a) < k < f'(b)$

To show that $\exists c \in (a,b)$ such that $f'(c) = k$.

write $F(x) = f(x) - kx$

$\Rightarrow F'(x) = f'(x) - k$, Now we have

$F'(a) = f'(a) - k < 0$, as $f'(a) < k$

and $F'(b) = f'(b) - k > 0$, as $f'(b) > k$.

Thus $F'(a) < 0$ and $F'(b) > 0$. Hence by Darboux theorem,

$\exists c \in (a,b)$ s.t. $F'(c) = 0$

$\Rightarrow f'(c) - k = 0 \Rightarrow f'(c) = k$.

Similarly we can show another case when $f'(a) > f'(b)$.

Corollary 2. If $f(x) \ne 0, , \forall x \in (a,b)$ then $f'(x)$ retains the same sign, positive or negative in (a,b).

Proof :- If possible, let $x_1, x_2 \in (a,b)$ such that $f'(x_1)$ and $f'(x_2)$ have opposite signs, where $x_1 < x_2$, then by Darboux theorem,

$$\exists c \in (x_1, x_2) \subset (a,b) \text{ suct that } f'(c) = 0.$$

which is contrary to the hypothesis. Hence $f'(x)$ retains the same sign.

Illustrative Examples

Example 1. Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

Solution. Firstly, we see the continuity of the function $f(x)$ at $x = 0$

We have $f(0) = |0| = 0$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} |h|$$
$$= \lim_{h\to 0} h = 0$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} |-h|$$
$$= \lim_{h\to 0} h = 0$$

Hence $f(0+0) = f(0) = f(0-0)$. So $f(x)$ is continuous at $x = 0$. Now we see the differentiability of $f(x)$ at $x = 0$. We have $Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-0}{-h} = 1$ and $Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-0}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1 \Rightarrow Rf'(0) \neq Lf'(0)$

Hence $f(x)$ is not differentiable at $x = 0$.

Example 2. Prove that the function $f(x) = |x| + |x-1|$ is not differentiable at $x = 0$ and $x = 1$.

Solution. Here we see that

(i) $|x| = -x$ and $|x-1| = 1-x$ when $x < 0$

(ii) $|x| = x$ and $|x-1| = 1-x$ when $0 \leq x \leq 1$

(iii) $|x| = x$ and $|x-1| = x-1$ when $x > 1$

Hence, the given function can be written as

$$f(x) = -x + 1 - x = 1 - 2x, x < 0$$
$$= x + 1 - x = 1, 0 \leq x \leq 1$$
$$= x + x - 1 = 2x - 1, x > 1$$

Now first we see the differentiability $f(x)$ at $x = 0$

We have

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$
$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{1-2(-h)-1}{-h} = \lim_{h\to 0} \frac{2h}{-h} = -2$$

Thus $Rf'(0) \neq Lf'(0)$. Therefore the given function is not differentiable at $x = 0$. Now we see the differentiability of $f(x)$ at $x = 1$.

We have $Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{[2(1+h)-1]-1}{h} = \lim_{h\to 0} \frac{2+2h-2}{h} = 2$ and $Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{-h} = 0$

Thus $Rf'(1) \neq Lf'(1)$. Therefore the given function is not differentiable at $x = 1$.

Example 3. Show that the function

$$f(x) = \begin{Bmatrix} x\tan^{-1}\left(\frac{1}{x}\right), x \neq 0 \\ 0 \qquad\qquad\quad , x = 0 \end{Bmatrix}$$

is not differentiable at $x = 0$.

Solution. Here we have

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h\tan^{-1}\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} \tan^{-1}\left(\frac{1}{h}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

and $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h\tan^{-1}\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h\to 0} \tan^{-1}\left(-\frac{1}{h}\right) = -\tan^{-1}\infty = -\frac{\pi}{2}$$

$$\Rightarrow Rf'(0) \neq Lf'(0)$$

Hence f(x) is not differentiable at $x = 0$.

Example 4. If $\emptyset(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$ and $\emptyset(0) = 0$. Show that $\emptyset'(x)$ exists for all values of x but $\emptyset'(x)$ is discontinuous at $x = 0$ and $\emptyset''(x)$ does not exists at origin.

Solution.$\emptyset(x) = x^2 \sin\left(\frac{1}{x}\right), x \neq 0$....................(i)

$$\emptyset(0) = 0$$

So $\emptyset'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), x \neq 0$ ....(ii)

First Part – To show that $\emptyset'(x)$ exists $\forall x$

$$R\emptyset'(0) = \lim_{h\to 0} \frac{\emptyset(h)-\emptyset(0)}{h}$$

$$= \lim_{h\to 0} \frac{\emptyset(h) - 0}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$$

and $\quad L\emptyset'(0) = \lim_{h\to 0} \frac{\emptyset(0-h)-\emptyset(0)}{-h}$

$$= \lim_{h\to 0} \frac{\emptyset(-h) - \emptyset(0)}{-h} = \lim_{h\to 0} \frac{-h^2 \sin\left(\frac{1}{h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus $\quad R\emptyset'(0) = L\emptyset'(0) = 0 \Rightarrow \emptyset'(0) = 0$

$\Rightarrow \emptyset'(x)$ exists at $x = 0$

But $\emptyset'(x)$ exists $\forall x$ s.t. $x \neq 0$.............. by (ii)

$\Rightarrow \emptyset'(x)$ exists $\forall x$.

Second Part. To show that $\emptyset'(x)$ is discontinuous at $x = 0$

Here $\emptyset'(0 + h) = 2h \sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right)$............by (ii)

$\Rightarrow \lim_{h\to 0} \emptyset'(0 + h)$ does not exists for $\lim_{h\to 0} \cos\left(\frac{1}{h}\right)$ does not exists.

Hence $\emptyset'(x)$ is discontinuous at $x = 0$.

Third Part. If a function is discontinuous at $x = 0$ then it will not be differentiable at $x = 0$. From the second part $\emptyset'(x)$ is discontinuous at $x = 0$. So $\emptyset'(x)$ is not differentiable at $x = 0$ i.e. $\emptyset''(x)$ does not exists at $x = 0$.

Example 5. Draw the graph of $y = |x - 1| + |x - 2|$ in the interval $[0,3]$ and discuss the continuity and differentiability of the function in this interval.

Solution. Let $y = f(x)$ then

(i) $y = f(x) = 1 - x + 2 - x = 3 - 2x$ when $0 \leq x \leq 1$

(ii)$y = f(x) = x - 1 + 2 - x = 1$ when $1 \leq x \leq 2$

(iii) $y = f(x) = x - 1 + x - 2 = 2x - 3$ when $2 \leq x \leq 3$

Hence the graph of the function consists of three straight line segments $y = 3 - 2x, y = 1, y = 2x - 3$ in the three intervals $[0,1], [1,2], [2,3]$ respectively.

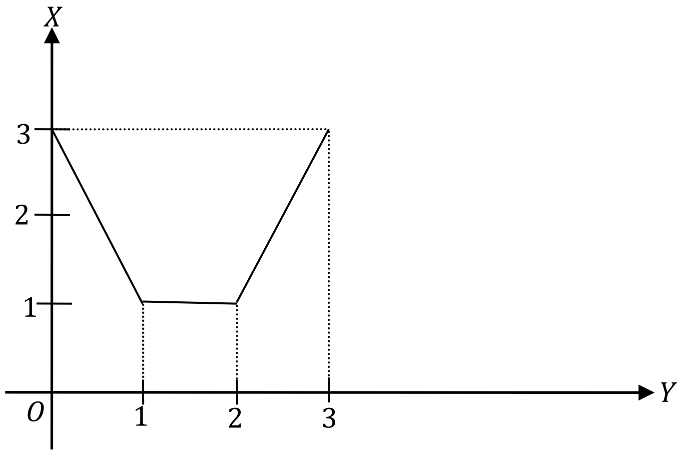

Fig. 3.9.1

From the graph of the function it is clear that the function is continuous throughout $[0,3]$ but it is not differentiable at $x = 1, 2$.

To test at $x = 1$

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[3-2(1-h)]-1}{-h}$$

$$= \lim_{h\to 0}\frac{2h}{-h} = -2$$

Thus $Rf'(1) \neq Lf'(1)$ so f is not differentiable at $x = 1$.

To test at $x = 2$

$$Rf'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{[2(2+h)-3]-1}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$$

$$Lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{1-1}{-h} = 0$$

Thus $Rf'(2) \neq Lf'(2)$ so f is not differentiable at $x = 2$.

Example 6. Let $f(x) = x.\frac{e^{yx}-e^{-yx}}{e^{yx}+e^{-yx}}$, $x \neq 0$

$$f(0) = 0.$$

Show that $f(x)$ is continuous but not differentiable at $x = 0$.

Solution. It is given that $f(0) = 0$; Now $f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$$= \lim_{h\to 0} h.\frac{e^{yh}-e^{-yh}}{e^{yh}+e^{-yh}}$$

$$= \lim_{h\to 0} h.\frac{1-e^{-2h}}{1+e^{-2h}}$$ [dividing the numerator and denominator by e^{yh}]

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

and $f(0-h)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0}-h.\frac{e^{-yh}-e^{+yh}}{e^{-yh}+e^{+yh}}=\lim_{h\to 0}-h.\frac{e^{-2h}-1}{e^{-2h}+1}=0$$

Since $f(0+0)=f(0-0)=f(0)$. Hence the function is continuous at $x=0$.

Now $Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\left[h.\frac{e^{yh}-e^{-yh}}{e^{yh}+e^{-yh}}\right]/h$$

$$=\lim_{h\to 0}\frac{1-e^{-2h}}{1+e^{-2h}}=\frac{1-0}{1+0}=1$$

And $Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)}{-h}$

$$=\lim_{h\to 0}\left[-h.\frac{e^{-yh}-e^{yh}}{e^{-yh}+e^{yh}}\right]/(-h)$$

$$=\lim_{h\to 0}\frac{e^{-2h}-1}{e^{-2h}+1}=-1$$

Since $Rf'(0)\neq Lf'(0)$, the function is not differentiable at $x=0$.

Example 7. A function f is defined by

$$f(x)=x^p\cos\left(\frac{1}{x}\right),x\neq 0;f(0)=0$$

what conditions should be imposed on p so that f may be

(i) continuous at $x=0$.

(ii) differentiable at $x=0$.

Solution. We have

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\left[(0+h)^p\cos\frac{1}{(0+h)}\right]$$

$$=\lim_{h\to 0} h^p\cos\left(\frac{1}{h}\right)\text{......................(i)}$$

$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\left[(0-h)^p\cos\frac{1}{(0-h)}\right]$$

$$=\lim_{h\to 0}(-h)^p\cos\left(\frac{1}{h}\right)\text{......................(ii)}$$

Now if $f(x)$ is continuous at $x=0$, then

i.e. the limits given by (i) and (ii) must both tend to zero. This is possible only if $p>0$, which is the required condition.

Now $Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h^p\cos\left(\frac{1}{h}\right)-0}{h}=\lim_{h\to 0} h^{p-1}\cos\left(\frac{1}{h}\right)\text{..................(iii)}$$

and $Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^p \cos\left(\frac{1}{h}\right)-0}{-h}$

$$= \lim_{h\to 0} -(-1)^p h^{p-1} \cos\left(\frac{1}{h}\right)$$

Now if $f'(x)$ exists at $x = 0$ then we must have

$Rf'(0) = Lf'(0)$ and this possible only if $p - 1 > 0$

i.e. $p > 1$ which gives $Rf'(0) = Lf'(0) = 0$.

Hence in order that f is differentiable at $x = 0$, p must be greater than 1.

Example 8. Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.

Solution. Since $f(x)$ is an even function,

so $f(-x) = f(x) \forall x$

$f'(0)$ exists $\Rightarrow Rf'(0) = Lf'(0) = f'(0)$

Now $f'(0) = Rf'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}, h > 0$

$= \lim_{h\to 0} \frac{f(-h)-f(0)}{h}$ [Since $f(-x) = f(x)$]

$$= -\lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$= -Lf'(0) = -f'(0)$

$\Rightarrow 2f'(0) = 0 \Rightarrow f'(0) = 0$

Thus we conclude that if $f(x)$ is an even function and $f'(0)$ exists then $f'(0) = 0$.

3.10. SUMMARY

In this unit we have defined the concept of Derivative at a point, Progressive and Regressive derivatives, Differentiability in an interval, Algebra of derivatives, The chain rule of Differentiablity, Derivative of inverse function and Darboux theorem.

- A function $f(x)$ is said to be differentiable at the point $x = c \in (a, b)$ iff
    - $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ i.e. $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists finitely.
- Right hand derivative of $f(x)$ at $x = c$ is denoted by $Rf'(c) = f'(c+0) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}, h > 0$
- Left hand derivative of $f(x)$ at $x = c$ is denoted by $Lf'(c) = f'(c-0) = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}, h > 0$
- and $f'(c)$ exists iff
    - $Lf'(c)$ and $Rf'(c)$ both exists, and
    - $Lf'(c) = Rf'(c)$
- $f(x)$ is said to be differentiable at $x = c$
- If given $\in > 0\ \exists \delta > 0$ s.t.

- $x \in (c - \delta, c + \delta) \Rightarrow f'(c) - \in < \frac{f(x) - f(c)}{x - c} < f'(c) + \in$
- The derivative of a function $f(x)$ at $x = a$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $x = a$, makes with x-axis

$$\left[\frac{d}{dx} f(x)\right]_{x=a} = \tan \psi$$

- Continuity is necessary but not a sufficient condition for differentiablilty
- e.g. $y = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

3.11. GLOSSARY

i. Sets-Well defined collection of objects.
ii. Continuous-sketch its curve on a graph without lifting your pen even once.
iii. Discontinuity-lack of continuity.

CHECK YOUR PROGRESS

1. The function $y = |x|$ is differentiable at every point of R. T/F
2. If a function $f(x)$ is continuous at $x = a$, it must also be differentiable at $x = a$. T/F
3. If a function $f(x)$ is differentiable at $x = a$, it must be continuous at $x = a$. T/F
4. If a function $f(x)$ is differentiable at $x = a$, it may or may not be continuous at $x = a$. T/F
5. The function $y = |x - 1|$is continuous at every point of R.T/F
6. The function $f(x)$ is said to be differentiable at $x = a$ if $\lim_{x \to a} \frac{f(x) - \cdots .....}{x - a}$ exists.T/F
7. If $f(x) = \sin x$, then
8. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$ ....
9. The function $f(x) = |x|$ is differentiable at every point of R except at $x =$..................
10. Continuity is a necessary but not a .... condition for the existence of a finite derivative.
11. The right hand derivate of $f(x)$ at $x = a$ is given by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{............}, h > 0$

3.12. REFERENCES

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3.14. TERMINAL QUESTIONS

1. A function ϕ is defined as follows :

   $\phi(x) = -x$ for $x \leq 0, \phi(x) = x$for $x \geq 0$.

   Test the character of the function at $x = 0$ as regards continuity and differentiability.

2. If $f(x) = \frac{x}{1+e^{yx}}$, $x \neq 0, f(0) = 0$, show that f is continuous at $x = 0$ but $f'(0)$ does not exist.

3. A function ϕ is defined as :

   $\phi(x) = 1 + x$ if $x \leq 2$,

   $\phi(x) = 5 - x$ if $x > 2$

   Test the continuity and differentiability of the function at $x = 2$.

4. Discuss the continuity and differentiability of the following function :

   $f(x) = x^2$ for $x < -2$

   $f(x) = 4$ for $-2 \leq x < 2$

   $f(x) = x^2$ for $x > 2$.

5. Examine the following curve continuity and differentiability

   $y = x^2$ for $x \leq 0$

   $y = 1$ for $0 < x \leq 1$

   $y = \frac{1}{x}$ for $x > 0$

6 Show that $f(x) = |x-1|, 0 \le x \le 2$ is not differentiable at $x = 1$. It is continuous in $[0,2]$.

7 The function $f(x) = |x-1|$ is not differentiable at
i) $x = 0$ ii) $x = -1$ iii) $x = 1$ iv) $x = 2$

8 The function $f(x) = |x+3|$ is not differentiable at
i) $x = 3$ ii) $x = -3$ iii) $x = 0$ iv) $x = 1$

9 The function $f(x) = x\sin\left(\frac{1}{x}\right), x \ne 0$ and $(0) = 0$, at $x = 0$ is
i)Continuous and differentiable ii) Continuous but not differentiable iii) Discontinuous and not differentiable iv) None of these

10 A function $f(x)$ is differentiable at $x = a$ if i) $Rf'(a) = Lf'(a)$ ii) $Rf'(a) = 0$ iii) $Lf'(a) = 0$ iv) $Rf'(a) \ne Lf'(a)$

11 The function $f(x) = \begin{cases} 2+x, x \ge 0 \\ 2-x, x < 0 \end{cases}$ is i) Discontinuous at $x = 0$ ii) Continuous but not differentiable at $x = 0$ iii) Continuous and differentiable at $x = 0$ iv) None of these

3.15. ANSWERS

ANSWER CHECK YOUR PROGRESS

SCQ1. F
SCQ2. F
SCQ3. T
SCQ4. F
SCQ5. F
SCQ6. $f(a)$
SCQ7. $cos\ x$
SCQ8. $x = 0$
SCQ9. $f'(x)$
SCQ10. Sufficient

ANSWER TO TERMINAL QUESTIONS

(TQ-1) Continuous at $x = 0$ but not differentiable at $x = 0$.
(TQ-3) Continuous but not differentiable at $x = 2$.
(TQ-4) Continuous but not differentiable at $x = -2, 2$.
(TQ-5) Discontinuous and non differentiable at $x = 0$, continuous and non differentiable at $x = 1$.
(TQ-6) Yes
(TQ-7) h
(TQ-8) (iii)
(TQ-9) (ii)
(TQ-10) (ii)
(TQ-11) (i)

UNIT–4:- MEAN VALUE THEOREMS

CONTENTS:-

4.1 INTRODUCTION

The mean value theorem makes the geometrically possible claim that a differentiable function f on an interval $[a,b]$ will at some point attain at some point attain a slope equal to the slope of the line through the end points $(a,f(a))$ and $(b,f(b))$. More, precisely,
$f'(c)=\frac{f(b)-f(a)}{b-a}$. For at least one point $c\in(a,b)$. This theorem is used to prove statements about a function on an interval starting from local hypotheses about derivatives at points of the interval. The mean value theorem in its modern form was stated and proved by **Augustin Louis Cauchy** in **1823**. Many variations of this theorem have been proved since then. Later in **1691**, **Michel Rolle** proved a special case of mean value theorem So it was named after him, Rolle's mean value theorem. In this unit we have explained the Rolle's theorem, Lagrange's theorem, Cauchy mean value theorem, Taylor theorem with Lagrange's form of remainder, Taylor's theorem with Cauchy's

form of remainder. We have also described the geometrical and algebraic interpretation of Mean value theorems.

4.2 OBJECTIVES

After completion of this unit the learner will be able to understand:

- **i.** Rolle's theorem and its geometrical interpretation.
- **ii.** Lagrange's mean value theorem.
- **iii.** Cauchy's theorem.
- **iv.** Mean value theorems of higher derivatives.

4.3 ROLLE'S THEOREM

If $f(x)$ is a real valued function defined in the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$.

(ii) $f(x)$ is differentiable in the open interval (a, b).

(iii) $f(a) = f(b)$, then there exists at least one value of x say c where $a<c<b$, such that $f'(c) = 0$.

Proof. Continuity of $f(x)$ in $[a, b]$ implies that $f(x)$ is bounded in $[a, b]$ and attains its bound at least once in $[a, b]$. Let $\sup f(x) = M$ and $\inf f(x) = m$, and let $f(c_1) = m, f(c_2) = M$ where $c_1, c_2 \in [a, b]$. Evidently, $M \geq m$.

Now two different cases arise:

Case I. When $M = m$. Then

$$f(c_1) = f(c_2) \Rightarrow f(x) = \text{constant}$$

$$\Rightarrow f'(x) = 0 \quad \forall x \in [a, b]$$

$$\Rightarrow f'(c) = 0 \quad c \in [a, b]$$

Case II. When $M>m$. In this case at least one of the bounds is different from $f(a) = f(b)$. we suppose first that $M \neq f(a) = f(b)$, *i.e.*, $f(c_2) \neq f(a) = f(b)$. This means that c_2 is different from a and b, *i.e.*, $c_2 \in [a, b]$ or $a < c_2 < b$.

Now

$$M = f(c_2) = \sup f(x) \Rightarrow f(x) \leq f(c_2) \quad \forall x \in [a, b]$$

(1)

Therefore $f(c_2 + h) \leq f(c_2) \quad \forall h > 0, c_2 + h \in [a, b]$

$$\Rightarrow \frac{f(c_2 + h) - f(c_2)}{h} \leq 0$$

It follows that

$$\underset{h\to 0}{Limit}\frac{f(c_2+h)-f(c_2)}{h}\le 0\, i.e.,\ R\, f^{'}(c_2)\le 0 \qquad (2)$$

Again (1) implies that

$f(c_2-h)\le f(c_2) \quad \forall h>0,\ c_2-h\in[a, b]$

Therefore, $f(c_2-h)-f(c_2)\le 0$

$$\Rightarrow \frac{f(c_2+h)-f(c_2)}{-h}\ge 0$$

It follows that

$$\underset{h\to 0}{Limit}\frac{f(c_2-h)-f(c_2)}{-h}\ge 0\, i.e.,\ L\, f^{'}(c_2)\ge 0 \qquad (3)$$

But if *f* is differentiable in (*a*, *b*) and $c_2\in[a, b]$ so that *f* is differentiable at *x* = c_2 *i. e.*,

$$L\, f^{'}(c_2)=R\, f^{'}(c_2)=f^{'}(c_2) \qquad (4)$$

Using (4) in (2) and (3) we find that

$f^{'}(c_2)\le 0$ and $f^{'}(c_2)\ge 0\Rightarrow f^{'}(c_2)=0$

Similarly if we suppose that $m\neq f(a)=f(b)$ then by making parallel arguments, we can prove that $f^{'}(c_1)=0$. Replacing c_1 and c_2 by *c*, we have the result.

Note1. There may be more than one point like *c* at which $f^{'}(x)=0$.

Note 2. Rolle's theorem does not hold good if

(i) *f* (*x*) is discontinuous in the closed interval [*a*, *b*].

(ii) *f* (*x*) does not exists at some point in (*a*, *b*).

(iii) $f(a)\neq f(b)$.

Note 3. The hypothesis of Rolle's theorem can not be weakened.

Example. If $f(x)=1-|x|,\ -1\le x\le 1,$ then $f(1)=f(-1)=0$ and f is continuous on [-1, 1]. Also $f^{'}(x)$ exist $\forall x\in(-1, 1)$ except *x* =0. Thus *f* satisfies all the condition of Rolle's theorem except that *f* is not differentiable at *x* = 0. For this *f* (*x*), there is no *c* in (-1, 1) for which $f^{'}(c)=0$.

4.3.1 GEOMETRICAL REPRESENTATION OF ROLLE'S THEOREM

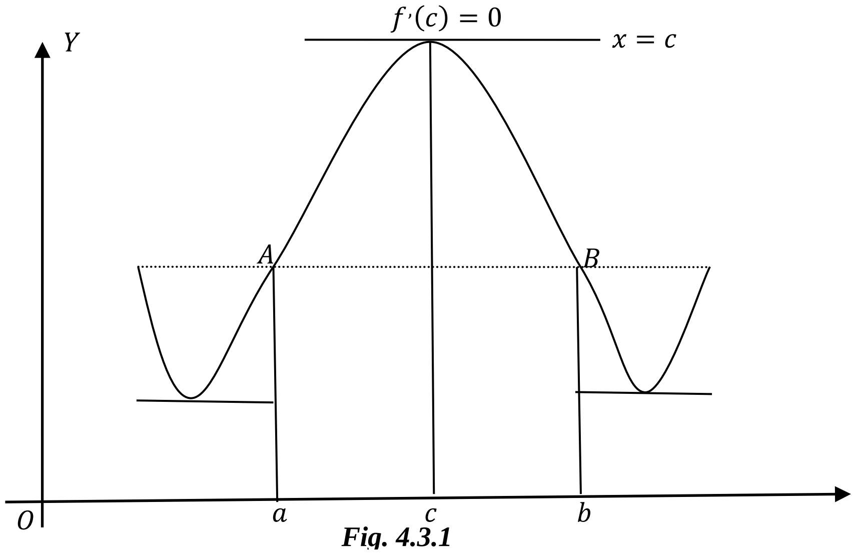

Fig. 4.3.1

Geometrically Rolle's theorem means that is the curve $y = f(x)$ is continuous from $x = a$ to $x = b$; has a definite tangent at each point of (a, b) and the ordinates at the extremities are equal then there exists at least one point between a and b at which the tangent is parallel to x-axis.

4.3.2 ALGEBRAIC REPRESENTATION OF ROLLE'S THEOREM

Rolle's theorem leads to a very important result in the theory of equations. Algebraically, Rolle's theorem means that if $f(x)$ is a polynomial function in x and $x = a$ and $x = b$ are two roots of the equation $f(x) = 0$, then there is at least one root of the equation $f'(x) = 0$ which lies between a and b.

4.4. LAGRANGE'S MEAN VALUE THEOREM

If a real valued function $f(x)$ defined on $[a, b]$ such that

(i) $f(x)$ is continuous on $[a, b]$.

(ii) $f(x)$ is differentiable in (a, b).

Then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof. Let us define a function $f(x)$ defined by

$$F(x) = f(x) + A.x \qquad (1)$$

Where A is a constant to be determined such that $F(a) = F(b)$.

Now

(i) Since $f(x)$ is continuous on $[a, b]$ and Ax is continuous on $[a, b]$, therefore $F(x)$ is also continuous on $[a, b]$. (Since sum of two continuous functions is again continuous).

(ii) Similarly $F(x)$ is differentiable in (a, b).

(iii) $F(a) = F(b) \Rightarrow f(a) + A.a = f(b) + A.b$

$$\Rightarrow -A = \frac{f(b) - f(a)}{b - a} \qquad (2)$$

Hence $F(x)$ satisfy all the conditions of Rolle's theorem on $[a, b]$ and consequently there exists $c \in (a, b)$ such that $f'(x) = 0$, this gives

$$F'(c) = f'(c) + A = 0 \Rightarrow -A = f'(c) \qquad (3)$$

Now from (2) and (3), we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Lagrange's mean value theorem is also known as **first mean value theorem**.

Note 1. Another form (alternative form) of Lagrange's mean value theorem. Let $b - a = h$, $c = a + \theta h$, then $h > 0$ and $a < c < b$ and this implies that $a < a + \theta h < a + h \Rightarrow 0 < \theta h < h \Rightarrow 0 < \theta < 1$.

Now $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ becomes $f'(a + \theta h) = \dfrac{f(a + h) - f(a)}{h}$ *i.e.*,

$$f(a + h) = f(a) + h f'(a + \theta h), \ 0 < \theta < 1.$$

Note 2. The hypothesis of the Lagrange's mean value theorem can not be weakened as

Example. Let $f(x) = |x|, \ -1 \le x \le 2$. Here f is continuous [-1, 2] and differentiable at all the points of (-1, 2) except $x = 0$ (so that second condition is violated). Now

$$f'(x) = \begin{cases} -1, & -1 < x < 0 \\ +1 & 0 < x < 2 \end{cases}$$

Also $f'(x) \neq \dfrac{f(2) - f(-1)}{2 - (-1)}$ for any $x \in (-1, 2)$.

4.4.1 GEOMETRICAL INTERPRETATION OF LAGRANGE'S MEAN VALUE THEOREM

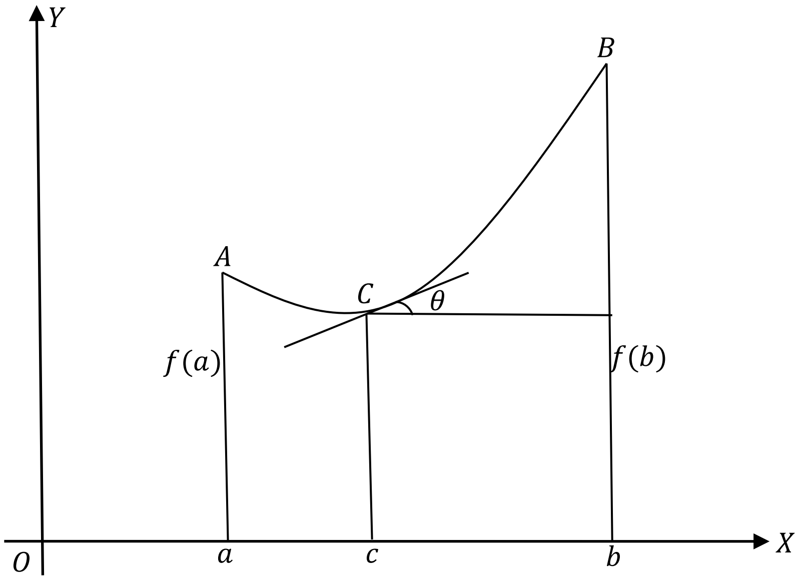

Fig.4.4.1.

If the curve $y = f(x)$ is continuous from $x = a$ to $x = b$ and has a definite tangent at each point on the curve between $x = a$ and $x = b$. Then, geometrically the first mean value theorem means that there is at least one point between $x = a$ and $x = b$ on the curve where the tangent to the curve parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$.

Let AB be the graph of the function $y = f(x)$ then the co-ordinate of the points A and B are given by $(a, f(a))$ and $(b, f(b))$. If the chord AB makes an angle θ with x – axis, then

$$\tan\theta = \frac{f(b) - f(a)}{b-a} = f'(c) \text{ where } a<c<b.$$

Corollary 1. If $f'(x) = 0 \; \forall \; x \in (a, b)$, then f (x) is constant in $[a, b]$. Thus if a function has differential coefficient which vanishes for all values of x in $[a, b]$, then the function is constant.

Proof. Given that $f'(x) = 0 \; \forall \; x \in [a, b]$. Let $a+h$ be any point of $[a, b]$. Then by first mean value theorem,

$$f(a+h) - f(a) = h f'(a+\theta h), \qquad 0 < \theta < 1,$$

$$= h \times 0 = 0 \; as \; f'(a+\theta h) = 0, \qquad a + \theta h \in [a,b]$$

Thus

$$f(a+h) = f(a) \quad \forall \; a+h \in [a,b]$$

$$i.e., \; f(x) = \text{constant} \quad \forall \; x \in [a,b]$$

Corollary 2. If two functions $f(x)$ and $g(x)$ be differentiable in (a, b) and if $f'(x) = g'(x), \; \forall \; x \in [a, b]$, then $f(x) - g(x)$ is constant.

Proof. It is given that

$f'(x) = g'(x) \quad \forall\ x \in [a, b]$ (1)

To prove that $f(x) - g(x)$ is constant.

Let $G(x) = f(x) - g(x)$ then $G'(x) = f'(x) - g'(x) = 0$ [by (1)]. This implies that

$G'(x) = 0, \ \forall\ x \in [a, b]$ (2)

Let $a + h$ be any point of (a, b). Then, by first mean value theorem

$$G(a+h) - G(a) = hG'(a+\theta h), \quad 0 < \theta < 1,$$
$$= 0 \ \text{[by (2)]}$$
$$\Rightarrow G(a+h) = G(a) \quad \forall\, a+h \in [a,b]$$
$$\Rightarrow G(x) = \text{constant} \quad \forall\, x \in [a,b]$$
$$\Rightarrow f(x) - g(x) = \text{constant} \quad \forall\, x \in [a,b].$$

Note 1. The result $f(b) - f(a) = (b-a) f'(c)$, is also known as formula for finite increment.

Note 2. For $f(b) - f(a)$, the Lagrange's mean value theorem yields Rolle's theorem

4.5 CAUCHY'S MEAN VALUE THEOREM

Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ such that

(i) $f(x)$ and $g(x)$ are continuous on $[a, b]$

(ii) $f(x)$ and $g(x)$ are differentiable on (a, b) and

(iii) $g'(x) \neq 0, \ \forall\ x \in (a,b)$,

Then there exists a point $c \in (a,b)$, such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. We observed that $(iii) \Rightarrow g(a) = g(b)$.

For, if $g(a) = g(b)$, then $g(x)$ satisfies all the conditions of Rolle's theorem and hence $g'(x) = 0$ for some $x \in (a,b)$, which is contrary to (iii). Let us define a function on $[a, b]$ by

$F(x) = f(x) + A.g(x)$ (1)

Where A is a constant to be determined such that

$F(a) = F(b)$ (2)

Now the function $F(x)$ is the sum of two continuous and differentiable functions, therefore

(i) $F(x)$ is continuous in $[a, b]$.

(ii) $F(x)$ is differentiable in (a, b).

(iii) $F(a) = F(b)$.

Then by Rolle's theorem $\exists$ some $c \in (a,b)$, such that $F'(c) = 0$. Here

$F'(x) = f'(x) + A.g'(x)$

$F'(c)=0 \Rightarrow f'(c)+A.g'(c)=0$

$$\Rightarrow -A=\frac{f'(c)}{g'(c)} \qquad (3)$$

Now $F(a)=F(b) \Rightarrow f(a)+A.g(a)=f(b)+A.g(b)$

$$\Rightarrow -A=\frac{f(b)-f(a)}{g(b)-g(a)} \qquad (4)$$

From (3) and (4) we get

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

4.5.1 ANOTHER FORM OF CAUCHY'S MEAN VALUE THEOREM

If we put $b = a + h$, then c can be written as $a+\theta h$ where $\theta \in R$ such that $0<\theta<1$, then Cauchy's mean value theorem can be put in the form –

If $f(x)$ and $g(x)$ are continuous in $[a, a + h]$ and are differentiable in $(a, a + h)$ and $g'(x) \neq 0,\ \forall\ x \in (a,b)$, then $\exists \theta \in R: 0<\theta<1,$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)},\ 0<\theta<1.$$

Note. If $g(x) = x$, then Cauchy's mean value theorem reduces to Lagrange's mean value theorem.

4.5.2 GEOMETRICAL INTERPRETATION OF CAUCHY'S MEAN VALUE THEOREM

Cauchy's mean value theorem can be interpreted geometrically to mean that the tangents to the curve $y = f(x)$ and $y = kg(x)$ where $k=\dfrac{f(b)-f(a)}{g(b)-g(a)}$, at a certain point $c \in (a,b)$ are parallel.

4.6 TAYLOR'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER

If f (x) is a single real valued function defined on $[a, a + h]$ of x such that

(i) All the derivatives of $f(x)$ up to $(n-1)^{th}$ order are continuous in $[a, a + h]$.

(ii) $f^{n}(x)$ exists in $(a, a + h)$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

, where $0 < \theta < 1$.

Proof. Consider the function $F(x)$ defined by

$$F(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ... + \frac{(a+h-x)^{n-1}}{n-1!} f^{n-1}(x)$$

$$+ \frac{(a+h-x)^n}{n!} A ....(1)$$

Where A is a constant to be determined such that
$F(a+h) = F(a)$
Subjecting (1) to this condition we get

$$F(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} A$$

and $F(a+h) = f(a+h)$ then we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} A ....(2)$$

Now by assumption, all the functions $f, f', f'', ..., f^{n-1}$ are continuous in closed interval $[a, a+h]$ and differentiable in $(a, a+h)$. Also $(a+h-x)$, $\frac{(a+h-x)^2}{2!}, ..., \frac{(a+h-x)^{n-1}}{n-1!}, \frac{(a+h-x)^n}{n!}$ all being polynomials are continuous in $[a, a+h]$ and differentiable in $(a, a+h)$. Also A is constant.
This means $F(x)$ is continuous in $[a, a+h]$ and differentiable in $(a, a+h)$. Now differentiating (1) with respect to x.

$$F'(x) = f'(x) + [(a+h-x) f''(x) - f'(x)] + \left[\frac{(a+h-x)^2}{2!} f'''(x) - \frac{2.(a+h-x)}{2!}.f''(x)\right]$$

$$+ .... + \left[\frac{(a+h-x)^{n-1}}{n-1!} f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{n-1!} f^{n-1}(x)\right] - \frac{n.(a+h-x)^{n-1}}{n!} A$$

Simplifying, we get

$$F'(x) = \frac{(a+h-x)^{n-1}}{n-1!} f^n(x) - \frac{n.(a+h-x)^{n-1}}{n!} A$$

$$= \frac{(a+h-x)^{n-1}}{n-1!} \left[f^n(x) - A\right]$$

Since $F(x)$ satisfies all the conditions of Rolle's theorem in $[a, a+h]$, we have
$F'(a+\theta h) = 0$ where $0 < \theta < 1$, therefore

$$\frac{(a+h-a-\theta h)^{n-1}}{n-1!}\left[f^{n}(a+\theta h)-A\right]=0$$

$\Rightarrow A = f^{n}(a+\theta h)$

Putting the value of A in equation (2) we get

$$f(a+h)= f(a)+(h)\,f^{'}(a)+\frac{h^2}{2!}f^{''}(a)+....+\frac{h^{n-1}}{n-1!}f^{n-1}(a)+\frac{h^n}{n!}f^{n}(a+\theta h)$$

where $0<\theta<1$.

Here $(n+1)^{th}$ term on the right i.e., $\frac{h^n}{n!}f^{n}(a+\theta h)$ is called the Lagrange's remainder after n terms in Taylor's expression $f(a+h)$ in the ascending powers of h.

4.7 TAYLOR'S THEOREM WITH CAUCHY'S FORM OF REMAINDER

If f (x) is a single real valued function defined on $[a, a+h]$ of $x$such that

(i) All the derivatives of $f(x)$ up to $(n-1)^{th}$ order are continuous in $[a, a+h]$.

(ii) $f^{n}(x)$ exists in $(a, a+h)$, then

$$f(a+h)= f(a)+h\,f^{'}(a)+\frac{h^2}{2!}f^{''}(a)+....+\frac{h^{n-1}}{n-1!}f^{n-1}(a)+\frac{h^n}{n-1!}(1-\theta)^{n-1}\,f^{n}(a+\theta h),$$

$0<\theta<1$.

Proof. Consider the function $F(x)$ defined by

$$F(x)= f(x)+(a+h-x)\,f^{'}(x)+\frac{(a+h-x)^2}{2!}f^{''}(x)+....+\frac{(a+h-x)^{n-1}}{n-1!}f^{n-1}(x)$$
$$+(a+h-x)\,A................(1)$$

Where A is a constant to be determined such that

$F(a+h)=F(a)$

Subjecting (1) to this condition we get

$$F(a)= f(a)+h\,f^{'}(a)+\frac{h^2}{2!}f^{''}(a)+....+\frac{h^{n-1}}{n-1!}f^{n-1}(a)+h\,A$$

and $F(a+h)= f(a+h)$ then we have

$$f(a+h)= f(a)+h\,f^{'}(a)+\frac{h^2}{2!}f^{''}(a)+....+\frac{h^{n-1}}{n-1!}f^{n-1}(a)+h\,A.........(2)$$

Now by assumption, all the functions $f, f^{'}, f^{''},...., f^{n-1}$ are continuous in closed interval $[a, a+h]$ and differentiable in $(a, a+h)$. Also $(a+h-x)$, $\frac{(a+h-x)^2}{2!},....,\frac{(a+h-x)^{n-1}}{n-1!},\frac{(a+h-x)^n}{n!}$ all being

polynomials are continuous in [a, a + h] and differentiable in (a, a + h). Also A is constant.
This means F (x) is continuous in [a, a + h] and differentiable in (a, a + h). Now differentiating (1) with respect to x.

$$F'(x) = f'(x) + [(a+h-x) f''(x) - f'(x)] + \left[\frac{(a+h-x)^2}{2!} f'''(x) - \frac{2.(a+h-x)}{2!}.f''(x)\right]$$

$$+ .... + \left[\frac{(a+h-x)^{n-1}}{n-1!} f^n(x) - \frac{(n-1)(a+h-x)^{n-2}}{n-1!} f^{n-1}(x)\right] - A$$

Simplifying, we get

$$F'(x) = \frac{(a+h-x)^{n-1}}{n-1!} f^n(x) - A$$

Since F (x) satisfies all the conditions of Rolle's theorem in [a, a + h], we have

$F'(a+\theta h) = 0$ where $0 < \theta < 1$, therefore

$$\frac{(a+h-a-\theta h)^{n-1}}{n-1!} f^n(a+\theta h) - A = 0$$

$$\Rightarrow A = \frac{(1-\theta)^{n-1} h^{n-1}}{n-1!} f^n(a+\theta h)$$

Putting the value of A in equation (2) we get

$$f(a+h) = f(a) + (h) f'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{n-1!} f^n(a+\theta h)$$

where $0 < \theta < 1$.

Here (n + 1)th term on the right *i.e.*, $\frac{h^n (1-\theta)^{n-1}}{n-1!} f^n(a+\theta h)$ is called the Cauchy's remainder after n terms in Taylor's expression f (a + h) in the ascending powers of h.

Corollary 1. If we take the interval [0, x] instead of [a, a + h], so change a = 0 and h = x in Taylor's theorem with Lagrange's form of remainder we get

$$f(x) = f(0) + x f'(x) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{n-1!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

which is known as Maclaurin's theorem or Maclaurin's development of f (x) in [0, x] with Lagrange's form of remainder $\frac{x^n}{n!} f^n(\theta x)$ after n terms.

Corollary 2. If we change a = 0 and h = x in Taylor's theorem with Cauchy's form of remainder we get

$$f(x) = f(0) + x f'(x) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{n-1!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-1}}{n-1!} f^n(\theta x)$$

which is known as Maclaurin's theorem or Maclaurin's development with Cauchy's form of remainder. The $(n + 1)^{th}$term $\frac{x^n(1-\theta)^{n-1}}{n-1!} f^n(\theta x)$ is known as remainder.

Illustrative Examples

Example 1. Discuss the applicability of Rolle's theorem for $f(x) = 2 + (x-1)^{2/3}$ in [0, 2].

Solution. We have $f(x) = 2 + (x-1)^{2/3}$. Here $f(0) = 3$ and $f(2) = 3$. It shows that the third condition of Rolle's theorem is satisfied.Since $f(x)$ is an algebraic function of x, so it is continuous in [0, 2]. Thus the first condition of Rolle's theorem is also satisfied. Now

$$f'(x) = \frac{2}{3}\frac{1}{(x-1)^{1/3}}$$

We see that at $x = 1$, $f'(x)$ does not exists and $x = 1 \in (0,2)$. So the second condition of Rolle's theorem is not satisfied. Hence the Rolle's theorem is not applicable in the given function.

Example 2. Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.

Solution. We have $f(x) = x^3 - 6x^2 + 11x - 6$, which is polynomial in x of degree 3 and so it is continuous and differentiable for all real values of x. Now f (x) = 0 gives

$$x^3 - 6x^2 + 11x - 6 = 0$$
$$\Rightarrow x^3 - x^2 - 5x^2 + 5x + 6x - 6 = 0$$
$$\Rightarrow x^2(x-1) - 5x(x-1) + 6\,(x-1) = 0$$
$$\Rightarrow (x-1)(x^2 - 5x + 6) = 0$$
$$\Rightarrow (x-1)(x-3)(x-2) = 0$$
$$\Rightarrow x = 1,\ 2,\ 3$$
$$\Rightarrow f(1) = 0,\ f(2) = 0,\ f(3) = 0$$

If we take [1, 3], then all the three conditions of Rolle's theorem are satisfied. So there exists at least one vale of x in (1, 3) for which $f'(x) = 0$.Now

$$f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0$$
$$x = \frac{12 \pm \sqrt{144-132}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$
$$x = 2 + \frac{1}{\sqrt{3}},\ 2 - \frac{1}{\sqrt{3}} \in (1,3).$$

Hence Rolle's theorem is verified.

If we take [1, 2]then, the point $x = 2 - \frac{1}{\sqrt{3}} \in (1,2)$ and $f^{'}(x) = 0$ at this point. And if we take[2, 3], then $x = 2 + \frac{1}{\sqrt{3}} \in (2,3)$ for which $f^{'}(x) = 0$ at this point.

Example 3. Verify Rolle's theorem in the case of functions:

(i) $f(x) = \sin x, \quad x \in [0, \pi]$

(ii) $f(x) = (x-a)^m (x-b)^n$, where m and n are positive integers and $x \in [a,b]$.

(iii) $f(x) = x(x+3)e^{-x/2}, \quad x \in [-3, 0]$

Solution.

(i) The function $f(x) = \sin x$ is continuous in $[0, \pi]$and also differentiable in $(0, \pi)$. Also

$f(0) = f(\pi) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied. Hence at least one value of x in $(0, \pi)$ such that $f^{'}(x) = 0$. Now

$$f^{'}(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$$

Since $\frac{\pi}{2} \in (0, \pi)$. Hence Rolle's theorem is verified.

(ii) We have $f(x) = (x-a)^m (x-b)^n$, where m and n are positive integers. Here $f(x)$ is a polynomial in x of degree $m + n$. So $f(x)$ is continuous in [a, b] and is differentiable in (a, b). Also $f(a) = f(b) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied so there exists at least one point x in (a, b) such that $f^{'}(x) = 0$. Now

$$f^{'}(x) = (x-a)^m n.(x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n = 0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{n(x-a) + m(x-b)\} = 0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{-na - mb + (m+n)x\} = 0$$

$$\Rightarrow x = a,\ x = b,\ x = \frac{mb+na}{(m+n)}$$

We see that $x = \frac{mb+na}{(m+n)} \in (a,b)$ which divide the interval in the ratio of m : n. Hence Rolle's theorem is verified.

(iii) Here $f(x) = x(x+3)e^{-x/2}$. So

$$f^{'}(x) = (x^2 + 3x)e^{-x/2} . \left(-\frac{1}{2}\right) + (2x+3)e^{-x/2}$$
$$= \left(-\frac{1}{2}\right)(x^2 - x - 6)e^{-x/2}$$

which exist for every value in [-3, 0]. Hence *f* (*x*) is differentiable in (-3, 0) and so it is continuous in [-3, 0]. Also *f* (-3) = *f* (0) = 0. Hence all the conditions of Rolle's theorem are satisfied. So, there exist at least one point in (-3, 0) for which $f^{'}(x) = 0$. Now

$$f^{'}(x) = 0 \Rightarrow \left(-\frac{1}{2}\right)(x^2 - x - 6)e^{-x/2} = 0$$
$$\Rightarrow (x^2 - x - 6) = 0$$
$$\Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3, -2$$

The value $x = -2 \in (-3, 0)$ for which $f^{'}(x) = 0$. Hence Rolle's theorem is verified.

Example 4. If $f : [-2, 1] \to R$ is defined by $f(x) = |x|$, examine the validity of Lagrange's mean value theorem.

Solution. Here *a* = -2, *b* = 1 and $f(x) = |x|$. So

(i) *f* (*x*) is continuous in [-2, 1]

(ii) *f* is not differentiable in (-2, 1), for *f* is not differentiable at *x* = 0.

Hence the conditions of Lagrange's mean value theorem are not satisfied. Now

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{1-2}{1+2} = -\frac{1}{3}$$

To search $c \in (-2, 1)$ such that $f^{'}(c) = -\frac{1}{3}$. We have

$$f(x) = |x| \Rightarrow f(x) = \begin{cases} -x, & if \;\; x \in [-2, 0] \\ x, & if \;\; x \in [0, 1] \end{cases}$$

$$\Rightarrow f^{'}(x) = \begin{cases} -1, & if \;\; x \in [-2, 0] \\ 1, & if \;\; x \in [0, 1] \end{cases}$$

Hence in either case, $f^{'}(x) \neq -\frac{1}{3}$ for any $x \in (-2, 1)$. Thus neither the hypothesis nor the conclusion is valid.

Example 5. If $f(x) = (x-1)(x-2)(x-3)$ and *a* = 0, *b* = 4, find '*c*' using Lagrange's mean value theorem.

Solution. The function $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$, so

$f(a) = f(0) = -6$

$f(b) = f(4) = 6$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{6-(-6)}{4-0} = \frac{12}{4} = 3$$

Also $f'(x) = 3x^2 - 12x + 11$ gives $f'(c) = 3c^2 - 12c + 11$. Putting these values in Lagrange's mean value theorem we get

$$\frac{f(b)-f(a)}{b-a} = f'(c), \qquad a < c < b$$

$3 = 3c^2 - 12c + 11$

$\Rightarrow 3c^2 - 12c + 8 = 0$

$$\Rightarrow c = \frac{12 \pm \sqrt{144-96}}{6} \Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

as both the values of c lies in (0, 4), so both of these values of c are the required values of c.

Example 6. Compute the value of θ in the first mean value theorem $f(x+h) = f(x) + h f'(x+\theta h)$, *if* $f(x) = ax^2 + bx + c$

Solution. Here the function $f(x) = ax^2 + bx + c$, so $f(x+h) = a(x+h)^2 + b(x+h) + c$ and $f'(x) = 2ax + b \Rightarrow f'(x+\theta h) = 2a(x+\theta h) + b$. Substituting all these values in $f(x+h) = f(x) + h f'(x+\theta h)$

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x+\theta h) + b] \qquad \text{(A)}$$

The relation (*A*) is true for all values of *x*. Hence when $x \to 0$ we get

$$ah^2 + bh + c = c + h[2a\theta h + b]$$

$$\Rightarrow ah^2 = 2a\theta h^2$$

$$\Rightarrow \theta = \frac{1}{2}$$

Hence $\theta = \frac{1}{2}$.

Example 7. Verify Cauchy's mean value theorem for the functions x^2 and x^3 in [1, 2].

Solution. Let $f(x) = x^2$ and $g(x) = x^3$, then $f(x)$ and $g(x)$ are continuous in [1, 2] and differentiable in (1, 2). Also $g'(x) = 3x^2 \neq 0$ for any $x \in (1, 2)$. Hence by Cauchy's mean value theorem there exists at least one $c \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{4-1}{8-1} = \frac{f^{'}(c)}{g^{'}(c)} = \frac{2c}{3c^2} = \frac{2}{3c} \Rightarrow \frac{3}{7} = \frac{2}{3c}$$

$$\Rightarrow 9c = 14 \Rightarrow c = \frac{14}{9} \in (1,2).$$

Example 8. If in the Cauchy's mean value theorem, we write

(i) $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, then c is the geometric mean of a and b. And if

(ii) $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, then c is the harmonic mean of a and b.

Solution.

(i) Cauchy's mean value theorem states

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{'}(c)}{g^{'}(c)} \qquad (1)$$

Given $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, so that $f^{'}(x) = \frac{1}{2\sqrt{x}}$ and $g(x) = -\frac{1}{2.x^{3/2}}$. Putting these values in (1) we get

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2.c^{3/2}}} = -\frac{c^{3/2}}{c^{1/2}} = -c$$

$-\sqrt{ab} = -c \Rightarrow c = \sqrt{ab}$. Therefore c is the geometric mean of a and b.

(ii) Given $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, so that $f^{'}(x) = -\frac{1}{x^2}$ and $g(x) = -\frac{2}{x^3}$. Putting these values in (1) we get

$$\frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{b^2}-\frac{1}{a^2}} = \frac{-\frac{1}{c^2}}{-\frac{2}{c^3}} = \frac{c}{2}$$

$$\frac{ab}{(a+b)} = \frac{c}{2}$$

$$c = \frac{2ab}{(a+b)}.$$

Therefore c is the harmonic mean of a and b.

4.8 SUMMARY

In this unit we have explain the Rolle's theorem Lagrange's mean value theorem,Cauchy's theorem,and its geometrical interpretation. In this unit also discussed Mean value theorems of higher derivatives.

1. **Rolle's theorem:**

If $f(x)$ is a real valued function defined in the closed interval $[a, b]$ such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$.
(ii) $f(x)$ is differentiable in the open interval (a, b).
(iii) $f(a) = f(b)$, then there exists at least one value of x say c where $a<c<b$, such that $f'(c) = 0$.

2. Lagrange's mean value theorem

If a real valued function $f(x)$ defined on $[a, b]$ such that

(i) $f(x)$ is continuous on $[a, b]$.
(ii) $f(x)$ is differentiable in (a, b).

Then there exists a point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

3. Cauchy's mean value theorem

Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ such that

(i) $f(x)$ and $g(x)$ are continuous on $[a, b]$
(ii) $f(x)$ and $g(x)$ are differentiable on (a, b) and
(iii) $g'(x) \neq 0, \quad \forall\ x \in (a,b)$,

Then there exists a point $c \in (a,b)$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

4.9 GLOSSARY

i. Sets-Well defined collection of objects
ii. Continuous-sketch its curve on a graph without lifting your pen even once
iii. Discontinuity-lack of continuity

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CHECK YOUR PROGRESS

1. Rolle's theorem is applicable for $f(x) = \sin x$ in $[0, 2\pi]$.T/F

2. Rolle's theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.T/F

3. Lagrange's mean value theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.T/F

4. Rolle's theorem is not applicable for $f(x) = x(x+2)e^{-x/2}$ in $[-2, 0]$.T/F

5. The value of 'c' of Lagrange's mean value theorem for the function $f(x) = 2x^2 + 3x + 4$ in $[1, 2]$ is $c = 5/4$.T/F

4.12 TERMINAL QUESTIONS

(TQ-1) Verify Lagrange's mean value theorem for the function $f : [-1, 1] \to R$ given by $f(x) = x^3$.

(TQ-2) Verify Rolle's theorem for the following functions:

(i) $f(x) = (x-4)^5(x-3)^4$ in [3, 4]

(ii) $f(x) = x^3 - 4$ in [-2, 2]

(iii) $f(x) = e^x[\sin x - \cos x]$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

(iv) $f(x) = 10x - x^2$ in [0, 10]

(TQ-3) Discuss the applicability of Rolle's theorem for the function

$$f(x) = \begin{cases} x^3 - 4, & if \quad 0 \le x \le 1 \\ 3 - x, & if \quad 1 \le x \le 2 \end{cases}$$

(TQ-4) Verify the truth of the Rolle's theorem for the function $f(x) = |x-2|$ in [1, 3]. Justify your answer with correct reason.

(TQ-5) If $f(x) = \dfrac{1}{x}$ in [-1, 1], will the Lagrange's mean value theorem be applicable to $f(x)$?

(TQ-6) Find c of the Lagrange's mean value theorem if $f(x) = x(x-1)(x-2); a = 0, b = \dfrac{1}{2}$.

(TQ-7) Find c of the Lagrange's mean value theorem when

(i) $f(x) = x^3 - 3x - 2$ in [-2, 3]

(ii) $f(x) = 2x^2 + 3x + 4$ in [1, 2]

(iii) $f(x) = x(x-1)$ in [1, 2]

(TQ-8) If $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(\theta x)$, find the value of θ as $x \to 1$, $f(x)$ being $(x-1)^{5/2}$.

(TQ-9) If $f(x+h) = f(x) + h f'(x) + \dfrac{h^2}{2!} f''(x + \theta h)$, find the value of θ as $x \to a$, $f(x)$ being $(x-a)^{5/2}$.

(TQ-10) Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3, ∞].

FILL IN THE BLANKS

(TQ-11) In Rolle's theorem if f (x) is continuous in [a, b], differentiable in (a, b) and f (a) = f (b) then ....

(TQ-12) In Lagrange's mean value theorem if f (x) is continuous in [a, b], differentiable in (a, b) then there exists $c \in (a, b)$ such that ....

(TQ-13) In Cauchy's mean value theorem if $f\ (x)$ and $g\ (x)$ are continuous in $[a,\ b]$, differentiable in $(a,\ b)$ and $g^{'}(x) \neq 0\ \forall\, x \in (a,\ b)$ then $\exists c \in (a,\ b)$ such that ....

(TQ-14) The remainder term in the Taylor's theorem with Lagrange's form of remainder is ....

(TQ-15) The remainder term in the Taylor's theorem with Cauchy's form of remainder is ....

MULTIPLE CHOICE QUESTIONS

(TQ-16) The value of 'c' of Rolle's theorem for the function $f(x) = e^x \sin x$ in $[0,\ \pi]$ is given by

a) $c = \dfrac{3\pi}{4}$ ***b***) $c = \dfrac{\pi}{4}$

c) $c = \dfrac{\pi}{2}$ ***d***) $c = \dfrac{5\pi}{6}$

(TQ-17) The value of 'c' of Lagrange's mean value theorem for the function $f(x) = x(x-1)$ in $[1,\ 2]$ is given by

a) $c = \dfrac{5}{4}$ ***b)*** $c = \dfrac{3}{2}$

c) $c = \dfrac{7}{4}$ ***d)*** $c = \dfrac{11}{6}$

(TQ-18) The function $f(x) = \sin x$ is increasing in the interval

a) $[0,\ \pi]$ ***b***) $\left[0,\ \dfrac{\pi}{2}\right]$

c) $\left[\dfrac{\pi}{4},\ \dfrac{3\pi}{4}\right]$ ***d***) $\left[\dfrac{\pi}{2},\ \pi\right]$

(TQ-19) Out of the following functions, tell the function for which the conditions of Rolle's theorem are satisfied.

a) $f(x) = |x|$ in [-1, 1] ***b***) $f(x) = x^2$ in [2, 3]

c) $f(x) = \sin x$ in $[0,\ \pi]$ ***d***) $f(x) = \tan x$ in $[0,\ \pi]$

(TQ-20) In Taylor's theorem with Cauchy's form of remainder, the remainder term is

a) $\dfrac{h^n}{n!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ ***b***) $\dfrac{h^n}{n-1!}(1-\theta)^{n-1} f^{(n-1)}(a+\theta h)$

c) $\dfrac{h^n}{n-1!}(1-\theta)^n f^{(n)}(a+\theta h)$ ***d***) $\dfrac{h^n}{n-1!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$

4.13 ANSWERS

CHECK YOUR PROGRESS

SCQ1. T
SCQ2. F
SCQ3. F
SCQ4. F
SCQ5. F

TERMINAL QUESTIONS

(TQ-1) Lagrange's mean value theorem is truly verified.
(TQ-2) Rolle's theorem is verified in each case.
(TQ-3) The given function is not differentiable at x = 1 and so Rolle's theorem is not applicable to the given function in [0, 2].
(TQ-4) The function does not satisfy the second condition of the Rolle's theorem that $f(x)$ must be differentiable in (1, 3).
(TQ-5) Not applicable.
(TQ-6) $c = 1 - \dfrac{\sqrt{21}}{6}$.
(TQ-7) (i) $\pm\sqrt{7/3}$ (ii) $3/2$ (iii) $3/2$.
(TQ-8) $\theta = \dfrac{9}{25}$.
(TQ-9) $\theta = \dfrac{64}{225}$.
(TQ-10) Rolle's theorem is verified.
(TQ-16) $\exists c \in (a, b)$ such that $f'(c) = 0$
(TQ-17) $\dfrac{f(b)-f(a)}{b-a} = f'(c)$
(TQ-18) $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f'(c)}{g'(c)}$
(TQ-19) $\dfrac{h^n}{n!} f^{(n)}(a+\theta h)$
(TQ-20) $\dfrac{h^n}{n-1!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$

(TQ-21) (a)
(TQ-22) (b)
(**TQ-23)** (b)
(TQ-24) (c)
(TQ-25) (d)

BLOCK – II
EXPANSION OF FUNCTIONS AND INDETERMINATE FORMAND INTEGRALS

UNIT-5:- INDETERMINATE FORMS

CONTENTS:-

5.1 INTRODUCTION

Indeterminate Forms is found in English as a chapter title in 1841 in An Elementary Treatise on Curves, Functions, and Forces by Benjamin Pierce. Forms such as $\frac{0}{0}$ are called singular values and singular forms in in 1849 in An Introduction to the Differential and Integral Calculus, 2nd ed., by James Thomson. In this unit we shall study some unusual forms. Generally, these are undefined forms, so are called "Indeterminate forms". We cannot find their actual value, but we try to find their limiting value with different methods.

There are seven standard indeterminate forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty\ , \infty-\infty, 0^0, \infty^0 \text{ and } 1^\infty$$

Let us discuss the case of $\frac{0}{0}$

Suppose we have $\frac{x}{x}, x \in \mathbb{R}$.. We generally say that

$\frac{x}{x} = 1$ … (1)

$\frac{0}{x} = 0$ … (2)

$\frac{x}{0} = \pm\infty$, depending upon the sign of x. …(3)

If we put $x = 0$ in equation (1), (2) and (3), we get

$\frac{0}{0} = 1$
$\frac{0}{0} = 0$
$\frac{0}{0} = +\infty\ or\ -\infty$

You might have confused what is happening!
Actually division by '0' is not permissible. It means it is impossible to define the case$\frac{0}{0}$!
Again we know that

$$1^2 = 1 \times 1 = 1$$
$$1^3 = 1 \times 1 \times 1 = 1$$
$$\ldots$$
$$\ldots$$
$$1^n = \underbrace{1 \times 1 \times \ldots \times 1}_{n\ times} = 1$$

Generally we say that if we multiply 1 with itself any number of times, it will be 1. But it is true only for finite n. We shall discuss ahead that the limiting value of 1^∞ goes towards exponential (e) !

Also we can discuss another three possible cases as $(x > 0)$

$x - x = 0$ …(4)
$\infty - x = +\infty$ …(5)
$x - \infty = -\infty$ …(6)

But when we take $x = \infty$ in the equation (4), (5) and (6), we get the absurd results,

$$\infty - \infty = 0$$
$$\infty - \infty = +\infty$$
$$\infty - \infty = -\infty$$

So, we have discussed some of the cases, which justify that why these forms are indeterminate!

5.2 OBJECTIVES

After complition of this unit, we shall understand

- **i.** The nature of indeterminate forms.
- **ii.** Standard indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty, 0^0, \infty^0$, and 1^∞.
- **iii.** The evaluating procedures of limits of different indeterminate forms by Algebraic Methods and L'Hospital's Rule.
- **iv.** L'Hospital's Rule for $\frac{0}{0}$and $\frac{\infty}{\infty}$.
- **v.** Exponential forms.
- **vi.** Limits of other indeterminate forms.

5.3 THE EVALUATING PROCEDURES OF LIMITS OF DIFFERENT INDETERMINATE FORMS

Algebraic Methods

In cases where the expansion of functions involved are known, or some of the limits are known, algebraic method may be used to solve the problems. The following expansions should be remembered:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty,\ |x| < 1$
- $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \infty\right),\ |x| < 1$
- $sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \infty$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \infty$
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \infty$
- $(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n a^n$
- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots.$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots,\ |x| < 1$
- $a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \frac{x^3}{3!} (\log a)^3 + \cdots$

5.4 L'HOSPITAL'S RULE

We shall discuss Taylor's series in the Block II inspired by this result, the famous French mathematicians De L'Hospital devised a method to find the values of $\lim_{x \to a} \frac{f(x)}{g(x)}$ in the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Case I: Let us see how it works, with the help of Taylor's series. We take the form $\frac{0}{0}$.

Let us take $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(a) = 0,\ g(a) = 0$ and both functions are indefinitely differentiable at $x = a$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a + (x-a))}{g(a + (x-a))} \left[\frac{0}{0}\ form\right]$$

$$= \lim_{x\to a}\left\{\frac{f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots}{g(a)+\frac{(x-a)}{1!}g'(a)+\frac{(x-a)^2}{2!}g''(a)+\cdots}\right\}$$

$$= \lim_{x\to a}\left\{\frac{\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots}{\frac{(x-a)}{1!}g'(a)+\frac{(x-a)^2}{2!}g''(a)+\cdots}\right\} \quad as\ f(a)=0$$

$$= g(a)$$

$$= \lim_{x\to a}\left\{\frac{(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots}{(x-a)g'(a)+\frac{(x-a)^2}{2!}g''(a)+\cdots}\right\}$$

Taking $(x-a)$ as common factor, we get

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

$$= \lim_{x\to a}\left\{\frac{f'(a)+\frac{(x-a)^1}{2!}f''(a)+\cdots}{g'(a)+\frac{(x-a)^1}{2!}g''(a)+\cdots}\right\} \qquad (1)$$

$=\frac{f'(a)}{g'(a)}$, provided both $f'(a)$ and $g'(a)$ are simultaneously not zero.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \qquad (2)$$

If $f'(a)=0=g'(a)$, then again from equation (1), we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left\{\frac{\frac{(x-a)^1}{2!}f''(a)+\frac{(x-a)^2}{3!}f'''(a)+\cdots}{\frac{(x-a)^1}{2!}g''(a)+\frac{(x-a)^2}{3!}g'''(a)+\cdots}\right\}$$

$$= \lim_{x\to a}\left\{\frac{\frac{(x-a)^1}{2!}\left[f''(a)+\frac{(x-a)^2}{3}f'''(a)+\cdots\right]}{\frac{(x-a)^1}{2!}\left[g''(a)+\frac{(x-a)^2}{3}g'''(a)+\cdots\right]}\right\}$$

$$= \lim_{x\to a}\left\{\frac{\left[f''(a)+\frac{(x-a)^2}{3}f'''(a)+\cdots\right]}{\left[g''(a)+\frac{(x-a)^2}{3}g'''(a)+\cdots\right]}\right\}$$

$=\frac{f''(a)}{g''(a)}$, provided both $f''(a)$ and $g''(a)$ are simutaneously not zero.

$$\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$$

Continuing in this way, we can get

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

Similarly, we can find the same expression in case of $\frac{\infty}{\infty}$.

Note:(1) L'Hospital's rule is totally different from the quotient law of differentiation. There is a solid logical base that why we only differentiate numerator and denominator directly, instead of using quotient law of differentiation.

(2) It must be clearly remembered that L'Hospital's method be used only in the situations of $\frac{0}{0}$ and $\frac{\infty}{\infty}$ not in other cases.

(3) In L'Hospital's rule, numerator $f(x)$ and denominator $g(x)$ are to be differentiated separately.

(4) It may be helpful for learners that $\log_e 1 = 0, \log_e 0 = -\infty, \log_e \infty = +\infty, e^0 = 1, e^{-\infty} = 0$, $e^{\infty} = \infty$.

Ex.1 Evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x}$

Sol. Clearly, $\lim\limits_{x\to 0} \frac{\sin x}{x}$ is a $\frac{0}{0}$ form.

Algebraic Method:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \infty\right]}{x} \\ &= \lim_{x\to 0} \frac{x\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \infty\right]}{x} \\ &= \lim_{x\to 0} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \infty\right] \\ &= 1. \end{aligned}$$

L'Hospital'sRule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Note: In second method, dash ($'$) above $\sin x$ and x represents the first derivative with respect to x (variable with respect to the limit has been taken).

Ex.2 Evaluate $\lim\limits_{x\to 0} \frac{\log x}{x-1}$

Sol. Clearly, $\lim\limits_{x\to 0} \frac{\log x}{x-1}$ is a $\left(\frac{0}{0}\right)$ form.

Algebraic Method:

$$\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{\log\big(1 + (x-1)\big)}{(x-1)}$$

$$= \lim_{x\to 1} \frac{\left\{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots\right\}}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)\left\{1-\frac{(x-1)^1}{2}+\frac{(x-1)^2}{3}-\frac{(x-1)^3}{4}+\cdots\right\}}{(x-1)}$$

$$= \lim_{x\to 1} \left\{1-\frac{(x-1)^1}{2}+\frac{(x-1)^2}{3}-\frac{(x-1)^3}{4}+\cdots\right\}$$

$$= 1$$

L'Hospital's Rule:

$$\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{(\log x)'}{(x-1)'} = \lim_{x\to 1} \frac{\left(\frac{1}{x}\right)}{1} = 1.$$

Ex.3 Find $\lim_{x\to 0} \frac{x-\sin x}{x^2}$

Sol. **Algebraic Method:**

$$\lim_{x\to 0} \frac{(x-\sin x)}{x^3} = \lim_{x\to 0} \frac{\left\{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\infty\right)\right\}}{x^3} \qquad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{\left\{\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots\infty\right\}}{x^3}$$

$$= \lim_{x\to 0} \frac{x^3\left\{\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots\infty\right\}}{x^3}$$

$$= \lim_{x\to 0} \left\{\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots\infty\right\}$$

$$= \frac{1}{3!} = \frac{1}{6}.$$

L'Hospital's Rule:

$$\lim_{x\to 0} \frac{(x-\sin x)}{x^3} = \lim_{x\to 0} \frac{(x-\sin x)'}{(x^3)'}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{3x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)'}{(3x^2)'}$$

$$= \lim_{x\to 0} \frac{\{-(-\sin x)\}}{6x}$$

$$= \lim_{x\to 0} \frac{(\sin x)'}{(6x)'}$$

$$= \lim_{x\to 0} \frac{(\cos x)}{6}$$

$$= \frac{1}{6}.$$

Ex.4 Find $\lim_{x\to 0} \frac{e^x-1}{x}$

Sol. **Algebraic Method:**

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{\left\{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right\}-1}{x} \qquad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{\left\{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right\}}{x}$$

$$= \lim_{x\to 0} \frac{x\left\{1 + \frac{x^1}{2!} + \frac{x^2}{3!} + \cdots\right\}}{x}$$

$$= \lim_{x\to 0} \left\{1 + \frac{x^1}{2!} + \frac{x^2}{3!} + \cdots\right\}$$

$$= 1.$$

L'Hospital's Method:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{(e^x - 1)'}{(x)'} = \lim_{x\to 0} \frac{e^x}{1} = 1.$$

Ex.5 Evaluate $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$

Sol. **Algebraic Method:**

$$\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2} \qquad \mathbf{\frac{0}{0}\ form}$$

$$= \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \left(\frac{1}{2!} + \frac{1}{3}\right)x^3 + termscontaininghigherpowersofx}{x^2}$$

$$= \lim_{x\to 0} \frac{x^2\left\{\frac{1}{2} - \frac{5}{6}x + termscontainingx\right\}}{x^2}$$

$$= \lim_{x\to 0} \left\{\frac{1}{2} - \frac{5}{6}x + terms\ containing\ x\right\}$$

$$= \frac{1}{2}.$$

L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2} \qquad \mathbf{\frac{0}{0}\ form}$$

$$= \lim_{x\to 0} \frac{\{x\cos x - \log(1+x)\}'}{(x^2)'}$$

$$= \lim_{x\to 0} \frac{1.\cos x - x(\sin x) - \frac{1}{(1+x)}}{2x} \qquad \mathbf{\frac{0}{0}\ form)}$$

$$= \lim_{x\to 0} \frac{(\cos x - x\sin x - \frac{1}{(1+x)})'}{(2x)'}$$

$$= \lim_{x\to 0}\frac{-\sin x-(1.\sin x+x.\cos x)+\dfrac{1}{(1+x)^2}}{2}$$

$$= \lim_{x\to 0}\frac{-2\sin x-x.\cos x+\dfrac{1}{(1+x)^2}}{2}$$

$$=\frac{1}{2}.$$

Note: Observe that L'Hospital's rule is sometimes easier than the algebraic method. We will explain next examples only by L'Hospital's rule.

Ex.6 Evaluate $\lim\limits_{x\to 0}\dfrac{\log(1-x^2)}{\log\cos x}$

Sol. $\lim\limits_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}$ $\quad\frac{\mathbf{0}}{\mathbf{0}}$ **form**

$$= \lim_{x\to 0}\frac{\{\log(1-x^2)\}'}{(\log\cos x)'}$$

$$= \lim_{x\to 0}\frac{\left\{\dfrac{-2x}{(1-x^2)}\right\}}{\left\{\dfrac{-\sin x}{\cos x}\right\}}$$

$$= \lim_{x\to 0}\frac{2x\cos x}{(1-x^2)\sin x}$$

$$= \lim_{x\to 0}\frac{2(1.\cos x-x.\sin x)}{(-2x).\sin x+(1-x^2).\cos x}$$

$$= \lim_{x\to 0}\frac{2\cos x-2x\sin x}{-2x\sin x+\cos x-x^2\cos x}$$

$$=\frac{2}{1}=2$$

Ex.7 Find $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$

Sol. $\lim\limits_{x\to 0}\frac{(1+x)^n-1}{x}=\lim\limits_{x\to 0}\frac{\{(1+x)^n-1\}'}{\{x\}'}$

$$= \lim_{x\to 0}\frac{n(1+x)^{n-1}}{1}$$

$$= n$$

Ex.8 Evaluate $\lim\limits_{x\to 0}\dfrac{a^x-x^a}{x^x-a^a}$

Sol. $\lim\limits_{x\to a}\frac{a^x-x^a}{x^x-a^a}=\lim\limits_{x\to a}\frac{(a^x-x^a)'}{(x^x-a^a)'}=\lim\limits_{x\to a}\frac{a^x\log a-a.x^{a-1}}{x^x(\log x+1)}$

$$=\frac{a^a\log a-a.a^{n-1}}{a^a\log a+a^a}=\frac{a^a(\log a-1)}{a^a(\log a+1)}=\frac{\log a-1}{\log a+1}$$

Note: The first derivate of x^x in above example calculated as follows:

$$y = x^x$$

Taking logarithms

$$\log y = x \log x$$

Now differentiating both sides with respect to x

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \log x + 1$$

$$\left(\frac{dy}{dx}\right) = \mathrm{y}(\log x + 1) = x^x(\log x + 1).$$

Ex.9 Evaluate $\lim\limits_{x\to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$

Sol. $\lim\limits_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x}$

$$= \lim_{x\to 0} \frac{(5\sin x - 7\sin 2x + 3\sin 3x)'}{(\tan x - x)'}$$

$$= \lim_{x\to 0} \frac{5\cos x - 7\times 2\cos 2x + 3\times 3\cos 3x}{sec^{\,2}\, x - 1}$$

$$= \lim_{x\to 0} \frac{(5\cos x - 14\cos 2x + 9\cos 3x)'}{(sec^{\,2}\, x - 1)'}$$

$$= \lim_{x\to 0} \frac{-5\sin x + 14.2\sin 2x - 9.3\sin 3x}{2\sec x\tan x}$$

$$= \lim_{x\to 0} \frac{-5\cos x + 28.2\cos 2x - 27.3\cos 3x}{2(\sec x\, sec^2 x + \sec x\tan x\tan x)}$$

$$= \frac{-5 + 56 - 81}{2} = \frac{-30}{2} = -15\,.$$

Ex.10 Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

Sol. $\lim\limits_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim\limits_{x\to 0} \frac{(e^x - e^{\sin x})'}{(x - \sin x)'}$

$$= \lim_{x\to 0} \frac{e^x - \cos x\,.\,e^{\sin x}}{1 - \cos x}$$

$$= \lim_{x\to 0} \frac{(e^x - \cos x\,.\,e^{\sin x})'}{(1 - \cos x)'}$$

$$= \lim_{x\to 0} \frac{e^x - \{\cos x\,.\cos x.\, e^{\sin x} + (-\sin x).\, e^{\sin x}\}}{\sin x}$$

$$= \lim_{x\to 0} \frac{\{e^x - \cos^2 x\,.\,e^{\sin x} + \sin x.\, e^{\sin x}\}'}{(\sin x)'}$$

$$= \lim_{x\to 0} \frac{e^x - \left\{2\cos x\,.(-\sin x).\, e^{\sin x} + \cos^3 x\,.\,e^{\sin x}\right\} + \{\cos x.\; e^{\sin x} + \sin x\cos x.\, e^{\sin x}\}}{\cos x}$$

$$= \lim_{x\to 0} \frac{e^x + 3\sin x\cos x.\; e^{\sin x} - \cos^3 x\,.\,e^{\sin x} + \cos x.\; e^{\sin x}}{\cos x}$$

$$= \frac{1 - 1 + 1}{1} = 1.$$

Case II: Form $\frac{\infty}{\infty}$

Ex.11 Evaluate $\lim\limits_{n\to\infty} \frac{n^2+5}{n^2+4n+3}$

Sol. Clearly, $\lim\limits_{n\to\infty} \frac{n^2+5}{n^2+4n+3}$ is a $\frac{\infty}{\infty}$ form.

Algebraic Method:

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4n+3} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{5}{n^2}\right)}{n^2\left(1+\frac{4}{n}+\frac{3}{n^2}\right)} = \lim_{n\to\infty} \frac{\left(1+\frac{5}{n^2}\right)}{\left(1+\frac{4}{n}+\frac{3}{n^2}\right)} = 1 .$$

L'Hospital's Method:

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4n+3}$$
$$= \lim_{n\to\infty} \frac{(n^2+5)'}{(n^2+4n+3)'}$$
$$= \lim_{n\to\infty} \frac{2n}{2n+4} \quad \text{(Again } \tfrac{\infty}{\infty} \text{ form)}$$
$$= \lim_{n\to\infty} \frac{(2n)'}{(2n+4)'}$$
$$= \lim_{n\to\infty} \frac{2}{2}$$
$$= 1 .$$

Ex.12 Evaluate $\lim\limits_{x\to 0} \frac{\log x}{\cot x}$

Sol. This is of the form $\frac{\infty}{\infty}$. We have therefore,

$$\lim_{x\to 0} \frac{\log x}{\cot x} = \lim_{x\to 0} \frac{(\log x)'}{(\cot x)'} = \lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{-cosec^2 x} \left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to 0} \frac{-\sin^2 x}{x} \left(\frac{0}{0}\ form\right)$$
$$= \lim_{x\to 0} \frac{-2\sin x \cos x}{1} = 0.$$

Ex.13 Find $\lim\limits_{x\to \frac{\pi}{2}} \frac{\log(x-\frac{\pi}{2})}{\tan x}$

Sol. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\log(x-\frac{\pi}{2})}{\tan x}$ is a $\frac{\infty}{\infty}$ form.

We have,

$$\lim_{x\to\frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{[\log\left(x-\frac{\pi}{2}\right)]'}{(\tan x)'} = \lim_{x\to\frac{\pi}{2}} \frac{\left(\frac{1}{\left(x-\frac{\pi}{2}\right)}\right)}{sec^2 x}\left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{cos^2 x}{\left(x-\frac{\pi}{2}\right)}\ (\frac{0}{0}\ form)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(cos^2 x)'}{\left(x-\frac{\pi}{2}\right)'}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-2\cos x \sin x}{1}$$

$$= 0.$$

Ex.14 Evaluate $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)}$

Sol. $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x-e^a)}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to a} \frac{(\log(x-a))'}{(\log(e^x-e^a))'}$

$$= \lim_{x\to a} \frac{\left(\frac{1}{x-a}\right)}{\left(\frac{1}{e^x-e^a}\right)e^x}$$

$$= \lim_{x\to a} \frac{e^x - e^a}{(x-a)e^x}\left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to a} \frac{(e^x-e^a)'}{[(x-a)e^x]'}$$

$$= \lim_{x\to a} \frac{e^x}{(x-a)e^x + e^x}$$

$$= \lim_{x\to a} \frac{e^x}{[(x-a)+1]e^x}$$

$$= \lim_{x\to a} \frac{1}{[(x-a)+1]}$$

= 1

Ex.15 Find $\lim_{x\to\infty} \frac{e^x + 3x^3}{4e^x + 4x}$

Sol. $\lim_{n\to\infty} \frac{e^x+3x^3}{4e^x+4x}\left(\frac{\infty}{\infty}\ form\right)$

$$= \lim_{n\to\infty} \frac{(e^x+3x^3)'}{(4e^x+4x)'}$$

$$= \lim_{n\to\infty} \frac{e^x+9x^2}{4e^x+4}\left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{n\to\infty} \frac{(e^x+9x^2)'}{(4e^x+4)'}$$

$$= \lim_{n\to\infty} \frac{e^x+18x^1}{4e^x}\left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{n\to\infty} \frac{(e^x + 18x^1)'}{(4e^x)'}$$
$$= \lim_{n\to\infty} \frac{(e^x + 18)}{4e^x}\left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{n\to\infty} \frac{e^x}{4e^x}$$
$$= \frac{1}{4}.$$

Ex.16 Evaluate $\lim_{x\to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$

Sol. We have,

$$\lim_{x\to 0} \frac{\log(tan^2 2x)}{\log(tan^2 x)}\left(\frac{\infty}{\infty} form\right) = \lim_{x\to 0} \frac{2\log(\tan 2x)}{2\log(\tan x)}\left(\frac{\infty}{\infty} form\right)$$
$$= \lim_{x\to 0} \frac{(\log(\tan 2x))'}{(\log(\tan x))'} = \lim_{x\to 0} \frac{\left(\frac{1}{\tan 2x}\right).\,2\,sec^2 2x}{\left(\frac{1}{\tan x}\right).\ sec^2 x}$$
$$= \lim_{x\to 0} \frac{2\tan x\, cos^2 x}{\tan 2x\, cos^2 2x} = \lim_{x\to 0} \frac{2\sin x \cos x}{\sin 2x \cos 2x}$$
$$= \lim_{x\to 0} \frac{\sin 2x}{\sin 2x \cos 2x} = \lim_{x\to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1.$$

Ex.17 Evaluate $\lim_{x\to 0} \frac{\log(\sin\ x)}{\cot x}$

Sol. We have,

$$\lim_{x\to 0} \frac{\log(\sin x)}{\cot x}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to 0} \frac{(\log(\sin x))''}{(\cot x)'}$$
$$= \lim_{x\to 0} \frac{\frac{1}{\sin x}.\cos x}{-cosec^2 x} = \lim_{x\to 0}\left(-\frac{\cos x}{\sin x}.sin^2 x\right)$$
$$= \lim_{x\to 0}(-\cos x.\sin x) = 0.$$

Ex.18 Find $\lim_{n\to\infty} \frac{x^n}{e^x}$, where n is a positive integer.

Sol. We have,

$$\lim_{x\to\infty} \frac{x^n}{e^x}\left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to\infty} \frac{(x^n)'}{(e^x)'} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to\infty} \frac{(nx^{n-1})'}{(e^x)'} = \lim_{n\to\infty} \frac{n(n-1)x^{n-2}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{n\to\infty} \frac{(n(n-1)x^{n-2})'}{(e^x)'}$$
$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)x^{n-3}}{e^x}\left(\frac{\infty}{\infty}\ form\right)$$

Repeating this process, we get

$$= \lim_{n\to\infty} \frac{(n(n-1)(n-2)\ldots n\ factors)}{e^x}$$

$$= \lim_{n\to\infty} \frac{n!}{e^x} = \frac{n!}{e^\infty} = \frac{n!}{\infty} = 0.$$

Ex.19 Find $\lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x}$

Sol. We have,

$$\lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x} \left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to 0} \frac{(\log \sin 2x)'}{(\log \sin x)'}$$
$$= \lim_{x\to 0} \frac{\left(\frac{2}{\sin 2x} . \cos 2x\right)}{\left(\frac{1}{\sin x} . \cos x\right)}$$
$$= \lim_{x\to 0} \frac{2 \cot 2x}{\cot x} \left(\frac{\infty}{\infty} form\right)$$
$$= \lim_{x\to 0} \frac{(2 \cot 2x)'}{(\cot x)'}$$
$$= \lim_{x\to 0} \frac{-4\ cosec^2\ 2x}{-cosec^2\ x} \left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to 0} \frac{4\ sin^2 x}{sin^2 2x}$$
$$= \lim_{x\to 0} \frac{4\ sin^2 x}{(2 \sin x \cos x)^2}$$
$$= \lim_{x\to 0} \frac{1}{cos^2 x} = 1.$$

Ex.20 Find $\lim_{x\to\infty} \frac{\log x}{a^x}, a > 1.$

Sol. We have,

$$\lim_{x\to\infty} \frac{\log x}{a^x} \left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to\infty} \frac{(\log x)'}{(a^x)'}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{a^x \log a}$$
$$= \frac{1}{\log a} \lim_{x\to\infty} \frac{1}{x\ a^x}$$
$$= \frac{1}{\log a} \times 0 = 0.$$

5.5 OTHER INDETERMINATE FORMS

Forms $0 \times \infty$ and $\infty - \infty$

These forms can easily be reduced either to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply previous method.

(I) Consider $\lim\limits_{x\to a} f(x).g(x)$, when $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$, then

$$\lim_{x\to a} f(x).\,g(x)\;(0\times\infty\; form) = \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)}\left(\frac{0}{0}\; form\right)$$

Or

$$\lim_{x\to a} f(x).\,g(x)\;(0\times\infty\; form) = \lim_{x\to a}\frac{g(x)}{\left(\frac{1}{f(x)}\right)}\left(\frac{\infty}{\infty}\; form\right).$$

Consider $\lim\limits_{x\to a} f(x)-g(x)$,

when $\lim\limits_{x\to a} f(x)=\infty$, and $\lim\limits_{x\to a} g(x)=\infty$, then

$$\lim_{x\to a} f(x)-g(x)\;(\infty-\infty\; form)$$

$$= \lim_{x\to a}\frac{f(x).\,g(x)[f(x)-g(x)]}{f(x).\,g(x)}$$

$$= \lim_{x\to a}\frac{\left(\frac{f(x)-g(x)}{f(x).\,g(x)}\right)}{\left(\frac{1}{f(x).\,g(x)}\right)}$$

$$= \lim_{x\to a}\frac{\left(\frac{f(x)}{f(x).\,g(x)}-\frac{g(x)}{f(x).\,g(x)}\right)}{\left(\frac{1}{f(x).\,g(x)}\right)}$$

$$= \lim_{x\to a}\frac{\left(\frac{1}{g(x)}-\frac{1}{f(x)}\right)}{\left(\frac{1}{f(x).\,g(x)}\right)}\left(\frac{0}{0}\; form\right)$$

Ex.21 Evaluate $\lim\limits_{x\to 0} x\log x = 0$.

Sol. We have,

$$=\lim_{x\to 0} x\log x\,(0\times\infty\; form)$$

$$= \lim_{x\to 0}\frac{\log x}{\left(\frac{1}{x}\right)}\left(\frac{\infty}{\infty}\; form\right) = \lim_{x\to 0}\frac{(\log x)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{x}\right)}{-\left(\frac{1}{x}\right)^2}$$

$$= \lim_{x\to 0}(-x) = 0.$$

Ex.22. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-\tan x)$.

Sol. We have,

$$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)(\infty-\infty\; form) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right)\left(\frac{0}{0}\; form\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)'}{(\cos x)'}$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} = 0.$$

5.6 THE FORMS $0^0, 1^\infty$ AND ∞^0

These indeterminate forms can be made to depend upon one of the previous forms.

Consider, $y = f(x)^{g(x)}$, taking logarithm

$$\log y = g(x) \log f(x)$$

In any of the above forms, $\log y$ takes the form $0 \times \infty$.

Ex.23 Evaluate $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Sol. Let $\left(1+\frac{a}{x}\right)^x$ $(1^\infty form) = L$

$$\log L = \lim_{x\to\infty}\left\{x \log\left(1+\frac{a}{x}\right)\right\} (0\times\infty\ form)$$
$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{a}{x}\right)}{\left(\frac{1}{x}\right)} \left(\frac{0}{0}\ form\right)$$
$$= \lim_{x\to\infty} \frac{(1+\frac{a}{x})^{-1}.(-a\,x^{-2})}{-x^{-2}}$$
$$= \lim_{x\to\infty} a\left(1+\frac{a}{x}\right)^{-1} = a.$$

Therefore, $\log L = a \implies L = e^a$.

Ex.24 Evaluate $\lim_{x\to 0}(\cos ecx)^{\frac{1}{\log x}}$ $(\infty^0\ form)$

Sol. Let $y = \lim_{x\to 0}(cosec\ x)^{\frac{1}{\log x}}$ $(\infty^0\ form)$

$$\log y = \lim_{x\to 0}\frac{1}{\log x}(\log cosec\ x) \qquad \left(\frac{\infty}{\infty}\ form\right)$$
$$= \lim_{x\to 0}\frac{(\frac{1}{cosec\ x})(-cosec\ x \cot x)}{\frac{1}{x}}$$
$$= \lim_{x\to 0}\frac{-x}{\tan x}\left(\frac{0}{0}\ form\right)$$
$$= \lim_{x\to 0}\frac{-(x)'}{(\tan x)'}$$
$$= \lim_{x\to 0}\frac{-1}{sec^2 x}$$
$$= -1.$$

Therefore, $y = e^{-1} = \frac{1}{e}$.

Ex.25 Evaluate $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Sol. Let $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$ $(1^\infty\ form) = y.$

$$\log y = \lim_{x \to 0} \frac{\log \cos x}{x^2} \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x \to 0} \frac{-\tan x}{2x}$$

$$= -\frac{1}{2} \lim_{x \to 0} \frac{\tan x}{x} = \frac{-1}{2}.$$

Hence, $y = e^{\frac{-1}{2}}$.

5.7 EVALUATION OF EXPONENTIAL LIMITS OF THE FORM 1^∞

To evaluate the exponential form 1^∞, we use the following procedures:
If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \{1 + f(x)\}^{\frac{1}{g(x)}} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

...........(7)

Or,

When $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$ then

$$\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} (1 + f(x) - 1)^{g(x)} = e^{\lim_{x \to a} (f(x)-1) g(x)}$$

...............(8)

Ex.26 Evaluate $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

Sol. Here, $f(x) = x$, and $g(x) = x$.

Clearly, $\lim_{x \to a} f(x) = 0, \lim_{x \to a} g(x) = 0$

Hence, $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e^{\lim_{x \to 0} \frac{x}{x}}$ {using equation (7)}

$$= e.$$

Ex.27 Find $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

Sol. Here, $f(x) = \frac{1}{x}$, and $g(x) = \frac{1}{x}$. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} =$ 0 and $\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1}{x} = 0$. Hence,

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e^{\lim_{x \to \infty} \frac{f(x)}{g(x)}} = e^{\lim_{x \to \infty} \frac{x}{x}} = e.$$

Ex.28 Find $\lim_{x \to 0} (1 + \lambda x)^{\frac{1}{x}}$.

Sol. Here, $f(x) = 1 + \lambda x$, and $g(x) = \frac{1}{x}$.

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + \lambda x) = 1$, and $\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x} = \infty$.

Hence, $\lim_{x\to0}(1+\lambda x)^{\frac{1}{x}} = e^{\lim_{x\to0}\frac{\lambda x}{x}} = e^{\lambda}$.

Ex.29 Find $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^{x}$.

Sol. We have,

$f(x) = 1+\frac{\lambda}{x}$and $g(x) = x$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(1+\frac{\lambda}{x}) = 1$ and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x = \infty$.

Hence, $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^{x} = e^{\lim_{x\to a}\frac{\lambda}{x}.\,x} = e^{\lambda}$.

Ex.30 Evaluate: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$

Sol. We have, $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x} = e^{\lim_{x\to\infty}\frac{2}{x}.x} = e^{2}$.

Ex.31 Evaluate: $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

Sol. As $x\to\infty$, $\lim_{x\to\infty}\frac{x+6}{x+1} = 1$ and $(x+4)\to\infty$

$$\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$$

$$= \lim_{x\to\infty}\left(1+\left(\frac{x+6}{x+1}-1\right)\right)^{x+4} = \lim_{x\to\infty}\left(1+\frac{5}{x+1}\right)^{x+4}$$

$$= e^{\lim_{x\to\infty}\left(\frac{5}{x+1}\right)(x+4)}$$

$$= e^{\lim_{x\to\infty} 5.\left(\frac{x+4}{x+1}\right)}$$

$$= e^{5\lim_{x\to\infty}\left(\frac{x+4}{x+1}\right)}$$

Consider, $\lim_{x\to\infty}\left(\frac{x+4}{x+1}\right)\left(\frac{\infty}{\infty} form\right)$

$$= \lim_{x\to\infty}\frac{(x+4)'}{(x+1)'} = \lim_{x\to\infty}\frac{1}{1} = 1.$$

Now, $e^{5\lim_{x\to\infty}\left(\frac{x+4}{x+1}\right)} = e^{5.(1)} = e^{5}$.

Ex.32 Evaluate: $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^{x}$

Sol. $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^{x} = \lim_{x\to\infty}\left(1+\left(\frac{x-3}{x+2}-1\right)\right)^{x}$

$$= \lim_{x\to\infty}\left(1+\frac{-5}{x+2}\right)^{x}$$

$$= e^{\lim_{x\to\infty}\left(\frac{-5}{x+2}\right)x}$$

$$= e^{-5\lim_{x\to\infty}\frac{x}{x+2}}$$

Consider, $\lim_{x\to\infty}\frac{x}{x+2}\left(\frac{\infty}{\infty} form\right) = \lim_{x\to\infty}\frac{(x)'}{(x+2)'} = \lim_{x\to\infty}\frac{1}{1} = 1$.

Now, $e^{-5\lim_{x\to\infty}\frac{x}{x+2}} = e^{-5}$.

Ex.33 Evaluate: $\lim_{x\to1}(\log_3 3x)^{\log_x 3}$

Sol. $\lim_{x\to1}(\log_3 3x)^{\log_x 3} = \lim_{x\to1}(\log_3 3+\log_3 x)^{\log_x 3}$

$$= \lim_{x\to 1}(1+\ \log_3 x)^{\frac{1}{\log_3 x}}$$
$$= e^{\lim\limits_{x\to 1}\log_3 x \times \frac{1}{\log_3 x}} = e^1.$$

5.8 SUMMARY

In this unit, we are familiar with the seven standard indeterminate forms. We studied the limit of $\frac{f(x)}{g(x)}$ as $x \to a$, in general, equal to the limit of the numerator divided by the limit of the denominator. But when these two limits are both zero, this limit reduces to $\frac{0}{0}$ (meaningless). This does not imply that $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is meaningless or it does not exist. Now, According to our plan we understood the concept of limit (in previous unit) and evaluation of limits of different forms and then the existence of limits. After understanding this unit, we summarize the results to problems of evaluation of limits as follows:

(1) **Algebraic Limits:** Limits of algebraic forms are further sub-classified as

(i) $\frac{\mathbf{0}}{\mathbf{0}}$ $\boldsymbol{form}$ **:** These form are based on

(a) Factorization Method: In this method numerators and denominators are factorized. The common factors are cancelled and the rest output is result.

(b) Rationalization Method: When we have fractional powers on expressions in numerator or denominator or in both, rationalize it. After rationalization the terms are factorized which on cancellation gives the result.

(c) Standard Formula:
$\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}, where\ \ n\ is\ rational\ number\ .$

(d) **L'Hospital's Rule:**
$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$, provided the later limit exists. But, if it again take form $\frac{0}{0}$, then
$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$ and this process is continued till $\frac{0}{0}$ form is removed.

(ii) $\frac{\infty}{\infty}$ $\boldsymbol{form}$ **:**

(a) These types of problems are solved by taking the highest power of the terms tending to infinity as common numerator and denominator. After that they are cancelled and the rest output is the result.

(b) **L'Hospital's Rule:**
$\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$, provided the later limit exists. But, if it again takes $\frac{\infty}{\infty}$, formthen
$\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to a}\frac{f''(x)}{g''(x)}$ and this process is continued till $\frac{\infty}{\infty}$ form is removed.

(iii) $\infty-\infty$ ***form*** **:**

(a) Such problems are simplified (rationalization etc.) first, thereafter they generally acquire $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

(2) Trigonometric Limits:

(i) To evaluate trigonometric limits the following results are very important.

(a) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

(b) $\lim\limits_{x\to 0}\frac{\tan x}{x}=1$

(c) $\lim\limits_{x\to 0}\cos x=1$

(d) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=0.$

(ii) **Expansion Method:** Sometimes following expansions are useful for evaluating the trigonometric limits.

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\infty.$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\infty.$$
$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots\infty.$$
$$\sin^{-1}x=x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots\infty.$$
$$\sin hx=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\infty.$$
$$\cos hx=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\infty.$$

(iii) If $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then apply L'Hospital's Rule.

(3) Logarithmic Limits:

(i) **Expansion Method:** To evaluate logarithmic limit the following expansions are useful.

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\infty,\quad |x|<1.$$
$$\log(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\infty\right),\quad |x|<1.$$

(ii) If $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then apply **L'Hospital's Rule.**

(4) Exponential Limits:

(i) Expansion Method:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty.$$

(ii) **Standard Formula:** Sometimes we use the following results.

(a) $\lim_{x\to 0} \frac{e^x-1}{x} = 1.$

(b) $\lim_{x\to 0} \frac{a^x-1}{x} = 1.$

(iii) $\mathbf{1^\infty}$ ***form***:If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\{1+f(x)\}^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

Or, When $\lim_{x\to a} f(x) = 1$,and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a}(1+f(x)-1)^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}.$$

(iv) L'Hospital's Rule.

(5) Some Important limits:

(i) $\lim_{x\to\infty} a^x = \begin{cases} \infty, & if\ a > 1 \\ 1, & if\ a = 1 \\ 0, & if\ 0 \le a < 1 \end{cases}$

(ii) If m and n are positive integers and a_0, $b_0$are non-zero real numbers, then

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}$$

$$= \begin{cases} 0, & m < n \\ \frac{a_0}{b_0}, & m = n \\ \infty, & m > n\ when\ a_0b_0 > 0 \\ -\infty, & m > n\ when\ a_0b_0 < 0 \end{cases}$$

(iii) Sometimes taking logarithm of both sides is useful in case of the form$0^0, 1^\infty\ and\ \infty^0$ (see Example 24). In this case $\log y$ generally acquire the form $0 \times \infty$, which on simplifying gives the $\frac{0}{0}$ or$\frac{\infty}{\infty}$ form.

5.9 GLOSSARY

Exponential:We know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots .$$

Replacing$x$ by $\frac{1}{n}$, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots .$$

Taking limit as $\to \infty$, we have exponential e as

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. The value of e is lies between 2 and 3($e \approx 2.718$).

Indeterminate form: When the function involves the independent variable in such a manner that for a certain assigned value of that variable its value cannot be found by simple substituting that value of the variable, the function is said to take an indeterminate form.

L'Hospital, Guillaume François Antoine, Marquis de(1661–1704) French mathematician who in 1696 producedthe first textbook on differential calculus. This, and asubsequent book on analytical geometry, were standard textsfor much of the eighteenth century. The first contains**L**'Hospital's rule, known to be due to Johann Bernoulli, who isthought to have agreed to keep the Marquis de L'Hôpitalinformed of his discoveries in return for financial support.

CHECK YOUR PROGRESS

1. 0^0 is an Indeterminate Form. True\False
2. $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 1$ True\False
3. $\lim\limits_{x\to 0}\left[\frac{1-cosx}{x^2}\right] = \frac{1}{2}$ True\False
4. $\lim\limits_{x\to 0+} x^{sinx} = 1$, where $I = (0,\infty)$ True\False
5. Let f be a differentiable on $(0,\infty)$ and suppose that $\lim\limits_{x\to\infty}\left(f(x)+f'(x)\right) = L$ then $\lim\limits_{x\to\infty} f(x) = L$. True/False

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5.12 TERMINAL QUESTIONS

Choose only one correct option.

(TQ-1) $\lim_{x\to\infty}(0.752)^x$

(i) 0 .

(ii) $+\infty$.

(iii) $-\infty$

(iv) None of the above.

(v)

(TQ-2) $\lim_{x\to 0}(cot\ x\ )^{sin\ x}$

(i) 0 .

(ii) + 1.

(iii) -1.

(iv) None of the above.

(TQ-3) $\lim_{n\to x}\frac{x}{x}, x \neq 0$

(i) 0

(ii) 1

(iii) is an indeterminate form.

(iv) Cannot be found.

(TQ-4) Rational function $\frac{f(x)}{g(x)}, g(x) \neq 0$ is

(i) Always an indeterminate form as x tends to ∞.
(ii) Always a determinate form as x tends to ∞.
(iii) May be determinate or indeterminate form as x tends to ∞.
(iv) Nothing can be said.

(TQ-5) $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$, limit on R.H.S exists; is

(i) True in case $f(a) = g(a) = \infty, a \neq 0$.
(ii) False
(iii) True
(iv) True when $f(0) = g(0) = 0$.

(TQ-6) $\lim_{x\to 0} x^{-1} sin\ x$

(i) 1
(ii) 0
(iii) Does not exist.
(iv) Not finite.

(TQ-7) $\lim_{x\to 0} \frac{ax^2+9}{bx^2+8}$

(i) $\frac{9}{8}$
(ii) $\frac{a}{b}$.
(iii) Depending on a not b.
(iv) Is an indeterminate form.

(TQ-8) $\lim_{x\to 0} \frac{ax^2+9x}{bx^2+8x}$

(i) $\frac{9}{8}$
(ii) $\frac{a}{b}$.
(iii) Depending on a not b.
(iv) Is an indeterminate form.

(TQ-9) $\lim_{x\to 0} \frac{ax^2}{bx^2}, a \neq 0\ and\ b \neq 0,$ is

(i) $\frac{9}{8}$
(ii) $\frac{a}{b}$.
(iii) Depending on a and b.
(iv) Is an indeterminate form.

(TQ-10) $\lim_{x\to 0} x^x$

(i) 0
(ii) -1
(iii) $+1$
(iv) ∞.

(TQ-11) Which are indeterminate forms?......................................
....
....

(TQ-12) How do you solve indeterminate forms of limits?

....
....
....

5.13 ANSWERS:-

CHECK YOUR PROGRESS

SCQ1. True
SCQ2. False
SCQ3. True
SCQ4. True
SCQ5. True

TERMINAL QUESTIONS (TQ'S)
(TQ-1) (i)
(TQ-2) (ii)
(TQ-3) (ii)
(TQ-4) (iii)
(TQ-5) (iv)
(TQ-6) (i)
(TQ-7) (i)
(TQ-8) (i)
(TQ-9) (ii)
(TQ-10) (iii)

UNIT-6:- SUCCESSIVE DIFFERENTIATION

CONTENTS

6.1 INTRODUCTION

Generally we differentiate a function $y = f(x)$ one or two times. But sometimes we need n^{th} derivative of that function. In this chapter we shall try to find some general expressions of some standard functions. Besides this, the evaluation of the n^{th}derivative of product of two functions is one of the targets of this chapter. Dutch mathematician Leibnitz developed a method for product, which we shall explore here. Actually, British mathematician Newton and Dutch (Netherland's citizen) Leibnitz were contemporary and both were instrumental in the development of calculus. But Newton got almost all the credit and is considered as 'Father of Calculus'. Leibnitz could not get due credit in mathematical fraternity. Here we shall learn one of the famous works of Leibnitz.

6.2 OBJECTIVES:

After completion of this unit, we shall understand

- **i.** n^{th} Differential coefficient of y(dependent variable) with respect to x(independent variable).
- **ii.** The evaluating procedures of n^{th}derivative of a function .
- **iii.** Some standard results on n^{th} differential coefficient.

iv. Successive derivatives with the help of partial fractions.
v. Leibnitz's theorem for n^{th}derivative of product of two functions.

6.3 SUCCESSIVE DIFFERENTIATION

Let $y = f(x)$ be a differentiable function. Then its first derivative with respect to x is written as $\frac{dy}{dx} = f'(x)$Similarly, second derivative with respect to x is$\frac{d^2y}{dx^2} = f''(x)$.In general, $n\,th$ derivative with respect to x is$\frac{d^ny}{dx^n} = f^{(n)}(x)$.
Let us write$\frac{dy}{dx} = Dy = y_1, \frac{d^2y}{dx^2} = D^2y = y_2, and\ so\ on.$
So in general we can write $\frac{d^ny}{dx^n} = D^ny = y_n$.
The expression D^ny also follows the laws of indices i.e.

$$D^ny_r = y_{n+r};\ r = 0, 1, 2, 3, \dots\ .$$

In particular $D^3y_2 = y_{3+2} = y_5 =$ fifth derivative of y with respect to x.

Let us calculate derivatives of some simple functions. We shall find first, second, third, … , $n\,th$order derivatives to justify the name "successive derivative".

Ex.1 **Find all the possible derivatives of the function $\boldsymbol{f(x) = ax^3 + bx^2 + cx + d}$.**

Sol. Here, $f(x) = ax^3 + bx^2 + cx + d$.
Therefore, $f'(x) = 3ax^2 + 2bx^1 + c$;
$f''(x) = 6ax^1 + 2b$;
$f'''(x) = 6a$;
$f^{iv}(x) = 0$; and all higher derivatives are evidently zero.

Ex.2 **Find the $\mathbf{n}^{\mathbf{th}}$ derivative of the function $\boldsymbol{f(x) = x^n}$.**

Sol. Here, $f(x) = x^n$.
First derivative $f'(x) = nx^{n-1}$;
Second derivative $f''(x) = n(n-1)x^{n-2}$;
Third derivative $f'''(x) = n(n-1)(n-2)x^{n-3}$;
Fourth derivative $f^{iv}(x) = n(n-1)(n-2)(n-3)x^{n-4}$;
In general, n^{th} derivative

$$f^{(n)}(x) = n(n-1)(n-2)(n-3)\dots\big(n-(n-2)\big)\big(n-(n-1)\big)x^{n-n}$$
$$= n(n-1)(n-2)(n-3)\dots 2.1$$
$$= n!$$

All higher derivatives are evidently zero.

Ex.3 **Find the $\mathbf{n}^{\mathbf{th}}$ derivative of the function $\boldsymbol{f(x) = e^x}$.**

Sol. Here, $f(x) = e^x$;
First derivative $f'(x) = e^x$;

Second derivative $f''(x) = e^x$;
In general, nth derivative $f^n(x) = e^x$.
In this example, no higher derivativeis zero.

Ex.4 Find the nth derivative of the function $f(x) = \sin x$.

Sol. Here, $f(x) = \sin x$;

First derivative $f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right)$;

Second derivative $f''(x) = -\sin x = \sin\left(2.\frac{\pi}{2} + x\right)$;

Third derivative $f'''(x) = -\cos x = \sin\left(3.\frac{\pi}{2} + x\right)$;

Fourth derivative $f^{iv}(x) = \sin x = \sin\left(4.\frac{\pi}{2} + x\right)$;

In general, nth derivative $f^n(x) = \sin\left(n.\frac{\pi}{2} + x\right)$.

Ex.5 Find the nth derivative of the function $f(x) = \log x$.

Sol. Here, $f(x) = \log x$;

First derivative $f'(x) = \frac{1}{x}$;

Second derivative $f''(x) = (-1).\frac{1}{x^2}$;

Third derivative $f'''(x) = (-1)(-2).\frac{1}{x^3} = (-1)^2\ 1.2.\frac{1}{x^3}$;

Fourth derivative $f^{iv}(x) = (-1)(-2)(-3).\frac{1}{x^4} = (-1)^3\ 1.2.3.\frac{1}{x^4}$;

So, nth derivative $f^n(x) = (-1)^{n-1}\ 1.2.3.4 \ldots.(n-1).\frac{1}{x^n} = (-1)^{n-1}(n-1)!.\frac{1}{x^n}$.

Motivated by the above example, we here find some standard results on nth derivatives.

6.4 STANDARD RESULTS

(1) The n^{th} derivative of $\frac{1}{ax+b}$

Let$y = \frac{1}{ax+b} = (ax+b)^{-1}$. Then

$y_1 = (-1).a.(ax+b)^{-2}$,

$y_2 = (-1)(-2)a^2(ax+b)^{-3} = (-1)^2\ 1.2.a^2(ax+b)^{-3}$,

$y_3 = (-1)(-2)(-3)a^3(ax+b)^{-4} = (-1)^3 1.2.3.a^3(ax+b)^{-4}$;

In general, $y_n = (-1)^n 1.2.3 \ldots.n\ a^n(ax+b)^{-n-1} = \boldsymbol{(-1)^n\, n!\, a^n(ax+b)^{-n-1}}$.

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(2) The n^{th} derivative of $(ax+b)^m$

Let $y = (ax+b)^m$. Then

$y_1 = m.a.(ax+b)^{m-1}$,

$y_2 = m.(m-1).a^2.(ax+b)^{m-2}$,
$y_3 = m.(m-1).(m-2).a^3.(ax+b)^{m-3}$,
So,
$$\boldsymbol{y_n = m(m-1)(m-2)\ldots\big(m-(n-1)\big)a^n(ax+b)^{m-n}}$$

Note: In particular, if $m=-1$, then
$y_n = -1(-2)(-3)\ldots(-n)a^n(ax+b)^{-1-n} = (-1)^n n!\,a^n(ax+b)^{-1-n}$, gives standard result (1). So, Standard result (1) is particular form of standard result (2).

(3) The n^{th} derivative of e^{ax+b}

Let $y = e^{ax+b}$. Then
$y_1 = a.e^{ax+b}$,
$y_2 = a^2.e^{ax+b}$,
$y_3 = a^3.e^{ax+b}$,
So, $\boldsymbol{y_n = a^n.e^{ax+b}}$

Note: In particular**, n^{th} derivative of $e^{ax} = a^n e^{ax}$.**

(4) The n^{th} derivative of a^x

Let $y = a^x$. Then
$y_1 = a^x.\log a$,
$y_2 = a^x.(\log a)^2$,
$y_3 = a^x.(\log a)^3$,
So, $\boldsymbol{y_n = a^x.(\log a)^n}$**.**

(5) The n^{th} derivative of $\log(ax+b)$

Let $y = \log(ax+b)$. Then
$y_1 = a.(ax+b)^{-1}$,
$y_2 = (-1)a^2.(ax+b)^{-2}$,
$y_3 = (-1)(-2)a^3.(ax+b)^{-3} = (-1)^2 1.2.a^3.(ax+b)^{-3}$,
Therefore, $y_n = (-1)^{n-1}\,1.2.3\ldots.(n-1).a^n.(ax+b)^{-n}$
$\boldsymbol{y_n = (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n}}$.

Note: In particular,

n^{th} derivative of $\log x = (-1)^{n-1}(n-1)!\,(x)^{-n}$.

(6) The n^{th} derivative of $\sin(ax+b)$

Let $y = \sin(ax+b)$. Then
$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$,
$y_2 = -a^2\sin(ax+b) = a^2\sin\left(ax+b+2.\frac{\pi}{2}\right)$,
$y_3 = -a^3\cos(ax+b) = a^3\sin\left(ax+b+3.\frac{\pi}{2}\right)$,
So, $\boldsymbol{y_n = a^n\sin\left(ax+b+n.\frac{\pi}{2}\right)}$**.**

(7) The n^{th}derivative of $\cos(ax+b)$

Let $y=\cos(ax+b)$. Then

$y_1 = -a\sin(ax+b) = a\cos\left(ax+b+\frac{\pi}{2}\right)$,

$y_2 = -a^2\cos(ax+b) = a^2\cos\left(ax+b+2.\frac{\pi}{2}\right)$,

$y_3 = \ a^3\sin(ax+b) = a^3\cos\left(ax+b+3.\frac{\pi}{2}\right)$,

So, $\boldsymbol{y_n = a^n\cos\left(ax+b+n\,.\frac{\pi}{2}\right)}$.

(8) The n^{th} derivative of $e^{ax}\sin(bx+c)$

Let $y=e^{ax}\sin(bx+c)$. Then

$$y_1 = a.e^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$
$$= \ e^{ax}(a\sin(bx+c) + \ b\cos(bx+c))$$

Putting $a=r\cos\phi$ and $b=r\sin\phi$, we get

$$y_1 = \ e^{ax}(r\cos\phi\sin(bx+c) + r\sin\phi\cos(bx+c))$$

$y_1 = \ \ r\,e^{ax}\sin(bx+c+\phi),\ where\ r=\sqrt{a^2+b^2}\ and\ \phi = \tan^{-1}\frac{b}{a}$.

Similarly

$y_2 = r^2e^{ax}\sin(bx+c+2\phi)$,

$y_3 = r^3e^{ax}\sin(bx+c+3\phi)$,

So,

$$y_n = r^ne^{ax}\sin(bx+c+n\,\phi), where\ \ r=\sqrt{a^2+b^2}\ and\ \phi = \tan^{-1}\frac{b}{a}.$$

(9) The n^{th} derivative of $e^{ax}\cos(bx+c)$

$y_n = r^ne^{ax}\cos(bx+c+n\,\phi), where\ \ r = \sqrt{a^2+b^2}\ and\ \phi = \tan^{-1}\frac{b}{a}$.

Now we will see the importance of standard results in the evaluation of n^{th} differential coefficient.

Ex.6 Find the n^{th} differential coefficient of $\log[(ax+b)(cx+d)]$.

Sol. Let $y=\log[(ax+b)(cx+d)]$. Then

$$y = \log(ax+b) + \log(cx+d)$$

Then $D^ny = D^n[\log(ax+b)] + D^n[\log(cx+d)]$

By using Standard Result (5), we have

$$D^n[\log(ax+b)] = (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n}$$

Hence, $D^ny = \ (-1)^{n-1}(n-1)!\,a^n(ax+b)^{-n} + (-1)^{n-1}(n-1)!\,c^n(cx+d)^{-n}$

$$= \ (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right].$$

Ex.7 Find the n^{th} differential coefficient of $y = \sin 4x\ cos\ 2x$.

Sol. Let $y = \sin 4x\ \cos 2x = \frac{1}{2}[\sin 6x\ + \sin 2x]$

Then $D^n y = \frac{1}{2}[\ D^n(\sin 6x) + D^n(\sin 2x)]$

By using standard result (6), we have $D^n(\sin(ax+b)) = a^n \sin\left(ax + b + n\,.\frac{\pi}{2}\right)$

Therefore, $D^n y = 6^n \sin\left(6x + n\,.\frac{\pi}{2}\right) + 2^n \sin\left(2x + n\,.\frac{\pi}{2}\right)$.

Ex.8 Find the n^{th} differential coefficient of $y = \sin^3 x$.

Sol. Let $y = \sin^3 x$.

We know that $\sin 3x = 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin 3x$,

$y = \sin^3 x = \frac{1}{4}[3\sin x\ - \sin 3x]$,

Then $D^n y = \frac{1}{4}[3\ D^n(\sin x) - D^n(\sin 3x)]$

Using Standard Result (6),

$D^n(\sin(ax+b)) = a^n \sin\left(ax + b + n\,.\frac{\pi}{2}\right)$

$D^n y = \frac{1}{4}\left[3.\sin\left(x + n.\frac{\pi}{2}\right) - 3^n \sin\left(3x + n.\frac{\pi}{2}\right)\right]$.

Ex.9 **Find the n^{th} differential coefficient of $y = \frac{1}{(5x+4)}$.**

Sol. Let $y = \frac{1}{(5x+4)} = (5x+4)^{-1}$.

Standard result

(1), $D^n[(ax+b)^{-1}] = (-1)^n\ n!\, a^n (ax+b)^{-n-1}$.

Here, $a = 5, and\ b = 4$.

Hence, $D^n y = (-1)^n\ n!\, 5^n (5x+4)^{-n-1}$.

Ex.10 **Find the n^{th} differential coefficient of $y = cos\ x\ cos\ 2x\ cos\ 3x.$**

Sol. Let $y = \cos x\ \cos 2x \cos 3x = \frac{1}{2}\cos x\,(2\cos 2x \cos 3x)$

$$= \frac{1}{2}\cos x\,(\cos 5x + \cos x)$$

$$= \frac{1}{4}[2\cos x \cos 5x + 2\cos^2 x]$$

$$y\ = \frac{1}{4}[\cos 6x + \cos 4x + \cos 2x + 1]$$

Hence, $y_n = D^n y = \frac{1}{4}[D^n(\cos 6x) + D^n(\cos 4x) + D^n(\cos 2x) + D^n(1)]$.By using standard result (7), $D^n(\cos(ax+b)) = a^n \cos\left(ax + b + n\,.\frac{\pi}{2}\right)$.So, $y_n = \frac{1}{4}\left[6^n \cos\left(6x + n.\frac{\pi}{2}\right) + 4^n \cos\left(4x + n.\frac{\pi}{2}\right) + 2^n \cos\left(2x + n.\frac{\pi}{2}\right) + 0\right]$.

$y_n = \frac{1}{4}\left[6^n \cos\left(6x + n.\frac{\pi}{2}\right) + 4^n \cos\left(4x + n.\frac{\pi}{2}\right) + 2^n \cos\left(2x + n.\frac{\pi}{2}\right)\right].$

Ex.11 Find the n^{th} differential coefficient of $y = \sin mx + \sin nx$.

Sol. Since, $y = \sin mx + \sin nx$,

So, $y_n = D^n y = D^n(\sin mx) + \mathrm{D^n}(\sin nx)$

Using Standard result (6),

$D^n(\sin(ax+b)) = a^n \sin\left(ax + b + n\,.\frac{\pi}{2}\right).$

$$y_n = m^n \sin\left(mx + n\,.\frac{\pi}{2}\right) + n^n \sin\left(nx + n\,.\frac{\pi}{2}\right)$$

Ex.12 Find the n^{th} differential coefficient of $y = \sin mx + \cos mx$.

Sol. Since, $y = \sin mx + \cos mx$,

Then $y_n = D^n y = D^n(\sin mx) + D^n(\cos mx)$.

Using standard results (6) and (7),

$D^n(\sin(ax+b)) = a^n \sin\left(ax + b + n\,.\frac{\pi}{2}\right)$and

$$D^n(\cos(ax+b)) = a^n \cos\left(ax + b + n\,.\frac{\pi}{2}\right).$$

Hence,$y_n = m^n \sin\left(mx + n.\frac{\pi}{2}\right) + m^n \cos\left(mx + n.\frac{\pi}{2}\right)$.

$y_n = m^n\left[\sin\left(mx + n.\frac{\pi}{2}\right) + \cos\left(mx + n.\frac{\pi}{2}\right)\right],$

$y_n = m^n\left[\left(\sin\left(mx + n.\frac{\pi}{2}\right) + \cos\left(mx + n.\frac{\pi}{2}\right)\right)^2\right]^{\frac{1}{2}},$

$y_n = m^n\left[1 + 2\sin\left(mx + n.\frac{\pi}{2}\right)\cos\left(mx + n.\frac{\pi}{2}\right)\right]^{\frac{1}{2}},$

$y_n = m^n\left[1 + \sin 2\left(mx + n.\frac{\pi}{2}\right)\right]^{\frac{1}{2}},$

$y_n = m^n[1 + \sin(2mx + n.\pi)]^{\frac{1}{2}},$

$y_n = m^n[1 + \sin 2mx\ \cos n\pi + \cos 2mx \sin n\pi]^{\frac{1}{2}},$

$y_n = m^n[1 + (-1)^n \sin 2mx + 0.\cos 2mx]^{\frac{1}{2}},$

since $\cos n\pi = (-1)^n$ *and* $\sin n\pi = 0$.

$y_n = m^n[1 + (-1)^n \sin 2mx]^{\frac{1}{2}}.$

Ex.13 Find the n^{th} derivative of $y = e^x \cos^3 x$.

Sol. Here $= e^x \cos^3 x$.

Now, $\cos 3x = 4\cos^3 x - 3\cos x$

$4\cos^3 x = \cos 3x + 3\cos x$,

$\Longrightarrow \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$,

Hence, $y = \frac{1}{4}e^x \cos 3x + \frac{3}{4}e^x \cos x$,

$\Rightarrow y_n = D^n y = \frac{1}{4}D^n(e^x \cos 3x) + \frac{3}{4}D^n(e^x \cos x)$,

Using standard formula (9),

$D^n(e^{ax}\cos(bx+c)) = r^n e^{ax}\cos(bx+c+n\phi), r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$.

Now, $D^n(e^x\cos 3x) = r^n e^x \cos(3x+n\phi), r = \sqrt{1^2+3^2}$.

and $\phi = \tan^{-1}\frac{3}{1}\left(\sqrt{10}\right)^n e^x\cos(3x+n\tan^{-1}3)$.

Similarly, $D^n(e^x\cos x) = r^n e^x\cos(x+n\phi)$,

$r = \sqrt{1^2+1^2}$ and $\phi = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

Therefore $D^n(e^x\cos x) = \left(\sqrt{2}\right)^n e^x\cos\left(x+n.\frac{\pi}{4}\right)$.

Therefore

$$y_n = \frac{1}{4}\left(\sqrt{10}\right)^n e^x\cos(3x+n\tan^{-1}3) + \frac{3}{4}\left(\sqrt{2}\right)^n e^x\cos\left(x+n.\frac{\pi}{4}\right).$$

6.5 SUCCESSIVE DERIVATIVES WITH THE HELP OF PARTIAL FRACTIONS

Sometimes, expressions are given in the form of quotient of polynomials i.e, in rational function forms. We use the method of partial fractions to separate those terms and then we can find relatively easily the n^{th} derivative.

Case (I): Let $y = \frac{x}{(x-2)(x-3)}$

In denominator, each bracket contains linear expression. You know that the degree of remainder is always less than that of divisor. So we have

(1) $\frac{x}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$

Taking LCM of RHS, we get

$$\frac{x}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow x = A(x-3)+B(x-2)$$

(2) $x = A(x-3)+B(x-2)$

Now we may have two methods to solve equation (2).

Method 1. (General Method):

$$x = Ax - 3A + Bx - 2B$$
$$= x(A+B) + (-3A-2B)$$

equating the corresponding coefficients, we get

$$A+B = 1 \ and \ -3A-2B = 0.$$

On solving, we get, $A = -2, and \ B = 3$.

Putting the values of A and B in equation (1), we have

$$y = \frac{-2}{(x-3)} + \frac{3}{(x-2)}.$$

Now we can find n^{th} derivative easily by using Standard Results.

Method 2. Putting $x = 3$ in equation (2) we get

$$3 = 0 + B \Rightarrow B = 3.$$

Now Putting $x = 2$ in equation (2) we get

$$2 = -A + 0 \Rightarrow A = -2.$$

Substituting the value of A and B in equation (2), we get

$$y = \frac{-2}{(x-3)} + \frac{3}{(x-2)}.$$

Case (II): If $y = \frac{\boldsymbol{x^2}}{\boldsymbol{(x-2)(x-3)}}$

If we multiply the factors of denominators, we get a polynomial of degree two. Now we know that for a proper rational function, degree of numerator must be less than that of the denominator.

So, $(x-2)(x-3) = x^2 - 5x + 6$

Now

(4) $y = \frac{x^2}{x^2+5x+6} = 1 + \frac{5x-6}{x^2-5x+6}$

Now take

(5) $\frac{5x-6}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$

Now solve as in case (I).

Case (III): If $y = \frac{(x+2)}{(x-2)(x-3)^2}$

Now the partial fractions

(6) $\frac{(x+2)}{(x-2)(x-3)^2} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$

The following table gives us an idea of the types of partial fractions to be taken for different types of proper rational algebraic functions:

Type of proper rational function	**Type of partial fractions**
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

$\frac{px^2+qx+r}{(x-a)^2(x-b)}, a \neq b$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ When x^2+bx+c cannot be factorized.	$\frac{A}{(x-a)}+\frac{Bx+C}{x^2+bx+c}$
$\frac{px^3+qx^2+rx+s}{(x^2+ax+b\,)(x^2+cx+d)}$ When x^2+ax+b, x^2+cx+d cannot ne factorized.	$\frac{Ax+B}{x^2+ax+b}+\frac{Cx+D}{x^2+cx+d}$

Ex.14 **Find the n^{th} differential coefficient of $y=\frac{x^3}{x^2-3x+2}$.**

Sol. Here, the given function is improper function.

Since, degree of denominator<degree of numerator.

Therefore,

$$y=\frac{x^3}{x^2-3x+2}=x+3+\frac{7x-6}{(x-1)(x-2)}.$$

$$\frac{7x-6}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}.$$

$$\Rightarrow 7x-6=A(x-2)+B(x-1).$$

$$\Rightarrow 7x-6=x(A+B)-2A-B$$

Comparing the corresponding coefficients, we get

$A+B=7$ and$-2A-B=-6$.

On solving, we get $A=\ -1, and\ B=8.$

Hence,

$$\frac{7x-6}{(x-1)(x-2)}=\frac{-1}{(x-1)}+\frac{8}{(x-2)}$$

Now,

$$y=x+3-\frac{1}{(x-1)}+\frac{8}{(x-2)}$$

$$y_n=D^ny=D^nx+D^n(3)-D^n[(x-1)^{-1}]+8D^n[(x-2)^{-1}],$$

If $n>1$, then

$$y_n=D^ny=0+0-(-1)^n n!\,(x-1)^{-n-1}+8(-1)^n n!\,(x-2)^{-n-1}.$$

{by using standard result (1), $D^n[(ax+b)^{-1}]=(-1)^n\,n!\,a^n(ax+b)^{-n-1}$}

Hence,

$$y_n=(-1)^n n!\left[\frac{-1}{(x-1)^{n+1}}+\frac{8}{(x-2)^{n+1}}\right].$$

Ex.15 **Find the n^{th} derivative of $y=\frac{1}{a^2-x^2}$.**

Sol. Here, $y=\frac{1}{a^2-x^2}=\frac{1}{(a+x)(a-x)}$.

$$y = \frac{1}{(a+x)(a-x)} = \frac{A}{(a+x)} + \frac{B}{(a-x)}$$
$$\Rightarrow 1 = A(a-x) + B(a+x)$$

Putting $x = a$, we get

$$1 = B(2a) \Rightarrow B = \frac{1}{2a}.$$

Putting $x = -a$, we get

$$1 = A(2a) \Rightarrow A = \frac{1}{2a}.$$

Hence,

$$y = \frac{1}{2a}\left[\frac{1}{(a+x)} + \frac{1}{(a-x)}\right]$$
$$= \frac{1}{2a}\left[\frac{1}{(x+a)} - \frac{1}{(x-a)}\right]$$

Then $y_n = D^n y = \frac{1}{2a}\left[D^n[(x+a)^{-1}] - D^n[(x-a)^{-1}]\right]$.

by using standard result (1),

$$D^n[(ax+b)^{-1}] = (-1)^n\, n!\, a^n (ax+b)^{-n-1}$$

$$y_n = \frac{1}{2a}[(-1)^n n!\,(x+a)^{-n-1} - (-1)^n n!\,(x-a)^{-n-1}].$$

$$y_n = \frac{(-1)^n n!}{2a}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right].$$

Ex.16 Find the n^{th} derivative of $\frac{x^2+4x+1}{x^3+2x^2-x-2}$.

Sol. Let $y = \frac{x^2+4x+1}{x^3+2x^2-x-2}$.

$$y = \frac{x^2+4x+1}{x^3+2x^2-x-2} = \frac{x^2+4x+1}{(x-1)(x+1)(x+2)},$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)},$$

$$\Rightarrow x^2+4x+1$$
$$= A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

Putting $x = -1$, we get

$$(-1)^2 - 4 + 1 = 0 + B(-2)(1) + 0 \Rightarrow B = 1.$$

Putting $x = 1$, we get

$$1^2 + 4 + 1 = A(2)(3) + 0 + 0 \Rightarrow A = 1.$$

Putting $x = -2$, we get

$$(-2)^2 - 8 + 1 = 0 + 0 + C(-3)(-1) \Rightarrow C = -1.$$

Substituting the value of $A, B\ and\ C$, we obtain

$$y = \frac{1}{(x-1)} + \frac{1}{(x+1)} - \frac{1}{(x+2)}.$$

Then

$$y_n = D^n y = D^n[(x-1)^{-1}] + D^n[(x+1)^{-1}] - D^n[(x+2)^{-1}].$$

$$y_n = (-1)^n n!\,(x-1)^{-n-1} + (-1)^n n!\,(x+1)^{-n-1} - (-1)^n n!\,(x+2)^{-n-1},$$

$$y_n = (-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right].$$

Use of De Moivre's Theorem in Partial fractions:

When we cannot break up the denominator of a given algebraic function into real linear factors, the partial fraction method can be used after resolving the denominator into its linear factors, real or imaginary.
The following example illustrates the use of De Moivre's theorem in evaluating n^{th} differential coefficient.

Ex.17 Find the n^{th} differential coefficient of $\frac{1}{x^2+a^2}$.

Sol. Let $y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)}$,

$$\frac{1}{(x+ia)(x-ia)} = \frac{A}{(x+ia)} + \frac{B}{(x-ia)}$$
$$\Rightarrow 1 = A(x-ia) + B(x+ia)$$

Putting $x = ia$, we get $1 = B(2ia) \Rightarrow B = \frac{1}{2ia}$.

Putting $x = -ia$, we get $1 = A(-2ia) \Rightarrow A = \frac{-1}{2ia}$.

Now we have

$$y = \frac{1}{2ia}\left[\frac{1}{(x-ia)} - \frac{1}{(x+ia)}\right]$$

$$y_n = D^n y = \frac{1}{2ia}\left[D^n[(x-ia)^{-1}] - D^n[(x+ia)^{-1}]\right]$$
$$= \frac{1}{2ia}[(-1)^n n!\,(x-ia)^{-n-1} - (-1)^n n!\,(x+ia)^{-n-1}]$$
$$y_n = \frac{(-1)^n n!}{2ia}[(x-ia)^{-n-1} - (x+ia)^{-n-1}].$$

Let $x = r\cos\phi \text{ and } a = r\sin\phi, \text{ so that } \phi = \tan^{-1}\frac{a}{x}$.

Then

$$y_n = \frac{(-1)^n n!}{2ia}[(r\cos\phi - i.r\sin\phi)^{-n-1} - (r\cos\phi + i.r\sin\phi)^{-n-1}]$$

$$y_n = \frac{(-1)^n n!}{2ia\, r^{n+1}}\left[(\cos\phi - i\sin\phi)^{-(n+1)} - (\cos\phi + i\sin\phi)^{-(n+1)}\right]$$

$$y_n = \frac{(-1)^n n!}{2ia\, r^{n+1}}[\{\cos(n+1)\phi + i\sin(n+1)\phi\} - \{\cos(n+1)\phi - i\sin(n+1)\phi\}]$$

$$y_n = \frac{(-1)^n n!}{2ia\, r^{n+1}}[2i\sin(n+1)\phi]$$

{By using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$}

Putting $r = \frac{a}{\sin\phi}$, we get

$$y_n = \frac{(-1)^n n!}{2a.\,a^{n+1}}.\sin(n+1)\phi\,.\sin^{n+1}\phi$$
$$\Rightarrow\ y_n = \frac{(-1)^n n!}{2\,a^{n+2}}.\sin(n+1)\phi\,.\sin^{n+1}\phi\ \text{, where } \phi = \tan^{-1}\frac{a}{x}.$$

6.6 LEIBNITZ'S THEOREM

This theorem is useful for finding the n^{th} differential coefficient of a product.

Theorem 1. If u and v are any two functions of x such that all their desired differential coefficient exists, then the n^{th}differential coefficient of their product is given by

$$D^n(uv) = (D^n u).v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + \cdots + {}^nC_r D^{n-r}u.D^r v \ldots\ldots\ldots \ldots + u\,D^n v.$$

or

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots.. + {}^nC_n u v_n = \sum_{r=0}^{n} {}^nC_r u_{n-r} v_r$$

Proof. We shall prove this theorem by the principle of mathematical induction.We know that first derivative of the product of two function is given by $D(uv) = Du.v + u.Dv$.
Thus the theorem is true for $n = 1$.
Suppose that the theorem is true for n. Then,

$$D^n(uv) = (D^n u).v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + \cdots + \cdots + {}^nC_r D^{n-r}u.D^r v \quad \ldots + u\,D^n v.$$

Differentiating with respect to$x$, we get

$$D^{n+1}(uv) = D[(D^n u).v] + {}^nC_1 D[D^{n-1}u.Dv] + {}^nC_2 D[D^{n-2}u.D^2v] + \ldots + {}^nC_r D[D^{n-r}u.D^r v] + \cdots + D[u.D^n v].$$
$$= \{D^{n+1}u.v + D^n u.Dv\} + {}^nC_1\{D^n u.Dv + D^{n-1}u.D^2v\} + \ldots + {}^nC_r\{D^{n-r+1}u.D^r v + D^{n-r}u.D^{r+1}v\} + \cdots + \{Du.D^n v + u.D^{n+1}v\}.$$

Rearranging the terms, we get

$$D^{n+1}(uv) = D^{n+1}u.v + (1 + {}^nC_1)D^n u.Dv + \cdots + ({}^nC_r + {}^nC_{r+1})D^{n-r}u.D^{r+1}v + \cdots + u.D^{n+1}v.$$

We know that ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

Hence $D^{n+1}(uv) = D^{n+1}u.v + {}^{n+1}C_1\, D^n u.Dv + \cdots + {}^{n+1}C_{r+1}\, D^{n-r}u.D^{r+1}v + \cdots + u.D^{n+1}v.$

Thus, if the theorem is true for any value of n it is also true for the next value $(n+1)$. But we have already seen that theorem is true for $n=1$. Hence it must be true for $n=2$ and so for $n=3$, and so on. Thus the theorem is true for all positive integral values of n.

Note: While applying Leibnitz's theorem if we observe that one of the two functions is such that all its differential coefficients after a certain stage become zero, then we should take the function as the second function.

Ex.18 Find the n^{th} differential coefficient of $x \cos x$.

Sol. Observe that the second and higher derivatives of x are all zero, therefore for the sake of convenience we shall take x as the second function.

By Leibnitz's theorem

$D^n(uv) = (D^n u).v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + \cdots + {}^nC_r D^{n-r}u.D^r v + \quad \ldots + u\, D^n v.$

$$\Rightarrow D^n(\cos x\,.x) = (D^n \cos x).x$$
$$= \cos\left(x + n.\frac{\pi}{2}\right)x.$$

Ex.19 If $y = x^n \log x$, show that $y_{n+1} = \frac{n!}{x}$.

Sol. We have $y = x^n \log x$.

$$y_1 = x^n \frac{1}{x} + n\,x^{n-1}\log x$$
$$xy_1 = x^n + n\,x^n \log x$$
$$xy_1 = x^n + ny.$$

Now differentiating n times with respect to x, and using Leibnitz's theorem, we get

$D^n(y_1 x) = D^n x^n + n\,D^n y.$

$(D^n y_1).x + {}^nC_1(D^{n-1}y_1).(Dx) = n! + ny_n,$

$y_{n+1}.x + n\,y_n.1 = n! + ny_n,$

$\Rightarrow y_{n+1} = \frac{n!}{x}.$

Ex.20 Find the n^{th} differential coefficient of $x^3 e^{ax}$.

Sol. Choosing for the sake of convenience, x^3 to be the second function.

$D^n(x^3 e^{ax}) = a^n e^{ax}.x^3 + {}^nC_1 \quad a^{n-1}e^{ax}.3x^2 + {}^nC_2\, a^{n-2}e^{ax}.6x + {}^nC_3\, a^{n-1}e^{ax}.6.$

$= a^n x^3 e^{ax} + 3\,{}^nC_1 \quad a^{n-1}x^2 e^{ax} + 6\,{}^nC_2 \quad a^{n-2}xe^{ax} + 6\,{}^nC_3\, a^{n-1}e^{ax}.$

Ex.21 If $y = e^{a\sin^{-1}x}$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Sol. Since, $y = e^{a\sin^{-1}x}$,

We have $y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$

Or $(1-x^2)y_1^2 = a^2y^2$

Differentiating with respect to x, we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 2a^2y\,y_1$$

Or $(1-x^2)y_2 - xy_1 = a^2y$

Applying Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = a^2y_n.$$
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

Ex.22 If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, then prove that
$$(x^2-1)y_{n+2} + (2n+1)xy_n + (n^2-m^2)y_n = 0.$$

Sol. $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x \Rightarrow y^{\frac{2}{m}} + 1 = 2x\,y^{\frac{1}{m}}$

$$\Rightarrow y^{\frac{2}{m}} - 2x\,y^{\frac{1}{m}} + 1 = 0.$$

Therefore, $y^{\frac{1}{m}} = \frac{2x\pm\sqrt{4x^2-4x}}{2} = \left(x \pm \sqrt{x^2-1}\right)$

$$\Rightarrow y = \left(x \pm \sqrt{x^2-1}\right)^m.$$

Case (I): Taking positive sign, $y = \left(x + \sqrt{x^2-1}\right)^m$

Differentiating it with respect to x,

$$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left[1 + \frac{1}{2}.\frac{1}{\sqrt{x^2-1}}.2x\right]$$
$$= m\left(x+\sqrt{x^2-1}\right)^{m-1}\left[1 + \frac{x}{\sqrt{x^2-1}}\right]$$
$$= m\left(x+\sqrt{x^2-1}\right)^{m-1}\left[\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right]$$
$$= m\left(x+\sqrt{x^2-1}\right)^{m}\left[\frac{1}{\sqrt{x^2-1}}\right]$$
$$\Rightarrow \sqrt{x^2-1}\,.y_1 = m\left(x+\sqrt{x^2-1}\right)^m$$
$$\Rightarrow \sqrt{x^2-1}\,.y_1 = my$$
$$\Rightarrow (x^2-1)y_1^2 = m^2y^2$$

Case (II): Taking negative sign, $y = \left(x - \sqrt{x^2-1}\right)^m$

Differentiating it with respect to x,

$$y_1 = m\left(x-\sqrt{x^2-1}\right)^{m-1}\left[1 - \frac{1}{2}.\frac{1}{\sqrt{x^2-1}}.2x\right]$$
$$= m\left(x-\sqrt{x^2-1}\right)^{m-1}\left[1 - \frac{x}{\sqrt{x^2-1}}\right]$$
$$= m\left(x-\sqrt{x^2-1}\right)^{m-1}\left[\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right]$$
$$= -m\left(x-\sqrt{x^2-1}\right)^{m}\left[\frac{1}{\sqrt{x^2-1}}\right]$$
$$\sqrt{x^2-1}y_1 = -m\left(x-\sqrt{x^2-1}\right)^m$$
$$\Rightarrow \sqrt{x^2-1}y_1 = -my$$
$$\Rightarrow (x^2-1)y_1^2 = m^2y^2\,.$$

Thus, we get the same value of y_1 for positive and negative signs of (1), thus,$(x^2-1)y_1^2 = m^2y^2$.

Differentiating it again with respect to x,

$$(x^2-1)2y_1y_2 + 2xy_1^2 = m^2 2yy_1$$
$$\Rightarrow (x^2-1)y_2 + xy_1 - m^2y = 0.$$

Ex.23 If $y = a\cos(\log x) + b\sin(\log x)$, show that
$$x^2y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)y_n = 0.$$

Sol. We have $y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$.
$$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x).$$
Differentiating again, we have
$$xy_2 + y_1 = \frac{-a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$$
$$x^2y_2 + xy_1 = -\{a\cos(\log x) + b\sin(\log x)\} = -y$$
Therefore, $x^2y_2 + xy_1 + y = 0$.

Applying Leibnitz's Theorem, we have
$$D^n(x^2y_2) + D^n(xy_1) + D(y) = 0.$$
$[x^2y_{n+2} + {}^nC_1\ 2x.y_{n+1} + {}^nC_2 y_n.2] + 2\,[y_{n+1}.x + {}^nC_1\ y_n.1] = 0$.

$(1+x^2)y_{n+2} + 2nx.y_{n+1} + n(n-1)y_n + 2x\, y_{n+1} + 2n\, y_n = 0$.

or
$$(1+x^2)y_{n+2} + 2(n+1)x\, y_{n+1} + n(n+1)y_n = 0.$$

Ex.24 If $y = tan^{-1}x$, find $(y_n)_0$.

Sol. We have

(1) $y = tan^{-1}x$.

(2) $y_1 = \frac{1}{(1+x^2)}$

(3) $(1+x^2)y_1 - 1 = 0$.

Differentiating (3), we get

(4) $(1+x^2)y_2 + 2xy_1 = 0$.

Differentiating n times by Leibnitz's theorem, we get
$$D^n[(1+x^2)y_2] + 2D^n[xy_1] = 0.$$
$$D^ny_2.(1+x^2) + n.D^{n-1}y_2.2x + \frac{n(n-1)}{2!}.D^{n-2}y_2.2 + 2[D^ny_1.x + n.D^{n-1}y_1.Dx] = 0$$
$\Rightarrow y_{n+2}.(1+x^2) + 2n.y_{n+1}x + n(n-1)y_n + 2y_{n+1}.x + 2n.y_n.1 = 0$

(5) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$.

Putting $x = 0$ in (1), (2) and (4), we get
$$(y)_0 = 0,\ \ (y_1)_0 = 1, and\ (y_2)_0 = 0.$$
Also putting$x = 0$ in equation (5), we get
$$(1+0)(y_{n+2})_0 + 0 + n(n+1)(y_n)_0 = 0$$
(6) $(y_{n+2})_0 = -n(n+1)(y_n)_0$.

Putting $n-2$ in place of n in equation (6), we get

$$(y_n)_0 = -(n-2)(n-1)(y_{n-2})_0$$

Since, from equation (6), we have

$$(y_{n-2})_0 = -(n-3)(n-4)(y_{n-4})_0$$

So,

$$(y_n)_0 = [-(n-2)(n-1)][-(n-3)(n-4)](y_{n-4})_0.$$

Now there are two cases:

Case (I). When n is even, we have

$$(y_n)_0 = [-(n-2)(n-1)][-(n-3)(n-4)] \dots [-3.2](y_2)_0 = 0.$$

Case (II). When n is odd, we have

$$(y_n)_0 = [-(n-2)(n-1)][-(n-3)(n-4)] \dots [-4.3][-2.1](y_1)_0$$

$$(y_n)_0 = (-1)^{\frac{n-1}{2}}(n-1)!, since\ (y_1)_0 = 1.$$

Suppose we have to find $(y_7)_0$

So $n = 7$, an odd number. From case (II), we get

$$(y_7)_0 = (-1)^3(7-1)! = -6!.$$

Now we are in a stage to conclude this unit here.

6.7 SUMMARY

In this unit, we studied the n^{th} differential coefficient of a function, i.e., actual meaning of the successive differentiation. We have seen some standard results on n^{th}differential coefficient are very useful for evaluating n^{th} derivative of various families of functions. Basic trigonometric identities play an important role for finding n^{th} differential coefficient of the trigonometric functions.We also studied the use of partial fraction in the successive differentiation of algebraic functions such as rational functions. When denominator can be factorized into real linear factors, then we get partial fractions form easily. But when denominator cannot be factorized into real linear factors, then factorize the denominator into linear factors, real or imaginary. De Moivre's theorem plays an important task in such types of problems.We have studied the famous work of Leibnitz (Leibnitz's theorem). This theorem provides us a general method for finding the n^{th} derivative of the product of two functions.

6.8 GLOSSARY

i. ***Function:***A function f from S to T, where S and T are non-empty sets, is a rule that associates with each element of S (the domain) a unique element of T (the codomain).

ii. ***Factorial:*** **For** a positive integer n, the notation $n!$ (read as 'n factorial') is used for the product $n(n-1)(n-2) \dots \times 2 \times 1$.Thus $4! = 4 \times 3 \times 2 \times 1 = 24$.

iii. ***De Moivre, Abraham***(1667–1754) Prolific mathematician, born in France, who later settled in England. In De Moivre's Theorem, he is remembered for his use of complex numbers in trigonometry. But he was also the author of two notable early works on probability. His Doctrine of Chances of 1718, examines numerous problems and develops a number of principles, such as the notion of independent events and the product law. Later work contains the result known as Stirling's formula and probably the first use of the normal frequency curve.

iv. ***<u>De Moivre's theorem</u>*:** From the definition of multiplication (of a complex number), it follows that (cos $\theta 1$ + i sin $\theta 1$)(cos $\theta 2$ + i sin $\theta 2$) = cos ($\theta 1$ + $\theta 2$) + i sin ($\theta 1$ + $\theta 2$). This leads to the following result known as De Moivre's Theorem, which is crucial to any consideration of the powers $z^n$of a complex number z:Forall positive integers n,$(cos\theta + i\ sin\ \theta)^n = \cos n\theta + i\ sin\ n\theta = e^{in\theta}$.

v. ***<u>Leibnitz, Gottfried Wilhelm</u>***(1646–1716) A great mathematician, philosopher, scientist and writer on a wide range of subjects, who was, with Newton, the founder of the calculus. Newton's discovery of differential calculus was perhaps ten years earlier than Leibniz's, but Leibniz was the first to publish his account, written independently of Newton, in 1684. Soon after, he published an exposition of integral calculus that included the Fundamental Theorem of Calculus. He also wrote on other branches of mathematics, making significant contributions to the development of symbolic logic, a lead which was not followed up until the end of the nineteenth century.

CHECK YOUR PROGRESS

1. $D^2(x^4e^{5x})$
(i) $e^{5x}(25x^4 + 40x^3 + 12x^2)$. .
(ii) $4x^3e^{5x} + 5e^{5x}x^4$.
(iii) $4x^3e^{5x} + 20x^2e^{5x}$
(iv) None of the above.

2. 10^{th} differential coefficient of e^x
(i) $10e^x$.
(ii) $10!\,e^x$
(iii) 0.
(iv)e^x.

3. $D^n[(5x + 12)^{-1}]$
(i) $(-1)^{n-1}n!\,(5x + 12)^{n+1}$
(ii) $(-1)^n n!\,5^n(5x + 12)^{-n-1}$.
(iii) $(-1)^n n!\,12^n(5x + 12)^{-n-1}$.
(iv) $(-1)^n n!\,5^n(5x + 12)^{-n+1}$.

4. $D^{250}[\sin px]$
(i) $\sin(px + n.\frac{\pi}{2})$.
(ii) $\sin(nx + p.\frac{\pi}{2})$.
(iii) $\sin(px + 250.\frac{\pi}{2})$.
(iv) Nothing can be said.

5. $D^n[\log x]$

(i) $(-1)^n n!\, x^{-n}$.

(ii) $(-1)^{n-1}(n-1)!\, x^{-n}$

(iii) $(-1)^{n-1}(n-1)!\, x^{n}$

(iv) $(-1)^n (n-1)!\, x^{-n}$.

6. $D^n\left[\frac{1}{x^2-a^2}\right]$

(i) $\frac{1}{2a}(-1)^n n!\{(x+a)^{-(n+1)} + (x-a)^{-(n+1)})\}$

(ii) 0

(iii) $\frac{1}{2a}(-1)^n n!\{(x-a)^{-(n+1)} + (x+a)^{-(n+1)})\}$

(iv) $\frac{1}{2a}(-1)^n n!\{(x-a)^{-(n+1)} - (x+a)^{-(n+1)})\}$

7. $If\ y = \mathrm{A}\sin mx\ + Bcos\ mx,$ then

(i) $y_2 - my = 0.$

(ii)$y_2 + m^2 y = 0.$

(iii) $y_2 - m^2 y = 0.$

(iv) $y_2 + my = 0.$

8. $If\ x = a(\cos\theta + \theta\sin\theta),\ \ y = a(\sin\theta - \theta\cos\theta)$, then y_1 is

(i) $\cos\theta$

(ii) $\sin\theta$

(iii) $\tan\theta$

(iv) $\cot\ \theta$

9. $D^n[\sin 2x \sin 3x]$, is

(i) $\frac{1}{2}\left[\cos\left(x + n.\frac{\pi}{2}\right) - 5^n\cos\left(5x + n.\frac{\pi}{2}\right)\right]$

(ii) $\frac{1}{2}\left[\cos\left(x + \frac{\pi}{2}\right) - 5^n\cos\left(5x + n.\frac{\pi}{2}\right)\right]$

(iii) $\frac{1}{2}\left[\cos\left(x + n.\frac{\pi}{2}\right) + 5^n\cos\left(5x + n.\frac{\pi}{2}\right)\right]$

(iv) $\frac{1}{2}\left[\cos\left(x + n.\frac{\pi}{2}\right) - 5^n\cos\left(5x + \frac{\pi}{2}\right)\right]$.

10. $D^4(x^3\log x)$

(i) 0

(ii) $\frac{6}{x}$

(iii) $\frac{-6}{x}$

(iv) 1

6.9 REFRENCES

- **i.** Joseph Edwards, "Differential Calculus for Beginners", Macmillan and Co., Ltd., New York; 1896.
- **ii.** Gorakh Prasad, "Text-Book on Differential Calculus", Pothishala Private Ltd., Allahabad; 1936.
- **iii.** Tom M. Apostol, "Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra", John Wiley & Sons; 1967.
- **iv.** Amit M. Agarwal, "Differential Calculus", Arihant Prakashan, Meerut.

6.10 SUGGESTED READING

- **i.** Differential Calculus for Beginners by Joseph Edwards.
- **ii.** Text-Book on Differential Calculus by Gorakh Prasad.
- **iii.** Calculus by R. Kumar.
- **iv.** Krishna's Text Book on Calculus by A. R. Vasistha.
- **v.** 12th class Mathematics Book by R. D. Sharma.
- **vi.** Pragati's Calculus by Sudhir K. Pundir.
- **vii.** Lectures on Basic Courses (1-2) on NPTEL website.
- **viii.** Leibnitz's work on Wikipedia.

6.11 TERMINAL QUESTIONS

1. If $p^2 = a^2\cos\theta + b^2\sin^2\theta$, prove that

$$p + p_2 = \frac{a^2b^2}{p^3}.$$

2. Find the n^{th} differential coefficient of $e^{ax}\sin bx$ and deduce the n^{th} differential coefficient of $\sin x \sin bx$.

3. If $y = \sin^{-1} x$, prove that$(1 + x^2)y_2 - xy_1 = 0$.
Also prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.

4. If $y = x^2e^x$, show that

$$y_n = \frac{1}{2}.n.(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

5. If $y = \cos(\log x)$, prove that

$$x^2y_{n+2} + (2n + 1)xy_{n+1} + (n^2 + 1)y_n = 0.$$

6. If $\cos^{-1}(\frac{y}{b}) = \log\left(\frac{x}{n}\right)^n$, prove that

$$x^2y_{n+2} + (2n + 1)xy_{n+1} + 2n^2y_n = 0.$$

7. If $y = Ae^{-kt}\cos(pt + c)$, show that

$$y_2 + 2ky + n^2 y = 0,$$
where $n^2 = p^2 + k^2$.

8. Prove that the value of the n^{th} differential coefficient of $\frac{x^3}{x^2-1}$ for $x = 0$ is zero if n is even, and is $-n!$ if n is odd and greater than 1.

9. Find the n^{th} derivative of $x^2 \sin x$ at $x = 0$.

10. Find the n^{th} derivative of following functions:

(i) $e^{2x} + e^{-2x}$. **(ii)** $\sin 2x \sin 3x$.

6.12 ANSWERS

ANSWER CHECK YOUR PROGRESS

SCQ1. (i)
SCQ2. (iv)
SCQ3 (ii)
SCQ4 (iii)
SCQ5 (ii)
SCQ6 (iv)
SCQ7 (ii)
SCQ8 (iii)
SCQ9 (i)
SCQ10 (ii)

ANSWER OF TERMINAL QUESTIONS

TQ9 At $x = 0$,

$$\frac{d^n}{dx^n}(x^2 \sin x) = n(n-1)\sin\left((n-2)\frac{\pi}{2}\right) = (n-n^2)\sin\frac{n\pi}{2}$$

TQ10 (i) $2^n[e^{2x} + (-1)^n e^{-2x}]$

(ii) $\frac{1}{2}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{2}5^n\cos\left(5x + \frac{n\pi}{2}\right)$

UNIT 7:- EXPANSION OF FUNCTION OF ONE VARIABLE

CONTENTS:-

7.1 INTRODUCTION

A Scottish mathematician Colin Maclaurin (1698-1746) was the first person in the history of mathematics, who equaled an indefinitely differentiable function with a convergent series at the origin. A series is called convergent if its sum is finite and unique. Almost all the expansions of functions which we know are actually the expansions at the origin.After that a British mathematician Brook Taylor raised the question:

"If we have an indefinitely differentiable function $y = f(x)$, differentiable at $x = a, a \in \mathbb{R}$, Can we have a similar expression at that point?"

The answer to this question is '**Yes**' and we know this expression as Taylor's series. Actually Taylor's series is generalization of Maclaurin's series.

7.2 OBJECTIVES

After completion of this unit learner will be able to understand

i. Expansion of a function in an infinite power-series.
ii. Maclaurin's Theorem
iii. Taylor's Theorem
iv. Maclaurin's Theorem

7.3 MACLAURIN'S THEOREM

Theorem: Let f(x) be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$ and can be expanded as an infinite series in x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots.$$

Proof: Suppose

(1) $f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots.$

Since, $f(x)$ is differentiable term by term any number of times, then by successive differentiation, we have

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots$$

$$f''(x) = 2.1.A_2 + 3.2.A_3x + 4.3\ A_4x^2 + \cdots$$

$$f'''(x) = 3.2.1.A_3 + 4.3.2\ A_4x + \cdots$$

Putting $x = 0$, we get

$$f(0) = A_0, \qquad f'(0) = A_1, \qquad f''(0) = 2!\,A_2, \qquad f'''(0) = 3!\,A_3, \ \ldots$$

$$\Rightarrow A_0 = f(0), \qquad A_1 = f'(0), \qquad A_2 = \frac{f''(0)}{2!}, \qquad A_3 = \frac{f'''(x)}{3!}, \ \ldots$$

Substituting all these values in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Note: Maclaurin's expansion of $f(x)$ fails if any of the functions $f(x), f'(x), f''(x), \ldots$ becomes infinite or discontinuous at any point of the interval $[0, x]$.

7.4 TAYLOR'S THEOREM

Theorem: Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[a, a + h]$, assuming that $f(a + h)$ can be expanded as an infinite power series in h, we have

$$f(a + h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots.$$

Proof : Suppose

(1) $f(a+h\,)=A_0+A_1h+A_2h^2+A_3h^3+\cdots.$

Since, the expansion (1) be differentiable term by term any number of times with respect to h. Then by successive differentiation with respect to h, we have

$$f'(a+h)=A_1+2A_2h+3A_3h^2+4A_4h^3+\cdots.$$

$$f''(a+h)=2.1\,A_2+3.2.A_3h+4.3A_4h^2+\cdots.$$

$$f'''(a+h)=3.2.1\,A_3+4.3.2.A_4h+\cdots,$$

$$f^{iv}(a+h)=4.3.2.1\,A_4+\cdots,and\ so\ on.$$

Putting $h=0$ in each of the above relations, we get

$$f(a)=A_0,\qquad f'(a)=A_1,$$
$$f''(a)=2!\,A_2,\qquad f'''(a)=3!\,A_3,\qquad f^{iv}(a)=4!\,A_4$$

In general, $f^n=n!\,A_n$.

$$\Rightarrow A_0=f(a),\ \ A_1=f'(a),\qquad A_2=\frac{1}{2!}f''(a),\ \ A_3=\frac{1}{3!}f'''(a),$$
$$\ldots\quad ,A_n=\frac{1}{n!}f^n(a),and\ \ so\ on.$$

Substituting these values of $A_0,A_1,A_2,A_3,\ldots,A_n,\ldots$ in (1), we get

$$f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots+\frac{h^n}{n!}f^n(a)$$
$$+\cdots.$$

This is Taylor’s theorem.

Remark: (I) Another useful form of Taylor’s theorem is obtained on replacing h by $(x-a)$. Thus

$$f(x)=f\big(a+(x-a)\big)$$

$$\Rightarrow f(x)=f(a)+(x-a)\,f'(a)+\frac{(x-a)^2}{2!}f''(a)$$
$$+\frac{(x-a)^3}{3!}f'''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)+\cdots.$$

which is an expansion of $f(x)$ as power series in $(x-a)$.

(II) Since, Taylor’s theorem is given by

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots.$$

If we put $a = 0$, we get

$$f(x) = f(0) + x\,f'(0) + \frac{(x)^2}{2!}f''(0) + \cdots + \frac{(x)^n}{n!}f^n(0) + \cdots.$$

which is Maclaurin's theorem.

In this unit we observed that Binomial, Exponential, Logarithmic and other well known expansions are all particular cases of one general theorem, i.e., Taylor's Theorem. But in many cases it is not possible to find such an expansion for a function.

In this unit, we obtained formal expansion of a function $\boldsymbol{f(x)}$ without giving any idea of ranges of values of $\boldsymbol{x}$ for which the expansion is valid. In general,

Taylor's Infinite Series: If a function $f(x)$ possesses derivatives of all orders in the interval $[a, a+h]$. Then for every positive integer n, we have

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n.$$

Where $R_n = \frac{h^n}{n!}f^n(a+\theta h), (0 < \theta < 1)$.

$$f(a+h) = S_n + R_n.$$

Wher$S_n = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$.

Let us suppose $R_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(a+h)$

$$\Rightarrow f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \cdots.$$

If R_n does not tends to zero as $n \to \infty$, then $\lim_{n\to\infty} S_n \neq f(a+h)$.

Now consider, **Maclaurin's series**

"If a function $f(x)$ possesses derivatives of all orders in the interval $[0, x]$. Then for every positive integer n, we have

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n.$$

Where $R_n = \frac{h^n}{n!}f^n(\theta x), (0 < \theta < 1)$.

$$f(x) = S_n + R_n.$$

Where $S_n = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0)$.

Let us suppose $R_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(x)$

$$\Rightarrow f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \cdots.$$

If R_n does not tends to zero as $n \to \infty$, then $\lim_{n\to\infty} S_n \neq f(x)$."

From the above discussion, we can conclude if any of the function $f(x), f'(x), f''(x), \ldots$ becomes infinite or does not exists for any value of x in the given interval or if R_n does not tends to zero as $n \to \infty$. Then Taylor's theorem or Maclaurin's theorem fails to expand $f(a+h)$ in an infinite power series. Thus, before expand a given function as an infinite Taylor's theorem, it is essential to examine the behavior of R_n as $n \to \infty$, which is not a simple task. Therefore we obtained the expansion by assuming the possibility of expanding it in an infinite series (i.e., by assuming $R_n \to 0$ as $n \to \infty$).

Sometimes, we want only few terms of an expansion, then it is more convenient to use the Binomial, Exponential or the Logarithmic theorems, or the well-known expansions of $\sin x$ and $\cos x$. Occasionally it is very easy to derive an expansion by differentiating a known series.

7.5. SOLVED EXAMPLE

Example 1. Expand $\sin x$ by Maclaurin's theorem.

Solution: Let $f(x) = \sin x$.

By Maclaurin's theorem, we expand $f(x)$ as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Now, we have to find $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0), \dots$

Since $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$.

$f'(x) = \cos x \Rightarrow f'(0) = 1$.

$f''(x) = -\sin x \Rightarrow f''(0) = 0$.

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$.etc.

$f^n(x) = \sin\left(x + n.\frac{\pi}{2}\right) \quad \Rightarrow \quad f^n(0) = \begin{cases} 0 & if\ n = 2m \\ (-1)^m & if\ n = 2m+1 \end{cases}$.

Substituting the values of $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0), \dots$

$$\sin x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.(-1) + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.(-1)^2 + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots.$$

Example 2. Expand $f(x) = e^x$ by Maclaurin's theorem.

Solution: Let $f(x) = e^x$.

By Maclaurin's theorem, we expand $f(x)$ as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Now, we have to find $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0), \dots$

Since $f(x) = e^x \Rightarrow f(0) = e^0 = 1$.

$f'(x) = e^x \Rightarrow f'(0) = 1.$

$f''(x) = e^x \Rightarrow f''(0) = 1.$

$f'''(x) = e^x \Rightarrow f'''(0) = 1.$

etc. In general

$f^n(x) = e^x \Rightarrow f^n(0) = 1.$

Substituting the values of $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0)$, ... We have $e^x = 1 + x.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.1 + \cdots + \frac{x^n}{n!}.1 + \cdots.$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Example 3. Expand $\cos x$by Maclaurin's theorem.

Solution: Let $f(x) = \cos x$.

By Maclaurin's theorem, we expand $f(x)$ as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Now, we have to find $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0), \dots$

Since $f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1.$

$f'(x) = -\sin x \Rightarrow f'(0) = 0.$

$f''(x) = -\cos x \Rightarrow f''(0) = -1.$

$f'''(x) = \sin x \Rightarrow f'''(0) = 1.$

etc. In general,

$f^n(x) = \cos\left(x + n.\frac{\pi}{2}\right) \quad \Rightarrow \quad f^n(0) = \begin{cases} (-1)^m & if\ n = 2m \\ 0 & if\ n = 2m+1 \end{cases}.$

Substituting the values of $f(0), f'(0), f''(0), f'''(0), \dots, f^n(0), \dots$

$$\cos x = 1 + x.0 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.(-1)^2 + \frac{x^5}{5!}.0 + \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots.$$

Example 4. Expand $(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{n}}$.

Solution: Let $f(x)=(1+x)^n$. Since,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+\frac{x^m}{m!}f^m(0)+\cdots.$$

{Observe that we made distinction between n^{th} term of the Maclaurin's series (by replacing n by m) and n used in question}

Now, we have to find $f(0), f'(0), f''(0), f'''(0), \dots, f^m(0), \dots$

$f(x)=(1+x)^n \Rightarrow f(0)=1$.

$f'(x)=n(1+x)^{n-1} \Rightarrow f'(0)=n$.

$f''(x)=n(n-1)(1+x)^{n-2} \Rightarrow f''(0)=n(n-1)$.

....

In general, $f^m(x)=n(n-1)(n-2)\dots(n-m+1)(1+x)^{(n-m)}$

$$\Rightarrow f^m(0)=n(n-1)(n-2)\dots(n-m+1).$$

Substituting the suitable values in Maclaurin's series, we get

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\dots(n-m+1)}{m!}x^m+\cdots.$$

This is known as Binomial series.

Example 5. Expand $\boldsymbol{log}\ (\mathbf{1}+\boldsymbol{x})$.

Solution: Let $f(x)=\log(1+x)$. Since,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+\frac{x^m}{m!}f^m(0)+\cdots.$$

Now, $f(x)=\log(1+x) \Rightarrow f(0)=\log 1=0$.

$f'(x)=\frac{1}{(1+x)} \Rightarrow f'(0)=1$.

$f''(x)=\frac{(-1)}{(x+1)^2} \Rightarrow f''(0)=-1!$.

$f'''(x)=\frac{(-1)^2 2.1}{(x+1)^3} \Rightarrow f'''(x)=2!$.

and so on. In general $f^n(x) = \frac{(-1)^{(n-1)}(n-1)!}{(x+1)^n} \Rightarrow f^n(0) = (-1)^{n-1}(n-1)!$.Substituting the suitable values

$$\log(1+x) = 0 + x.1 + \frac{x^2}{2!}.(-1!) + \frac{x^3}{3!}.(2!) + \cdots + \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \cdots.$$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots.$$

Example 6. Expand a^x.

Solution. Let $f(x) = a^x$. Then $f(0) = 1$.

$$f'(x) = a^x \log a \Rightarrow f'(0) = \log a.$$

$$f''(x) = a^x(log\ a)^2 \Rightarrow f''(0) = (\log a)^2.$$

$$f'''(x) = a^x(\log a)^3 \Rightarrow f'''(0) = (\log a)^3.$$

And, so on. In general,

$$f^n(x) = a^x(\log a)^n \Rightarrow f^n(0) = (\log a)^n.$$

Now by Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^m}{m!}f^m(0) + \cdots.$$

So,

$$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots + \frac{x^n}{n!}(\log a)^n + \cdots.$$

Example 7. Expand $\tan x$.

Solution: Let $y = \tan x$. Then $(y)_0 = 0$.

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + (y)^2 \Rightarrow (y_1)_0 = 1 + (y)_0^2 = 1 + 0 = 1,$$

$$y_2 = 2yy_1 \Rightarrow (y_2)_0 = 2(y)_0(y_1)_0 = 2.0.1 = 0,$$

$$y_3 = 2y_1y_1 + 2yy_2 \Rightarrow (y_3)_0 = 2(y_1)_0^2 + 2(y)_0.(y_2)_0 = 2.(1)^2 + 2.0.0 = 2,$$

Similarly $(y_4)_0 = 0,\ and\ (y_5)_0 = 16$, and so on.

Using Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

We get,

$$f(x) = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.16 + \cdots.$$

$$\Rightarrow \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

Example 8. Expand $\sin^{-1} x$.

Solution: Let $y = \sin^{-1} x$. Then $y_1 = \frac{1}{\sqrt{1-x^2}}$

$$(1 - x^2)y_1^2 - 1 = 0.$$

Differentiating again, we have

$$(1 - x^2)2y_1y_2 - 2xy_1^2 = 0.$$

$$(1 - x^2)y_2 - xy_1 = 0.$$

{Note: Here $2y_1 \neq 0$.}

Now differentiating n- times by Leibnitz's rule, we get

$$(1 - x^2)y_{n+2} - (2n + 1)xy_1 - n^2y_n = 0.$$

Putting $x = 0$ in the above relations, we get

$$(y)_0 = 0, \quad (y_1)_0 = 1, \quad (y_2)_0 = 0, \ \ and\ (y_{n+2})_0 = n^2(y_n)_0.$$

Putting $n = 1, 2, 3, \ldots,$ we have $(y_3)_0 = 1^2,\ (y_4)_0 = 2^2.0 = 0\,, (y_5)_0 = 3^2.1^2, (y_6)_0 = 0, (y_7)_0 = 5^2.3^2.1^2,\ and\ so\ on.$

Now by Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\Rightarrow \sin^{-1} x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.1^2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.3^2.1^2 + \frac{x^6}{6!}.0 + \frac{x^7}{7!}.5^2.3^2.1^2 + \cdots.$$

$$\Rightarrow \sin^{-1} x = x + \frac{x^3}{3!}.1^2 + \frac{x^5}{5!}.3^2.1^2 + \frac{x^7}{7!}.5^2.3^2.1^2 + \cdots.$$

$$\Rightarrow \sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7!} + \cdots.$$

Example 9. Show that the first five terms in the power series for $\log(1 + \sin x)$ are

$$x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24}.$$

Solution: Let $y = \log(1 + \sin x)$. Then $(y)_0 = \log 1 = 0$.

$y_1 = \frac{\cos x}{(1+\sin x)} \Rightarrow (y_1)_0 = 1,$

$y_2 = \frac{-1}{1+\sin x} \Rightarrow (y_2)_0 = -1,$

$y_3 = \frac{\cos x}{(1+\sin x)}.\frac{1}{(1+\sin x)} = -y_1 y_2 \Rightarrow (y_3)_0 = 1.$

$y_4 = -y_1 y_3 - y_2^2 \Rightarrow (y_4)_0 = -1.(1) - (-1)^2 = -2.$

$y_5 = -y_1 y_4 - y_2 y_3 - 2y_2 y_3 = -y_1 y_4 - 3y_2 y_3 \Rightarrow (y_5)_0 = 5.$

By using Maclaurin's series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Substituting the values, we get

$$\log(1 + \sin x) = 0 + x.1 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}.1 + \frac{x^4}{4!}.(-2) + \frac{x^5}{5!}.(5) + \cdots.$$

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \cdots.$$

Therefore, the first five terms in the power series for $\log(1 + \sin x)$ are

$$x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24}.$$

Example 10. Expand $\sec x$.

Solution: Let $y = \sec x$. Then $(y)_0 = 1$.

$$y_1 = \sec x \tan x \Rightarrow (y_1)_0 = 0.$$

$$y_2 = \sec x . \sec^2 x + \sec x . \tan x . \tan x$$
$$= \sec^3 x + \sec x . (\sec^2 x - 1) = 2\sec^3 x - \sec x$$

$$\Rightarrow y_2 = 2y^3 - y \Rightarrow (y_2)_0 = 2.1^3 - 1 = 1.$$

$$y_3 = 6y^2y_1 - y_1 \Rightarrow (y_3)_0 = 6.1^2.0 - 0 = 0.$$

$$y_4 = 6y^2y_2 + 12yy_1 - y_2 \Rightarrow (y_4)_0 = 6.1^2.1 + 12\,.1.0 - 1$$
$$= 5, \textit{and so on}.$$

By using Maclaurin's series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\sec x = 1 + x.0 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.5 + \cdots.$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots.$$

Example 11. Expand $\log \sec x$.

Solution: Let $y = \log \sec x$. Then $(y)_0 = 0$.

$$y_1 = \tan x \Rightarrow (y_1)_0 = 0.$$

$$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 \Rightarrow (y_2)_0 = 1 + 0^2 = 1.$$

$$y_3 = 2y_1y_2 \Rightarrow (y_3)_0 = 0.$$

$$y_4 = 2y_1y_3 + 2y_2^2 \Rightarrow (y_4)_0 = 2.$$

$$y_5 = 2y_1y_4 + 2y_2y_3 + 4y_2y_3 \Rightarrow y_5 = 2y_1y_4 + 6y_2y_3 \Rightarrow (y_5)_0 = 0.$$

$$y_6 = 2y_1y_5 + 2y_2y_4 + 6y_2y_4 + 6y_3^2 = 2y_1y_5 + 8y_2y_4 + 6y_3^2$$
$$\Rightarrow (y_6)_0 = 8.1.2 = 16.$$

Now by Maclaurin's theorem

$$y = (y)_0 + x.(y_1)_0 + \frac{x^2}{2!}.(y_2)_0 + \frac{x^3}{3!}.(y_3)_0 + \cdots + \frac{x^n}{n!}.(y_n)_0 + \cdots.$$

$$\log(\sec x) = 0 + x.0 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.2 + \frac{x^5}{5!}.0 + \frac{x^6}{6!}.16 + \cdots.$$

$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots.$$

Example 12. Expand $e^x \sec x$,

Solution: Let $y = e^x \sec x$. Then $(y)_0 = 1$.

$$y_1 = e^x \sec x + e^x \sec x\,.\tan x = y + y\tan x \Rightarrow (y_1)_0 = 1,$$

$$y_2 = y_1 + y_1 \tan x + y.\sec^2 x \Rightarrow (y_2)_0 = 2.$$

$$y_3 = y_2 + y_2 \tan x + 2y_1 \sec^2 x + 2y \sec^2 x \,.\tan x \Rightarrow (y_3)_0 = 2 + 2 = 4,$$

And so on.

Substituting these values in Maclaurin's theorem

$$y = (y)_0 + x.(y_1)_0 + \frac{x^2}{2!}.(y_2)_0 + \frac{x^3}{3!}.(y_3)_0 + \cdots + \frac{x^n}{n!}.(y_n)_0 + \cdots.$$

We get,

$$e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \cdots.$$

In the above examples, we expanded following functions:

i. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots.$

ii. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$

iii. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots.$

iv. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-m+1)}{m!}x^m + \cdots.$

v. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots.$

vi. $a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots + \frac{x^n}{n!}(\log a)^n + \cdots.$

vii. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$

viii. $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7!} + \cdots.$

These expansions are very useful for finding the limit of indeterminate forms (see unit- Indeterminate Forms).

Example 13. Expand $\log\sin(x+h)$ in powers of h by Taylor's theorem.

Solution:Let $f(x+h) = \log\sin(x+h)$

$$\Rightarrow f(x) = \log\sin x,$$

$$f'(x) = \frac{1}{\sin x}.\cos x = \cot x,$$

$$f''(x) = -cosec^2 x,$$

$$f'''(x) = 2cosec\, x.\,cosec\, x \cot x,$$

$$\ldots\ldots\ldots$$

Now by Taylor's theorem, we have

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots.$$

Substituting the values of $f(x), f'(x), f''(x), f'''(x), \ldots$

$$\Rightarrow \log\sin(x+h) = \log\sin x\; + h\cot x - \frac{h^2}{2}cosec^2x + \frac{h^3}{3}cosec^2x\;cot\;x + \cdots.$$

Example 14. Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$ by using Taylor's series.

Solution: Let $f(x) = \sin x$. Then $f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$

Now, expanding $f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$ by Taylor's theorem in powers of $\left(x-\frac{\pi}{2}\right)$, we get

$$f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 f''\left(\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 f'''\left(\frac{\pi}{2}\right) + \cdots.$$

Now, $f(x) = \sin x \Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$$

$$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0.$$

$$f^{iv}(x) = \sin x \Rightarrow f^{iv}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

Putting all these values in (1), we get

$$\sin x = 1 - \left(x-\frac{\pi}{2}\right).0 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3.0 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \cdots.$$

$$\Rightarrow \sin x = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 + \cdots.$$

Example 15. If $f(x) = x^3 + 8x^2 + 15x - 24$, calculate the value of $f\left(\frac{11}{10}\right)$ by Taylor's series.

Solution: Let $f(x) = x^3 + 8x^2 + 15x - 24$. Since $f\left(\frac{11}{10}\right) = f\left(1 + \frac{1}{10}\right)$

So, $f(1) = 0. f'(x) = 3x^2 + 16x + 15 \Rightarrow f'(1) = 34.$

$$f''(x) = 6x + 16 \Rightarrow f''(1) = 22.$$

$$f'''(x) = 6 \Rightarrow f'''(1) = 6.$$

$$f^{iv}(x) = 0 \Rightarrow f^{iv}(1) = 0.$$

By Taylor's theorem,

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots.$$

Put $x = \frac{11}{10}, a = \frac{1}{10}$

$$f\left(\frac{11}{10}\right) = f\left(1 + \frac{1}{10}\right)$$
$$= f(1) + \frac{1}{10} f'(1) + \frac{1}{10^2}.\frac{1}{2!} f''(1) + \frac{1}{10^3}.\frac{1}{3!}.f'''(1) + \cdots.$$

$$\Rightarrow f\left(\frac{11}{10}\right) = 0 + \frac{1}{10}.34 + \frac{1}{10^2}.\frac{1}{2}.22 + \frac{1}{10^3}.\frac{1}{6}.6$$
$$= 3.4 + 0.11 + 0.001 = 3.511.$$

Example 16. Prove that

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \cdots.$$

Solution: First we observe that we are expand $\sin(x+h)$in powers of h.

So, let $f(x) = \sin x$. Then $f(x+h) = \sin(x+h)$.

By using Taylor's theorem

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots.$$

Since $f(x) = \sin x$

$$\Rightarrow f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f'''(x) = -\cos x, \ldots, \textit{and so on.}$$

Substituting these values in Taylor's theorem, we get

$$\sin(x+h) = \sin x + h.\cos x + \frac{h^2}{2!}.(-\sin x) + \frac{h^3}{3!}.(-\cos x) + \cdots.$$

$$\Rightarrow \sin(x+h) = \sin x + h.\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \cdots.$$

Example 17. Expand $2x^3 + 7x^2 + x - 1$ in powers of $(x-2)$.

Solution: Let $f(x) = 2x^3 + 7x^2 + x - 1$.

We have $f(x) = f[2 + (x-2)]$.

$$f(x) = 2x^3 + 7x^2 + x - 1 \Rightarrow f(2) = 45.$$

$$f'(x) = 6x^2 + 14x + 1 \Rightarrow f'(2) = 53.$$

$$f''(x) = 12x + 14 \Rightarrow f''(2) = 38.$$

$$f'''(x) = 12 \Rightarrow f'''(2) = 12.$$

In general, $f^n(x) = 0, when\ n \geq 4 \Rightarrow f^n(2) = 0, when\ n \geq 4$.

Using Taylor's theorem

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots.$$

$$\Rightarrow f(x) = f[2 + (x-2)]$$
$$= f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2)$$

Substituting the values of $f(2), f'(2), f''(2), and\ f'''(2)$

[Notice that $f^n(2) = 0, when\ n \geq 4$]

$$f(x) = 45 + (x-2).53 + \frac{(x-2)^2}{2!}.38 + \frac{(x-2)^3}{3!}.12$$

$$\Rightarrow f(x) = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

Example 18. Find $\sqrt{17}$ to four decimal places by using Taylor's theorem.

Solution. Let $f(x) = \sqrt{x}$.

Now we have to find $f(17)$

Since $f(17) = f(16+1)$

Consider Taylor's theorem

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots.$$

$$f(17) = f(16+1)$$
$$= f(16) + 1.f'(16) + \frac{1^2}{2!}.f''(16) + \frac{1^3}{3!}f'''(16) + \frac{1^4}{4!}f^{iv}(16) + \cdots.$$

Since $f(x) = \sqrt{x} \Rightarrow f(16) = 4$.

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(16) = \frac{1}{8}$.

$f''(x) = \frac{1}{2}.\frac{-1}{2}.(x)^{\frac{-3}{2}} = \frac{-1}{4}(x)^{\frac{-3}{2}} \Rightarrow f''(16) = \frac{-1}{4.(4^3)} = \frac{-1}{4^4}$.

$f'''(x) = \frac{1}{2}.\frac{-1}{2}.\frac{-3}{2}.(x)^{\frac{-5}{2}} \Rightarrow f'''(16) = \frac{3}{8.(4^5)}$.

Substituting the values, we get

$$\sqrt{17} = 4 + 1.\frac{1}{8} + \frac{1}{2!}.\frac{-1}{4^4} + \frac{1}{3!}.\frac{3}{8.(4^5)}.$$

$$\Rightarrow \sqrt{17} = 4 + 0.125 - 0.00195 + 0.00002$$

$$\sqrt{17} = 4.12307 \approx 4.123.$$

Note: In this example we have $f(x) = \sqrt{x}$, and we take $f(17) = f(16+1)$, and we expand it by Taylor's theorem in powers of $h = 1$ and derivatives are taken at $a = 16$for the sake of convenience. Because we can find the exact value of the function and its derivative at $a = 16$. Suppose if we take $f(17) = f(15+2)$, it will be very difficult to find the value of the function and its derivative at $x = 15$.

Example 19. Expand e^x in powers of $(x-1)$.

Solution: Let $f(x)=e^x$. We are to expand $f(x)$ in powers of $(x-1)$.

Hence, $f(x)=f[1+(x-1)]$

We know that

$$f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots+\frac{h^n}{n!}f^n(a)+\cdots.$$

$$\Rightarrow f(x)=f[1+(x-1)]$$
$$=f(1)+(x-1)\,f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\cdots+\frac{(x-1)^n}{n!}f^n(1)+\cdots.$$

Now we are to find $f(1), f'(1), f''(1), f'''(1), \dots, f^n(1), \dots$. For this

$$f(x)=e^x\Rightarrow f(1)=e.$$

$$f'(x)=e^x\Rightarrow f'(1)=e.$$

$$f''(x)=e^x\Rightarrow f''(1)=e.$$

$$f'''(x)=e^x\Rightarrow f'''(1)=e.$$

In general, $f^n(x)=e^x\Rightarrow f^n(1)=e$.

Substituting these values in above expansion, we get

$$e^x=e+(x-1)e+\frac{(x-1)^2}{2!}e+\frac{(x-1)^3}{3!}e+\cdots+\frac{(x-1)^n}{n!}e+\cdots.$$

$$\Rightarrow e^x=e\left[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots+\frac{(x-1)^n}{n!}+\cdots\right].$$

Example 20. Show that $\boldsymbol{log\,(x+h)=log\,h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\cdots}$.

Solution: Firstly observe that we are to expand $\log(x+h)$ in ascending powers of x.

So, let $f(h)=\log h$.

Then $f(h+x)=\log(h+x)$.

By Taylor's theorem

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots.$$

Now we have,

$$f(h+x) = f(h) + x\,f'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots + \frac{x^n}{n!}f^n(h) + \cdots.$$

Where $f(h) = \log h$.

$$f'(h) = \frac{1}{h};\ \ f''(h) = \frac{(-1)}{h^2};\ \ f'''(h) = \frac{2}{h^3};\ \ etc.$$

Substituting these values in Taylor's expansion, we get

$$log\ (x+h) = log\ h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots.$$

Example 21. Prove that $\boldsymbol{f(mx) = f(x) + (m-1)xf'(x) + \frac{1}{2!}(m-1)^2x^2f''(x) + \cdots}$.

Solution: Since, we are to expand $f(mx)$ in powers of $(m-1)x$.

So, $f(mx) = f[x + (m-1)x]$

By Taylor's Theorem

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots.$$

$$\Rightarrow f(mx) = f[x + (m-1)x]$$

$$= f(x) + (m-1).x.f'(x) + \frac{1}{2!}(m-1)^2.x^2f''(x) + \cdots.$$

Example 22. Expand $\boldsymbol{log\ sin\ x}$ in powers of $\boldsymbol{(x-a)}$.

Solution: Now we are to expand $f(x)$ in powers of $(x-a)$. For this,

$$f(x) = f[a + (x-a)]$$

By Taylor's theorem,

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots.$$

$$\Rightarrow f(x) = f[a + (x - a)]$$
$$= f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \cdots.$$

Since $f(x) = \log \sin x \Rightarrow f(a) = \log \sin a$.

$$f'(x) = \frac{1}{\sin x}.\cos x = \cot x \Rightarrow f'(a) = \cot a.$$

$$f''(x) = -cosec^2 x \Rightarrow f''(a) = -cosec^2 a,$$

$$f'''(x) = 2cosec\, x.cosec\, x \cot x \Rightarrow f'''(a) = 2cosec^2 a \cot a.$$

Substituting these values in Taylor's theorem, we get

$$\log \sin x = \log \sin a + (x - a) \cot a + \frac{(x - a)^2}{2!} (-cosec^2 a) + \frac{(x - a)^3}{3!} (2cosec^2 a \cot a) + \cdots.$$

$$\Rightarrow \log \sin x = \log \sin a + (x - a) \cot a - \frac{(x - a)^2}{2!} (cosec^2 a) + \frac{(x - a)^3}{3!} (2cosec^2 a \cot a) + \cdots.$$

Example 23. Expand $\cos^3 x$ in powers of x.

Solution: Let $f(x) = \cos^3 x = \frac{1}{4}[\cos 3x + 3 \cos x]$

We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots$.

$$f(x) = \frac{1}{4}[\cos 3x + 3 \cos x]$$
$$= \frac{1}{4}\left\{\left[1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \cdots + (-1)^m \frac{3^{2m}x^{2m}}{(2m)!} + \cdots\right] + 3\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots\right]\right\}$$

$\Rightarrow \cos^3 x = \frac{1}{4}\left\{\left[(1 + 3) - (3^2 + 3)\frac{x^2}{2!} + (3^4 + 3)\frac{x^4}{4!} - \cdots + (-1)^m (3^{2m} + 3)\frac{x^{2m}}{(2m)!} + \cdots\right]\right\}$.

Example 24. Expand $e^{\sin x}$ as far as the term containing x^4.

Solution: We have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots$

$$\Rightarrow \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots.$$

$$\begin{aligned} e^{\sin x} &= e^{\left(x - \frac{x^3}{6} + \cdots\right)} \\ &= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 \\ &\quad + \frac{1}{6}\left(x - \frac{x^3}{6} + \cdots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6} + \cdots\right)^4 + \cdots \end{aligned}$$

$$\Rightarrow e^{\sin x} = 1 + x - \frac{x^3}{6} + \cdots + \frac{1}{2}\left(x^2 - \frac{1}{3}x^4 + \cdots\right) + \frac{1}{6}(x^3 + \cdots) + \frac{1}{24}(x^4 + \cdots) + \cdots$$

$$\Rightarrow e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots.$$

Example 25. Expand $\tan x$ in powers of x as far as term involving x^5.

Solution: We know that

$$\tan x = \frac{\sin x}{\cos x}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots; \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots;$$

$$\Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

By actual division, we get

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

Example 26.Expand $\boldsymbol{sec^2\, x}$.

Solution: We know thattan $x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

Differentiating with respect to x

$$\sec^2 x = 1 + \frac{1}{3}.3x^2 + \frac{2}{15}.5x^4 + \cdots.$$

$$\Rightarrow \sec^2 x = 1 + x^2 + \frac{2}{3}x^4 + \cdots.$$

Example 27. Expand $(1+x)^{-1}$.

Solution: We know that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots.$$

Differentiating with respect to x

$$\frac{1}{(1+x)} = 1 - \frac{2\,x}{2} + \frac{3x^2}{3} - \cdots + (-1)^n\frac{nx^{n-1}}{n} + \cdots.$$

$$\Rightarrow (1+x)^{-1} = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + \cdots.$$

7.6 SUMMARY

In this unit, we studied the expansion of an indefinitely differentiable function $f(x)$ in an infinite power series by Maclaurin's and Taylor's Theorems. We have seen that such an expansion is not always possible. It is **necessary that $\boldsymbol{R_n} \to \mathbf{0}$ as** $\to \infty$, but to examine the nature of R_n as $n \to \infty$ is difficult. So, we obtained the formal proofs of Maclaurin's and Taylor's theorems without bothering about the nature of R_n as $n \to \infty$. In this unit, we obtained the expansion by assuming the possibility of expanding it in an infinite series (i.e., by assuming $R_n \to 0$ as $n \to \infty$).

We also studied that if any of the function $f(x), f'(x), f''(x), \ldots$ becomes infinite or do not exist for any value of x in the given interval, then Taylor's and as well as Maclaurin's theorems fail to expand the function. For example: The function $\log x$ does not possesses Maclaurin's expansion because it is not defined at $x = 0$. Another such function is $\cot x$.

We also found that the well- known expansions such as Binomial, Exponential, Logarithmic and Trigonometric expansions are special cases to Taylor's Theorem.We have seen some expansions of functions in power series which are easily calculated by using these well-known expansions or by differentiating or integrating the well-known expansion. These expansions are also very useful for evaluating the indeterminate forms.

7.7 GLOSSARY

i. ***Function:***A function f from S to T, where S and T are non-empty sets, is a rule that associates with each element of S (the domain) a unique element of T (the codomain).

ii. ***Power series***:A series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$ inascending powers of x, with coefficients

$a_0, a_1, a_2, \dots, a_n, \dots$ is apower series in x. For example, the geometric series $1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ is a power series.

iii. **Maclaurin, Colin**(1698–1746) Scottish mathematician whowas the outstanding British mathematician of the generationfollowing Newton's. He developed and extended the subjectcalculus. His textbook on the subject contains important original results, but the Maclaurin series, which appears in it, is just a special case of the Taylor series known considerablyearlier. He also obtained notable results in geometry andwrote a popular textbook on algebra.

iv. **Taylor, Brook**(1685–1731) English mathematician whocontributed to the development of calculus. His text of 1715contains what has become known as the Taylor series.

CHECK YOUR PROGRESS

Choose only one correct option.

1. $(1 + \mathrm{x})^{-1}$

(v) $1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + \cdots$. .

(vi) $1 + x + x^2 + \cdots + x^{n-1} + \cdots$..

(vii) $x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + \cdots$.

(viii) None of the above.

2. If $f(x) = a^x$, then $f^n(0)$

(v) 0. .

(vi) $a^x \log a$.

(vii) $(\log a)^n$

(viii) 1.

3. The coefficient of x^3 in the expansion of $\sin^{-1} x$ is

(v) 0.

(vi) $\frac{1}{6}$.

(vii) $\frac{1}{2}$.

(viii) 1.

4. Coefficient of $(x-1)^2$ in the expansion of $3x^2 - 6x + 3$ is

(v) 0.

(vi) 1.

(vii) 3.

(viii) Nothing can be said.

5. Let $y = e^{ax}\sin bx$. Then $(y_2)_0$ is

(v) b^3.

(vi) $e^{ax}\sin 2bx$.

(vii) 0.

(viii) $2ab$.

6. The coefficient of n^{th} term in the expansion of e^{mx}Maclaurin's series

(v) $\frac{m^n}{n!}$

(vi) 0

(vii) $\frac{1}{n!}$

(viii) e^x.

7. If $y = \sin mx$, then (y_n)

(i) $\cos mx$.

(ii) $\sin(m+n)x$.

(iii) $\sin(mx + n.\frac{\pi}{2})$.

(iv) 0.

8. If $y = \log(1+x)$. Then coefficient of x^nin Maclaurin's theorem is

(i) $\frac{1}{n}$.

(ii) $\frac{(-1)^n}{n}$.

(iii) $\frac{1}{n!}$.

(iv) $\frac{(-1)^n}{n!}$.

9. In the expansion of $\cos x$, coefficient of x^3 is

(i) 0.

(ii) $\frac{1}{3}$.

(iii) $\frac{1}{3!}$

(iv) 1..

10. $D^4(x^3)$

(v) 0

(vi) 1.

(vii) $\frac{-6}{x}$

(viii) ∞.

7.8 REFERENCES

i. Joseph Edwards, "Differential Calculus for Beginners", Macmillan and Co., Ltd., New York; 1896.
ii. Gorakh Prasad, "Text-Book on Differential Calculus", Pothishala Private Ltd., Allahabad; 1936.
iii. Tom M. Apostol, "Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra", John Wiley & Sons; 1967.
iv. Amit M. Agarwal, "Differential Calculus", Arihant Prakashan, Meerut.

7.9 *SUGGESTED READING*

i. Differential Calculus for Beginners by Joseph Edwards.
ii. Text-Book on Differential Calculus by Gorakh Prasad.
iii. Calculus by R. Kumar.
iv. Krishna's Text Book on Calculus by A. R. Vasistha.
v. 12th class Mathematics Book by R. D. Sharma.
vi. Pragati's Calculus by Sudhir K. Pundir.
vii. Lectures on Basic Courses (1-2) on NPTEL website.
viii. Leibnitz's work on Wikipedia.

7.10 *TERMINAL QUESTIONS:*

(1) Prove the following

(a) $e^{x+h} = e^x + he^x + \frac{h^2}{2!}e^x + \frac{h^3}{3!}e^x + \cdots$.

(b) $(x+h)^{-1} = \frac{1}{x}\left[1 - \frac{h}{x} + \frac{h^2}{x^2} - \frac{h^3}{x^3} + \cdots\right]$.

(c) $\log\cos(x+h) = \log\cos x - htan\, x - \frac{h^2}{2}\sec^2 x - \frac{h^3}{3}\sec^2 x \tan x + \cdots$.

(d) $\tan(x+h) = \tan x + h.\sec^2 x + h^2\sec^2 x \tan x + \cdots$.

(e) $\tan^{-1}(x+h) = \tan^{-1} x + \frac{h}{(1+x^2)} - \frac{xh^2}{(1+x^2)^2} + \cdots$.

(2) Prove that: $e^{(\sin x)} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$.

(3) Prove that : $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$.

(4) Prove that : $\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots$.

(5) Find the Maclaurin's theorem with Lagrange's form of remainder for $f(x) = \cos x$.

(6) Assuming the validity of expansion, expand $\log_e x$ in powers of (x-1) and hence find an approximate value of $\log_e 1.1$.

7.11 *ANSWERS:*

Answer of Check your progress:

CHQ1: (i)
CHQ2: (iii)
CHQ3: (ii)
CHQ4: (iii)
CHQ5: (iv)
CHQ6: (i)
CHQ7: (iii)
CHQ8: (ii)
CHQ9: (i)
CHQ10: (ii)

Answer of Terminal questions:

TQ5: $-\sin\left(n\pi+\frac{\pi}{2}\right)\sin(\theta x)=-\cos n\pi\sin(\theta x)=(-1)^{n+1}[\sin(\theta x)]$

TQ6: $f(1.1)=0.1-\frac{0.1^2}{2}+\frac{0.1^3}{3}-\frac{0.1^4}{4}+......\cong 0.095305$.

UNIT 8- MAXIMA AND MINIMA

CONTENTS

8.1 OBJECTIVES

After reading this unit learners will be able

- **i.** To understand the concept of maxima and minima
- **ii.** To find the maximum and minimum points
- **iii.** To find maximum and minimum values of the function
- **iv.** How to use derivative to find maxima minima
- **v.** To work out simple problems on maxima and minima

8.2 INTRODUCTION

In this unit we will study about the stationary points and its types. Concept of maxima and minima has been explained. It will be shown how differentiation can be used to find maxima and minima. Behaviour of gradient for maximum points and minimum points has been explained with the help of graph. Maxima and minima hold importance in practical life also. A manufacturer will think of increasing his profit similarly he wants to minimize his loss. So in this unit we have tried to explain the important properties of maxima and minima including important related theorems and examples.

8.3 STATIONARY POINTS

Let y=f(x) be a given function. Then dy/dx of the function is called the gradient of the given function and the points of the function at which dy/dx=0 are called stationary points and the tangent at these points is parallel to X-axis. In the Fig 8.1, A, B, C are the points having tangent parallel to X- axis, so their gradient i.e. dy/dx=0.Consequently these are the stationary points.

TYPES OF STATIONARY POINTS

Stationary points are of three types

- **i.** Maximum points
- **ii.** Minimum points
- **iii.** Points of inflection

Definitions of these points are given on the next page. Fig 8.1 is showing three points A, B, C which are maximum, point of inflection and minimum points respectively.

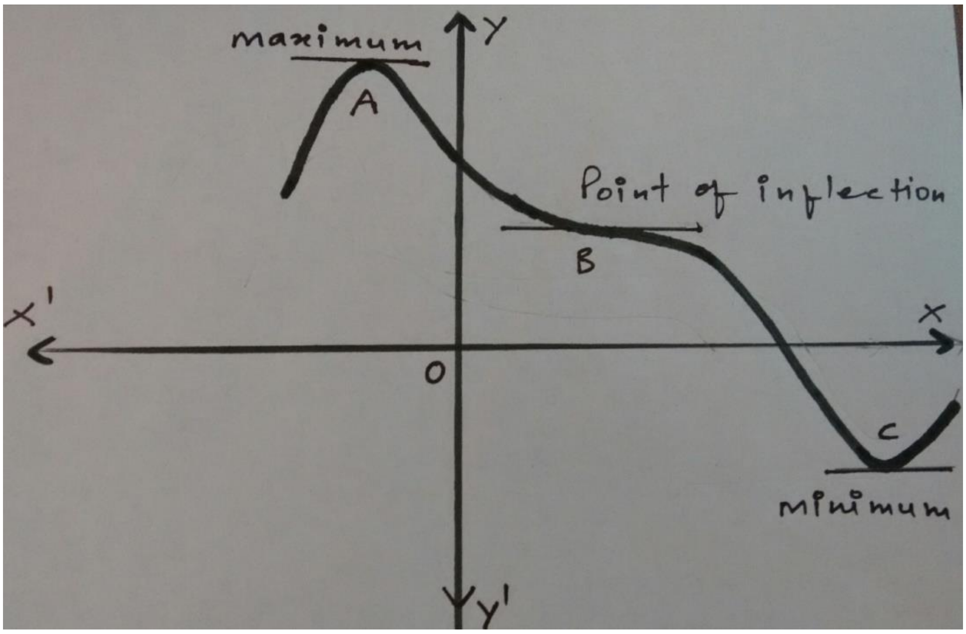

Fig 8.3.1

Maximum Points-

Let us see the behavior of gradient for maximum points.Gradient is positive before the maximum point, zero at the maximum point and negative after the maximum point. Analyze it, we will see that value of gradient i.e. dy/dx is decreasing with respect to x. Meaning of this statement is that d/dx(dy/dx)= d^2y/dx^2 is negative i.e. rate of change of dy/dx with respect to x is negative. See the Fig 8.3.2 to understand above mentioned facts.

KEY POINT: if dy/dx=0 at a point and $d^2y/dx^2<0$ there ,then that point must be maximum

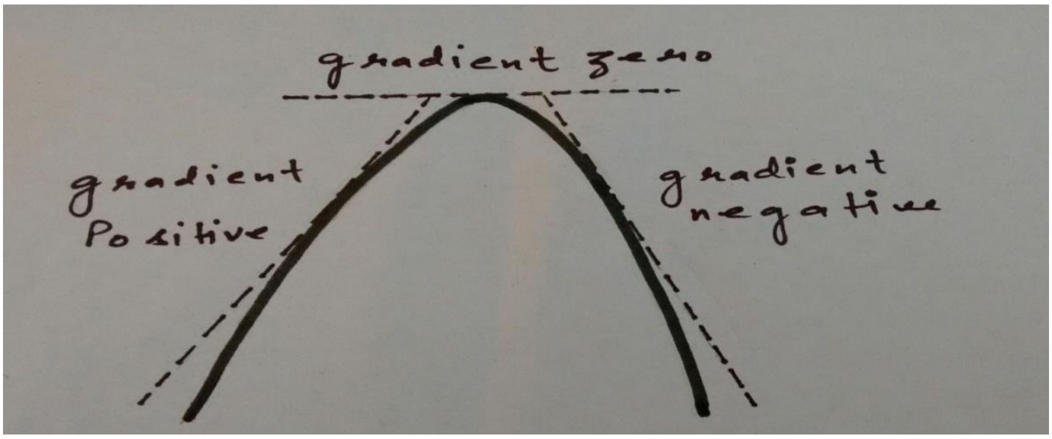

Fig 8.3.2

Minimum Point-

In minimum point just before minimum point gradient is negative , at minimum point gradient is zero and after minimum point gradient is positive. Here we can analyze that gradient i.e. dy/dx is increasing with respect to x. So the rate of change of dy/dx is positive i.e. d^2y/dx^2 is positive.

Key point: If dy/dx=0 at a point and $d^2y/dx^2>0$ there, then that point must be minimum

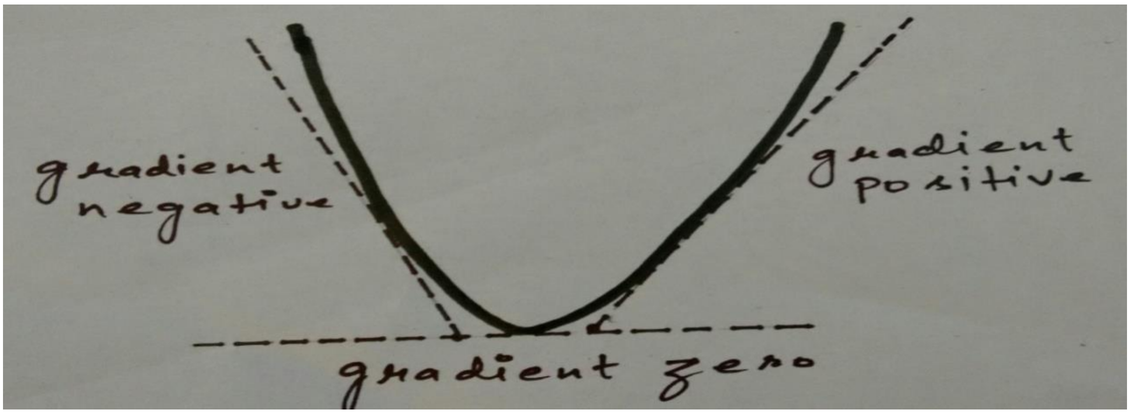

Fig 8.3.3

Point of Inflection-

These are the points where dy/dx= 0 and $d^2y/dx^2=0$ and $d^3y/dx^3\neq0$. Two figures are given below to clear the concept of points of inflection

Key point: If dy/dx=0 and $d^2y/dx^2=0$ and $d^3y/dx^3\neq0$, then the point is point of inflection

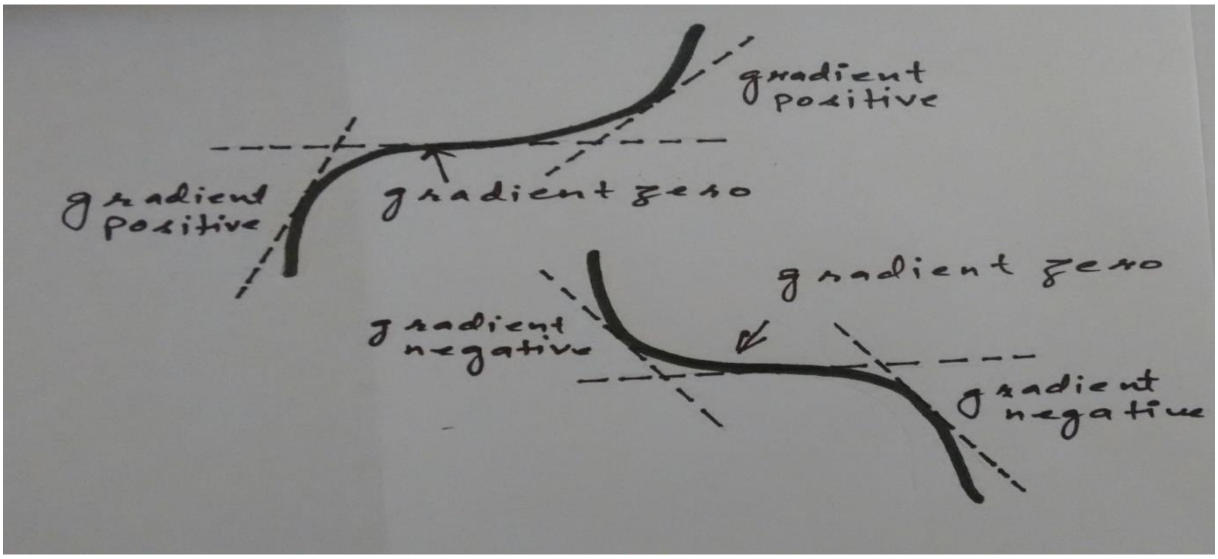

Fig 8.3.4

For points of inflection we can also use points of inflexion. Both the spellings are correct.

8.4 ABSOLUTE MAXIMA AND MINIMA

The function f has an absolute maximum or global maximum at a point c if f(c) ≥f(x) for all x in D .Here D is the neighbourhood of c and f(c) will be called the maximum value of f on the neighbourhood D. The function f has an absolute minimum or global minimum at c if f(c) ≤f(x) for all x in D.Here D is neighbourhood of D and f(c) is called the minimum value of f on the neighbourhood D. Maximum and minimum values of f on D are called extreme values of f.

Neighbourhood: Let x be a point on the number line. Then the interval (x-a,x+a) is called the neighbourhood of x.

8.5 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

When we see the graph of a continuous function, it increases and decreases alternatively. In a continuous function if value of a function increases to a certain point and then it begins to decrease, we call that a point of maximum and the value of function at that point is called the maximum value.

Similarly, when value of a continuous function decreases to a certain and then decreases the point is called minimum point and the value of the function at this point is called minimum value..

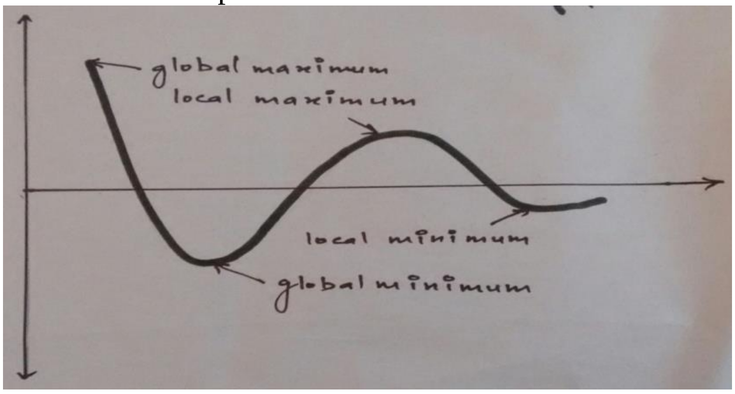

Fig 8.5.1

8.6 LOCAL MAXIMA AND LOCAL MINIMA

Have you noticed the previous page fig 8.4. From the fig. we can notice that a function may have more than one maximum or minimum value. So, in case of continuous function we have minimum (maximum) value in an interval and these values are not absolute or global minimum (maximum) of the function. This is the reason that sometimes we call the values as local maxima or minima.

Properties of maxima and minima

- Between two equal values of a function there must lie at least one maximum and minimum
- If dy/dx changes from +ve to –ve as x passes through a point a, then function y=f(x) is a maximum at the point a
- If dy/dx changes from –ve to +ve as x passes through a point a, then function y=f(x) is a minimum at the point a
- If sign of dy/dx does not change when x passes through a point a, then point a is neither maximum nor minimum at x=a

Method to find Maxima and Minima

Step 1: Firstly find dy/dx

Step 2: Put dy/dx=0 and find the value of x.These x will be stationary points.

Step 3: Now find d^2y/dx^2.

Step 4: To check whether the stationary points from step 1 are maxima or minima put these points one by one in d^2y/dx^2. If value of d^2y/dx^2 is negative then the point is maxima and if d^2y/dx^2 is positive then the point is minima.

Step 5: Suppose $d^2y/dx^2 = 0$ and $d^3y/dx^3 \neq 0$ then x is a point of inflection.

Step 6: If d^2y/dx^2= and $d^3y/dx^3=0$ then d^4y/dx^4, if its negative then x is maximum point and if its positive then x is a minimum point.
Step 7: If $d^4y/dx^4=0$ then find d^5y/dx^5 and so on.
Step 8: Repeat the above steps for each root of the equation dy/dx=0 i.e. f'(x)=0.

8.7 EXAMPLES

Ex.1 Find the maxima and minima for the function $y = 4x - x^2$.

Sol. Firstly let us find $dy/dx = 4 - 2x$

Now for stationary points $dy/dx = 0$

i.e. $4 - 2x = 0$…………………..(1)

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

Differentiating $dy/dx = 4 - 2x$ once again we get, $d^2y/dx^2 = -2$

This is negative which is suggesting a maximum point

Now substitute $x = 2$ into$y = 4x - x^2$, we get

$$y = 8 - 4 = 4$$

So,maximum point is (2,4).

Ex.2 Find maxima and minima for $y = 2 + 3x^2 - x^3$

Sol. Firstly $dy/dx = 6x - 3x^2$ …………(1)

For stationary points

$$dy/dx = 0$$

$$6x - 3x^2 = 0$$

After factorizing,$3x\ (2 - x) = 0$

$x = 0$ or $x = 2$.

Differentiating (1) we get, $d^2y/dx^2 = 6 - 6x$,

whose value ispositive when $x = 0$ and when $x = 2$ value is negative.

Substituting the values of x into $y = 2 + 3x^2 - x^3$

We get, $x = 0$ gives $y = 2$ and $x = 2$ gives $y = 6$.

So,Minimum point is (0, 2)and maximum point is (2, 6).

Ex.3 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$.

Sol.$f(x) = x^3 - 3x^2 - 9x$.

Differentiate w.r.t. x, we get

$$f'(x) = 3x^2 - 6x - 9$$

$$= 3(x^2 - 2x - 3)$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow 3(x^2 - 2x - 3) = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x - 3)(x + 1) = 0$$

$$x = 3, -1$$

$$x = 3, x = -1$$

Now, $f'(x) = 3x^2 - 6x - 9$

Differentiating w.r.t. x, we get

$$f''(x) = 6x - 6$$
$$= 6(x - 1)$$

To check $x = 3$

$f''(3) = 6(3 - 1) = 6(2) = 12$ =positive value

Therefore $x = 3$ is a minimum point

and minimum value is $f(3) = 3^3 - 3(3)^2 - 9(3)$

$$= 27 - 3(9) - 27$$
$$= -27$$

To check $x = -1$

$f''(-1) = 6(-1 - 1) = 6(-2) = -12$ = negative value

Therefore,$x = -1$ is a maximum point.

and maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1)$

$$= -1 - 3 + 9$$
$$= -4 + 9 = 5$$

Ex.4 Find all local maxima and minima of the function $f(x) = 2x^3 - 3x^2 - 12x + 8.$

Sol. Here $f(x) = 2x^3 - 3x^2 - 12x + 8.$

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x + 1)(x - 2)$$

Now, put $f'(x) = 6(x + 1)(x - 2) = 0$

So we get, $x = -1$ and $x = 2$

In the next step we will check the value of f''(x) for each $x = -1$ and $x = 2$.

$$f''(x) = 12x - 6$$
$$= 6(6x - 1)$$

At $x = -1$, $f''(-1) = 6(6(-1) - 1)$

$$= 6(-6 - 1) = (6)(-7) = -42,$$

which is a negative value so point x=-1 is a point where local maximum exists

At $x = 2$, $f''(2) = 6(6(2) - 1) = 6(12 - 1) = 6(11) = 66$, which is a positive value so point $x = 2$ is a point where local minimum exists

Ex.5 Find the maximum and minimum value of the function

$$f(x) = x^5 - 5x^4 + 5x^3 - 10$$

Sol. To find maxima and minima let us put $f'(x) = 0$, then by solving $f'(x) = 5x^4 - 20x^3 + 15x^2 = 0$, we get x= 3, 1 and 0 as stationary points.

Now $f''(x) = 20x^3 - 60x^2 + 30x$ and $f''(1) = -10$.
Hence $f(x)$has a maximum value at $x = 1$.

Similarly the value of $f''(3) = -90$. Hence $f(x)$ has a maximum value at $x = 3$.

Now at $x = 0$, $f''(0) = 0$, and $f'''0 \neq 0$. So at $x = 0$.
Function $f(x)$ has neither maximum nor a minimum, it's a point of inflection.

Ex.6 Find all the points of local maxima and local minima of the function

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Sol. Firstly,$f'(x) = 6x^2 - 12x + 6$

Now after putting this value equal to zero,we get, f'(x)=0

$\Rightarrow 6(x^2 - 2x + 1) = 0$

$\Rightarrow 6(x-1)(x-1) = 0$

There is only one stationary point $x = 1$.

Now finding $f''(x) = 12(x-1)$

Put $x = 1$, we get$f''(1) = 0$ and $f'''(x) = 12$,

so $f'''(1) = 12 \neq 0$, so,$x = 1$ is a point of inflexion.

Ex.7 Find the local minimum value of the function $f(x) = 3 + |x|$, where x is a real number.

Sol. Value of $|x| \geq 0$

So minimum value of $|x| = 0$

Minimum value of $f(x) = 3 +$ minimum value of $|x|$

$= 3 + 0 = 3$

So, minimum value of $f(x) = 3$

We cannot find any maximum value.

8.8 SUMMARY

In this unit we studied about the stationary points.There are three types of stationary points maxima, minima and inflection points. The method we studied here to find these points is called second derivative test.In second derivative test firstly we find stationary points by putting dy/dx=0.Then if second derivative at the point is negative then point is maxima and if second derivative is positive at the stationary point,the point is minima. Gradient is positive before the maximum point, zero at the maximum point and negative after the maximum point. In minimum point just before minimum point gradient is negative , at minimum point gradient is zero and after minimum point gradient is positive. Alsoif dy/dx=0 at a point and $d^2y/dx^2<0$ there ,then that point must be maximum.If dy/dx=0 at a point and $d^2y/dx^2>0$ there, then that point must be minimum. To explain all the concepts graphs have been used and examples have been used.

Remark- Extreme points are the stationary points at which the function attains either local maximum or local minimumvalues.Extreme values are both local maximum and local

minimum values of the function f(x). So, a function attains an extreme value at point x=a if f (a) is either
a local minimum or a local maximum.

8.9 GLOSSARY

i. Stationary point.
ii. Maxima.
iii. Minima.
iv. Inflection point.
v. Extreme point.
vi. Extreme Values.
vii. Derivative.
viii. Function.

CHECK YOUR PROGRESS

i. The critical point of the function $y = x^2$ is...................
ii. The points of maxima and minima of a function:y = $2x^3 - 3x^2 + 6$........
iii. The function $x^2 \log x$ in the interval (1, e) has a point of maximum or minimum. True/False
iv. There can be any number of absolute maxima and absolute minima for a function within the entire domain. True/False
v. The function $f(x) = x^{-x}$, attains a maximum value at $x = \frac{1}{e}$. True/False

8.10 REFERENCES

(i) www.nuffieldfoundation.org/sites/default/files/.../FSMQ%20Stationary%20points.pdf.
(ii) *download.nos.org/srsec311new/L.No.25.pdf*
(iii) www.mathcentre.ac.uk/resources/uploaded/mc-ty-maxmin-2009-1.pdf
(iv) www.maths.usyd.edu.au › Teaching program › Junior › MATH1011 › Quizzes

8.11 SUGGESTED READING

(i) Mathematical Analysis by S.C. Malik and Savita Arora.
(ii) Real Analysis by Krishan Prakashan.
(iii) Real Analysis by Gupta and Goyal
(iv) Theory of maxima and minima by Harris Hancock

8.12 TERMINAL QUESTIONS

TQ1. Find all local maximum and minimum points for the function $f(x) = x^3 - x$

TQ2. Find all local maximum and minimum points for $f(x) = sinx + cosx$.

TQ3. find all local maximum and minimum points (x, y),

$$f(x) = \begin{cases} x-3, x < 3 \\ x^3, 3 \le x \le 5 \\ 1/x, x > 5 \end{cases}$$

TQ4. Find the local extrema of $f(x) = |x| + |x-1|$.

TQ5. Explain why the function $f(x) = 1/x$ has no local maxima or Minima?

....

....

8.13 ANSWER

CHECK YOUR PROGRESS:

CYQ1.Maxima.
CYQ2.$x = 1$.
CYQ3.False.
CYQ4.False.
CYQ5. True.

TERMINAL QUESTIONS:

TQ1. local minimum at $x = \sqrt{3}/3$, local maximum at $x = -\sqrt{3}/3$

TQ2. local maximum when $x = \frac{\pi}{4}$ and also when $x = \frac{\pi}{4} \pm \frac{\pi}{4}$. local minima

at $x = 5\frac{\pi}{4} + 2k\pi$ for every integer k.

TQ3. Local maximum at $x = 5$.

TQ4. Local min of 1 at every point of [0,1], local max of 1 at every point of (0,1).

UNIT 9:- INTEGRAL

CONTENTS

9.1 INTRODUCTION

In the previous unit we studied absolute minima, maximum and minimum values of a function, local maxima and local minima.The given unit is about the idea of integration, and also about the technique of integration. We explain how it is done in principle, and then how it is done in practice. Integration is a problem of adding up infinitely many things, each of which is infinitesimally small. Doing the addition is not recommended. The whole point of calculus is to offer a better way.

The problem of integration is to find a limit of sums. The key is to work backward from a limit of differences (which is the derivative).

We can integrate $g(x)$ if it turns up as the derivative of another function $f(x)$. The integral of $g(x) = \cos x$is $f = \sin x$. The integral of $g(x) = x$is $f = \frac{1}{2}x^2$. Basically, $f(x)$ is an "antiderivative".If we don't find a suitable $f(x)$, numerical integration can still give an excellent answer.

We could go directly to the formulas for integrals, which allow learners to compute areas under the most amazing curves.

November 23, 1616-
October 28, 1703.
Fig. John Wallis
(Reference: https://www.britannica.com/biography/John-Wallis)

Fig 9.1.1

9.2 OBJECTIVES

In this Unit, learners will able to Antiderivatives

- **i.** Analyze Definite Integrals
- **ii.** Define Properties of Integral,
- **iii.** Construct Fundamental theorem

9.3 ANTIDERIVATIVES

The symbol of integration$\int$,was inventedLeibnizto represent the integral.It is a stretched-out$S$, from the Latin word for sum. This symbol is a powerful reminder of the wholeconstruction: Sum approaches integral, S approaches $\int$, and rectangular areaapproaches curved area.

$$curved\ area\ = \int g(x)dx = \int \sqrt{x}dx$$

The rectangles of base Δx lead to this limit-the integral of $\sqrt{x}$.The "dx" indicatesthat Δx approaches zero. The heights g_j of the rectangles are the heights $g(x)$ of thecurve. The sum of g_j times Δx approaches "$the\ integral\ of\ f\ of\ x\ dx.$"You can imaginean infinitely thin rectangle above every point, instead of ordinary rectangles abovespecial points.

We now find the area under the square root curve. The "limits of integration" are0and9. The lower limit is $x = 0$, where the area begins.

The upper limit is $x = 9$, since we stop after Nine.The area of the rectangles is a sum of base Δxtimes heights $\sqrt{x}$.The curved area is the limit of this sum. i.e. limit is the integral of $\sqrt{x}$ from 0 to 9

$= \lim_{x \to 0}\left[\sqrt{\Delta x}(\Delta x) + \sqrt{2\Delta x}(\Delta x) + \cdots + \sqrt{9\Delta x}(\Delta x)\right] = \int_{x=0}^{x=9} \sqrt{x}\, dx$ ..(2)

What is $f(x)$? Instead of the derivative of $\sqrt{x}$,we need its "antiderivative." We haveto find a function $f(x)$ whose derivative is $\sqrt{x}$.The derivative of x^n is nx^{n-1}-'-now we need theantiderivative. Since the derivative lowers the exponent, the antiderivative raises it. Wego from $x^{\frac{1}{2}}$ to $x^{\frac{1}{3}}$. But then the derivative is $\frac{3}{2}x^{\frac{1}{2}}$. It contains an unwantedfactor $\frac{3}{2}$ To cancel that factor, put $\frac{2}{3}$into the antiderivative:$f(x) = \frac{2}{3}x^{\frac{3}{2}}$has the required derivative$g(X) = x^{\frac{1}{2}} = \sqrt{x}$.

ExampleThe antiderivative of $x^2$is $\frac{1}{3}x^3$. This is the area under the parabola $g(x) = x^2$. The area out to $x = 2$ is $\frac{1}{3}(2)^3 = \frac{8}{3}$.

Indefinite Integrals and definite Integrals: Now we will discuss two different kinds of integrals. They both use the antiderivative $f(x)$. The definite one involves the limits $a$and b, the indefinite one doesn't:

The indefinite integral of a function x^2 is a function $f(x) = \frac{x^3}{3}$

The definite integral of a function $x^2$from x = 0 to x = 9 is the number $f(9) - f(0) = \frac{9^3}{3} - 0 = 243.$

The definite integral must be 243. But the indefinite integral is not necessarily$\frac{x^3}{3}$

We can change $f(x)$ by a constant without changing its derivative

The following functions are also antiderivatives:

$f(x) = \frac{x^3}{3} + 1, f(x) = \frac{x^3}{3} + 2,$or in general $f(x) = \frac{x^3}{3} + C$, where C is arbitrary constant

The indefinite integral isthe most general antiderivative (with no limits):

Example: Find an antiderivatives $f(x)$ for $g(x)$ of following. Ten compute the definite integral $\int_0^1 g(x)dx = f(1) - f(0)$

(i) $5x^4 + 4x^3$ (ii) $x + 12x^2$ (iii) $\sin x + \sin 2x$ (iv) $\sec^2 x + 1$

Proof (i) The antiderivative of $5x^4 + 4x^3$ is $\int 5x^4 + 4x^3 dx = x^5 + 12x^2$.

$$\int_0^1 g(x)dx = f(1) - f(0) = (5(1)^4 + 4(1)^3) - (5(0)^4 + 4(0)^3)$$
$$= 9$$

(ii) The antiderivative of $x + 12x^2$ is $\int x + 12x^2 dx = \frac{x^2}{2} + \frac{12x^3}{3} = \frac{x^2}{2} + 4x^3$.

$$\int_0^1 g(x)dx = f(1) - f(0) = \left(\frac{1^2}{2} + 4(1)^3\right) - \left(\frac{0}{2} + 4(0)^3\right) = \frac{9}{2}$$

(iii) The antiderivative ofsin $x + \sin 2x$ is $\int \sin x + \sin 2x \, dx = -\cos x - \frac{\cos 2x}{2}$.

$$\int_0^1 g(x)dx = f(1) - f(0)$$
$$= \left(-\cos(-1) - \frac{\cos\big(2(-1)\big)}{2}\right) - \left(\cos 0 - \frac{\cos\big(2(0)\big)}{2}\right)$$
$$= \left(-\cos 1 + \frac{\cos 2}{2} - \left(1 - \frac{1}{2}\right)\right) = -\cos 1 + \frac{\cos 2}{2} - \frac{1}{2}$$

(iv) The antiderivative ofsec$^2 x + 1$ is $\int (\sec^2 x + 1)dx = \tan x$.

$$\int_0^1 g(x)dx = f(1) - f(0) = \tan 1 - \tan 0 = \tan 1$$

9.4 DEFINITE INTEGRAL AND ITS PROPERTIES

Sometimes in geometrical and other applicationof integral calculus it becomes necessary to find the difference in the values of an integral of a function f(x) for two given values of the variable x, say a and b. The difference is called the definite integral of f(x) from a to b or between the limits a and b.

This definite integral is denoted by $\int_a^b f(x)dx$ and is read as "the integral of f(x) w.r.t. to x between a and b."

It is often written as $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Where F(x) is an integral of f(x), F(b) is the value of F(x) at x=-b, F(a) is the value of F(x) at x=a and a and b are lower and upper limit of integration respectively.

Fundamental Theorem of integral Calculus:

Let f is integrable over the interval [a,b] and φ be a differentiable function on interval [a,b] such that $\varphi'(x) = f(x) for\ all\ x\ in\ [a,b]$. Then

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$$

Fundamental Properties of definite integrals

Property 1:$\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e value of definite integral doe not change with change in variable.

Proof. Let $\int_a^b f(x)dx = \varphi(x)$.

Then

$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$ ……………………………….(1)

and

$\int_a^b f(t)dt = \varphi(b) - \varphi(a)$ ……………………………….(2)

From equation (1) and (2), we get

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Property 2: $\boldsymbol{\int_a^b f(x)dx = -\int_b^a f(x)dx}$

Proof. Let $\int_a^b f(x)dx = \varphi(x)$.

Then

$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$ ……………………………….(1)

Also

$$-\int_b^a f(x)dx = -\big(\varphi(a) - \varphi(b)\big)$$

$\Rightarrow -\int_b^a f(x)dx = \varphi(b) - \varphi(a)$ ……………………………….(2)

From equation (1) and (2), we get

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property 3: $\boldsymbol{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \ where \ a < c < b}$

Proof. Let $\int_a^b f(x)dx = \varphi(x)$.

Then

$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$ ……………………………….(1)

Now

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \varphi(b) - \varphi(c) + \varphi(c) - \varphi(a)$$

$$\Rightarrow \int_a^c f(x)dx + \int_c^b f(x)dx = \varphi(b) - \varphi(a)$$

From equation (1) and (2), we get

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Property 4: $\boldsymbol{\int_0^a f(x)dx = \int_0^a f(a-x)dx}$

Proof. Let $I = \int_0^a f(x)dx$

Let $x = a - t \Rightarrow dx = -dt$

We can see that when $x = 0$ then $t = a$ and $x = a$ then $t = 0$.

Therefore

$$I = \int_a^0 f(a-t)(-dt) = -\int_a^0 f(a-t)dt$$

Using Property 2, we get

$$I = \int_0^a f(a-t)dt$$

Again using Property 1, we get

$$I = \int_0^a f(a-x)dx$$

Thus, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Odd function: A function $f(x)$ is said to be odd function of x if $f(-x) = -f(x)$.

Even function: A function $f(x)$ is said to be even function of x if $f(-x) = f(x)$.

Property 5: (i)$\boldsymbol{\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx}$ **if** $\boldsymbol{f}$ **is even function.**

(ii) $\boldsymbol{\int_{-a}^a f(x)dx = 0}$**if**$\boldsymbol{f}$ **is odd function.**

Proof. Let $I = \int_{-a}^a f(x)dx$

From Property 4, we can write given definite integral as

$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$ ………………………..(1)

Let

$$I_1 = \int_{-a}^0 f(x)dx$$

Put $x = -t \Rightarrow dx = -dt$
We can see that when $x = -a$ then $t = a$ and $x = 0$ then $t = 0$.
Therefore

$$I_1 = \int_a^0 f(-t)(-dt)$$

Thus $I_1 = -\int_a^0 f(-t)(dt)$
Using Property 2, we get
$I_1 = \int_0^a f(-t)(dt)$ …………………………………..(2)
(i) If f is even function then $f(-t) = f(t)$, hence

$$I_1 = \int_0^a f(t)dt$$

Using Property 1, we get

$$I_1 = \int_0^a f(x)dx$$

Therefore $\int_{-a}^0 f(x)dx = \int_0^a f(x)dx$…………………………..(3)
From equation (1) and (3), we get

$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

Hence $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
If f is odd function then $f(-t) = -f(t)$, hence

$$I_1 = \int_0^a -f(t)dt$$

Using Property 1, we get

$$I_1 = -\int_0^a f(x)dx$$

Therefore $\int_{-a}^0 f(x)dx = -\int_0^a f(x)dx$…………………………..(4)
From equation (1) and (4), we get

$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx$$

$$= 0$$

Hence $\int_{-a}^a f(x)dx = 0$.

Property 6: $(\boldsymbol{i})\ \int_0^{2a} \boldsymbol{f(x)dx} = \boldsymbol{2}\int_0^a \boldsymbol{f(x)dx}$ **if** $\boldsymbol{f(2a-x)} = \boldsymbol{f(x)}$

$$(\boldsymbol{ii})\int_0^{2a} \boldsymbol{f(x)dx} = \boldsymbol{2}\int_0^a \boldsymbol{f(x)dx}\ \textbf{if}\ \boldsymbol{f(2a-x)} = -\boldsymbol{f(x)}$$

Proof. Using Property 3, we get

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

Putting $x = 2a - t \Rightarrow dx = -dt$ in the second integral and changing the limit, we get

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^0 f(2a-t)(-dt)$$

Using Property 2, we get

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-t)dt$$

Using Property 1 in the second integral, we get

$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$ …………..(1)

(i) $f(2a-x) = f(x)$

Hence,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

(ii) $f(2a-x) = -f(x)$

Hence,

$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$ **.**

NOTE:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Ex 9.1. Solve $\int_0^{\pi} \frac{x}{\alpha^2\, cos^2x+\beta^2 sin^2x} dx$

Sol. Let $I = \int_0^{\pi} \frac{x}{\alpha^2\, cos^2x+\beta^2 sin^2x} dx$……………………….(1)

Using property 4, we get

$$I = \int_0^{\pi} \frac{\pi - x}{\alpha^2\, cos^2(\pi-x) + \beta^2 sin^2(\pi-x)} dx$$

$\Rightarrow$ $\boldsymbol{I} = \int_0^{\pi} \frac{\pi-x}{\alpha^2\, cos^2x+\beta^2 sin^2x} dx$ …………………………(2)

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{\alpha^2\, cos^2x + \beta^2 sin^2x} dx + \int_0^{\pi} \frac{\pi - x}{\alpha^2\, cos^2(\pi-x) + \beta^2 sin^2(\pi-x)} dx$$

$$= \int_0^{\pi} \frac{x + \pi - x}{\alpha^2\, cos^2(\pi-x) + \beta^2 sin^2(\pi-x)} dx$$

Therefore

$2I = \pi \int_0^{\pi} \frac{1}{\alpha^2\, cos^2x+\beta^2 sin^2x} dx$……………………….(3)

If $f(x) = \frac{1}{\alpha^2\, cos^2x+\beta^2 sin^2x}$

then$f(\pi - x) = \frac{1}{\alpha^2\, cos^2(\pi-x)+\beta^2 sin^2(\pi-x)}$

$$\Rightarrow f(\pi - x) = \frac{1}{\alpha^2\, cos^2x + \beta^2 sin^2x}$$

Hence $f(x) = f(\pi - x)$

Using Property 6 in equation (3), we get

$$2I = 2\pi \int_0^{\pi/2} \frac{1}{\alpha^2\, cos^2x + \beta^2 sin^2x} dx$$

Now dividing the numerator and denominator by cos^2x, we get

$$I = \pi \int_0^{\pi/2} \frac{1/\cos^2 x}{\frac{\alpha^2\, cos^2x}{\cos^2 x} + \frac{\beta^2 sin^2x}{\cos^2 x}} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{\alpha^2 + \beta^2 \tan^2 x} dx$$

Putting $\beta \tan x = t \Rightarrow \beta \sec^2 x\, dx = dt$, changing the limit, we get

$$I = \frac{\pi}{\beta} \int_0^{\infty} \frac{dt}{\alpha^2 + t^2} = \frac{\pi}{\beta}\frac{1}{\alpha}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty}$$

$$= \frac{\pi}{\beta}\frac{1}{\alpha}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{\pi}{\beta}\frac{1}{\alpha}\frac{\pi}{2} = \frac{\pi^2}{2\alpha\beta}$$

Ex 9.2. Solve $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Sol. Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$……………………….(1)

Using property 4, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$\Rightarrow\ I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x} dx$ ………………………….(2)

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x\ \sin x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} \frac{(x + \pi - x)\sin x}{1 + \cos^2 x} dx$$

Therefore

$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$ ……………………….(3)

If $f(x) = \frac{\sin x}{1+\cos^2 x}$

then$f(\pi - x) = \frac{\sin(\pi-x)}{1+\cos^2(\pi-x)}$

$$\Rightarrow f(\pi - x) = \frac{\sin x}{1 + \cos^2 x}$$

Hence $f(x) = f(\pi - x)$

Using Property 6 in equation (3), we get

$$2I = 2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

Putting $\cos x = t \Rightarrow -\sin x\, dx = dt$, changing the limit, we get

$$I = -\pi \int_1^{0} \frac{dt}{1 + t^2} = -\pi[\tan^{-1} t]_1^0$$

$$= -\pi[\tan^{-1} 0 - \tan^{-1} 1]$$

$$= \frac{\pi^2}{4}$$

Ex 9.3. Prove that $\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx = \pi\left(\frac{\pi}{2} - 1\right)$

Sol. Let $I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$……………………….(1)

Using property 4, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)} dx$$

$\Rightarrow\ I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x} dx$ ………………………….(2)

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x\ \sin x}{1 + \sin x} dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x} dx$$

$$= \int_0^{\pi} \frac{(x + \pi - x)\sin x}{1 + \sin x} dx$$

Therefore

$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$ ……………………….(3)

Now multiplying the numerator and denominator by $(1 - \sin x)$, we get

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,(1 - \sin x),}{(1 + \sin x)(1 - \sin x)} dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \frac{\pi}{2}\left[\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos^2 x} dx\right]$$

$$= \frac{\pi}{2}\left[\int_0^{\pi} \tan x \sec x\, dx - \int_0^{\frac{\pi}{2}} \tan^2 x\, dx\right]$$

$$= \frac{\pi}{2}\left[\int_0^{\pi} \tan x \sec x\, dx - \int_0^{\frac{\pi}{2}} (\sec^2 x - 1) dx\right]$$

$$= \frac{\pi}{2}[[\sec x]_0^{\pi} - [\tan x - x]_0^{\pi}]$$

$$= \frac{\pi}{2}\left[(-1 - 1) - [0 - 0 - (\pi - 0)]_0^{\frac{\pi}{2}}\right]$$

$$= \frac{\pi}{2}(-2 + \pi) = \pi\left(\frac{\pi^2}{2} - 1\right)$$

Ex 9.4. Prove that $\int_0^{\pi/2} \log(\tan x)\, dx = 0$

Proof. Let $I = \int_0^{\pi/2} \log(\tan x)\, dx$……………………….(1)

Using property 4, we get

$$I = \int_0^{\pi/2} \log \tan(\frac{\pi}{2} - x)\, dx$$

$\Rightarrow\ I = \int_0^{\frac{\pi}{2}} \log \cot x\, dx$……………………….(2)

Adding equation (1) and (2), we get

$$2I = \int_0^{\pi/2} \log \tan x\ \ dx + \int_0^{\frac{\pi}{2}} \log \cot x\, dx$$

$$= \int_0^{\pi} (\log \tan x + \log \cot x) dx$$

Therefore

$2I = \int_0^{\pi} (\log (\tan x\ \ .\cot x) dx$ ……………………….(3)

$2I = \int_0^{\pi} \log 1\, dx$ ……………………….(3)

$$2I = 0 \Rightarrow I = 0$$

Hence $\int_0^{\pi/2} \log(\tan x)\ dx = 0$

9.5 DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let $f(x)$ be a single valued continuous function defined in the interval (a,b) and let the interval (a, b) be divided into n equal parts each of length h such that $nh = b - a$, then we define

$$\int_a^b f(x)\, dx = lim_{n\to\infty}{}_{h\to 0} h[f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + (n - 1)h]$$

where $nh \to b - a$

or we can say

$\int_a^b f(x)\, dx = \lim_{h\to 0} h \sum_{k=0}^{n-1} f(a + kh)$where $nh \to b - a$

Ex 9.5. Find by summation $\int_a^b \sin x\, dx$

Proof. We can see that $f(x) = \sin x$

$\Rightarrow \int_a^b f(x) dx$

$$= lim_{n\to\infty}{}_{h\to 0} h[\sin(a) + \sin(a + h) + \sin(a + 2h) + \cdots + \sin(a + (n - 1)h]$$

$= lim_{n\to\infty}{}_{h\to 0} h\left[\frac{\sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{h}{2}\right)} . \sin\left(a + \frac{1}{2}(n - 1)h\right)\right]$ (from trigonometric identities)$= lim_{n\to\infty}{}_{h\to 0} h\left[\frac{\sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{h}{2}\right)} . \sin\left(a + \frac{1}{2}(n - 1)h\right)\right]$

Now $nh = b - a$

Hence $\int_a^b f(x) dx\ = lim_{n\to\infty}{}_{h\to 0} h\left[\frac{\sin\frac{b-a}{2}}{\sin\left(\frac{h}{2}\right)} . \sin\left(a + \frac{1}{2}(b - a - h)\right)\right]$

$$= lim_{n\to\infty}{}_{h\to0} h\left[\frac{\sin\left(a+\frac{1}{2}(b-a-h)\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$= \sin\frac{b-a}{2}\ lim_{n\to\infty}{}_{h\to0}\frac{2.h/2}{\sin h/2}\sin\left(a+\frac{1}{2}(b-a-h)\right).$$

$$= \sin\frac{b-a}{2}\ 2.\sin\left(a+\frac{1}{2}(b-a)\right)$$

$$= 2\sin\frac{b-a}{2}\sin\left(\frac{a+b}{2}\right)$$

$$= cos\ a - cos\ b$$

9.6 SUMMATION OF SERIES WITH THE HELP OF DEFINITE INTEGRAL

The definition of a definite integral as the limit of a sum helps us to evaluate the limit of the sums of some special type of series.

As we know that

$$\int_a^b f(x)dx = \lim_{h\to0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h]$$

$= \lim_{h\to0} h\sum_{k=0}^{n-1} f(a+kh)$where$nh = b-a$

Putting $a = 0$ and $b = 1$, therefore $h = \frac{1}{n}$, we get

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(\frac{k}{n})$$

Ex 9.6.Prove that the limit of the sum $\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}$ where $n$is indifinetly increased is log 3.

Sol. Here we can see that

General term of the series is $\frac{1}{n+k}$ and k varies 0to $2n$.

Now

$$\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{\mathbf{1}}{\boldsymbol{n}+\boldsymbol{k}} = \lim_{n\to\infty}\sum_{k=0}^{2n}\frac{\mathbf{1}}{\boldsymbol{n}\left(\mathbf{1}+\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)}$$

Taking $\frac{1}{n}$ outside the sign of summation, we have

$$\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\frac{\mathbf{1}}{\left(\mathbf{1}+\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)}$$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)$where$\boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right) = \frac{\mathbf{1}}{\left(\mathbf{1}+\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)}$

We can see that when $k = 0$ then $\frac{k}{n} = 0$ and $k = 2n$ then $\frac{k}{n} = \frac{2n}{n} = 2.$

Thus when $n \to \infty$ then $\frac{k}{n}$ tends to 0 and 2 respectively.

By replacing $\frac{k}{n}$ with $x, 1/n$with dx and $\lim_{n\to\infty}\sum by$ the sign of integration$\int$, also taking limit of x from 0 to 2, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\frac{\mathbf{1}}{\left(\mathbf{1}+\boldsymbol{\frac{k}{n}}\right)}=\int_0^2\frac{1}{1+x}dx$$
$$=|\log(1+x)|_0^2$$
$$=\log 3-\log 1=\log 3$$

Ex 9.7.Prove that the limit of the sum $\frac{1}{nb}+\frac{1}{nb+1}+\frac{1}{nb+2}+\cdots+\frac{1}{nm}$ where $n$is indifinetly increased is $\log\frac{m}{b}$.

Sol. Here we can see that

General term of the series is $\frac{1}{nb+k}$ and k varies 0to $(m-b)n$.

Now

$$\lim_{n\to\infty}\sum_{k=0}^{(m-b)n}\frac{1}{nb+k}=\lim_{n\to\infty}\sum_{k=0}^{(m-b)n}\frac{1}{n(b+\frac{k}{n})}$$

Taking $\frac{1}{n}$ outside the sign of summation, we have

$$\lim_{n\to\infty}\sum_{k=0}^{(m-b)n}\frac{1}{n(b+\frac{k}{n})}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{(m-b)n}\frac{1}{(b+\frac{k}{n})}$$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{(m-b)n}f\left(\frac{k}{n}\right)$where$f\left(\frac{k}{n}\right)=\frac{1}{(b+\frac{k}{n})}$

We can see that when $k=0$ then $\frac{k}{n}=0$ and $k=(m-b)n$ then $\frac{k}{n}=\frac{(m-b)n}{n}=m-b.$

Thus when $n\to\infty$ then $\frac{k}{n}$ tends to 0 and $m-b$respectively.

By replacing $\frac{k}{n}$ with $x, 1/n$ with dx and $\lim_{n\to\infty}\sum by$ the sign of integration$\int$, also taking limit of x from 0 to $m-b$, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{(m-b)n}\frac{1}{(b+\frac{k}{n})}=\int_0^{m-b}\frac{1}{b+x}dx$$
$$=|\log(b+x)|_0^{m-b}$$
$$=\log m-\log b=\log\frac{m}{b}$$

Ex 9.8. Find the limit of $\left\{\frac{n!}{n^n}\right\}^{\frac{1}{n}}$ where n tends to ∞.

Sol. Let $M=\lim_{n\to\infty}\left\{\frac{n!}{n^n}\right\}^{\frac{1}{n}}$

$$=\lim_{n\to\infty}\left\{\frac{\mathbf{1.2.3\ldots\ldots n}}{\boldsymbol{n.n\ldots\ldots.n}(\boldsymbol{n}\ \mathbf{times})}\right\}^{\frac{1}{n}}$$
$$=\lim_{n\to\infty}\left\{\frac{\mathbf{1}}{\boldsymbol{n}}.\frac{\mathbf{2}}{\boldsymbol{n}}.\frac{\mathbf{3}}{\boldsymbol{n}}\ldots\ldots\frac{\boldsymbol{n}}{\boldsymbol{n}}\right\}^{\frac{1}{n}}$$

Now taking log ion both side we get

$$\log M = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{n}} [\log \frac{\mathbf{1}}{\boldsymbol{n}} + \log \frac{2}{n} + \log \frac{3}{n} + \cdots + \log \frac{n}{n}]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \log \frac{\boldsymbol{k}}{\boldsymbol{n}}$$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)$where$\boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right) = \log \frac{\boldsymbol{k}}{\boldsymbol{n}}$

We can see that when $k = 0$ then $\frac{k}{n} = 0$ and $k = n$ then $\frac{k}{n} = \frac{n}{n} = 1$.

Thus when $n \to \infty$ then $\frac{k}{n}$ tends to 0 and 1 respectively.

By replacing $\frac{k}{n}$ with $x, 1/n$ with dx and $\lim_{n\to\infty} \sum by$ the sign of integration$\int$, also taking limit of x from 0 to 1, we get

$$\log M = \int_0^1 \log x \, dx$$

$$|x log\, x|_0^1 - \int_0^1 x . \frac{1}{x} \, dx$$

$$= 0 - |x|_0^1$$

$$= -1$$

Hence $\log M = -1$

$\Rightarrow M = e^{-1} = \frac{1}{e}$ i.e. $\lim_{n\to\infty} \left\{\frac{\boldsymbol{n}!}{\boldsymbol{n}^{\boldsymbol{n}}}\right\}^{\frac{\mathbf{1}}{\boldsymbol{n}}} = \frac{1}{e}$.

Ex 9.9.Find $\boldsymbol{\lim_{n\to\infty}} \left[\left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{2}}}\right)\left(\mathbf{1} + \frac{\mathbf{2}^{\mathbf{2}}}{\boldsymbol{n}^{\mathbf{2}}}\right)\left(\mathbf{1} + \frac{\mathbf{3}^{\mathbf{2}}}{\boldsymbol{n}^{\mathbf{2}}}\right) \ldots . \left(\mathbf{1} + \frac{\boldsymbol{n}^{\mathbf{2}}}{\boldsymbol{n}^{\mathbf{2}}}\right)\right]^{\frac{\mathbf{1}}{\boldsymbol{n}}}$

Sol..Let $M = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\left(1 + \frac{3^2}{n^2}\right) \ldots . \left(1 + \frac{n^2}{n^2}\right)\right]^{\frac{1}{n}}$

Now taking log ion both side we get

$$\log M = \lim_{n\to\infty} \frac{\mathbf{1}}{\boldsymbol{n}} [\log\left(1 + \frac{1}{n^2}\right) + \log\left(1 + \frac{2^2}{n^2}\right) + \log\left(1 + \frac{3^2}{n^2}\right) + \cdots + \log\left(1 + \frac{n^2}{n^2}\right)]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \log\left(1 + \frac{k^2}{n^2}\right)$$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right)$where$\boldsymbol{f}\left(\frac{\boldsymbol{k}}{\boldsymbol{n}}\right) = \log\left(1 + \frac{k^2}{n^2}\right)$

We can see that when $k = 0$ then $\frac{k}{n} = 0$ and $k = n$ then $\frac{k}{n} = \frac{n}{n} = 1$.

Thus when $n \to \infty$ then $\frac{k}{n}$ tends to 0 and 1 respectively.

By replacing $\frac{k}{n}$ with $x, 1/n$ with dx and $\lim_{n\to\infty} \sum by$ the sign of integration$\int$, also taking limit of x from 0 to 1, we get

$$\log M = \int_0^1 \log(1 + x^2) \, dx$$

$$|x \log(1 + x^2)|_0^1 - \int_0^1 x . \frac{2x}{1 + x^2} \, dx$$

$$= \log 2 - 2 \int_0^1 \frac{x^2}{1 + x^2} \, dx$$

$$= \log 2 - 2\left[\int_0^1 \frac{1+x^2}{1+x^2}\,dx - \int_0^1 \frac{1}{1+x^2}\,dx\right]$$
$$= \log 2 - 2[|x|_0^1 - |\tan^{-1} x|_0^1]$$
$$= \log 2 - 2\left[1 - \frac{\pi}{4}\right]$$
$$= \log 2 - \log e^{2\left[1-\frac{\pi}{4}\right]}$$
$$= \log \frac{2}{e^{2\left[1-\frac{\pi}{4}\right]}}$$

Hence $\log M = \log 2e^{\frac{\pi}{2}-2}$

$\Rightarrow M = 2e^{\frac{\pi}{2}-2}$

i.e. $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots.\left(1+\frac{n^2}{n^2}\right)\right]^{\frac{1}{n}} = 2e^{\frac{\pi}{2}-2}$.

CHECK YOUR PROGRESS

(CQ 1) If $f(-x) = f(x)$ then $\int_{-1}^{1} f(x)dx =$ ____________

(CQ 2) If $f(2a - x) = -f(x)$ then $\int_{a}^{2a} f(x)dx =$ ____________

(CQ 3) If $f(-x) = -f(x)$, then $\int_{-1}^{1} f(x)dx =$ ____________

(CQ 4) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} dx =$ ____________

(CQ 5) $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+(\frac{r}{n})} = \log 2$ (T/F)

(CQ 6) $\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx = 0$ (T/F)

(CQ 7) Let f is integrable over the interval $[a, b]$ and φ be a differentiable function on interval [a,b] such that $\varphi'(x) = f(x) for\ all\ x\ in\ [a, b]$. Then $\int_a^b f(x)dx = \varphi(b) - \varphi(a)$(T/F)

9.7 SUMMARY

In this unit we studied antiderivatives, definite integrals, properties of integral, fundamental theorem and summation of series with the help of definite integral.

9.8 GLOSSARY

- **i.** **Set-** a well defined collection of elements
- **ii.** **Integral**-express the area under the curve of a graph of the function

9.9 REFERENCES

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iii. https://fl01000126.schoolwires.net/cms/lib/FL01000126/Centricity/Domain/261/FDWK_3ed_Ch05_pp262-319.pdf

9.10 SUGGESTED READINGS

i. Howard Anton, I. Bivens and Stephan Davis (2016). Calculus (10th edition). WileyIndia.

ii. George B. Thomas Jr, Ross L.Finney (1998), Calculus and Analytical Geometry, Adison Wiley Publishing Company.

iii. James Stewart (2012). Multivariable Calculus (7th edition). Brooks/Cole. Cengage.

iv. S.C. Malik and Savita Arora (2021). Mathematical Analysis (6th edition). New Age International Private Limited.

9.11 TERMINAL QUESTION

Long Answer Questions

(TQ 1) Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots.+\frac{1}{2n}\right]$

(TQ 2) Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+m}+\frac{1}{n+2m}+\cdots.+\frac{1}{n+nm}\right]$

(TQ 3) Show that $\int_0^{\infty}(\cot^{-1}x)^2dx = \pi\log 2$

(TQ 4) Show that $\int_0^{\pi/2}\frac{1}{1+\tan x}dx = \frac{\pi}{4}$

(TQ 5) Show that $\int_0^{\pi} x\sin^3 x\, dx = \frac{2\pi}{3}$

.**Fill in the blanks**

(TQ 6) $\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}$ is ________________.

(TQ 7) $\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n+r} =$ ______________.

9.12 ANSWERS

(CQ 1) $2\int_0^1 f(x)dx$ (CQ 2) 0 (CQ 3) 0

(CQ 4) $\pi/4$ (CQ 5) T (CQ 6) F

(CQ 7) T

(TQ 6) $\frac{\pi}{4}$ (TQ 7) $\int_0^1\frac{1}{1+x}dx$

BLOCK III
ASYMPTOTES AND DOUBLE AND TRIPLE INTREGRALS

UNIT 10:-ASYMPTOTES

CONTENTS:

10.1 INTRODUCTION

This word is derived from the Greek word ‘**asumptotos**’, which means “**not following together”.**Some curves are limited in extent (e.g. circle, ellipse etc.). For such curves, every tangent has a usual meaning when x tends to some finite value.
There is another family of curves which extend up to infinity e.g. hyperbola, parabola, exponential curve etc. see the case of $y = \frac{1}{x}$.
You can observe the tangent at P_1.

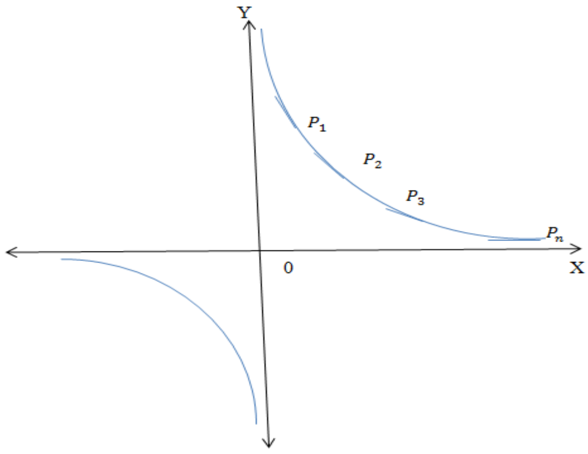

Fig.10.1.1

Similarly tangents at $P_2, P_3, \ldots$ .These tangents are slowly becoming parallel to x −axis. Now think, when $n \to \infty$, which line can be considered as the tangent at P_n on $y = \frac{1}{x}$?.You might have observed that for such curves:

Tangents are tending to a fixed line which is at a finite distance (here zero) from the origin. This forms the basic concept of asymptote.

Now we can formally define asymptote:

"A straight line at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve."

Here we shall study vertical, horizontal and oblique asymptotes, depending on their orientations.

<u>Vertical Asymptote:</u> The line $x = a$ is a vertical asymptote of the curve $y = f(x)$ if at least one of the following is true:

(i) $\lim_{x \to a^-} f(x) = \pm\infty$

(ii) $\lim_{x \to a^+} f(x) = \pm\infty$.

Example 1:$f(x) = \frac{x}{x-a}$ has a vertical asymptote $x = \lambda$!

Note:(1) The function $f(x)$ may or may not be defined at $x = a$i.e. functional value $f(a)$ does not affect the asymptote, i.e, functional value $f(a)$does not affect the asymptote.

Example 2:$f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2, & x \le 0 \end{cases}$ has a limit of $+\infty$ as $x \to 0^+$. $f(x)$ has the vertical asymptote $x = 0$, though $f(0) = 2$. The graph of this function intersects the vertical asymptote once at $(0,2)$.

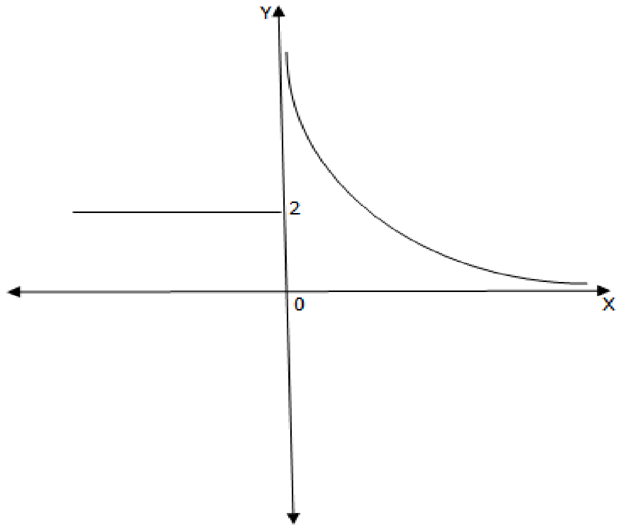

Fig. 10.1.2

(2) It is impossible for the graph of a function to intersect a vertical asymptote (actually any vertical line) in more than one point.
(3) If the graph of a function $y = f(x)$ is continuous then it is impossible that its graph intersects any vertical asymptote.
(4) A common example of a vertical asymptote is the case of a rational function at a point x such that the denominator is zero and numerator is non-zero.

Example 3: $f(x) = \frac{2x}{(x-1)(x-2)(x-3)}$

Here $x = 1, x = 2,\ x = 3$ are vertical asymptotes.

(5)Rational function: It is the quotient of two polynomials. For example:$f(x) = \frac{g(x)}{h(x)}$ or $f(x) = \frac{2x+3}{(x+1)(x-2)}$ etc.

<u>Horizontal Asymptote:</u> These are horizontal lines which approach the graph of the function $y = f(x)$ when $x \to \pm\infty$.

Example 4:

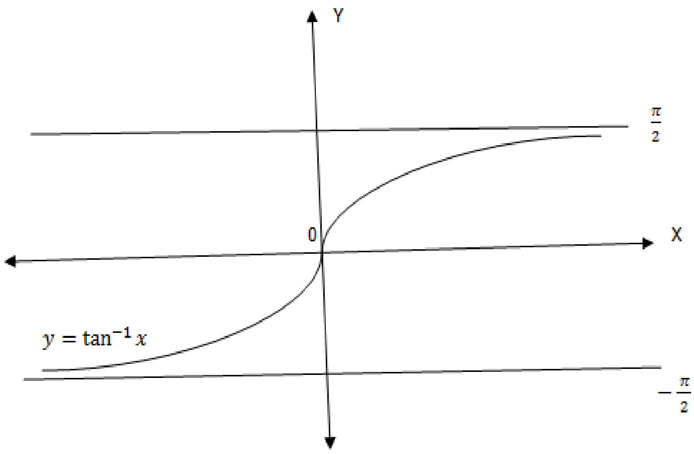

Fig.10.1.3

Here $x = \pm\frac{\pi}{2}$ are asymptotes of $y = \tan^{-1} x$, when $x \to \pm\infty$. The horizontal line $y = a$ is a horizontal asymptote of the function $y = f(x)$ if either $\lim_{x\to-\infty} f(x) = c$ or $\lim_{x\to+\infty} f(x) = c$.

Oblique Asymptote: A linear asymptote (or simply asymptote) is called oblique if it is neither parallel to x −axis nor y −axis. So $y = mx + c,\ m \neq 0$ may be an oblique asymptote.

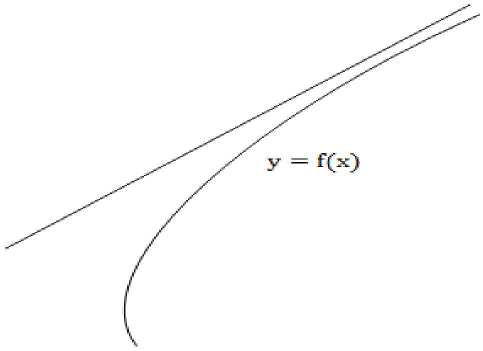

Fig. 10.1.4

Horizontal and Oblique Asymptote for Rational Functions

$\mathbf{deg}(N_r)$ −de g(D_r)	Asymptotes in general	Example	Asymptote for example
< 0	$y = 0$	$f(x) = \frac{1}{x^2+1}$	$y = 0$
$= 0$	$y = \textit{the ratio of leading coe}$	$f(x) = \frac{2x^2+7}{3x^2+x+}$	$y = \frac{2}{3}$
$= 1$	$y = \textit{quotient of the euclidea division of the numerator the denominator.}$	$f(x) = \frac{x^2+x+1}{x}$	$y = x + 1$
> 1	$\textit{none}$	$f(x) = \frac{2x^4}{3x^2+1}$	$\textit{no linear asympto}$

Transformation of Known Functions: If a known function has an asymptote (e.g. $y = 0$ for $y = e^x$), then the translations of it also have an asymptote→

(i) If $x = a$ is a vertical asymptote of $f(x)$, then $x = a + h$ is a vertical asymptote of $f(x - h)$.

(ii) If $y = c$ is a horizontal asymptote of (x) , then $y = c + k$ is an horizontal asymptote of $f(x) + k$.

(iii) If $y = mx + c$ is an asymptote of $f(x)$, then $y = \lambda mx + \lambda c$ is an asymptote of $\lambda f(x)(\lambda \in R)$.

10.2 OBJECTIVES

In this unit, we will understand:

i. The meaning of the term "Asymptotes".

ii. Existence of Asymptotes.

iii. Non-existence of Asymptotes.

iv. Procedures for finding the Asymptotes in various cases.

v. Intersection of a curve with its asymptotes.

10.3 DETERMINATION OF ASYMPTOTES

The equation of a line which is neither parallel to x −axis nor to y −axis is $y = mx + c, \quad m \neq 0$.

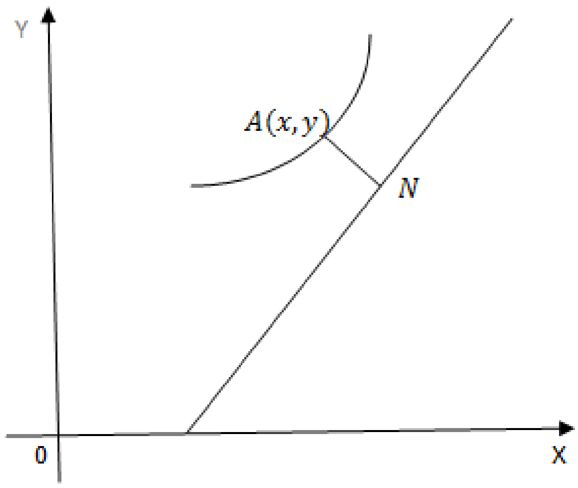

Fig. 10.3.1

Let $A(x, y)$ be a point on an infinite branch of the curve $f(x, y) = 0$. Let $p = AN$ be the perpendicular distance of any point $A(x, y)$ on the infinite branch of a given curve. Then

$$p = \frac{|y - mx - c|}{\sqrt{1 + m^2}}$$

If $y = mx + c$ is an asymptote, then $AN \to 0$ as $A \to \infty$ along the curve i.e. when $x \to \infty$.

So, we have

$$\lim_{x\to\infty} (y - mx - c) = 0 \Rightarrow \lim_{x\to\infty} (y - mx) = c \quad \text{..(1)}$$

We can write, $\frac{y}{x} - m = (y - mx)\left(\frac{1}{x}\right)$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{y}{x}-m\right)=\lim_{x\to\infty}\left[(y-mx)\left(\frac{1}{x}\right)\right]$$

$$=\lim_{x\to\infty}(y-mx)\lim_{x\to\infty}\left(\frac{1}{x}\right)$$

(as product of limits is equal to limit of products.)

$$= c\,(0) = 0$$

$\Rightarrow \lim_{x\to\infty}\left(\frac{y}{x}\right)=m$ ………………………………………….(2)

So from equations (1) and (2), we conclude that for an oblique asymptote→

(i) $\lim_{x\to\infty}\left(\frac{y}{x}\right)$ in $f(x,y)=0$ represents the slope m.

(ii) $\lim_{x\to\infty}(y-mx)$ represents $'c'$ in $y=mx+c$.

(A)

Note:

(1) The values of y for different branches of the curve $f(x,y)=0$ will be different for a given value of x. So we may obtain various different values of $'m'$ and correspondingly several different values of $\lim_{x\to\infty}(y-mx)$.

Thus a curve may have more than one asymptote.

(2) This method can determine all the asymptotes except those which are parallel to y −axis. To determine those asymptote, we start with the equation $x=my+d$ which can represent every straight line not parallel to x −axis and so that when $y\to\infty$, $m=\lim_{y\to\infty}\left(\frac{x}{y}\right)$ and $d=\lim_{y\to\infty}(x-my)$.

(3) The asymptotes not parallel to any axis can be obtained either way.

Example 5: Examine the Folium $x^3+y^3-3axy=0$ for asymptotes.

Solution: The given equation is of third degree

(1) $x^3+y^3-3axy=0$

Dividing both sides by x^3, we get

$$1+\left(\frac{y}{x}\right)^3-3a\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=0$$

Let $x\to\infty$. Then by reminding $\lim_{x\to\infty}\left(\frac{y}{x}\right)=m$, we get

$1+m^3-0=0 \Rightarrow (m+1)(m^2-m+1)=0.$

So the only real value of $m=-1$.

Put $m=-1$ in $y=mx+p$ where $p\to c$ when $x\to\infty$.

$$\Rightarrow y=p-x.$$

Putting $y=p-x$ in equation (1), we get

$$x^3+(p-x)^3-3ax\,(p-x)=0$$

$\Rightarrow 3(p+a)x^2-3(p^2+ap)x+p^3=0$, which is of the second degree in x.

Dividing by x^2, we get

$$3(p+a)-3(p^2+ap)\frac{1}{x}+p^3.\frac{1}{x^2}=0.$$

When $x \to \infty, p \to c$. So we get $3(c+a) = 0 \Rightarrow c = -a$.
Hence $y + x + a = 0$ is the only asymptote of this curve.

Branches of a curve: If y has two or more values for every value of x, it is usually possible to suppose that this is a case where two or more distinct are given.

But it is generally more convenient to regard the curves corresponding to these distinct functions, not as different curves, but as different branches of one curve. In general, each branch has its own asymptote.

Example 6: If $y^2 - 2xy - 1 = 0$, then $y = x \pm \sqrt{x^2+1}$.
So we obtain $y = x + \sqrt{x^2+1}$ and $y = x - \sqrt{x^2+1}$ as two branches of the curve $y^2 - 2xy - 1 = 0$, and each branch will have an asymptote.

10.4 ASYMPTOTES PARALLEL TO Y-AXIS

The asymptote parallel to y –axis are obtained by equating to zero the real linear factors in the coefficient of the highest power of $'y'$, in the equation of the curve.

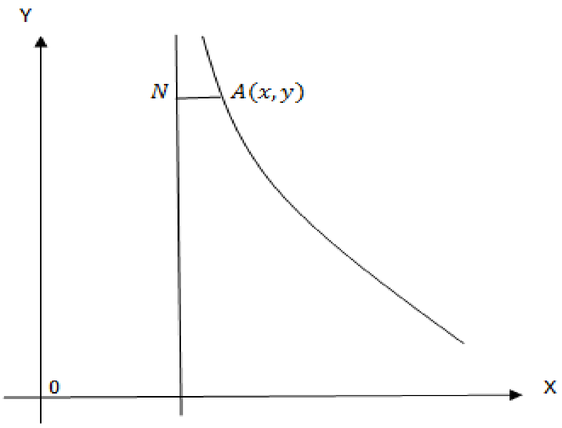

Fig. 10.4.1

Proof: Let $x = k$ be an asymptote of the curve.

Here only $y \to \infty$ as a point $A(x, y)$ recedes to infinity along the curve.

The distance AN of any point $A(x, y)$ on the curve is equal to $x - k$.

$\lim_{y \to \infty} (x - k) = 0 \Rightarrow \lim_{y \to \infty} x = k$, which provides the value of k.

Arranging the equation of the curve in descending powers of y, so that it takes the form

(1) $y^m \phi(x) + y^{m-1}\phi_1(x) + y^{m-2}\phi_2(x) + \cdots = 0$

Where $\phi(x), \phi_1(x), \ \phi_2(x), \ldots$ etc are polynomials in x.

Dividing the equation (1) by y^m, we get

(2) $$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \cdots = 0$$

Let $y \to \infty$.

Let us write $\lim_{y\to\infty} x = k$. The equation (2) gives, $\phi(k) = 0$, so that k is a root of the equation $\phi(x) = 0$.

Let k_1, k_2 be the roots of $\phi(x) = 0$. Then the asymptotes parallel to Y −axis are $x = k_1$, $x = k_2$ etc.

We know that $(x - k_1), (x - k_2)$ etc. are the factors of $\phi(x)$ which is the coefficient of the heighest power y^m of y in the given equation.

10.5 ASYMPTOTE PARALLEL TO X −AXIS

In the same way, the asymptotes, parallel to X −axis, are obtained by equating zero the real line factors in the coefficient of the highest power of x, in the equation of the curve.

Example 7: Find the asymptotes of the curve $x^2y^2 = 9x^2 + 4y^2$ parallel to the axes.

Solution: We have

(1) $x^2y^2 = 9x^2 + 4y^2$

$$\Rightarrow x^2(y^2 - 9) - 4y^2 = 0$$

So asymptotes parallel to x −axis are given by, $y^2 - 9 = 0$

$\Rightarrow y = \pm 3$.

Similarly from equation (1), $y^2(x^2 - 4) - 9x^2 = 0$.

So asymptotes parallel to Y −axis are given by $x^2 - 4 = 0$

$\Rightarrow x = \pm 2$.

10.6 ASYMPTOTES OF THE GENERAL ALGEBRAIC CURVE

Suppose equation of the curve be

(1) $\{a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \cdots + a_{n-1}yx^{n-1} + a_nx^n\} + \{b_1y^{n-1} + b_2y^{n-2}x + \cdots + b_{n-1}yx^{n-2} + b_nx^{n-1}\} + \{\, cy^{n-2} + \cdots\} + \cdots = 0$

(2) $x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots = 0$

Where $\phi_r\left(\frac{y}{x}\right)$ is an expression of the r^{th} degree in $\left(\frac{y}{x}\right)$.

Determination of "m"

Dividing by x^n, we can write

(3) $\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots = 0$

Now taking limit $x \to \infty$, (and excluding the case of asymptotes parallel to the Y −axis i.e. the case in which $\lim_{x\to\infty}\left(\frac{y}{x}\right) \to \infty$); we get

(4) $\phi_n(m) = 0$

Where $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$.

On solving equation (4), we select only real values of "m", which give slopes of asymptotes $y = mx + c$.

Determination of "c": Now differentiating equation (3) with respect to x, we obtain

$$\left\{\phi_n'\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots\right\}\left(\frac{xy'-y}{x^2}\right) - \frac{1}{x^2}\phi_{n-1}\left(\frac{y}{x}\right) - \frac{2}{x^3}\phi_{n-2}\left(\frac{y}{x}\right) - \cdots = 0.$$

Multiplying by x^2, we get,

$$\left\{\phi_n'\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots\right\}(xy' - y) - \phi_{n-1}\left(\frac{y}{x}\right) - \frac{2}{x}\phi_{n-2}\left(\frac{y}{x}\right) - \cdots = 0.$$

Now taking $x \to \infty$ and using equation (A), we get

(5) $$c\phi_n'(m) + \phi_{n-1}(m) = 0.$$

This equation determines one value of c for each value of m found from equation (4).

Hence the asymptotes are $y = mx + c$.

Alternative Method: (1) If we substitute $y = mx + c$ in equation (2) of the last article and solve, we get

(1) $$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{2!x^2}\phi_n''(m) + \cdots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi_{n-1}'m + \cdots\right\} + \cdots = 0$$

Now, if we equate to zero the coefficients of the two highest powers of x, we get equations (4) and (5) of the last article. Hence we have the following rule for determining the asymptotes:

(i) Put $y = mx + c$ in the equation.

(ii) Equate to zero the coefficient of the two highest powers of x and determine m and c from these.

(iii) If $(m_1, c_1), (m_2, c_2), \ldots$ are the values of m and c thus obtained, the asymptotes are $y = m_1x + c_1$; $y = m_2x + c_2$, etc.

(2) We observe that $\phi_n(m)$ can be obtained at once by putting $x = 1$ and $y = m$ in the highest degree terms of the equation of the curve. Similarly $\phi_{n-1}(m)$ can be obtained by putting $x = 1$ and $y = m$ in the $(n-1)^{th}$ degree terms. Hence we get the asymptotes more quickly.

Example 8: Find the asymptotes of

$$y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

Solution: Putting $y = mx + c$ in the above expression, we get

$$(mx+c)^3 - x^2(mx+c) - 2x(mx+c)^2 + 2x^3 - 7x(mx+c) + 3(mx+c)^2 + 2x^2 + 2x + 2(mx+c) + 1 = 0.$$

Or $x^3(m^3 - m - 2m^2 + 2) + x^2(3m^2c - c - 4mc - 7m + 3m^2 + 2) + \cdots = 0$.

Therefore $'m'$ and $'c'$ are given by

$\phi_n(m) = m^3 - m - 2m^2 + 2 = 0$ and

$\phi_{n-1}(m) = c(3m^2 - 4m - 1) + 3m^2 - 7m + 2 = 0$.

From $\phi_n(m) = 0$, we get $(m-1)(m+1)(m-2) = 0 \Rightarrow m = 1, -1, 2$.

From $\phi_{(n-1)}(m) = 0$, we get $c = -\frac{(3m^2-7m+2)}{(3m^2-4m-1)}$.

If $m = 1, c = -1$; $m = -1, c = -2, m = 2, c = 0$.

Hence the asymptotes are

$y = x - 1; y = -x - 2; y = 2x$.

Method(II): Putting $x = 1$ and $y = m$ in the third degree terms and equating to zero, we get

$$\phi_n(m) = m^3 - 2m^2 - m + 2 = 0$$

and $\phi_n(m) = c(3m^2 - 4m - 1) + (3m^2 - 7m + 2) = 0$.

Now solve as usual.

10.7 ASYMPTOTE MIGHT NOT EXIST

If one or more values of m, found from $\phi_n(m) = 0$, make $\phi_n'(m) = 0$, but $\phi_{n-1}(m) \neq 0$; the equation for determining the corresponding value of c becomes

(1) $$0.c + \phi_{n-1}(m) = 0.$$

We can calculate $'c'$ only if its coefficient is non-zero. i.e, the equation in $'c'$ was $Fc + G = 0$, where $\lim_{x\to\infty} F = 0$ and $\lim_{x\to\infty} G = \phi_{n-1}(m)$.

Hence $\lim_{x\to\infty} c = +\infty$, or $-\infty$, and this corresponds to the case when the tangent goes further and further away as $x \to \infty$. (See the definition of asymptote).

Example 9: Find the asymptote of the curve $y^2 = 4ax; a \neq 0$.

Solution: Putting $y = mx + c$, we get

$$(mx+c)^2 - 4ax = 0$$
$$m^2x^2 + 2mcx + c^2 - 4ax = 0$$
$$m^2x^2 + 2x(mc - 4a) + c^2 = 0$$

Putting the coefficient of x^2 and x to zero, we get

$\phi_n(m) = m^2 = 0 \Rightarrow m = 0$,

And $\phi_{n-1}(m) = mc - 4a = 0 \Rightarrow 0.c - 4.a = 0$, which is impossible as $a \neq 0$.

Hence $y^2 = 4ax$ has no asymptote.

10.8 TWO PARALLEL ASYMPTOTES

If any value of m, found from $\phi_n(m) = 0$, has repeated values i.e., say $m = \alpha, \alpha$, then there may be two parallel asymptotes. Then by theory of equations,
$\phi_n(\alpha) = 0 = \phi_n'(\alpha)$. Generally to find $'c'$, we use
$c\phi_n'(m) + \phi_{n-1}(m) = 0$.
Here, we shall get $0.c + \phi_{n-1}(m) = 0$
For the finite $'c'$, we consider $\phi_{n-1}(m) = 0$.
Now we differentiate equation (3) twice with respect to x(of the article, "asymptotes of the general algebraic curves) and solving similarly, we get

$$(1)\frac{1}{2}c^2\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m) = 0.$$

On solving above expression, we shall obtain two different $'c'$ for one m i.e. parallel asymptotes.
Note: If we get $m = \alpha, \alpha, \alpha$, then c will be obtained from

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + c\phi_{n-2}'(m) + \phi_{n-3}(m) = 0.$$

You can now generalize it.

Example 10: Find the asymptotes of

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$$

Solution: Putting $x = 1, y = m$ in 3rd degree terms, we get

$$\phi_3(m) = 1 - m - m^2 + m^3 = 0$$
$$\Rightarrow m = -1, 1, 1.$$

Also $\phi_2(m) = 2 - 4m^2 + 2m$.
To determine c, we have $c\phi_3'(m) + \phi_2(m) = 0$
$\Rightarrow c(-1 - 2m + 3m^2) + (2 - 4m^2 + 2m) = 0$.
For $m = -1$, we get $c = 1$.
So $y = -x + 1$ or $y + x - 1 = 0$ is an asymptote.
For $m = 1, 1$, we use the equation

$$\left(\frac{c^2}{2}\right)\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$
$$\Rightarrow \left(\frac{c^2}{2}\right)(-2 + 6m) + c(2 - 8m) + (1 + m) = 0.$$

Putting $m = 1$ and solving, we get

$$c^2 - 3c + 1 = 0 \Rightarrow c = \frac{3 \pm \sqrt{5}}{2}.$$

Hence $y = x + \frac{3+\sqrt{5}}{2}$ and $y = x + \frac{3-\sqrt{5}}{2}$ are two other asymptotes.

10.9 TOTAL NUMBER OF ASYMPTOTES

As the equation for determining m, viz $\phi_n(m) = 0$, is of degree n, and so by 'Fundamental theorem of algebra' it has n roots (whether real or imaginary).
So 'a curve of degree n has at most n asymptotes'. It may have less than n asymptotes (even no asymptote), but not more than n asymptotes.

Note: (i) If some of the roots of $\phi_n(m) = 0$ are complex, corresponding to those values of m, there will be no real asymptote.
(ii) There might be no asymptote corresponding to even a real root e.g. $y^2 = 4ax$ (Observe it)!

Historical Note: The meaning of the word 'asymptote' has changed a couple of times.When Appolonius first used it (2200 Years ago), it meant any straight line that did not meet a given curve. With that meaning, a hyperbola has two asymptotes.That definition was used until the 19th centuary. The concept of asymptote required a curve to get closer to the asymptotic straight line but never cross it as it approached it. (It could cross it somewhere else).

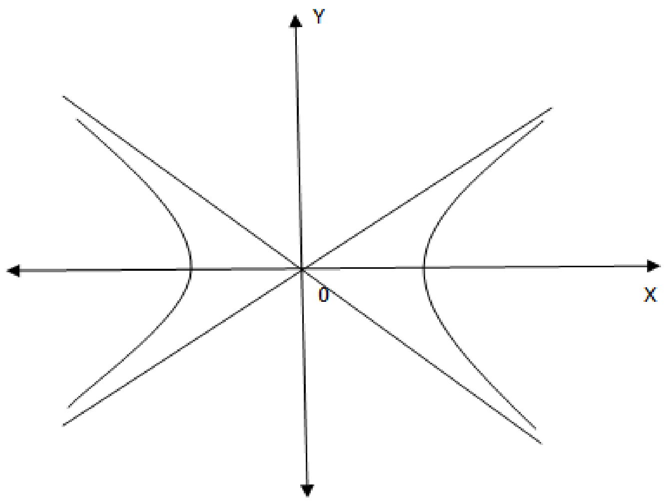

Fig. 10.9.1

As you go further along the curve, the curve gets closer and closer to the asymptote. In fact, if you gofar enough, the distance to the asymptote will be halved.This concept of halving the distance defined limits as used by Euclid and others until the modern era. Then, a broader definition of limits, the $\epsilon - \delta$ definition was developed.With the mew definition, a curve could be asymptotic to a line even if it crossed it just so long as you can keep the curve as close as you like to the line if you can go far enough.Thus the curve below has the same two asymptotes as the hyperbola above.

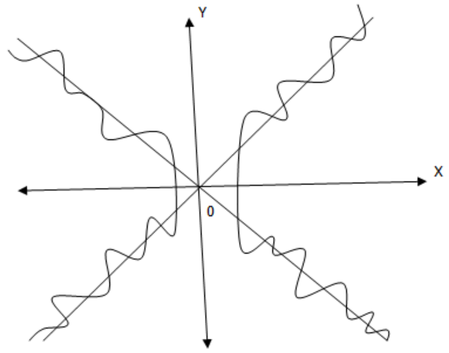

Fig. 10.9.2

10.10 INTERSECTION OF A CURVE AND ITS ASYMPTOTES

Theorem: Any asymptote of a curve of the n^{th}degree cuts the curve in $(n-2)$ points.

Proof: Let $y=mx+c$ be an asymptote of a rational algebraic curve of degree $'n'$ whose equation is

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\cdots=0.$$

To find the points of intersection, we have to solve the two equations simultaneously.

(1) $\Rightarrow x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\cdots=0$

Expanding each term by Taylor's theorem and arranging in descending powers of x, we have

(2) $[\phi_m(m)]x^n+[c\phi_n'(m)+\phi_{n-1}(m)]x^{n-1}+\left[\frac{1}{2}c^2\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\cdots=0.$

Since $y=mx+c$ is an asymptote, therefore
$\phi_n(m)=0$ and $c\phi_n'(m)+\phi_{n-1}(m)=0.$
So equation (2) reduces to

$$\left[\frac{1}{2}c^2\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\cdots=0;$$

which is of the $(n-2)$ degree and therefore gives $(n-2)$ values of $x$and consequently, the asymptote cuts the curve in $(n-2)$ points.

Geometric Explanation: A straight line cuts another line at most at one point. A straight line cuts a quadratic curve $y=ax^2+bx+c, a\neq 0$ at most two points.

Similarly, in general, a straight line cuts a curve of degree n at most in n points.
As one of these points of intersection is kept fixed (say, B), and another point of intersection is made to tend to it (i.e. A $\to$ B), the straight line AB tends to the tangent at the B.
Hence a tangent (and therefore, as a particular case, an asymptote will, in general, cut the curve in $(n-2)$ points.

Corollaries: (1) Thus n asymptotes will cut the curve in $n(n-2)$ points.
(2) If the equation of a curve of the n^{th} degree can be put in the form

$$F_n + F_{n-2} = 0$$

Where F_{n-2} is of degree $(n-2)$ at most and F_n consists of n, non-repeated linear factors, then the $n(n-2)$ points of intersection of the curve and its asymptotes lie on the curve, $F_{n-2} = 0$.

This follows from the fact that the joint equation of the asymptotes is $F_n = 0$. So that the points of intersection satisfy the equations $F_n = 0$ and $F_n + F_{n-2} = 0$ and consequently they satisfy the equation the equation $F_{n-2} = 0$.

(3) For a cubic curve, $n = 3$. Therefore asymptotes cut the curve in $3(3-2) = 3$ points which lie on a curve of degree $3 - 2 = 1$, i.e. the three points of intersection of a cube curve and its asymptotes lie on a straight line.
(4) For a bi-quadratic (or quartic) curve, $n = 4$. So asymptotes cut the curve in $4(4-2) = 8$ points, which lie on a curve of degree $4 - 2 = 2$ i.e. the eight points of intersection of a quartic curve and its asymptote lie on a conic.

Example 11: Prove that the points of intersection of the curve $2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0,$ and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.
Solution: Given curve is

(1) $$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$$

You can easily find the three asymptotes $y - x + 1 = 0$; $y + x - 2 = 0$; $y - 2x = 0$.
Combined equation of asymptotes is

$$(y - x + 1)(y + x + 2)(y - 2x) = 0$$

(2) $$\Rightarrow y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2y - 4x = 0.$$

Multiplying this by $'2'$ and subtracting from equation (1), we get $8x + 2y + 1 = 0$, on which the points of intersection must lie.

10.11 ASYMPTOTES BY EXPANSION

Theorem: If a curve can be written as

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \cdots,$$

then $y = mx + c$ is an asymptote.

Proof: Let us consider the equation of curve as

(1) $$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \cdots$$

Where the series $\frac{A}{x} + \frac{B}{x^2} + \cdots$ is convergent for sufficiently large values of x.

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = m - \frac{A}{x^2} - \cdots.$$

So the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Or $Y - \left(mx + \frac{A}{x} + \frac{B}{x^2} + \cdots\right) = (m - \frac{A}{x^2} - \frac{2B}{x^3} - \cdots)(X - x)$

(2) $$\Rightarrow Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \cdots\right)X + c + \frac{2A}{x} + \frac{3B}{x^2} + \cdots$$

Let $x \to \infty$, then from (2), we have

$$Y = mX + c$$

On generalizing, we get the asymptote $y = mx + c$.

Ex. 12 Find the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution:

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \Rightarrow y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

$$y = \pm\frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}}$$

So for large x, $1 > \frac{a^2}{x^2}$.

Now using the Binomial expansion formula

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots,$$

Where $|x| < 1, n \in R$, we get

$$y = \pm\frac{bx}{a}\left[1 - \frac{1}{2}.\frac{a^2}{x^2} - \frac{1}{8}.\frac{a^4}{x^4} + \cdots\right]$$

$$y = \pm\left[\frac{bx}{a} - \frac{ab}{2x} - \frac{1}{8}.\frac{a^3b}{x^3} - \cdots\right]$$

When $x \to \infty$, we have $y = \pm \frac{bx}{a}$.
Hence asymptotes are $y = \pm \frac{ba}{x}$.

10.12 A USEFUL METHOD OF FINDING ASYMPTOTES OF ALGEBRAIC CURVES

Suppose the equation of the curve of degree n, be

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots = 0.$$

Case (I): Let $(y - ax)$ be a non-repeated factor of the n^{th} degree terms of the equation to the curve. Then the equation to the curve can be written as

(1) $(y - ax)F_{n-1} + P_{n-1} = 0.$

Where F_{n-1} contains only terms of degree $(n-1)$, and P_{n-1} contains terms of various degrees, none of which is of a degree higher than $(n-1)$.

Writing (1) as $y - ax + \frac{P_{n-1}}{F_{n-1}} = 0.$

Taking the limit of $\frac{P_{n-1}}{F_{n-1}}$ as $x \to \infty$, we shall get the equation of the asymptote, since the curve approaches the asymptote when $x \to \infty$.

This limit can be easily found if we remember that $\lim_{x\to\infty}\left(\frac{y}{x}\right) = a$.

Hence the asymptote corresponding to the factor $(y - ax)$ is

$$y - ax + \lim_{x\to\infty, \frac{y}{x}\to a}\left(\frac{P_{n-1}}{F_{n-1}}\right) = 0.$$

Example 13: Find the asymptotes of the curve given by

$$y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

Solution: Factorizing the third degree terms, the equation to the curve can be written as

$$(y - x)(y + x)(y - 2x) - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

Hence one asymptote is

$$y - x = \lim_{x\to\infty, \frac{y}{x}\to 1} \frac{(7xy - 3y^2 - 2x^2) + terms\ of\ lower\ degree}{(y + x)(y - 2x)}$$

$$= \lim_{x\to\infty, \frac{y}{x}\to 1} \frac{\left(7\frac{y}{x} - 3\frac{y^2}{x^2} - 2\right) + terms\ which\ tend\ to\ zero}{\left(\frac{y}{x} + 1\right)\left(\frac{y}{x} - 2\right)}$$

$$= \frac{7 - 3 - 2}{(1 + 1)(1 - 2)} = -1.$$

So one asymptote is $y - x + 1 = 0$.
Similarly, a second asymptote is

$$y + x = \lim_{x\to\infty, \frac{y}{x}\to -1} \frac{7\left(\frac{y}{x}\right) - 3\left(\frac{y}{x}\right)^2 - 2}{\left(\frac{y}{x} - 1\right)\left(\frac{y}{x} - 2\right)} = -2$$

$$\text{or } y + x + 2 = 0.$$

Similarly third asymptote can easily be calculated.

Case (II): Let the terms of the n^{th} degree in the equation to the curve contain $(y - ax)^2$ as a factor, and suppose $(y - ax)$ is not a factor of the $(n-1)^{th}$ degree terms.

Then proceeding as in case (I) , we find that $\lim\limits_{x\to\infty, \frac{y}{x}\to a} (y - ax)^2 \to +\infty \; or - \infty.$

Hence there is no asymptote in this case.

Case (III): Let the equation to the curve be of the form

$$(y - ax)^2 F_{n-2} + (y - ax) G_{n-2} + P_{n-2} = 0,$$

Where F_{n-2} and G_{n-2} contain only terms of degree $(n - 2)$, and P_{n-2} contain terms none of which is of a higher degree than $(n - 2)$.

Dividing by F_{n-2} and taking limits as $x \to \infty$ and $\frac{y}{x} \to a$, we get an equation of the form

$$(y - ax)^2 + B(y - ax) + C = 0,$$

Which, on solving for $(y - ax)$, gives us two asymptotes of the form $y - ax = c_1$, and $y - ax = c_2$.

Case (IV): We can proceed in the same way if the n^{th} degree terms contain $(y - ax)^3$, or a higher power of $(y - ax)$, as a factor.

<u>Note:</u> If the equation to the curve is of the form

(2) $(ax + by + c)P_{n-1} + Q_{n-1} = 0$

Where P_{n-1} and Q_{n-1} contain terms none of which is of a higher degree than $(n - 1)$, and P_{n-1} contains at least one term of degree $n - 1$(to ensure that the equation of the curve is of degree n), a little consideration (or working out a few examples) will show that the asymptote corresponding to the factor $(ax + by + c)$ is

$$(ax + by + c) + \lim_{x\to\infty, \frac{y}{x}\to\left(\frac{-a}{b}\right)} \frac{Q_{n-1}}{P_{n-1}} = 0,$$

and that a similar modification can be made in the other cases.

Thus we need not transform an equation of the form (2) into an equation of the form (1) as a preliminary to finding out the asymptotes.

Example 14: Find the asymptotes of

$$y^3 + x^2 y + 2xy^2 - y + 1 = 0.$$

Solution: By factorizing the terms of degree 3, the equation to the curve can be written as

$$y(y + x)^2 - y + 1 = 0.$$

The asymptotes corresponding to the factor $(y + x)^2$ are

$$(y + x)^2 + \lim_{x\to\infty, \frac{y}{x}\to -1} \left(\frac{-y + 1}{y}\right) = 0,$$

$$(y + x)^2 = -\lim_{x\to\infty, \frac{y}{x}\to -1} \frac{\left(\frac{-y}{x} + \frac{1}{x}\right)}{\left(\frac{y}{x}\right)} = \frac{1 + 0}{1} = 1$$

$$\Rightarrow y + x = \pm 1.$$

So asymptotes are $y + x - 1 = 0$ and $y + x + 1 = 0$. The third asymptote can easily be calculated as $y = 0$.

Example 15: Find the asymptotes of

$$(x - y - 1)^2(x^2 + y^2 + 2) + 6(x - y - 1)(xy + 7) - 8x^2 - 2x - 1 = 0.$$

Solution: Dividing by the coefficient of $(x - y - 1)^2$, and taking limits we see that the asymptotes parallel to $(x - y - 1 = 0)$ are

$$(x - y - 1)^2 + 6(x - y - 1) \lim_{x\to\infty, \frac{y}{x}\to 1} \left(\frac{xy + 7}{x^2 + y^2 + 2}\right) + \lim_{x\to\infty, \frac{y}{x}\to 1} \left(\frac{-8x^2 - 2x - 1}{x^2 + y^2 + 2}\right) = 0,$$

$$(x - y - 1)^2 + 6(x - y - 1) \lim_{x\to\infty, \frac{y}{x}\to 1} \left(\frac{\frac{y}{x} + \frac{7}{x^2}}{1 + \left(\frac{y}{x}\right)^2 + \frac{2}{x^2}}\right) + \lim_{x\to\infty, \frac{y}{x}\to 1} \left(\frac{-8 - \frac{2}{x} - \frac{1}{x^2}}{1 + \left(\frac{y}{x}\right)^2 + \frac{2}{x^2}}\right) = 0.$$

$$\Rightarrow (x - y - 1)^2 + 6(x - y - 1).\frac{1}{2} - 4 = 0$$

$$(x - y - 1)^2 + 3(x - y - 1) - 4 = 0$$

$$\Rightarrow x - y - 1 = \frac{-3 \pm \sqrt{9 + 16}}{2} = 1, -4.$$

Hence two asymptotes are $x - y - 2 = 0$ and $x - y + 3 = 0$.

The other two asymptotes are imaginary since the remaining linear factors of the fourth degree terms in the equation to the curve are imaginary.

10.13 ASYMPTOTES BY INSPECTION

Theorem: If the equation of a curve of the n^{th} degree can be put in the form $F_n + F_{n-2} = 0$; where F_{n-2} is of degree $(n - 2)$ at the most, then every linear factor of F_n, when equated to zero will give an asymptote, provided that, no straight line obtained by equating to zero any other linear factor of F_n is parallel to it or coincident with it.

Proof: Let $ax + by + c = 0$, be a non-repeated factor of F_n. We write $F_n = (ax + by + c)F_{n-1}$, where F_{n-1} is of degree $(n - 1)$. The asymptote, parallel to $ax + by + c = 0$ is $(ax + by + c) + \lim \frac{F_{n-2}}{F_{n-1}} = 0$, where $x \to \infty$ and $\frac{y}{x} \to \frac{-a}{b}$. For the determination of the limit $\left(\frac{F_{n-2}}{F_{n-1}}\right)$, we divide the numerator as well as the denominator by x^{n-1},

$$\lim_{x,y\to\infty,\frac{y}{x}\to\frac{-a}{b}}\left(\frac{-F_{n-2}}{F_{n-1}}\right)=\lim\left[\frac{1}{x}-\frac{\left(\frac{F_{n-2}}{x^{n-2}}\right)}{\left(\frac{F_{n-1}}{x^{n-1}}\right)}\right]=0.$$

Since $\lim\left(\frac{1}{x}\right)=0, \lim\left(\frac{F_{n-2}}{x^{n-2}}\right)$ exists and is finite and $\lim\left(\frac{F_{n-1}}{x^{n-1}}\right)$exists and is finite and non-zero.

Therefore, $ax+by+c=0$ is an asymptote.

Example 16: Find the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solution: By the above proposition, the asymptotes must be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \Rightarrow y=\pm\frac{b}{a}x.$$

10.14 POSITION OF A CURVE WITH RESPECT TO AN ASYMPTOTE

Theorem (i) The curve $y=mx+c+\frac{A}{x}+\frac{B}{x^2}+\cdots$ lies above or below the asymptote $y=mx+c$ in the right half of the plane according as $A>0$ or $A<0$.

(ii) In the left half of the plane, it lies above or below the asymptote $y=mx+c$ according as $A<0$ or $A>0$.

(iii) If $A=0$, the curve lies above or below the asymptote according as $B>0$ or $B<0$.

Proof: (i) Let y_1 and $y_2$denote the ordinates of the curve and the asymptote respectively, for a given value of x.

Then $y_1=mx+c+\frac{A}{x}+\frac{B}{x^2}+\cdots$.

$$y_2=mx+c.$$

(1) $$\Rightarrow y_1-y_2=\frac{A}{x}+\frac{B}{x^2}+\cdots.$$

For large values of x, the term on the right hand side is $\frac{A}{x}$ and determines the sign of y_1-y_2.

$y_1-y_2>0$ or <0 according as $\frac{A}{x}>0$ or <0.

(i) In the right half of the plane, $x>0$, So that $y_1-y_2>0$ or <0 according as $A>0$ or $A<0$.
$\Rightarrow$ the curve is above or below the asymptote according as $A>0$ or $A<0$.

(ii) In the left half-plane, $x<0$.
So $y_1-y_2>0$ or <0, according as $A<0$ or $A>0$.

(iii) If $A=0$, the predominant term on the RHS of (1) is $\frac{B}{x^2}$ so that for all x, other than zero. $y_1-y_2>0\ or<0$, according as $B>0$ or $B<0$.

<u>Note:</u>Above discussion is very useful in curve tracing.

Note: The method of substituting $y = mx + c$ and equating to zero the coefficients of the two highest powers of x applies only to algebraic curves.

10.15 ASYMPTOTES TO NON-ALGEBRAIC CURVES

In the case of non-algebraic curves, asymptotes can be found in simple cases by applying the definition, or by the expansion of y in negative powers of x.

Example 17: Find the asymptotes of $y = \tan x$.

Solution: Here $y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$.

Hence the tangent at (x, y) is

$$Y - \tan x = (X - x)\sec^2 x$$

(1) $$\Rightarrow Ycos^2 x - \sin x \cos x = X - x$$

Now as $x \to \frac{\pi}{2}$ from the left, $y \to \infty$ and the distance of (x, y) from the origin tend to infinity.

Hence, to obtain the asymptote, we must take the limit of (1) as $x \to \frac{\pi}{2}$.

$\Rightarrow Y.0 - 0 = X - \frac{\pi}{2}$,

$\Rightarrow X = \frac{\pi}{2}$is one asymptote.

Since $y = \tan x$ is a periodic function with period π. So other asymptotes are $X = -\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, ...

10.16 ASYMPTOTES IN POLAR CO-ORDINATES

Theorem: The polar equation of any line is $p = rcos\,(\theta - \alpha)$, where, p is the length of the perpendicular from the pole to the line and $'\alpha'$ is the angle which this perpendicular makes with the initial line.

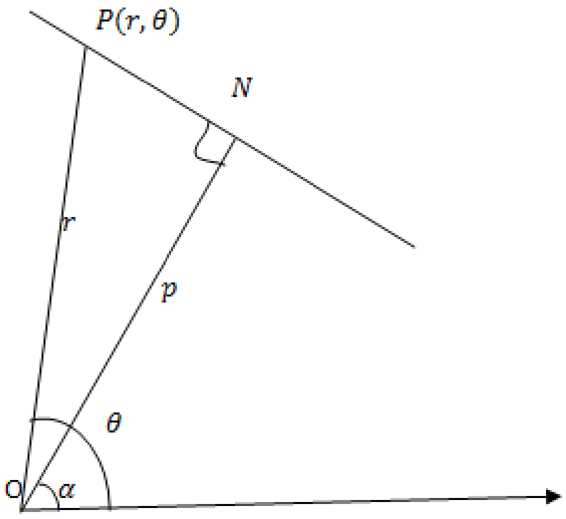

Fig. 10.16.1

Proof: Let ON be the perpendicular on the given line.

Given, ON = $p, < XON = \alpha$.

If $P(r, \theta)$ be any point on the given line, we have $< NOP = \theta - \alpha$.

Now $\frac{ON}{OP} = \cos(\theta - \alpha) \Rightarrow \frac{p}{r} = \cos(\theta - \alpha)$

$\Rightarrow p = r\cos(\theta - \alpha)$.

<u>Determination of the asymptote of the curve $r = f(\theta)$:</u>

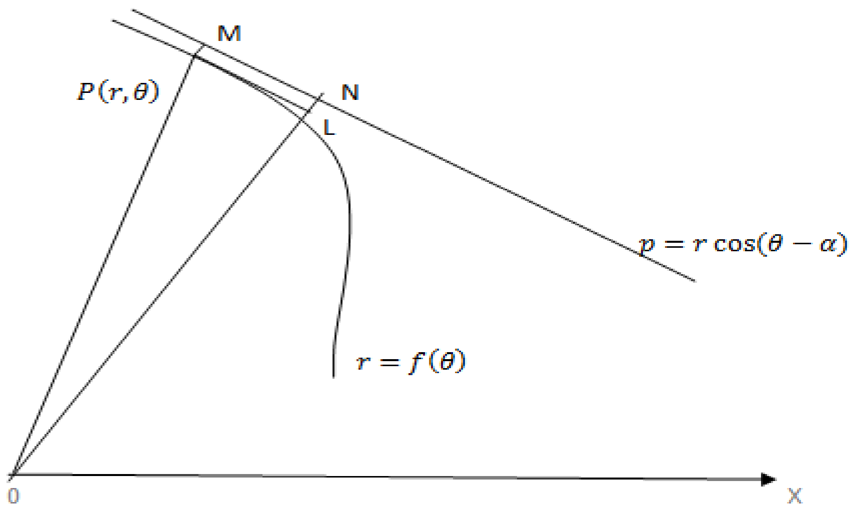

Fig. 10.16.2

Let $P(r, \theta)$ be any point on $r = f(\theta)$. We draw ON, perpendicular to the line $p = r\cos\,(\theta - \alpha)$.

Now we draw $PL \perp ON$ and $PM\ \perp$ to the given line.

So $PM = LN = ON - OL = p - OP\cos(\theta - \alpha)$

(1) $\Rightarrow PM = p - r\cos\,(\theta - \alpha)$

Let us put $r = \frac{1}{u}$

when $r \to \infty$, we have $u \to 0$.

Also we suppose, when $r \to \infty$ (or $u \to 0$), $\theta \to \theta_1$.
So, we have $PM \to 0$.
From equation (1)

$$0 = p - \lim_{\theta \to \theta_1}\left[\frac{\cos(\theta - \alpha)}{u}\right]$$

$$p = \lim_{\theta \to \theta_1}\left[\frac{-\sin(\theta-\alpha)}{\frac{du}{d\theta}}\right], \qquad \textit{using L'hospital Rule}$$

$$p = \frac{+\sin(\theta_1 - \alpha)}{\lim_{\theta \to \theta_1}\left(-\frac{du}{d\theta}\right)}.$$

From polar co-ordinate geometry, in this situation $\theta_1 - \alpha = \frac{\pi}{2}$. So

$$p = \frac{\sin\left(\frac{\pi}{2}\right)}{\lim_{\theta \to \theta_1}\left(-\frac{du}{d\theta}\right)}$$

(2) $$\Rightarrow p = \lim_{\theta \to \theta_1}\left(-\frac{d\theta}{du}\right)$$

So the equation of asymptote is

$$p = r\cos\,(\theta - \alpha) \Rightarrow \lim_{\theta \to \theta_1}\left(-\frac{d\theta}{du}\right) = r\cos(\theta - \alpha)$$

$$= r\cos\,\left[\theta - \left(\theta_1 - \frac{\pi}{2}\right)\right] = r\cos\left[\frac{\pi}{2} - (\theta_1 - \theta)\right]$$

(3) $\lim_{\theta \to \theta_1}\left(-\frac{d\theta}{du}\right) = r\sin(\theta_1 - \theta)$

<u>Working Rule:</u>

(i) Substitute $r = \frac{1}{u}$ in the equation of the curve.

(ii) Solve the equation for θ when $u \to 0$.

(iii) Let $\theta = \theta_1$ be such a value.

(iv) Find $p = \lim_{\theta \to \theta_1}\left(-\frac{d\theta}{du}\right)$.

(v) Now desired equation is $p = r\sin\,(\theta_1 - \theta)$.

Example 18: Find the asymptote of the curve $r = \frac{a}{\left(\frac{1}{2} - \cos\theta\right)}$.

Solution: Let $u = \frac{1}{r}$.

(1) $$\Rightarrow u = \frac{1}{a}\left(\frac{1}{2} - \cos\theta\right)$$

When $u \to 0$, we have $\frac{1}{2} - \cos\theta \to 0$

$$\Rightarrow \cos\theta \to \frac{1}{2}$$

$$\theta \to \pm\frac{\pi}{2}.$$

Suppose $\theta_1 = \pm\frac{\pi}{3}$.

Now $\frac{du}{d\theta} = \frac{1}{a}\sin\theta \Rightarrow -\frac{d\theta}{du} = -\frac{a}{\sin\theta}$

<u>Case (i):</u> When $\theta_1 \to \frac{\pi}{3}$, $\lim_{\theta \to \frac{\pi}{3}} \left(-\frac{d\theta}{du}\right) = \lim_{\theta \to \frac{\pi}{3}} \left(-\frac{a}{\sin\theta}\right) = -\frac{2a}{\sqrt{3}}$

So, asymptotes will be

$-\frac{2a}{\sqrt{3}} = r\sin\left(\frac{\pi}{3} - \theta\right)$ or $4a = r\left(\sqrt{3}\sin\theta - 3\cos\theta\right)$.

<u>Case (ii):</u> when $\theta \to -\frac{\pi}{3}$, $\lim_{\theta \to -\frac{\pi}{3}} \left(-\frac{d\theta}{du}\right) = \lim_{\theta \to -\frac{\pi}{3}} \left(-\frac{a}{\sin\theta}\right) = +\frac{2a}{\sqrt{3}}$

So asymptote will be

$$\frac{2a}{\sqrt{3}} = r\sin\left(-\frac{\pi}{3} - \theta\right)$$

$$4a = r(\sqrt{3}\sin\theta + 3\cos\theta)\,.$$

Example 19: Find the asymptote of $r = \frac{a\theta^2}{\theta - 1}$.

Solution: Putting $r = \frac{1}{u}$ in the equation, we get

$$u = \frac{\theta - 1}{a\,\theta^2}$$

When $u \to 0$, we have $\theta \to 1$. Let $\theta_1 = 1$. Also

$$\frac{du}{d\theta} = \frac{(a\theta^2)1 - (\theta - 1)2a\theta}{(a\theta^2)^2} = \frac{a\theta^2 - 2a\theta^2 + 2a\theta}{a^2\theta^4} = \frac{2a\theta - a\theta^2}{a^2\theta^4}$$

$$= \frac{2 - \theta}{a\,\theta^3}$$

So

$$\frac{d\theta}{du} = \frac{a\theta^3}{2 - \theta}$$

Hence,$p = \lim_{\theta \to \theta_1} \left(-\frac{d\theta}{du}\right) = \lim_{\theta \to 1} \left(\frac{a\theta^3}{\theta - 2}\right) = -a$.

Now the equation of the asymptote is $p = r\sin(\theta_1 - \theta)$

$$\Rightarrow -a = r\sin(1 - \theta)$$

$$a = r\sin(\theta - 1).$$

10.17.MISCELLANEOUS EXAMPLES

Example 20: Find the equation of the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

And which passes through the points $(0,0), (1,0)$ and $(0,1)$.

Solution: We write $F_3 = x^3 - 6x^2y + 11xy^2 - 6y^3$

$$F_3 = (x - y)(x - 2y)(x - 3y)$$

and, $$F_1 = x + y + 1.$$

So the equation of the curve can be written in the form

$$F_3 + F_1 = 0,$$

Where F_3 has non-repeated linear factors.

Thus $F_3 = 0$ is the joint equation of the asymptotes of the cubic.

The general equation of that cubic will be

$$F_3 + ax + by + c = 0$$

(1) $\Rightarrow (x-y)(x-2y)(x-3y)+ax+by+c=0.$

This curve passes through $(0,0),(1,0)$ and $(0,1)$.
So we have $0+0+c=0 \Rightarrow c=0$
$(1)+a=0 \Rightarrow a=-1.$
$-6+b=0 \Rightarrow b=6.$
Hence the required cubic is $x^3-6x^2y+11xy^2-6y^3-x+6y=0.$

Example 21: Find the asymptotes of the curve

$$y^2=\frac{x(x-a)(x-2a)}{(x+3a)}$$

and determine on which side of the asymptotes, the curve lies.

Solution: We have

$$y=\pm\sqrt{\frac{x(x-a)(x-2a)}{(x+3a)}}$$

$$y=\pm x\left(1-\frac{a}{x}\right)^{\frac{1}{2}}\left(1-\frac{2a}{x}\right)^{\frac{1}{2}}\left(1+\frac{3a}{x}\right)^{-\frac{1}{2}}$$

$$y=\pm x\left[1-\frac{a}{2x}+\cdots\right]\left[1-\frac{a}{x}+\cdots\right]\left[1-\frac{3a}{2x}+\cdots\right]$$

$$y=\pm x\left[1-\frac{a}{x}-\frac{a}{2x}+\frac{a^2}{2x^2}+\cdots\right]\left[1-\frac{3a}{2x}+\cdots\right]$$

$$y=\pm x\left[1-\frac{3a}{2x}-\frac{3a}{2x}+\frac{9a^2}{4x^2}+\frac{a^2}{2x^2}-\frac{3a^3}{4x^3}+\cdots\right]$$

$$y=\pm x\left[1-\frac{3a}{x}-\frac{11a^2}{2x^2}+\cdots\right]$$

$$y=\pm\left[x-3a-\frac{11a^2}{2x}+\cdots\right].$$

When $x\to\infty$, we have two asymptotes
$y=\pm(x-3a)$ i.e. $y-x+3a=0,\ y+x-3a=0.$

<u>**Discussion:**</u> The difference between the ordinate of the curve and that of the asymptote
$y=x-3a$ being $\frac{11a^2}{2x}\ldots$. We see that the curve lies above the asymptote when $x>0$ and below it when $x<0$.Similarly, it may be seen that the curve lies below the second asymptote when $x>0$ and above it when <0.

Example 22: Find the oblique asymptotes of the curve $x^3-xy^2+y^2=0$ and find the position of the curve relative to them.

Solution:$x^3=(x-1)y^2 \Rightarrow y=\pm\sqrt{\frac{x^3}{x-1}}$

$$y = \pm x\left(1-\frac{1}{x}\right)^{-\frac{1}{2}}$$
$$= \pm x\left[1+\frac{1}{2x}+\frac{3}{8x^2}+\cdots\right]$$
$$y = \pm\left[x+\frac{1}{2}+\frac{3}{8x}+\cdots\right].$$

Thus the curve has two branches whose equations can be written as

(1) $y = x+\frac{1}{2}+\frac{3}{8x}+\cdots$, and

(2) $y = -x-\frac{1}{2}-\frac{3}{8x}-\cdots$.

From (1), we find that $y = x+\frac{1}{2}$is an asymptote to this branch (say A). From (2), we find that $y = -x-\frac{1}{2}$ is an asymptote to this branch (say B).For branch A, $y_1-y_2 = \frac{3}{8x}+\cdots$. So the curve is above the asymptote when $x > 0$ and below the asymptote when $x < 0$.

For the branch B, $y_1-y_2 = -\frac{3}{8x}-\cdots$. So the curve is below the asymptote when $x > 0$ above the asymptote when $x < 0$.

Example 23: Show that the four asymptotes of the quadratic curve

$$(x^2-y^2)(x^2-4y^2)+2x^3-3x^2y-5xy^2+6y^3+y^2-3xy+1 = 0$$

cut the curve in points which lie on a circle of unit radius.

Solution: The asymptotes of the curve $(x^2-y^2)(x^2-4y^2)+2x^3-3x^2y-5xy^2+6y^3+y^2-3xy+1 = 0$, are

$$x-y = 0$$
$$x+y+1 = 0$$
$$x-2y = 0$$
$$x+2y+1 = 0.$$

The joint equation of the asymptotes is

$$F_4 = (x-y)(x+y+1)(x-2y)(x+2y+1)$$
$$F_4 = (x^2-y^2)(x^2-4y^2)+2x^3-3x^2y-5xy^2+6y^3+x^2-3xy+2y^2 = 0.$$

The equation of the curve can be re-arranged and written as

$$F_4-(x^2+y^2-1) = 0.$$

The eight points of the intersection of $F_4 = 0$, and $F_4-(x^2+y^2-1) = 0$ lie on the unit circle, $x^2+y^2-1 = 0$.

10.18 SUMMARY

In this unit, we studied asymptote to a curve as a straight line at a finite distance from origin, which cuts a curve in two points at infinite distances from origin and yet is not itself wholly at infinity. Historically, some sources include the requirement that the curve may not cross the line infinitely often, but this is unusual for modern authors. We also studied about the three kinds of asymptotes: horizontal, vertical and oblique asymptotes. We understood the calculating procedures for finding the asymptotes in various cases and found that "Asymptotes convey information about the behavior of curves in the large, and determining the asymptotes of a function is an important step in sketching its graph".

10.19 GLOSSARY

i. **Infinity:** A value greater than any fixed bound, denoted by ∞.

ii. **Tangent (to a curve):** Let P be a point on a (plane) curve. Then the tangent to the curve at P is the line through P that touches the curve at P.

iii. **Normal (to a curve):**Let P be a point on a curve in the plane. Then the normal at P is the line through P perpendicular to the tangent at P.

iv. **Homogeneous Polynomial:** An algebraic expression in x and y, in which sum of powers of x and y in each term is same, is called a homogeneous. e.g.$f(x,y) = ax + by \text{ or } ax^2 + 2hxy + by^2$; $a, h, b \in R$ etc.

We can write it as $f(x,y) = x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right] = x^2\,\phi\left(\frac{y}{x}\right)$

Where $\phi(t) = a + 2ht + bt^2$. Similarly, we can write a homogeneous expression as $f(x.y) = x^n\phi\left(\frac{y}{x}\right)$ with degree n.

CHEECK YOUR PROGRESS

1. The n asymptotes of a curve of the n^{th} degree cut it in
   (a) 2 points.
   (b) n points.
   (c) $n(n-2)$ points.
   (d) $(n-1)$ points.
2. The asymptote of $xy^2 = 4a^2(2a - x)$ is
   (a) $x = 0$.
   (b) $y = 0$.
   (c) $x + y = 0$.
   (d) $x - y = 0$.
3. The number of asymptotes of a curve of the n^{th} degree can-not exceed:
   (a) $(n-1)$.
   (b) n.
   (c) $(n-2)$.
   (d) $(n+1)$.
4. The asymptotes for the curve $y = \frac{x^2+1}{x-3}$ is
   (a) $x = 1$.
   (b) $x + 3 = 0$.
   (c) $x = 3$.
   (d) None of these.
5. The number of asymptotes of the curve $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ is
   (a) 2.
   (b) 3.
   (c) 4.

6. The curve $y^2 = x$ has
(a) 1 asymptote.
(b) 2 asymptotes.
(c) 3 asymptotes.
No asymptotes.
7. The asymptotes, parallel to the axes for the curve $x^2y^2 = a^2(x^2 + y^2)$ are
(a) $x = \pm a;\ \ y = \pm a.$
(b) $x = a;\ \ x = -a.$
(c) $y = a;\ \ y = -a.$
(d) $x = \pm a.$
8. A closed curve has
(a) No asymptote
(b) One asymptote
(c) Infinitely many asymptotes
(d) n −asymptotes.
9. The number of asymptotes of a curve of n^{th} degree is at most n. True\False.
10. The parabola $y^2 = 4ax$ possesses real asymptote. True\False.

10.20 REFERENCES

i. Joseph Edwards, “Differential Calculus for Beginners”, Macmillan and Co., Ltd., New York; 1896.
ii. Gorakh Prasad, “Text-Book on Differential Calculus”, Pothishala Private Ltd., Allahabad; 1936.

iii. Tom M. Apostol, “Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra”, John Wiley & Sons; 1967.

iv. Hari Kishan, R. K. Shrivastav, “Calculus”, Ram Prasad and Sons, Bhopal; 2004-05.

v. Asymptotes on Wikipedia.

10.21 SUGGESTED READING

i. Differential Calculus for Beginners by Joseph Edwards.

ii. Text-Book on Differential Calculus by Gorakh Prasad.

iii. Calculus by R. Kumar.

iv. Krishna’s Text Book on Calculus by A. R. Vasistha.

v. 12^{th} class Mathematics Book by R. D. Sharma.

vi. Pragati’s Calculus by Sudhir K. Pundir.

vii. Lectures on Basic Courses (1-2) on NPTEL website.

viii. Asymptotes on Wikipedia.

10.22TERMINAL QUESTIONS

TQ1: Find the asymptotes of the curve $x^2(x-y)+ay^2=0$.
TQ2: Find the asymptotes of the curve $y^2x-a^2(x-a)=0$.
TQ3: Find the Asymptotes of the curve $xy(x+y)=a(x^2-a^2)$.
TQ4: Show that the asymptotes of the cubic $x^3-xy^2-2xy+2x-y=\ 0$ cut the curve again in points which lie on the line.
TQ5: Find the asymptotes of the curve and their postion with regard to the curve $x^3+y^3=\ 3ax^2$.

10.23ANSWERS

CHECK YOUR PROGRESS

CYQ1. c
CYQ2. a
CYQ3. b
CYQ4. c
CYQ5. c
CYQ6. d
CYQ7. a
CYQ8. a
CYQ9. True.
CYQ10. False.

TERMINAL QUESTIONS

TQ1: $x - y + a = 0.$
TQ2: $x = 0. y = \pm a.$
TQ3: $y = a, x = 0, x + y + a = 0.$
TQ5: $x + y = a$. The curve lies above or below the asymptote according as x is positive or negative.

UNIT:-11-ENVELOPE AND EVOLUTE

CONTENTS

11.1 INTRODUCTION

We will first discuss the basic information about tangents, normals and curvature of a curve at a point.

A brief survey of tangents and normals:

Let us consider a curve $y = f(x)$ and P and Q be two distinct points on it. Let point P slides on the curve towards Q. When it reaches at R, chord PQ becomes RQ. Finally when $P \to Q$, chord $PQ \to$ tangent MN at Q. The slope of tangent MN at Q is defined as

$$\left(\frac{dy}{dx}\right)_Q = \tan\psi = [f'(x)]_{at\ Q}.$$

Now equation of tangent at a point $A(\alpha, \beta)$ is

$$y - \beta = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} (x - \alpha).$$

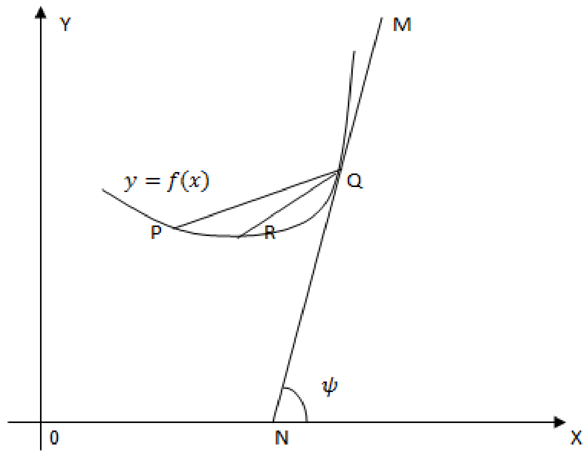

Fig. 11.1.1

<u>Note:</u>

(1) If equation of curve is given in parametric form i.e. $x = x(t),\ y = y(t)$, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

(2) If the equation of curve is given in implicit form i.e. you can separate x and y i.e. $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{\left(-\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}.$$

<u>Angle of Intersection:</u>

Let two curves $y = f(x)$ and $y = g(x)$ intersect at point P. By solving both equations, find the coordinates of P. Now draw tangents to both curves. Let m_1 and m_2 be their slopes. Then angle between curves=angle between tangents at P

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

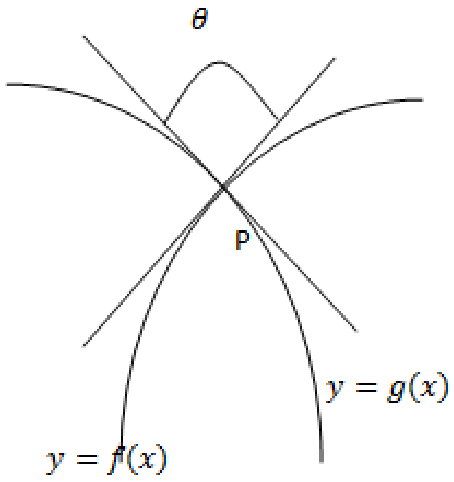

Fig. 11.1.2

Normal:

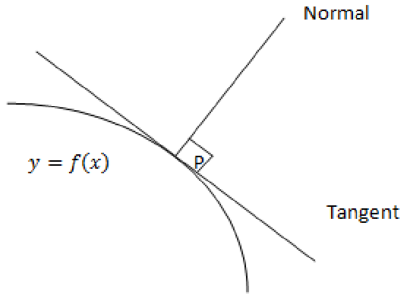

Fig. 11.1.3

The normal to a curve at any point is the straight line which passes through that point and is at right angles to the tangent to the curve at that point.

$$\text{Slope of normal} = \frac{-1}{\text{slope of corresponding tangent}} = \frac{-1}{\left(\frac{dy}{dx}\right)_P}.$$

Polar Coordinates:

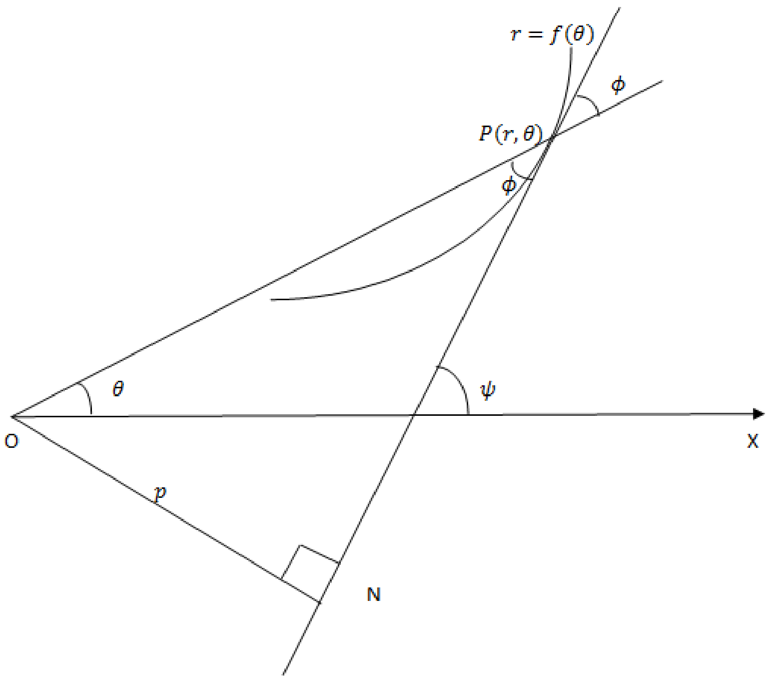

Fig. 11.1.4

By the property of triangle, $\psi = \theta + \phi$

Also,

(i) $\tan\phi = \frac{rd\theta}{dr}$

(ii) $\sin\phi = \frac{rd\theta}{ds}$

(iii) $\cos\phi = \frac{dr}{ds}$

(iv) Length of perpendicular from the pole on tangent $= ON = p = r\sin\phi$.

(v) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(vi) $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(vii) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

(viii) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Pedal Equations: The relation between p and r for a given curve is called its pedal equation i.e. $p = f(r)$ or $r = g(p)$.

A Brief Survey of Curvature

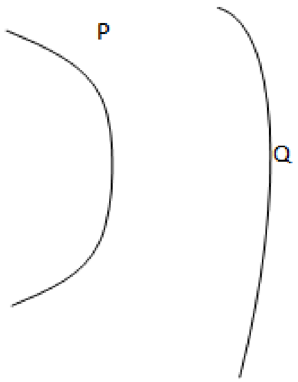

Fig. 11.1.5

Here we have shown two curves P and Q. Curve P bends more sharply than the other i.e. curve P has a greater curvature than the other. But in order to get a quantitative estimate of curvature. We define it.

Definition: Let P be a given point on a given curve, and Q any other point on it. Let the normals at P and Q intersect at N. If N tends to a definite position c as $Q \to P$, then c is called the centre of curvature of the curve at P.

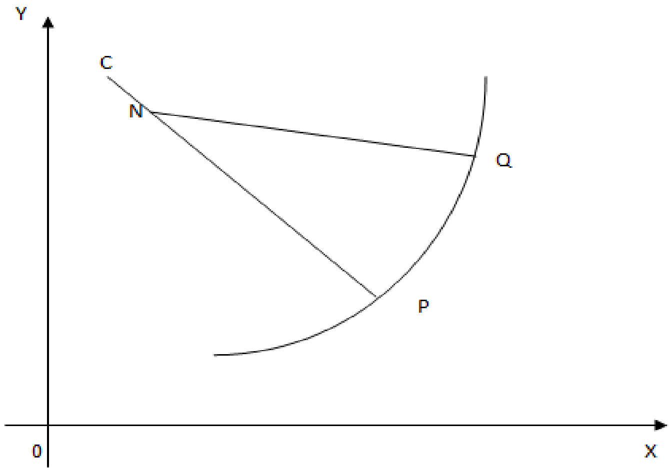

Fig. 11.1.6

Here N must tend to $'c'$ whether $Q \to P$ from the right or from the left. The reciprocal of the distance CP is called the curvature of the curve at P. The circle with its centre at c and radius CP is called the circle of curvature of the curve at P. The distance $'CP'$ is called the 'radius of curvature' of the curve at P, denoted by ρ.Any chord, drawn through P, of the circle of curvature at P, is called a chord of curvature.

Formulae for radius of curvature:

$$\rho = \frac{ds}{d\psi} \qquad \text{(intrinsic form)}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \text{(Cartesian form)}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \text{(Polar form)}$$

Centre of Curvature:

Let $C(\alpha, \beta)$ be centre of the circle at P on the curve $y = f(x)$. Then

$$\rho = r\frac{dr}{dp}(pedal\ form)$$

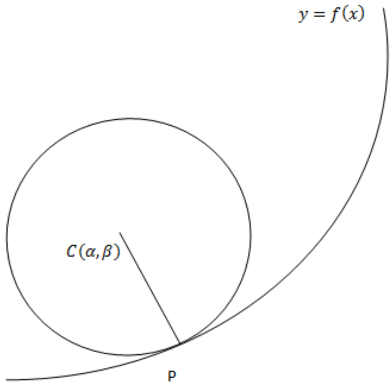

Fig. 11.1.7

$$\alpha = x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$$

$$\beta = y + \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$$

11.2 OBJECTIVES

In this unit, we will understand

i. The meaning of terms ‘Envelope’ and ‘Evolute’.
ii. Method of finding the Envelope.
iii. Geometrical significance of the Envelope.
iv. The relation between Envelope and Evolute of a Curve.

11.3 ENVELOPES

Family of Curves: Let us consider $x^2 + y^2 = \lambda^2$, $\lambda \in R$. It represents a family of concentric circles with varying radii. For a particular circle, λ has a fixed value, which is called ‘parameter’.

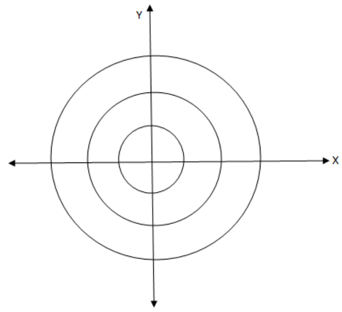

Fig. 11.3.1

Similarly $y = mx + \frac{1}{x}$, $m \neq 0$ also represents a family of straight lines.In general, if $F(x, y, \alpha)$ is an expression involving x, y and α, the curves corresponding to the equation $F(x, y, \alpha) = 0$ constitute a family of curves.

Envelope: A curve (i) which touches each member of a family of curves, and (ii) at each point is touched by some member of the family, is called the envelope of that family of curves.

Example: From Co-ordinate geometry, we know that all straight lines whose equation is of the form $y = mx + \frac{a}{m}$ touch the parabola $y^2 = 4ax$.

Also this parabola $y^2 = 4ax$ has at every point a tangent which is of the form $y = mx + \frac{a}{m}$.

Hence, we infer that the envelope of the family of straight lines $y = mx + \frac{a}{m}$ is the parabola $y^2 = 4ax$.

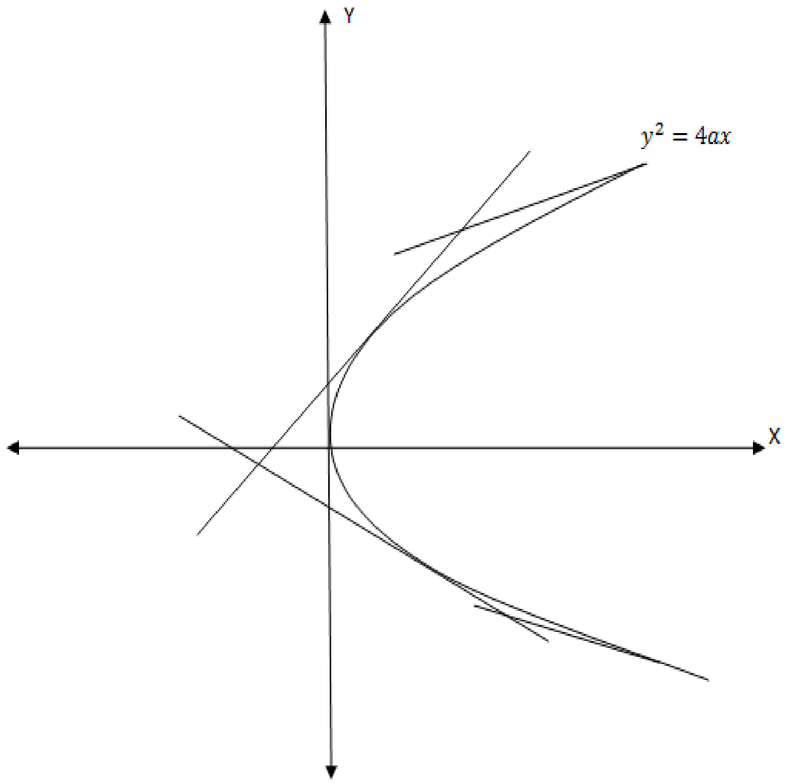

Fig. 11.3.2

<u>Another Definition:</u> If $F(x, y, \alpha) = 0$ represents a family of curves whose parameter is $'\alpha'$, and if the curves $F(x, y, \alpha) = 0$ and $F(x, y, \alpha + h) = 0$ cut in a point which tends to a definite point P as $h \to 0$, the locus of P (for varying values of α) is called the envelope of the family.

11.4 METHOD OF FINDING THE ENVELOPE

Let the given family of curves be

(1) $$f(x, y, \alpha) = 0$$

Suppose $'\alpha'$ have a particular value, then above equation represents one member of the family.

Suppose another member of the family be

(2) $$f(x, y, \alpha + h) = 0$$

The coordinates of the point of intersection, say P_1 of (1) and (2) will satisfy the equation

$$f(x, y, \alpha + h) - f(x, y, \alpha) = 0.$$

Dividing by h, we get

$$\frac{f(x, y, \alpha + h) - f(x, y, \alpha)}{h} = 0.$$

Taking limit as $h \to 0$, we see that coordinates of the point P to which P_1 tends as $h \to 0$, satisfy the equation

(3) $\frac{\partial f(x,y,\alpha)}{\partial \alpha} = 0.$

Also the coordinates of P must satisfy (1), because P is a point on (1). If we now eliminate $'\alpha'$ between (1) and (3), we shall get an equation which the coordinates of P will satisfy for all values of $'\alpha'$ i.e. the result of eliminating $'\alpha'$ between (1) and (3) will be the locus of P.

Working Method: The equation of the envelope of the family of curves $f(x,y,\alpha)=0$; where α is the parameter, is obtained by eliminating α between the equations

$$f(x,y,\alpha)=0 \text{ and } \frac{\partial f(x,y,\alpha)}{\partial \alpha}=0.$$

Ex.1 Find the envelope of the straight lines $\left(\frac{x}{a}\right)\cos\theta+\left(\frac{y}{b}\right)\sin\theta=1$, the parameter being $'\theta'$ and interpret the result geometrically.

Sol. The equation of the given family of straight lines is

$$\left(\frac{x}{a}\right)\cos\theta+\left(\frac{y}{b}\right)\sin\theta=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating partially with respect to parameter $'\theta'$, we get

$$\left(-\frac{x}{a}\right)\sin\theta+\left(\frac{y}{b}\right)\cos\theta=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By eliminating θ between equations (1) and (2), we will get the envelope of the family of straight lines (1).So squaring and adding equation (1) and (2), we get

$$\frac{x^2}{a^2}(\cos^2\theta+\sin^2\theta)+\frac{y^2}{b^2}(\sin^2\theta+\cos^2\theta)=1$$

$$\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which is the required envelope.

<u>**Geometrical Interpretation:**</u> The equation (3) represents an ellipse whose centre is origin. Whatever may be the value of θ, (i) the straight line (1) always touches the ellipse (3) and (ii) the ellipse (3) is also touched at each point by some straight line belonging to the family (1).

Ex.2 Find the envelope of the family of straight lines $ax\sec\theta-by\,cosec\,\theta=a^2-b^2$, where parameter is θ.

Sol. The equation of the given family of straight lines is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating (1) partially with respect to θ, we get

$$\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By eliminating θ between (1) and (2), we get the required envelope.

From (2) , $\tan^3\theta=-\left(\frac{by}{ax}\right) \Rightarrow \tan\theta=-\left(\frac{by}{ax}\right)^{\frac{1}{3}}$

So, $\sin\theta=\frac{(by)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}, \quad \cos\theta=-\frac{(ax)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}$

Substituting these values in (1), we get

$$\pm\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}=a^2-b^2,$$

$$\Rightarrow\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{3}{2}}=a^2-b^2,$$

$\Rightarrow (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$, which is the equation of the required envelope.

11.5 ELIMINATION IN THE CASE OF A QUADRATIC

In case the equation $f(x,y,\alpha)=0$ is a quadratic in parameter α, say

$$A\alpha^2+B\alpha+C=0,$$

where A,B and C are functions of x and y, the result of differentiation partially with respect to α is

$$2A\,\alpha+B=0.$$

Putting $\alpha=-\frac{B}{2A}$ in the equation (1), we get

$$A\left(-\frac{B}{2A}\right)^2+B\left(-\frac{B}{2A}\right)+C=0$$

$$\frac{B^2}{4A}-\frac{B^2}{2A}+C=0\Rightarrow -\frac{B^2}{2A}+C=0$$

$$\Rightarrow B^2-4AC=0.$$

This is the required envelope.

Ex.3 **Find the envelope of the family of straight lines $y=mx+\frac{a}{m}$.**

Sol. Equation can be written as

$m^2x-my+a=0$ ……………………………………………..(1)

Here m is parameter.

Using the discriminant relation $'B^2-4AC=0'$, we get

$$(-y)^2-4xa=0$$

Or $$y^2=4ax.$$

Second Method:

Differentiating equation (1) partially with respect to m, we get

$$0=x-\frac{a}{m^2}\Rightarrow m=\sqrt{\frac{a}{x}}$$

Putting m in equation (1), we get

$$y=x\sqrt{\frac{a}{x}}+a\sqrt{\frac{x}{a}}$$

$$\Rightarrow y=\sqrt{ax}+\sqrt{ax}$$

$$y=2\sqrt{ax}$$

$$Y^2=4ax.$$

Note: The method using $'B^2 - 4AC = 0'$ is used only when we have a quadratic equation in parameter. For other cases, we shall use general method.

Ex.4 Find the envelope of the family of circles $(x-c)^2 + y^2 = R^2$, where c is parameter.

Sol. The given family of circles is

$(x-c)^2 + y^2 = R^2$…………………………(1)

Differentiating equation (1) partially with respect to $'c'$, we get

$$-2(x-c) = 0$$

$(x-c) = 0$……………………………………………..(2)

Eliminating c between equations (1) and (2), we get the envelope of the family (1).

So by putting $x - c = 0$ in equation (1), we get

$y^2 = R^2$ or $y = \pm R$.

Hence the envelope of the family (1) consists of the straight lines $y = \pm R$.

Ex.5 Find the envelope of the family of straight lines $y = mx + \sqrt{a^2m^2 + b^2}$, where parameter is '$m$'.

Sol. The equation of the given family of straight lines is

$$y - mx = \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow (y - mx)^2 = a^2m^2 + b^2$$

$$\Rightarrow (x^2 - a^2)m^2 - (2xy)m + (y^2 - b^2) = 0.$$

……………………………….(1)

This is a quadratic equation in the parameter $'m'$.

So the required envelope is obtained by equating to zero the discriminant of (1). So,

$$(-2xy)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$\Rightarrow x^2b^2 + y^2a^2 = a^2b^2$$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is an ellipse.

Ex.6 Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the two parameters a, b are connected by the relation $ab = c^2$, c being a constant.

Sol. We shall eliminate one parameter, say b.

$b = \frac{c^2}{a}$ ....……………………………………………..………(1)

The equation of the given family of straight lines is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$(1) \Rightarrow \frac{x}{a} + \frac{ay}{c^2} = 1$$

$$\Rightarrow c^2x + a^2y = ac^2$$

$$\Rightarrow (y)a^2 + (-c^2)a + (xc^2) = 0.$$

This is a quadratic equation in the parameter a. So envelope will be

$$(-c^2)^2 - 4y(xc^2) = 0.$$

$$\Rightarrow c^2 = 4xy \qquad or \qquad xy = \frac{c^2}{4}.$$

It is a rectangular hyperbola.

11.6 GEOMETRICAL SIGNIFICANCE OF THE ENVELOPE:

Theorem 1 The envelope of a family of curves touches each member of the family.

Proof. Let any member of the family be

$f(x, y, \alpha) = 0$………………………………(1)

Where α is constant and equal to α_1, say.

The equation of the envelope is the result of eliminating α between

$f(x, y, \alpha) = 0 \qquad and \qquad \frac{\partial f(x,y,\alpha)}{\partial \alpha} = 0.$ ……(2)

Thus the equation of the envelope may be regarded as (1)

$f(x, y, \alpha) = 0$ …………………………………………..(3)

in which α is not a constant, but a function of x and y given by

$\frac{\partial f(x,y,\alpha)}{\partial \alpha} = 0$ ....………………………………………(4)

Consider now the point P on (1), where P is the limiting position to which the intersection of $f(x, y, \alpha_1) = 0$, and $f(x, y, \alpha_1 + h) = 0$, tends as $h \to 0$. This point P lies on the curve (1) and also on the envelope (2).

The tangent at P to the curve (1) has the gradient $\frac{dy}{dx}$, given by

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} = 0$ ………………………………………….(5)

Where, in the differentiation, α is kept constant and equal to α_1.

But the tangent at P to the envelope has the gradient $\frac{dy}{dx}$ given by

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} + \left\{\frac{\partial f}{\partial \alpha}.\frac{d\alpha}{dx}\right\}_{\alpha=\alpha_1} = 0$ …………………………..(6)

because α is not constant for the envelope.

But in virtue of equation (4), which is satisfied at every point of the envelope, (6) reduces to (5); i.e., the gradients of the tangents to the curve and the envelope at the common point P are the same.

⇒ the curve and the envelope have the same tangent at P.

⇒ they touch each other at P.

Note:

(1) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero, the value of $\frac{dy}{dx}$ cannot be found from equation (5) or (6), and the above argument would break down.

So the preposition might not be true for such points. If $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ at some point, then there is a singular point.

(2) If the given family of curves is a family of straight lines or a family of conics, we have no singular points.

Hence the envelope of a family of straight lines or of conics touches each member of the family at all their common points without exception.

11.7 EQUIVALENCE OF TWO DEFINITIONS OF ENVELOPE

The propositions of the last article enables us to infer at once that in general the two definitions of an envelope would give us the same curve, with the exception that second definition might in certain cases give us a curve the whole or a part of which is <u>not</u> an envelope in the sense of the first definition.

For example, if the curve $f(x, y, \alpha) = 0$ is the curve C_1 (which has a cusp at P) and C_2 is the curve $f(x, y, \alpha + h) = 0$, it is evident that as $h \to 0, P_1 \to P$.

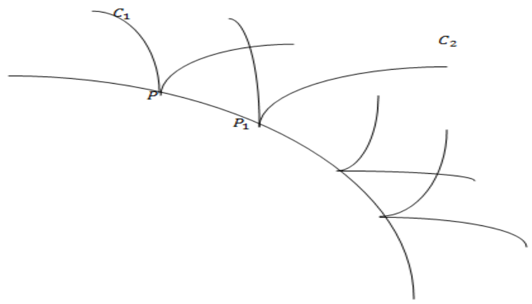

Fig. 11.7.1

Hence the result of eliminating α between $f = 0$ and $\frac{\partial f}{\partial \alpha} = 0$ will be, or will at least include, the locus of cusps.But from the figure it is evident that the loci of the cusps will not touch C_1 and C_2 or the other members of the family.

There are other loci (besides the locus of the cusps) which are sometimes obtained in the process of finding the envelope by eliminating α between $f = 0$ and $\frac{\partial f}{\partial \alpha} = 0$.

<u>Note:</u> If the equation to a family of curves is not given, but the law is given in accordance with which any member of the family can be obtained, the equation to the family must first be found in a suitable form.

Ex.7 Find the envelope of the circles drawn upon the radii vectors of the ellipse

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ **as diameter.**

Sol. From geometry, any point on the ellipse is $P(\mathrm{a}\cos\theta, b\sin\theta)$.

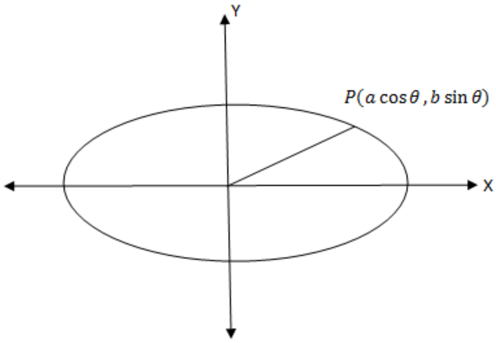

Fig. 11.7.2

Now we know that if (x_1, y_1) and (x_2, y_2) are end points of a diameter of a circle, then the equation of that circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

So equation of a circle whose diameter is OP, will be

$$(x - 0)(x - a\cos\theta) + (y - 0)(y - b\sin\theta) = 0$$

$\Rightarrow x^2 + y^2 - ax\cos\theta - by\sin\theta = 0.$ …………(1)

This is the family of circles drawn upon the radii vectors of given ellipse. Here θ is the parameter.

So differentiating equation (1) partially with respect to θ, we get

$ax\sin\theta - by\cos\theta = 0$/…………………………………….(2)

To get the required envelope, we have to eliminate θ from the equations (1) and (2).

From (1)

$ax\cos\theta + by\sin\theta = x^2 + y^2$ ……………………………(3)

Squaring and adding equation (2) and (3), we get

$$a^2x^2 + b^2y^2 = (x^2 + y^2)^2.$$

Ex.8 Obtain the envelope of the family of the curves given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2 - \alpha^2} = 1,$$

where α is the parameter.

Sol. On solving the given equation,

$$(\alpha^2)^2 + (y^2 - x^2 - k^2)\alpha^2 + x^2k^2 = 0.$$

This is a quadratic equation in α^2. So its envelope will be

$$(y^2 - x^2 - k^2)^2 - 4x^2k^2 = 0.$$

$$\Rightarrow y^2 - x^2 - k^2 = \pm 2kx.$$

Ex.9 Find the envelope of the family of straight lines $x\cos\alpha + y\sin\alpha = a$, the parameter being α.

Sol. The equation of the given family of straight lines is

$x\cos\alpha + y\sin\alpha = a$ …………(1)

Differentiating partially with respect to α, we get

$-x\cos\alpha + y\cos\alpha = 0$ ………(2)

Squaring and adding equations (1) and (2), we get

$x^2 + y^2 = a^2$

which is the required envelope.

Geometrical Interpretation:

$x^2 + y^2 = a^2$ is the equation of a circle whose centre is (0,0) and radius a. This circle is the envelope of the family of straight lines $x\,cos\,\alpha + y\,sin\,\alpha = a.$ So for each value of α, the straight line $x\,cos\,\alpha + y\,sin\,\alpha = a$ touches the circle $x^2 + y^2 = a^2$.

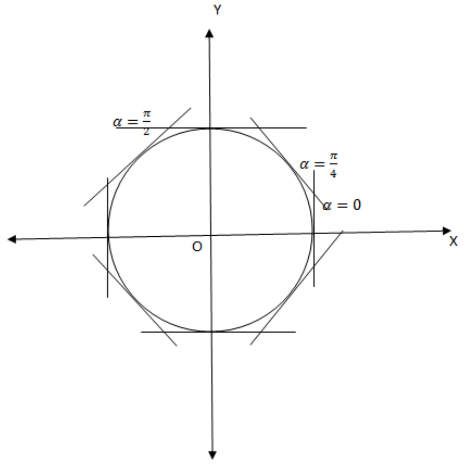

Fig. 11.7.3

Conversely, the circle $x^2 + y^2 = a^2$ is touched at each point by some straight line belonging to the family $x\cos\alpha + y\sin\alpha = a$.

Ex.10 Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse.

Sol.

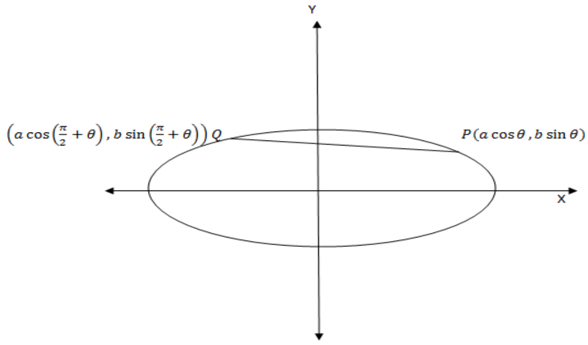

Fig. 11.7.4

Let the equation of the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let $P(a\cos\theta, b\sin\theta)$ and $Q\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$ i.e. $Q\,(-a\sin\theta, b\cos\theta)$ be end points of conjugate diameters of ellipse.

Now equation of PQ line is

$$y - b\sin\theta = \frac{b(\sin\theta - \cos\theta)}{a(\cos\theta + \sin\theta)}(x - a\cos\theta)$$

$\Rightarrow (ay + bx)\cos\theta + (ay - bx)\sin\theta = ab.$ ………….(1)

Here θ is the parameter.

Differentiating equation (1) partially with respect to θ, we get

$-(ay + bx)\cos\theta + (ay - bx)\cos\theta = 0.$ ……………(2)

Squaring and adding equation (1) and (2), we get

$$(ay + bx)^2 + (ay - bx)^2 = a^2b^2$$

$$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 1$$

$$or \qquad \frac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{y^2}{\left(\frac{b}{\sqrt{2}}\right)^2} = 1,$$

which is the required envelope and is a similar ellipse.

Ex.11 Find the envelope of the circles drawn on the radii vectors of the parabola $y^2 = 4ax$ as diameter.

Sol. Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$.

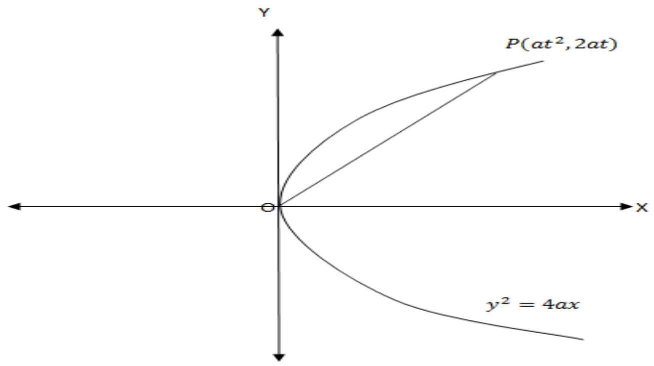

Fig. 11.7.5

Now equation of circle considering OP as diameter is

$$(x - 0)(x - at^2) + (y - 0)(y - 2at) = 0$$

$x^2 - axt^2 + y^2 - 2ayt = 0$ ….……………………….(1)

We have to find the envelope of the family of circles (1), where $'t'$ is the parameter.

$$\Rightarrow (-ax)t^2 + (-2ay)t + (x^2 + y^2) = 0.$$

This is a quadratic equation in t. So, envelope is

$$(-2ay^2) - 4(-ax)(x^2 + y^2) = 0.$$

$$\Rightarrow ay^2 + x(x^2 + y^2) = 0.$$

Ex.12 Show that the radius of curvature of the envelope of the family of lines $x\cos\alpha + y\sin\alpha = f(\alpha)$, is $f(\alpha) + f''(\alpha)$.

Sol. The given equation of the family of lines is

$x\cos\alpha + y\sin\alpha = f(\alpha)$ ……………………(1)

Where α is the parameter.

Differentiating equation (1) partially with respect to α, we get

$-x\sin\alpha + y\cos\alpha = f'(\alpha).$ ……………….(2)

To find the radius of curvature of the envelope of the given family of lines, we solve equation (1) and (2) to obtain

$x = f(y)\cos\alpha - f'(\alpha)\sin\alpha$……………(3)

$y = f(\alpha)\sin\alpha + f'(\alpha)\cos\alpha.$ ……………..(4)

So,

$$\frac{dx}{d\alpha} = f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha - f''(\alpha)\sin\alpha - f'(\alpha)\cos\alpha$$
$$= -[f(\alpha) + f''(\alpha)]\sin\alpha.$$
$$\frac{dy}{d\alpha} = f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha + f''(\alpha)\cos\alpha - f'(a)\sin\alpha$$
$$= [f(\alpha) + f''(\alpha)]\cos\alpha.$$

Since,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\alpha}\right)}{\left(\frac{dx}{d\alpha}\right)} = -\cot\alpha.$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\alpha) = -\frac{d}{d\alpha}(-\cot\alpha).\frac{d\alpha}{dx}$$
$$= \frac{cosec^2\,\alpha}{\left(\frac{dx}{d\alpha}\right)} = \frac{cosec^2\alpha}{-[f(\alpha) + f''(\alpha)]\sin\alpha}.$$

So, radius of curvature will be

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = -\frac{[1+\cot\alpha]^{\frac{3}{2}}}{cosec^3\alpha}[f(\alpha) + f''(\alpha)]$$
$$\rho = -[f(\alpha) + f''(\alpha)].$$

Since radius of curvature is distance only, so neglecting negative sign, we have

$$\rho = f(\alpha) + f''(\alpha).$$

Ex.13 Find the envelope of the straight lines

$\frac{x}{a} + \frac{y}{b} = 1,$……………………………………………(1)

where the parameters a and b are related by the equation

$a^n + b^n = c^n,$…………………………………………(2)

c being a constant.

Sol. Let us consider a and b as functions of some other parameter t.

Differentiating (1) and (2) with respect to t, considering x and y as constant, we get

$\frac{x}{a^2}.\frac{da}{dt} + \frac{y}{b^2}.\frac{db}{dt} = 0$ and $a^{n-1}.\frac{da}{dt} + b^{n-1}.\frac{db}{dt} = 0.$

Equating the values of $\frac{\left(\frac{da}{dt}\right)}{\left(\frac{db}{dt}\right)}$ from both equations, we get

$$\frac{\left(\frac{x}{a^2}\right)}{a^{n-1}} = \frac{\left(\frac{y}{b^2}\right)}{b^{n-1}}$$

$$\Rightarrow \frac{\left(\frac{x}{a}\right)}{a^n} = \frac{\left(\frac{y}{b}\right)}{b^n} = \frac{\frac{x}{a}+\frac{y}{b}}{a^n+b^n} = \frac{1}{c^n}$$

(Since $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$)

$$\Rightarrow \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} = \frac{1}{c^n} \Rightarrow \ a^{n+1} = x.c^n; b^{n+1} = y.c^n.$$

Putting these values in equations (2), we get

$$(c^n x)^{\frac{n}{n+1}} + (c^n y)^{\frac{n}{n+1}} = c^n$$

$$\Rightarrow x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$$

which is the required envelope.

11.8 EVOLUTE OF A CURVE

Let us consider a curve $y = f(x)$. We take some points $P_1, P_2, P_3, \ldots$ etc on the curve and draw corresponding circle of curvatures with respective centres C_1, C_2, C_3 etc.

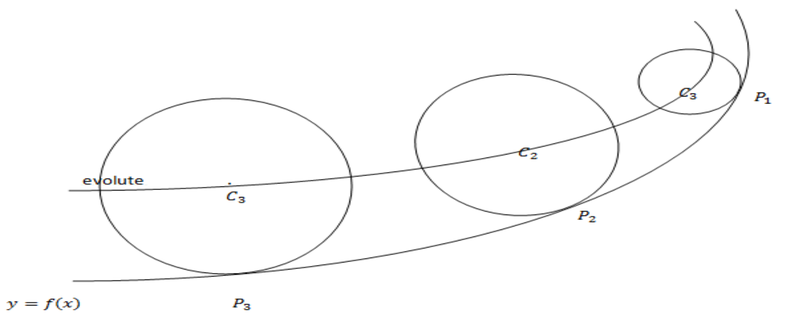

Fig. 11.8.1

Now generalize this concept i.e. if we draw circles of curvatures at every point of curvature, and join their centres, we shall get the locus of centre of curvature at an arbitrary point of $y = f(x)$. This locus is called evolute.

11.9 EVOLUTE AS THE ENVELOPE OF THE NORMALS

The centre of curvature of a curve for a given point P (on it) is the limiting position of the intersection of the normal at P with the normal at any other consecutive point Q as $Q \to P$.

So by the definition of envelope, the envelope of the normals to a curve is the evolute of that definition.

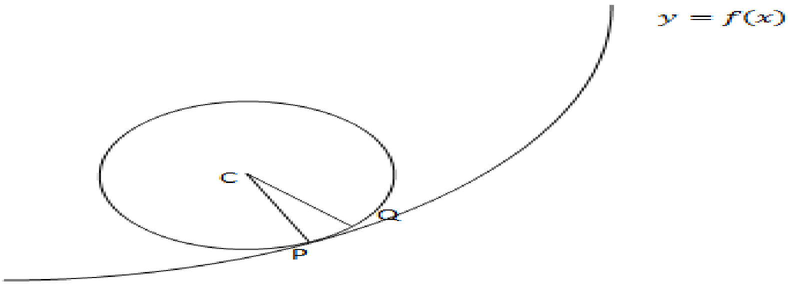

Fig. 11.9.1

Definition (II): The evolute of a curve is the envelope of the normals to that curve.

Theorem: The normal at any point of a curve is a tangent to its evolute touching at the corresponding centre of curvature

Proof: The coordinates (α, β) of the centre of curvature for any point $P(x, y)$ on the given curve are given by $\alpha = x - \rho \sin\psi$; $\beta = y + \rho\cos\psi$.

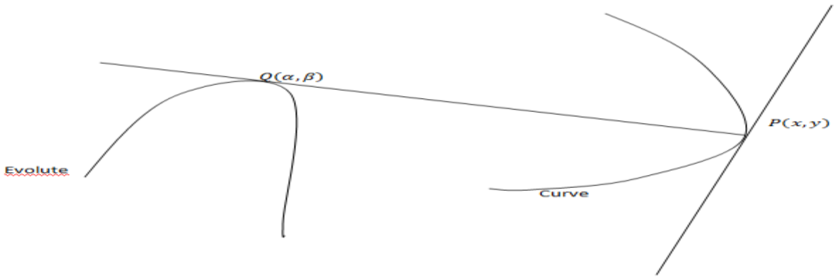

Fig. 11.9.2

Differentiating these with respect to x, we get

$$\frac{d\alpha}{dx} = 1 - \rho\cos\psi . \frac{d\psi}{dx} - \sin\psi . \frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\psi} . \frac{dx}{ds} . \frac{d\psi}{dx} - \sin\psi . \frac{d\rho}{dx}$$

$\Rightarrow \frac{d\alpha}{dx} = -\sin\psi . \frac{d\rho}{dx}$ ……………………(1)

Also, $\frac{d\beta}{dx} = \frac{dy}{dx} - \rho\sin\psi . \frac{d\psi}{dx} + \cos\psi . \frac{d\rho}{dx}$

$$= \frac{dy}{dx} - \frac{ds}{d\psi} . \frac{dy}{ds} . \frac{d\psi}{dx} + \cos\psi . \frac{d\rho}{dx}$$

$\Rightarrow \frac{d\beta}{dx} = \cos\psi . \frac{d\rho}{dx}$ ………………………………(2)

From equation (1) and (2), we have

$\frac{d\beta}{d\alpha} = -\cot\psi$…………………………………(3)

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent at Q to the evolute and $-\cot\psi$ is the slope of the normal PQ at P to the given curve.

These two slopes are equal and Q is a common point on both the lines. Hence the tangent at Q to the evolute and $-\cot\psi$ is the slope of the normal PQ at P to the given curve.

These two slopes are equal and Q is a common point on both the lines. Hence the tangent at Q to the envelope and the normal at P to the given curve touches its evolute at the corresponding point.

Ex.14 Find the evolute of the parabola $y^2 = 4ax$.

Sol. From Co-ordinate geometry, we know that equation of normal at $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$ to the curve $y^2 = 4ax$ is

$y = mx - 2am - am^3$ ……………………………(1)

where, m is the parameter.

Now envelope of (1) is the evolute of $y^2 = 4ax$. Differentiating equation (1) partially with respect to m, we get

$$0 = x - 2a - 3am^2$$

$$\Rightarrow m = \sqrt{\left(\frac{x-2a}{3}\right)}$$

Substituting this value in equation (1) and solving, we get $27\,ay^2 = 4(x-2a)^3$, which is the required evolute.

Second Method: Curve is $y^2 = 4ax$

Differentiating with respect to x, we get

$$2y\left(\frac{dy}{dx}\right) = 4a \qquad or \qquad \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx} = \frac{2a}{\sqrt{4ax}} = \frac{\sqrt{a}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{3}{2}} = -\frac{\sqrt{a}}{2x\sqrt{x}}$$

Suppose (α, β) be the centre of curvature for the point (x, y). Then

$$\alpha = x - \frac{\left(\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]\right)}{\frac{d^2y}{dx^2}} = x - \frac{\left(\frac{\sqrt{a}}{\sqrt{x}}\left[1 + \frac{a}{x}\right]\right)}{\left(-\frac{1}{2}.\frac{\sqrt{a}}{x\sqrt{x}}\right)}$$

$\alpha = 3x + 2a$ ……………………………………………(2)

Also

$$\beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = y + \frac{\left[1 + \frac{a}{x}\right]}{-\frac{1}{2x}.\sqrt{\frac{a}{x}}}$$

$\Rightarrow \beta = -2a^{-\frac{1}{2}}x^{\frac{3}{2}}$. ……………………………………..(3)

If we eliminate $'x'$ from the expressions of α and β, we will get an expression only in α and β. So putting $x = \frac{\alpha - 2a}{3}$ in equation (3), we get

$$\beta = -\frac{2}{\sqrt{a}}\left(\frac{\alpha - 2a}{3}\right)^{\frac{3}{2}}.$$

On squaring,

$$\beta^2 = \frac{4}{a}\left(\frac{\alpha - 2a}{3}\right)^3$$
$$\Rightarrow 27\, a\, \beta^2 = 4\,(\alpha - 2a)^3.$$

Hence locus of centre of curvature is $27ay^2 = 4(x-2a)^3$, which is the required evolute.

Ex.15 Find the evolute of the ellipse

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}.$$

Sol. The given ellipse in parametric form is $x = a\cos\theta$, $y = b\sin\theta$, where θ is the parameter.

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = bcos\ \theta$$
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{bcos\ \theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta.$$

So, slope of the normal to given ellipse at the point $(a\ \cos\theta, b\sin\theta) = \frac{a\ \sin\theta}{b\cos\theta}$.

Hence equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is

$$y - bsin\ \theta = \frac{a\ \sin\theta}{b\cos\theta}(x - a\cos\theta)$$
$$\Rightarrow \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

Now the evolute of the given ellipse is the envelope of the family given by equation (1). We have done it in envelope section, which is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

Length of arc of an evolute:

Theorem: The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

Proof. Let s be the length of the arc of the given curve measured from some fixed point A on the curve up to $P(x, y)$ and σ the length of the arc of the evolute measured from some fixed point on it up to $C(\alpha, \beta)$.

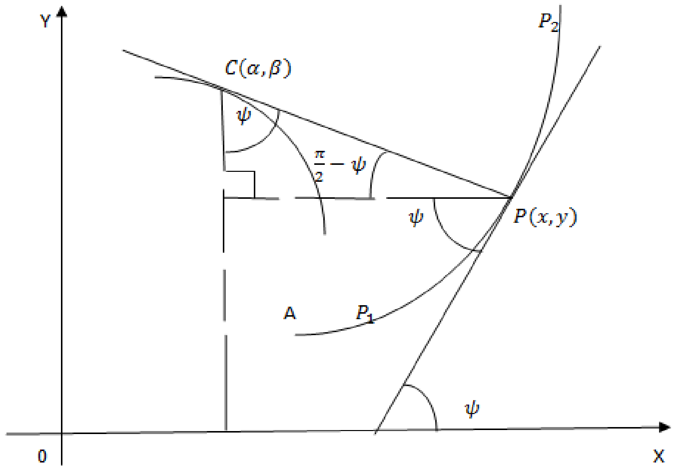

Fig. 11.9.2

It is obvious that if $C(\alpha, \beta)$ is the centre of curvature, corresponding to $P(x, y)$, then

$\alpha = x - \rho \sin\psi$, $\quad \beta = y + \rho\cos\psi$ ....(1)

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \rho\cos\psi\left(\frac{d\psi}{ds}\right) - \sin\psi\left(\frac{d\rho}{ds}\right)$$

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \frac{ds}{d\psi}.\frac{dx}{ds}.\frac{d\psi}{ds} - \sin\psi\left(\frac{d\rho}{ds}\right)$$

$$= \frac{dx}{ds} - \frac{dx}{ds} - \sin\psi\left(\frac{d\rho}{ds}\right)$$

$$\Rightarrow \frac{d\alpha}{ds} = -\sin\psi\left(\frac{d\rho}{ds}\right).$$

Now

$$\frac{d\beta}{ds} = \frac{dy}{ds} - \rho\sin\psi\left(\frac{d\psi}{ds}\right) + \cos\psi\left(\frac{d\rho}{ds}\right)$$

$$= \frac{dy}{ds} - \frac{ds}{d\psi}.\frac{dy}{ds}.\frac{d\psi}{ds} + \cos\psi\left(\frac{d\rho}{ds}\right)$$

$$\frac{d\beta}{ds} = \cos\psi\left(\frac{d\rho}{ds}\right).$$

Since we know that

$$\frac{d\rho}{ds} = \sqrt{\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2} = \sqrt{1 + \left(\frac{d\beta}{d\alpha}\right)^2} \,.\, \frac{d\alpha}{ds} = \frac{d\sigma}{d\alpha}.\frac{d\alpha}{ds} = \frac{d\sigma}{ds}.$$

$$\Rightarrow \frac{d\rho}{d\sigma} = 1 \quad or \quad d\rho = d\sigma$$

$$\Rightarrow d(\rho - \sigma) = 0.$$

$$\Rightarrow \ \rho - \sigma = constant = c, say.$$

Hence $\rho_2 - \rho_1 = \sigma_2 - \sigma_1$, where ρ_1 and ρ_2 are the values of ρ for any two points P_1 and P_2 on the curve and σ_1 and σ_2 are the corresponding values of σ.

11.10 INVOLUTES

If one curve is the evolute of another, then the latter is called an involute of the former. Thus if the curve $C_1C_2C_3$ is the evolute of the curve $P_1P_2P_3$, then $P_1P_2P_3$ is an involute of $C_1C_2C_3$.

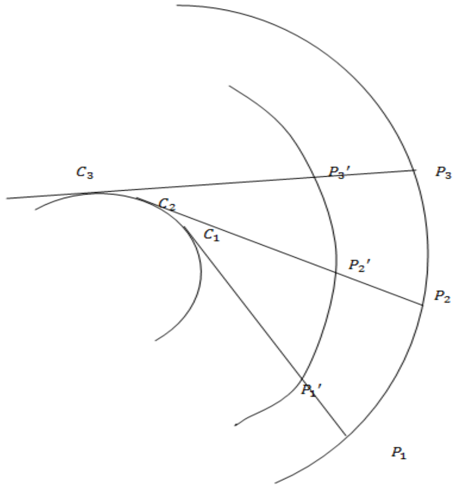

Fig. 11.10.1

Theorem: Every curve has an infinite number of involutes.

Proof: Let C_1 and C_2 be the centres of curvature of the curve $P_1P_2P_3$ at P_1 and P_2 respectively, then by the last article

$$C_1P_1 + arc\ C_2C_1 = C_2P_2.$$

Hence, if a thread were wrapped round the curve $C_3C_2C_1$ and were presented from slipping, it is evident that when the thread is unwrapped, (being kept taut all the time) the point on the thread which was at P_1 to begin with will describe the curve $P_1P_2P_3$.

This explains why the curve $C_1C_2C_3$ is called the evolute of the curve $P_1P_2P_3$.

Obviously, any point on the thread will describe an involute of the curve $C_1C_2C_3$.

Thus every curve has an infinite number of Involutes.

Parallel Curves: If the curves $P_1P_2P_3$ and $P_1'P_2'P_3{}'$ are both involutes of the same curve, then they are called parallel curves; because the distance between them measured along their common normal is constant.

11.11 SUMMARY

We studied that an envelope of a family of curves in the plane is a curve that is tangent to each member of the family at some point, and these points of tangency together form the whole envelope. Classically, a point on the envelope can be thought of as the intersection of two "infinitesimally adjacent" curves, meaning the limit of intersections of nearby curves.We also noticed thatthe **evolute** of a curve is the locus of all its centers of curvature. That is to say that when the center of curvature of each point on a curve is drawn, the resultant shape will be the evolute of that curve. The evolute of a circle is therefore a single point at its center. Equivalently, an evolute is the envelope of the normals to a curve.Evolutes are closely connected to involutes: A curve is the evolute of any of its involutes.

11.12GLOSSARY

i. ***Parameter:*** A variable that is to takedifferent values, thereby giving different values to certain other variables.

ii. ***Quadratic Equation:***A quadratic equation in the unknown x is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are given real numbers, with $a \neq 0$.

CHECK YOUR PROGRESS

Choose the correct answer for each question:

1. Curvature of circle of radius r is
(a) 0.
(b) r.
(c) $\frac{1}{r}$.
(d) $2r$.

2. If $s = c \sec \psi$then ρ is
(a) $c \sec \psi$
(b) $c \sec^2 \psi \tan \psi$
(c) $c \sec \psi \tan \psi$
(d) None

3. The locus of the centre of curvature for a curve is:
(a) Envelope
(b) Evolute
(c) Radius of curvature
(d) None of these.

4. The envelope of the normals to the curve is

a) Evolute.
b) Curvature.
c) Envelope.
d) None of these.

5. Envelope of the family of circles $(x-c)^2+y^2=r^2$, where the parameter being c, is

(a) $y=\pm r$.
(b) $y=0$.
(c) $x=0$.
(d) $xy=0$.

6. The equation of the evolute of the parabola $y^2=2ax$ is

(a) $y=\ \pm a$.
(b) $y=0$.
(c) $x=0$.
(d) $27a\,y^2=8(x-a)^3$.

7. Envelope of the family of curves of the form $A\lambda^2+B\lambda+C=0$is

(a) $A\lambda+B=0$.
(b) $B^2-4AC=0$.
(c) $B-C=0$.
(d) None of these.

8. The centre of curvature at $(1,2)$ for the curve $y^2=4x$ is

(a) $(2,5)$
(b) $(2,-5)$
(c) $(5,-2)$
(d) $(-5,-2)$

9.The envelope of the family of curves

(a) Touches each member of the family.
(b) Intersects each member of the family.
(c) Touch one member and intersects each member of the family.
(d) None of these.

10. Envelope of the family of straight line $y=mx+a/m$ is:

(a) $y^2=2ax$
(b) $x^2=4ay$
(c) $xy=8(x-a)^3$
(d) None of the above

11.13 REFERENCES

i. Joseph Edwards, "Differential Calculus for Beginners", Macmillan and Co., Ltd., New York; 1896.
ii. Gorakh Prasad, "Text-Book on Differential Calculus", Pothishala Private Ltd., Allahabad; 1936.
iii. Tom M. Apostol, "Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra", John Wiley & Sons; 1967.
iv. Hari Kishan, R. K. Shrivastav, "Calculus", Ram Prasad and Sons, Bhopal; 2004-05.
v. Asymptotes on Wikipedia.

11.14 SUGGESTED READING

i. Differential Calculus for Beginners by Joseph Edwards.
ii. Text-Book on Differential Calculus by Gorakh Prasad.
iii. Calculus by R. Kumar.
iv. Krishna's Text Book on Calculus by A. R. Vasistha.
v. 12th class Mathematics Book by R. D. Sharma.
vi. Pragati's Calculus by Sudhir K. Pundir.
vii. Lectures on Basic Courses (1-2) on NPTEL website.
viii. Asymptotes on Wikipedia.

11.15 TERMINAL QUESTIONS

TQ1: Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

TQ2: Show that the chord of curvature through the pole of the cardioid $r = a(1 - cos\theta)$ is $\frac{4}{3}r$.

TQ3. Find the envelope of the curve $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$ when $a^n + b^n = c^n$.

TQ4. Define the envelope, evolute and radius of curvature

...

...

TQ5. Find the envelope of the circles which pass through the origin and whose centres lie on $x^2 - y^2 = a^2$.

11.16ANSWERS

CHECK YOUR PROGRESS

CYQ1. (c)
CYQ2. (c)
CYQ3. (b)
CYQ4. (c)
CYQ5. (a)
CYQ6. (d)
CYQ7. (b)
CYQ8. (c)
CYQ9. (a)
CYQ10. (b)

TERMINAL QUESTIONS

TQ1. Evolute $\left(\frac{a}{x}\right)^{\frac{2}{3}} - \left(\frac{b}{y}\right)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.
TQ3. $(x)^{\overline{\frac{mn}{(m+n)}}} + y^{\overline{\frac{mn}{(m+n)}}} = c^{\overline{\frac{mn}{(m+n)}}}$.

UNIT-:12 INTEGRATION AND VOLUME AND SURFACE OF SOLID OF REVOLUTION

CONTENTS:-

12.1 INTRODUCTION

The solid generated by revolving an area about a fixed straight line lying in its plane is known as a solid of revolution.

Volume of revolution:- The volume generated by revolving an area about a fixed straight line in its plane is known as a volume of revolution.
Surface of revolution:-The surface generated by revolving an arc about a fixed straight line lying in its plane is known a surface of revolution.

Axis of revolution:- The fixed straight line about which an area or an arc revolves is known as axis of revolution.

In this unit we are defined about Volume of solid of revolution, Volume of a solid by double integration, Surface of Revolution, Surface revolution (Parametric Form), Surface revolution (Polar Form), Area of the surface by double integration, and Theorems Pappus (or Guldin).

12.2 OBJECTIVES

The objective of this topic is to at the end of this topic learner will be able to

- **i.** Volume of solid of revolution.
- **ii.** Surface of revolution.
- **iii.** Cartesian, parameteric and polar in form of volume and surface of revolution.
- **iv.** Area of surface by double integration.
- **v.** Pappus theorem of volume and surface of revolution.

12.3 VOLUME OF A SOLID OF REVOLUTION

Case I: When the axis of revolution is the x-axis

The volume of the solid generated by the revolution of the area bounded by the curve y = f(x)the x-axis and the two ordinates x = a and x = b axis is given by$V = \pi \int_a^b y^2 dx = \pi \int_a^b y[f(x)]^2 dx$.

Proof:- Let the Cartesian equation of the curve is $y = f(x)$ and let AC and BD be the two ordinates $x = a$ and $x = b$ respectively.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve. From P and Q draw PM and QN Perpendicular and the x-axis. Further from P and Qdraw PS and QR perpendiculars to QN and PM produced respectively.

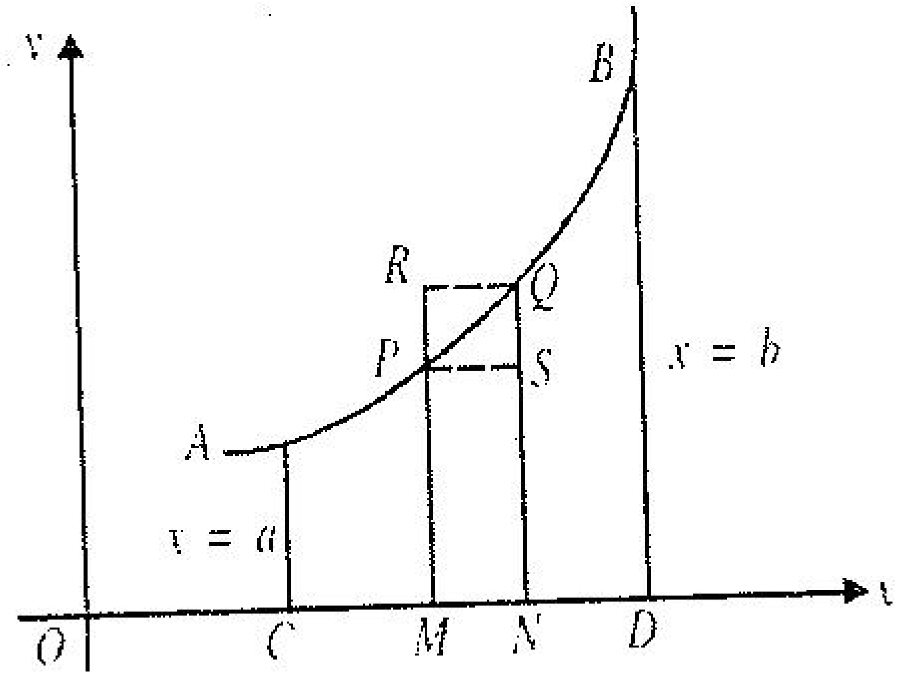

Fig 12.3.1

Let the volume of the solids generated by the revolution of areas $ACMPA$ and $ACNQA$ about the x −axis be v and $V + \delta V$respectively.

So δV is the volume of the solid generated by the revolution of the elementary area $PMNQP$ about the $x -$ axis and it lies between the volumes generated by the revolution of the rectangles $MNSP$and $MNQR$ about the $x -$ axis .

Now, $PM = y, QN = y + \delta y \text{ and } MN = \delta x$.

The volume generated by revolving the area $MNSP$ is $= \pi y^2 \delta x$ and

The volume generated by revolving the area MNQR is $= \pi(y + \delta y)^2 \delta x$

Since the volume δv i.e. the volume generated by revolving the area $PMNQP$ lies between the volume generated by the areas MNSP and MNQR Therefore $\pi y^2 \delta x < \delta v < \pi(y + \delta y)^2 \sqrt{x}$.

$$i.e. \pi y^2 < \frac{\delta v}{\delta x} < \pi(y + \delta y)^2$$

In the limiting case when $Q \to P, \delta x \to 0 \text{ and } \delta y \to 0 \text{ If follows that}$

$$\frac{dv}{dx} = \pi y^2$$

$$i.e. dv = \pi y^2 dx$$

Integrating this with respect to x between the limits $x = a \text{ and } x = b \text{ we get}$

$$\int_{x=a}^{b} dv = \int_{a}^{b} \pi y^2 dx$$

$$(value\ of\ v\ at\ x = b) - (value\ of\ v\ at\ x = a) = \int_{a}^{b} \pi y^2 dx$$

i.e. The volume of the solid generated by the revolution of the area between, arc AB, the x-axis and the ordinates x-a,x-b about the x-axis is given by

$$V = \pi \int_{a}^{b} y^2 dx = \pi \int_{a}^{b} [f(x)]^2 dx$$

Case II When the axis of revolution is the y-axis.

Proceeding exactly as in Case I above, we can prove the following: $x = f(y)$, the y-axis and the abscissa y=c and y=d is

$$V = \int_{c}^{d} x^2 dy = \int_{c}^{d} [f(y)]\ ^2 dy$$

Case III: When the axis of revolution is any straight line:

Let the axis of revolution is any straight line AB (different from x and y-axis) and CD be the arc of the curve.

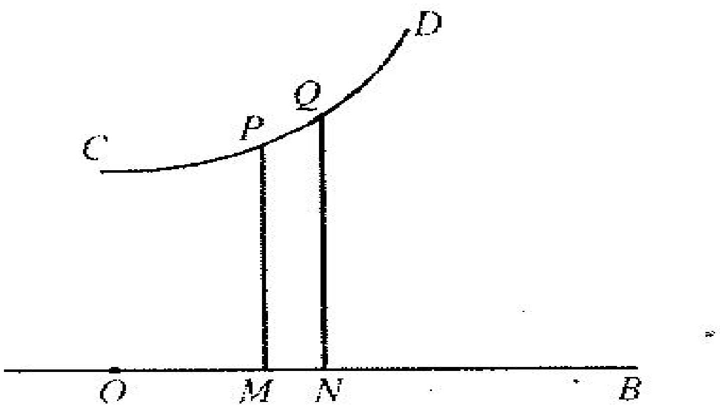

Fig 12.3.2

Let PM be the length of the perpendi point P on the arc $CD$to the axis of revolution AB and O is the fixed point on the axis AB. Then the volume generated is given by

$$V = \pi \int\ \int\ (PM)\ ^2 d(OM)$$

Illustrative Examples

Ex.1 Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ about\ x - axis.$

Or

Find the volume of Prolate spheroid.

Sol. The solid generated by revolving the ellipse about the major axis i.e. the x-axis, is called a prolate spheroid.

The equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

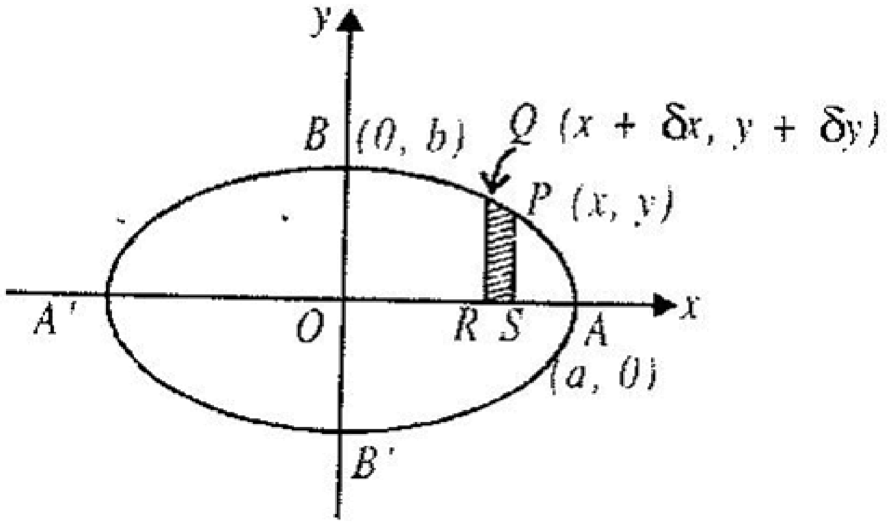

Fig. 12.3.3

The solid generated by revolving the area $ABA'OA$ about the x −axis. Let us take an elementary strip PQRS perpendicular to x-axis. Since the ellipse is symmetrical about the y −axis, therefore for the portion of the ellipse lying in the first quadrant, x varies from o to a.

Hence, the required volume

$$= 2.\pi\int_0^a y^2dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx$$

$$= 2\pi\frac{b^2}{a^2}\int_0^a (a^2-x^2)dx$$

$$= 2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right] \quad = \frac{2\pi b^2}{a^2}(a^3-\frac{a^3}{3})$$

$$= \frac{4}{3}\pi ab^2$$

Hence volume of prolate spheroid is $V=\frac{4}{3}\pi ab^2$

Ex.2 **Find the volume of the solid generated by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the y-axis.**

Or

Find the volume of the oblate spheroid.

Sol. The solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the minor axis i.e. y-axis is called the oblate spheroid.
The given equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

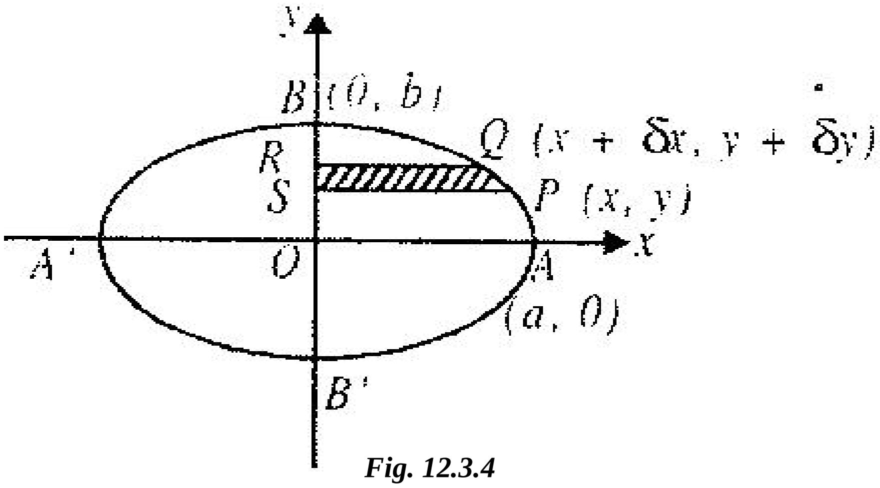

Fig. 12.3.4

The solid generated by the area $BAB'OB$ about y −axis.
Since the ellipse is symmetrical about the y-axis, therefore the portion of the ellipse lying in the first quadrant, y varies from o to b.
Hence the required volume

$$= 2.\pi\int_0^b x^2 dy$$

$$= 2\pi\int_0^b \frac{a^2}{b^2}(b^2-y^2)dy$$

$$= 2\pi\frac{a^2}{b^2}\int_0^b (b^2-y^2)dy$$

$$= 2\pi\frac{a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right] = 2\pi\frac{a^2}{b^2}(b^3-\frac{b^3}{3})$$

$$V=\frac{4}{3}\pi a^2 b$$

Hence the volume of the oblate spheroid is

$$V=\frac{4}{3}\pi a^2 b$$

Ex.3 Prove that the volume of solid generated by the revolution of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$minor axis is the geometric mean

ofthose generated by the revolution of an ellipse and of the auxiliary circle about the major axis.

Sol. The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The volume of the solid formed by the revolution of the ellipse about the major axis (prolate spheroid) is-

$$v_1=\frac{4}{3}\pi b^2 a$$

The volume of the solid generated by the revolution of the ellipse about the minor axis (oblate spheroid) is

$$v_2=\frac{4}{3}\pi a^2 b$$

The equation of the auxillary circle of the ellipse

$\frac{x^2}{a}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$

The volume of the solid generated by the revolution of the auxiliary circle about major axis is

$$v_3=2\pi\int_0^a y^2dx=2\pi\int_0^a (a^2-x^2)dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a=2\pi.\frac{2}{3}a^3$$

$$v_3=\frac{4}{3}\pi a^3$$

Hence geometric mean of v_1 *and* v_3

$$=\sqrt{\frac{4}{3}\pi b^2 a.\frac{4}{3}\pi a^3}$$

$$=\frac{4}{3}\pi a^2 b=v_2$$

Hence $v_2=\sqrt{v_1v_3}$

Ex.4 Show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$

Sol. A sphere is generated by the revolution of a semicircular area about its bounding diameter. Equation of a circle with radius a, whose centre is at origin is $x^2+y^2=a^2$

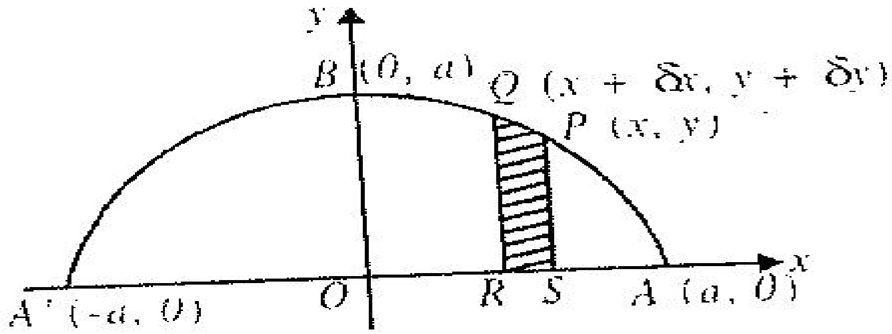

Fig. 12.3.5

Let AA' be the bounding diameter about which the semicircle revolves. Since the circle is symmetrical about y axis so we will take the area of revolution of only in positive quadrant and twice it. Take the elementary strip PSRQ. Where P(x, y) and $Q(x+\delta x, y+\delta y) we\ have\ PS = y\ and\ RS = \delta x$

Now volume of the elementary disc formed by revolving the strip PMNQ about the diameter AA' is

$$\pi(PS)^2RS = \pi y^2\delta x$$

$$= \pi(a^2 - x^2)\delta x$$

Hence the required volume of the sphere is

$$2\pi\int_0^a (a^2 - x^2)dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi.\frac{2}{3}a^3$$

$$V = \frac{4}{3}\pi a^3$$

Ex.5 The Area between a parabola and its latus rectum revolves about it's directrix. Find the ration of the volume of the ring thus obtained to the volume of the sphere whose diameter is the latus rectum.

Sol. Let the equation of the parabola is $y^2 = 4ax$

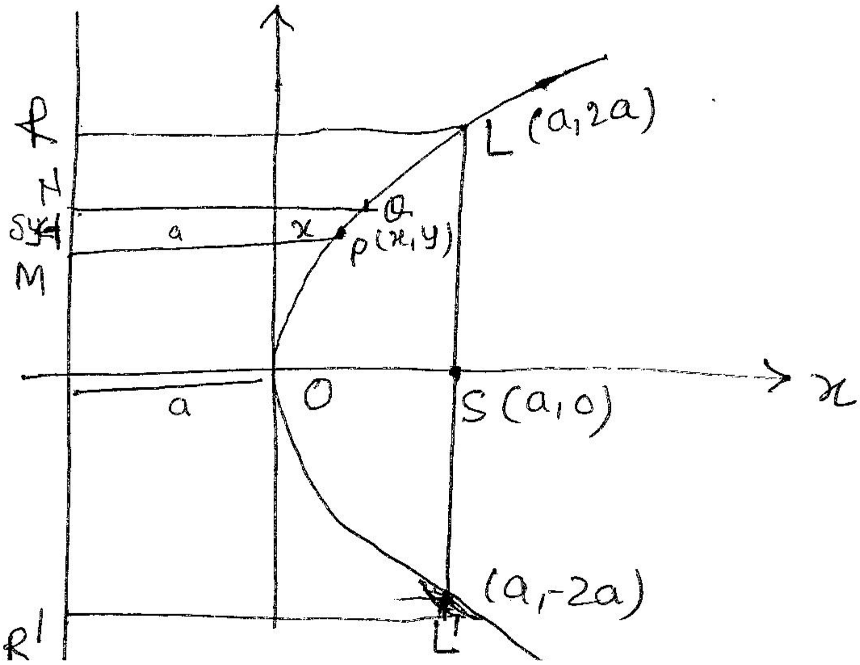

Fig. 12.3.6

Then the directrix is the line $x = -a$. Let LL' be the letus rectum. The area $LOL'SL$ is revolved about the directrix. The volume of the ring thus obtained $(v) = v_1 - v_2$ where v_1 is the volume of the cylinder formed by the revolution of the rectangle $LL'R'R$ about directrix and v_2 is the volume of the red formed by the revolution of the arc LOL' about directrix.

Now volume V_1 *of the cylender* $= \pi r^2 h$

$=\pi(LR)^2 LL'$

$=\pi(2a)^2.4a$

$= 16\pi a^3$

To find the volume v_2 of the reel consider an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc OL and PM, QN are perpendiculars from P and Q on directrix.

Now, we have $PM = a + x$ *and* $MN = \delta y$

Hence, volume $V_2 = 2\int_0^{2a} \pi(a + x)\quad {}^2 dy$

$=2\int_0^{2a} \pi(a^2 + 2ax + x^2)dy$

$=2\int_0^{2a} \pi[a^2 + 2a.\frac{y^2}{4a} + \left(\frac{y^2}{4a}\right)^2]dy$

$=2\pi\int_0^{2a} \pi[a^2 +.\frac{y^2}{2} + \frac{y^4}{16a^2}\quad]dy$

$=2\pi\left[a^2 y + \frac{1}{3}\frac{y^3}{2} + \frac{y}{16a^2 5}{}^5\right]_0^{2a}$

$$= 2\pi\left[2a^3 + \frac{4}{3}a^3 + \frac{2a^3}{5}\right]$$

$$v_2 = 2a^3\pi\frac{56}{15} = \frac{112\pi a^3}{15}$$

Hence volume of the ring $= v_1 - v_2$

$$16\pi a^3 - \frac{112\pi a^3}{15}$$

$$v = \frac{128\pi a^3}{15}$$

Now volume of the sphere whose diameter is the letus rectum i.e. 4a i.e. radius of sphere is 2a.

$$v' = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(2a)^3 = \frac{32\pi a^3}{3}$$

Hence the required ratio $= \frac{v}{v'}$

$$= \frac{128\pi a^3/15}{32\pi a^3/3} = \frac{4}{5}$$

Ex.6 **Find the volume of the solid generated by revolution of the loop $y^2(a + x) = x^2(3a - x)$ about the x –axis.**

Sol. Equation of the curve is

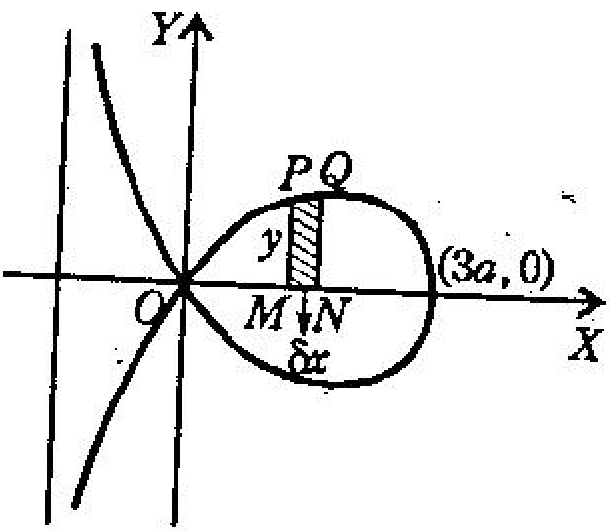

Fig. 12.3.7

The given curve is symmetrical about x-axis.

Putting $y = 0$,we get

$x^2(3a - x) = 0 \Rightarrow x = 0$ and $x = 3a$. Hence the loop is formed between $x = 0$ and $x = 3a$.

Asymptote parallel to y-axis is $x = -a$.

Since the curve is symmetrical about x –axis. So the volume generated by the revolution of the whole loop about the axis is the same as the volume generated by the revolution of the upper half loop about x-axis.

Take the elementary strip MNQ where P is the point and Q is the point

Then, we have $PM = y$ and $MN = \delta x$.

Now the volume of the elementary disc formed by revolving the elementary strip $PMNQ$ about the x −axis is $\pi(PM)^2MN = \pi y^2 \delta x$.

Hence the volume generated by the loop is $\pi\int_0^{3a} y^2 dx$.

$$= \pi \int_0^{3a} \frac{x^2(3a-x)}{a+x} dx$$

$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{a+x}\right] dx$$

$$= \pi \left[\frac{-x^3}{3} + 2ax^2 - 4a^2x + 4a^3\log(a+x)\right]$$

$$= \pi[-(9a^3 - 0) + (18a^3 - 0) - 4a^2x + 4a^3\log(a+x)]$$

$$= \pi\int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{a+x}\right] dx$$

$$= \pi\int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{a+x}\right] dx$$

$$= \pi \int_0^{3a} \left[-\frac{x^3}{3} + 2ax^2 - 4a^2x + 4a^3 log(a+x)\right]_0^{3a}$$

$$= \pi[-9a^3 + 18a^3 - 12a^3 + 4a^3(log4a - loga)]$$

$$= \pi[-3a^3 + 4a^3 log4]$$

$$= \pi a^3[8log2 - 3]$$

$$V = 4a^3[8log2 - 3]$$

Ex7.Find the volume of the solid generated by the revolution of the cissoid $y^2(2a - x) = x^3$ *about its asymptote.*

Sol. The curve $y^2(2a - x) = x^3$ is symmetric about x −axis. Putting $x = 0,\ y = 0$, we find the curve passes through the origin. Tangent at origin y=0, Hence origin is cusp. Asymptote parallel to y-axis is x=2a.

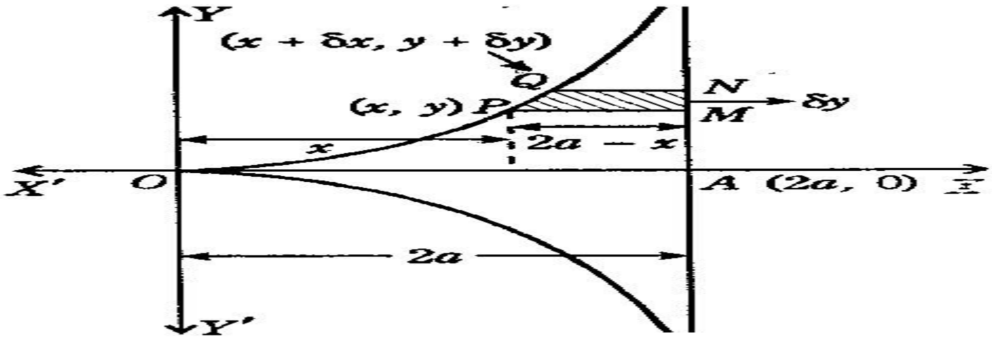

Fig. 12.3.8

Take an elementary strip $PMNQ$ ⊥to the asymptote $x = 2a$. Where p is the point (x, y) and Q is the point ($x + \delta x, y + \delta y$)

We have $PM = 2a - x$, $MN = \delta y$

Now volume of the elementary disc formed by the revolution of the strip about the line $x = 2a$ is.

$\pi PM^2 MN = \pi(2a - x)^2 \delta y$

Hence required volume

$$V = 2\int_0^{\infty} \pi(2a - x)^2 dy$$

$The\ curve\ is\ y^2(2a - x) = x^3$

$$We\ have\ \ y^2 = \frac{x^3}{2a - x}$$

Differentiating with respect to x we get

$$2y\frac{dy}{dx} = \frac{(2a - x)3x^2 - x^3(-1)}{(2a - x)^2} = \frac{2(3a - x)x^2}{(2a - x)^2}$$

$$\frac{dy}{dx} = \frac{(3a - x)x^2}{(2a - x)^2}\frac{}{y}$$

$$= \frac{(3a - x)x^2}{(2a - x)^2}\frac{(2a - x)\quad y^2}{x\quad {}^3/_2}$$

$$dy = \frac{(3a - x)x^2}{(2a - x)^{3/2}}\frac{x\quad y^2\ dx}{}$$

Also when y=0, x=0 and when $y \to \infty, x \to 2a$

Hence

$$V = 2\pi\int_0^{2a}(2a - x)^2\frac{(3a - x)x^2}{(2a - x)^{3/2}}\frac{x\quad y^2\ dx}{}$$

$$= 2\pi\int_0^{2a}(2a - x)^{y^2} x\quad y^2 (3a - x)dx$$

Now put $x = 2a\sin\ ^2\theta\ so\ dx = 4asin\theta cos\theta d\theta$ also when $x = 0, \theta = 0\ and\ when\ x = 2a, \theta = \frac{\pi}{2}$

Therefore

$$V = 2\pi\int_0^{\pi/2} (2a - 2a\sin^2\theta)^{1/2}(2a\sin^2\theta)^{1/2}(3a - 2a\sin^2\theta)\, 4a\sin\theta\cos\theta d\theta$$

$$= 2\pi \int_0^{\pi/2} 8a^2\cos^2\theta\sin^2\theta[3 - 2\sin^2\theta]\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2}[3\sin^2\theta\cos^2\theta - 2\sin^4\theta\cos^2\theta]\, d\theta$$

$$= 16\pi a^3\left[\frac{3.\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{2\Gamma 3} - \frac{2.\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\Gamma 4}\right]$$

$$= 16\pi a^3\left[\frac{3.\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.2.1} - \frac{2.\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3.2.1}\right]$$

$$V = 2\pi^2a^3$$

12.3.1 VOLUME OF A SOLID OF REVOLUTION (B-PARAMETRIC FORM)

Let the curve is given by parametric equations $x = \emptyset(t)\, and\ y = \varphi(t)$

(a) The volume of the solid generated by revolving the area bounded by the curve, the x-axis and the ordinates $x = a, y = b$ about the axis of x is given by

$$v = \pi\int_a^b y^2dx = \pi\int_{t^1}^{t^2}[\varphi(t)]^2\frac{dx}{dt}dt$$

$where\ t_1\ and\ t_2\ are\ the\ values\ of\ t\ corresponding\ to\ x = a\ and\ x = b\ resectively.$

(b) The volume of the solid generated by revolving the area bounded by the curve $x = f(y)$ the ordinates, the y –axis and the abscissa $y = c$ to $y = d$ about the axis of y is

$$V = \pi\int_c^d x^2dy = \pi\int_{t_1}^{t_2}\{\emptyset(t)\}^2\frac{dy}{dt}dt$$

where $t_1\ and\ t_2\ are\ the\ values\ of\ t\ corresponding\ to\ y = c\ and\ y = d\ respectively.$

Ex.8.Show that the volume of the solid generated by the revolution of the curve $(a - x)y^2 = a^2x$ about its asymptote is $\frac{1}{2}\pi^2a^3$.

Sol. The curve $(a - x)y^2 = a^2x$

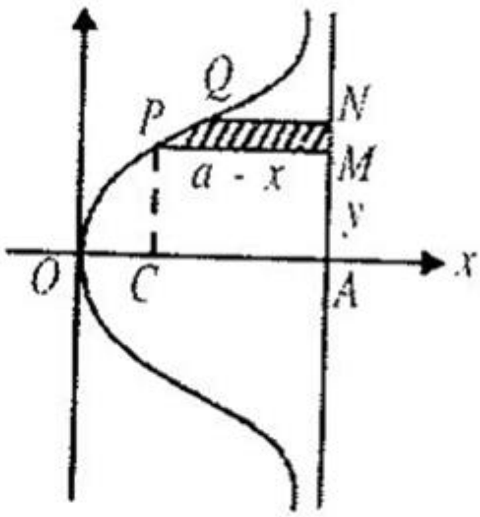

Fig. 12.3.8

is symmetrical about x −axis. It passes through origin and tangent at origin is $x = 0$ i.e. y-axis

The asymptote of the curve is $x = a$.

Let $P(x, y)$ be any point on the curve

Draw perpendicular PM from p to the asymptote.

$PM = oa - oc = a - x, MN = \delta y$

Hence the required volume

$= \int_{-\infty}^{\infty} \pi(PM)^2 d(AM)$

[Since $(a-x)y^2 = a^2x$]

$$so, x = \frac{ay^2}{a^2+y^2}$$

$$= 2\int_0^{\infty} \pi(a-x)^2 dy$$

$$= 2\pi \int_0^{\infty} \left[a - \frac{ay^2}{a^2+y^2}\right]^2 dy = 2\pi \int_0^{\infty} \frac{a^6}{(a^2+y^2)^2} dy$$

$Put\ y = tan\theta\ so\ that\ dy = sec^2\theta d\theta$

$Then\ we\ have, where\ y = 0 \Rightarrow 0\ \ and\ when\ y = \infty \Rightarrow \theta = \frac{\pi}{2}$

Required volume

$$V = 2\pi a^6 \int_0^{\pi/2} \frac{asec^2\theta d\theta}{a^4 sec^4\theta}$$

$$= 2\pi a^3 \int_0^{\pi/2} cos^2\theta d\theta$$

$$= 2\pi a^3 \frac{\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2.1}$$

$$V = \frac{1}{2}\pi^2 a^3$$

Hence the required volume is V=$\pi^2 a^3/2$

<u>Illustrative Examples of parametric curves</u>

Ex. 9. Find the volume of the solid generated by the revolution of the curve $x = acos^3 t, y = asin^3 t$ abou the x −axis

OR

Find the volume of the spindle shaped solid generated by revolving the astorid $x^{2/3} + y^{2/3} = a^{2/3}$ $about\ the\ x\ axis.$

Sol.

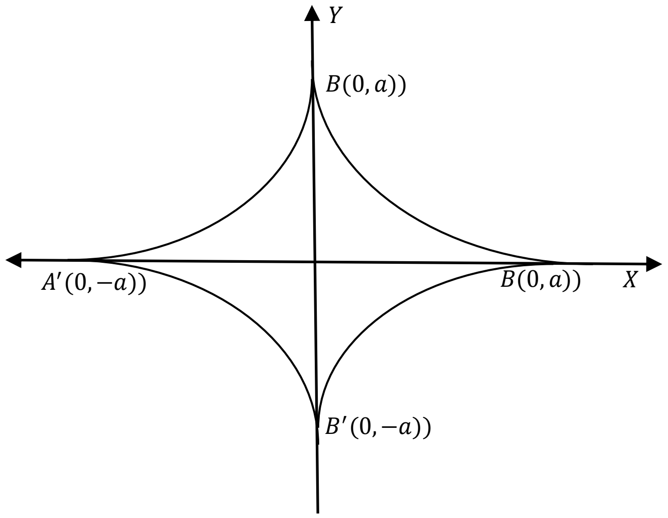

Fig. 12.4.1

$Since\ x = acos^3 t$

$$\frac{dx}{dt} = 3acos^2 t(-sint)$$

The volume of the solid generated by revolving the area ABA'OA about the x-axis i.e.

$2 \times volume\ of\ the\ solid\ generated\ by\ revolving\ the\ area\ ABOA\ about\ the\ x - axis.$

$$Hence\ the\ required\ volume\ v = \int_{x=0}^{a} \pi y^2 dx$$

$When\ x = 0\ \Rightarrow cos^3 t = 0 \Rightarrow t = \pi/_2\ and$

$When\ x = a \Rightarrow cos^3 t = 1 \Rightarrow t = 0$

Hence $V = 2\int_{t=\pi/_2}^{0} y^2 \frac{dx}{dt} dt$

$=2\int_{\pi/_2}^{0} (asin^3 t)\quad {}^2 3acos^2 t(-sint)dt$

$=6\pi a^3 \int_0^{\pi/2} sin^7 tcos^2 tdt$

$$= 6\pi a^3 \frac{\Gamma 4 \Gamma\, ^3/_2}{2\Gamma \frac{11}{2}}$$

$$= 6\pi a^3 \frac{3.2.1.\frac{1}{2}\sqrt{\pi}}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}$$

$$V = \frac{32}{105}\pi a^3$$

Ex.10. Find the volume of the solid generated by the revolution of the curve $x = 2asin^2t, y = 2a\frac{sin^3t}{cost}$ about its asymptote.

Sol. Eliminating t between x and y we get the equation of curve $y^2(2a-x) = x^3$ which is a cissoid. The curve is symmetrical about x-axis and x=2a is the asymptote of the curve.

Hence $V = 2\int_{y=0}^{\infty}(2a-x)\quad^2 dy$

$=2\int\ (2a-x)\quad^2\frac{dy}{dt}dt$

$Here\ y = 2asin^3t\quad /cost$

$$\frac{dy}{dt} = 2a\left[\frac{cost3sin^2tcost - sin^3t(-sint)}{cos^2t}\right]$$

$$= 2a\left[\frac{3sin^2tcos^2t + sin^4t}{cos^2t}\right]$$

Also when y=0 $\Rightarrow 2a\frac{sin^3t}{cost} = 0 \Rightarrow sint = 0 \Rightarrow t = 0$

$$When\ y = \infty \Rightarrow 2a\frac{sin^3t}{cost} = \infty \Rightarrow cost = 0 \Rightarrow t = \pi/2$$

$$Putting\ the\ values\ of\ x,y\ \frac{dy}{dt}\ in\ \ (A) we\ get$$

$$V = 2\int_0^{\pi/2}\pi(2a - 2asin^2t)\quad^2 2a\frac{(3sin^2tcos^2t + sin^4t)}{cos^2t}dt$$

$$=2\pi\int_0^{\pi/2}8a^3cos^2t(3sin^2tcos^2t + sin^4t)dt$$

$$= 16\pi a^3\left[\int_0^{\pi/2}3sin^2tcos^4tdt + \int_0^{\pi/2}sin^4tcos^2tdt\right]$$

$$=16\pi a^3\left[\frac{3.\Gamma 3/2\,\Gamma 5/2}{2\Gamma 4} + \frac{\Gamma 5/2\,\Gamma 3/2}{2\Gamma 4}\right]$$

$$= 16\pi a^3\left[\frac{3.\frac{1}{2}\sqrt{\pi}\frac{3}{2}\frac{1}{2}\sqrt{4}}{2.3.2.1} + \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3.2.1}\right]$$

$$V = 2\pi^2a^3$$

Ex.11 Prove that the volume of the reel formed by the revolution of the cycloid

$x = a(\theta + sin\theta), y = a(1 - cos\theta) about\ the\ tangent\ at\ the\ vertex\ is\ \pi^2 a^2$

Sol. The given cycloid is symmetrical about y-axis, and the tangent at the vertex is the x-axis. The reel is formed by revolving the area BOACD about the x-axis. The vertex is the origin and $\theta\ varies\ from\ 0\ to\ \pi\ for\ the\ area\ OACO.$

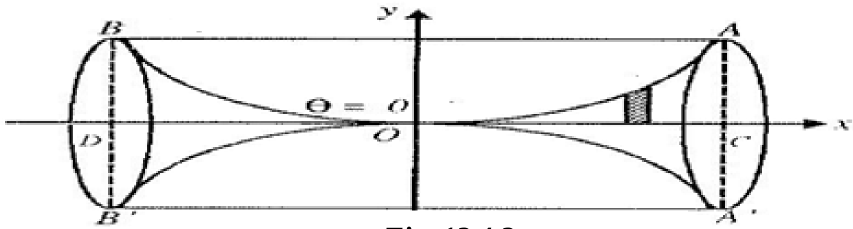

Fig. 12.4.2

Hence the required volume

$$V = 2\int_0^{\pi} \pi y^2 dx$$

$$= 2\int_0^{\pi} \pi y^2 \frac{dx}{d\theta} dx \qquad [x = a(\theta + sin\theta)\ \frac{dx}{d\theta} = a(1 + cos\theta)]$$

$$= 2\pi \int_0^{\pi} a^2(1 - cos\theta)^2 a(1 + cos\theta)d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (2sin^2 \theta/_2)^2 cos^2 \theta/_2\ d\theta$$

$$Put\ \theta/_2 = \emptyset$$

$$d\theta = 2d\emptyset\ when\ \theta = 0 \Rightarrow s\ \emptyset = 0\ when\ \theta = \pi \Rightarrow \emptyset = \theta/_2$$

$$V = 32\pi a^3 \int_0^{\pi/2} sin^4\emptyset cos^2\emptyset d\emptyset$$

$$= 32\pi a^3 \frac{\Gamma 5/_2 \Gamma 3/_2}{2\Gamma 4}$$

$$= 32\pi a^3 \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3.2.1}$$

$$V = \pi^2 a^3$$

12.3.2 VOLUME OF A SOLID OF REVOLUTION (POLAR FORMS)

Let the equation of the curve in polar form is

$r = f(\theta,) then$

(a) $If\ the\ portion\ of\ the\ curve\ lying\ between\ the\ points\ x = a\ and\ x = b\ and\ the\ curve\ revolves\ about\ x - axis\ i.e., the\ initial\ line.$

$The\ volume\ of\ the\ solid\ generated\ is$

$$V=\int_{x=a}^{b}\pi y^2 dx=\pi\int_{\theta=\gamma}^{\beta} y^2\frac{dx}{d\theta}$$

Where

γ *and* β *are the values of* θ *at the points* $x=a$ *and* $x=b$ *respectively.*

But $x=r\cos\theta, y=r\sin\theta$. *so*

$$V=\pi\int_{\gamma}^{\beta}[r\sin\theta]^2\frac{d}{d\theta}(r\cos\theta).d\theta$$

(c) If the portion of the curve lying between the points $y=c$ *and* $y=d$, *and the curve revolves about* $y-axis$ *i.e., the line* $\theta=\frac{\pi}{2}$, *(line| to the initial line.)Then the volume of the solid generated is*

$$V=\int_{y=c}^{d}\pi x^2 dy=\pi\int_{\theta=\gamma}^{\beta} x^2\frac{dy}{d\theta}d\theta$$

$$V=\pi\int_{\theta=\gamma}^{\beta}[r\cos\theta]^2\frac{d(r\sin\theta)d\theta}{d\theta}$$

<u>Alternative method</u>

The volume of the solid generated by the revolution of the area bounded by the curve $r=f(\theta)$ *and the radii vectors* $\theta=\gamma$ *and* $\theta=\beta$

(i) About the initial line $\theta=0$ $(i.e, x-axis)$

$$V=\frac{2}{3}\pi\int_{\gamma}^{\beta} r^3\sin\theta d\theta$$

(ii) About the line $\theta=\frac{\pi}{2}$ $(i.e, y-axis)$

$$V=\frac{2}{3}\pi\int_{\gamma}^{\beta} r^3\cos\theta d\theta$$

<u>Illustrative Examples</u>

Ex.12.The cardiod $r=a(1+\cos\theta)$ revolves about the initial line. Find the volume of solid generated

Sol. The curve $r=a(1+\cos\theta)$ is symmetrical about the initial line.

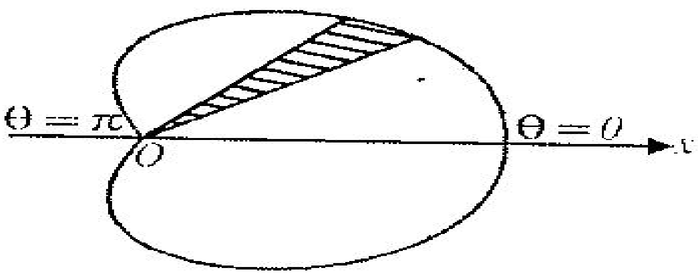

Fig. 12.5.1

And $\theta = 0 \Rightarrow r = 2a, \theta = {}^{\pi}/_{2} \Rightarrow r = a \; and \; \theta = \pi \Rightarrow r = 0$

So for the curve $\theta = 0 \; to \; \theta = \pi, above \; the \; initial \; line.$

Hence volume of the solid generated by revolving the area ABOA about the initial line is

$$V = \frac{2}{3}\pi \int_0^{\pi} r^3 sin\theta d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} a^3 (1 + cos\theta)^3 sin\theta d\theta \; [r = a(cos\theta + 1)]$$

$$= \frac{2}{3}a^3\pi \int_0^{\pi} a^3 (1 + 2cos^2 {}^{\theta}/_{2} - 1)^3 . 2sin {}^{\theta}/_{2} \, cos {}^{\theta}/_{2} \, d\theta$$

$$= \frac{32}{3}\pi \int_0^{\pi} a^3 sin {}^{\theta}/_{2} \, cos^7 {}^{\theta}/_{2} \, d\theta$$

Put ${}^{\theta}/_{2} = \emptyset$ so $d\theta = 2d\emptyset$ and when $\theta = 0, \emptyset = 0$ and when $\theta = \pi, \emptyset = {}^{\pi}/_{2}$

$$V = \frac{64}{3}\pi a^3 \int_0^{\pi/2} sin\emptyset cos^7 \emptyset d\emptyset$$

$$= \frac{64}{3}\pi a^3 \frac{\Gamma 1 \Gamma 4}{2\Gamma 5}$$

$$= \frac{64}{3}\pi a^3 \frac{\Gamma 1 \Gamma 4}{2.4.\Gamma 4}$$

$$V = \frac{8}{3}\pi a^3$$

Ex.13Prove that volume of the solid generated by revolving the curve lemniscates $r^2 = a^2 cos2\theta \; about \; the \; line \; \theta = \frac{\pi}{2}$

Sol. The curve $r^2 = a^2 cos2\theta$ is symmetrical about the initial line and the pole.

Put $r = 0, cos2\theta = 0 \Rightarrow 2\theta \pm {}^{\pi}/_{2} \Rightarrow \theta = \pm {}^{\pi}/_{4}$

Hence $\theta = \pm \frac{\pi}{4}$ are tangents at pole.

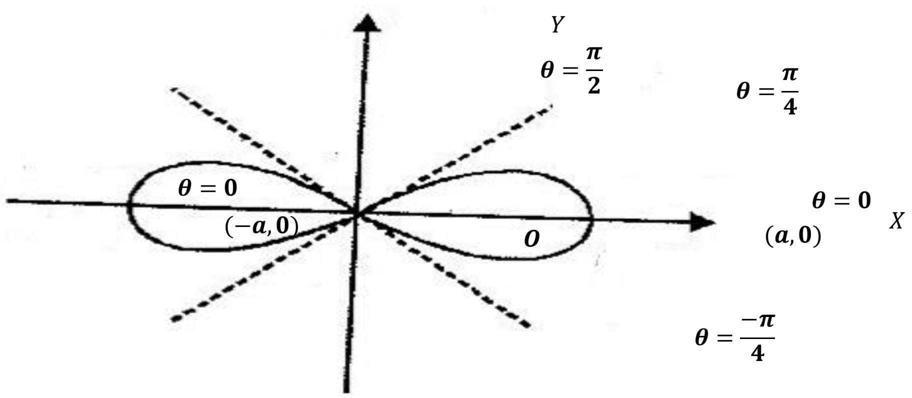

Fig. 12.5.2

$When\ \theta = 0, r^2 = a^2 \Rightarrow r \pm a$

Hence for the upper half of loop $\theta\ varies\ from\ 0\ to\ {}^{\pi}/_{4}$

Hence the required volume (revolving about $\theta = {}^{\pi}/_{2}$)

$$= 2 \times \frac{2}{3}\pi \int_0^{\pi/4} r^3 cos\theta d\theta$$

$$= \frac{4}{3}\pi \int_0^{\pi/4} a^3 (cos2\theta)^{3/2} cos\theta d\theta$$

Put $\sqrt{2}sin\theta = sin\emptyset\ so\ \sqrt{2}cos\theta d\theta = d\emptyset$

$Also\ when\ \theta = 0, \emptyset = 0\ and\ when\ \theta = {}^{\pi}/_{4}, \emptyset = {}^{\pi}/_{2}$

V $\quad = \frac{4}{3\sqrt{2}}\pi a^3 \int_0^{\pi/2} (1 - sin^2\emptyset)^{3/2} cos\emptyset d\emptyset$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \int_0^{\pi/2} cos^4\emptyset d\emptyset$$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \frac{\Gamma 5/2\, \Gamma 1/2}{2\Gamma 3}$$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2.2.1}$$

Thus $\qquad V = \frac{\pi^2 a^3}{4\sqrt{2}}$

12.4 VOLUME OF A SOLID BY DOUBLE INTEGRATION

Let the area $dydz\ on\ the\ yz\ plane\ (i.e.\, x = 0)$. Through the each point on the boundary of this small area, draw lines parallel to x-axis and thus construct a small cylinder whose base is the area $dydz$ and generators parallel to x-axis.

This volume of the cylinder $= xdydz$

Hence, Volume of the solid = $\iint xdydz$

Considering area

$z =$

$0\ and\ constructing\ cylender\ as\ above\ by\ drawing\ lines\ parallel\ to\ z - axis\ we\ can\ have\ the\ volume\ of\ solids$

$$V = \iint dxdy$$

(3) In a similar way considering area $dydz$ on the plane y=0 and construct cylinder as above by drawing lines parallel to y-axis.

$$Volume\ of\ the\ solid\ V = \iint ydxdz$$

Ex.14Find the volume of the sphere $x^2+y^2+z^2=a^2$

Sol. Equation of the sphere is

$x^2+y^2+z^2=a^2$

Projection of the sphere on $yz\ plane\ (i.e, x=0) is\ the\ circle\ x^2+y^2+z^2=a^2$

In the positive octant y varies from o to a and z varies from o to $\sqrt{a^2-y^2}$

Hence volume of the solid varies lies in positive octant $=\int_{y=0}^{a}\int_{z=0}^{\sqrt{a^2-y^2}} xdydz$

Hence total volume of the sphere is

$$V=8\int_{y=0}^{a}\int_{z=0}^{\sqrt{a^2-y^2}} xdydz$$

Hence total volume of the sphere is $V=8\int_0^a\int_{z=0}^{\sqrt{a^2-y^2}} xdydz$

$$=8\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2-z^2dydz}$$

$$=8a\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{\left(1-\frac{y^2}{a^2}\right)-\frac{z^2}{a^2}}\,dydz$$

$$put\quad \frac{z}{a}=\sqrt{1-\frac{y^2}{a^2}}sin\theta, we\ get$$

$$V=8a^2\int_0^a\int_{\theta=0}^{\pi/2}\left(1-\frac{y^2}{a^2}\right)cos^2\theta dyd\theta$$

$$=8a^2\int_0^a\left(1-\frac{y^2}{a^2}\right)cos^2\theta dyd\theta$$

$$=8a^2\int_0^a\left(1-\frac{y^2}{a^2}\right)\frac{1}{2}\frac{1}{2}\pi dy$$

$$=2\pi a^2\left[y-\frac{y^3}{3a^2}\right]_0^a=\frac{4}{3}\pi a^3$$

Ex.15 Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol. Equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Projection of the curve on the xy- plane (i.e. z=0)

Is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ or\ y^2=\frac{b^2}{a^2}(a^2-x^2)$

Or $y=\pm\frac{b\sqrt{(a^2-x^2)}}{a}$

Hence limits of y varies from o to $\frac{b\sqrt{(a^2-x^2)}}{a}$ in the positive octant.
X varies from 0 to a.
Hence volume of the ellipsoid lying in the positive octant is $\iint \ zdxdy$

Since the ellipsoid is symmetrical in all eight octants. Hence total volume of the ellipsoid is

$$V = 8\int_{x=0}^{a}\int_{y=0}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} zdxdy$$

$$= 8\int_0^a\int_0^{\frac{b\sqrt{(a^2-x^2)}}{a}} c\sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}dxdy$$

$$Put\ \frac{y}{b} = \sqrt{1-\frac{x^2}{a^2}}sin\theta$$

$$V = 8\int_0^a\int_0^{\pi/2} cb\left[1-\frac{x^2}{a^2}\right]cos^2\theta dxd\theta$$

$$= 8bc\int_0^a\left(1-\frac{x^2}{a^2}\right)\frac{1}{2}\frac{\pi}{2}dx$$

$$= 2\pi bc\left[x-\frac{x^3}{3a^2}\right] = 2\pi bc\left[a-\frac{a}{3}\right]$$

$$Thus \qquad V = \frac{4}{3}\pi bc$$

12.5 SURFACE OF REVOLUTION (CARTESIAN FORM)

12.5.1 WHEN THE AXIS OF REVOLUTION IS THE X-AXIS:-

The area of the surface generated by revolving an area bounded by the curve
$y = f(x,)the\ x-axis\ and\ the\ ordinates\ x = a\ and\ x = b, about\ the\ x-axis\ is$

$$s = 2\pi\int_{x=a}^{b} yds\ \ i.e.s = 2\pi\int_{x=a}^{a} y\frac{ds}{dx}dx$$

Where s is the length of the arc measured from a fixe point to any point (x,y) on the curve.

Hence $\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$

Proof:- Let the equation of the curve in Cartesian form is $y = f(x) and\ let\ AC\ and\ BD\ are\ given$ ordinates $x = a\ and\ x = b. Let\ P(x, y) and\ Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve.

From P and Q draw PM and QN perpendiculars to x-axis. Further, from P and Q draw PS and QR perpendiculars to PM and QN respectively.

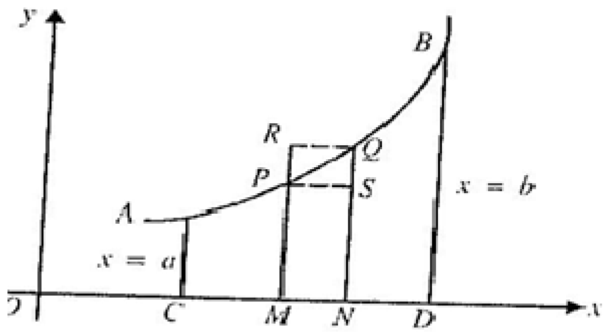

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Fig 12.5.1

$Let\ AP = s\ and\ arc\ AQ = s + \delta x\ so\ that\ arc\ PQ = \delta s$(a being a fixed point).Let the area of the surface of a solid generated by revolving the arc AP and AQ about x-axis be A and δA then δA is the area of the surface of the solid generated by the revolution of the arc PQ about x-axis. While revolving about x-axis, the line PS and QR generates cylinders. Let suppose that the area of curved surface of these cylinders are$2\pi y \delta x$and $2\pi(y + \delta y)\delta s$respectively and assume that $PS = AR = arc\ PQ$. Since P and Q are neighbouring points.So, the surface generated by revolving the arc PQ about x-axis lies between these two surfaces.

$$i.e., 2\pi y \delta x < \delta a < 2\pi(y + \delta y)\delta s$$

$$2\pi y < \frac{\delta a}{\delta s} < 2\pi(y + \delta y)$$

In limiting case when $\theta \to P, we\ have\ \delta y \to 0, \delta s \to 0\ we\ get$

$$\frac{dA}{ds} = 2\pi y$$

$$dA = 2\pi y ds$$

Integrating between the limits $x = a \ and \ x = b \ we \ get$

$$\int_{x=a}^{b} dA = \int_{a}^{b} 2\pi y ds$$

$$i.e. [A]_{x=a}^{x=b} = \int_{x=a}^{b} 2\pi y ds$$

i.e, Area of the surface generated by the curve AB.

$$= \int_{a}^{b} 2\pi y ds \ i.e.$$

$$S = 2\pi \int_{x=a}^{b} y.\frac{ds}{dx} dx$$

$$S = 2\pi \int_{x=a}^{b} \left[y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right].dx$$

$$S = 2\pi \int_{x=a}^{b} \left[y.\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right] dx$$

12.5.2 WHEN THE AXIS OF REVOLUTION IS THE Y-AXIS:-

Proceeding in the same manner we can show that:
The area of the surface of a solid generated by revolving the curve $x = f(y)$, the y-axis and the abscis $y = c \ and \ y = d \ is$

$$S = 2\pi \int_{y=c}^{d} x ds = 2\pi \int_{y=c}^{d} x\frac{ds}{dy} dy \ i.e.$$

$$S = 2\pi \int_{c=c}^{d} f(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2} dy$$

12.5.3 WHEN THE AXIS OF REVOLUTION IS ANY STRAIGHT LINE

Let the axis of revolution is any straight line AB (which is not x-axis or y-axis) and CD be the arc of the curve. Let PM be the length of the perpendicular drawn from A any point P on the arc Cd to the axis of revolution AB.

The area of the surface generated by revolving the curve CD about AB is

$$S = 2\pi \int_a^b (PM)d(AB)$$

Ex. 16 Find the surface area of a sphere of radius a.

Sol. The sphere is generated by revolving a semi-circle or radius a about is bounding diameter

Equation of the semicircle

$$x^2 + y^2 = a$$

Differentiating with respect to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -{}^{x}/_{y}$$

$$Thus\ \frac{ds}{dx}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\sqrt{1 + \frac{x^2}{y^2}} = \sqrt{\frac{x^2 + y^2}{y^2}} = \frac{a}{y} \Rightarrow ds = \frac{a}{y}dx$$

X varies from –a to +a for the semi-circle

Hence the required surface area is

$$2\pi \int_{-a}^{a} yds = 2\pi \int_{-a}^{a} y\frac{a}{y}dx$$

$$= 2\pi a \int_{-a}^{a} dx = 4\pi a^2$$

Ex. 17 Prove that the surface of the prolate spheroid formed by revolution of the ellipse of eccentricity e about its major axis is equal to

$$= 2 \times (Area\ of\ the\ ellipse)\left(\sqrt{1 - e^2} + \frac{1}{e}Sin^{-1}e\right)$$

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating (1) with respect to x we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \text{ i.e.} \frac{dy}{dx}=\frac{-b^2}{a^2}\frac{x}{y}$$

$$so\ \frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{b^4}{a^4}\frac{x^2}{y^2}}$$

$$=\sqrt{1+\frac{b^4x^2}{a^4\frac{b^2}{a^2}(a^2-x^2)}}$$

$$=\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}$$

$$=\sqrt{1+\frac{a^2(1-e^2)x^2}{a^2(a^2-x^2)}}\ [Since b^2=a^2(1-e^2)]$$

$$\frac{ds}{dx}=\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since ellipse is symmetrical about both the axes and for upper half of the ellipse, x varies from –a to a

Hence the surface area generated is

$$S=2\pi\int_{x=-a}^{a} yds=2\pi\int_{-a}^{a} y.\frac{ds}{dx}.dx$$

$$=2\pi\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}dx, \qquad [using\ (1) and\ (2)]$$

$$=\frac{4\pi b}{a}\int_0^a\sqrt{a^2-e^2x^2}dx \qquad \left[\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx \text{ if } f(x) \text{ is even function.}\right]$$

$$=\frac{4\pi be}{a}\int_0^a\sqrt{\left(\frac{a}{e}\right)^2-x^2dx}$$

$$=\frac{4\pi be}{a}\left[\frac{x}{2}\sqrt{\left(\frac{a}{e}\right)^2-x^2}+\frac{1}{2}\frac{a^2}{e^2}Sin^{-1}\left(\frac{x}{a/e}\right)\right]_0^a$$

$$=\frac{2\pi be}{a}\left[a\sqrt{\frac{a^2}{e^2}-a^2}+\frac{a^2}{e^2}Sin^{-1}(e)\right]$$

$$=\frac{2\pi be}{a}\left[\frac{a^2\sqrt{1-e^2}}{e}+\frac{a^2}{e^2}Sin^{-1}(e)\right]$$

Thus $\quad S=2\pi ab\left[\sqrt{1-e^2}+\frac{1}{e}Sin^{-1}(e)\right]$

Since we know that area of the ellipse (1) is πab

Thus $S = 2 \times Area\ of\ the\ ellipse \left[\sqrt{1-e^2} + \frac{1}{e} Sin^{-1}(e)\right]$

12.6 SURFACE REVOLUTION (PARAMETRIC FORM)

Let the parametric equation of the curve be
$x = \emptyset(t) and\ y = \varphi(t)$

(A) If the area bounded by the curve, the x-axis and the ordinates at the points where $t = a\ and\ t = b$ is revolved about x-axis, then the surface area of the solid formed is given by

$$S = 2\pi \int_{t=a}^{b} yds = 2\pi \int_{t=a}^{t=b} y \frac{ds}{dt} dt$$

$$S = 2\pi \int_{t=a}^{t=b} \varphi(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$S = 2\pi \int_{t=0}^{b} \varphi(t) \sqrt{\left(\frac{d\emptyset}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2}\, dt$$

$$as \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(B) Similarly, If the area bounded by the curve, the and the abscissa at the points where t=a and t=b is revolved about y-axis, then the surface area the solid formed is

$$S = 2\pi \int_{t=a}^{b} xdx = 2\pi \int_{t=a}^{b} \emptyset(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, .dt$$

$$S = 2\pi \int_{t=a}^{b} \emptyset(t) \sqrt{\left(\frac{d\emptyset}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2}\, .dt$$

<u>Illustrative Examples.</u>

Ex. 18 Find the surface of the solid generated by the revolution of the astroid $x = a\,cos^3t, y = a\,sin^3t$ or $x^{2/3} + y^{2/3} = a^{2/3}$ about x –axis.

Sol. The parametric equations of the curve arc
$x = a\cos^3 t\,,\ y = a\sin^3 t$, so

$\frac{dx}{dt} = -3a\cos^2 \mathrm{t} \sin \mathrm{t}$ and $\frac{dy}{dt} = 3a\sin^2 \mathrm{t} \cos \mathrm{t}$

Now, $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}$

$$= \sqrt{9a^2 \cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)}$$
$$= 3\,\mathrm{a} \cos t \sin t$$

The given curve is symmetrical about both the axes. Also the curve in the first quadrant, t varies from $0\ to\ {}^{\pi}/_{2}$

Hence the required surface area is

$$S = 2 \times 2\pi \int_0^{\pi/2} y \frac{ds}{dt} dt$$
$$= 4\pi \int_0^{\pi/2} a \sin^3 t\,.\,3\,\mathrm{a} \cos t \sin t$$
$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\ \ dt$$
$$= 12\pi a^2 \int_0^{\pi/2} Sin^4 t Cos t dt$$
$$= 12\pi a^2 \frac{\Gamma^5/_2 \Gamma 1}{2\Gamma^7/_2}$$
$$= 12\pi a^2 \frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}$$

Thus $\quad S = \frac{12}{5}\pi a^2$

Ex. 19 If an ellipse of eccentricity e and semi-major axis revolve about its minor-axis. Show that the surface of the spheroid thus generated is

$$2\pi a^2 \left[1 + \frac{1-e^2}{2e} log\left(\frac{1+e}{1-e}\right)\right]$$

Sol. The parametric equations of the ellipse be

$x = a \cos t$ ……………………………………….(1)

$x = b \sin t$ ………………………………………..(2)

$where\ b^2 = a^2(1 - e^2)$

Now differentiating (1) and (2) with respect to t, we get

$\frac{dx}{dt} = -a \sin t$ and $\frac{dy}{dt} = b \cos t$

Since, $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$$

$= \sqrt{a^2 \sin^2 t + a^2\,(1 - e^2) \cos^2 t}$ [since $b^2 = a^2(1 - e^2)$]

In the first quadrant we have when $y = 0, \theta = 0$ *when* $y = b, \theta = {}^{\pi}/_{2}$]

$$\frac{ds}{dt} = a\sqrt{1 - e^2 \cos^2 t}$$

Hence surface area of the (prolate spheroid) by revolving the ellipse about minor axis is

$S = 2 \times$ *Surface area generated by revolving the arc of the ellipse in first quadrant revolve*

$$= 2 \times 2\pi \int_0^{\pi/2} x\, ds$$

$$= 2 \times 2\pi \int_0^{\pi/2} x \frac{ds}{dt} dt$$

$$= 4\pi \int_0^{\pi/2} a\cos t\ a\sqrt{1 - e^2\cos^2 t}\, dt$$

$$= 4\pi \int_0^{\pi/2} a\cos t\ a\sqrt{1 - e^2(1-\sin^2 t)}\, dt$$

$$= 4\pi \int_0^{\pi/2} a\cos t\ a\sqrt{(1 - e^2) + e^2\sin^2 t)}\, dt$$

Let $e\sin t \quad = u \Rightarrow e\cos t\ dt = du$

Also when $t = 0 \Rightarrow u = 0$ and when $t = \pi/2 \Rightarrow u = e$

Then we have

$$S = 4\pi a^2 \int_0^e \sqrt{(1-e^2) + u^2} \frac{du}{e}$$

$$\frac{4\pi a^2}{e} \int_0^e \sqrt{(1-e^2) + u^2}\, du$$

$$\frac{4\pi a^2}{e} \left[\frac{u}{2}\sqrt{(1-e^2) + u^2} \quad + \frac{1}{2}(1-e^2)\log\left[u + \sqrt{(1-e^2)+u^2}\right]_0^e\right]$$

$$\frac{4\pi a^2}{e} \left[\frac{e}{2} + \frac{1}{2}(1-e^2)log(e+1) - \frac{1}{2}(1-e^2)log\sqrt{1-e^2}\right]$$

$$\frac{4\pi a^2}{e} \left[\frac{e}{2} + \frac{1}{2}(1-e^2)log\left(\frac{1+e}{\sqrt{1-e^2}}\right)\right]$$

$$2\pi a^2 \left[1 + \frac{(1-e^2)}{e} log\left(\sqrt{\frac{1+e}{1-e}}\right)\right]$$

$$S = 2\pi a^2 \left[1 + \frac{(1-e^2)}{2e} log\frac{1+e}{1-e}\right]$$

Ex. 20 Find the surface of the solid formed by the revolution about x-axis of the loop of the curve $x = t^2, y = t - \frac{1}{3}t^3$

Sol. The parametric equations of the curve are

$x = t^2, y = t - \frac{1}{3}t^3,$ so

$$\frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 1 - t^2$$

Hence $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4t^2 + (1-t^2)^2}$

$=\sqrt{(1+t^2)^2} = 1 + t^2$

$$Put\ y = 0 \Rightarrow t - \frac{1}{3}t^3 = 0 \Rightarrow t\left[1 - \frac{1}{3}t^2\right] = 0$$

$t=0 \Rightarrow t=\pm\sqrt{3}$

Hence the loop is formed between t=0 and t=$\sqrt{3}$

Hence required surface area

$$S=2\pi\int_{t=0}^{\sqrt{3}} yds=2\pi\int_0^{\sqrt{3}} y\frac{ds}{dt}dt$$

$$=2\pi\int_0^{\sqrt{3}}\left(t-\frac{1}{3}t^3\right)(1+t^2)dt$$

$$=\frac{2\pi}{3}\int_0^{\sqrt{3}}(3t+2t^3-t^5)dt$$

$$=\frac{2\pi}{3}\left[3\frac{t^2}{2}+\frac{1}{2}t^4-\frac{t^6}{6}\right]_0^{\sqrt{3}}$$

$$=\frac{2}{3}\pi\left[\frac{9}{2}+\frac{9}{2}-\frac{9}{2}\right]$$

$$=3\pi$$

Hence the required surface area is 3π

12.7 SURFACE OF REVOLUTION (POLAR FORM)

Let the polar equation of the given curve be $f(\theta)=r$ then the surface generated by the revolution of the arc of the curve between the raddi vector $\theta=\alpha \; and \; \theta=\beta$ about the initial line is given by

$$S=2\pi\int_{\theta=\alpha}^{\theta=\beta} yds=2\pi\int_\alpha^\beta (r\sin\theta)\frac{ds}{d\theta}d\theta$$

Where $\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

Illustrative Examples

Ex 21 Find the surface of the solid formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about the initial line.

Sol. The equation of the cardioids is $r=a(1+\cos\theta)$

Differentiating with respect to θ

$$\frac{dr}{d\theta}=-a\sin\theta$$

$$since \quad \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$=\sqrt{a^2(1+cos\theta)^2+a^2\sin^2\theta}$$

$$=a\sqrt{1+\cos^2\theta+2cos\theta+\sin^2\theta}$$

$$=a\sqrt{2}\sqrt{2cos^2\theta/_2}$$

Thus $\frac{ds}{d\theta} = 2a\cos{}^{\theta}/_{2}$

The cardioid is symmetrical about the initial line. Also, for the upper half of the curve, θ varies from $0\ to\ \pi$

Hence the required surface is

$$S = 2\pi\int_0^{\pi} y ds = 2\pi\int_0^{\pi}(r\sin\theta)\frac{ds}{d\theta}d\theta[y = r\sin\theta]$$

$$2\pi\int_0^{\pi} a(1+\cos\theta).\sin\theta\,.\,2a\cos{}^{\theta}/_{2}\,d\theta$$

$$2\pi\int_0^{\pi} 2a^2a(1+\cos{}^2{}^{\theta}/_{2}-1).2\sin{}^{\theta}/_{2}.\cos{}^{\theta}/_{2}\cos{}^{\theta}/_{2}\,d\theta$$

$$16\pi a^2\int_0^{\pi}\sin{}^{\theta}/_{2}.\cos^4{}^{\theta}/_{2}\,d\theta$$

$${}^{\theta}/_{2} = \emptyset \text{ so } d\theta = 2d\emptyset$$

Also, when $\theta = 0, \emptyset = 0$ and when $\theta = \pi, \emptyset = {}^{\pi}/_{2}$

Hence $s = 16\pi a^2\int_0^{\pi/2}\text{Sin}\emptyset\cos^4\emptyset.2d\emptyset$

$$S = 32\pi a^2\int_0^{\pi/2}\text{Sin}\emptyset\cos^4\emptyset.d\emptyset$$

$$= 32\pi a^2\frac{\Gamma 1\Gamma{}^5/_2}{2\Gamma{}^7/_2}$$

$$= 32\pi a^2.\frac{1.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}{2.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}$$

Thus $S = \frac{32}{5}\pi a^2$

Ex. 22 Find the surface of the solid generated by the revolution of the lemniscates $r^2 = a^2\cos 2\theta$ about a tangent at the pole.

Sol. The given equation of the lemniscates

$r^2 = a^2\cos 2\theta$

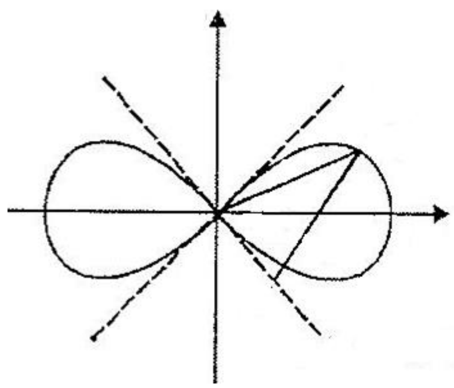

Fig.12.7.1

$$2r\frac{dr}{d\theta} = -2a^2 \sin\ 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 Sin2\theta}{r}$$

Also

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2 cos2\theta + \frac{a^4 \sin^2 2\theta}{r^2}}$$

$$= \sqrt{a^2 cos2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}}$$

$$\frac{ds}{d\theta} = \frac{a}{\sqrt{cos2\theta}}$$

Also for the curve $r^2 = a^2 \cos 2\theta$

Tangent at the poles are given by $r = 0$ so $\cos 2\theta = 0, 2\theta = \pm\frac{\pi}{2}, 2\theta = \pm\frac{\pi}{4}$

Let $P(r,\theta)$ be any point on the curve and let OC be one of the tangent's at the pole and $\angle AOC = \frac{\pi}{4}$.

From P draw PM, perpendicular to the tangent OC. Then,

$$\angle POM = \angle POA + \angle AOC = \theta + \frac{\pi}{4}$$

$$\frac{PM}{OP} = \angle POM$$

$$PM = OP \sin \angle POM = r\, sin\left(\theta + {}^{\pi}/_{4}\right)$$

Since $\theta \pm \frac{\pi}{4}$ are the tangents at pole. Therefore for the loop $\theta\ varies\ from -\frac{\pi}{4}\ to\ \frac{\pi}{4}$

Hence the required surface is

$$S = 2 \times 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} PM.\frac{ds}{d\theta}d\theta$$

$$S = 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} rsin(\theta).\frac{ds}{d\theta}d\theta$$

$$S = 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\sqrt{cos2\theta}\sin\left(\theta + \frac{\pi}{4}\right)\frac{a}{\sqrt{cos2\theta}}d\theta$$

$$=4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin\left(\theta+\frac{\pi}{4}\right) d\theta$$

$$=4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin\left(\theta+\frac{\pi}{4}\right) d\theta$$

$$=\frac{4\pi a^2}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (sin\theta + cos\theta) d\theta$$

$$= 2\sqrt{2}\pi a^2 \left[\int_{-\pi/4}^{\pi/4} \sin\theta \; d\theta + \int_{-\pi/4}^{\pi/4} \cos\theta \; d\theta\right]$$

$$= 4\sqrt{2}\pi a^2 \int_0^{\pi/4} \cos\theta \; d\theta = 4\sqrt{2}\pi a^2 \frac{1}{\sqrt{2}}$$

[as $\int_{-a}^{a} f(x)\; dx = 0$ if $f(-x) = -f(x)$

and $\int_{-a}^{a} f(x)\; dx = 2 \int_0^a f(x)\, if\; f(-x) = f(x)$]

Therefore, $S = 4\pi a^2$.

12.8 AREA OF THE SURFACE BY DOUBLE INTEGRATION

Let the equation of the surface is $z = f(x, y)$.Consider a point $P(x, y, z)$ on the surface surrounding the point. Consider an element of area δs of the surface.

Let $\delta x . \delta y$ be the projection of the area δs on xy plane. Then

$$\delta x \, \delta y = \delta s \cos\gamma$$

where γ is the angle between tangent plane to the given surface at $P(x, y, z)$ and xy plane $(z = 0)$.

Then

$$\sec\gamma = \sqrt{[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2]}$$

From (1)

$\delta s = \delta x . \delta y . sec\gamma$

$$= \delta x \delta y \sqrt{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]}$$

Hence the required surface is

$$= \iint \sqrt{[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2]}.dxdy$$

Where the limits of x and y are to be taken from the region of projection of the given surface on the plane $z = 0$.

Illustrative Exampless

Ex. 23. Find the area of the surface of the sphere $x^2+y^2+z^2=a^2$ included between the cylinder $x^2+y^2=a\ \ x$

Sol. Equation of the sphere is $x^2+y^2+z^2=a^2$ surface is $z^2=a^2-x^2-y^2 so,$

$$2z\frac{\delta z}{\delta x}=-2x \text{ or } \frac{\delta z}{\delta x}=-\frac{x}{z}$$

$$2z\frac{\delta z}{\delta y}=-2y \text{ or } \frac{\delta z}{\delta x}=-\frac{y}{z}$$

Also the projection of the given surface on the plane z=0 is $x^2+y^2+z^2=a^2$

Hence the required area of the surface

$$=\iint\sqrt{[1+\left(\frac{\partial x}{\partial c}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$

$$4\iint\sqrt{\left[1+\frac{x^2}{z^2}+\frac{y^2}{z^2}\right]}dxdy$$

Over half of the circle $x^2+y^2=ax$

$$=4\iint\sqrt{\frac{x^2+y^2+z^2}{z^2}}dxdy$$

$$S=4\iint\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$$

Over half of the circle

$x^2+y^2=ax \text{ or } r^2=ar\cos\theta\ \ i.e, r=a\cos\theta\ \ changing\ to\ polars.$

$$S=4\int_0^{\frac{\pi}{2}}\int_{r=0}^{acos\theta}\frac{rd\theta dr}{\sqrt{a^2-r^2}}$$

$$=-4a\int_0^{\pi/2}\left[\sqrt{a^2-r^2}\right]_0^{aCos\theta}d\theta$$

$$=-4a\int_0^{\frac{\pi}{2}}(asin\theta-a)d\theta$$

$$=4a^2[\theta+Cos\theta]_0^{\pi/2}$$

$$=4a^2\left[\frac{\pi}{2}-1\right]$$

Hence the required surface is S $=4a^2\left[\frac{\pi}{2}-1\right]$

Ex. 24 Find the surface of $x^2 + z^2 = a^2$ that lies inside the cyclinder $x^2 + y^2 = a^2$

Sol. The equation of the surface is $z^2 = a^2 - x^2$ so we have

$$2z\frac{\partial z}{\partial x} = -2x \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{x}{z}\ , \text{also } \frac{\partial z}{\partial y} = 0$$

Also the projection of the given surface on the plane $z = 0$ is $x^2 = a^2$ so $x = \pm a$

Hence limits of x is first octant are from 0 to a.

Also for the cylinder $x^2 + y^2 = a^2$, we have $y^2 = a^2 - x^2$

Hence the required surface is

$$S = 8\iint \sqrt{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]}\,dxdy \qquad [Total\ surface = 8 \times surface\ in\ first\ octant]$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \left[\sqrt{1 + \left(-\frac{x}{z}\right)^2 + 0}\right] dxdy$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \left[\sqrt{\frac{x^2+z^2}{z^2}}\right] dxdy$$

$$= 8a\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \frac{a}{z}\,dxdy$$

$$= 8a\int_{x=0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \frac{a}{z}\,dxdy$$

$$= 8a\int_{x=0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2}}\,dxdy$$

$$= 8a\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}}[y]$$

$$= 8a\int_{0}^{a} dx$$

$$S = 8a^2$$

Hence the required surface is $8a^2$

12.9 THEORMES PAPPUS (OR GULDIN)

Pappus, a great mathematician given his theorems of volumes and surfaces of solid of revolution in the end of third century. These theorems was discovered by P guldin over one thousand year later.

12.9.1 THE THEOREM OF PAPPUS FOR THE VOLUME

"If a plane closed curve revolves through any angle about an axis in its plane which the curve does not intersect, then the volume generated is equal to the product of the area of the close curve and the length of the path of its centroid."

Proof.

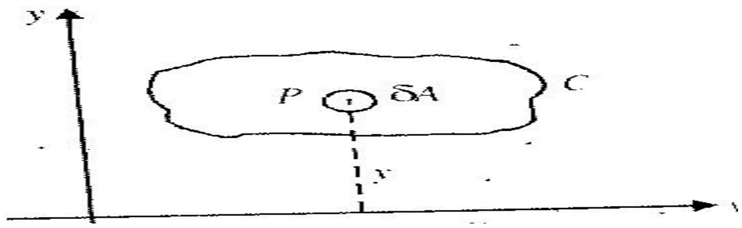

Fig. 10.11.1

Let OX be the axis of rotation, and C be the plane closed curve of Area A. Take a point $\boldsymbol{P(x,y)}$ on the curve and an elementary area $\boldsymbol{\delta A}$ of the curve surrounding the point. If the curve revolves about OX through an angle $\boldsymbol{\alpha}$, the volume generated by element $\boldsymbol{\delta A}$is $\boldsymbol{\gamma . y\delta A}$.Hence the volume generated by the whole area A is $\int \boldsymbol{\gamma y\delta A} = \gamma \frac{\int y\delta A}{\int \delta A} \int \boldsymbol{\delta A}$

Where the integration is taken over the whole area A. $= \boldsymbol{\gamma \bar{y} A}$, where $\bar{\boldsymbol{y}}$ is the distance of the centroid of the area inclosed by C from OX.

V =length of the path of the centroid × area enclosed by the curve.

12.9.2 THE THEOREM OF PAPPUS FOR THE SURFACE

If a plane arc revolves through any angle about an axis in its plane which the arc does not intersect, then the area of the surface generated is equal to the product of the length of arc and the length of path of its centroid.

Proof:-

Let OX be the axis of rotation and AB be an arc of lengths of the given curve.

Let $P(x,y)$ and $Q(\boldsymbol{x+\delta x, y+\delta y})$ be any two neighbouring points the curve. Let arc $\boldsymbol{PQ = \delta s}$.

If the arc AB revolves about OX through an angle $\boldsymbol{\gamma}$then the area of the surface generated by the element PZ is $\boldsymbol{\gamma y \delta x}$ of the surface

Hence the area of the surface generated by the whole arc AB

$$= \int \boldsymbol{\gamma y \delta s}$$

$$\gamma . \frac{\int \boldsymbol{yds}}{\int \boldsymbol{ds}} . \int \boldsymbol{ds}$$

where the integration is taken over the arc AB.

$= \boldsymbol{\gamma . \bar{y} s}$ where $\bar{\boldsymbol{y}}$ is the distance of the centroid of the arc AB from OX.

$\int S$ = length of the path of the centroid × length of arc AB

Note. The closed curves or arc in these theorems should not cross the axis of revolution.

Illustrative Examples

Ex. 24 Find the volume and surface area of the anchor ring generated by the revolving of a circle of radius a about an axis in its own plane distance b from the centre $(b > a)$

Sol. The centroid of the area of a circle and its circumference are both

at the centre c. By the theorem of Pappus for the volume, the required volume of the anchor ring

= Area of Circle × length of the path of its centroid

$$V = (\pi a^2).\,2\pi b$$

i.e. $V = 2\pi^2 a^2 b$

Further, by the theorem of Pappus for the surface, the required surface area of the anchor ring.

= (circumference of the circle ×length of the path of its centroid)

$= 2\pi a.\,2\pi b = 4\pi^2 ab$

$= 4\pi^2 ab$

Ex. 25 Find the volume of the ring generated by the revolution of the cardioid $r = a(1 + cos\theta)$ about the line $r\cos\theta + a = 0$, given that the centroid of the cardioids is at a distance $\frac{5a}{6}$ from the pole.

Sol. We known that are of the cardioids is $\frac{3}{2}\pi a^2$

The given line of revolution is $r\cos\theta + a = 0$ i.e, $x + a = 0 or x = -a$

Let G be the centoid of the area of the cardioids.

It is given that $OG = \frac{5}{6}a.$ Thus the length of the perpendicular from the G on the line of revolution is

$$GN = a + \frac{5}{6}a = \frac{11}{6}a$$

The circumference of the circle $= 2\pi GN$

$$= 2\pi \times \frac{11}{6}a = \frac{11}{3}\pi a$$

Hence the required volume is

V = (Area of the cardioids) × (circumference of the circle generated by its centroid)

$$= \frac{3}{2}\pi a^2 \times \frac{11}{3}\pi a$$

$$V = \frac{11}{2}\pi^2 a^3$$

12.10 SUMMARY

In this unit following topic were discussed:

1. The volume of the solid generated by the revolution of the arc bounded by the curve $y = f(x)$, the axis and the ordinates $x = a$ and $x = b$ is

$$V = \pi \int_{x=a}^{b} y^2\, dx$$

2. The volume of the solid generated by the revolution of area bounded by the curve $x = f(y)$the $y - axis$and the abscissa$y = a$ and $y = b$ is

$$V = \pi \int_{y=a}^{b} x^2\, dy$$

3. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radiivectors$\theta = \alpha$and $\theta = \beta$.

   (a) About the initial line $\theta = 0$ is $V = \frac{2}{3}\pi \int_{\alpha}^{\beta} r^3 \sin\theta\, d\theta$

   (ii) About the line $\theta = \frac{\pi}{2}$ is $V = \frac{2}{3}\pi \int_{\alpha}^{\beta} r^3 \cos\theta\, d\theta$

4. The area of the surface generated by revolving about the x-axis an area bounded by the curve $y = f(x)$, the $x -$ axis and the line $x = a$ and $x = b$ is

$s = 2\pi \int_{x=a}^{b} y\, ds = 2\pi \int_{x=a}^{b} y\, \frac{ds}{dx} dx$, where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

5. The area of the surface generated by revolving about the y-axis an area bounded by the curve$x = f(y)$, the $y -$axis and the two abscissa $y = a$ and $y = b$ is

$$S = 2\pi \int_{x=a}^{b} x ds = 2\pi \int_{y=a}^{b} x.\frac{ds}{dy} dy, \qquad where\ \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$for\ the\ polar\ curves\ r = f(\theta)$

6. For the polar curves $r = f(\theta)$The surface generated by the revolution about the initial line of the arc intercepted between the radii vectors $\theta = \gamma\ and\ \theta = \beta$ is

$$S = 2\pi \int_{\theta=\gamma}^{\beta} y ds = 2\pi \int_{\theta=\gamma}^{\beta} (r sin\theta) \frac{ds}{d\theta} d\theta$$

where $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

7. If a plane closed curve revolves through any angle about an axis in its plane which the curve does not intersect, then the volume generated is equal to the product of the area of the closed curve and the length of the path of its centroid.(pappus theorem for volume)
8. If a plane arc revolves through any angle about an axis in its plane which the arc does not intersect, then the area of the surface generated is equal to the product of the length of the arc and the length of the path of its centroid.

12.11 GLOSSARY

i. **Circumference**-the perimeter of a circle or ellipse
ii. **Volume**-quantity of three-dimensional space enclosed by a closed surface.
iii. **Cardioid-** a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius

CHECK YOUR PROGRESS

Fill in the blanks in the following:

1. The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$ about x-axis is $\int_a^b \ldots\ldots dx$.
2. The volume of a hemisphere of radius b is……………..
3. The curved surface of a hemisphere of radius a is……………….
4. The area of the surface of revolution formed by revolving the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is (…………….)πa^2.

12.12 REFERENCES

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ii. Gorakh Prasad, “Text-Book on Differential Calculus”, Pothishala Private Ltd., Allahabad; 1936.
iii. Tom M. Apostol, “Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra”, John Wiley & Sons; 1967.
iv. Hari Kishan, R. K. Shrivastav, “Calculus”, Ram Prasad and Sons, Bhopal; 2004-05.

12.13SUGGESTED READINGS

i. Differential Calculus for Beginners by Joseph Edwards.
ii. Text-Book on Differential Calculus by Gorakh Prasad.
iii. Calculus by R. Kumar.
iv. Krishna's Text Book on Calculus by A. R. Vasistha.
v. Pragati's Calculus by Sudhir K. Pundir.

12.14TERMINAL QUESTIONS

EXERCISE

1. Find the volume of a hemisphere of radius 'a'.
2. Find the volume of the right circular cone of height h and base of radius a.
3. Find the volume of the solid generated by the revolution of the curve $y = \frac{a^3}{a^2+x^2}$ about its asymptote.
4. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2(2a-x)(a-x)$ about the axis.
5. Find the volume of the solid formed by revolving one loop of the curve $r^2 = a^2cos2\theta$ about the initial line.
6. Find the surface area of the solid by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.
7. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - sin\theta), y = a(1 - cos\theta)$ about the x-axis.
8. The area of the cardioids $x = a(1 + cos\theta)$included between- $\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, is rotated about the line $\theta = \frac{\pi}{2}$. Find the area of surface generated.
9. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y = a$, find the volume of the solid generated.
10. Find the curved surface of a hemisphere of radius a.

OBJECTIVE TYPE QUESTIONS

Multiple choice type questions:

11. Curved surface of a hemisphere of radius a is

a)πa^2b)$2\pi a^2$c)$\frac{4}{3}\pi a^2$d)$5\pi a^2$

12. The volume of the solid generated by revolving the cardioids $r = a(a + cos\theta)$ about the initial line is

a)$\frac{8}{3}\pi a^2$ b)$\frac{4}{3}\pi a^2$ c)$\frac{2}{3}\pi a^2$d) none of these

13. The area of the surface of revolution formed by revolving the curve $\boldsymbol{r = 2a\cos\theta}$ about the initial line is

a) $8\pi a^2$ b) $6\pi a^2$ c) $4\pi a^2$ d) $2\pi a^2$

14. The surface of the solid generated by the revolution about x −axis of the area between the curve $y = f(x)$, the x −axis and the ordinates $x = a$ and $x = b$ is

a) $\int_{x=0}^{b} 2\pi y ds$ b) $\int_{x=a}^{b} 2\pi y dx$ c) $\int_{x=a}^{b} 2\pi x ds$ d) $\int_{x=0}^{b} ds$

15. The volume of the solid generated by the revolution of the loop of the curve $y^2 = x^2(2 - x)$ about x −axis

a) $\frac{8\pi}{3}$ b) 2π d) $\frac{4\pi}{3}$ d) $\frac{2\pi}{3}$

True or false type questions

Write “T” or “F” according as the following statement is True or false:

16. The volume generated by the revolution of the area lying between the curve $\boldsymbol{x = f(y)}$, the y −axis and the line $y = a$ and $y = b$ about y −axis is given by $\boldsymbol{\int_a^b \pi x^2 dy}$

17. The oblate spheroid is generated by revolving the ellipse $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ about its major axis.

18. The surface area of the segment of a sphere of radius a and height h is $\boldsymbol{2\pi a^2 h}$

19. The area of the surface of the solid formed by the revolution of the cardioid $\boldsymbol{r = a(1 + \cos\theta)}$ about the initial line is $\boldsymbol{32\pi a^2}$

12.15ANSWERS

Check your Problem:

CHQ1. y^2
CHQ2. $\frac{2}{3}\pi b^3$
CHQ3. $2\pi a^2$
CHQ4. $\frac{12}{5}$

Terminal questions:

(TQ-1) $\frac{2}{3}\pi a^3$
(TQ-2) $\frac{1}{3}\pi^2 h$

(TQ-3) $\frac{1}{2}\pi^2 a^3$

(TQ-4) $\frac{23}{60}\pi a^3$

(TQ-5) $\frac{\sqrt{2}}{24}\pi a^3\{3\log(1+\sqrt{2})-\sqrt{2}\}$

(TQ-6) $32\pi\{1+\log(2+\sqrt{3})\}$

(TQ-7) $\frac{64}{3}\pi a^2$

(TQ-8) $\frac{48\sqrt{2}}{5}\pi a^2$

(TQ-9) $\frac{8\sqrt{2}}{15}\pi a^3$

(TQ-10) $2\pi a^2$

(TQ-11) b

(TQ-12) a

(TQ-13) c

(TQ-14) a

(TQ-15) c

(TQ-16) T

(TQ-17) F

(TQ-18) F

(TQ-19) F

UNIT:-13 BETA AND GAMMA FUNCTIONS

CONTENTS

13.1 INTRODUCTION

As introduced by the Swiss mathematician Leonhard Euler in18th century, gamma function is the extension of factorial function to real numbers. Beta function (also known as Euler's integral of the first kind) is closely connected to gamma function; which itself is a generalization of the factorial function.

The definite Integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m,n > 0$…....(11.1.1) is known as Beta function and denoted by$B(m,n)$ [read as "Beta m,n"]
where$m$ and n are positive integer or fraction. The Beta function is also known as Eularian integral of first kind. The integral is convergent if and only if $m,n > 0$
Thus $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, for\ m,n > 0$
The improper integral $\int_0^\infty e^{-x}x^{n-1}dxforn > 0$……….(11.1.2) where n is a positive real number, is known as Gamma function, and denoted by Γn (Read as 'Gamma n").The gamma function is also known as Eulerian integral of second kind. The integral is convergent for $n > 0$.Thus,$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$

13.2 OBJECTIVES

In this unit, we shall understand

- **i.** Beta functions and its properties
- **ii.** Gamma functions and its properties
- **iii.** Relation between Beta and Gamma functions
- **iv.** Important results from Beta and Gamma functions

13.3 SYMMETRICAL PROPERTY OF BETA FUNCTION

Theorem 1.To show that $B(m,n) = B(n,m)$

Proof:

We know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m,n > 0$…(11.3.1)

$= \int_0^1 (1-x)(1-x)^{m-1}[1-(1-x)]^{n-1}dx$

[Since $\int_0^a f(xdx) = \int_0^a f\,(a-x)dx$]

$$= \int_0^1 (1-x)^{m-1}x^{n-1}dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$= B(n\,, m)$[By the definition of Beta function]

This shows that Beta function is symmetrical in $m$and n.

13.4 EVALUATION OF BETA FUNCTION

We know that

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m,n > 0$………………(11.3.1)

Then the three following cases arises:

Case 1. When n is a positive integer.

Integrating by parts, taking $(1-x)^{n-1}$ as the first function, we have

$$B(m,n) = \int_0^1 (1-x)^{n-1}x^{m-1}dx$$

$$= \left[(1-x)^{n-1}.\frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1)\frac{x^m}{m}dx$$

$$= 0 + \frac{(n-1)}{m}\int_0^1 x^m(1-x)^{n-2}dx$$

Again integrating by parts,

$$B(m,n)=\frac{(n-1)(n-2)}{m(m+1)}\int_0^1 x^{m+1}(1-x)^{n-3}dx$$

Repeating this process again and again, we get

$$B(m,n)=\frac{(n-1)(n-2)(n-3)\ldots\ldots 3.2.1}{m(m+1)(m+2)\ldots\ldots(m+n-2)}\int_0^1 x^{m+n-2}dx$$

$$B(m,n)=\frac{(n-1)(n-2)\ldots\ldots\ldots\ldots 3.2.1}{m(m+1)\ldots\ldots\ldots\ldots(m+n-2)}\left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1$$

$$B(m,n)=\frac{(n-1)(n-2)\ldots.3.2.1)}{m(m+1)\ldots\ldots\ldots\ldots(m+n-1)}.1$$

$$B(m,n)=\frac{\Gamma(\mathrm{n}-1)}{\mathrm{m}(\mathrm{m}+1)\ldots\ldots(\mathrm{m}+\mathrm{n}-1)}$$

Case 2.When m is a positive integer.

Since Beta function is symmetrical in m and n i.e.$B(m,n)=B(n,m)$.

we have $B(m,n)=\frac{\Gamma(\mathrm{m}-1)}{n(n+1)(n+2)\ldots.(n+m-1)}$

(by interchanging m and n in equation)

Case 3.When both m and n are positive integers.

Let us take of case I

$$B(m,n)=\frac{\Gamma(\mathrm{n}-1)}{\mathrm{m}(\mathrm{m}+1)\ldots\ldots(\mathrm{m}+\mathrm{n}-1)}$$

Multiplying the numerator and denominator by$\Gamma(\mathrm{m}-1)$

$$B(m,n)=\frac{\Gamma(\mathrm{n}-1)\,\Gamma(\mathrm{m}-1)}{\Gamma\mathrm{m}-1\,.\,\mathrm{m}(\mathrm{m}+1)\ldots\ldots\ldots(\mathrm{m}+\mathrm{n}-1)}$$

Therefore,$B(m,n)=\frac{\Gamma(\mathrm{m}-1)\,\Gamma(\mathrm{n}-1)}{\Gamma(\mathrm{m}+\mathrm{n}-1)}$

13.5 ANOTHER FORM OF BETA FUNCTION

Form 1:Show that

$$\boldsymbol{B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx \quad m,n} > 0$$

Proof. We know that,

$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ ……………….(13.5.1)

Put $x=\frac{1}{1+y}\Rightarrow 1-x=1-\frac{1}{1+y}=\frac{y}{1+y}$and

$dx=-\frac{1}{(1+y)^2}dy$When $x=0\Rightarrow\frac{1}{1+y}=0\Rightarrow 1+y=\infty\Rightarrow$ $y=\infty\ and$

When $x=1\Rightarrow\frac{1}{1+y}=1\Rightarrow 1+y=1\Rightarrow y=0$

Putting all these value in (11.4.1), we get

$$B(m,n)=\int_\infty^0\left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$\Rightarrow B(m,n) = -\int_{\infty}^{0} \frac{y^{n-1}}{(1+y)m+n} dy$$

$$\Rightarrow B(m,n) = \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy \left[\int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

$$\Rightarrow B(m,n) = \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx \qquad As\left[\int_a^b f(x)dx = \int_b^a f(t)dt\right]$$

$\Rightarrow B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ [As $B(m,n) = B(n,m)$]

Form 2.Show that

$$\boldsymbol{B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx}$$

Proof. We know that

$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ [From Form1]

By using the property of definite integral

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx; a \le c \le b$, we get

$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)m+n} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ ……(11.4.2)

Now solving $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$……………………(11.4.3)

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

When $x \to 1$ then $t \to 1$ and $x \to \infty$ then $t \to 0$

Putting these values in (11.4.3), we get

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt$$

$$= \int_0^1 -\frac{t^{m+n}}{(1+t)^{m+n}.t^{m+1}} dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt$$

$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$ [Because $\int_a^b f(t)dt = \int_a^b f(x)dx$]

Now putting the value of $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$ in equation (11.4.2), we get

$$B(m,n) = \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Hence, $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

Form 3. Show that

$$\boldsymbol{B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}dx}{(ax+b)^{m+n}}}$$

Proof. We know that,

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Putting $y = \frac{ax}{b} \Rightarrow dy = \frac{a}{b} dx$

When $y \to 0$then $x \to 0$and $y \to \infty$ then $x \to \infty$, we get

$$B(m,n) = \int_0^\infty \frac{(\frac{a}{b}x)^{m-1}}{\left(1 + \frac{ax}{b}\right)^{m+n}} dx$$

$$= a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$$

Form 4. Show that.

$$\boldsymbol{\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}B(m,n)}$$

Proof. We know that

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$……………………….(11.4.4)

Putting $x = \frac{y-b}{a-b} \Rightarrow dx = \frac{dy}{a-b}$

When $x \to 0$ then$y \to b$ and $x \to 1$ then $y \to a$

Substituting these values in equation 11.4.4 we get.

$$B(m,n) = \int_0^1 \left(\frac{y-b}{a-b}\right)^{m-1} \left(\frac{a-y}{a-b}\right)^{n-1} . \frac{dy}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (y-b)^{m-1} (a-y)^{n-1} dy$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx \qquad [\int_a^b f(x)dx = \int_a^b f(t)dt]$$

Hence, $\int_b^a (x-b)^{m-1} (a-x)^{n-1} dx = (a-b)^{m+n-1} B(m,n)$

13.6 PROPERTIES OF GAMMA FUNCTION

Show that

i) $\boldsymbol{\Gamma(n+1) = n\,\Gamma(n)}$ **when** $\boldsymbol{n} > 0$

ii) $\boldsymbol{\Gamma(n) = (n-1)!}$ **where n is a positive integer.**

Proof. (i) We know that

$\Gamma n = \int_0^\infty e^{-x}.e^{n-1}dx, n$...……..…(11.5.1)

$$\Gamma(n+1) = \int_0^\infty e^{-x}.x^{(n+1)-1}dx$$

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x}x^n dx$$

Integrating by parts taking x^n as first function we get

$\Gamma n + 1 = [x^n(-e^x)]_0^\infty - \int_0^\infty nx^{n-1}(-e^{-x})dx$…………………….…..…(11.5.2)

Now, $\lim_{x\to\infty}[e^{-x}x^n] = \lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^r}{r!}+\cdots\frac{x^n}{n!}+\cdots}$

Dividing numerator and denominator with x^n,we get

$\lim_{x\to\infty}[e^{-x}x^n] = \lim_{x\to\infty}\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}2!}+\cdots+\frac{1}{x^{n-r}r!}+\cdots+\frac{1}{n!}+\frac{x}{n+1!}+\cdots} = 0$……(11.5.3)

Using value of equation (11.5.3) in equation (11.5.2), we get

$$\Gamma n + 1 = 0 + n\int_0^\infty e^{-x}x^{n-1}dx$$

$\Gamma n + 1 = n\Gamma n$ which proves the result (i)

This relation is known as **reduction formula**.

(ii) We know that $\Gamma n + 1 = n\Gamma n$

Replacing n by $n-1$, we get

$$\Gamma n = (n-1)\Gamma n - 1$$

Similarly, $\Gamma n - 1 = (n-2)\Gamma n - 2, \Gamma n - 2 = (n-3)\Gamma n - 3$ etc.

Hence if n is a positive integer, then proceeding as above, we get

$$\Gamma n = (n-1)(n-2)(n-3)\ldots.3.2.1.\Gamma 1$$

Now $\Gamma 1 = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = [-e^{-x}]_0^\infty = 1$

Hence $\Gamma n = (n-1)(n-2)(n-3)\ldots..3.2.1 = (n-1)!$

It may be remember that $\Gamma 0 = \infty$ and $\Gamma(-n) = \infty$ where n is a positive integer.

13.7 ANOTHER FORMS OF GAMMA FUNCTION.

Form 1.Show that

$$\frac{\boldsymbol{\Gamma n}}{\boldsymbol{c^n}} = \int_0^\infty \boldsymbol{e^{-cy} y^{n-1} dy}$$

Proof. We have $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$……………………………………………(13.7.1)

Put $x = cy$ so that $dx = cdy$; when $x = 0$ then $y = 0$ and when $x = \infty$ then $y = \infty$

$$\Gamma n = \int_0^\infty e^{-cy} (cy)^{n-1} cdy$$

$$= \int_0^\infty e^{-cy} c^n y^{n-1} dy$$

Hence, $\int_0^\infty e^{-cy} y^{n-1} dy = \frac{\Gamma n}{c^n}$

Form 2.Show that

$$\boldsymbol{\Gamma n = \int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy}$$

Proof. We know that $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$…………………………………………..(13.7.2)

Put $e^{-x} = y \Rightarrow x = \log\frac{1}{y}$

Differentiate both side w.r.t. x, we get

$-e^{-x} dx = dy$; when $x = 0$ then $y = 1$ and when $x = \infty$ then $y = 0$

Substitute these values in equation (11.6.2), we get

$$\Gamma n = \int_1^0 -\left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy$$

Hence, $\Gamma n = \int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy$

Form 3.Show that

$$\int_0^\infty \mathbf{e}^{-(y)^{\mathbf{1}/\mathbf{n}}} \mathbf{dy} = \boldsymbol{\Gamma}\mathbf{n} + \mathbf{1}$$

Proof. We know that

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\text{Put } x^n = y \Rightarrow x = y^{\frac{1}{n}}$$

Differentiate both side w.r.t. x, we get

$n x^{n-1} dx = dy$; when $x = 0$ then $y = 0$ and when $x = \infty$ then $y = \infty$, we get

$$\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} \frac{dy}{n}$$

$$\Rightarrow n\Gamma n = \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

$$\Rightarrow \Gamma n + 1 = \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

13.8 RELATION BETWEEN BETA AND GAMMA FUNCTIONS.

1. Show that

$$\mathbf{B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}, m > 0, n > 0}$$

Proof. We know that from the transformation of Gamma function

$\Gamma m = z^m \int_0^\infty e^{-zx} x^{m-1} dx$ ………………………………(13.8.1)

$$= \int_0^\infty z^m e^{-zx} x^{m-1} dx$$

Multiplying both the sides by$e^{-z} z^{n-1}$, we get

$\Gamma m \, e^{-z} z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} dx$ ……………(13.8.2)

Now integrating both the sides of equation (13.8.2) with respect to z from 0 to ∞, we get

$$\Gamma m \int_0^\infty e^{-z} z^{n-1} dx = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] x^{m-1} dx$$

$\Rightarrow \Gamma m \, \Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx$(from transformation of Gamma function)

$$= \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$$

$\Rightarrow \Gamma m \, \Gamma n = \Gamma(m+n) \, B(m,n)$ (fromTransormation of Beta function)

Hence, $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

2. Show that

$$\int_0^{\pi/2} \mathbf{\sin^{2n-1} \cos^{2n-1} \theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma m + \Gamma n}, m > 0, n > 0}$$

Proof. We know that

$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ ……………………(13.8.3)

$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m + n}$ ……………………………(13.8.4)

From equations (13.8.3) and (11.8.4), we get

$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$…………………………(13.8.5)

Let $x = \sin^2\theta$ so that $dx = 2 \sin\theta \cos\theta \, d\theta$

When $x \to 0$ then $\theta \to 0$ and when $x \to 1$ then $\theta \to \frac{\pi}{2}$

Therefore,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$=\int_0^{\pi/2} \sin^{2m-2}\theta(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

$=$

$2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$……………………….(13.8.6)

From equation(13.8.5) and equation(13.8.6) , we get

$$2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\frac{\Gamma m\,\Gamma n}{2\,\Gamma(m+n)}$$

Substituting $2m-1=p$and $2n-1=q$, the result can also be put in the form. $\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

13.9 SOME IMPORTANT RESULTS FROM BETA AND GAMMA FUNCTIONS

1. Show that$\Gamma(\mathbf{n})\,\Gamma(\mathbf{1-n})=\frac{\pi}{\sin n\pi}$, where $0<n<1$

Proof. We have $B(m,n)=$

$\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$…………………….………………………(13.9.1)

and$B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, m>0, n>0$ …………….(13.9.2)

Now, from equations (11.9.1) and (11.9.2), we get

$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$ …………..…………….(13.9.3)

Putting $m+n=1$ or $m=1-n$ in (13.9.3), we get

$$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma 1}=\int_0^\infty\frac{x^{n-1}}{1+x}dx \text{ where } 0<n<1$$

But we know that$\Gamma 1=1$ and$\int_0^\infty\frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin n\pi}$where $0<n<1$

Hence$\Gamma(n)\,\Gamma(n-1)=\frac{\pi}{\sin n\pi}, 0<n<1$

2. Show that$\Gamma\frac{1}{2}=\sqrt{\pi}$

Proof. We know that B(m,n)=$\int_0^1 x^{m-1}(1-x)^{n-1}dx$…(13.9.4)

$B(m,n)=\frac{\Gamma m\Gamma n}{\Gamma m+n}$…………………………(13.9.5)

Now, from equations (13.9.4) and (13.9.5), we get

$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ …….(13.9.6)

Putting $m=n=\frac{1}{2}$in (13.9.5), we get

$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 1}=\left(\Gamma\frac{1}{2}\right)^2$ (because $\Gamma 1=1$).........(13.8.7)

Now, Putting $m=n=\frac{1}{2}$in (13.8.4), we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx$$

$$\Rightarrow \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

Now, put $x = \sin^2\theta$ so $dx = 2\sin\theta\cos\theta\,d\theta$

Also when $x \to 0$ then $\theta \to 0$ and when $x \to 1$, $\theta \to \frac{\pi}{2}$, we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\pi/2} \frac{1}{\sin\theta}.\frac{1}{\cos\theta} 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$$

Substituting the value of $B\left(\frac{1}{2},\frac{1}{2}\right)$in equation (13.8.7), we get

$$\left(\Gamma\frac{1}{2}\right)^2 = \pi$$

Hence, $\Gamma\frac{1}{2} = \sqrt{\pi}$

3. Showthat

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Proof. Let $A = \int_0^\infty e^{-x^2}dx$

Let$x^2 = t \Rightarrow 2xdx = dt \Rightarrow dx = \frac{1}{2^x}dt = \frac{1}{2}t^{-1/2}dt$…(13.8.8)

Also when $x \to 0$ then $t \to 0$ and when $x \to \infty$ then $t \to \infty$

Putting these values in equation (13.8.8), we get

$$A = \int_0^\infty e^{-t}\frac{1}{2}t^{-1/2}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}t^{1/2-1}dt$$

$$= \frac{1}{2}\,\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Hence, $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$

4. Show that

$$\mathbf{B(n,n)} = \frac{\sqrt{\pi}\,\Gamma n}{2^{2n-1}\,\Gamma n + \frac{1}{2}}$$

Proof. We know that $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$…(13.8.9)

Also, $B(n,n) = \int_0^1 x^{n-1}(1-x)^{n-1}dx$ ………(13.8.10)

Let $x = sin^2\theta \Rightarrow dx = 2\,sin\,\theta\,cos\,\theta\,d\theta$

When $x \to 0$ then $\theta \to 0$ and when $x \to 1$ then $\theta \to \frac{\pi}{2}$

Using these values in equation (13.8.10), we get

$$B(n,m) = \int_0^{\pi/2} 2.sin^{2n-1}\theta\;cos^{2n-1}\theta\;d\theta$$

$$= \int_0^{\pi/2} \frac{2(2\sin\theta\cos\theta)^{2n-1}}{2^{2n-1}} d\theta$$

$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta$……………..(13.8.11)

Let $2\theta = t \Rightarrow d\theta = \frac{dt}{2}$,also when $\theta \to 0$then$t \to 0$and when $\theta \to \frac{\pi}{2}$then$t \to \pi$. Using these values in equation (13.8.11), we get when

$$B(n,n) = \frac{1}{2^{2n-1}} \int_0^{\pi} \sin^{2n-1} t dt$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} t dt$$

$\{\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$)when$f(2a-x) = f(x)\}$

$$= \frac{2}{2^{2n-1}} \cdot \frac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(n+\frac{1}{2}\right)} \left[\because \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

Hence,$B(n,n) = \frac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$

5. Legendre-Duplication formula

Show that

$$\boldsymbol{\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n), n} > 0$$

Proof. From the above article, we have

$$B(n,n) = \frac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$$

$= \frac{\sqrt{\pi}}{2^{2n-1}} \frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$…………(13.8.12)

and$B(n,n) = \frac{\Gamma(n)\,\Gamma(n)}{2\,\Gamma(n+n)}$ $\left[B(m,n) = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(m+n)}\right]$……...(13.8.13)

From (13.8.12) and (13.8.13) we have

$$\frac{\Gamma n\,\Gamma n}{\Gamma 2n} = \frac{\sqrt{\pi}}{2^{2n-1}} \frac{\Gamma n}{\Gamma n + \frac{1}{2}}$$

Hence $\Gamma n\,\Gamma n + \frac{1}{2} = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma 2n$

6. Show that

$\boldsymbol{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\ldots\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{2^{\frac{n-1}{2}}\pi^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}}$, **where n is a positive integer**.

Proof. Let $A=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\ldots\ldots\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$

$=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\ldots\ldots\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$…….(13.8.14)

Writing in the reverse order we get.

$A=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)$………..(13.8.15)

Multiplying (11.8.14) and (11.8.15), we get

$$A^2=\left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right]\ldots\ldots\ldots\left[\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(\frac{2}{n}\right)\right]\left[\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right]$$

$$=\frac{\pi}{\sin\frac{\pi}{n}}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}\ldots\ldots\ldots\ldots\frac{\pi}{\sin\frac{(n-2)\pi}{n}}\cdot\frac{\pi}{\sin\frac{(n-1)\pi}{n}}$$

$\left[\because \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}\text{for } 0<n<1\right]$

Therefore, $A^2=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\sin\frac{(n-2)\pi}{n}\sin\frac{(n-1)\pi}{n}}$………………(13.8.16)

Now we use the following trigonometric identity

$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots\ldots\ldots\ldots$(13.8.17)

Taking limit as $\theta\to 0$in equation (11.8.17), we get

$$\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta}=\lim_{\theta\to 0}\left\{2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots\ldots\sin\left(\theta+\left(\frac{n-1}{n}\right)\pi\right)\right\}$$

$$\Rightarrow\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta}=\lim_{\theta\to 0}\left\{2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\ldots\sin\left(\frac{n-1}{n}\right)\pi\right\}$$

$$\lim_{\theta\to 0}\left\{2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\ldots\sin\left(\frac{n-1}{n}\right)\pi\right\}=\lim\Rightarrow\frac{\sin n\theta}{\sin\theta}$$

$$=\lim_{\theta\to 0}n\frac{\frac{\sin n\theta}{n\theta}}{\frac{\sin\theta}{\theta}}$$

$$=n\frac{\lim_{\theta\to 0}\frac{\sin n\theta}{n\theta}}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=n$$

Therefore, $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\ldots\sin\left(\frac{n-1}{n}\right)\pi=\frac{n}{2^{n-1}}$ ….(13.8.18)

Using equation (13.8.18) in equation (13.8.16), we get

$$A^2 = \frac{\pi^{n-1}}{\frac{n}{2^{n-1}}} = \frac{2^{n-1}\pi^{n-1}}{n} \Rightarrow A = \frac{2^{\frac{n-1}{2}}\pi^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$$

7. Show that

(a) $\int_0^\infty e^{-ax}\cos bx . x^{m-1}dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}}\cos m\theta$

(b) $\int_0^\infty e^{-ax}\sin bx . x^{m-1}dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}}\sin m\theta$

Proof. We know that

$$\int_0^\infty e^{-px}x^{m-1}dx = \frac{\Gamma(m)}{p^m} \qquad m > 0, p > 0$$

put $p = a - ib$, we get

$$\int_0^\infty e^{-(a-ib)x}.x^{m-1}dx = \frac{\Gamma m}{(a-ib)^m}$$

$$\text{or } \int_0^\infty e^{-ax}e^{ibx}x^{m-1}dx = \frac{\Gamma m[a+ib]^m}{[(a-ib)(a+ib)]^m}$$

$$\text{or } \int_0^\infty e^{-ax}.e^{ibx}x^{m-1}dx = \frac{\Gamma m(a+ib)^m}{(a^2+b^2)^m} \ldots\ldots\ldots\ldots\ldots..(2)$$

$$[e^{ix} = \cos\theta + i\sin\theta]$$

Let $a + ib = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts we get

$$r\cos\theta = a \ldots\ldots\ldots.(3)$$

$$r\sin\theta = b \ldots\ldots\ldots.(4)$$

Squaring and adding (3) and (4), we get

$$r^2 = a^2 + b^2 \Rightarrow r \pm (a^2+b^2)^{1/2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Dividing (4) by (3), we get

$$\tan\theta = b/a \Rightarrow \theta = \tan^{-1}(b/a) \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now $(a+ib)^m = [r(\cos\theta + i\sin\theta)]^m$

$r^m[\cos m\theta + i\sin m\theta] \ldots\ldots\ldots\ldots \ldots.$ [By De'Moirer's theorem]

Substituting these values in equation (2)

$$\int_0^\infty e^{-ax}[\cos bx + i\sin bx]x^{m-1}dx = \frac{\Gamma m r^m[\cos m\theta + i\sin m\theta]}{(a^2+b^2)^m}$$

Equating real and imaginary parts on both the sides.

$$\int_0^\infty e^{-ax}\cos bx . x^{m-1}dx = \frac{\Gamma m. r^m\cos m\theta}{(a^2+b^2)^m}$$

$$\int_0^\infty e^{-ax}\sin bx. x^{m-1}dx = \frac{\Gamma m. r^m\sin m\theta}{(a^2+b^2)^m}$$

$$\int_0^\infty e^{-ax}\cos bx . x^{m-1}dx = \frac{\Gamma m(a^2+b^2)^{m/2}\sin m\theta}{(a^2+b^2)^m}$$

$$\int_0^\infty e^{-ax}\sin bx . x^{m-1}dx = \Gamma m.\frac{(a^2+b^2)^{m/2}\sin m\theta}{(a^2+b^2)^m}$$

$$\text{Hence.} \int_0^\infty e^{-ax}\cos bx . x^{m-1}dx = \frac{\Gamma m\cos m\theta}{(a^2+b^2)^{m/2}} \ldots\ldots\ldots\ldots(7)$$

$$\int_0^\infty e^{-ax}\sin bx\,.\,x^{m-1}dx = \frac{\Gamma m \sin m\theta}{(a^2+b^2)^{m/2}} \ldots\ldots\ldots\ldots\ldots (8)$$

Where $\theta = \tan^{-1}\left({}^{b}/_{a}\right)$

Deduction (i) Let $a = 0$, then from (5)and (6)we get

$$r = b \text{ and } \theta = \tan^{-1}\frac{b}{0} = \tan^{-}\infty = {}^{\pi}/_{2}$$

$$\int_0^\infty x^{m-1}\cos bx dx = \frac{\Gamma m}{b^m}\cos\frac{m\pi}{2} \ldots . (9)$$

$$\int_0^\infty x^{m-1}\sin bx dx = \frac{\Gamma m}{b^m}\sin\frac{m\pi}{2} \ldots\ldots\ldots\ldots (10)$$

(ii) Let $m = 1$, then $\Gamma m = \Gamma n = 1$, then from (7)and (8)we get.

$$\int_0^\infty e^{-ax}\cos bx dx = \frac{\cos\theta}{(a^2+b^2)^{1/2}} \ldots\ldots .. (11)$$

$$\text{and } \int_0^\infty e^{-ax}\sin bx dx = \frac{\sin\theta}{(a^2+b^2)^{1/2}} \ldots . (12)$$

But $\tan\theta = {}^{b}/_{a}$ so $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$

Then from (11) and (12) we get

$$\int_0^\infty e^{-ax}\cos bx dx = \frac{a}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\sin bx dx = \frac{b}{a^2+b^2}$$

8. To show that

$$\int_a^b (\mathbf{x}-\mathbf{a})^{\mathbf{p}}(\mathbf{b}-\mathbf{x})^{\mathbf{q}}\mathbf{dx} = (\mathbf{b}-\mathbf{a})^{\mathbf{p+q+1}}\frac{\mathbf{p}!\,\mathbf{q}!}{(\mathbf{p}+\mathbf{q}+\mathbf{1})!}$$

where p and q are positive integers.

Proof:- Let $A = \int_a^b (x-a)^p(b-x)^q dx$

Putting $x = a\cos^2\theta + b\sin^2\theta$ so that

$$dx = 2a\cos\theta(-\sin\theta)\,d\theta + 2b\sin\theta\cos\theta d\theta$$
$$= 2(b-a)\sin\theta\cos\theta\, d\theta$$

when $x = a$ then $a = a\cos^2\theta + b\sin^2\theta$

$$\Rightarrow a - a\cos^2\theta = b\sin^2\theta$$
$$\Rightarrow\ a - a\cos^2\theta = b\sin^2\theta$$
$$a\sin^2\theta - b\sin^2\theta = 0$$
$$(a-b)\sin^2\theta = 0$$
$$\sin^2\theta = 0 \Rightarrow \theta = 0$$

When $x = b$ then

$$b = a\cos^2\theta + b\sin^2\theta$$
$$b(1-\sin^2\theta) = a\cos^2\theta$$
$$(b-a)\cos^2\theta = 0$$
$$\cos^2\theta = 0\ \Rightarrow \theta = {}^{\pi}/_{2}$$

and $x - a = a\cos^2\theta + b\sin^2\theta - a$

$$= b\sin^2\theta - a(1-\cos^2\theta)$$

$$= \mathrm{b}\sin^2\theta - \mathrm{a}\sin^2\theta$$
$$= (\mathrm{b}-\mathrm{a})\sin^2\theta$$
$$\mathrm{b} - \mathrm{x} = \mathrm{b} - \mathrm{a}\cos^2\theta - \mathrm{b}\sin^2\theta$$
$$= \mathrm{b}(1-\sin^2\theta) - \mathrm{a}\cos^2\theta$$
$$= \mathrm{b}\cos^2\theta - \mathrm{a}\cos^2\theta$$
$$= (\mathrm{b}-\mathrm{a})\cos^2\theta$$

Putting all these values in equation (1) we get

$$\mathrm{A} = \int_0^{\pi/2} (\mathrm{b}-\mathrm{a})^{\mathrm{p}}\sin^{2\mathrm{p}}\theta(\mathrm{b}-\mathrm{a})^{\mathrm{q}}\theta\,\sin^{2\mathrm{q}}\theta\; 2(\mathrm{b}-\mathrm{a})\sin\theta\cos\theta$$
$$= 2(\mathrm{b}-\mathrm{a})^{\mathrm{p+q+1}} \int_0^{\pi/2} \sin^{2\mathrm{p}+1}\theta\cos^{2\mathrm{q}+1}\theta d\theta$$
$$= 2(\mathrm{b}-\mathrm{a})^{\mathrm{p+q+1}}.\frac{\mathrm{p}!\,\mathrm{q}!}{(\mathrm{p}+\mathrm{q}+1)!}\left[\text{from } \int_0^{\pi/2}\sin^{\mathrm{m}}\theta\cos^{\mathrm{n}}\theta d\theta = \frac{\Gamma\frac{\mathrm{m}+1}{2}\Gamma\frac{\mathrm{n}+1}{2}}{2\Gamma\frac{\mathrm{m}+\mathrm{n}+2}{2}}\right]$$
$$= (\mathrm{b}-\mathrm{a})^{\mathrm{p+q+1}}\frac{\mathrm{p}!\,\mathrm{q}!}{(\mathrm{p}+\mathrm{q}+1)!}, \text{if } \mathrm{p}+1 > 0 \; and \; q+1 > 0$$
$$[\text{Since } \Gamma\mathrm{n}+1 = \mathrm{n}!]$$
$$= (\mathrm{b}-\mathrm{a})^{\mathrm{p+q+1}}\frac{\mathrm{p}!\,\mathrm{q}!}{(\mathrm{p}+\mathrm{q}+1)!}, \qquad \mathrm{p} \geq 1 \text{and } \mathrm{q} \geq 1$$

which is true[p and q are positive integers]

Hence$\int_{\mathrm{a}}^{\mathrm{b}}(\mathrm{x}-\mathrm{a})^{\mathrm{b}}(\mathrm{b}-\mathrm{x})^{\mathrm{q}}d\mathrm{x} = (\mathrm{b}-\mathrm{a})^{\mathrm{p+q+1}}\frac{\Gamma\mathrm{p}\Gamma\mathrm{q}}{\Gamma\mathrm{p+q+1}}$where p and q are positive integers.

Illustrative Examples

Ex. 1.Evaluate the integrals

(i)$\int_0^\infty \frac{xdx}{1+x^6}dx(ii)\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}$

Sol. (i):- Let I= $\int_0^\infty \frac{xdx}{1+x^6}$

$Put\; x^6 = y \Rightarrow x = y^{1/6}$so that $dx = \frac{1}{6}y^{-5/6}dy$

$$I = \frac{1}{6}\int_0^\infty \frac{y^{1/6}y^{-5/6}}{1+y}dy$$

$=\frac{1}{6}\int_0^\infty \frac{y}{(1+y)}^{-2/3} dy$

$$= \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{(1+y)}dy$$
$$= \frac{1}{6}\int_0^\infty \frac{y^{\frac{1}{3}-1}}{1+y}dy = \frac{1}{6}\int_0^\infty \frac{y^{\frac{1}{3}-1}}{(1+y)^{1/3+2/3}}dy$$

$= \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right)$ $\left[B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}\right]$

$= \frac{1}{6} \frac{\Gamma \frac{1}{2} \Gamma \frac{2}{3}}{\Gamma \frac{1}{3} + \frac{2}{3}} = \frac{1}{6} \frac{\Gamma \frac{1}{3} \Gamma \frac{2}{3}}{\Gamma 1}$

$$= \frac{1}{6} \Gamma \frac{1}{3} \Gamma 1 - \frac{1}{3} \left[\Gamma n \Gamma 1 - n = \frac{\pi}{sin n\pi}\right]$$

$$= \frac{1}{6} \frac{\pi}{\sqrt{3/2}}$$

$$= \frac{\pi}{3\sqrt{3}}$$

Let $I = \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx$

$$= \int_0^\infty \frac{x^4}{(1+x)^{15}} dx + \int_0^\infty \frac{x^9}{(1+x)^{15}} dx$$

$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^\infty \frac{x^9}{(1+x)^{15}} dx$

$= B(5,10 + B(10,5)$

$= 2B(5,10)[As\ B(m,n) = B(n,m)]$

$= 2 \frac{\Gamma 5 \Gamma 10}{\Gamma 15} = 2. \frac{4.3.2.1 \Gamma 10}{14.13.14.11. \Gamma 10}$

$$= \frac{1}{5005}$$

Ex. 2. Evaluate $\int_0^\infty x^2 e^{-x^2} dx$

Sol. $I = \int_0^\infty x^2 e^{-x^2} dx$

Let $x^2 = t \Rightarrow x = t^{1/2} \Rightarrow dx = \frac{1}{2} t^{1/2 - 1} dx$

When $x = 0, t = 0$ and $x = \infty, t = \infty$, we have

$$I = \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2} \int_0^\infty t e^{-t} t^{-1/2} dt$$

$= \frac{1}{2} \int_0^\infty e^{-t} t^{\frac{3}{2} - 1} dt$

$= \frac{1}{2} \Gamma \frac{3}{2} = \frac{1}{2} \frac{1}{2} \Gamma \frac{1}{2}$

$= \frac{\sqrt{\pi}}{4}$

Ex. 3. Prove that $\int_0^a \frac{1}{(a^n - x^n)^{1/n}} dx = \frac{\pi}{n} cosec \frac{\pi}{n}$

Sol. Let $I = \int_0^a \frac{dx}{(a^n - x^n)^{1/n}}$

Now putting $x^n = a^n t$ or $x = a t^{1/n}$ so that $dx = \frac{1}{n} a t^{1/n - 1} dt$

when $x = 0, t = 0$, when $x = a, t = 1$ we have

$$I = \int_0^1 \frac{1}{(a^n - a^n t)^{1/n}} . \frac{1}{n} a t^{1/n - 1} dt$$

$= \frac{1}{n} . a . \frac{1}{a} \int_0^a (1-t)^{1 - 1/n}\ t^{1/n - 1} dt$

$$= \frac{1}{n} \int_0^a t^{\frac{1}{n} - 1} (1-t)^{1 - \frac{1}{n}} dt$$

$= \frac{1}{n} B\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ [By definition of $B(m,n)$]

$$= \frac{1}{n}\frac{\Gamma\frac{1}{n}\Gamma 1-\frac{1}{n}}{\Gamma\frac{1}{n}+1-\frac{1}{n}}\left[since\ B(m,n) = \frac{\Gamma m\Gamma n}{\Gamma m+n}\right]\text{and}\frac{1}{n}\Gamma\frac{1}{n}\Gamma 1-\frac{1}{n}[\Gamma 1 = 1]$$

$$= \frac{1}{n}\frac{\pi}{Sin\ \frac{\pi}{n}}\left[\Gamma n\ \Gamma 1 - n = \frac{\pi}{Sin n\pi} \qquad 0 < n < 1\right]$$

$$= \frac{\pi}{n}cosec\frac{\pi}{n}$$

Ex. 4. Evaluate $\int_0^2 x(8-x^3)^{1/3}dx$

Sol. Let $I = \int_0^2 x(8-x^3)^{1/3}dx$

Let $x^3 = 8t\ or\ x = 2t^{1/3}$

so that $dx = \frac{2}{3}t^{1/3-1}dt$

when $x = 0, t = 0$ and when $x = 2, t = 1$

then we have

$$I = \int_0^1 2t^{1/3}(8-8t)^{1/3}\frac{2}{3}t^{1/3-1}dt$$

$$=\frac{8}{3}\int_0^1 t^{2/3-1}(1-t)^{4/3-1}dt$$

$$= \frac{8}{3}B\left(\frac{2}{3},\frac{4}{3}\right)$$

$$=\frac{8}{3}\frac{\Gamma\frac{2}{3}\Gamma 4/3}{\Gamma 2/3+4/3}$$

$$=\frac{8}{3}\frac{\Gamma 2/3\Gamma 4/3}{\Gamma 2} = \frac{8}{3}\frac{\Gamma 2/3\Gamma 4/3}{1\Gamma 1}$$

$$= \frac{8}{3}\Gamma\frac{2}{3}\frac{1}{3}\Gamma\frac{1}{3}[\Gamma n = n-1\Gamma n-1]$$

$$= \frac{8}{9}\Gamma 1-\frac{1}{3}\Gamma\frac{1}{3}$$

$$=\frac{8}{9}\frac{\pi}{sin^{\pi}/3} = \frac{8}{9}\frac{\pi}{\sqrt{3/2}}\left[\Gamma n\ \Gamma 1-n = \frac{\pi}{Sin n\pi}\ for\ 0 < n < 1\right]$$

$$= \frac{16}{9}\frac{\pi}{\sqrt{3}}$$

Ex. 5. Express $\Gamma 1/6$ in terms of $\Gamma\frac{1}{3}$

Sol. We know that $\Gamma n\ \Gamma n+\frac{1}{2} = \frac{\sqrt{\pi}}{2n-1}\Gamma 2n \ldots\ldots\ldots\ldots (1)$

[duplication formula]

$\Gamma n\ \Gamma 1-n = \pi/\sin n\pi \quad 0 < n < 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Putting $n = \frac{1}{6}$in(1), we get

$\Gamma\frac{1}{6}\Gamma\frac{1}{6}+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2.\frac{1}{6}-1}}\Gamma 2\frac{1}{6}$

$\Rightarrow \Gamma\frac{1}{6}\Gamma 2/3 = \frac{\sqrt{\pi}}{2^{-2/3}}\Gamma\frac{1}{3}$

Now putting $n = \frac{1}{3}$in(3) we get

$\Gamma\frac{1}{3}\Gamma 1-\frac{1}{3} = \frac{\sqrt{\pi}}{2^{-2/3}} = \frac{\pi}{\sqrt{3/2}}$

$$\Rightarrow\ \Gamma\frac{1}{3}\Gamma\frac{2}{3}=\frac{2\pi}{\sqrt{3}}$$

$$\Rightarrow\ \frac{\Gamma 2}{3}=\frac{2\pi}{\sqrt{3}.\Gamma\frac{1}{3}}$$

Substituting the value of $\Gamma\frac{2}{3}$in (3) we have

$$\Rightarrow \Gamma\frac{1}{6}=\frac{\sqrt{\pi}}{2^{-2/3}}\frac{\Gamma\frac{1}{3}}{\frac{2\pi}{\sqrt{3}\,\Gamma\frac{1}{3}}}$$

$$\Rightarrow \Gamma\frac{1}{6}=\frac{\sqrt{3}}{2^{1/3}}\frac{\left[\Gamma\frac{1}{3}\right]^2}{\sqrt{\pi}}$$

Ex. 6.Find the value of$\Gamma\frac{1}{9}\Gamma\frac{2}{9}\Gamma\frac{3}{9}\ldots\ldots\ldots\ldots.\Gamma\frac{8}{9}$

Sol. We known that

$$\Gamma\frac{1}{n}\Gamma\frac{2}{n}\ldots\Gamma\frac{n-1}{n}=\frac{(2\pi)^{\frac{(n-1)}{2}}}{n^{1/2}}$$

Putting $n=9$

$$\Gamma\frac{1}{9}\Gamma\frac{2}{9}\Gamma\frac{3}{9}\ldots\ldots\ldots\ldots.\Gamma\frac{8}{9}=\frac{(2\pi)^4}{9^{1/2}}=\frac{16\pi^4}{3}$$

Ex. 7.Show that$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x,\quad -1<2x<1$

Sol.$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{2}-x\right)\ \Gamma\left(\frac{1}{2}-x\right).\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right)$

$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right)$

$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}+x\right)\Gamma(1-\left(\frac{1}{2}+x\right))$

$=\left(\frac{1}{4}-x^2\right).\frac{\pi}{\sin\pi\left(\frac{1}{2}+x\right)}\left[\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi},\quad 0<n<1\right]$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\pi\left(\frac{\pi}{2}+x\pi\right)}$$

$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\cos\pi x}$

$=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x$

Hence$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x$

Ex. 8. Show that $\int_0^\infty\cos\left(bz^{\frac{1}{n}}\right)dz=\frac{1}{b^n}\cos\frac{n\pi}{2}\Gamma(n+1)$

Sol. Let $I=\int_0^\infty\cos\left(bz^{\frac{1}{n}}\right)dz$

Put$z^{\frac{1}{n}}=x\ i.e.z=x^n$,so that $dz=nx^{n-1}dx$ when $z=0,x=0$ and when $z=\infty,x=\infty,$ then

$$I = \int_0^\infty \cos bxn \,.\, x^{n-1} dx$$
$$= n \int_0^\infty x^{n-1} \cos bx\, dx$$
$$= n.\frac{\Gamma n}{b^n} cos \frac{n\pi}{2} \qquad [\int_0^\infty x^{m-1} cosbxdx = \frac{\Gamma m}{b^m} cos \frac{m\pi}{2}]$$
$$\Rightarrow \int_0^\infty \cos\left(bz^{\frac{1}{n}}\right) dz = \frac{1}{b^n} \cos\frac{n\pi}{2} \Gamma(n+1)$$

Ex. 9. Prove that $\int_0^\infty \frac{x^c dx}{c^x} = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, \quad c > 1$

Sol. We know that

$$c^x = e^{\log c^x} \quad = e^{x \log c}$$

Now we have $\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \frac{x^c}{e^{x \log c}} dx$

$$= \int_0^\infty x^c . e^{-x \log c} dx$$

Putting $x \log c = t \Rightarrow x = \frac{t}{\log c} \Rightarrow dx = \frac{dt}{\log c,}$

when $x = 0,\ t = 0$ and when $x = \infty, t = \infty,$ then

$$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t} \frac{dt}{\log c}$$
$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^c dt$$
$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^{c+1-1} dt$$

$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}$ [Since$\left(\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx\right)$]

Therefore, $\int_0^\infty \frac{x^c dx}{c^x} = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

13.10 SUMMARY

In this we have explained the following topic:

1. Beta Function $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx. \quad m,n > 0$ and is symmetrical in m and n.
2. Gamma Function $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \qquad n > 0$
3. $\Gamma 1 = 1, \Gamma \frac{1}{2} = \sqrt{\pi}$
4. $B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$
5. $\int_0^\infty sin^p\theta\, cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$
6. $\Gamma n = c^n \int_0^\infty e^{-cx} x^{n-1} dx$ (other forms of Gamma function)
7. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+1}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$(other form of Beta function)

8. $\Gamma n\,\Gamma 1 - n = \frac{\pi}{\text{Sinn}\pi}, 0 < n < 1$
9. Legendre-Duplication formula

$$\Gamma n \Gamma n + \frac{1}{2} = \frac{\sqrt{\pi}}{2^{2n-1}} \Gamma 2n$$

10. If n is an integer then

$$\Gamma\frac{1}{n}.\Gamma\frac{2}{n}.\Gamma\frac{3}{n}\ldots\ldots\ldots\ldots\Gamma\frac{n-2}{n}\Gamma\frac{n-1}{n} = \frac{(2\pi)^{\frac{(n-1)}{2}}}{n^{1/2}}$$

11. $\int_0^\infty e^{-ax}\cos(bx)\,x^{m-1}dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}}\cos m\theta$
12. $\int_0^\infty e^{-ax}\sin(bx)\,x^{m-1}dx = \frac{\Gamma m}{(a^2+b^2)^{m/2}}\cos m\theta$

13.11 GLOSSARY

i. Set.
ii. Function.
iii. Beta Function.
iv. Gamma Function.

Check your Progress

Fill in the blanks

1. For $m, n > 0, \int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}} =$ _______
2.If n ispositive integer then$\Gamma n =$_____
3.For $a > 0, n > 0, \int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma n}{a^n}$
4.$B(m,n) = \frac{\Gamma m\,\Gamma n}{\_\_\_\_}$

Write “T” for true and ‘F’ for false statements.

5.$\Gamma 5 = 120$
6.$\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \pi\sqrt{2}$
7.$\int_0^\infty e^{-x}x^{1/2}dx = \Gamma\frac{1}{2}$
8.$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} = 1$

13.12 REFERENCES

i. Joseph Edwards, “Differential Calculus for Beginners”, Macmillan and Co., Ltd., New York; 1896.
ii. Gorakh Prasad, “Text-Book on Differential Calculus”, Pothishala Private Ltd., Allahabad; 1936.

iii. Tom M. Apostol, “Calculus Volume- 1: One Variable Calculus With An Introduction To Linear Algebra”, John Wiley & Sons; 1967.
iv. Hari Kishan, R. K. Shrivastav, “Calculus”, Ram Prasad and Sons, Bhopal; 2004-05.

13.13 SUGGESTED READINGS

i. Differential Calculus for Beginners by Joseph Edwards.
ii. Text-Book on Differential Calculus by Gorakh Prasad.
iii. Calculus by R. Kumar.
iv. Krishna’s Text Book on Calculus by A. R. Vasistha.
v. Pragati’s Calculus by Sudhir K. Pundir.

13.14 TERMINAL QUESTIONS

(TQ-1) Evaluate $\int_0^\infty x^8\left(\frac{x^8(1-x^6)}{(1+x)^{24}}dx\right.$

(TQ-2) Prove that $\int_\infty^0 x^{2n-1}e^{-ax^2}dx = \frac{\Gamma n}{2a^n}$

(TQ-3) Evaluate $\int_0^\infty e^{-\gamma x^2}dx, \gamma > 0$

(TQ-4) Prove that $\int_0^1 \frac{1}{\sqrt{1-x^n}}dx = \frac{\Gamma 1/n}{\Gamma\frac{1}{2}+\frac{1}{n}}, \frac{\sqrt{\pi}}{n}$

(TQ-5) Prove that

$$\int_0^{\pi/2} tan^n x dx = \frac{\pi}{2} sex\frac{n\pi}{2}, -1 < n < 1$$

(TQ-6) Prove that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{4\sqrt{2}}$$

(TQ-7) Evaluate $\int_0^\infty cos(c^2x^2)\,dx$

(TQ-8) Prove that $\int_{-\infty}^\infty cos\left(\frac{\pi}{2}x^2\right)dx = 1$

(TQ-9) Prove that $\int_0^1 \log\Gamma(x) \quad dx = \frac{1}{2}log2\pi$

(TQ-10) Prove that $\int_0^\infty \frac{sinbz}{z}dz = \frac{\pi}{2}$

13.15 ANSWERS

CHECK YOUR PROGRESS

CYQ1- 0
CYQ2-$(n-1)!$
CYQ3- a^n
CYQ4-$\Gamma(m+n)$
CYQ5- F

CYQ6- T
CYQ7- F
CYQ-8 F

TERMINAL QUESTIONS

(TQ-1) 0
(TQ-3) $\frac{1}{2\sqrt{\gamma}}\Gamma\left(\frac{1}{2}\right)$
(TQ-7) $\frac{1}{2c}\sqrt{\pi/2}$

BLOCK-IV
FUNCTIONS OF SEVERAL VARIABLES

UNIT14:- PARTIAL DIFFERENTIATION

CONTENTS

14.1 INTRODUCTION:

In mathematics, the partial derivative of any function having several variables is its derivative with respect to one of those variables where the others are held constant. Partial derivatives are useful in analyzing surfaces for maximum and minimum points and give rise to partial differential equations. The modern partial derivative notation was created by Adrien-Marie Legendre (1786), although he later abandoned it; Carl Gustav Jacob Jacobi reintroduced the symbol in 1841.In mathematics, differential calculus is a subfield of calculus that studies the rates at which quantities change. It is one of the two traditional divisions of calculus, the other being integral calculus the study of the area beneath a curve. The primary objects of study in differential calculus are the derivative of a function, related notions such as the differential, and their applications. The derivative of a function at a chosen input value describes the rate of change of the function near that input value. The process of finding a derivative is called differentiation**.** in this topic we have covered Function of two variables and Homogeneous function.

Adrien-Marie Legendre,
(September 18, 1752-
January 10, 1833)
Fig.14.1.
Ref:
https://www.bridgemanimages.com/en/noartistknown/adrien-marie-legendre-french-revolution/nomedium/asset/2596140

14.2 *OBJECTIVES:*

At the end of this topic student will be able to understand:

(i) Limit and continuity of two variable function.
(ii) Partial Derivatives
(iii) Definition ofHomogeneous function.
(iv) Euler's Theorem and application of Euler's Theorem.
(v) Composite Function and Change of variables

14.3 *FUNCTION OF TWO VARIABLES:*

If to each point (x , y) of a certain of xy – plane, there is assigned a real number z, then z is known to be a function of two variable x and y.

Examples: f(x , y) = 3x + 5y +3 , g(x , y) = $x^2 + y^2$ are the function of two variables.and h(x) = $4x^5 - 7x^2 + 9$, j(y) = 11x are the function of one variables.

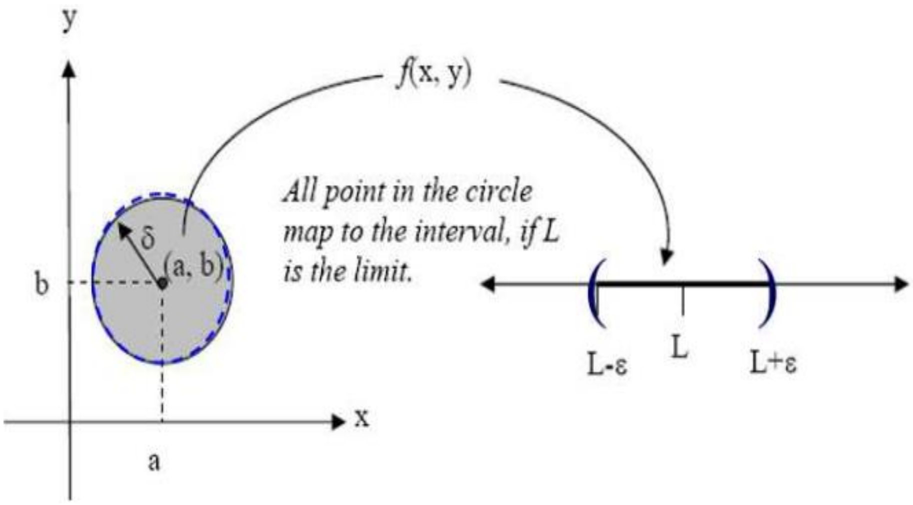
14.3.1 LIMIT OF A FUNCTION OF TWO VARIABLES:

Recall from The Limit of a Function the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing *a*. We say that the function $f(x)$ has a limit *l* at *a*, if for every $\varepsilon > 0$, there exists a $\delta > 0$ depending on ε, such that if $0 < |x - a| < \delta$ for all *x* in the domain of *f*, then $|f(x) - l| < \varepsilon$. In symbol, we write $\lim_{x \to a} f(x) = l$.

Geometrically, the definition says that for any $\varepsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that any point inside the interval $(a - \delta, a + \delta)$ is mapped to a point inside the interval $(l - \varepsilon, l + \varepsilon)$.

A similar definition extends to functions in two variables: We say that *L* is the limit of a function $f(x, y)$ at the point (*a*, *b*), written $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if $f(x, y)$ is as close to *L* as we please whenever the distance from the point (*x*, *y*) to the point (*a*, *b*) is sufficiently small, but not zero. Using $\varepsilon\delta$ definition we say that *L* is the limit of $f(x, y)$ as (*x*, *y*) approaches (*a*, *b*) if and only if for every given $\varepsilon > 0$ we can find a $\delta > 0$ such that for any point (*x*, *y*) where $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ we have $|f(x, y) - L| < \varepsilon$ i.e., to say that *L* is the limit of $f(x, y)$ as $(x, y) \to (a, b)$ means that for any given $\varepsilon > 0$, we can find an open punctured disk (i.e., without the center and the boundary) centered at (*a*, *b*) such that for any point (*x*, *y*) inside the disk the difference $f(x, y) - L$ is within ε, i.e., $L - \varepsilon < f(x, y) < L + \varepsilon$. Below figure illustrates this.



As in the case of functions of one variable, limits of functions of two variables possess the following properties:

1. The limit, if it exists, is unique.

2. The limit of a sum, difference, product, is the sum, difference, product of limits.
3. The limit of a quotient is the quotient of limits provided that the limit in the denominator is not zero.

NOTE: In the case of functions of one variable, if a function $f(x)$ has a limit l at $x = a$ then the limit of $f(x)$ as x approaches a from either the left or right must be l. Similar situation occurs for a function $f(x, y)$ of two variables with the difference that the point (x, y) can approach (a, b) in infinite directions. Hence, if we can find two directions toward (a, b) with two different limits then the function has no limit as $(x, y) \to (a, b)$.

14.3.2 CONTINUITY OF A FUNCTION OF TWO VARIABLES:

We can now define the continuityof a function of two variables in terms of limit. Intuitively, we expect our definition to support the idea that there are no breaks or gaps in the function if it is continuous. The continuity of functions of two variables is defined in the same way as for functions of one variable:

A function $f(x, y)$ is continuous at the point (a, b) if the following two conditions are satisfied:

1. Both $f(a,b)$ and $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exist; and
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a,b)$.

A function is continuous on a region R in the xy-plane if it is continuous at each point in R. A function that is not continuous at (a, b) is said to be discontinuous at (a, b).

Like the case of functions of one variable, the following results are true for multivariable functions:

1. The sums, products, quotients (where denominator function is not zero), and compositions of continuous functions are also continuous.
2. Polynomial functions are continuous.
3. Rational functions are continuous in their domain.

NOTE:

1. $f(x,y)$ is said to be continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ irrespective of the path along with $x \to a, y \to b$.
2. It should not be assumed that the path along which the point (x, y) tends to (a, b) is immaterial, because $\lim_{x\to a}\{\lim_{y\to a} f(x,y)\}$ is not always equal to $\lim_{y\to a}\{\lim_{x\to a} f(x,y)\}$.

ILLSTRATIVE EXAMPLES

Example 1. Let $f(x,y) = x^2 + y^2$. Is $\lim_{(x,y)\to(1,1)} f(x,y) = 3$?

Solution. Let $\varepsilon = 0.1$. Is there $\delta > 0$ such that all the points (x, y) inside the open disk with radius δ and centered at (1, 1) satisfy $\left|x^2 + y^2 - 3\right| < 0.1$ or equivalently $2.9 < x^2 + y^2 < 3.1$? Clearly, any such open disk will share points with the open disk centered at (1, 1) and with radius 0.2. But any point (x, y) in this disk satisfies $(x-1)^2 + (y-1)^2 = 0.2^2$ or $x^2 + y^2 - 2(x+y) = 0.04$. Since 0.8 <$x$< 1.2 and 0.8 <$y$< 1.2, we find 3.2 < 2($x$ + y) < 4.8. This implies that $f(x,y) = x^2 + y^2 < (1.2)^2 + (1.2)^2 = 2.88 < 2.9$. Hence any point in the disk centered at (1, 1) and radius 0.2 will fall outside the interval (2.9, 3.1). In particular, this is true for any point in the disk centered at (1, 1) and radius δ. We conclude that $\lim_{(x,y)\to(1,1)} f(x,y) \neq 3$.

Alternatively, $\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} x^2 + y^2$

$$= \lim_{x\to1}[\lim_{y\to1}(x^2 + y^2)] = \lim_{x\to1}(x^2 + 1) = 2 \neq 3.$$

Example 2. Find $\lim_{(x,y)\to(1,2)} \frac{x^2 y}{x^4 + y^2}$.

Solution. $\lim_{(x,y)\to(1,2)} \frac{x^2 y}{x^4 + y^2} = \frac{1^2 2}{1^4 + 2^2} = \frac{2}{5}$.

Example 3. Find $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x - y}$.

Solution. Here, we can not put the limit directly as we get 0 in the denominator. Therefore, we try to rewrite the fraction to simplify it. So,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x - y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x^2+xy+y^2)}{x-y}$$

$$= \lim_{(x,y)\to(0,0)} (x^2+xy+y^2) = 0$$

Example4. Show that $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous at (0,0).

Solution. The given function is clearly continuous everywhere except at (possibly) (0, 0). Let's check its continuity at (0, 0). Let $\varepsilon > 0$. Can we find a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, then $|f(x,y) - 0| < \varepsilon$.

Using the fact that $x^2 < x^2 + y^2$, i.e. $\left|\dfrac{x^2}{x^2+y^2}\right| \le 1$. Therefore, we have

$$|f(x,y) - 0| = \left|\frac{x^2 y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right||y| \le \sqrt{x^2+y^2} < \delta = \frac{\varepsilon}{2} < \varepsilon.$$

Hence, $f(x,y)$ is continuous at (0, 0).

Alternatively,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2 y}{x^2+y^2}\right) = 0.$$

And, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0} \dfrac{x^2 y}{x^2+y^2}\right) = 0$, i.e. limit is same as (*x*, *y*) approaches (0, 0) either first along *y*=0 then along with *x*=0 or first along *x*=0 then along with *y*=0, which shows that limit does not depend upon the path by which (*x*, *y*) approaches to (0, 0). Also, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Therefore, $f(x,y)$ is continuous at (0, 0).

Example 4. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Solution. Here, let us calculate the limit when (x, y) approach (0, 0) along different paths:

(i). along the x-axis, i.e.y = 0:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x.0}{x^2+0^2} = 0$$

(ii). along the y-axis, i.e., x = 0:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0.y}{0^2+y^2} = 0 .$$

(iii). along the line straight line path $y = mx$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left.\frac{xy}{x^2+y^2}\right|_{y=mx} = \lim_{x\to 0} \frac{x.mx}{x^2+(mx)^2} = \frac{m}{1+mx^2} ,$$

which is different for different value of mi.e. limit depends upon the path.

Further, we obtained a different limit. So, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Example 5. Show that $f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is discontinuous at (0,0).

Solution. The given function is clearly continuous everywhere except at (possibly) (0, 0). Now let's look at the limit of $f(x,y)$ as (x, y) approaches (0, 0) along two different paths. First, let's approach (0, 0) along the x-axis, i.e., y = 0:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0} = 1 .$$

Now, let's approach (0, 0) along the y-axis, i.e., x = 0:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0 .$$

Since the limit is not the same along the two different directions, we conclude that $f(x, y)$ is discontinuous at (0, 0).

CHECK YOUR PROGRESS

True or false Questions

Problem 1: $\lim_{(x,y)\to(0,0)} (x+y) = 0$

Problem 2: if a function f(x , y) is continuous at (a , b) then $\lim_{(x,y)\to(a,b)}$ does not exist.

Problem 3: Every rational function is continuous in their domain.

14.4 PARTIAL DERIVATIVE:

Let z = f(x , y) be a function of two independent variables x and y.

Then the partial derivative z with respect to x is the ordinary derivative of z with respect to x when y regarded as a constant and is denoted by

f_x or $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$

Thus if $\lim_{h\to 0} \frac{f(a+h,b)-f(a,\ b)}{h}$ exist, then this limit is called the partial derivative of f(x , y) with respect to x at (a , b) and is denoted by

$f_x(a , b)$ or $\left(\frac{\partial z}{\partial x}\right)_{(a,\,b)}$ or $\left(\frac{\partial f}{\partial x}\right)_{(a,\,b)}$

Similarly, if $\lim_{k\to 0} \frac{f(a,b+k)-f(a,\ b)}{k}$ exist, then this limit is called the partial derivative of f(x , y) with respect to y at (a , b) and is denoted by

$f_y(a , b)$ or $\left(\frac{\partial z}{\partial y}\right)_{(a,\,b)}$ or $\left(\frac{\partial f}{\partial y}\right)_{(a,\,b)}$

Let f:X $\to \mathbb{R}$ and X $\subseteq \mathbb{R}^2$. If the function f has partial derivatives at each point of X then f is partially differentiable on X.

Note: We have by definition

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,\ b)}{h} = \lim_{x\to a} \frac{f(x,b)-f(a,\ b)}{x-a}$$

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,\ b+k)-f(a,\ b)}{k} = \lim_{y\to b} \frac{f(a,y)-f(a,\ b)}{y-a}$$

Example 1: Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at (1 , 2) if f(x , y) = $2x^2 - xy + 2y^2$.

Solution: We have $\left(\frac{\partial f}{\partial x}\right)_{(1,\,2)} = \lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h}$

$$= \lim_{h\to 0} \frac{\{2(1+h)^2-(1+h).2+2.2^2\}-\{2.1^2-1.2+2.2^2\}}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+2h}{h} = \lim_{h\to 0}(2h+2) = 2$$

and $\left(\frac{\partial f}{\partial y}\right)_{(1\,,\,2)} = \lim_{k\to 0} \frac{f(1\,,2+k)-f(1\,,2)}{k}$

$$= \lim_{k\to 0} \frac{\{2-(2+k)+2\,(2+k)^2\}-\{2-2+8\}}{k}$$

$$= \lim_{k\to 0} \frac{2\,k^2+7k}{k} = \lim_{k\to 0}(2k+7) = 7.$$

Example 2: find $f_x\,(0\,,0)$ and $f_y\,(0\,,0)$ if

$$f(x) = \begin{cases} \left(\dfrac{x^2 - xy}{x+y}\right), & (x\,,y) \neq (0\,,0) \\ 0\quad, & (x\,,y) = (0\,,0) \end{cases}$$

Solution: we have

$$f_x\,(0\,,0) = \lim_{h\to 0} \frac{f(0+h\,,0)-f(0\,,\,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2-h.0}{h+0}-0}{h} = \lim_{h\to 0} 1 = 1$$

$$f_y\,(0\,,0) = \lim_{k\to 0} \frac{f(0\,,0+k)-f(0\,,\,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = \lim_{h\to 0} 0 = 0.$$

Example 3. Find the first order partial derivatives of $u = \tan^{-1}\dfrac{x^2+y^2}{x+y}$.

Solution: We have, $u = \tan^{-1}\dfrac{x^2+y^2}{x+y}$.

Now differentiating u with respect to x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\dfrac{x^2+y^2}{x+y}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2 + (x^2+y^2)^2} \cdot \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{1}{(x+y)^2 + (x^2+y^2)^2} \cdot \frac{(x+y)2x - (x^2+y^2)1}{1}$$

$$= \frac{x^2 + 2xy - y^2}{(x+y)^2 + (x^2+y^2)^2}.$$

Now differentiating *u* with respect to *y*, we get

$$\frac{\partial u}{\partial y} = \frac{1}{1 + \left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2 + (x^2+y^2)^2} \cdot \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$= \frac{1}{(x+y)^2 + (x^2+y^2)^2} \cdot \frac{(x+y)2y - (x^2+y^2)1}{1}$$

$$= \frac{y^2 + 2xy - x^2}{(x+y)^2 + (x^2+y^2)^2}.$$

Example 4. If $z = e^{ax+by} f(ax - by)$, show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Solution: We have, $z = e^{ax+by} f(ax - by)$ ….(1)

Now differentiating (1) partially with respect to *x*, we get

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by).a + e^{ax+by}.af(ax - by)$$

$$\Rightarrow b\frac{\partial z}{\partial x} = abe^{ax+by}[f'(ax - by) + f(ax - by)] \quad \text{….(2)}$$

Now differentiating (1) partially with respect to *y*, we get

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by).(-b) + e^{ax+by}.bf(ax - by)$$

$$\Rightarrow a\frac{\partial z}{\partial y} = abe^{ax+by}[-f'(ax-by) + f(ax-by)] \quad ....(3)$$

Adding equations (2) and (3), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by}[2f(ax-by)]$$

$$\Rightarrow b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Example 5. If $u = (x^2+y^2+z^2)^{-1/2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. (Laplace Equation)

Solution: Differentiating *u* partially with respect to *x*, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= -x(x^2+y^2+z^2)^{-3/2}.$$

Again differentiating partially with respect to *x*, we get

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}(x).(x^2+y^2+z^2)^{-3/2} - x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2}$$

$$= -1.(x^2+y^2+z^2)^{-3/2} - x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}.2x$$

$$= (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2).$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(2y^2-x^2-z^2)$ and $\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2)$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Example 6. If $x^x y^y z^z = c$, show that at $x = y = z, \frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Solution: Given $x^x y^y z^z = c$, which defines *z* as a function of *x* and *y*. Taking logarithmic on both sides, we get

$$x \log x + y \log y + z \log z = \log c$$

Differentiating partially with respect to *y*, we get

$$y \frac{1}{y} + 1.\log y + z \frac{1}{z} \frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0.$$

$$\Rightarrow 1 + \log y + (1 + \log z) \frac{\partial z}{\partial y} = 0 \qquad \ldots.(1)$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z} \qquad \ldots.(2)$$

Similarly, $\frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z}$. ….(3)

Now differentiating equation (1) partially with respect to *x*, we get

$$(1 + \log z) \frac{\partial^2 z}{\partial x \partial y} + \left(\frac{1}{z} \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1 + \log z)} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}. \qquad \ldots.(4)$$

At $x = y = z$, from equations (2) and (3), we get

$\frac{\partial z}{\partial y} = -1$ and $\frac{\partial z}{\partial x} = -1$.

From equation (4), at $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1 + \log x)} (-1)(-1) = -\frac{1}{x(\log e + \log x)}$$

$$= -\frac{1}{x(\log ex)} = -(x \log ex)^{-1}.$$

Example 7. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: Differentiating *u* partially with respect to *x*, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}.$$

Similarly, we find $\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$ and $\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$.

Adding, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{x^3 + y^3 + z^3 - 3xyz}$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz)} = \frac{3}{x+y+z}.$$

Now, $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

14.5 HOMOGENEOUS FUNCTION:

In mathematics, a homogeneous function is one with multiplicative scaling behaviour i.e. if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor. For example, a homogeneous real-valued function of two variables *x* and *y* is a real-valued function that satisfies the condition $f(rx, ry) = r^k f(x, y)$ for some constant *k* and all real numbers *r*. The constant *k* is called the degree of homogeneity.

Alternatively, a real valued function $f(x, y)$ is said to be homogeneous of degree (or order) *k* in the variables *x* and *y* if it can be expressed in the form $x^k \phi\left(\frac{y}{x}\right)$ or $y^k \phi\left(\frac{x}{y}\right)$.

Let consider a function $f(x, y) = \frac{x+y}{\sqrt{x+y}}$, then

1. $f(rx, ry) = \frac{rx+ry}{\sqrt{rx+ry}} = r^{1/2} \frac{x+y}{\sqrt{x+y}} \Rightarrow f(rx, ry) = r^{1/2}.f(x, y)$,

   which implies that $f(x, y) = \frac{x+y}{\sqrt{x+y}}$ is a homogeneous function of degree $\frac{1}{2}$ in variables *x* and *y*.

2. $f(x, y) = \frac{x+y}{\sqrt{x+y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\sqrt{\left(1+\frac{y}{x}\right)}} = x^{1/2}\phi\left(\frac{y}{x}\right)$, which indicates that

   $f(x, y) = \frac{x+y}{\sqrt{x+y}}$ is a homogeneous function of degree $\frac{1}{2}$ in variables *x* and *y*.

3. $f(x, y) = \frac{x+y}{\sqrt{x+y}} = \frac{y\left(\frac{x}{y}+1\right)}{\sqrt{y}\sqrt{\left(\frac{x}{y}+1\right)}} = y^{1/2}\phi\left(\frac{x}{y}\right)$, which clarifies that

   $f(x, y) = \frac{x+y}{\sqrt{x+y}}$ is a homogeneous function of degree $\frac{1}{2}$ in variables *x* and *y*.

Similarly, a function $f(x,y,z)$ is said to be homogeneous of degree (or order) *k* in the variables *x*, *y* and *z* if it can be expressed in the form $x^k\phi\left(\frac{y}{x},\frac{z}{x}\right)$ or $y^k\phi\left(\frac{x}{y},\frac{z}{y}\right)$ or $z^k\phi\left(\frac{x}{z},\frac{y}{z}\right)$. Alternative test for this is $f(rx,ry,rz)=r^k f(x,y,z)$.

Note: A homogeneous function is not necessarily continuous as shown by this example

function f defined by $f(x,y)=x$ if $xy>0$ and $f(x,y)=0$ if $xy\le 0$. This function is homogeneous of degree 1.

i.e. $f(rx,ry)=rf(x,y)$

for any real numbers r, x, y. It is discontinuous at y = 0, x≠0.

14.5.1 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

Statement: If *u* is a homogeneous function of degree *n* in variables *x* and *y*, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$.

Proof: Since *u* is a homogeneous function of degree *n* in *x* and *y*, it can be written as

$$u=x^n\phi\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \text{ and}$$

$$\frac{\partial u}{\partial y}=x^n\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=x^{n-1}\phi'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}=nx^n\phi\left(\frac{y}{x}\right)-x^{n-1}y\phi'\left(\frac{y}{x}\right) \text{ and } y\frac{\partial u}{\partial y}=x^{n-1}y\phi'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)=nu.$$

Euler's theorem can be extended to a homogeneous function of any number of variables. Therefore, if u is be homogeneous function of degree n in variables x, y and z, then statement of Euler's theorem is $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$.

14.5.2 EXAMPLES BASED ON HOMOGENEOUS FUNCTION:

Example1. Verify Euler's theorem for the function $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$.

Solution. Given $u(x,y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, then

$$u(rx,ry)=r^0\left(\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right)=r^0u(x,y),$$

Therefore, u is a homogeneous function of degree 0 in x and y, and as per Euler's theorem we have to verify: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0\times u=0$.

Now $$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}.\frac{1}{y}+\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}$$

And $$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}.\left(-\frac{x}{y^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=-\frac{x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y}=-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

Thus, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$, which verified Euler's theorem.

Example2. If u is a homogeneous function of degree n in x and y, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$.

Solution. Since u is homogeneous function of degree n in x and y, then by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \;\ldots..(1)$$

Differentiate (1) partially w.r.t.x, we get

$$1.\frac{\partial u}{\partial x}+x.\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \;\ldots..(2)$$

Differentiate (1) partially w.r.t.y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+1.\frac{\partial u}{\partial y}+y.\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y} \;\ldots..(3)$$

Multiplying (2) by x, (3) by y, adding and then using the result $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n.nu-nu$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u\,.$$

Example3. If $F(u)=V(x,y)$, where V is a homogeneous function of degree n in x and y, then show that:

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{F'(u)}$, and

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1]$, where $\phi(u) = n\dfrac{F(u)}{F'(u)}$.

Solution. Since $V = F(u)$ is a homogeneous function of degree *n*, then by Euler's theorem, we have

$$x\frac{\partial}{\partial x}[F(u)] + y\frac{\partial}{\partial y}[F(u)] = nF(u)$$

$$\Rightarrow xF'(u)\frac{\partial u}{\partial x} + yF'(u)\frac{\partial}{\partial y} = nF(u)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}$$

Let $n\dfrac{F(u)}{F'(u)} = \phi(u)$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \phi(u) \text{ .....(1)}$$

Differentiate (1) partially w.r.t.*x*, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \phi'(u)\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = [\phi'(u) - 1]\frac{\partial u}{\partial x} \text{ .....(2)}$$

Differentiate (1) partially w.r.t.*y*, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\phi'(u)\frac{\partial u}{\partial y}$$

$$\Rightarrow y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial y \partial x} = [\phi'(u) - 1]\frac{\partial u}{\partial y} \text{ .....(3)}$$

Multiplying (2) by *x*, (3) by *y* and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [\phi'(u) - 1]\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

Using (1), we have

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1].$$

Example4. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, then show that:

(i) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u$, and

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$.

Solution. (i). Here, $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$ is not a homogeneous function but if we take $\tan u=\dfrac{x^3+y^3}{x-y}=V(x,y)$, then $F(u)=\tan u$ is homogeneous of degree 2 as $V(rx,ry)=r^2V(x,y)$. Thus by Euler's theorem

$$x\frac{\partial}{\partial x}[\tan u]+y\frac{\partial}{\partial y}[\tan u]=2\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sin u}{\cos u}.\frac{1}{\sec^2 u}=2\sin u\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

(ii). Now using the result as proved in previous example, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1], \text{ where } \phi(u)=n\frac{F(u)}{F'(u)}.$$

Here, $\phi(u)=2\dfrac{\tan u}{\sec^2 u}=\sin 2u$ and $\phi'(u)=2\cos 2u$, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u[2\cos 2u-1]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\sin 2u\cos 2u-2\sin 2u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-2\sin 2u\ .$$

Example5. If $u=\log_e\left(\frac{x^4+y^4}{x+y}\right)$, then show that: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$.

Solution. Here, u is not a homogeneous function. Let $e^u=\frac{x^4+y^4}{x+y}$, which is homogeneous function of degree 3. Thus by Euler's theorem

$$x\frac{\partial}{\partial x}[e^u]+y\frac{\partial}{\partial y}[e^u]=3e^u$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

Example6. If $u=\sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, then show that: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0$.

Solution. Here, u is not a homogeneous function. But, let

$$\sin u=\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}=V(x,y,z)\text{, then}$$

$$V(rx,ry,rz)=\frac{r(x+2y+3z)}{r^4\sqrt{x^8+y^8+z^8}}=r^{-3}.V(x,y,z)$$

Therefore, $F(u)=\sin u$ is a homogeneous function of degree -3 in variables *x*, *y* and *z*. Thus by Euler's theorem, we have

$$x\frac{\partial}{\partial x}[\sin u]+y\frac{\partial}{\partial y}[\sin u]+z\frac{\partial}{\partial z}[\sin u]=-3\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z}=-3\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}+3\tan u=0\ .$$

CHECK YOUR PROGRESS

True or false Questions

Problem 4: $\frac{\partial y}{\partial x}$ is called second order Partial derivative.

Problem 5: $\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)=\frac{\partial^2 y}{\partial x^2}$.

Problem 6: $f(x,y)=\dfrac{x+y}{\sqrt{x+y}}$ is a homogeneous function of degree 1 in variables x and y.

Problem 7. If f (x , y) = 2y, then $\frac{\partial f}{\partial x}=0$.

14.6 COMPOSITE FUNCTION:

(i) if u = f(x , y) , where x = φ (t) , y = Ψ (t) , then u is called a composite function of the single variable t and we can find $\frac{du}{dt}$.

(ii) if z = f(x , y) , where x = φ (u , v) , then z is called a composite function of two variables u and v so that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

14.6.1 DIFFERENTIATION OF COMPOSITE FUNCTIONS:

If u is composite function of t, defined by the relations u = f(x , y) ; x = φ (t) ,y = Ψ (t) , then

$$\mathbf{\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$$

Proof. Here, u = f(x , y) …(1)
let δt be an increment in t and δx , δy , δu the corresponding increments in x, y and u respectively. then , we have

$u + \delta u = f(x + \delta x, y + \delta y)$ …(2)

subtracting (1) from (2), we get

$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$

$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t} \quad \ldots(3)$$

as $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}, \ \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \ \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$\lim_{\delta t \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$

and $$\lim_{\delta t \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

$\therefore$ from (1), $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$\frac{du}{dt}$ is called the total derivative of u to distinguish it form the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial x}$.

Cor.1. if u = f(x , y , z) and x , y , z are function of t ,then u is a composite function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Cor.2. if z = f(x , y) and x , y are function of u and v , then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \ ; \ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Cor.3. if u = f(x , y), where y = $\varphi(x)$, then since x = Ψ(x) , u is a composite function of x.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \Longrightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

Cor.4. if we are given an implicit function f(x , y) = c , then u = f(x , y) , where u = c using cor.3 , we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

But $\frac{du}{dx} = 0$ $\quad \therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$ or $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

ILLUSTRATIVE EXAMPLES

Example 1. If u =$\sin^{-1}(x-y)$, x = 3t , y = 4t^3 , show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

Solution: The given equations define u as a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} .3 + \frac{1}{\sqrt{1-(x-y)^2}} (-1) . 12\, t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Example 2. If u = $x^2 - y^2 + \sin yz$, where y = e^x and $z = \log x$; find $\frac{du}{dx}$.

Solution: $\frac{du}{dx} = \frac{\partial u}{\partial y} . \frac{dy}{dx} + \frac{\partial u}{\partial z} . \frac{dz}{dx} + \frac{\partial u}{\partial x}$

$$= (-2y + z \cos yz)e^x + (y \cos yz) \frac{1}{x} + 2x$$

$$= \{-2e^x + \log x \cos(e^x \log x)\}\, e^x + \left\{e^x \cos(e^x \log x) \frac{1}{x}\right\} + 2x$$

$$= 2(x - e^{2x}) + e^x \cos(e^x \log x)(\log x + \frac{1}{x}).$$

Example 3. If u = f(r , s) and r = x + y, s = x – y ; Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Solution: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} . \frac{\partial s}{\partial x}$

$$= \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \ldots(1) \qquad (\because \frac{\partial r}{\partial x} = \frac{\partial s}{\partial x} = 1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} . \frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \ldots(2) \qquad (\because \frac{\partial r}{\partial y} = 1,\ \frac{\partial s}{\partial y} = -1)$$

Adding (1) and (2) , we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$$

14.7 CHANGE OF VARIABLES:

The difference between the dependent and independent variables is well known to us. Sometimes, it is desirable, particularly dealing with the solutions of the differential equations, to change the independent variable into the dependent variable or into another variable which it is connected by a relation.

14.7.1 CHANGE OF INDEPENDENT VARIABLE INTO DEPENDENT VARIABLE:

Let y = f(x). Then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \left(\frac{dx}{dy}\right)^{-1} \ldots(1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)^{-1}$$

$$= \frac{d}{dy}\left(\frac{dx}{dy}\right)^{-1}\frac{dy}{dx}$$

$$= -\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\frac{dy}{dx}$$

$$= -\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\left(\frac{dx}{dy}\right)^{-1} = -\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2} \ldots(2)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[-\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}\right] = -\frac{d}{dy}\left[\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}\right]\frac{dy}{dx}$$

$$= \left[-\left(\frac{dx}{dy}\right)^{-3}\frac{d^3x}{dy^3} - 3\left(\frac{dx}{dy}\right)^{-4}\left(\frac{d^2x}{dy^2}\right)^2\right]\left(\frac{dx}{dy}\right)^{-1}$$

$$= -\left(\frac{dx}{dy}\right)^{-4}\frac{d^3x}{dy^3} + 3\left(\frac{dx}{dy}\right)^{-5}\left(\frac{d^2x}{dy^2}\right)^2 \quad \ldots$$

(3)

And so on.

14.7.2 TO CHANGE OF INDEPENDENT VARIABLE x INTO ANOTHER VARIABLE t WHERE x = f(t):

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = \left(\frac{dx}{dt}\right)^{-1}\frac{dy}{dt} \ldots(1)$$

$$\Rightarrow \quad \frac{d}{dx} \equiv \left(\frac{dx}{dt}\right)^{-1}\frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{dx}{dt}\right)^{-1}\frac{d}{dt}\left[-\left(\frac{dx}{dt}\right)^{-1}\frac{dy}{dt}\right]$$

$$= \left(\frac{dx}{dt}\right)^{-1}\left[-\left(\frac{dx}{dt}\right)^{-1}\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)(-1)\left(\frac{dx}{dt}\right)^{-2}\frac{d^2x}{dt^2}\right]$$

$$= \left(\frac{dx}{dt}\right)^{-3}\left[\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right]$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dx}{dt}\right)^{-1}\frac{d}{dt}\left[\frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}\right]$$

$$= \frac{\left(\frac{dx}{dt}\right)^3\left[\frac{dx}{dt}\frac{d^3y}{dt^3} - \frac{d^3x}{dt^3}\frac{dy}{dt}\right] - \left[\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right]3\left(\frac{dx}{dt}\right)^2\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)^6}$$

$$= \left(\frac{dx}{dt}\right)^{-5}\left[\left(\frac{dx}{dt}\frac{d^3y}{dt^3} - \frac{d^3x}{dt^3}\frac{dy}{dt}\right)\frac{dx}{dt} - 3\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right)\frac{d^2y}{dx^2}\right] \quad \ldots(3)$$

And so on.

ILLUSTRATIVE EXAMPLES

Example1. Show that equation $\frac{d^2y}{dx^2} = a$, may be written as in the form $\frac{d^2x}{dy^2} + a\left(\frac{dx}{dy}\right)^3 = 0$.

Solution: $\frac{d^2y}{dx^2} = a$

$$\Rightarrow \left(\frac{dx}{dt}\right)^{-3}\frac{d^2x}{dy^2} = a$$

$$\Rightarrow -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3} = a$$

$$\Rightarrow \frac{d^2x}{dy^2} + a\left(\frac{dx}{dy}\right)^3 = 0.$$

Example 2. Change the independent variable from x to y in $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$.

Solution: $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^{-2}\right\}^{3/2}}{\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}}$

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{-\left(\frac{dx}{dy}\right)^3\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}} = -\frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}.$$

Example 3. Transform the equation $\frac{d^2y}{dx^2} + \cot x\,\frac{dy}{dx} + 4y\,cosec^2x = 0$ by substitution z = 2log tan(x/2) .

Solution: z = 2log tan(x/2)

$$\therefore \frac{dz}{dx} = \frac{2}{\tan(x/2)}\sec^2\frac{x}{2}.\frac{1}{2} = \frac{2}{\sin x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{2}{\sin x}\frac{dy}{dz}$$

$$\Rightarrow \frac{d}{dx} = \frac{2}{\sin x}\frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{2}{\sin x}\frac{d}{dz}\left[\frac{2}{\sin x}\frac{dy}{dz}\right]$$

$$= \frac{2}{\sin x}\left[\frac{2}{\sin x}\frac{d^2y}{dz^2} - \frac{2\,cosx}{sin^2x}\frac{sinx}{2}\frac{dx}{dz}\frac{dy}{dz}\right]$$

$$= \frac{2}{\sin x}\left[\frac{2}{\sin x}\frac{d^2y}{dz^2} - \frac{2\,cosx}{sin^2x}\frac{sinx}{2}.\frac{dy}{dz}\right]$$

$$= \frac{4}{sin^2x}\frac{d^2y}{dz^2} - \frac{2\,cosx}{sin^2x}\frac{dy}{dz}$$

Substituting these values in the given equation, we get

$$\frac{4}{sin^2x}\frac{d^2y}{dz^2} - \frac{2\,cosx}{sin^2x}\frac{dy}{dz} + \cot x.\frac{2}{\sin x}\frac{dy}{dz} + 4y\,cosec^2x = 0$$

$$\Rightarrow 4\,cosec^2x\frac{d^2y}{dz^2} + 4y\,cosec^2x = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = 0\ .$$

14.7.3 CHANGE OF BOTH DEPENDENT AND INDEPENDENT VARIABLES:

The relations between Cartesian co-ordinates (x , y) and polar coordinates (r , θ) of any point are x = r $\cos\theta$, $y = r\sin\theta$

$\therefore$ We have $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\left(\frac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\frac{dr}{d\theta}\right)\cos\theta - r\sin\theta}$

$$= \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}\left(\frac{\left(\frac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\frac{dr}{d\theta}\right)\cos\theta - r\sin\theta}\right)}{\frac{dx}{d\theta}}$$

$$= \frac{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}$$

Example 4. To show that, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2.$

Solution: $x^2 + y^2 = r^2$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$\Rightarrow\quad x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$…(1)

$\frac{y}{x} = \tan\theta$

$\Rightarrow \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} = sec^2\theta\frac{d\theta}{dt}$

$\Rightarrow x\frac{dy}{dt} - y\frac{dx}{dt} = x^2 sec^2\theta\frac{d\theta}{dt} = r^2 cos^2\theta sec^2\theta\frac{d\theta}{dt}$

$= r^2\frac{d\theta}{dt}$ …(2)

Squaring and adding (1) and (2) , we get

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2.$

Example 5. Transform to polars the formula $p = \frac{x\frac{dy}{dx} - y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$.

Solution: Given $p = \frac{x\frac{dy}{dx} - y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

Multiplying the N^r and D^r by dx/dt

$= \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{r^2\frac{d\theta}{dt}}{\sqrt{\left(\frac{dr^2}{dt}\right) + r^2\left(\frac{d\theta}{dt}\right)^2}}$

$\therefore \frac{1}{p^2} = \frac{\left(\frac{dr^2}{dt}\right) + r^2\left(\frac{d\theta}{dt}\right)^2}{\left(r^2\frac{d\theta}{dt}\right)^2} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$

CHECK YOUR PROGRESS

True or false Questions

Problem 8: If u is composite function of t, defined by the relations u = f (x , y) ;

$x = \varphi$ (t) , $y = \Psi$ (t) , then $\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} . \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} . \frac{\mathbf{dy}}{\mathbf{dt}}$

Problem 9: If u = f(r , s) and r = x + y, s = x – y ; then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Problem 10: If $u = \sin^{-1}(x - y)$, x = 3t , y = $4t^3$, then $\frac{du}{dt} = \frac{8}{\sqrt{1-t^2}}$.

14.8 SUMMARY:

A partial derivative is the derivative of a multi-variable function with respect to a single variable. The other variables in the function are treated as constants. Partial derivatives give the rate of change of the function as one variable changes.Limit, Continuity, partial derivatives and Homogeneous Functions are the main topic of Differential Calculus. This topics covered definition and examples.

14.9 GLOSSARY:

- **i.** Sets
- **ii.** Function
- **iii.** Limit
- **iv.** Continuity
- **v.** Partial Differentiation
- **vi.** Homogeneous Function
- **vii.** Euler's theorem

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- **v.** Gilbert Strang (1991). Calculus. Wellesley-Cambridge Press.

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- **ii.** George B. Thomas Jr, Ross L.Finney (1998), Calculus and Analytical Geometry, Adison Wiley Publishing Company.
- **iii.** James Stewart (2012). Multivariable Calculus (7th edition). Brooks/Cole. Cengage.
- **iv.** S.C. Malik and Savita Arora (2021). Mathematical Analysis (6th edition). New Age International Private Limited.

14.12 TERMINAL QUESTIONS:

Q 1. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that:

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$ and

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}$.

Q 2. If $\log u=\left(\frac{x^3+y^3}{3x+4y}\right)$, then show that: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$.

Q3. If $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$, prove that:

$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u$.

Q 4. Show that the function

$$f(x)=\begin{cases}\left(\frac{x-y}{x+y}\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is discontinuous at the origin.

Q 5. If z = e^{ax+by}f(ax – by) , Show that b$\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}$ = 2abz.

Q 6. Find the first order partial derivatives of the following functions:
(i) u = y^x (ii) u = $\log(x^2+\ y^2)$

Q 7. If z = u^2 + v^2, u = = r cos θ $, v=r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial y}{\partial \theta}$.

Q 8. Transform the equation $x^2\frac{d^2y}{dx^2}$ + x $\frac{dy}{dx}$ + y = 0 by the substitution of x = e^x.

Q 9. If z is a function of x and y, where x = e^u + e^{-v} and y = e^{-u} - e^v , show that
$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

Q 10. If u = x^2y + y^2z + z^2x , show that $u_x+u_y+u_z=(x+y+z)^2$.

14.13ANSWERS:

CHECK YOUR PROGRESS

CYQ1. True
CYQ2. False
CYQ3. True
CYQ4 False
CYQ5 True
CYQ6 False
CYQ7 True
CYQ8. True

CYQ9 True
CYQ10. False

TERMINAL QUESTIONS

TQ 6. (i) $y^x \log y$ (ii) xy^{x-1}
TQ 7. 2r , 0
T**Q 8.**$\frac{d^2y}{dx^2}$ + y = 0

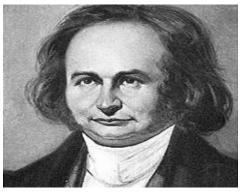
UNIT 15:- EXPANSION OF FUNCTION IN TWO VARIABLES AND JACOBIAN

CONTENTS:-

15.1 *INTRODUCTION:*

In this section we want to go over some of the basic ideas about the expansion of functions of two variables, the expansion of functions of two variables by Taylor's theorem. The origins of Taylor series expansion can be traced back to the 18th century, when English mathematician Brook Taylor first introduced the concept in his 1715 book "Methodus Incrementorum Directa et Inversa" (Direct and Inverse Methods of Incrementation).Jacobian is a functional determinant, useful in transformation of variables from cartesian to polar, cylindrical and spherical polar co-ordinates in multiple integrals. Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804 – 1851) who made significant contributions to Mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Carl Gustav Jacob Jacobi
(German Mathematician
1804 – 1851)
Fig 15.1



Ref: ***https://en.wikipedia.org/wiki/Carl_Gustav_Jacob_Jacobi***

15.2 OBJECTIVES:

At the end of this topic learner will be able to understand:

(i) Taylor's theorem for a function of two variables.
(ii) Maclaurin's series
(iii) Jacobians.

15.3 TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES:

The **Taylor series formula** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715.A function can be approximated by using a finite number of terms of its Taylor series. Taylor's theorem gives quantitative estimates on the error introduced by the use of such an approximation. The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial.The Taylor series of a function is the limit of that function's Taylor polynomials as the degree increases, provided that the limit exists. A function may not be equal to its Taylor series, even if its Taylor series converges at every point. A function that is equal to its Taylor series in an open interval (or a disc in the complex plane) is known as an analytic function in that interval.

Using Taylor's theorem for a function $f(x)$ of single variable x, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ...$$

Now let $f(x, y)$ be a function of two variables x and y. If y is kept as constant, then $f(x, y)$ reduces in a function of single variable x. Therefore, using Taylor's theorem, we have

$$f(x+h, \boldsymbol{y}+\boldsymbol{k}) = f(x, \boldsymbol{y}+\boldsymbol{k}) + h\frac{\partial}{\partial x}f(x, \boldsymbol{y}+\boldsymbol{k}) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x, \boldsymbol{y}+\boldsymbol{k}) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x, \boldsymbol{y}+\boldsymbol{k}) + ...$$

….(1)

Now keeping x as constant and applying Taylor's theorem for a function of single variable y, we have

$$f(\boldsymbol{x}, y+k) = f(\boldsymbol{x}, y) + k\frac{\partial}{\partial y}f(\boldsymbol{x}, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(\boldsymbol{x}, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(\boldsymbol{x}, y) + ...$$

….(2)

Using equation (2), we can write equation (1) as

$$f(x+h,y+k)=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]$$

$$+h\frac{\partial}{\partial x}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]$$

$$+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]+\ldots.$$

$$f(x+h,y+k)=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]$$

$$+\left[h\frac{\partial}{\partial x}f(x,y)+hk\frac{\partial^2}{\partial x\partial y}f(x,y)+h\frac{k^2}{2!}\frac{\partial^3}{\partial x\partial y^2}f(x,y)+h\frac{k^3}{3!}\frac{\partial^4}{\partial x\partial y^3}f(x,y)+\ldots.\right]$$

$$+\left[\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x,y)+\frac{h^2}{2!}k\frac{\partial^3}{\partial x^2\partial y}f(x,y)+\frac{h^2}{2!}\frac{k^2}{2!}\frac{\partial^4}{\partial x^2\partial y^2}f(x,y)+\frac{h^2}{2!}\frac{k^3}{3!}\frac{\partial^5}{\partial x^2\partial y^3}f(x,y)+\ldots.\right]$$

$$+\left[\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x,y)+\frac{h^3}{3!}k\frac{\partial^4}{\partial x^3\partial y}f(x,y)+\frac{h^3}{3!}\frac{k^2}{2!}\frac{\partial^5}{\partial x^3\partial y^2}f(x,y)+\frac{h^3}{3!}\frac{k^3}{3!}\frac{\partial^6}{\partial x^3\partial y^3}f(x,y)+\ldots.\right]+\ldots.$$

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial x\partial y}+\frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3}+\frac{h^2}{2!}k\frac{\partial^3 f}{\partial x^2\partial y}+h\frac{k^2}{2!}\frac{\partial^3 f}{\partial x\partial y^2}+\frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right)+\ldots.$$

$$=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3}+3h^2k\frac{\partial^3 f}{\partial x^2\partial y}+3hk^2\frac{\partial^3 f}{\partial x\partial y^2}+k^3\frac{\partial^3 f}{\partial y^3}\right)+\ldots.$$

or

$$f(x+h,y+k)=f(x,y)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\frac{1}{4!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^4 f+\ldots.$$

NOTE:

1. Replacing *x* by *a* and *y* by *b*, we get

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}\right)f(a,b)$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3}{\partial x^3}+3h^2k\frac{\partial^3}{\partial x^2\partial y}+3hk^2\frac{\partial^3}{\partial x\partial y^2}+k^3\frac{\partial^3}{\partial y^3}\right)f(a,b)+....$$

And, now putting $h = x\text{-}a$ and $k = y\text{-}b$, we get

$$f(x,y)=f(a,b)+\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)+$$

$$\frac{1}{2!}\left((x-a)^2\frac{\partial^2}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2}{\partial x\partial y}+(y-b)^2\frac{\partial^2}{\partial y^2}\right)f(a,b)$$

$$+\frac{1}{3!}\left((x-a)^3\frac{\partial^3}{\partial x^3}+3(x-a)^2(y-b)\frac{\partial^3}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3}{\partial x\partial y^2}+(y-b)^3\frac{\partial^3}{\partial y^3}\right)f(a,b)+....$$

or

$$f(x,y)=f(a,b)+\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^2f(a,b)$$

$$+\frac{1}{3!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^3f(a,b)+....$$

which is called the Taylor's series expansion of $f(x, y)$ about (a, b). This form of Taylor's series is practically applicable to find the expansion of given two variables functions. This form of Taylor's series is sometimes called Taylor's series expansion of $f(x, y)$ in powers of $(x–a)$ and $(y–b)$.

2. Putting $a = 0$ and $b = 0$ in above equation, we get

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2f(0,0)$$

$$+\frac{1}{3!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^3f(0,0)+....$$

or

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x^2\frac{\partial^2}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)f(0,0)$$

$$+\frac{1}{3!}\left(x^3\frac{\partial^3}{\partial x^3}+3x^2y\frac{\partial^3}{\partial x^2\partial y}+3xy^2\frac{\partial^3}{\partial x\partial y^2}+y^3\frac{\partial^3}{\partial y^3}\right)f(0,0)+....$$

This is called **Maclaurin's series** expansion of $f(x, y)$, which is a special case of Taylor's series.

ILLSTRATIVE EXAMPLES

Example 1. Find the Taylor's series expansion of f(x, y) = $\tan^{-1}\frac{y}{x}$ about (1, 1) upto and inclusive of seconddegree terms. Hence compute f(1.1, 0.9) approximately.

Solution: Given $f(x,y)=\tan^{-1}\frac{y}{x} \Rightarrow f(1,1)=\tan^{-1}1=\frac{\pi}{4}$

$$f_x(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2} \Rightarrow f_x(1,1)=-\frac{1}{2}$$

$$f_y(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2} \Rightarrow f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(x,y)=-y(-1)\frac{2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1)=\frac{1}{2}$$

$$f_{xy}(x,y)=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1)=0$$

$$f_{yy}(x,y)=x(-1)\frac{2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1)=-\frac{1}{2}.$$

Similarly,

$f_{xxx}(1,1)=-\frac{1}{2}$, $f_{xxy}(1,1)=-\frac{1}{2}$, $f_{xyy}(1,1)=\frac{1}{2}$ and $f_{yyy}(1,1)=\frac{1}{2}$.

Using Taylor's series expansion of $f(x, y)$ about (1, 1), we have

$$f(x,y)=f(1,1)+[(x-1)f_x(1,1)+(y-1)f_y(1,1)]+$$
$$\frac{1}{2!}[(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)]$$
$$+\frac{1}{3!}[(x-1)^3 f_{xxx}(1,1)+3(x-1)^2(y-1)f_{xxy}(1,1)+3(x-1)(y-1)^2 f_{xyy}(1,1)+(y-1)^3 f_{yyy}(1,1)]+....$$

$$\Rightarrow \tan^{-1}\frac{y}{x}=\frac{\pi}{4}+[(x-1)\left(-\frac{1}{2}\right)+(y-1)\left(\frac{1}{2}\right)]+$$
$$\frac{1}{2!}[(x-1)^2\left(\frac{1}{2}\right)+2(x-1)(y-1).0+(y-1)^2\left(-\frac{1}{2}\right)]$$
$$+\frac{1}{3!}[(x-1)^3\left(-\frac{1}{2}\right)+3(x-1)^2(y-1)\left(-\frac{1}{2}\right)+3(x-1)(y-1)^2\left(\frac{1}{2}\right)+(y-1)^3\left(\frac{1}{2}\right)]+....$$

$$\Rightarrow \tan^{-1}\frac{y}{x}=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2-\frac{1}{12}(x-1)^3$$
$$-\frac{1}{4}(x-1)^2(y-1)+\frac{1}{4}(x-1)(y-1)^2+\frac{1}{12}(y-1)^3+....$$

Now

$$f(1.1,0.9)=\frac{\pi}{4}-\frac{1}{2}(0.1)+\frac{1}{2}(-0.1)+\frac{1}{4}(0.1)^2-\frac{1}{4}(-0.1)^2-\frac{1}{12}(0.1)^3$$
$$-\frac{1}{4}(0.1)^2(-0.1)+\frac{1}{4}(0.1)(-0.1)^2+\frac{1}{12}(-0.1)^3+....$$
$$=0.6857 \text{ (Approximately)}$$

Example 2. Expand $f(x,y)=e^x\log(1+y)$ in powers of x and y upto third order partial derivative terms.

Solution: Given $f(x,y)=e^x\log(1+y)\Rightarrow f(0,0)=0$

$$f_x(x,y)=e^x\log(1+y)\Rightarrow f_x(0,0)=0$$
$$f_y(x,y)=\frac{e^x}{1+y}\Rightarrow f_y(0,0)=1$$
$$f_{xx}(x,y)=e^x\log(1+y)\Rightarrow f_{xx}(0,0)=0$$
$$f_{xy}(x,y)=\frac{e^x}{1+y}\Rightarrow f_{xy}(0,0)=1$$
$$f_{yy}(x,y)=-\frac{e^x}{(1+y)^2}\Rightarrow f_{yy}(0,0)=-1$$
$$f_{xxx}(x,y)=e^x\log(1+y)\Rightarrow f_{xxx}(0,0)=0$$
$$f_{xxy}(x,y)=\frac{e^x}{1+y}\Rightarrow f_{xxy}(0,0)=1$$
$$f_{xyy}(x,y)=-\frac{e^x}{(1+y)^2}\Rightarrow f_{xyy}(0,0)=-1$$
$$f_{yyy}(x,y)=\frac{2e^x}{(1+y)^3}\Rightarrow f_{yyy}(0,0)=2$$

Now Maclaurin's series expansion of $f(x, y)$ gives

$$f(x,y)=f(0,0)+[xf_x(0,0)+yf_y(0,0)]+\frac{1}{2!}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]$$
$$+\frac{1}{3!}[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)]+....$$
$$\Rightarrow e^x\log(1+y)=0+[x.0+y.1]+\frac{1}{2!}[x^2.0+2xy.1+y^2(-1)]$$
$$+\frac{1}{3!}[x^3.0+3x^2y.1+3xy^2(-1)+y^3.2]+....$$
$$=y+xy-\frac{1}{2}y^2+\frac{1}{2}x^2y-\frac{1}{2}xy^2+\frac{1}{3}y^3+....$$

Example 3. Expand $x^2y+3y-2$ in powers of (x-1) and (y+2) using Taylor's theorem

Solution: Let $f(x,y) = x^2y + 3y - 2$ and we know expansion of $f(x,y)$ in powers of (*x*-*a*) and (*y*-*b*) using Taylor's theorem is given by

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \ldots$$

Here $a = 1, b = -2$ and $f(x,y) = x^2y + 3y - 2$, then

$f(1,-2) = 1(-2) + 3(-2) - 2 = -10$

$f_x(x,y) = 2xy \Rightarrow f_x(1,-2) = -4$

$f_y(x,y) = x^2 + 3 \Rightarrow f_y(1,-2) = 4$

$f_{xx}(x,y) = 2y \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy}(x,y) = 2x \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy}(x,y) = 0 \Rightarrow f_{yy}(1,-2) = 0$

$f_{xxx}(x,y) = 0 \Rightarrow f_{xxx}(1,-2) = 0$

$f_{xyy}(x,y) = 0 \Rightarrow f_{xyy}(1,-2) = 0$

$f_{xxy}(x,y) = 2 \Rightarrow f_{xxy}(1,-2) = 2$

$f_{yyy}(x,y) = 0 \Rightarrow f_{yyy}(1,-2) = 0$ and all higher order partial derivatives vanish.

Substituting these values, we get the required Taylor's series

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \frac{1}{3!}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$

$$\Rightarrow x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

.

Example 4. Expand e^{ax+by} by using Maclaurin's Theorem upto the third term.

Solution: Since the expansion required in power of x, y the point (a,b) associated with (0,0) and the expansion of f (x,y) about (0,0) is given by;

$$f(x, y) = f(0,0) + \left[x f_{xx}(0,0) + y f_{yy}(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots\cdots\cdots$$

The function and its partial derivatives evaluated at (0,0) is as follows:

$$\begin{aligned}
f(x, y) &= e^{ax + by} &\to\ & 1\\
f_x &= ae^{ax + by} &\to\ & a\\
f_y &= be^{ax + by} &\to\ & b\\
f_{xx} &= a^2e^{ax + by} &\to\ & a^2\\
f_{xy} &= abe^{ax + by} &\to\ & ab\\
f_{yy} &= b^2e^{ax + by} &\to\ & b^2\\
f_{yyy} &= b^3e^{ax + by} &\to\ & b^3\\
f_{xxx} &= a^3e^{ax + by} &\to\ & a^3\\
f_{xyy} &= ab^2e^{ax + by} &\to\ & ab^2\\
f_{xxy} &= a^2be^{ax + by} &\to\ & a^2b
\end{aligned}$$

Substitute these value in the expansion of f (x,y), we get;

$$f(x, y) = 1 + (ax + by) + \frac{(ax + by)^2}{2!} + \frac{(ax + by)^3}{3!} + \cdots\cdots\cdots .$$

Example 5. Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in power of h, k up to end inclusive of the second degree terms.

Solution: Here $f(x + h, y + k) = \frac{(x+h)(y+k)}{x+h+y+k}$

Putting h = k = 0, we have $f(x, y) = \frac{xy}{x+y}$

$f_x = \frac{(x+h).y - xy.1}{(x+y)^2} = \frac{y^2}{(x+y)^2}$, $f_y = \frac{y^2}{(x+y)^2}$, by Symmetry

$f_{xx} = -\frac{2y^2}{(x+y)^3}$, $f_{xx} = -\frac{2x^2}{(x+y)^3}$,

$f_{xy} = \frac{(x+y)^2 .2x - x^2 .2(x+y)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$

$\therefore \frac{(x+h)(y+k)}{x+h+y+k} = f(x + h, y + k) = f(x, y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \ldots.$

$= \frac{xy}{x+y} + \left[h.\frac{y^2}{(x+y)^2} + k.\frac{x^2}{(x+y)^2}\right] + \frac{1}{2}\left[h^2 .\frac{-2y^2}{(x+y)^3} + 2hk\ \frac{2xy}{(x+y)^3} + k^2 .-\frac{2x^2}{(x+y)^3}\right] + \ldots$

$= \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}.h + \frac{x^2}{(x+y)^2}.k - \frac{y^2}{(x+y)^3}.h^2 + \frac{2xy}{(x+y)^3}.hk - \frac{x^2}{(x+y)^3}.k^2 + \ldots.$

CHECK YOUR PROGRESS

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True or false Questions

Problem 1: We can not expand $e^x \sin y$ in power of x and y using Taylor's theorem as far as terms of third degree.

Problem 2: If f(x , y) = $e^x \log(1+y)$ then f_{yy} (0 , 0) = -1.

Problem 3: The expansion of function of two variables by Taylor's theorem is unique.

15.4 JACOBIAN:

The Jacobian matrix contains information about the local behaviour of a function. The Jacobian matrix can be seen as a representation of some local factor of change. It consists of first order partial derivatives. If we take the partial derivatives from the first order partial derivatives, we get the second order partial derivatives, which are used in the Hessian matrix. The Hessian matrix is used for the Second Partial DerivativeTest with which we can test, whether a point x is a local maximum, minimum or a so called saddle point .

With the Jacobian matrix we can convert from one coordinate system into another. The Jacobian determinant is used when making a change of variables when evaluating a multiple integral of a function over a region within its domain. To accommodate for the change of coordinates the magnitude of the Jacobian determinant arises as a multiplicative factor within the integral. The Jacobian matrix was developed by Carl Gustav Jacob Jacobi (1804–1851), a German Jewish mathematician.

If u_1 , u_2 , … , u_n are function of n independent variables x_1 , x_2 , … , x_n then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \cdots & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \cdots & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \cdots & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Is called the Jacobian of $u_1, u_2, \dots, u_n$ with respect to $x_1, x_2, \dots, x_n$ and is denoted either by $\frac{\partial(u_1, u_2 \dots u_n)}{\partial(x_1, x_2 \dots x_n)}$ or by J $(u_1, u_2 \dots u_n)$. Thus if u and v are functions of two independent variables x and y, we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = J(u, v)$$

Similarly if u, v and w are functions of three independent variables x, y and z, we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J(u, v, w).$$

Properties of Jacobians (Chain Rules)

1. If u, v are functions of r, s where r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,v)}$.

Proof: Since u, v are composite functions of x, y

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} = u_r r_x + u_s s_x$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \dots (1)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} = v_r r_x + v_s s_x$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} = v_r r_x + v_s s_x$$

Now, $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} . \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$

$$= \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_x + v_s s_x \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad [\text{ using (1) }]$$

$$= \frac{\partial(u,v)}{\partial(x,y)}.$$

Note: if u_1, u_2, u_3 are functions of y_1, y_2, y_3 are functions of x_1, x_2, x_3 then $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)} . \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$

2. If J_1 is the Jacobian of u , v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then $J_1J_2 = 1$, i.e., $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Proof: Let u = u(x , y) and v = v(x , y) so that u and v are functions of x,
y.

Differentiating partially w.r.t. u and v, we get

$$1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} = u_x x_u + u_y y_u$$

$$0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} = u_x x_v + u_y y_v \ldots (1)$$

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} = v_x x_u + v_y y_u$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

Now $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \cdot \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

> **Note:** It can be extended to three variables as
> $\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1.$

3. If functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

Proof: It is given that theu, v and ware not independent variables, then there will be a relation F(u, v, w) = 0, which will connect these independent variables.

Differentiating this relation with respect to x, y and z, we get

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x} = 0 \quad \ldots. (1)$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial y} = 0 \quad \ldots. (2)$$

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial z} = 0 \quad \ldots. (3)$$

Eliminating $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ from (1) , (2) and (3) , we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

ILLSTRATIVE EXAMPLES

Example 1. If x = r sinθ Cosφ, y = r sinθ Sinφ , show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$ = r^2 Sinθ.

Solution: We have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta\ cos\varphi & r\ cos\theta\ cos\varphi & -rsin\theta\ sin\varphi \\ sin\theta\ sin\varphi & r\ cos\theta\ sin\varphi & r\ sin\theta\ cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

= cos θ (r^2 sin θ cosθ cos$^2\varphi$ + r^2 sin θ cosθ sin$^2\varphi$)
+ r sin θ(r sin$^2\theta$ cos$^2\varphi$ + r sin$^2\theta$ sin$^2\varphi$) , expanding the determinant along the third row

= r^2 sin θ cos$^2\theta$ + r^2 sin$^3\theta$ = r^2 sin θ(cos$^2\theta$ + sin$^2\theta$)
= r^2 sin θ.

Example 2. If u = $\frac{x^2+y^2+z^2}{x}$, v = $\frac{x^2+y^2+z^2}{y}$, w = $\frac{x^2+y^2+z^2}{z}$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1-\frac{y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix}$$

Applying $C_1 \rightarrow \frac{y}{x} C_2 + \frac{z}{x} C_3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{x^2+y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{x^2+y^2+z^2}{xy} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{x^2+y^2+z^2}{xz} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix}$$

$$= \frac{x^2+y^2+z^2}{x^2 .\ xy .xz}\begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy - \frac{x(x^2+z^2)}{y} & 2xz \\ 1 & 2xy & xz - \frac{x(x^2+y^2)}{z} \end{vmatrix}$$

$$= \frac{x^2+y^2+z^2}{x^2\ yz}\begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\frac{x(x^2+y^2+z^2)}{y} & 2xz \\ 0 & 2xy & -\frac{x(x^2+y^2+z^2)}{z} \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{x^2y^2z^2}{(x^2+y^2+z^2)^3}$$

Example 3. If x = r $\sin\theta$, y = r $\sin\theta$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$. also prove that JJ' = 1

Solution: $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

= r $(cos^2\theta + sin^2\theta)$ = r

Now, r = $\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Hence, $\frac{\partial(x,y)}{\partial(r,\theta)} . \frac{\partial(r,\theta)}{\partial(x,y)}$ = r. $\frac{1}{r}$ = 1

$\Longrightarrow$ JJ' = 1.

Example 4. Calculate the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

u = xyz, v = xy + yz + zx , w = x + y + z

Solution: $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$

(By $C_2 \longrightarrow C_2 - C_1$ And $C_3 \longrightarrow C_3 - C_1$) then

$$= \begin{vmatrix} yz & z(x-y) & y(x-z) \\ y+z & x-y & x-z \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} z(x-y) & y(x-z) \\ x-y & x-z \end{vmatrix}$$

$$= (x-y)(x-z)\begin{vmatrix} z & y \\ 1 & 1 \end{vmatrix}$$

$$= (x-y)(z-y)(x-z)$$

$$= (x-y)(y-z)(z-x) .$$

Example 5: Let x(u, v) = $u^2 - v^2$, y(u, v) = 2 uv. Therefore, find the Jacobian J (u, v).

Solution: Given thatx (u, v) = $u^2 - v^2$ and y (u, v) = 2 uv

We know that,

$$J(x, y) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2.$$

Example 6: Findthe Jacobian of p, q, r with respect to x, y, z given p = x + y + z, q = y + z, r = z.

Solution: We have to find

$$J = \frac{\partial(p,q,\ r)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

$\therefore \frac{\partial p}{\partial x} = 1, \frac{\partial p}{\partial y} = 1, \frac{\partial p}{\partial z} = 1, \frac{\partial q}{\partial x} = 0, \frac{\partial q}{\partial y} = 1, \frac{\partial q}{\partial z} = 1, \frac{\partial r}{\partial x} = 0, \frac{\partial r}{\partial y} = 0,$

$\frac{\partial r}{\partial z} = 1$, then we get

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding we get

J = 1(1 – 0) = 1.

CHECK YOUR PROGRESS

True or false Questions

Problem 4: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix}$.

Problem 5: If x = r sinθ Cosφ, y = r sinθ Sinφ , then $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sqrt{(1 - \cos^2\theta)}$.

Problem 6:The Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

u = x + 2y + z , v = x + 2y + 3z , w = 2x + 3y + 5z is 2.

Problem 7: ifu = $x^2 - y^2$, v = 2xy and x = r cosθ , y = r sinθ then $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Problem 8: Jacobian of u_1 , u_2 , … , u_n with respect to variables x_1 , x_2 , … , x_n is denotedby$\frac{\partial\,(u_1,u_2 \dots u_n)}{\partial\,(x_1,x_2 \dots x_n)}$ or by J $(u_1,\ u_2\ \dots\ u_n)$.

Problem 9: If functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

Problem 10: $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 0$

15.5 SUMMARY:

The Taylor series or Taylor expansion of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point.It is extensively used for the elaboration of mathematical series. A function can be approximated by using a finite number of terms of its Taylor series. Taylor's theorem provides quantitative estimations on the error which were introduced by the usage of such an approximation.Jacobian is the determinant of the Jacobian matrix. The matrix will contain all partial derivatives of a vector function. The main use of Jacobian is found in the transformation of coordinates. It deals with the concept of differentiation with coordinate transformation.

15.6 GLOSSARY:

i. Function of two variables
ii. Partial Derivatives
iii. Determinant
iv. Trigonometric functions
v. Functions
vi. Taylor’s Theorem
vii. Jacobian

15.7 REFERENCES:

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15.9 TERMINAL QUESTIONS:

Q 1. Find the first six terms of the expansions of the function $e^x \log(1+y)$ in a Taylor series in the neighbourhood of the point (0, 0).

Q 2. Expand e^{xy} at (1, 1).

Q 3. Expand $e^x \sin x$ in powers of x and y as far as terms of the third degree.

Q 4. If x = uv , y = $\frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,v)}$.

Q 5. If u = $\frac{y-x}{1+xy}$ and v = $\tan^{-1} y - \tan^{-1} x$, find $\frac{\partial(u,v)}{\partial(x,v)}$.

Q 6. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, then show that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$ = 4.

Q 7. If x = $e^v \sec u , y = e^v \tan u$, then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$.

Q 8. If u = x + 2y + z, v = x – 2y + 3z and w = 2xy – xz + 4yz – 2z^2 ,they are not independent. Also find the relation between u, v and w .

Q 9. Find the Jacobian of the functions $y_1 = (x_1 - x_2)(x_2 + x_3)$
$y_2 = (x_1 + x_2)(x_2 - x_3)$, $y_3 = x_3(x_1 - x_3)$, hence show that the functions are not independent. Find the relation between them.

Q 10. Use The Jacobian To Prove That The Functions u = x + y – z, v = x – y + z and w = x^2 + y^2 + z^2 – 2yz Are Not Independent Of One Another. Find The Relation Between Them.

15.10 ANSWERS

CHECK YOUR PROGRESS

CYQ1. False
CYQ2. True
CYQ3. True
CYQ 4. False
CYQ 5. True
CYQ 6. False
CYQ 7. True
CYQ 8. True
CYQ 9. True
CYQ 10. False

Terminal Questions:

TQ 1. $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \dots$
TQ 2. $e\left\{1 + (x-1) + (y-1) + \frac{1}{2!}((x-1)^2 + 4(x-1)(y-1) + (y-1)^2) + \cdots\right\}$
TQ 3. $y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \dots$
TQ 4. $\frac{(u-v)^2}{4uv}$
TQ 5. 0
TQ 7. $- e^{2v}\sec u$
TQ 8. $u^2 - v^2 = 4w$.
TQ 9. $y_1 + y_2 - 2y_3 = 0$
TQ 10. $(u+v)^2 + (u-v)^2 = 4w$.

E-mail:info@uou.ac.in
Website: https://www.uou.ac.in/

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University, Haldwani, Nainital-263139
Uttarakhand, India
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,