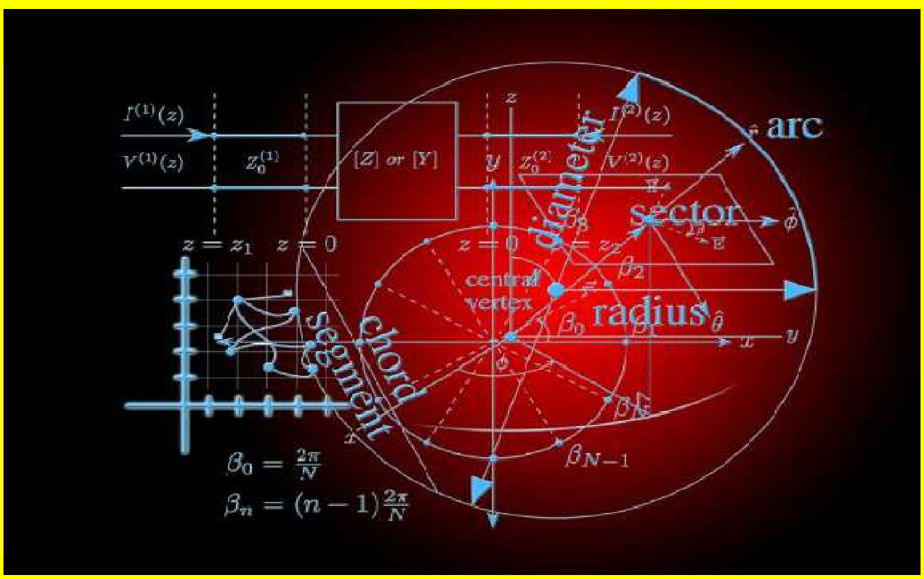
MSCPH501

M. Sc. Ist Semester

MATHMATICAL PHYSICS



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MATHMATICAL PHYSICS

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Contents

Course: MATHMATICAL PHYSICS **Course code: MSCPH501**

Credit: 3

UNIT 1: VECTOR

STRUCTURE:

1.0 Objective:

After reading this unit you will be able to understand:
* Basic idea about vector and vector types
* Vector representation, addition, subtraction
* Multiplication of vectors
* Scalar product, vector product and triple product
* Differentiation of vector
* Gradient, Divergence and curl
* Vector integration
* Gauss Divergence theorem
* Poisson's equation and Laplace's equation
* Green's theorem and Stoke's theorem
* Curvilinear coordinate systems

1.1 Introduction:

On the basis of direction, the physical quantities may be divided into two main classes.

1.1.1 Scalar quantities: The physical quantities which do not require direction for their representation. These quantities require only magnitude and unit and are added according to the usual rules of algebra. Examples of these quantities are: mass, length, area, volume, distance, time speed, density, electric current, temperature, work etc.

1.1.2 Vector quantities: The physical quantities which require both magnitude and direction and which can be added according to the vector laws of addition are called vector quantities or vector. These quantities require magnitude, unit and direction. Examples are weight, displacement, velocity, acceleration, magnetic field, current density, electric field, momentum angular velocity, force etc.

1.2 Vector representation:

Any vector quantity say A, is represented by putting a small arrow above the physical quantity like $\vec{A}$. In case of print text a vector quantity is represented by bold type letter like **A**. The vector can be represented by both capital and small letters. The magnitude of a vector quantity A is denoted by $|\vec{A}|$ or *mod A* or some time light forced italic letter *A*. We should understand following types of vectors and their representations.

1.2.1 Unit vector

A unit vector of any vector quantity is that vector which has unit magnitude. Suppose $\vec{A}\ is\ a\ vector$ then unit vector is defined as

$$\hat{A} = \frac{\vec{A}}{|A|}.$$

The unit vector is denoted by $\hat{A}$ and read as 'A *unit vector or A hat'*. It is clear that the magnitude of unit vector is always 1. A unit vector merely indicates direction only. In Cartesian coordinate system, the unit vector along x, y and z axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively as shown in figure 1.1.

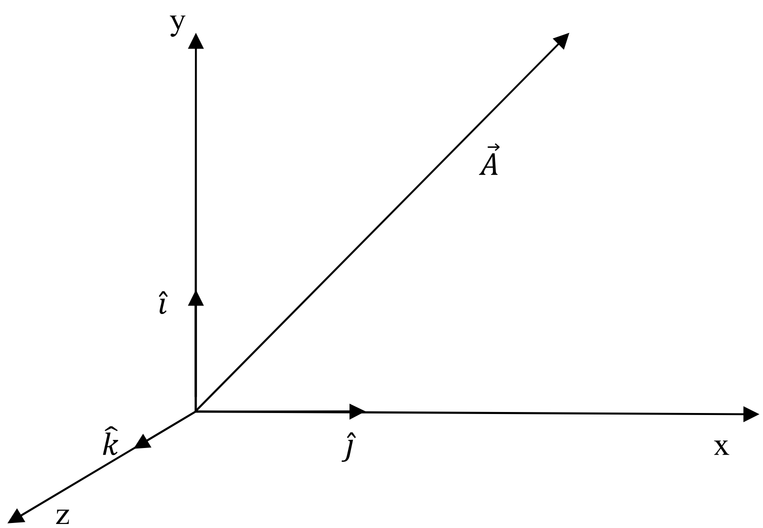

Figure 1.1: Vector representation

Any vector in Cartesian coordinate system can be represented as

$$A = \hat{\imath}\, A_x + \hat{\jmath}\, A_y + \hat{k}\, A_z .$$

Where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vector along x, y, z axis and, A_x, A_y, A_z are the magnitudes projections or components of $\vec{A}$ along x, y, z axis respectively.

The unit vector in Cartesian coordinate system can be given as:

$$\hat{A} = \frac{\hat{\imath}\, A_x + \hat{\jmath}\, A_y + \hat{k}\, A_z}{\sqrt{A_x{}^2 + B_x{}^2 + C_x{}^2}}.$$

1.2.2 Zero vector or Null vector:

A vector with zero magnitude is called zero vector or null vector. The condition for null vector is $|\vec{A}| = 0$.

1.2.3 Equal vectors:

If two vectors have same magnitude and same direction, the vectors are called equal vector.

1.2.4 Like vectors:

If two or more vectors have same direction, but may have different magnitude, then the vectors are called like vectors.

1.2.5 Negative vector:

A vector is called negative vector with reference to another one, if both have same magnitude but opposite directions.

1.2.6 Collinear vectors:

All the vectors parallel to each other are called collinear vectors. Basically collinear means the line of action is along the same line.

1.2.7 Coplanar vector:

All the vectors whose line of action lies on a same plane are called coplanar vectors. Basically coplanar means lies on the same plane.

1.2.8 Addition and subtraction of vectors:

The addition of two vectors can be performed by triangle law or parallelogram law. According to triangle law if a vector is placed at the head of another vector, and these two vectors represent two sides of a triangle then the third side or a vector drawn for the tail end of first to the head end of second represents the resultant of these two vectors. If vectors $\vec{A}$ and $\vec{B}$ are two vectors as shown in figure 1.4, then resultant $\vec{R}$ can be obtained by applying triangle law.

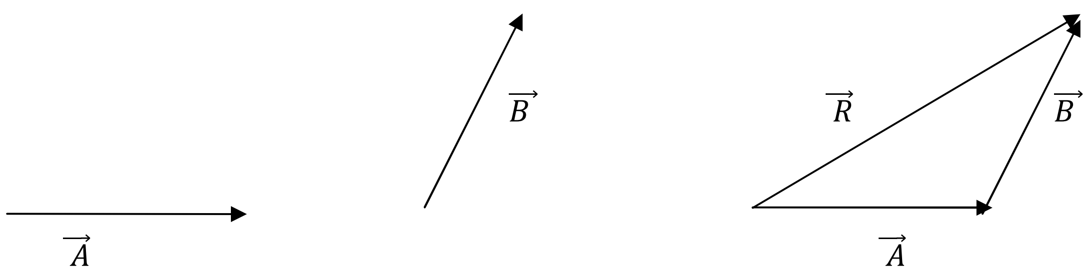

Figure 1.2: Addition of two vectors

If the angle between $\vec{A}$ and $\vec{B}$ is θ, and resultant $\vec{R}$ makes angle α with vector $\vec{A}$ then magnitude of $\vec{R}$ is

$$|R| = \sqrt{A^2 + B^2 + 2AB\cos\theta}\,.$$

The angle α is given as

$$\alpha = tan^{-1}\frac{B\sin\theta}{A + Bcos\,\theta}.$$

You should notice that all three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ are concurrent *i.e.* vectors acting on the same point O. The same addition can be shown by Figure 1.3

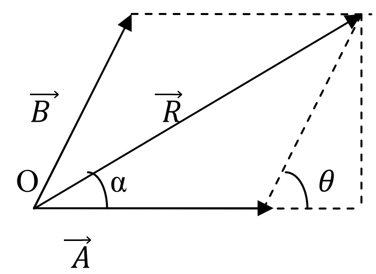

Figure 1.3: Addition of two vectors

Similarly the subtraction of a vector vectors $\vec{B}$ from another vector, $\vec{A}$ is the addition of vectors $\vec{A}$ and $(-\vec{B})$.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}).$$

1.2.9 Resolution of vector:

A vector can be resolved into two or more vectors and these vectors can be added in accordance with the polygon law of vector addition, and finally original vector can be obtained. If a vector is resolved into three components which are mutually perpendicular to each other, called rectangular components or mutually perpendicular components of a vector. These components are along the three coordinate axes x, y and z respectively as show in Figure 1.4.

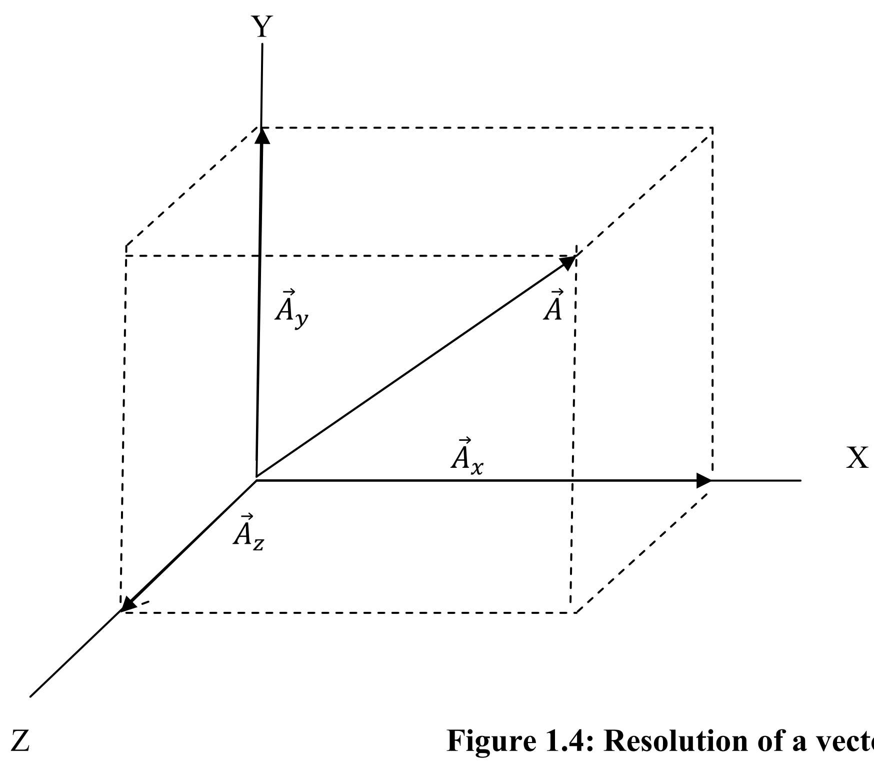

Figure 1.4: Resolution of a vector

If the unit vectors along x, y and x axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively then any vector $\vec{A}$ can be give as

$$\vec{A} = \hat{\imath}\,A_x + \hat{\jmath}\,A_y + \hat{k}\,A_z\ .$$

$\vec{A}$ constitutes the diagonal of a parallelepiped and, A_x, A_y and A_z are the edges along x, y and z axes respectively. $\vec{A}$ is the polynomial addition of A_x, A_y and A_z. The rectangular components A_x, A_y and A_z can be considered as orthogonal projections of vector $\vec{A}$ on x, y and z axis respectively. Mathematically, the magnitude of vector $\vec{A}$ can be given as:

$$\boldsymbol{A} = |\vec{A}| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}\ .$$

1.2.10 Position vector:

In Cartesian co-ordinate system the position of any point P(x, y, z) can be represented by a vector **r**, with respect to origin O, the vector **r** is called position vector of point P. Position vector is often denoted by $\vec{r}$. Figure1.5 shows the position vector of a point P(x, y, z) in Cartesian coordinate system. If we have two vectors $\vec{P}$ and $\vec{Q}$ with position vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ respectively such as

$\boldsymbol{r_1} = \hat{\imath}\,x_1 + \hat{\jmath}\,y_1 + \hat{k}\,z_1$

$\boldsymbol{r_2} = \hat{\imath}\,x_2 + \hat{\jmath}\,y_2 + \hat{k}\,z_2$.

Where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of point P and Q respectively.

Now the vector PQ can be given as

PQ = OQ – OP (∴ OP + PQ = OQ)

$\vec{r} = \vec{r}_2 - \vec{r}_1$.

Therefore, vector PQ = position vector of Q − position vector of P

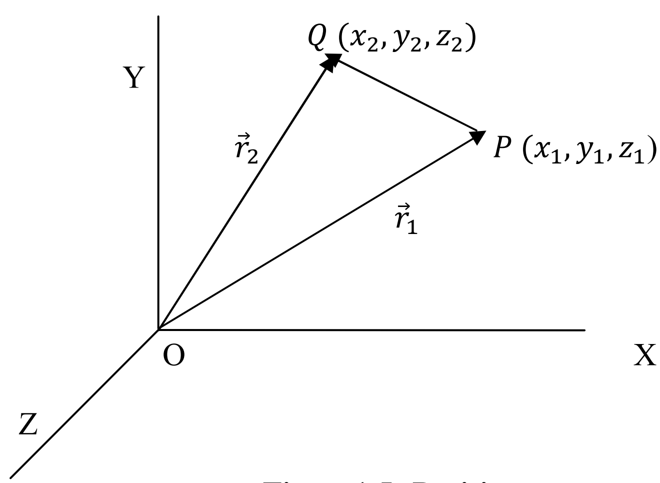

Figure1.5: Position vectors

1.3 Multiplication of vectors

1.3.1 Multiplication and division of a vector by scalar:

If a vector **P** is multiplied by a scalar quantity m then its magnitude becomes m times. For example if m is a scalar and $\vec{A}$ is a vector then its magnitude becomes m times.

Similarly, in case of division of a vector A by a non zero scalar quantity n, its magnitude becomes 1/n times.

1.3.2 Product of two vectors:

There are two distinct ways in which we can define the product of two vectors.

1.3.2.1 Scalar product or dot product:

Scalar product of two vectors **P** and **Q** is defined as the product of magnitude of two vectors P and Q and cosine of the angle between the directions of these vectors.

If θ is the angle between two vectors $\vec{P}$ and $\vec{Q}$, then dot product (*read as* $\vec{P}\ dot\ \vec{Q}$) of two vectors is given by-

$$\overrightarrow{P}.\vec{Q} = PQ\cos\theta = P\,(Q\cos\theta)$$

$$= P\,(projection\ of\ vector\ Q\ on\ P) = P.MN.$$

The Figure 1.6 shows the dot product. The resultant of dot product or scalar product of two vectors is always a scalar quantity. In physics the dot product is frequently used, the simplest example is work which is dot product of force and displacement vectors.

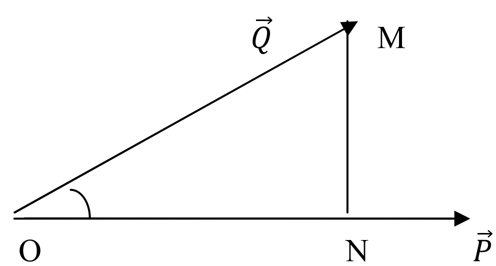

Figure1.6: Dot product of two vectors

Important properties of dot product

(i) Condition for two collinear vectors:

If two vectors are parallel or angle between two vectors is 0 or π, then vectors are called collinear. In this case

$$\overrightarrow{P}.\vec{Q} = PQ\cos 0^o = PQ.$$

Then the product of two vectors is same as the product of their magnitudes.

(ii) Condition for two vector to be perpendicular to each other:

If two vectors are perpendicular to each other then the angle between these two vectors is 90^0, then

$\vec{P}.\vec{Q} = PQcos\ 90^o = 0.$

Hence two vectors are perpendicular to each other if and only if their dot product is zero.

In case of unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we know that these vectors are perpendicular to each other then

$\hat{\imath}\,.\,\hat{\jmath} = \hat{\imath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

similarly

$\hat{\imath}\,.\,\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1.$

(iii) Commutative law holds:

In case of vector dot product the commutative law holds. Then

$\vec{P}.\vec{Q} = \vec{Q}.\vec{P}.$

(iv) Distributive property of scalar product:

If P, Q and R are three vectors then according to distributive law

$\vec{P}.(\vec{Q} + \vec{R}) = \vec{P}.\vec{Q} + \vec{P}.\vec{R}.$

Example 1.1: Show that vector $\vec{A} = 3i + 6j - 2k$ and $\vec{B} = 4i - \hat{\jmath} + 3k$ are mutually perpendicular.

Solution: If the angle between $\vec{A}$ and $\vec{B}$ is θ then

$\vec{A}\,.\,\vec{B} = A\,B\cos\theta$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{(3i+6j-2k)\,(4i-j+3k)}{\sqrt{(A_x^2 + A_y^2 + A_z^2)}\,\sqrt{(B_x^2 + B_y^2 + B_z^2}} = 0$$

$\cos\theta = 0, \theta = 90^0$. Therefore the vectors are mutually perpendicular.

Example 1.2: A particle moves from a point (3,-4,-2) meter to another point (5,-6, 2) meter under the influence of a force $\vec{F} = (-3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\,)$ N. Calculate the work done by the force.

Solution: Suppose the particle moves from point A to B. Then displacement of particle is given by

$$\vec{r} = position\ vetor\ of\ B - position\ vetor\ A$$
$$\vec{r} = [(5-3)i + (-6+4)j + (2+2)k]\text{ meter}$$
$$\vec{r} = (2i - 2j + 4k)\text{ meter.}$$

Work done = $\vec{F}.\vec{r}$ = $[(-3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\,).(\,2i - 2j + 4k)]$N meter =2 joule.

1.3.2.2 Vector product or Cross Product

The vector product or cross product of two vectors is a vector quantity and defined as a vector whose magnitude is equal to the product of magnitudes of two vectors and sine of angle between them.
If $\vec{A}$ and $\vec{B}$ are two vectors then cross product of these two vectors is denoted by $\vec{A} \times \vec{B}$ (*read as* $\vec{A}\ cross\ \vec{B}$) and given as

$$\vec{A} \times \vec{B} = ABsin\emptyset\, \hat{n} = \vec{C}.$$

Where $\emptyset$ is the angle between vectors $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ $(i.e. normal\ to\ the\ plane\ containg\ \vec{A}\ and\ \vec{B})$.
Suppose $\vec{A}$ is along x axis and $\vec{B}$ is along y axis then vector product can be considered as an area of parallelogram OPQR as shown is figure 1.7 in XY plane whose sides are $\vec{A}$ and $\vec{B}$ and direction is perpendicular to plane OPQR i.e. along z axis. The cross product $\vec{A}$and $\vec{B}$ is positive if direction of $\emptyset$ ($\vec{A}\ to\ \vec{B}$) is positive or rotation is anticlockwise as show in figure 1.8, and negative if the rotation of $\emptyset$ ($\vec{A}\ to\ \vec{\boldsymbol{B}}$) is clockwise (Figure 1.12).

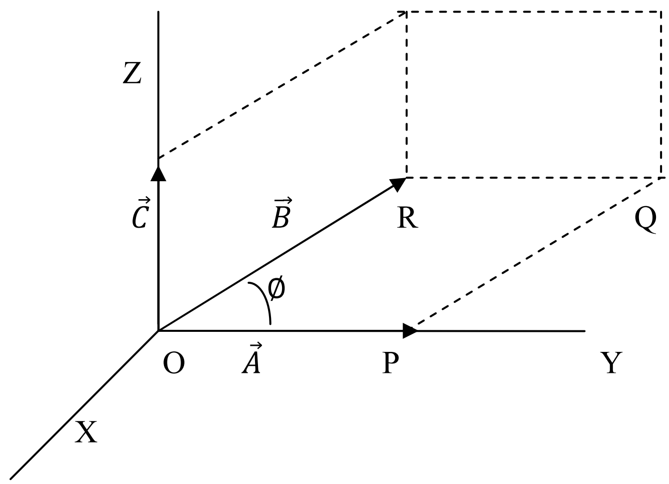

Figure 1.7 Vector product as area of parallelogram OPQR

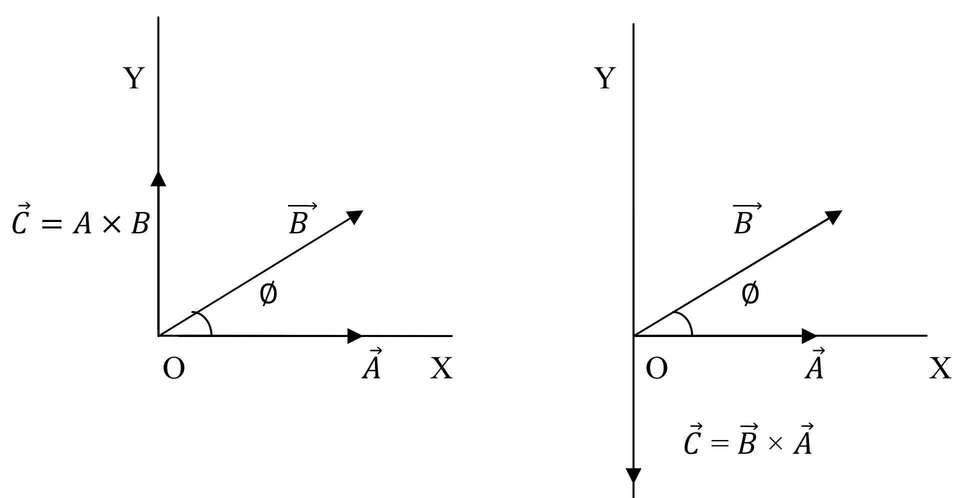

Figure1.8: Direction of vector product

Important properties of vector product

(i) **Commutative law does not hold**: From the definition of vector product of two vectors $\vec{A}$ and $\vec{B}$ the vector products are defined as

$\vec{A} \times \vec{B}$= AB sin$\emptyset\,\hat{n}$

$\vec{B} \times \vec{A}$ = AB sin$\emptyset$ $(-\hat{n})$ = $-$AB sin$\emptyset\,\hat{n}$ = $-\ \vec{A} \times \vec{B}$.

Since in case of $\vec{B} \times \vec{A}$ the angle of rotation becomes opposite to case $\vec{A} \times \vec{B}$, hence product becomes negative.

Therefore, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$.

(ii) **Distributive law holds**:

In case of vector product the distribution law holds.

$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

(iii) **Product of equal vectors**:

If two vectors are equal then the angle between them is zero, and vector product becomes

$$\vec{A} \times \vec{A} = |A||A| sin\emptyset\,\hat{n} = 0.$$

Hence the vector product of two equal vectors in always zero.

In case of Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$are unit vectors along x, y and z axes then

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0.$$

(iv) **Collinear vectors**: Collinear vectors are vectors parallel to each other. The angles between collinear vectors are always zero therefore

$$\vec{A} \times \vec{B} = |A||B| sin\emptyset\,\hat{n} = 0.$$

Thus, two vectors are parallel or anti-parallel or collinear if its vector product is 0.

(v) **Vector product of orthogonal vector** : If two vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other then angle between such vectors is $\emptyset = 90°$, therefore

$$\vec{A} \times \vec{B} = AB\ sin\emptyset\,\hat{n}$$

$$\vec{A} \times \vec{B} = |A||B|\,\hat{n}$$

In Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vector along x, y and z axes then

$$\boldsymbol{\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath}\ and\ \ \hat{k} \times \hat{\imath} = \hat{\jmath}}$$

$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}\ and\ \ \hat{\imath} \times \hat{k} = \hat{\jmath}.$$

(vi) **Determinant form of vector product**: If $\vec{A}$and $\vec{B}$ are two vectors given as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}.$$

Then,

$\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$

$$= A_xB_x\hat{\imath}\times\hat{\imath} + A_xB_y\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k} + A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_y\hat{\jmath}\times\hat{\jmath} + A_yB_z\hat{\jmath}\times\hat{k} + A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath} + A_zB_z\hat{k}\times\hat{k}$$

$$= A_xB_y\hat{k} - A_xB_z\hat{\jmath} - A_yB_x\hat{k} + A_yB_x\hat{\imath} + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}$$

(Since $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0 \; and \; \hat{\imath}\times\hat{k} = -\hat{\jmath}, \hat{\jmath}\times\hat{\imath} = -\hat{k}, \hat{k}\times\hat{\jmath} = -\hat{\imath}$)

$$= \hat{\imath}(A_yB_z - A_zB_y) - \hat{\jmath}\,(A_xB_z - A_zB_x) + \; k(A_xB_y - A_yB_x)$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Physical significance of vector product:

In physics, numbers of physical quantities are defined in terms of vector products. Some basic examples are illustrated below.

(i) **Torque:** Torque or moment of force is define as

$$\vec{\tau} = \vec{r}\times\vec{f}.$$

Where $\vec{\tau}$ is torque, $\vec{r}$ is position vector of a point P where the force $\vec{f}$ is applied. (Figure 1.9)

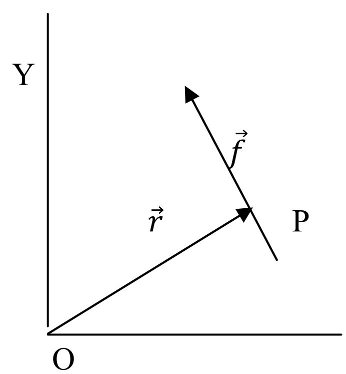

Figure 1.9

(ii) **Lorentz force on a moving charge in magnetic field**: if a charge q is moving in a magnetic field $\vec{B}$with a velocity $\vec{V}$at an angle with the direction of magnetic field then force $\vec{F}$experienced by the charged particle is give as;

$$\vec{F} = q(\vec{V}\times\vec{B})$$

This force is called Lorentz force and its direction is perpendicular to the direction of both velocity and magnetic field B.

(iii) **Angular Momentum**: Angular momentum is define as the moment of the momentum and given as:

$$\vec{L} = \vec{r}\times\vec{p}.$$

Where $\vec{r}$ is the radial vector of circular motion and $\vec{p}$ is the linear moment of the body under circular motion, and $\vec{L}$ is angular momentum along the direction perpendicular to both $\vec{r}$ and $\vec{p}$. The law of conservation of angular momentum is a significant property in all circular motions.

1.3.3. Product of three vectors:

If we consider three vectors$\vec{A}$, $\vec{B}$ and $\vec{C}$, we can define two types of triple products known as scalar triple product and vector triple product.

1.3.3.1 Scalar Triple product:

Let us consider three vectors $\vec{A}$, $\vec{B}$and $\vec{C}$ then the scalar triple product of these three vectors is defined as $\vec{A}.(\vec{B} \times \vec{C})$ and denoted as $[\vec{A}\ \vec{B}\vec{C}]$. This is a scalar quantity. If we consider $\overrightarrow{A}$, $\vec{B}$and $\vec{C}$ the three sides of a parallelepiped as shown in Figure 1.14 then $\vec{B} \times \vec{C}$ is a vector which represents the area of parallelogram OBDC which is the base of the parallelogram. The direction of $\vec{B} \times \vec{C}$ is naturally along Z axis (perpendicular to both $\vec{B}$ and $\vec{C}$). If $\emptyset$ is the angle between the direction of vectors ($\vec{B} \times \vec{C}$) and vector $\vec{A}$, then the dot product of vectors ($\vec{B} \times \vec{C}$) and vector $\vec{A}$ is given as (Figure 1.10)

$$\vec{A} \,.\, (\vec{B} \times \vec{C}) = |A||\vec{B} \times \vec{C}|cos\emptyset = ACos\emptyset(\vec{B} \times \vec{C}) = h.\,(\vec{B} \times \vec{C})$$

= Vertical height of parallelogram × area of base of parallelogram

= Volume of parallelogram = $[A\ B\ C]$.

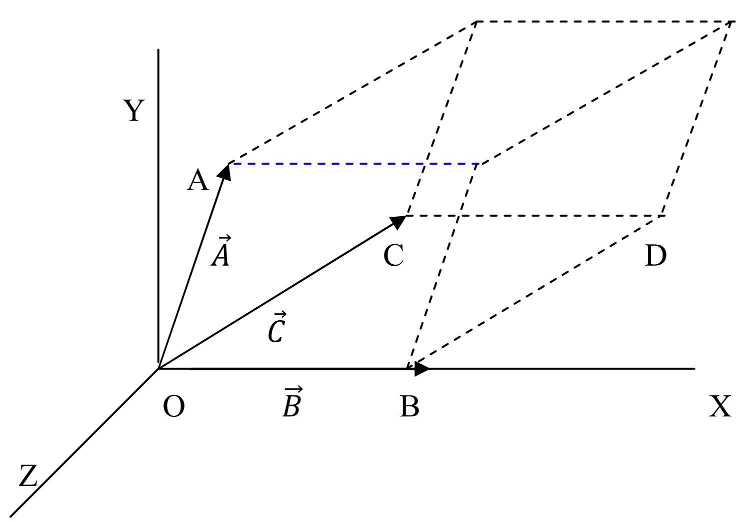

Figure 1.10: Volume of parallelogram represented by 3 vectors

Therefore, it is clear that $\vec{A}\,.(\vec{B}\times\vec{C})$ represents the volume of parallelepiped constructed by vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as its sides. Further, it is a scalar quantity as volume is scalar. It can also be noted that in case of scalar triple product the final product (volume of parallelepiped) remains same if the position of $\vec{A}, \vec{B}$ *and* $\vec{C}$ or dot and cross are interchanged.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B}) = (\vec{B}\times\vec{C}).\vec{A} = (\vec{C}\times\vec{A}).\vec{B} = (\vec{A}\times\vec{B}).\vec{C}.$$

Scalar triple product can also be explained by determinant as

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

In case of three vectors to be coplanar, it is not possible to construct a parallelepiped by using such three vectors as its sides; therefore the scalar triple product must be zero.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = 0.$$

1.3.3.2 Vector triple product:

The vector triple product of three vectors is define as

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}.$$

The vector triple product is product of a vector with the product of two another vectors. The vector triple product can be evaluated by determinant method as given below.

$$(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= i(B_yC_z - B_zC_y) - j(B_xC_z - B_zC_x) + k(B_xC_y - B_yC_x)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}.$$

As in cross product the vector $\vec{A}\times(\vec{B}\times\vec{C})$ will be perpendicular to plane containing vectors $\vec{A}$ and $(\vec{B}\times\vec{C})$. Since $(\vec{B}\times\vec{C})$ is itself in the direction perpendicular to plane containing $\vec{B}$ and $\vec{C}$, therefore the direction of $\vec{A}\times(\vec{B}\times\vec{C})$ will be along the plan containing $\vec{B}$ and $\vec{C}$, *hence is a linear* combination of $\vec{B}$ *and* $\vec{C}$.

1.4 Differentiation of vector:

Suppose $\vec{r}$ is the position vector of a particle situated at point P with respect to origin O. If particle moves with time, then vector $\vec{r}$ varies corresponding to time t, and $\vec{r}$ is said to be vector function of scalar variable t and represented as $\vec{r}$ =F(t) as shown in Figure 1.11

If P is the position of particle at time t then OP = $\vec{r}$.

If Q is the position of particle at time t+δt and position vector of Q is $(\vec{r} + \delta\vec{r})$

as $\overrightarrow{PQ} = \overrightarrow{OQ}$-$\overrightarrow{OP}$

$= \vec{r} + \delta\vec{r} - \vec{r}$.

In limiting case if $\delta t \to 0$ then $\delta\vec{r} \to 0$ and P tends to Q and the chord become the tangent at P. Differentiation is define as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}.$$

When the limit exists only then the function $\vec{r}$ is differentiable. If we further differentiate function with respect to t and hence it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities.

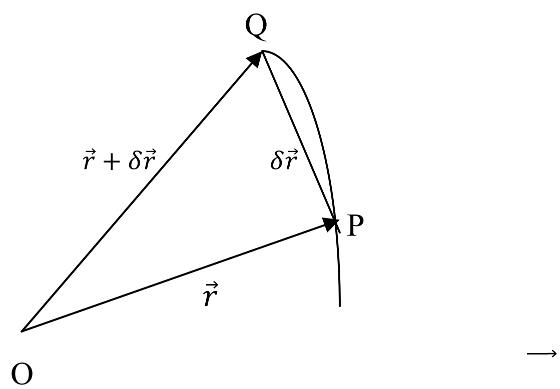

Figure 1.11

Properties of vector differentiation:

If $\vec{A}$ and $\vec{B}$ are two vectors, $\emptyset$ is a scalar field and $\vec{C}$ is a constant vector then

(1) $\frac{d}{dt}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$.

(2) $\frac{d}{dt}(A \times \emptyset) = \frac{d\vec{A}}{dt}\emptyset + \vec{A}\frac{d\emptyset}{dt}$.

(3) $\frac{d}{dt}(\vec{A}.\vec{B}) = \vec{A}.\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}.\vec{B}$.

(4) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$.

(5) $\frac{d\vec{C}}{dt} = 0$.

(6) $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$ where s is the scalar function of t.

(7) $\frac{d}{dt}(r^2) = \frac{d}{dt}(\vec{r}.\vec{r}) = \vec{r}\frac{d\vec{r}}{dt} + \vec{r}\frac{d\vec{r}}{dt} = 2\vec{r}\frac{d\vec{r}}{dt}$, where $\vec{r}$ is the position vector.

Example 1.3: A particle is moving along the curve x $= t^2 + 2$, y$= t^2 + 1$ and z$= 3t + 5$. Find the velocity and acceleration of particle along the direction 3i+2j+6k at time t=2.

Solution:
Curve is define as x $= t^2 + 2$, y$= t^2 + 1$ and z$= 3t + 5$.
The position vector of particle at any time t is given as

$$\bar{r} = xi + yj + zk$$

$$\bar{r} = (t^2 + 2)i + (t^2 + 1)j + (3t + 5)k$$

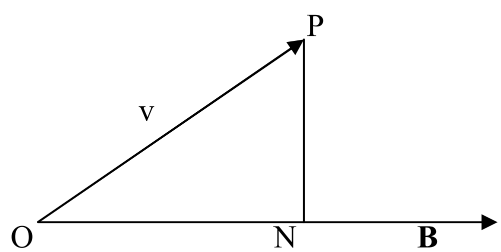

Figure 1.12

Velocity is given as

$$\frac{d\bar{r}}{dt} = 3t^2 i + 2tj + 3k$$

at t=2 velocity becomes

$$\frac{d\bar{r}}{dt} = 12i + 4j + 3k.$$

Component of the velocity along the direction $3i + 2j + 6k = \vec{B}\ (say)$

$$ON = |\bar{v}|\cos\theta\,.\hat{b} = |\bar{v}|\ \frac{\bar{v}.\bar{B}}{|\bar{v}||\bar{B}|}.\frac{B}{|\bar{B}|} = \frac{(\bar{v}.\bar{B})B}{|B|^2}$$

$$= \frac{(16\ i + 4j + 3k).(3i + 2j + 6k)}{3^2 + 2^2 + 6^2}(3i + 2j + 6k) = \frac{74}{49}(3i + 2j + 6k)$$

acceleration $\bar{a}$ can be given as $\bar{a} = \frac{d\bar{r}}{dt} = 6t\ i + 2j$

acceleration $\bar{a}$ at t=2 can be given as $\bar{a} = 12i + 2j$.

Component of acceleration along direction $\bar{B}$ is given as

$= |\bar{a}| \cos\theta \, . \, \hat{b} \; = \; |\bar{a}| \frac{\bar{a}.\bar{B}}{|\bar{a}|\,|B|} \frac{\bar{B}}{|B|} = \frac{(\bar{a}.\bar{B})\,\bar{B}}{|B|^2}$

$= \frac{(12i+2j).(3i+2j+6k)}{32^2+2^2+6^2}(3i+2j+6k)$

$= \frac{52}{49}(3i+2j+6k).$

1.4.1 Partial derivative:

If f is a vector function which depends on variable (x, y, z), then the partial derivatives are defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y, z) - f(x,y,z)}{\delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y, z) - f(x,y,z)}{\delta y}$$

$$\frac{\partial f}{\partial z} = \lim_{\delta z \to 0} \frac{f(x, y, z+\delta z) - f(x,y,z)}{\delta z}.$$

In case of partial derivatives with respect to a variable, all the other remaining variables are taken as constant.

Partial derivatives of second order are defined as:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x})$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right).$

1.4.2 Del operator:

The vector differential operator del is denoted by **∇** and defined as

$$\boldsymbol{\nabla} = \mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

1.4.3 Scalar and vector point functions:

(1) **Field**: Field is a region of the space defined by a function.

(ii) **Scalar point function**: A scalar function $\emptyset(x, y, z)$ defines all scalar point in the space. For example, gravitational potential is a scalar function defined at all gravitational fields in the space.

(iii) **Vector potential function**: If a vector function $\vec{F}$(x, y, z) defines a vector at every point in space then it is called vector point function. For example gravitational force is a vector function defined at a gravitational field in the space.

1.4.4 Gradient:

The gradient of a scalar function $\emptyset$ is defined as

$$\text{grad}\ \emptyset = \nabla\emptyset = (\mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\emptyset$$

$$= \mathrm{i}\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + k\frac{\partial\emptyset}{\partial z}$$

grad $\emptyset$ $is\ a\ vector\ qunatity.$

Total differential d$\emptyset$ of a scalar function $\emptyset(x, y, z)$ can also be expressed as,

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz.$$

Total differential d$\emptyset$ of a scalar function $\emptyset$ can be expressed as

$$\mathrm{d}\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$= (i\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + k\frac{\partial\emptyset}{\partial z})\ (idx + jdy + kdz)$$

$d\emptyset = (\vec{\nabla}\emptyset).\vec{dr} = |\nabla\emptyset||\mathrm{dr}|\cos\theta = (\vec{\nabla}\emptyset).\ \mathrm{dr}\ \hat{r}$, (where $\hat{r}$ is a unit vector along d$\vec{r}$)

also θ is angle between $\vec{\nabla}\emptyset$ and d$\vec{r}$ (The direction of displacement).

So, $\frac{d\emptyset}{dr} = (\vec{\nabla}\emptyset).\ \hat{r}$.

This, $\frac{d\emptyset}{dr}$ is the directional derevative of $\emptyset$. The rate of change is maximum if $\hat{r}$

is along $\vec{\nabla}\emptyset$ and $\hat{r}$ is zero.

Hence gradient of the sector field $\emptyset$ defines a vector field the magnitude of which is equal to the maximum rate of change of $\emptyset$ and the direction of which is the same, as the direction of displacement along with the rate of change is maximum.

Example 1.4: In the heat transfer, the temperature of any point in space is given by T=xy+yx+zx. Find the gradient of T in the direction of vector 4i-3k at a point (2, 2, 2).

Solution:

Temperature is define as

T=xy+yx+zx

gradient of temperature T is given as

$$grad\, T = \nabla T = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\partial\right)(xy + yz + zx).$$

$$\nabla T = i(y+z) + j(x+z) + k(x+y)$$

at point (2, 2, 2) the ∇T is $(4i + 4j + 4k)$.

The gradient T in the direction of vector 4i-3k is

$= (4i + 4j + 4k)$. Unit vector along $(4i - 3k)$

$=(4i + 4j + 4k).\frac{(4i-3k)}{\sqrt{4^2+3^2}}$

=4/5.

1.4.5 Physical significance of grad ∅ :

The physical significance of grad ∅ can be explained on the basis of surface defined by scalar field ∅. The value of ∅ remains constant on the surface S, as shown in Figure 1.13 and it is called a level surface or equi-scalar surface. Let us consider two surfaces S and S' defined by scalar function ∅ and ∅+d∅ respectively. Suppose $\vec{n}$ is normal to the surfaces S and S'. If the coordinates of point P and Q are (x, y, z) and (x+dx, y+dy, z+dz) then the distance between P and Q are

$$\vec{dr} = idx + jdy + kdz$$

as the definition of differentiation

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$= \left(\frac{\partial\emptyset}{\partial x}i + \frac{\partial\emptyset}{\partial y}y + \frac{\partial\emptyset}{\partial z}k\right).(dx\, i + dy\, j + dz\, k)$$

$$d\emptyset = \vec{\nabla}\emptyset . d\vec{r}.$$

If we consider the point Q approaches to P and finally lies on P then

$$d\emptyset = 0$$

$$\vec{\nabla}\emptyset . d\vec{r} = 0.$$

Where $\nabla\emptyset$ $and\ dr\ are\ perpendiular\ to\ each\ other.$

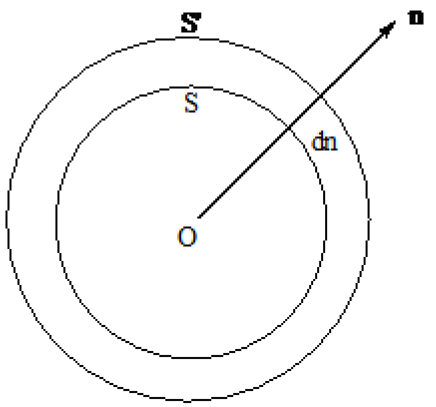

Figure 1.13: physical significance of grad

Therefore, $\nabla\emptyset$ is a vector which is perpendicular to the surface S.

If $\vec{n}$ is normal on the surface S and d$\vec{n}$ represents the distance between surfaces S to S' then $dn = dr\ cos\ \theta = \hat{n}. d\vec{r}$

and $d\emptyset = \frac{\partial\emptyset}{\partial n} dn = \frac{\partial\emptyset}{\partial n}\hat{n}. d\vec{r}$.

By using equation (1), $\vec{\nabla}\emptyset . \vec{d}r = \frac{\partial\emptyset}{\partial n}\hat{n}. d\vec{r}$

$$\vec{\nabla}\emptyset = \frac{\partial\emptyset}{\partial n}\hat{n} .$$

Thus, $\nabla\emptyset$ is defined as a vector whose magnitude is rate of change of $\emptyset$ along normal to the surface and direction is along the normal to the surface.

Example1.5: Find the directional derivative of a scalar function $\emptyset(x, y, z) = x^2 + xy + z^2$ at the point A (2,-1,-1) in the direction of the line AB where coordinate of B are (3, 2, 1).
Solution:
The component of $\nabla\emptyset$ along the direction of a vector $\vec{A}$ is called directional derivative of $\emptyset$ and given as $\nabla\emptyset . \hat{A}$
Now $\nabla\emptyset = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + xy + z^2)$
$=(2x + y)i + x\, j + 2z\, k$
gradient at point A (2,-1,-1)

$\nabla\emptyset = 3i + 2\,j - 2\,k.$

The vector $\overrightarrow{AB} = position\ vector\ of\ B - position\ vector\ of\ A$

$= (3i + 2j + k) - (2i - j - k) = i + 3j.$

Directional derivative of $\emptyset$ in the direction of AB is

$$\vec{\nabla}\emptyset\,.\widehat{AB} = (3i + 2j - 2k).\frac{(i + 3j)}{\sqrt{1+9}} = \frac{9}{\sqrt{10}}.$$

1.4.6 Divergence of Vector:

The divergence is defined as dot product of del operator with any vector point function $\vec{f}$ or any vector $\bar{F}$ and given as,

div. $\vec{f} = \nabla.\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\,(if_x + jf_{y+}kf_z)$ where $\vec{f} = if_x + jf_{y+}kf_z$

$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$

Since divergence of a vector $\vec{f}$ is dot product of del operator $\vec{\nabla}$ and that vector $\vec{f}$, therefore it is a scalar quantity.

1.4.6.1 Physical Significance of Divergence:

On the basis of fluid dynamics or a fluid flow, the divergence of a vector quantity can be explained. Let us consider a parallelepiped of edges dx, dy and dz along the x, y, z directions as shown in figure 1.14.

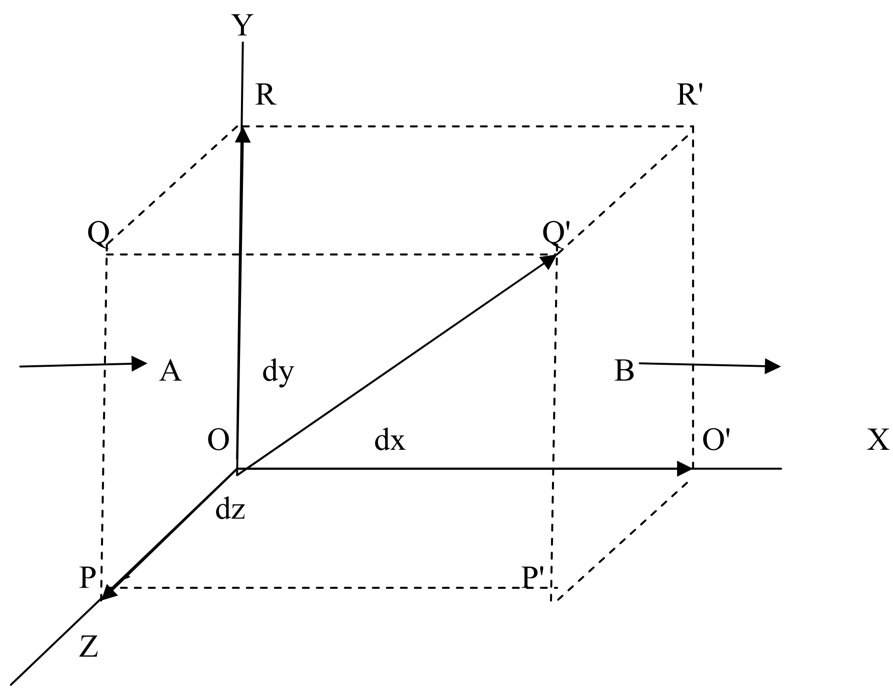

Figure 1.14: Physical Significance of Divergence

Let $\vec{v}$ is the velocity of fluid at A(x, y, z) and given as

$$\vec{v} = v_x i + v_y j + v_z k\,.$$

Where the $v_{x,} v_y, v_z$ *are the components of veolcity along x, y, z directions.*

Amount of fluid entering through the surface O'P'Q'R' per unit time is given as:

$$velocity \times area = v_x dydz.$$

Amount of fluid flowing out through the surface O'P'Q'R' per unit times is given as

= $v_{x+dx}\, dydz$

= $(v_x + \frac{\partial v_x}{\partial x} dx)dydz.$

Decrease in the amount of fluid in the parallelepiped along x axis per unit time.

=$v_x dydz - \left(v_x + \frac{\partial v_x}{\partial x} dx\right) dydz$

= $-\frac{\partial v_x}{\partial x} dxdydz.$

Negative sign shows, decrease in the amount of fluid inside the parallelepiped.

Similarly decrease of amount of fluid along y axis

=$-\frac{\partial v_y}{\partial y} dxdydz.$

Decrease of amount of fluid along z axis

= $-\frac{\partial v_z}{\partial z} dxdydz.$

Total amount of fluid decrease inside the parallelepiped per unit time= $-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz$).

Thus, the rate of loss of fluid per unit volume $= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

(We can ignore negative sign when we specify that the negative sign indicates decrease in the amount of fluid).

Further the rate of loss of fluid per unit volume

=$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(v_x i + v_y j + v_z k)$ = $\vec{\nabla}.\vec{v} = div\ \vec{v}$

Thus, *the divergence of velocity vector shows the rate of loss of fluid per unit timer per unit volume.*

If we consider fluid is incompressible, there is not any loss or gain in the amount of fluid, therefore $div\ v = 0$.

If the divergence of a vector is 0, then the vector function is called solenoidal.

Example 1.6: If u=$x^2+y^2+z^2$ and $\bar{r} =$ 2xi + 3yj + 2zk, then find the div (u$\bar{r}$).

Solution : Div (u $\bar{r}$) = $\nabla.(\text{u}\,\bar{r})$

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)(2xi+3yj+2zk)]$$

$$= i\frac{\partial}{\partial x}(x^2 2x)i+j\frac{\partial}{\partial y}(y^2 3y)j+k\frac{\partial}{\partial z}(z^2.2z)k$$

$$= 6x^2+9y^2+6z^2\,.$$

1.4.7 Curl

The curl of a vector $\vec{F}$ is defined as

Curl $\bar{F} = \nabla\times\vec{F}$ (where $\bar{F}$=$F_x i + F_y j + F_j k$)

=$(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z})\times\left(F_x i+F_y j+F_j k\right)$.

In terms of determinant of vector product

Curl $\bar{F}$=$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$.

Since curl is vector product of two vectors, therefore it is a vector quantity.

1.4.7.1 Physical significance of curl:

On the basis of angular velocity and linear velocity the curl can be explained.

Let us consider a particle moving with velocity $\bar{v}$ and $\bar{r}$ is the position vector of particle rotating around origin O. Let $\vec{\omega}$ is the angular velocity of particle then

$$curl\,\bar{v} = \nabla\times\bar{v}$$

$$= \nabla\times(\bar{\omega}\times\bar{r}) \qquad (\because\ \bar{v}=\bar{\omega}\times\bar{r})$$

$$= \nabla\left(\omega_x i+\omega_y j+\omega_z k\right)\times(xi+yj+zk)$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_y z - \omega_z y) i - (\omega_x z - \omega_z x) j + (\omega_x y - \omega_y x) k \right]$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_y z - \omega_z y & \omega_z x - \omega_x z & \omega_x y - \omega_y x \end{bmatrix}$$

$$curl\, \overline{v} = 2(\omega_x i + \omega_y j + \omega_z k) = 2\overline{\omega}\,.$$

Thus, the curl of linear velocity shows angular velocity which means rotation of particle. i.e. Curl of a vector quantity is connected with rotational properties of vector field. If curl of a vector is zero, $\nabla \times \overline{f} = 0$, there is no rotational property and $\overline{f}$ is called irrotational.

Example 1.7: Calculate the curl of a vector given by $\overline{F} = xyzi + 2x^2yj + (x^2z^2 - 2y^2)$k.

Solution:

$$curl\, \overline{F} = \nabla \times \overline{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times (xyzi + 2x^2yj + (x^2z^2 - y^2)k)$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 2x^2y & x^2z^2 - 2y^2 \end{bmatrix}$$

$$= -4yi - (2xz^2 - xy)j + (4xy - xz)k.$$

Example 1.8:

Show that $F = (y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}\ is\ irrolational.$

Solution:

$$curl\, F = \nabla \times F$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times [(y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}]$$

$$= 0.$$

Therefore, F is irrotational.

1.5 Vector integral:

1.5.1 Line Integral: The integral of a vector function F along a line or curve is called line integral.

Suppose $\vec{F}$(x, y, z) be a vector function and PQ is a curve and $\overrightarrow{dl}$ is a small length of curve as shown in figure 1.15 then line integral of vector $\vec{F}$ along a length $\overrightarrow{dl}$ is given as

$\int_l \vec{F}.\overrightarrow{dl}.$

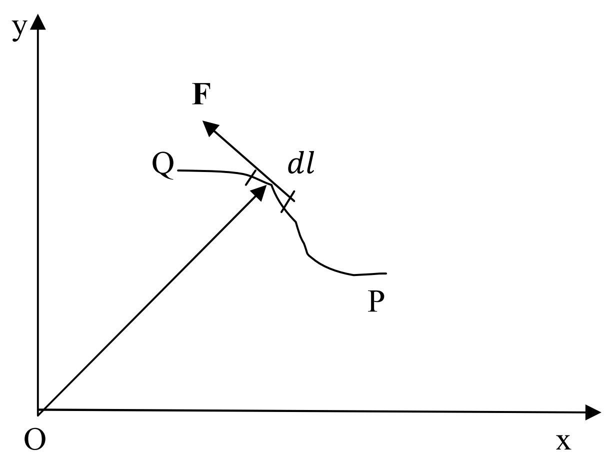

Figure 1.15: Line Integral

The integral may be closed or open depending on the nature of the curve whether closed or open. To compute the line integral of a function F, any method of integral calculus may be employed. In case of fore $\vec{F}$ acting on a particle along a curve PQ, the total work done can be calculated as line integral of force.

Work done= $\int_p^Q \vec{F}.\overrightarrow{dl}.$

1.5.2 Surface integral:

Similarly as line integral of F is a vector function and s is a surface, then surface integral of a vector function F over the surface s is given as

Surface integral=$\iint_s \vec{F}.\overrightarrow{dl}$.

The direction of surface integral is taken as perpendicular to the surface s.

If ds is written as ds = dxdy.

Surface integral=$\iint_s \vec{F}.\overrightarrow{ds} = \int_x \int_y F.dxdy$.

Surface integral represents flux through the surface S.

1.5.3 Volume integral:

If dV denotes the volume defined by dxdydz then the volume integration of a vector F is define as

Volume integral=$\int_V FdV = \int_x \int_y \int_z F.dxdydz.$

The volume integral can be explained in terms of total charge inside a volume. Suppose ρ is charge density of a volume dV then total charge inside the volume is given as q=$\int_v \rho\, dV$.

1.6 Vector identities:

If $\emptyset_1$ and $\emptyset_2$ are two scalar point functions and $\vec{A}$ and $\vec{B}$ are two vectors, then

$$\nabla(\emptyset_1 + \emptyset_2) = \nabla\emptyset_1 + \nabla\emptyset_2$$

$$\nabla(\emptyset_1\emptyset_2) = \emptyset_1\nabla\emptyset_2 + \emptyset_2\nabla\emptyset_1$$

$$div\,(\vec{A} + \vec{B}) = div\,\vec{A} + div\,\vec{B}$$

$$div\,(\vec{A}.\vec{B}) = \overrightarrow{A.}\,div\,B + \overrightarrow{B.}\,div\,\vec{A}$$

$$curl\,(\vec{A} + \vec{B}) = curl\,\vec{A} + curl\,\vec{B}$$

$$div\,(\emptyset\,\vec{A}) = \emptyset\,div\,\vec{A} + \vec{A}\,.grad\,\emptyset$$

$$curl\,(\emptyset\vec{A}) = \emptyset\,curl\,\vec{A} + grad\,\emptyset\,\times \vec{A}$$

$$div\,curl\,\vec{A} = 0$$

$$curl\,grad\,\,\emptyset = 0$$

$$div\,(\vec{A} \times \vec{B}) = \vec{B}.curl\,\vec{A} + \vec{A}.curl\,\vec{B}$$

$$curl\,curl\,\vec{A} = grad\,div\,\vec{A} - \nabla^2\,\vec{A}\,.$$

Example 1.9 : Prove that

(1) $div\,curl\,\vec{A} = 0$
(2) $curl\,grad\,\,\emptyset = 0.$

Solution:

(1) (1) $div\ curl\ \vec{A} = \nabla.\nabla \times \vec{A}$

$$= \nabla.\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \nabla.\left[i\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \text{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= 0.$$

(2) $curl\ grad\ \ \emptyset = \nabla \times \nabla\emptyset$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right) + j\left(\frac{\partial^2\ \emptyset}{\partial z\partial x} - \frac{\partial^2\emptyset}{\partial x\partial z}\right) + k\left(\frac{\partial^2\emptyset}{\partial x\partial y} - \frac{\partial^2\emptyset}{\partial y\partial x}\right) = 0.$$

Example 1.10:

Show that

(i) $div\left(\vec{A} \times \vec{B}\right) = \vec{B}.curl\ \vec{A} - \vec{A}\ .curl\ \vec{B}$

(ii) $curl\ curl\ \vec{A} = grad\ div\ \vec{A} - \nabla^2\ \vec{A}\ .$

Solution (i) $div\left(\vec{A} \times \vec{B}\right) =\ \nabla.(\vec{A} \times \vec{B})$

$$= (\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}).\ [(A_yB_z - A_zB_y)\ \hat{\imath} + (A_zB_x - A_xB_z)\ \hat{\jmath} + (A_x\ B_y - A_yB_x)\hat{k}]$$

$$= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$$

$$= B_x\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) + B_y\left(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right) + B_z\left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right) - A_x\left(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z}\right) - A_y\left(\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x}\right)$$

$$- A_z\left(\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y}\right)$$

$$= (B_x\hat{\imath} + B_y\hat{\jmath} + Bz\hat{k}).\ [\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right)\hat{\imath} + \left(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right)\hat{k}] -$$

$(A_x\hat{\imath} + A_y\hat{\jmath} + Az\hat{k}).\ [(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z})\hat{\imath} + (\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x})\hat{\jmath} + (\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y})\hat{k}]$

$= \overrightarrow{B}.\ \text{curl}\ \vec{A} - \overrightarrow{A}.\ \text{curl}\ \vec{B}$

$= \text{curl}\ \vec{A}\ .\ \vec{B} - \text{curl}\ \vec{B}\ .\ \vec{A}.$

Solution (ii)

$curl\ curl\ \bar{A} = \nabla \times (\nabla \times \bar{A})$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \left[i\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - j(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \text{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_y}{\partial y} - \frac{\partial A_y}{\partial z}\right) & (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) & (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\right] + j\left[\frac{\partial}{\partial z}\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$
$$+k\left[\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\right]$$

$$= i\left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x}\right] + j\left[\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y}\right]$$
$$+k\left[\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z}\right]$$

$$= \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x}\right) - \ (\frac{\partial^2 A_x}{\partial x^2} + \ \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2})\right]$$

$$= \sum i\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\ \right)\right]$$

$$= grad\ div\ \bar{A} - \nabla^2\ \bar{A}\ .$$

1.7 Gauss divergence theorem:

Gauss divergence theorem is a relation between surface integration and volume integration. The theorem states:

The surface integral of a vector filed $\vec{F}$ over a closed surface s is equal to the volume integral of divergence of $\vec{F}$ taken over the volume enclosed by surface s.

Mathematically $\iint_s \vec{F}.d\vec{s} = \iiint_v div\,\vec{F}\,dv$.

Mathematical proof:

Let us consider a vector $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$.

According to Gauss divergence theorem $\iint_s (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).ds = \iiint_v \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})dxdydz$

Or $\iint_s (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})\,ds = \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz$ (1)

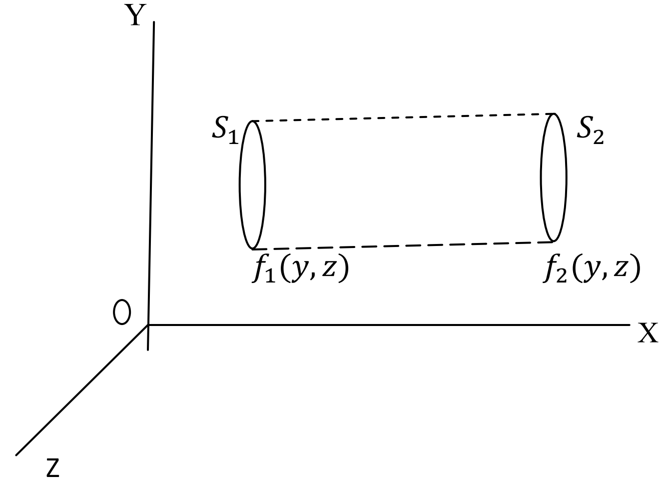

Figure 1.16: vector filed $\vec{F}$ over a closed surface and corresponding enclosed volume

Now we can prove equation (1)

Let us first evaluate $\iiint_v \frac{\partial F_1}{\partial x} dxdydz = \iint_s \left[\int_{x=f_1(y,z)}^{x=f_2(y,z)} \frac{\partial F_1}{\partial x} dx\right] dydz$

$$= \iint_s [F_1(x,y,z)]_{x=f_1(y,z)}^{x=f_2(y,z)} \text{dydz}$$

$$= \iint_R [F_1(f_2,y,z) - F_1(f_1,y,z)]dxdy \quad (2)$$

Now, the right portion of surface i.e. S_2 can be given as

$dydz = \hat{n}_2.ids_2$ where $\hat{n}_2$ is the direction of unit vector perpendicular to the surface.

Similarly the left portion of surface S_1 can be given as

$dydz = \hat{n}_1.ids_1$.

Putting the value of area in the factors of RHS of equation (2) we have

$$\iint_s \left[F_1(f_2,y,z)dydz = +\iint_{s_2} F_1\hat{n}_2.i\,ds_2\right]$$

$$\iint_s \left[F_1(f_1, y, z)dydz = -\iint_{S_1} F_1 \hat{n}_1 . i \; ds_1 \right].$$

Since the outward flux at surface S_2 is in the direction along the x axis and flux at surface S_1 is along the negative direction of x axis. Therefore, S_1 component is negative.

Putting the above values in equation (2) we have

$$\iiint_v \frac{\partial F_1}{\partial x} dxdydz = \iint_{S_2} F_1 \hat{n}_2 . i \; ds_2 + \iint_{S_1} F_1 \hat{n}_1 . i \; ds_2$$

$$\iiint_v \frac{\partial F_1}{\partial x} dv = \iint_S F_1 \hat{n} . i \; ds \; .$$

Since $\hat{n}_1$ and $\hat{n}_2$ are the direction perpendicular to yz plane that is along x axis shown by $\hat{n}$.

Similarly it can be shown that

$$\iiint_v \frac{\partial F_2}{\partial y} dv = \iint_S F_2 \hat{n} . j \; ds$$

and

$$\iiint_v \frac{\partial F_3}{\partial z} dv = \iint_S F_3 \hat{n} . k \; ds \; .$$

Adding all above terms

$$\iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dv = \iint_S (F_1 \hat{n} . i + F_2 \hat{n} . j + F_3 \hat{n} . k) \; ds$$

Or $\iiint_v (\nabla . F) dv = \iint_s \vec{F} . \overrightarrow{ds}$.

This is Gauss divergence theorem. The theorem relates the flux of a vector filed through a surface ($\vec{F} . \overrightarrow{ds}$) to the behavior of vector field ($\vec{\nabla} . \vec{F}$) inside the volume.

1.7.1 Deduction of Gauss law with Gauss Divergence theorem:

In electrostatics the Gauss law is one of the fundamental law and frequently used. This law is a result of Gauss theorem in electric field.

Statement: The total electric flux through a closed surface is equal to $\frac{1}{\epsilon_o}$ times total charge enclosed inside the surface.

Mathematically: $\iint_s \vec{E}.\overrightarrow{ds} = \frac{1}{\epsilon_o}$ (total charge inside the surface)

$$\iint_s \vec{E}.\overrightarrow{ds} = \frac{1}{\epsilon_o} \sum_i q_i.$$

Proof: Let us consider a charge q is situated at O, the origin of Cartesian coordinate system. Consider an imaginary surface called Gaussian surface around the charge q. The Gaussian surface may be of any shape but closed.

Consider a small surface ds on the Gaussian surface as shown is Figure 1.17. The distance (radial) of this surface is r from the origin and it subtends a solid angle $d\omega$ at the centre.

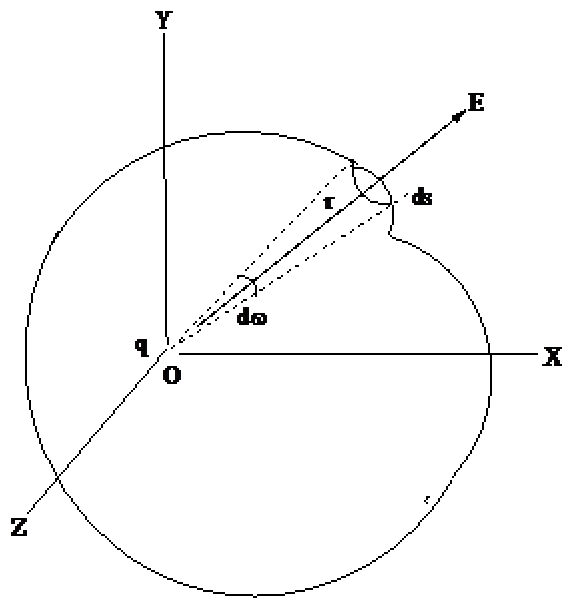

Figure 1.17: Gaussian surface

The electric flux through this small surface ds is

$$d\emptyset = \vec{E}.d\vec{s}\,.$$

The total electric flux through the whole surface

$$\emptyset = \iint_s \vec{E}\,.d\vec{s}\,.$$

Now the electric field on the surface ds is given by

$$E = \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}.d\vec{s}$$

where $\hat{r}$ is unit vector along the direction of $\vec{r}$. The flux $\emptyset$

$$\emptyset = \iint_s \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}.d\vec{s}$$

$$= \frac{1}{4\pi \in_0} \iint_s q.\frac{\hat{r}.\hat{n}\, d\vec{s}}{r^2}$$

$$where\ \hat{n}\ is\ unit\ vector\ perpendicualr\ to\ surface\ ds.$$

$$\emptyset = \frac{1}{4\pi \in_0} \iint_s \frac{q\ ds\ cos\theta}{r^2}\ .$$

Where $\hat{r}.\hat{n} = cos\theta$. Now $\frac{ds\ cos\theta}{r^2}$ is solid angle subtended by surface ds and denoted by $d\omega$.

$$\emptyset = \frac{1}{4\pi \in_0} \iint_s q.d\omega = \frac{1}{4\pi \in_0}.q.4\pi = \frac{q}{\in_0}.$$

Since total angle subtended by whole surface S at the centre is 4π.

Hence $\iint_s E.ds = \frac{1}{\in_0}\sum_i q_i$.

In case the charge in the closed surface is distributed in the volume V with volume charge density ρ then the statement can be given as

$$\emptyset = \iint_s E.ds = \frac{1}{\in_0}\iiint_v \rho\ dV\ .$$

1.7.2 Gauss law in differential form:

Gauss law in electrostatics is given as

$$\iint_s \vec{E}.d\vec{s} = \frac{1}{\in_0}\ (total\ charge\ inside\ surface\ s).$$

If ρ is volume charge density inside the volume and is enclosed by surface s then,

$$\iint_s \vec{E}.d\vec{s} = \frac{1}{\in_0}\iiint_v \rho dV$$

$$\iint_s \vec{E}.d\vec{s} = \iiint_v div\,\vec{E}\ dv.$$

Applying Gauss divergence theorem

$$\iiint_v div\,\vec{E}\;\;dV = \frac{1}{\epsilon_0}\iiint_v \rho dV$$

$$\iiint_v \left(div\,\vec{E} - \frac{\rho}{\epsilon_0}\right) dV = 0$$

$$div\,\vec{E} - \frac{\rho}{\epsilon_0} = 0$$

$$div\,\vec{E} = \frac{\rho}{\epsilon_0}.$$

This is called differential equation of Gauss law.

1.7.3: Poisson's equation and Laplace equation:

If we consider $\vec{E}$ as electric field and $\emptyset$ as electric potential then the electric field can be given as

$$\vec{E} = -\nabla\emptyset\,.$$

Now using differential form of Gauss law

$$div\,(-\nabla\emptyset) = \frac{\rho}{\epsilon_0}$$

$$\nabla(-\nabla\emptyset) = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\emptyset = -\frac{\rho}{\epsilon_0}.$$

This is called Poisson's equation. Poisson's equation is basically second order differential equation and operator ∇^2 is an operator defined as

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This is called Laplacian operator.

If there is no charge inside the volume i.e. ρ=0, then above equation becomes

$$\nabla^2\emptyset = 0.$$

This is called Laplace equation.

Example 1.11: If $\vec{r}$ is position vector of any point on the surface s whose volume is V, find $\iint_s \vec{r}.\,d\vec{s}$.

Solution:

$$\iint_s \vec{r}\,.\overrightarrow{ds} = \iiint_v div\,\vec{r}\,dV$$

$$= \iiint_v \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(ix + jy + kz)\,dV$$

$$= \iiint_v \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)\,dV$$

$$= \iiint_v 3\,dV = 3V.$$

Example 1.12:

Using Gauss divergence theorem find out $\iint_s \vec{A}\,.d\vec{s}$ where, $A = x^3i + y^3j + z^3k$ and s is a surface of a sphere defined by $x^2 + y^2 + z^2 = a^2$.

Solution:

$$\iint_s \vec{A}.d\vec{s} = \iiint_v \nabla.\vec{A}\,dV$$

$$= \iiint_v \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(x^3i + y^3j + z^3k)\,dV$$

$$= \iiint_v (3x^2 + 3y^2 + 3z^2)\,dV$$

$$= 3\iiint_v (x^2 + y^2 + z^2)\,dV$$

$$= 3\iiint_v a^2\,dV == 3a^2\iiint_v dV$$

$$= 3a^2\left(\frac{4}{3}\pi a^3\right) = \left(\frac{12}{3}\pi a^5\right).$$

1.8 Green's Theorem for a Plane:

Statement: If $\emptyset_1(x,y)$ and $\emptyset_2(x,y)$ are two scalar functions which are continuous and have continuous derivatives $\frac{\partial\emptyset_1}{\partial y}$ and $\frac{\partial\emptyset_2}{\partial x}$ over a region R bounded by simple closed curve c in x-y plane, as

$$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial\emptyset_2}{\partial x} - \frac{\partial\emptyset_1}{\partial y}\right)dxdy$$

Proof: Let us consider a close path ABCD denoted by curve c, and curve c divided into two parts curve c_1(ABC) and $c_2(CDA)$ as shown in Figure 1.18.

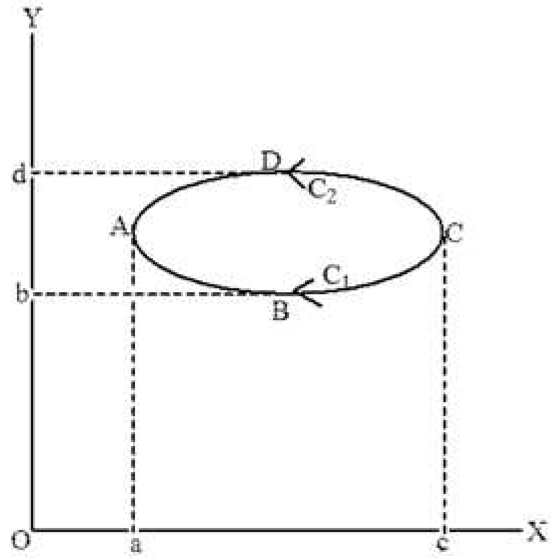

Figure 1.18: Close path ABCD denoted by curve C

The equation of curve c_1 is

$$y = y_1(x).$$

The equation of curve c_2 is

$$y = y_2(x).$$

First we calculate the value of

$$\iint_R \frac{\partial \emptyset_1}{\partial y} dxdy = \int_{x=a}^{x=c} \left[\int_{y=y_{1(x)}}^{y=y_2(x)} \frac{\partial \emptyset_1}{\partial y} dy \right] dx$$

$$= \int_a^c [\emptyset_1(x,y)]_{y_{1(x)}}^{y_2(x)} dx$$

$$= \int_a^c [\emptyset_1(x,y_2) - \emptyset_1(x,y_1)] dx$$

$$= -\int_c^a \emptyset_1(x,y_2) dx - \int_a^c \emptyset_1(x,y_1) dx$$

$$= -\left[\int_c^a \emptyset_1(x,y_2)dx + \int_a^c \emptyset_1(x,y_1)dx\right]$$

$$= -\left[\int_{c_2} \emptyset_1(x,y)dx + \int_{c_1} \emptyset_1(x,y)dx\right]$$

$$= -\oint_c \emptyset_1 (x,y)dx.$$

Thus $\oint_c \emptyset_1(x,y)dx = -\iint_R \frac{\partial \emptyset_1}{\partial y} dxdy.$

Similarly it can be proved that

$\oint_c \emptyset_2(x,y)dy = +\iint_R \frac{\partial \emptyset_2}{\partial x} dxdy.$

Adding above two equations

$$\oint_c (\emptyset_1(x,y)dx + \emptyset_2(x,y)dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

$$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy.$$

This is Green's theorem for a plane.

Example 1.13: A vector field $\vec{F}$ is given by $\vec{F} = siny\, i + x(1 + cosy)j$. Evaluate the line integral $\int_c \vec{F}.d\vec{r}$, where c is the circular path given by $x^2 + y^2 = a^2$.

Solution: The vector field F is given as

$\vec{F} = siny\; i + x(1 + cosy)j.$

Taking line integral along the curve c

$$\int_c F.dr = \int_c [sin\, y\; i + x(1 + cos\, y)j].(idx + jdy)$$

$$= \int_c (sinydx + x(1 + cosy)dy)$$

$$here\ \overline{r} = ix + jy \quad or \quad d\overline{r} = idx + jdy.$$

Now take $sin\, y = \emptyset_1$ and $x(1 + cos\, y) = \emptyset_2$.

On applying Green's theorem

$$\int_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R (\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y})dxdy$$

$$= \iint_R \left[\frac{\partial(x(1 + cos\, y))}{\partial x} - \frac{\partial\, sin\, y}{\partial y}\right] dxdy$$

$$= \iint_R [(1 + cosy) - cosy]dxdy$$

$$= \iint_R dxdy = \pi a^2 \,.$$

Since R is the region of circular path along xy plane given by $x^2 + y^2 = a^2$. Therefore the radius of circular path is a.

Example 1.14: Applying Green's theorem evaluate

$$\int_c [(x^2 + 3xy)dx + (x^2 + y^2)dy] \,.$$

Where c is a curve which form a square between the line $y = \pm 1 \; and \; x = \pm 1$.

Solution: Given integral is

$\int_c [(x^2 + 3xy)dx + (x^2 + y^2)dy]$.

Applying Green's theorem

$$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} \left[\frac{\partial (x^2+y^2)}{\partial x} - \frac{\partial}{\partial y}(x^2 + 3xy)\right] dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} [2x - 3x)] dxdy$$

$$= -\int_{-1}^{1} \int_{-1}^{1} xdxdy = -\int_{-1}^{1} xdx \int_{-1}^{1} dy$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{1} [y]_{-1}^{1} = -\frac{1}{2}(1^2 - 1^2)(1 + 1) = 0.$$

1.9 Stoke's Theorem:

Stoke's theorem transforms the surface integral of the curl of a vector into line integral of that vector over the boundary C of that surface.

Statement: The surface integral of the curl of a vector taken over the surface **s** bounded by a curve c is equal to the line integral of the vector A along the closed curve c.

Mathematically:

$$\iint_s Curl\vec{A}.d\vec{s} = \oint_c \vec{A}.d\vec{r}.$$

Since the curl A of a vector or vector function is along the normal to the surface, therefore the above statement may also be represented as

$$\iint_s curl\,\vec{A}.\hat{n}\,ds = \oint_c \vec{A}.d\vec{r}.$$

Where $\hat{n}$ is a unit vector perpendicular to the surface ds. Unit vector $\hat{n}$ can be given as

$$\hat{n} = \cos\alpha\, i + cos\beta\, j + cos\gamma\, k.$$

Proof: Let us consider a vector function A given as

$\vec{A}=A_x i + A_y j + A_z k$

and $\vec{r} = xi + yj + zk$

$d\vec{r} = idx + jdy + kdz.$

Using the Stoke's theorem $\int_c \vec{A}.\overrightarrow{dr} = \iint_s curl\,A.\hat{n}\,ds$

$\int_c \left(A_x i + A_y j + A_z k\right).(idx + jdy + kdz)$

$$= \iint_s \left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(A_x i + A_y j + A_z k\right)\right].(i\,cos\alpha + j\,cos\beta + k\,cos\gamma)ds$$

or

$\int_c \left(A_x dx + A_y dy + A_z dz\right) = \iint_s \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) k\right].(i\,cos\alpha + j\,cos\beta + k\,cos\gamma)\,ds$

or $\int_c \left(A_x dx + A_y dy + A_z dz\right) = \iint_s \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) cos\alpha + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) cos\beta + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) cos\gamma)\right] ds$

or

$$\int_c \left(A_x dx + A_y dy + A_z dz\right) = \iint_s \left[\left(\frac{\partial A_x}{\partial z} cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) + \left(-\frac{\partial A_y}{\partial z} cos\alpha + \frac{\partial A_y}{\partial x} cos\gamma\right) + \left(\frac{\partial A_z}{\partial y} cos\alpha - \frac{\partial A_z}{\partial x} cos\gamma\right)\right] ds. \quad (1)$$

Let us first prove the first term

$$\int_c A_x dx = \iint_s \left(\frac{\partial A_x}{\partial z} cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) ds. \quad (2)$$

Consider the A_x is function of (x, y, z) as $A_x(x, y, z)$ and $z = g(x, y)$ describes an equation of surface s, and ds is a small elementary part of this surface as shown in Figure 1.19.

$\int_C A_x(x,y,z)dx = \int_c A_x(x,y,g(x,y))dx$

$$= \int_c [A_x + \frac{\partial A_x\,(x,y,g(x,y))}{\partial y} dy]\ \text{dx}.$$

By using Green's theorem

$$\int_C A_x(x,y,z)dx = \iint_s (\frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}\frac{\partial g}{\partial y})\ dxdy. \qquad (3)$$

The direction cosines of the normal to the surface s are given as

$$\frac{Cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{Cos\beta}{-\frac{\partial g}{\partial y}} = \frac{Cos\gamma}{1}$$

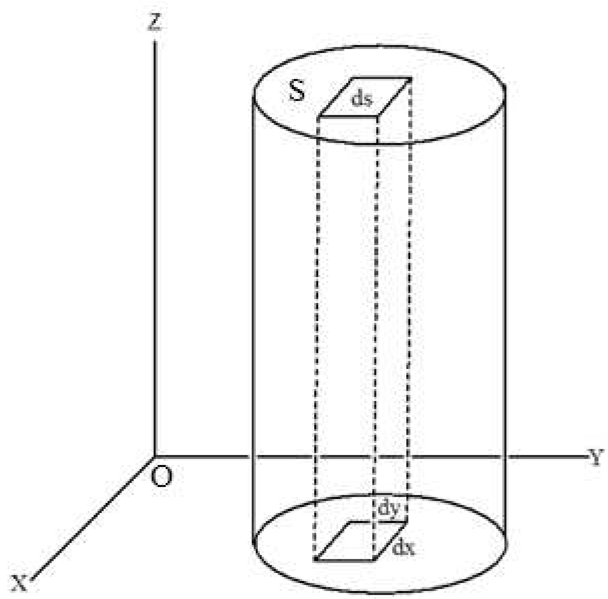

Figure 1.19: Small elementary part of surface

If the projection of ds on x-y plane is $ds\ cos\gamma$

Then $dxdy = ds\ cos\gamma$ or $ds = \frac{dxdy}{cos\gamma}$.

Putting this value on equation (2)

$$\iint_s \left(\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) ds = \iint_s \left(\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x Cos\gamma}{\partial y}\right) \frac{dxdy}{cos\gamma}$$

$$= \iint_s \left(\frac{\partial A_x}{\partial z}\frac{Cos\beta}{Cos\gamma} - \frac{\partial A_x}{\partial y}\right) dxdy$$

$$= \iint_s \left[\frac{\partial A_x}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial A_x}{\partial y}\right] dxdy$$

$$= -\iint \left[\frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}.\frac{\partial g}{\partial y}\right] dxdy.$$

Putting the value of R.H.S. from equation (3)

$$\iint_s \left(\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right)\ ds = \int_C A_x(x, y, z) dx$$

or $\int_C A_x(x, y, z) dx = \iint_s \left(\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right)\ ds.$ (4)

Similarly

$\int_c A_y dy = \iint_s \left(\frac{\partial A_y}{\partial x} Cos\gamma - \frac{\partial A_y}{\partial z} Cos\alpha\right) ds$ (5)

and $\int_c A_z dz = \iint_s \left(\frac{\partial A_z}{\partial y} Cos\gamma - \frac{\partial A_z}{\partial x} Cos\beta\right) ds.$ (6)

On adding above equations (4), (5) and (6)

$$\int_c \left(A_x dx + A_y dy + A_z dx\right) = \iint_s \left[\frac{\partial Ax}{\partial z} Cos\beta - \frac{\partial A_x}{\partial y} Cos\gamma + \frac{\partial A_y}{\partial x} cos\gamma - \frac{\partial A_y}{\partial z} Cos\alpha + \frac{\partial A_z}{\partial y} \cos\ r - \frac{\partial A_z}{\partial x} cos\beta\right] ds$$

Or $\int_c \vec{A}.\vec{dr} = \iint_s Curl\vec{A}.\vec{ds}$.

Hence Stoke's theorem is proved.

Example 1.15: Using Stoke's theorem evaluate

$\int_c [(2x - y)dx - yz^2 dy - y^2 zdz]$ *where c is the circle* $x^2 + y^2 = 1$, corresponding to the surface of a sphere of radius 1.

Solution:

The given integral

$$\int_c [(2x - y)dx - yz^2 dy - y^2 zdz]$$

$$= \int_c [(2x - y)i - yz^2 j - y^2 zk].(idx + jdy + kdz)$$

$$= \int_c \vec{A}.\overrightarrow{dr}.$$

Where $\vec{A} = (2x - y)i - yz^2 j - y^2 zk$ and $\overrightarrow{dr} = idx + jdy + kdz$

Using stokes theorem $\int_c \vec{A}.\overrightarrow{dr} = \iint_s \nabla \times \vec{A}.\ \overrightarrow{ds}$

$$= \iint_s \nabla \times \mathrm{A}.\hat{\mathrm{n}}\ \mathrm{ds} \qquad (1)$$

where $\hat{\mathrm{n}}$ = unit vector perpendicualr to surface ds

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yZ^2 & -y^2 z \end{vmatrix}$$

= i(-2zy+2yz)-j(0-0)+k(0+1)

= k.

Putting this value in equation (1)

$\int_c \vec{A}.d\vec{r} = \iint_s k.\hat{n}ds$.

Since ds is area of a circle described by $x^2 + y^2 = 1$ along xy plane, therefore direction of ds is along perpendicular to surface which is along z axis.

Thus, $\iint_s k.\hat{n}ds = \iint_s ds = \iint dxdy = \pi.$

Example 1.16: Verify Stoke's theorem for vector filed given by $\vec{F} = (3x - 2y)i + x^2 zj + y^2(z + 1)k$ for a plane rectangular area with corners at (0,0),(1,0) (1,2) and (0,2) in x-y plane.

Solution: the given function is

$\vec{F} = (3x - 2y)i + x^2 z\, j + y^2(z + 1)k.$

Since the vector field is applying in an area which is described in x-y plane only, therefore z=0 and function becomes

$$\vec{F} = (3x - 2y)i + y^2 k. \qquad (1)$$

According to Stoke's theorem

$$\int_c \vec{F}.\overrightarrow{dr} = \iint_s \nabla \times \vec{F}.\overrightarrow{ds}\,. \qquad (2)$$

The line integral along the close path described by rectangle OADC as shown in figure 3.5 and can be given as

$$\int_c \vec{F}.\overrightarrow{dr} = \int_{c_1} \vec{F}.\overrightarrow{dr} + \int_{c_2} \vec{F}.\overrightarrow{dr} + \int_{c_3} \vec{F}.\overrightarrow{dr} + \int_{c_4} \overrightarrow{F}.\overrightarrow{dr}.$$

Where C_1, C_2, C_3, C_4 are components of curve C.

$$\int_C F.dr = \int_0^1 [(3x-2y)i + y^2k].idx + \int_0^2 [(3x-2y)i + y^2k].jdy$$

$$+ \int_1^0 [(3x-2y)i + y^2k].idx + \int_2^0 [(3x-2y)i + y^2k].jdy$$

$$= \int_0^1 3xdx + \int_0^2 0\,dy + \int_1^0 (3x-4)dx + \int_2^0 0.dy$$

$$= \frac{3}{2} + 0 + \frac{5}{2} + 0 = 4$$

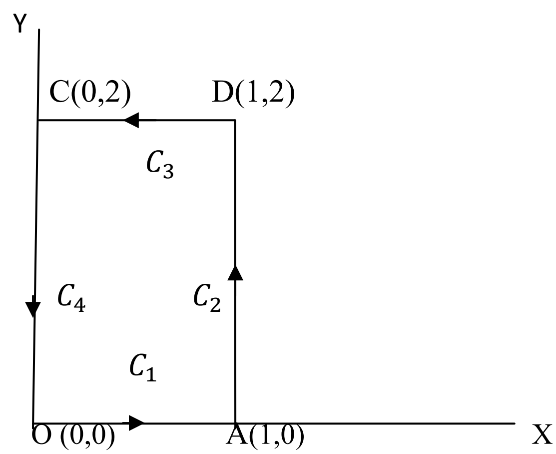

Figure 1.20

The L.H.S of equation (2) become 4 for given field. Now we calculate the R.H.S of equation (2)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-2y & 0 & y^2 \end{vmatrix}$$

$$= (2y\,i + 0 + 2k) = 2y\,i + 2k.$$

Now $\iint_s \nabla \times F.ds = \iint_s (2yi + 2k).\hat{n}\ dxdy$

$$= \iint_s 2dxdy = 2\iint_s dxdy$$

=2. Area of rectangle =2.2 = 4.

On comparing equation (3) and (4) the Stoke's theorem has been verified.

1.10 CURVILINEAR COORDINATES:

We are well familiar about Cartesian coordinate system in which any point or vector can be defined with the help of origin and three mutually perpendicular axes. Some time we need to use some another type of coordinate system which is more convenient for describing a system or solving an equation. In this unit we will study how the component of a vector can be formulated in another coordinate system called curvilinear coordinate system.

Let us consider a rectilinear Cartesian coordinate system in which a point is defined by a vector P(x,y,z). We can say the point P is determined by intersection of three mutually planes x=constant, y= constant and z= constant. Now we introduce another system of coordinates which is defined by the superposition of three another plans described by u_1 = constant, u_2 = constant, u_3 = constant. The family of these new plans is not necessarily parallel and mutually perpendicular. These new lanes are intersecting to each other at point P. These three new surfaces are called curvilinear surfaces. If these new surfaces are perpendicular to each other, then the coordinate defined by these plans called orthogonal curvilinear coordinate system. The three surfaces intersect each other at three curves or lines which are called coordinate lines. The three axis of this new coordinate system are defined by these coordinate lines.
Consider a point in the space is defined by P(x,y,z) in Cartesian coordinate system, and P(u_1, u_2,u_3), in curvilinear coordinates system as shown in Figure 1.21.

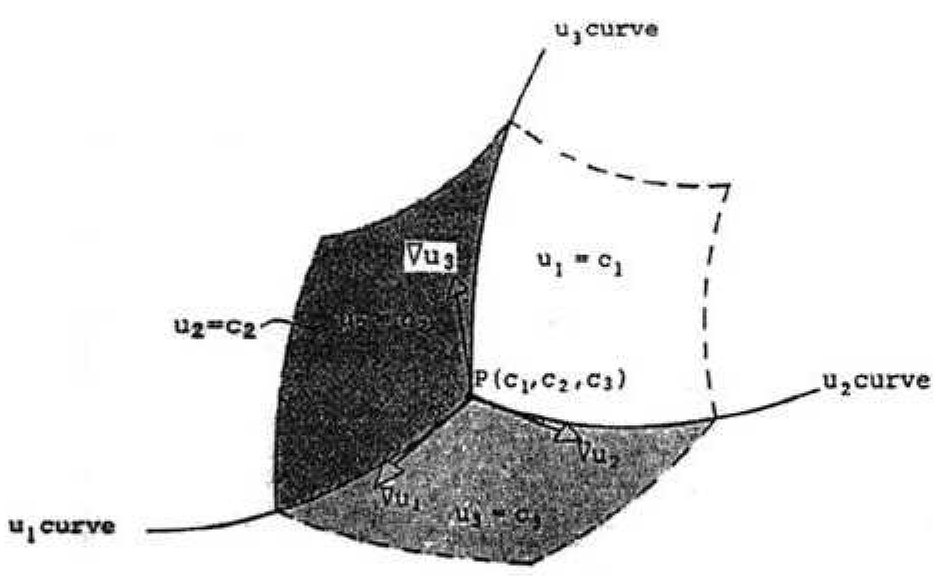

Figure 1.21: rectilinear Cartesian coordinate system

The transformation equations between the curvilinear coordinates and the Cartesian coordinates are

$$x = x(u_1, u_2, u_3)$$

$$y = y(u_1, u_2, u_3) \quad (1)$$
$$z = z(u_1, u_2, u_3).$$

The functions (1) are single-valued functions of u_1, u_2, and u_3 and are assumed to be continuously differentiable.

The set of eqs.(1) may be solved for u_1, u_2, u_3 in terms of x, y, z.

$$u_1 = u_1(x, y, z) \quad (2a)$$
$$u_2 = u_2(x, y, z) \quad (2b)$$
$$u_3 = u_3(x, y, z)\,. \quad (2c)$$

Here, u_1, u_2, u_3 are single- valued, continuously differentiable functions of x, y, and z.

The set of equations (1) and (2) define a one-to-one correspondence between each point (x, y, z) and the related set of values (u_1, u_2, u_3). The partial derivative of of equation 1 is given as

$$dx = \frac{\partial x}{\partial u_1}du_1 + \frac{\partial x}{\partial u_2}du_2 + \frac{\partial x}{\partial u_3}du_3 \quad (3a)$$

$$dy = \frac{\partial y}{\partial u_1}du_1 + \frac{\partial y}{\partial u_2}du_2 + \frac{\partial y}{\partial u_3}du_3 \quad (3b)$$

$$dz = \frac{\partial z}{\partial u_1}du_1 + \frac{\partial z}{\partial u_2}du_2 + \frac{\partial z}{\partial u_3}du_3\,. \quad (3c)$$

The distance between two points in the space with respect to curvilinear coordinate system is given by

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (4)$$

Putting the value from equation (3) and for simplicity 1, 2, 3 are denoted by $i, j = 1,2,3$ then expression for ds^2 can be represented as

$$ds^2 = \sum_{i,j}\left(\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j}du_i du_j\right). \quad (5)$$

Further for more simplification if

$$\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_j} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_j} = h_{ij}$$

Then

$$ds^2 = \sum_{i,j} h_{ij} du_i du_j.$$

For convenience we introduce a unit vector $\hat{w}_i$ normal to each surface $u_i = \text{const.}$ where $i = 1,2,3$.

Now for orthogonal curvilinear coordinate system in which surfaces always intersect to each other at right angles then

$$\text{as } \hat{w}_i.\hat{w}_i = 1\text{ , but } \hat{w}_i.\hat{w}_j = 0.$$

We have

$$h_{ii} = h_i;\;\; h_{jj} = h_j \text{ and } h_{kk} = \;h_k \;\text{ and }\; h_{ij} = 0 \text{ if } i \neq j\,.$$

Now above equation (5) becomes

$$ds^2 = (h_1du_1)^2 + (h_2du_2)^2 + (h_3du_3)^2 . \qquad (6)$$

In this orthogonal coordinate system h_1, h_2 and h_3 are called scale factor. The dimension of ds in space is of length. The product of h and u is of dimension of length however, the h and u may have any unit. The distance between two points in space along the coordinate line can be given as

$$ds_i = h_i du_i . \qquad (7)$$

Equation (7) indicates the distance between two points in the coordinate axis therefore surface element on the plane defined by coordinate axis ds_i and ds_j can be given as

$$d\mathbf{S}_{ij} = h_i du_i h_j du_j \qquad \text{where } i, j = 1,2,3 . \qquad (8)$$

Similarly the volume element in the orthogonal coordinate system defined by coordinate axis ds_i, ds_j and ds_k can be given as

$$dV_{ijk} = h_i du_i h_j du_j h_k du_k \qquad \text{where } i, j, k = 1,2,3 .$$

Differential operator in terms of orthogonal coordinate system:

Suppose we have three mutually perpendicular curvilinear planes defined by u_1 = constant, u_2 = constant, u_3 = constant. Now let us consider a scalar function $\psi(u_1, u_2, u_3)$ and a vector function $V = \hat{u}_1 V_1 + \hat{u}_2 V_2 + \hat{u}_3 V_3$ where $\hat{u}_1$, $\hat{u}_2$ and $\hat{u}_3$ are unit vector along the direction of curvilinear coordinates u_1, u_2 and u_3.

We know that del operator ∇ is a vector which give the maximum rate of change of space of a scalar function $\psi(u_1, u_2, u_3)$. In Cartesian coordinate system in we consider only one dimension then

$$\nabla \psi = \lim_{\delta x \to 0} \frac{\delta \psi}{\delta x} = \frac{\partial \psi}{\partial x} .$$

As del operator ∇ is a vector quantity, then we can introduce a unit vector along the direction of x and can write as

$$\nabla \psi = \hat{\imath} \frac{\partial \psi}{\partial x} .$$

In three dimensional case $\nabla \psi = \hat{\imath} \frac{\partial \psi}{\partial x} + \hat{\jmath} \frac{\partial \psi}{\partial y} + \hat{k} \frac{\partial \psi}{\partial z}$.

Similarly in curvilinear coordinate system for one dimensional case

$$\nabla \psi = \lim_{\delta s_1 \to 0} \frac{\delta \psi}{\delta s_1} = \frac{\partial \psi}{\partial s_1} = \frac{\partial \psi}{h_1 \partial u_1}$$

$$\nabla \psi = \hat{u}_1 \frac{\partial \psi}{h_1 \partial u_1} .$$

For three dimensional case

$$\nabla \psi = \hat{u}_1 \frac{\partial \psi}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial \psi}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial \psi}{h_3 \partial u_3} \qquad (9)$$

$$\nabla \equiv \hat{u}_1 \frac{\partial}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial}{h_3 \partial u_3} \qquad (10)$$

$\nabla \psi$ is nothing but gradient of a scalar function in curvilinear coordinate system.

Divergence: Similarly the divergence of a vector function in the space defined by curvilinear coordinates u_1 , u_2 and u_3 can be given as

$$\nabla . V = \nabla . (\hat{u}_1 V_1 + \hat{u}_2 V_2 + \hat{u}_3 V_3)$$

$$\nabla . V = \nabla . \hat{u}_1 V_1 + \nabla . \hat{u}_2 V_2 + \nabla . \hat{u}_3 V_3 \,. \qquad (11)$$

Since we know that

$$\text{div}\,(\phi A) = \phi \text{div} A + A \text{ Grad } \phi$$

$$\text{div}\,(\hat{u}_1 V_i) = V_i \nabla . \hat{u}_i + \hat{u}_i . \nabla V_i \,. \qquad (12)$$

Now we calculate $\nabla . \hat{u}_i$ as follow and we will put the value in above equation (12)

Since $\quad \text{curl}\left(\frac{\hat{u}_1}{h_1}\right) = \nabla \times \left(\frac{\hat{u}_1}{h_1}\right) = \nabla \frac{1}{h_1} \times \hat{u}_1 + \frac{1}{h_1} \nabla \times \hat{u}_1 \qquad (13)$

We know that $\nabla \times u_1 = \frac{\hat{u}_1}{h_1}$.

Since curl grad $u_1 = 0$ therefore $\nabla \times \frac{\hat{u}_1}{h_1} = 0$ and above equation (13) becomes

$$u_1 \times \nabla \left(\frac{1}{h_1}\right) = \frac{1}{h_1} \nabla \times u_1 \,. \qquad (14)$$

Using equation (10)

$$\nabla \left(\frac{1}{h_1}\right) = \frac{\hat{u}_1}{h_1} \frac{\partial}{\partial u_1} \frac{1}{h_1} + \frac{\hat{u}_2}{h_2} \frac{\partial}{\partial u_2} \frac{1}{h_1} + \frac{\hat{u}_3}{h_3} \frac{\partial}{\partial u_3} \frac{1}{h_1}$$

$$\nabla \left(\frac{1}{h_1}\right) = -\frac{\hat{u}_1}{{h_1}^3} \frac{\partial h_1}{\partial u_1} + \frac{\hat{u}_2}{{h_1}^2 h_2} \frac{\partial h_1}{\partial u_2} - \frac{\hat{u}_3}{{h_1}^2 h_3} \frac{\partial h_1}{\partial u_2} \,.$$

Putting this value in equation (14)

$$\nabla \times \hat{u}_1 = -\frac{\hat{u}_1 \times \hat{u}_1}{{h_1}^2} \frac{\partial h_1}{\partial u_1} + \frac{\hat{u}_1 \times \hat{u}_2}{h_1 h_2} \frac{\partial h_1}{\partial u_2} - \frac{\hat{u}_1 \times \hat{u}_3}{h_1 h_3} \frac{\partial h_1}{\partial u_2} \,.$$

Using the identity $u_i \times u_i = 0$

$$\nabla \times \hat{u}_1 = \frac{\hat{u}_2}{h_1 h_3} \frac{\partial h_1}{\partial u_3} - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial h_1}{\partial u_2} \qquad (15a)$$

similarly

$$\nabla \times \hat{u}_2 = \frac{\hat{u}_3}{h_2 h_1} \frac{\partial h_2}{\partial u_1} - \frac{\hat{u}_1}{h_2 h_3} \frac{\partial h_2}{\partial u_3} \qquad (15b)$$

$$\nabla \times \hat{u}_3 = \frac{\hat{u}_1}{h_3 h_2} \frac{\partial h_3}{\partial u_2} - \frac{\hat{u}_2}{h_3 h_1} \frac{\partial h_3}{\partial u_1} \,. \qquad (15c)$$

Now again using identity

$$\text{div}(A \times B) = B.\text{curl } A - A.\text{curl } B$$

$$\nabla \times \hat{u}_1 = \nabla.(\hat{u}_2 \times \hat{u}_3) = \hat{u}_3.\ (\nabla \times \hat{u}_2) - \hat{u}_2.\ (\nabla \times \hat{u}_3)$$

Putting the value of $\nabla \times \hat{u}_2$ and $\nabla \times \hat{u}_3$ from equation 15 we have

$$\nabla.\hat{u}_1 = \frac{1}{h_1 h_2 h_3}\frac{\partial(h_1 h_2)}{\partial u_1}.$$

From equation (12)

$$\nabla.(\hat{u}_1 V_1) = V_1 \nabla.\hat{u}_1 + \hat{u}_1 \nabla V_1$$

$$\nabla.(\hat{u}_1 V_1) = \frac{V_1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial u_1} + \hat{u}_1.\left(\frac{\hat{u}_1}{h_1}\frac{\partial V_1}{\partial u_1} + \frac{\hat{u}_1}{h_1}\frac{\partial V_1}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial V_1}{\partial u_3}\right)$$

$$\nabla.(\hat{u}_1 V_1) = \frac{V_1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial u_1} + \frac{1}{h_1}\frac{\partial V_1}{\partial u_1} = \frac{1}{h_1 h_2 h_3}\frac{\partial(V_1 h_2 h_3)}{\partial u_1}. \qquad (16a)$$

Similarly

$$\nabla.(\hat{u}_2 V_2) = \frac{1}{h_1 h_2 h_3}\frac{\partial(V_2 h_3 h_1)}{\partial u_2} \qquad (16b)$$

$$\nabla.(\hat{u}_3 V_3) = \frac{1}{h_1 h_2 h_3}\frac{\partial(V_3 h_1 h_2)}{\partial u_3}. \qquad (16c)$$

Combining all equation and using equation (11)

$$\text{div}\,V = \nabla.V == \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial u_1} + \frac{\partial(V_2 h_3 h_1)}{\partial u_2} + \frac{\partial(V_3 h_1 h_2)}{\partial u_3}\right]. \qquad (17)$$

Laplacian: The Laplacian operator is define as $\nabla.\nabla$ or denoted as ∇^2. Putting the value of $\nabla\psi$ from equation (9) in place of vector V in equation (11)

$$\nabla.\ \nabla\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left\{h_2 h_3 \frac{\partial\psi}{h_1\,\partial u_1}\right\} + \frac{\partial}{\partial u_2}\left\{h_3 h_1 \frac{\partial\psi}{h_2\,\partial u_2}\right\} + \frac{\partial}{\partial u_3}\left\{h_1 h_2 \frac{\partial\psi}{h_3\,\partial u_1}\right\}\right]$$

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left\{h_2 h_3 \frac{\partial\psi}{h_1\,\partial u_1}\right\} + \frac{\partial}{\partial u_2}\left\{h_3 h_1 \frac{\partial\psi}{h_2\,\partial u_2}\right\} + \frac{\partial}{\partial u_3}\left\{h_1 h_2 \frac{\partial\psi}{h_3\,\partial u_1}\right\}\right] \qquad (18)$$

Curl : Similarly the curl of a vector F in curvilinear coordinate system is given as

$$\text{curl}\,F = \nabla \times F = \nabla \times (\hat{u}_1 F_1 + \hat{u}_2 F_2 + \hat{u}_3 F_3)$$

We have vector F as $F = \hat{u}_1 F_1 + \hat{u}_2 F_2 + \hat{u}_3 F_3$

$$\nabla \times F = \nabla \times \hat{u}_1 F_1 + \nabla \times \hat{u}_2 F_2 + \nabla \times \hat{u}_3 F_3. \qquad (19)$$

We know that $\text{curl}\,(\phi A) = \phi\,\text{curl}\,A - A \times \text{grad}\,\phi$

$$\text{curl}\,(\hat{u}_1 F_1) = F_1\,(\nabla \times \hat{u}_1) - \hat{u}_1 \times \nabla F_1.$$

Substituting the value of $\nabla \times \hat{u}_1$ from equation (15) and ∇F_1 from equation (9)

$$\nabla \times (\hat{u}_1 F_1) = F_1\left[\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial u_2} - \frac{\hat{u}_3}{h_1h_2}\frac{\partial h_1}{\partial u_2}\right] - \hat{u}_1 \times \left[\frac{\hat{u}_1}{h_1}\frac{\partial V_1}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial V_1}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial V_1}{\partial u_3}\right]$$

$$= \left[\frac{\hat{u}_2 F_1}{h_1h_3}\frac{\partial h_1}{\partial u_3} - \frac{\hat{u}_3 F_1}{h_1h_2}\frac{\partial h_1}{\partial u_2} - \frac{\hat{u}_3}{h_2}\frac{\partial F_1}{\partial u_3} + \frac{\hat{u}_2}{h_3}\frac{\partial V_1}{\partial u_3}\right]$$

$$= \hat{u}_2\left[\frac{F_1}{h_1h_3}\frac{\partial h_1}{\partial u_3} + \frac{1}{h_1}\frac{\partial F_1}{\partial u_3}\right] - \hat{u}_3\left[\frac{F_1}{h_1h_2}\frac{\partial h_1}{\partial u_2} + \frac{1}{h_3}\frac{\partial F_1}{\partial u_2}\right]$$

$$= \frac{\hat{u}_2}{h_1h_3}\frac{\partial(F_1h_1)}{\partial u_3} - \frac{\hat{u}_3}{h_1h_2}\frac{\partial(F_1h_1)}{\partial u_2}$$

$$\nabla \times (\hat{u}_1 F_1) = \frac{\hat{u}_2}{h_1h_3}\frac{\partial(F_1h_1)}{\partial u_3} - \frac{\hat{u}_3}{h_1h_2}\frac{\partial(F_1h_1)}{\partial u_2}. \qquad (20a)$$

Similarly

$$\nabla \times (\hat{u}_2 F_2) = \frac{\hat{u}_3}{h_1h_2}\frac{\partial(F_2h_2)}{\partial u_1} - \frac{\hat{u}_1}{h_2h_3}\frac{\partial(F_2h_2)}{\partial u_3} \qquad (20b)$$

$$\nabla \times (\hat{u}_3 F_3) = \frac{\hat{u}_1}{h_2h_3}\frac{\partial(F_3h_3)}{\partial u_2} - \frac{\hat{u}_2}{h_3h_1}\frac{\partial(F_3h_3)}{\partial u_1}. \qquad (20c)$$

Substituting all values in equation (19)

$$\nabla \times F = \frac{\hat{u}_2}{h_1h_3}\frac{\partial(F_1h_1)}{\partial u_3} - \frac{\hat{u}_3}{h_1h_2}\frac{\partial(F_1h_1)}{\partial u_2} + \frac{\hat{u}_3}{h_1h_2}\frac{\partial(F_2h_2)}{\partial u_1} - \frac{\hat{u}_1}{h_2h_3}\frac{\partial(F_2h_2)}{\partial u_3} + \frac{\hat{u}_1}{h_2h_3}\frac{\partial(F_3h_3)}{\partial u_2} - \frac{\hat{u}_2}{h_3h_1}\frac{\partial(F_3h_3)}{\partial u_1}$$

$$\nabla \times F = \frac{\hat{u}_1}{h_2h_3}\left[\frac{\partial(F_3h_3)}{\partial u_2} - \frac{\partial(F_2h_2)}{\partial u_3}\right] + \frac{\hat{u}_2}{h_3h_1}\left[\frac{\partial(F_1h_1)}{\partial u_3} - \frac{\partial(F_3h_3)}{\partial u_1}\right] + \frac{\hat{u}_3}{h_1h_2}\left[\frac{\partial(F_2h_2)}{\partial u_1} - \frac{\partial(F_1h_1)}{\partial u_2}\right]. \qquad (21)$$

The determinant form of above equation is given as

$$\nabla \times F = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}. \qquad (22)$$

1.10.1 Spherical Coordinate System:

A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three coordinates (r, θ, ϕ) as radial distance r of that point from a fixed origin, its polar angle θ measured from a fixed zenith direction, and the azimuthal angle

ϕ of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. It can be seen as the three-dimensional version of the polar coordinate system. The radial distance is also called the radius or radial coordinate. The polar angle may be called colatitude, zenith angle, normal angle, or inclination angle. In physics (r, θ, φ) gives the radial distance, polar angle, and azimuthal angle,

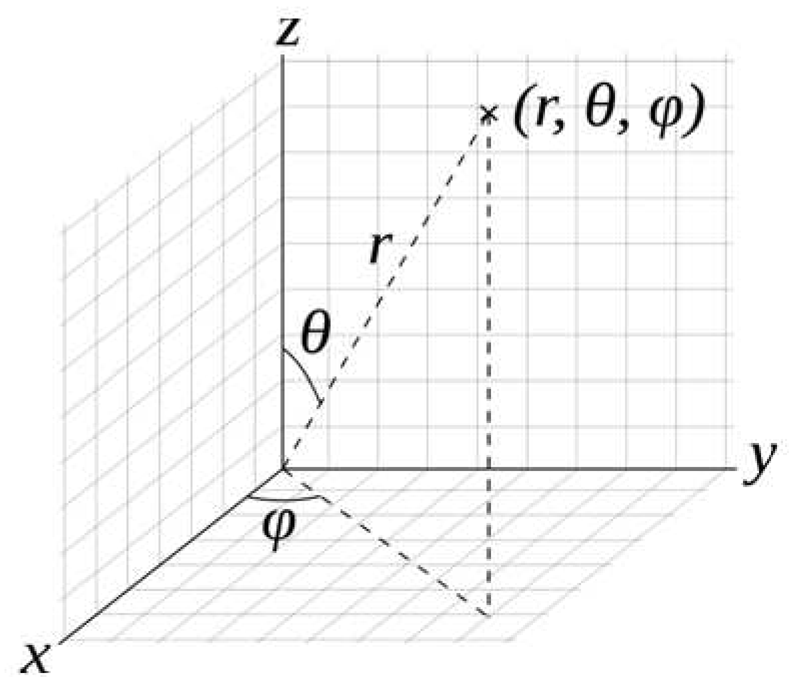

Figure 1.22: Spherical coordinate system in three-dimensional

If the coordinate of a point is given by (r, θ, ∅) in spherical coordinate system and (x,y,z) in Cartesian coordinate system. From the Figure 1.22

$$x = r\sin\theta\cos\emptyset \qquad (23a)$$

$$y = r\sin\theta\ \cos\emptyset \qquad (23b)$$

$$z = r\cos\theta \qquad (23c)$$

and $$r = \sqrt{x^2 + y^2 + z^2}. \qquad (24)$$

We know x is function of $(r, \theta, \emptyset)$ thus $x \equiv x(r, \theta, \emptyset)$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \emptyset}d\emptyset.$$

Partially differentiating equation (23a) with respect to r, θ, ∅ and putting the values in this equation we get

$$dx = \sin\theta\cos\emptyset\, dr + r\cos\theta\cos\emptyset\, d\theta - r\ \sin\theta\cos\emptyset\, d\emptyset \qquad (25)$$

Similarly y≡ $y(r,\theta,\emptyset)$ and

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \emptyset}d\emptyset$$

$$dy = \sin\theta\sin\emptyset\, dr + r\cos\theta\, csin\,\emptyset\, d\theta + r\ \sin\theta\cos\emptyset\, d\emptyset \qquad (26)$$

and z ≡ $z(r,\theta,\emptyset)$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \emptyset}d\emptyset$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta. \qquad (27)$$

We know the line element ds in Cartesian coordinate system is given as

$$ds^2 = dx^2 + dy^2 + dz^2. \qquad (28)$$

Substituting the value of dx, dy and dz from above equations 25,26 and 27

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta\ d\emptyset^2. \qquad (29)$$

Compare this equation (29) with standard curvilinear equation as given below

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2. \qquad (30)$$

We have

$$h_1 = 1\ and\ q_1 = r\ ;\ \ h_2 = r\ and\ q_2 = \theta\ ;\ h_3 = r\sin\theta\ \ and\ q_2 = \emptyset. \ \ (31)$$

Gradient:

Putting the coefficients in the equation of gradient in curvilinear coordinate

$$grad\ \psi = \nabla\ \psi = \hat{u}_1 \frac{\partial\psi}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial\psi}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial\psi}{h_3 \partial u_3}$$

$$grad\ \psi = \nabla\ \psi = \hat{u}_r \frac{\partial\psi}{1\,\partial r} + \hat{u}_\theta \frac{\partial\psi}{r\,\partial\theta} + \hat{u}_\emptyset \frac{\partial\psi}{r\sin\theta\,\partial\,\emptyset}$$

$$\nabla\ \equiv \hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{\partial}{r\,\partial\theta} + \hat{u}_\emptyset \frac{\partial}{r\sin\theta\,\partial\,\emptyset}\,. \qquad (32)$$

Divergence:

In orthogonal curvilinear equation divergence of a vector can be given as

$$div\, V = \nabla . V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial q_1} + \frac{\partial(V_2 h_3 h_1)}{\partial q_2} + \frac{\partial(V_3 h_1 h_2)}{\partial q_3}\right].$$

Putting the value of $h_1 h_2 h_3$ and $q_1\, q_2\, q_3$ from equation (31)

$$div\, V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial\left(V_r r^2 \sin\theta\right)}{\partial r} + \frac{\partial(V_\theta\, r \sin\theta)}{\partial\theta} + \frac{\partial(V_\emptyset\ r)}{\partial\emptyset}\right]$$

$$div\, V \equiv \frac{1}{r^2}\frac{\partial r^2}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\emptyset}. \qquad (33)$$

Laplacian :

In orthogonal curvilinear equation Laplacian of a funtion can be given as

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left\{h_2 h_3 \frac{\partial\psi}{h_1 \partial u_1}\right\} + \frac{\partial}{\partial u_2}\left\{h_3 h_1 \frac{\partial\psi}{h_2 \partial u_2}\right\} + \frac{\partial}{\partial u_3}\left\{h_1 h_2 \frac{\partial\psi}{h_3 \partial u_1}\right\}\right].$$

Putting the value of $h_1 h_2 h_3$ and $q_1\, q_2\, q_3$ from equation (31)

$$\nabla^2\psi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\emptyset}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\emptyset}\right)\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial\psi}{\partial\emptyset}. \qquad (34)$$

1.10.2 Cylindrical Coordinate system:

A cylindrical coordinate system is a three-dimensional coordinate system (r, Θ, z) that specifies point positions by the distance from a chosen reference axis z, the direction from the axis relative to a chosen reference direction, and the distance from a chosen reference plane perpendicular to the axis. The cylindrical coordinate system consist of a right circular cylinder having reference axis z, r is the perpendicular distance of a point from z axis and Θ is the angle r with respect to x axis as shown in figure . If (x,y,z) are Cartesian coordinate of point specified by (r, Θ, z).

Thus

$$x = r\cos\theta \qquad (35a)$$

$$y = r\sin\theta \qquad (35b)$$

$$z = z \qquad (35c)$$

$$x = x\,(r, \theta, z)$$

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial z} dz\,.$$

Partially differentiated equation (35) and putting the value in above equation

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta. \qquad (36a)$$

Similarly

$$dy = \sin\theta\, dr - r\cos\theta\, d\theta \qquad (36b)$$

$$dz = dz \qquad (36c)$$

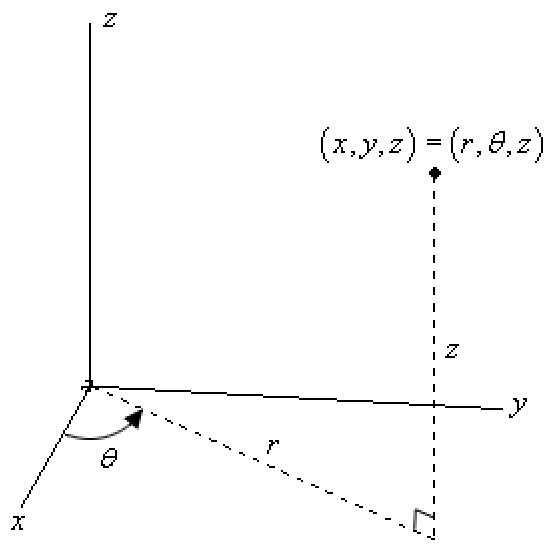

Figure 1.23: Cylindrical coordinate system in 3D

In Cartesian coordinate system, the line segment is given as

$$ds^2 = dx^2 + dy^2 + dz^2. \qquad (37)$$

Substituting the value of dx, dy and dz from above equations (36)

$$ds^2 = cos^2\theta dr^2 + r^2 sin^2\theta\, d\theta^2 - 2r\sin\theta\cos\theta\, dr\, d\theta + sin^2\theta dr^2 + r^2 cos^2\theta\, d\theta^2 + 2r\sin\theta\cos\theta\, dr\, d\theta + dz^2$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2. \qquad (38)$$

Compare this equation (29) with standard curvilinear equation as given below

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2.$$

We have

$$h_1 = 1 \; and \; q_1 = r \;\; ; \;\; h_2 = r \; and \; q_2 = \theta \;\; ; \; h_3 = 1 \; and \; q_3 = z. \qquad (39)$$

Now we can put the values of h and q and find out the value of gradient, curl and Laplacian.

Gradient:

Putting the coefficients in the equation of gradient in curvilinear coordinate

$$grad\,\psi = \nabla\,\psi = \hat{u}_1 \frac{\partial\psi}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial\psi}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial\psi}{h_3 \partial u_3}$$

putting the values of hand q

$$grad\,\psi = \nabla\,\psi = \hat{u}_r \frac{\partial\psi}{1\,\partial r} + \hat{u}_\theta \frac{\partial\psi}{r\,\partial\theta} + \hat{u}_z \frac{\partial\psi}{\partial z}$$

$$\nabla \;\equiv \hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{\partial}{r\,\partial\theta} + \hat{u}_z \frac{\partial}{\partial z}. \qquad (40)$$

Divergence:

In orthogonal curvilinear equation divergence of a vector can be given as

$$div\,V = \nabla\,.V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(V_1 h_2 h_3)}{\partial q_1} + \frac{\partial(V_2 h_3 h_1)}{\partial q_2} + \frac{\partial(V_3 h_1 h_2)}{\partial q_3}\right].$$

Putting the value of $h_1 h_2 h_3$ and $q_1\, q_2\, q_3$ from equation (39)

$$div\,V = \frac{1}{1.r.1}\left[\frac{\partial r\,V_r}{\partial r} + \frac{\partial(V_\theta\,)}{\partial\theta} + \frac{\partial(V_z\;r)}{\partial z}\right]$$

$$div\,V = \frac{1}{r}\frac{\partial\,(r\,V_r\,)}{\partial r} + \frac{1}{r}\frac{\partial(V_\theta\,)}{\partial\theta} + \frac{\partial V_z}{\partial z}$$

$$div\;\equiv \frac{1}{r}\frac{\partial\,r}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}. \qquad (41)$$

Laplacian :

In orthogonal curvilinear equation Laplacian of a function can be given as

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left\{h_2 h_3 \frac{\partial\psi}{h_1 \partial u_1}\right\} + \frac{\partial}{\partial u_2}\left\{h_3 h_1 \frac{\partial\psi}{h_2 \partial u_2}\right\} + \frac{\partial}{\partial u_3}\left\{h_1 h_2 \frac{\partial\psi}{h_3 \partial u_1}\right\}\right].$$

Putting the value of h_1, h_2, h_3 and $q_{1\,,}q_{2\,,}q_3$ from equation (39)

$$\nabla^2\psi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\mathrm{r}\,\frac{\partial\psi}{\partial \mathrm{r}}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\mathrm{r}}\,\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial \mathrm{z}}\left(\mathrm{r}\frac{\partial\psi}{\partial \mathrm{z}}\right)\right]$$

$$\nabla^2\psi = \frac{1}{\mathrm{r}}\left[\left(\mathrm{r}\,\frac{\partial^2\psi}{\partial \mathrm{r}^2}\right) + \frac{\partial\psi}{\partial \mathrm{r}} + \left(\frac{1}{\mathrm{r}}\,\frac{\partial^2\psi}{\partial\theta^2}\right) + \mathrm{r}\frac{\partial^2\psi}{\partial \mathrm{z}^2}\right]$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}^2}\,\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial \mathrm{z}^2}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial \mathrm{z}^2}. \qquad (42)$$

1.11 Summary:

1. Physical quantities are of two types, scalar and vector. The scalar quantities have magnitude only but no direction. The vector quantities have magnitude as well as direction.
2. Two vector quantities can be added with parallelogram law and triangle law. In parallelogram law, the resultant is denoted by the diagonal of parallelogram whose adjacent sides are represented by two vectors. In triangle law, we place the tail of second vector on the head of first vector, and resultant is obtained by a vector whose head is at the head of second vector and tail is at the tail of first vector.
3. For subtraction, we reverse the direction of second vector and add it with first vector.
4. In case of more than two vectors we simply use Polygon law of vector addition.
5. Any vector can be resolved into two or more components. By adding all components we can find the final vector.
6. If a vector makes angles α, β and γ with three mutual perpendicular axes x, y and z respectively then $cos\ \alpha, cos\ \beta$ and $cos\ \gamma$ are called direction cosines.
7. Scalar product of two vectors is defined as $\overrightarrow{P}.\ \vec{Q}\ = PQ\cos\theta$ which is a scalar quantity.
8. Vector product of two vectors is defined as $\vec{A}\times\vec{B} = ABsin\emptyset\,\hat{n}$ which is a vector quantity. The direction of vector is perpendicular to $\vec{A}$ and $\vec{B}$.
9. If two vectors are parallel to each other then they are said to be collinear. For collinear vectors $\overrightarrow{P}.\ \vec{Q}\ = PQ$ or $\vec{P}\times\vec{Q} = 0$.
10. If the angle between two vectors is 90^0, then vectors are called orthogonal. In this case $\overrightarrow{P}.\ \vec{Q}\ = 0$.
11. Cross product of two vectors can also be calculated by determinant. The determinant form of cross product is

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

12. Scalar triple product of three vectors can also be calculated by determinant. The determinant form of Scalar triple product is

$$\overrightarrow{A}.(\vec{B}\times\vec{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

13. Vector triple product is defined as

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}.\vec{C})\vec{B}-(\vec{A}.\vec{B})\vec{C}.$$

14. Differentiation and integration techniques are used to solve and explain many physical problems. Differentiation of a vector is defined as

$$\frac{d\vec{r}}{dt}=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}=\lim_{\delta t\to 0}\frac{\vec{r}(t+\delta t)-\vec{r}(t)}{\delta t}.$$

15. If we further differentiate function with respect t then it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities. If r is a position vector of a particle at time t then $\frac{d\vec{r}}{dt}$ denotes its velocity.

16. Partial derivative is defined as

$$\frac{\partial f}{\partial x}=\lim_{\delta x\to 0}\frac{f(x+\delta x,y,z)-f(x,y,z)}{\delta x}$$

In case of partial derivative with respect to a variable, all the other remaining variables are taken as constant.

17. Vector differential operator del is denoted by ∇ and defined as

$$\nabla=\mathrm{i}\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}.$$

18. The gradient of a scalar function ∅ is defined as

$$\text{grad}\,\emptyset=\nabla\emptyset=\left(\mathrm{i}\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\emptyset.$$

19. The divergence is dot product of del operator with any vector point function $\overrightarrow{f}$ and is given as

$$\text{div.}\,\vec{f}=\nabla.\vec{f}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(if_x+jf_{y+}kf_z)\ \text{ where } \vec{f}=\ if_x+jf_{y+}kf_z$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$$

20. The curl of a vector $\vec{F} = F_x i + F_y j + F_j k$ is defined as

Curl $\overline{F} = \nabla \times \vec{F} = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times (F_x i + F_y j + F_j k)$.

21. The integral of a vector function $\vec{F}$ along a line or curve is called line integral and given as $\int_l \vec{F}.\overrightarrow{dl}$.

22. If $\vec{F}$ is a vector function and s is a surface, then surface integral of a vector function $\vec{F}$ over the surface S is given as $\iint_s \vec{F}.d\vec{s}$.

23. If dV denotes the volume defined by dxdydz then the volume integration of a vector F is defined as $\int_V F dV = \int_x \int_y \int_z F.dxdydz$.

24. Gauss divergence theorem transforms surface integral into volume integral and vice-versa. The theorem states that the surface integral of a vector filed $\vec{F}$ over a closed surface s is equal to the volume integral of divergence of $\vec{F}$ taken over the volume enclosed by surface s.

$$\iint_s \vec{F}.\overrightarrow{ds} = \iiint_v div\,\vec{F}\,dv.$$

25. Gauss law is a result of Gauss theorem in electric field. According to this law the total electric flux through a closed surface is equal to $\frac{1}{\epsilon_o}$ times total charge enclosed inside the surface.

$$\iint_s E.ds = \frac{1}{\epsilon_o}\text{(total charge inside the surface)}.$$

26. Gauss law in differential form:

$$div\,E = \frac{\rho}{\epsilon_0}.$$

27. Poisson's equation and Laplace equation:

$$\nabla^2 \emptyset = -\frac{\rho}{\epsilon_0}$$

This is called Poisson's equation. Poisson's equation is basically second order differential equation and operator ∇^2 is an operator called Laplacian operator and defined as

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If there is no charge inside the volume i.e. ρ=0, then above equation becomes Laplace equation

$$\nabla^2\emptyset = 0.$$

28. Green's Theorem for a Plane: If $\emptyset_1(x, y)$ and$\emptyset_2(x, y)$ are two scalar functions which are continuous and have continuous derivatives $\frac{\partial\emptyset_1}{\partial y}$ *and* $\frac{\partial\emptyset_2}{\partial x}$ over a region R bounded by simple closed curve c in x-y plane, then

$$\oint_c(\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial\emptyset_2}{\partial x} - \frac{\partial\emptyset_1}{\partial y}\right)dxdy.$$

29. Stoke's Theorem: Stoke's theorem transforms the surface integral of the curl of a vector into line integral of that vector over the boundary C of that surface. According to this theorem the surface integral of the curl of a vector taken over the surface **s** bounded by a curve c is equal to the line integral of the vector A along the closed curve c.

$$\iint_s curl\,\vec{A}.\overrightarrow{ds} = \oint_c \vec{A}.\overrightarrow{ds}.$$

1.12 Glossary

Vector- Physical quantity with direction
Scalar quantities- Physical quantity without direction
Collinear – in same line or direction
Orthogonal- perpendicular to each other
Coplanar – on same plane Displacement – net change in location of a moving body.
Differentiation- instantaneous rate of change of a function with respect to one of its variables
Integration- The process of finding a function from its derivative. (Reverse of differentiation)
Partial derivative- derivative of a function with respect to a variable, if all other remaining variables are considered as constant
Operator – An *Operator* is a symbol that shows a mathematical operation.
del operator - vector differentiation operator
gradient- derivative of function.(rate of change of a function or slope)
divergence- rate at which density exits at a given region of space. (flux density)
Curl- describes the rotation of vector field.
line integral- Integration along a line.
surface integral- Integration along a surface.
volume integral- Integration along a volume.
Transformation- conversion

Flux – scalar product of a field vector and area
divergence- rate at which density exits in a given region of space. (flux density)
Curl- describes the rotation of vector field.

1.13 Reference Books:

1. Mathematical Physics – Satya prakash, Sultan Chand, Meerut
2. Mathematical Physics- H K Dass, S Chand and Company Ltd. New Delhi

1.14 Suggested readings:

1. Mathematical Methods for Physicists: Arfken.
2 Mathematical Methods for Physics: Wyle.
3. P.K. Chakrabarti and S.N. Kundu, A Text Book of Mathematical Physics, New Central Book Agency, Kolkata.
4. A.K. Ghatak, I.C. Goyal and S.H. Chua, Mathematical Physics Macmillan India, New Delhi.
5. B S Rajput, Mathematical Physics, Pragati Publication

1.15 Terminal questions

1.15.1 Short answer type questions

1. Define unit vector, like vector and equal vectors.
2. What are direction cosines? Give its significance.
3. What angle does the vector $3i + \sqrt{2}j + k$ make with y axis?
4. What is the condition for vector to be collinear?
5. Explain the difference between dot and cross products.
6. What is angular momentum? How the direction of angular momentum can be decided?
7. Give some examples of dot product in physics.
8. Give some examples of cross product in physics.
9. Define scalar triple product.
10. How the angle between two vectors can be obtained?
11. Define gradient of a scalar function $\emptyset$.
12. Show that $\nabla\emptyset$ is a vector whose magnitude is equal to maximum rate of change of $\emptyset$ with respect to space variable.
13. Show that $\nabla\emptyset$ is perpendicular to surface $\emptyset$.
14. Solve $\nabla\left(\frac{1}{\mathrm{r}}\right)$ for $r \neq 0$

15. If vector $\vec{F} = 6xzi - y^2\text{j} + \text{yzk}$ then calculate $\int_S \vec{F}.\hat{n}\, dS$ where S is the surface of a cube with boundaries $x = 0\ to\ x = 2,\quad \text{y}= 0\ to\ y = 2,\ \ z = 0\ to\ z = 2.$
16. Obtain the value $[grad\ \emptyset(\vec{r})] \times \vec{r}$
17. Find the area of parallelogram determined by the vectors $(i + 2\text{j} + 3\text{k})$ and $(-3i - 2\text{j} + 4\text{k})$.
18. Explain the physical significance of Gauss's divergence theorem.
19. If F is a scalar function which is solution of Laplace equation $\nabla^2 F = 0$ in a volume V bounded by the piecewise smooth surface S, then apply the Gauss theorem and show that
20. $\iint_S \hat{n}.\nabla F\, dS = 0$
21. Verify Green's theorem in a plane for $[(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is boundary of a region defined by $x = 0, y = 0, x + y = 1$
22. Prove that $\hat{n}.dS = 0$ and $\iint_S (\nabla \times \vec{F}\,) . \overrightarrow{dS} = 0$
23. If the line integral of a vector $\vec{A}$ around a closed curve is equal to the surface integral of the vector $\vec{B}$ taken over the surface bounded by the given closed curve then show that $\vec{B} = curl\,\vec{A}$.

1.15.2 Essay type questions

1. If $|\boldsymbol{A} + \boldsymbol{B}| = |\boldsymbol{A} - \boldsymbol{B}|$, show that **A** and **B** are perpendicular to each other.
2. What is the significance of dot product? Give the properties of cross product.
3. Show that $A = 5i + 2j + 4k\ and\ B = 2i + 3j - 4k$ are perpendicular to each other.
4. What is the vector product? Give the properties of vector product.
5. Find out the condition if two vectors are collinear.
6. Find the components of a vector along and perpendicular to the direction of another vector.
7. Define divergence of a vector function and its physical significance. Obtain the expression for the divergence of a vector $\vec{F}$.
8. Define curl of a vector function and its physical significance. Obtain the expression for the curl of a vector $\vec{F}$.
9. Prove that $\nabla \times (\vec{A} \times \vec{B}) = (\text{B}.\vec{\nabla})\text{A} - \overrightarrow{(\text{A}.\nabla)}\vec{B} + \vec{A}\, div\, \vec{B} - \vec{B}\, div\vec{A}$
10. Prove that any vector function can be expressed as the sum of lamellar vector and solenoidal vector.
11. Derive the equation of continuity
12. $\frac{\partial \rho}{\partial t} + \text{div J} = 0$
13. And show that how this equation express charge conservation.
14. Show that $\vec{u} \times \vec{v}$ is solenoidal if $\vec{u}$ and $\vec{v}$ are irrotational.
15. State and proof Gauss's divergence theorem.
16. State and prove Stoke's theorem in vector analysis.

17. State and prove Green's theorem in a plane.
18. Verify Green's theorem in a plane for $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is the boundary defined by $y = x^{1/2}; y = x^2$.

1.15.3 Numerical question

1. Calculate the dot product of vectors $\boldsymbol{A} = 6i + 7j + k\ and\ \boldsymbol{B} = i + 3j + 2k$.
2. A particle moves from the position $(3i + 3j + 2k) meter$ to another position $(-2i + 2j + 4k)\ meter$ under the influence of a force $\boldsymbol{F} = 3i + 2j + 4k$ newton. Calculate the work done by the force.
3. Obtain the projection of a vector $\boldsymbol{A} = 3i + 4j + 5k$ along a line which originates at a point (2, 2, 0) and passing through another point (-2, 4, 4).
4. Find the unit vector in the direction of resultant vectors of $\boldsymbol{A} = 6i + 7j + k\ and\ \boldsymbol{B} = i + 3j + 2k$.

UNIT - 2 MATRIX

STRUCTURE:

1.0 Objectives

After studying this unit, you should be able to-

- Knowledge on matrices
- Knowledge on matrix operations
- Matrix as a tool of solving linear equations with two or three unknowns
- Solve application problems that can be modeled by systems of linear equations.

1.1 Introduction: The understanding of matrices is essential in various field of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analyzing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management. In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

1.2 Matrices

1.2.1 Definition of a Matrix

"A rectangular array of real or complex numbers is called a matrix".

The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns. A matrix having m rows and n columns is said to have the order m× n.

A matrix A of order m× n can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}.$$

Where, a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.
In a more concise manner, we also denote the matrix A by $[a_{ij}]$ by suppressing its order.

Let A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$.
Then a_{11} = 1, a_{12} = 3, a_{13}= 7, a_{21}= 4, a_{22}= 5, and a_{23}= 6.

"A matrix having only one column is called a column vector; and a matrix with only one row is called a row vector".

Remarks:-

1. A matrix is a collection of objects of numbers over a field of numbers, the elements of the field being called the scalars.
2. It has no numerical value.
3. A matrix cannot be equal to a number.

1.2.2 Notations

Matrices are denoted by capital letters A, B, C,.. and their elements are denoted by the corresponding small letters a_{ij}, b_{ij}, c_{ij},……

Generally, we have used only a pair of brackets i.e. [ ] to denote a matrix, but a pair of parentheses i.e. ( ) and double bars i.e. || || are also sometimes used to indicate a matrix.

1.2.3 Order of a matrix

The order of a matrix is defined in terms of its number of rows and columns.
Order of a matrix = No. of rows × No. of columns.

1.2.4 Equality of two Matrices

Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ having the same order m× n are equal if $a_{ij} = b_{ij}$ for each i = 1, 2. . . m and j = 1, 2, . . . , n.
In other words, two matrices are said to be equal if they have the same order and their corresponding entries are equal.

1.2.5 Transpose of a Matrix: If in given m × n matrix A = $[a_{ij}]$, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A.
The transpose of A is denoted by A' or A^T.

For example, if A = $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 9 & 7 \\ 5 & 6 & 2 \end{bmatrix}$ and its transpose of A = $A' = \begin{bmatrix} 2 & 0 & 5 \\ 3 & 9 & 6 \\ 4 & 7 & 2 \end{bmatrix}$.

Thus, the transpose of a row vector is a column vector and vice-versa.

1.2.6 Conjugate of a matrix:-

Let A= $\begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$.

Conjugate of matrix A is $\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$.

1.2.7 Trace of a matrix:-The sum of all elements in the principal diagonal is called the trace of the matrix.

Trace of A= tr A= $\sum_{i=1}^{n} a_{ii}$.

Let A= $\begin{bmatrix} 2 & 8 & 0 \\ 4 & 3 & 7 \\ 3 & 6 & 9 \end{bmatrix}$.

The trace of matrix A is = 2+3+9=14.

1.3 Types of Matrices:

1.3.1 Zero matrix or Null matrix: A matrix in which each entry is zero, called a zero-matrix, denoted by 0.

For example,

A= $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix denoted by O or $O_{2\times 2}$.

1.3.2 Rectangular matrix: Any m×n matrix is called a rectangular matrix, if m≠n.

For example $A = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 8 & 4 \end{bmatrix}$.

1.3.3 Square matrix: A matrix having the number of rows equal to the number of columns is called a square matrix. Thus, its order is m× m (for some m) and is represented by m only.

For example A = $\begin{bmatrix} 3 & 7 & 2 \\ 4 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$ is a square matrix of order 3. In a square matrix, A = $[a_{ij}]$, of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal entries.

1.3.4 Diagonal matrix: A square matrix A = $[a_{ij}]$ is said to be a diagonal matrix if a_{ij}= 0 for i≠ j. In other words, all its non-diagonal elements are zero.

For example, A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

1.3.5 Identity or unit matrix: A square matrix A = $[a_{ij}]$ with a_{ij}=(1 if i = j and 0 if i≠ j is called the identity matrix, or in other words if all the diagonal elements are unity and diagonal elements are zero. It is denoted by I.

For example,

I= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is a unit matrix of order* 3, *and* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is a unit matrix of order* 2.

1.3.6 Triangular matrix: A square matrix A = $[a_{ij}]$, all of whose elements below the leading diagonal are zero, is said to be an upper triangular matrix or in other words a_{ij}= 0 for i> j.

For example- $A=\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 1\\ 0 & 0 & 6\end{bmatrix}$.

A square matrix A = $[a_{ij}]$, all of whose elements above the leading diagonal are zero, is said to be a lower triangular matrix or in other words $a_{ij}= 0$ for $i< j$.

For example- $A = \begin{bmatrix}2 & 0 & 0\\ 4 & 1 & 0\\ 5 & 6 & 7\end{bmatrix}$.

1.3.7 Single element matrix: A matrix $[a_{ij}]$ of order 1×1 is defined to be equal to a scalar 'a'.

1.3.8 Scalar matrix: A diagonal matrix, in which all the diagonal elements are equal to a scalar, is called a scalar matrix.

For example- $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$ is a scalar matrix of order 3 and is also written as diag [2, 2, 2].

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1.3.9 Symmetric and skew-symmetric matrix: A square matrix A is called symmetric if transpose of A i.e. A'= A and skew-symmetric if A'= −A.

For example, $\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$ *is symmetric and* $\begin{bmatrix}0 & -h & -g\\ h & o & -f\\ g & f & 0\end{bmatrix}$ *is skew symmetric.*

1.3.10 Orthogonal matrix: A square matrix A is said to be orthogonal if the product of the matrix A and the transpose matrix A' is an identity matrix i.e., AA'= A'A = I.

1.3.11 Nilpotent matrix: The matrices A for which a positive integer k exists such that A^k= 0 are called nilpotent matrices. The least positive integer k for which A^k= 0 is called the order of nilpotency.

Let $A = \begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}$, $A^2 = \begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

1.3.12 Idempotent matrix: The matrices that satisfy the condition that A^2 = A are called Idempotent matrices.

For example-

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

1.3.13 Involuntary matrix: A matrix A will be called an involuntary matrix, if $A^2 = I$(unit matrix).

1.3.14 Singular matrix: If the determinant of the matrix is zero, then the matrix is known as singular matrix e.g. A=$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular matrix because $|A| = 6 - 6 = 0$.

1.3.15 Unitary matrix: A square matrix A is said to be unitary if its product with Transpose of the conjugate gives the Identity matrix.

$$A^\theta A = I.$$

Where A^θ denotes the transpose of the conjugate of matrix A.

Let $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$, $A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$, $AA^\theta = I$.

1.3.16 Hermitian and skew-Hermitian matrix:-A square matrix A is called Hermitian matrix if transpose of the conjugate of A i.e. $A^\theta = (\bar{A})'$= A and skew-Hermitian if $A^\theta = (\bar{A})'$=−A.

1.4 Properties of Matrix:-

(a) The commutative law-

If A and B are two matrices of the same order, say m×n.

A+B = B+A

If $A = [a_{ij}]$ and $B = [b_{ij}]$ $i = 1,2, \dots \dots \dots, m$

j=1,2,… …… ….,n.

Then A+B = $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

= $[b_{ij} + a_{ij}]$ $since$ b_{ij} and a_{ij} are $scalars$

=$[b_{ij}] + [a_{ij}] = B + A$

i.e. the commutative law of addition holds.

(b) The associative law-

If A, B and C are three matrices of the same order, then

(A+B)+C=A+ (B+C)

Let $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$

(A+B)+C= $([a_{ij}] + [b_{ij}]) + [c_{ij}]$

= $([a_{ij} + b_{ij}] + [c_{ij}])$

= $[(a_{ij} + b_{ij}) + c_{ij}]$

$=[a_{ij} + (b_{ij} + c_{ij})], a_{ij}, b_{ij}$ and c_{ij} are $scalars.$
$= [a_{ij}] + ([b_{ij} + c_{ij}])$
=A+ (B+C)

i.e. the associative law of addition holds.

(c) The Distributive law-

If A and B are two matrices of the same order m×n and k is a scalar, then

k (A+B) = $k[a_{ij} + b_{ij}]$
$= [\ k(a_{ij} + b_{ij})]$
$= [ka_{ij}] + [kb_{ij}]$
$= k[a_{ij}] + k[b_{ij}],\ \ k\ is\ a\ scalar.$
= kA+kB.

The distributive law of addition holds.

(d) Existence of Additive identity-

If A be a matrix of any order, say, m×n and O a null matrix of the same order such that when it is added to A leaves it unchanged.

A+O = A

Then O is said to be the additive identity of A.

Proof: if $A = [a_{ij}]$ and O is a $null$ $matrix$

Then A+O= $[a_{ij} + O]$
$= [a_{ij}]$ since a zero added to any scalar leaves it unchanged.
= A.

Hence O is said to be an additive identity of A.

(e) Existence of Additive Inverse-

If A be a matrix of any order say m×n, and there exists a matrix –A of the same order such that if it is added to A, gives a null matrix O.

A+ (-A) =O

(-A) is said to be the additive inverse of A.

Let A=$[a_{ij}]$

-A= -$[a_{ij}] = [-a_{ij}]$

A+ (-A) = $[a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}]$ = 0.

Hence (-A) is said to be an additive inverse of A.

(f) The cancellation law-

If A, B and C are three matrices conformable for addition then the relation

A+B=A+C.

If B = C.

Let A= $[a_{ij}],\ B = [b_{ij}]$ and $C = [c_{ij}]$ i=1, 2,……..m
j=1, 2,……..n.

Then the relation

A+B = A+C

$a_{ij} + b_{ij} = a_{ij} + c_{ij}.$

Which yields, $b_{ij} = c_{ij}\ since\ a_{ij}, b_{ij}, c_{ij}$ all are scalars.
i.e. (i, j)th element of B=(i,j)th element of C, for all values of i and j.
As such B = C.
Hence the relation, "A+B = A+C" holds if and only if B = C.

1.4.1 Addition of Matrices: let A = $[a_{ij}]$ and B = $[b_{ij}]$ be are two m×n matrices. As the sum A + B is defined to be the matrix C = $[c_{ij}]$ with $c_{ij}= a_{ij} + b_{ij}$.

We define the sum of two matrices only when the order of the two matrices is same.

Thus if $A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

A+B = $\begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$.

1.4.2 Subtraction of Matrices:- let A = $[a_{ij}]$ and B = $[b_{ij}]$ be are two m×n matrices. Then the difference A - B is defined to be the matrix C = $[c_{ij}]$ with $c_{ij}= a_{ij} - b_{ij}$.

Thus if $A= \begin{bmatrix} 4 & 7 & 8 \\ 5 & 3 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \end{bmatrix}$,

A-B = $\begin{bmatrix} 4-1 & 7-2 & 8-5 \\ 5-3 & 3-1 & 6-4 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ 2 & 2 & 2 \end{bmatrix}$.

1.4.3 Multiplying a Scalar to a Matrix:- Let A = $[a_{ij}]$ be an m × n matrix. Then for any element k ∈ R, we define kA = $[ka_{ij}]$.
For example, if A =$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$
and k = 5, then 5A =$\begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}$.

1.4.4 Multiplication of Matrices: The multiplication of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.
Let A = $[a_{ij}]$ be an m × n matrix and B = $[b_{ij}]$ bean n × r matrix. Then the product AB is a matrix C = $[c_{ij}]$ of order m× r, with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

1.4.4.1 Properties of matrix multiplication: Suppose that the matrices A, B and C are so chosen that the matrix multiplications are defined.
1. Then (AB)C = A(BC). That is, the matrix multiplication is associative.
2. For any k ∈ R, (kA)B = k(AB) = A(kB).
3. Then A(B + C) = AB + AC. That is, multiplication distributes over addition.
4. If A is an n × n matrix then AI= IA = A, where I is identity matrix.

1.4.4.2 Determinant of a matrix: In linear algebra, the determinant is a value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A), det A, or |A|. It can be viewed as the scaling factor of the transformation described by the matrix.

In the case of a 2 × 2 matrix the specific formula for the determinant is:
$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$= ad – bc.

Similarly, suppose we have a 3 × 3 matrix A, and we want the specific formula for its determinant |A|:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = \text{aei-afh-bdi+bfg+cdh-ceg.}$$

Each determinant of a 2 × 2 matrix in this equation is called a **"minor"** of the matrix A. The same sort of procedure can be used to find the determinant of a 4 × 4 matrix, the determinant of a 5 × 5 matrix, and so forth.

1.5 Important Properties of Determinants

a. The value of a determinant is not altered if its rows are written as columns in the same order.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 6 & 7 \\ 1 & 2 & 0 \\ 4 & 1 & 5 \end{vmatrix}.$$

b. If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by –1.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & 5 \end{vmatrix} = -\begin{vmatrix} 6 & 2 & 1 \\ 3 & 1 & 4 \\ 7 & 0 & 5 \end{vmatrix}.$$

c. A common factor of all elements of any row (or column) can be placed before the determinant.

$$\begin{vmatrix} 3 & 8 & 1 \\ 5 & 4 & 2 \\ 1 & 12 & 3 \end{vmatrix} = 4\begin{vmatrix} 3 & 2 & 1 \\ 5 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix}.$$

d. If each element of a row (or a column) of a determinant can be expressed as a sum of two, the determinant can be written as the sum of two determinants.

$$\begin{vmatrix} 3 & 1 & 4 \\ 6 & 2 & 1 \\ 7 & 0 & 5 \end{vmatrix} = \begin{vmatrix} -1+4 & 1 & 4 \\ 3+3 & 2 & 1 \\ 5+2 & 0 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 4 \\ 3 & 2 & 1 \\ 5 & 0 & 5 \end{vmatrix} + \begin{vmatrix} 4 & 1 & 4 \\ 3 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix}.$$

1.6 Minor of a matrix: The determinant corresponding to any r$\times r$ submatrix of an m$\times n$ matrix A is called a minor of order r of the matrix A. For example, if

A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then its minorsare $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$.

1.7 Cofactors of a matrix: If A is a square matrix, (3 × 3) for example, then the minor of element $a_{ij}$is denoted by $M_{ij}$and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A.

The number $(-1)^{i+j}M_{ij}$is denotd by $C_{ij}$and is called the cofactor of elementa_{ij}.

Example, Let A = $\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$.

The minor of element a_{12} is

$M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix}$= 2(8) − 6(1) = 10.

Then the cofactor of $a_{12}$is

$$C_{12} == (-1)^{1+2}M_{12} = (-1) \times 10 = -10.$$

1.8 Adjoint of a matrix:-Let A=$[a_{ij}]$ be a square matrix of order n and let A_{ij}denote the cofactor of a_{ij} in the determinant$|A|$. The transpose of the matrix $[A_{ij}]$ is, then defined as the adjoint of A and is denoted by Adj (A).

Thus, if A= $\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$

Then $[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} \ldots & & A_{1n} \\ A_{21} & A_{22}\cdots & & A_{2n} \\ \ldots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & & \cdots A_{nn} \end{bmatrix}$

Adj A = $[A_{ij}]' = \begin{bmatrix} A_{11} & A_{21}\cdots & & A_{n1} \\ A_{21} & A_{22}\cdots & & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$.

1.9 Properties of adjoint:-If A=$[a_{ij}]$ is a square matrix of order n, then

(i) adj A′= (adj A)′
(ii) adj A*= (adj A)′
(iii) Adjoint of a symmetric (Hermitian) matrix is symmetric (Hermitian).
(iv) The adjoint of the product of square matrices is the product of their adjoint matrices taken in reverse order i.e., adj(AB) = adjB.adj A.

1.10 The inverse or reciprocal of a matrix:-
Let A be a square matrix of order n. If there exists a square matrix B of the same order n, such that

AB = BA = I, where I is identity matrix.

Then B is called inverse of A and is denoted by A^{-1}.
Thus, $AA^{-1} = A^{-1}A = I$.

We know that A (adj A) =$|A|$=I

$$\frac{A(adj\ A)}{|A|} = I.$$

Hence, $A^{-1} = \frac{adj\ A}{|A|}$ [since, $AA^{-1} = I$].

1.11 The Rank of a Matrix:
The maximum number of linearly independent rows in a matrix A is called the row rank of A, and the maximum number of linearly independent columns in A is called the column rank of A.
If A is an m× n matrix, or
A matrix 'A' is said to be of rank r, if and only if:
(i) There exist at least one non-zero minor of order r.
(ii) Every minor of order (r+1) and higher, vanishes.

1.12 Normal Form (Canonical Form): By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the normal form of A:

$$(i) I_r (ii) [I_r \quad 0] (iii) \begin{bmatrix} I_r \\ 0 \end{bmatrix} (iv) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

The number r so obtained is called the rank of A and we write ρ (A) = r.

Ex. Find the rank of A= $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$.

Sol. Since rank of a matrix is not altered by elementary operation, therefore, we have

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by} R_2 = R_2 - 2R_1 \text{ and } R_3 = R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 = R_2 + R_3 \ and\ then\ R_3 = R_3 - R_1.$$

No. of non-zero rows are 2, which shows that every minor of 3rd order is zero, while a minor of second order i.e., $\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$. Hence rank of A is 2.

Ex. Reduce the matrix A to its normal form, where $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

and hence find the rank of A.

Sol. A$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} [R_2 = R_2 - 4R_1; R_3 = R_3 - 3R_1; R_4 = R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad [R_3 = R_3 - R_2]$$

$$C_2 = C_2 - 2C_1, \quad C_3 = C_3 + C_1, \;\; C_4 = C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= C_3 + \frac{6}{7}C_2; \;\; C_4 = C_4 - \frac{11}{7}C_2$$

$$C_4 = C_4 + 2C_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left[R_2 = -\frac{1}{7}R_2; R_3 = -\frac{1}{2}R_3\right].$$

Hence**,** Rank of A = 3.

1.13 Eigen Values:-Let A is a matrix of order 3× 3, X is a column vector and Y is also a column vector.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$AX = Y. \qquad \ldots\ldots. (1)$$

Here, column vector X is transformed into the column vector Y by means of the square matrix A.
Let X is a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation Y=AX transforms X into a scalar multiple of itself i.e. λX.

$$AX = Y = \lambda X$$
$$AX-\lambda IX = 0$$
$$(A-\lambda I)X = 0. \qquad \ldots\ldots. (2)$$

Thus, the unknown scalar λ is known as an Eigen value of the matrix A and the corresponding non- zero vector X as Eigen vector.
Eigen values are also called characteristic values or proper values or latent values.

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}.$$

(a) Characteristic Polynomial: The determinant $|A - \lambda I|$ when expanded will give a polynomial, which is called characteristic polynomial of matrix A.

For example; $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(6 - 5\lambda + \lambda^2 - 2) - 2(2 - \lambda - 1) + 1(2 - 3 + \lambda)$

$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5.$

(b) Characteristic equation:-The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A e.g.

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

(c) Characteristic roots or Eigen values:- The roots of characteristic equation $|A - \lambda I|$=0 are called characteristic roots of matrix A. e.g.

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

Eigen values are $\lambda = 1,1,5$.

1.14 CAYLEY- HAMILTON THEOREM:

Statement- A square matrix satisfies its own characteristic equation i.e., if A is an n×m matrix whose characteristic equation is

$$\lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots C_n I_n = 0.$$

Putting λ=A in the above equation, we have

$$A^n + C_1A^{n-1} + C_2A^{n-2} + \cdots C_n I_n = 0.$$

Ex. Verify Cayley-Hamilton theorem for A=$\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ $and\ Find\ A^{-1}$.

Sol. we know that

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1-\lambda & 4\\2 & 3-\lambda\end{bmatrix} = 0\right|$$

$$(1-\lambda)(3-\lambda) - (4)(2) = 0$$

$$P(\lambda) = \lambda^2 - 4\lambda - 5I = 0.$$

Replace λ with A

P(A)= $A^2 - 4A - 5I = 0$ ………(1)

$$A^2 = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix} = \begin{bmatrix}1+8 & 4+12\\2+6 & 8+9\end{bmatrix} = \begin{bmatrix}9 & 16\\8 & 17\end{bmatrix}.$$

Now from (1)

$$P(A) = \begin{bmatrix}9 & 16\\8 & 17\end{bmatrix} - 4\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9-4-5 & 16-16-0\\8-8-0 & 17-12-5\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Cayley-Hamilton theorem is verified.

$$|A| = 3 - 8 = -5 \neq 0\ hence\ A^{-1} exists.$$

To find inverse, multiplying eq. (1) withA^{-1}, we get

$$A^{-1}(A^2 - 4A - 5I) = A^{-1}A^2 - 4AA^{-1} - 5IA^{-1} = 0$$

$$= A - 4I - 5A^{-1} = 0$$

$$A - 4I = 5A^{-1}$$

$$A^{-1} = \frac{1}{5}(A - 4I)$$

$$A^{-1} = \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}.$$ **Ans.**

Ex. Verify Cayley-Hamilton theorem for A=$\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$.

Sol.

$$|A - \lambda I| = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix} = 0$$

$$(7-\lambda)[(-1-\lambda)(-1-\lambda) - 4] - 2[-6(-1-\lambda) - 12] - 2[-12 - 6(-1-\lambda)] = 0$$

$$(7-\lambda)[\lambda^2 + 2\lambda - 3] - 2[6\lambda - 6] + 2[6\lambda - 6] = 0$$

$$7(\lambda^2 + 2\lambda - 3) - \lambda(\lambda^2 + 2\lambda - 3) = 0$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

Or $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

Now replace λ by A

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots\ldots\ldots\ldots (1)$$

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$A^3 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}.$$

Now from (1), we have

$$\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 5\begin{bmatrix} 25 & 8 & -8 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 79-125+49-3 & 26-40+14 & -26+40-14 \\ -78+120-42 & -25+35-7-3 & 26-40+14 \\ 78-120+42 & 26-40+14 & -25+35-7 \end{bmatrix} = 0.$$

Hence cayley-hamilton theorem is verified.

1.15 Summary: Hence this chapter deals with the matrices and its properties. Matrix is a rectangular array of elements, which are very helpful to deal with several variables at once. We can perform number of operations by organizing the elements in terms of rectangular arrays of numbers. Then we have found that matrices themselves can under certain conditions be added, subtracted and multiplied hence they will follow the set of algebraic rules. Another operation on the matrices is transpose by just reversing the transpose and columns. In another section we have discussed the various types of matrices like unit matrix, zero matrix, diagonal matrix etc. Matrices find many applications in scientific fields and apply to practical real life problems as well, thus making an essential concept for solving many practical problems.

1.16 References:

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2. H.K. Das, Rama Verma. *Mathematical Physics, S. Chand.*
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1.17 EXERCISE

1. Write the minors and cofactors of each element of the following determinants and also evaluate the determinant in each case:

(i) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

2. Matrices A and B are such that

3A-2B=$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ and -4A+B=$\begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$

Find A and B. **Ans:** $A = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$

3. If A=$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ choose α and β so that $(\alpha I + \beta A)^2 = A$. **Ans:** α=β=$\pm\frac{1}{\sqrt{2}}$

4. (i) Show that the matrix $\frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$ is idempotent.

(ii) Show that if A is idempotent, then

$$(1 + A)^n = 1 + (2^n - 1)A.$$

5. Prove that $\frac{1}{2}\begin{bmatrix} 1+i & i-1 \\ 1+i & 1-i \end{bmatrix}$ is unitary.

6. Show that adj$(kI_n) = k^{n-1}I_n$, where k is a scalar.

7. Find the adjoint and then inverse of the matrix

A=$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ **Ans**: $\frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$.

8. Reduce the following matrices into normal form and find the Rank:

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$

Ans: (i) 3 (ii) 3 (iii) 3

9. Find the eigen values of the matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$. **Ans:**λ= 1,2,5

10. Use Cayley-Hamilton Theorem to find the inverse of the matrix

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **Ans:** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Choose the correct alternative:

1. Transpose of a rectangular matrix is a
   (i) rectangular matrix
   (ii) diagonal matrix
   (iii) square matrix
   (iv) scalar matrix.

2. Additive inverse of a matrix A is
   (i) $\frac{adj(A)}{|A|}$ (ii) A^2
   (ii) $|A|$ (iv) A

3. The number of non-zero rows in an echelon form is called?
   i) rank of a matrix
   (iii) cofactor of the matrix
   (iv) reduced echelon form
   (v) conjugate of the matrix.

4. Rank of the matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ is
   (i) 0 (ii) 1 (iii) 3 (iv) 2.

5. Which of the following matrices are Hermitian:

   (i) $\begin{bmatrix} 1 & 2+i & 3-i \\ 2+i & 2 & 4-i \\ 3+i & 4+i & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2i & 3 & 1 \\ 4 & -1 & 6 \\ 3 & 7 & 2i \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 2-i & 5+2i \\ 2+i & 1 & 2-5i \\ 5-2i & 2+5i & 2 \end{bmatrix}$

   (iv) $\begin{bmatrix} 0 & i & 3 \\ -7 & 0 & 5i \\ 3i & 1 & 0 \end{bmatrix}$.

6. If λ is an Eigen value of the matrix M then for the matrix (M-λI), which of the following statement is correct?
   (i) Skew-symmetric (ii) Non singular (iii) Singular (iv) None of these.

7. A square matrix is idempotent if:
   (i) A'=A (ii) A'= -A (iii) $A^2 = A$ $(iv) A^2 = I.$

8. If A and B are matrices, then which from the following is true?

(i) $AB \neq BA$

(ii) $(A')' \neq A$

(iii) $A + B \neq B + A$

(iv) all are true.

9. Two matrices A and B are multiplied to get BA if

(i) no of rows of A is equal to no. of columns of B

(ii) no of columns of A is equal to columns of B

(iii) both have same order

(iv) both are rectangular.

10. A matrix having m rows and n columns with m ≠ n is said to be a

(i) scalar matrix

(ii) identity matrix

(iii) square matrix

(iv) rectangular matrix.

Ans: **(1)** (i) **(2)** (i) **(3)** (i) **(4)** (iv) **(5)** (iii) **(6)** (iii) **(7)** (iii) **(8)** (i) **9** (ii) **10**(iv)

UNIT 3: Complex Analysis

STRUCTURE:

3.0 Objectives

After studying this unit, you should be able to-

- Knowledge on Complex Numbers
- Knowledge on operation of fundamental laws of algebra on complex numbers
- Function of a complex variable
- Types of Singularities and Residues
- Evaluation of residues and Integrals

3.1 INTRODUCTION

Cantor, Dedekind and Weierstrass etc., extended the concept of rational numbers to a larger field known as real numbers which constitute rational as well as irrational numbers. But, the number system solely based on real numbers is not sufficient for all mathematical needs. There is no real number, rational or irrational, which satisfies the equation $x^2+1 = 0$. It was, therefore, felt necessary by Euler Gauss, Hamilton, Cauchy, Riemann and Weierstrass etc. to extend the field of real numbers to the still large field of complex numbers. Euler for the first time introduced the symbol *i* with the property $i^2=-1$ and then Gauss introduced a number of the form $\alpha+i\beta$, which satisfies every algebraic equation with real coefficients. Such a number $\alpha+i\beta$ with $i=\sqrt{-1}$ and α, β being real, is known as a *complex number.*

3.2 DEFINITIONS

3.2.1 Complex Numbers: "An ordered pair of real numbers such as (x, y) is termed as a complex number." If we write

$$z=(x, y) \text{ or } x+iy, \text{ where } i=\sqrt{-1}, \text{ then}$$

x is called the real part of z and y is called the imaginary part of the complex number z and denoted by,

$$x= R_z \text{ or } R(z) \text{ or } Re(z)$$

$$y= I_z \text{ or } I(z) \text{ or } Im(z).$$

3.2.2 Equality of complex numbers: Two complex numbers (x,y) and (x′,y′) are equal if x=x′ and y=y′.

3.2.3 Modulus and Argument of a complex number: If z=x+iy be a complex number then, If we introduce polar co-ordinates (r, θ), we have $x = r\cos\theta$ and $y = r\sin\theta$ and then from equation $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .Here, r is the modulus of the complex number x+iy and is denoted by $|x+iy|$ or arg (z).Argument of z; $\theta = \tan^{-1}(\frac{y}{x})$

3.3 OPERATION OF FUNDAMENTAL LAWS OF ALGEBRA ON COMPLEX NUMBERS

Taking three complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ we define the following operations:

3.2.1 Addition: The sum of two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ is defined as a complex number $z = (z_1 + z_2) = (x_1 + x_2, y_1 + y_2)$ such that its real part is the sum of real parts and imaginary part is the sum of imaginary parts of the given numbers.

3.3.1 Subtraction:

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$.

Multiplication: we have $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

i.e., $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

3.3.2 Conjugate complex numbers :

If z= x+iy, then its conjugate complex number is $\bar{z} = x - iy$

Evidently, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2.$$

3.3.3 Modulus properties:

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$|z_1 + z_2| \ge |z_1| - |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

3.4 Function of a complex variable: All the elementary functions of real variables may be extended into the complex plane replacing the real variable x by the complex variable z. Before giving a formula definition of functions of a complex variable, let us define some useful terms.

3.5 Set of points: The set of points in Argand diagram is a collection of points finite or infinite in number.

3.6 Neighborhood of a point: Let ‘a’ be a point in the Argand diagram. A set of all the z points such that $|z - a| < \epsilon$, where ϵ, is an arbitrary chosen small positive number, is defined as neighborhood of point ‘a’.

3.7 Limit point of a set: A point ‘a’ every neighborhood of which contains a point of set S other than ‘a’ is defined as the limit point of the set S of points in the Argand plane. For example, each point on the circumference of circle$|z| = r$ is a limit point of set$|z| < r$. These points do not belong to the set. But each point inside the circle is also a limit point that belongs to the set. Thus the limit point of a set may not necessarily be the point of the set. If ‘a’ is a limit point of the set S such that in the neighborhood of ‘a’, there exist entirely the point of the set S, it is defined as interior or inner point. If all the points in the neighborhood do not belong to the set S, it is said to be the boundary limit point.
A set is said to be closed if all its limit points (inner or boundary points) belong to the set. If a set consists of entirely the interior points, it is known to be an open set.

3.8 Domain: If every pair of points of a set of points in Argand diagram can be connected by a polygonal are every point of which is the point of the set then the set is said to be domain or region. Open domain is open connected set of points. When the boundary points of the set are also added to an open domain, it becomes a closed domain.
We may now give a formula definition of a function of complex variables. Let x and y be a pair of real variables such that z= x+iy, and let u and v be a pair of real functions such that w=u+iv, then w is said to be the function of complex variable z and written as w=*f*(z), if to every value of z in a certain domain D, there correspond one or more definite value of w. In case w has only one value for each value of z in the given domain D, w is said to be uniform or single valued function of z and if it takes more than one value for some or all value of z in D, then w is known as a many valued or multiple valued function of z. thus the function w=u+iv of complex variable z=x+iy is ordered pair of real functions of real variable,

i.e. $w = f(z) = u(x,y) + iv(x,y)$.

3.9 Analytic function: A function *f*(z) is said to be analytic at a point z=a, if *f*(z)is differentiable not only at ‘a’ but at every point of some neighborhood of ‘a’. A function *f*(z) is analytic in a domain if it is analytic at every point of the domain D of the function.
The points at which the function is not differentiable are called singular points or a singularity of the function.An analytic function is also known as “holomorphic”, “regular”, and “monogenic”.

3.10 Cauchy-Riemann equation

A necessary condition for a function *f*(z) such that w= *f*(z)= u(x,y)+ iv(x,y) to be analytic in domain D is that u and v satisfy Cauchy- Riemann equation given by

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \, and \, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Or $u_x=v_y$ and $u_y=-v_x$

These two equations are called the Cauchy- Riemann differential equations.

3.11 Harmonic Function: A function u(x, y) is called harmonic function if its first and second order partial derivatives are continuous and it satisfy Laplace equation

$$\text{i.e., } \nabla^2 u=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

3.12 Polar form of Cauchy-Riemann equation- If *f(*z)= u+iv is an analytic function and z= $re^{i\theta}$ then the Cauchy-Riemann equations are given by-

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \, and \, \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof:-

Let *f* (z) = u+iv is an analytic function, so $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \, and \, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ ….(1)

For polar co-ordinate system, we know that- x= r cosθ, y = r sinθ and $\theta=\tan^{-1}(\frac{y}{x})$

$$r^2=x^2+y^2$$

$$r=(x^2+y^2)^{1/2}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta, \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta$$

$$\frac{\partial \theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad \text{....(2)}$$

Similarly, $\frac{\partial u}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$, ...(3)

$$\frac{\partial v}{\partial x}=\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{...(4)}$$

$$\frac{\partial v}{\partial y}=\sin\theta\frac{\partial v}{\partial r}+\frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{...(5)}$$

Substituting these values in equation (1), we get

$$\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}=\sin\theta\frac{\partial v}{\partial r}+\frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{... (6)}$$

$$\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}=-\cos\theta\frac{\partial v}{\partial r}+\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{... (7)}$$

Multiplying (6) by cosθ, (7) by sinθ and adding, we get $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$

Again multiplying (6) by sinθ and (7) by cosθ and subtracting, we get $\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$

Hence polar form of Cauchy- Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

3.13 CAUCHY INTEGRAL FORMULA:

If f (z) is analytic within and on a closed contour c and 'a' is any point within c. Then,

$$f(a)=\frac{1}{2\pi i}\int\frac{f(z)}{z-a}dz$$

Proof- Let z=a , is a point within a closed contour c. Draw a circle γ, with centre at the point z=a and radius ρ such that it lies entirely within c.

Consider a function $\varphi(z)=\frac{f(z)}{z-a}$ is analytic in region between γ and c.

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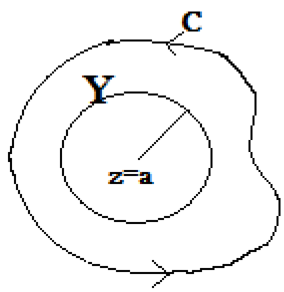


As we know- $\int_c \varphi(z)dz = \int_\gamma \varphi(z)dz$

$$\int_c \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz$$

$$\int_c \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz$$

$$\int_c \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)+f(a)-f(a)}{z-a}dz$$

$$\int_c \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)-f(a)}{z-a}dz + \int_\gamma \frac{f(a)}{z-a}dz \qquad ...(1)$$

Now, equation of circle, $|z-a| = \rho$

$z - a = \rho e^{i\theta}$ (Since, $e^{i\theta}$ =1)

$dz = i\rho e^{i\theta} d\theta$ and $0 \le \theta \le 2\pi$

So, $\int_\gamma \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a)}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$

$$= \int_{o}^{2\pi} if(a) d\theta = 2\pi i f(a)$$

Putting this value in equation (1)

$$\int_{c} \frac{f(z)}{z-a} dz = \int_{\gamma} \frac{f(z)-f(a)}{z-a} dz + 2\pi i f(a)$$

$$\int_{c} \frac{f(z)}{z-a} dz - 2\pi i f(a) = \int_{\gamma} \frac{f(z)-f(a)}{z-a} dz$$

Taking modulus on both sides

$$\left| \int_{c} \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| = \left| \int_{\gamma} \frac{f(z)-f(a)}{z-a} dz \right| \le \int_{\gamma} \frac{|f(z)-f(a)|}{z-a} |dz|$$

Now by the definition of continuity,

$|f(z) - f(a)| \langle \varepsilon, |z-a| = \rho$ and $\int_{\gamma} |dz = perimeter = 2\pi\rho|$

Hence,

$$\left| \int_{c} \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| \langle 2\pi\varepsilon$$

Making ϵ → 0, we get

$$\left| \int_{c} \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| \langle 0$$

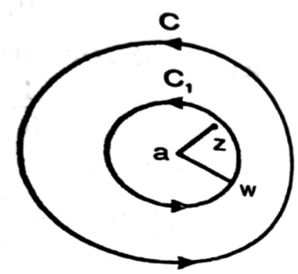
$$\left| \int_{c} \frac{f(z)}{z-a} dz - 2\pi i f(a) \right| = 0$$

$$\frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i} \int_{c} \frac{f(z)}{z-a} dz$$

This is Cauchy integral formula.

3.14 TAYLOR SERIES: If a function *f*(z) is analytic at all points inside a circle C, with its centre at the point a and radius R, then at each point z inside C.



$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(z-a)^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{f^n(a)}{n!}(z-a)^n + \ldots\ldots$$

PROOF: Take any point z inside C. Draw a circle C_1 with centre a, enclosing the point z. Let w be a point on circleC_1.

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{(w-a)} \frac{1}{(1-\frac{z-a}{w-a})} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

Applying binomial theorem

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\ldots\ldots\ldots\ldots+\left(\frac{z-a}{w-a}\right)^n+\ldots\ldots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a}+\frac{(z-a)}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\ldots\ldots+\frac{(z-a)^n}{(w-a)^{n+1}}+\ldots\ldots\ldots\ldots \qquad \ldots\ldots (1)$$

As $|z-a| < |w-a| \rightarrow \frac{|z-a|}{|w-a|} < 1$

so the series converges uniformly. Hence the series is integrable.

Multiplying eq.(1) by *f(w)*.

$$\frac{f(w)}{w-z}=\frac{f(w)}{w-a}+\frac{(z-a)f(w)}{(w-a)^2}+\frac{(z-a)^2 f(w)}{(w-a)^3}+.......+\frac{(z-a)^n f(w)}{(w-a)^{n+1}}+....$$

On integrating with respect to "w" we get

$$\int\frac{f(w)}{w-z}dw=\int_{c_1}\frac{f(w)}{w-a}dw+(z-a)\int_{c_1}\frac{f(w)}{(w-a)^2}dw+(z-a)^2\int_{c_1}\frac{f(w)}{(w-a)^3}dw+....+(z-a)^n\int_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

We know that,

$\int_{c_1}\frac{f(w)}{w-z}dz=2\pi i f(z)$ and $\int_{c_1}\frac{f(w)}{w-a}dz=2\pi i f(a)$

$\int_{c_1}\frac{f(w)}{(w-a)^2}dz=2\pi i f'(a)$ and so on.

Substituting these values in (2) we get

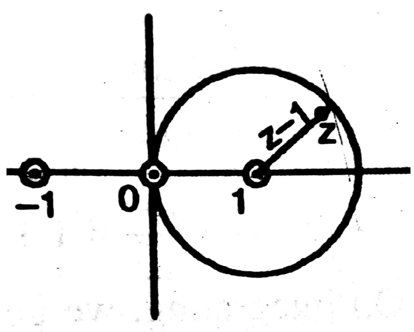
$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(z-a)^2}{2!}+.........................\frac{f^n(a)}{n!}(z-a)^n+......$$

This is Taylor's series.

Examples

Que: Find Taylor expansion of $f(z)=\frac{2z^2+1}{z^2+z}$ about the point z=1.

Ans:-



$f(z)=\frac{2z^3+1}{z(z+1)}$, singularities are given by z=0, -1

If centre of the circle is at z=1, then the distance of the singularities z=0 and z=-1 from the centre are 1 and 2. Hence, if a circle is drawn with centre z=1 and radius 1, then within the circle $|z-1|=1$, the given function $f(z)$ is analytic and therefore, it can be expanded in a Taylor series within the circle $|z-1|=1$.

$$\frac{2z^3+1}{z(z+1)} = 2z-2+\frac{1}{z+1}+\frac{1}{z}$$

$$= 2z-2+\frac{1}{z-1+2}+\frac{1}{z-1+1} \qquad [|z-1|]<1]$$

$$= 2z-2+\frac{1}{2}\left(1+\frac{z-1}{2}\right)^{-1}+[1+(z-1)]^{-1}$$

$$= 2z-2+\frac{1}{2}\left[1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+....\right]+[1-(z-1)+(z-1)^2-(z-1)^3+.....]$$

$$= 2z-2+\frac{3}{2}-\frac{3}{2}\left(\frac{z-1}{2}\right)+\frac{9}{8}(z-1)^2-\frac{17}{16}(z-1)^3+....$$

Which is required expansion.

Que: Expand cos z in a Taylor series about $z=\pi/4$.

Sol. Here $f(z)$= cos z, $f'(z)$= -sin z, $f''(z)$= -cos z, $f'''(z)$= sin z, …..

Here cos z= $f(z)$

$$= f\left(\frac{\pi}{4}\right)+\left(z-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+...$$

$$= \frac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3+...\right]$$

Which is required expansion.

Que: Expand the function $\left(\frac{\sin z}{z-\pi}\right)$ about $z=\pi$.

Sol. - Putting z-π= t, we have

$$\frac{\sin z}{z-\pi} = \frac{\sin(\pi + t)}{t} = -\frac{\sin t}{t}$$

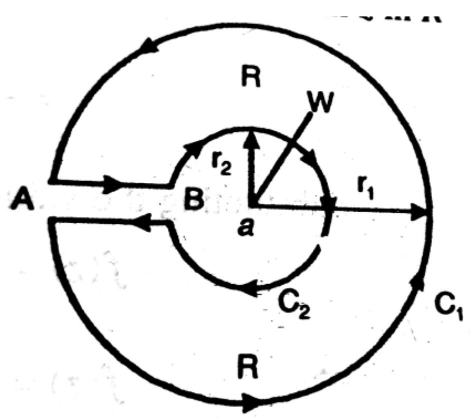
$$= -\frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - ......\right) = -1 + \frac{t^3}{3!} - \frac{t^5}{5!} + ...$$

$$= -1 + \frac{(z-\pi)^3}{3!} - \frac{(z-\pi)^5}{5!} + ...$$

Which is required expansion.

3.15 LAURENT'S SERIES: If we are required to expand *f*(z) about a point where *f*(z) is not analytic, then it is expanded by Laurent's series and not by Taylor's series.

Statement: If *f*(z) is analytic on c_1 and c_2 and the annular region R bounded by the two concentric circles c_1 and c_2 of radii r_1 and $r_2 (r_2 < r_1)$ and with centre at a, then for all z in R.



$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ...a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^n}$$

Where $a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^{n+1}} dw,$

$$b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{(w-a)^{-n+1}} dw$$

Proof: By introducing a cross cut AB, multi-connected region R is converted to a simply connected region. Now *f*(z) is analytic in this region.
Now by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{AB} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{BA} \frac{f(w)}{w-z}dw$$

Integral along c_2 is clockwise, so it is negative. Integrals along AB and BA cancel.

$$f(z) = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{w-z}dw \qquad \text{….(1)}$$

for the first integral, $\frac{f(w)}{w-z}$ can be expanded exactly as in Taylor's series as z lies on c_1.

$$\frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^2}dw + \frac{(z-a)^2}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^3}dw + ...$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + ... \qquad \textbf{…..(2)}$$

$$\left[a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^{n+1}}dw \right]$$

In the second integral, z lies on c_2. Therefore

$$|w-a| < |z-a| \text{ or } \frac{|w-a|}{|z-a|} < 1$$

So here $\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)}$

$$= -\frac{1}{z-a}\frac{1}{\left(1-\frac{w-a}{z-a}\right)} = -\frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

Using binomial expansion

$$= -\frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \ldots + \left(\frac{w-a}{z-a}\right)^{n+1} + \ldots\right]$$

Multiplying by $-\frac{f(w)}{2\pi i}$, we get

$$-\frac{1}{2\pi i}\frac{f(w)}{w-z} = \frac{1}{2\pi i}\frac{f(w)}{z-a} + \frac{1}{2\pi i}\frac{(w-a)f(w)}{(z-a)^2} + \frac{1}{2\pi i}\frac{(w-a)^2}{(z-a)^3}f(w) + \ldots$$

$$= \frac{1}{(z-a)}\frac{1}{2\pi i}f(w) + \frac{1}{2\pi i}\frac{1}{(z-a)^2}\frac{f(w)}{(w-a)^{-1}} + \frac{1}{2\pi i}\frac{1}{(z-a)^3}\frac{f(w)}{(w-a)^{-2}} + \ldots..$$

Integrating, we have

$$-\frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{w-z}dw = \left(\frac{1}{z-a}\right)\frac{1}{2\pi i}\int_{c_2} f(w)dw + \frac{1}{2\pi i}\frac{1}{(z-a)^2}\int_{c_2}\frac{f(w)}{(w-a)^{-1}}dw + \frac{1}{2\pi i}\frac{1}{(z-a)^3}\int_{c_2}\frac{f(w)}{(w-a)^{-2}}dw + \ldots$$

$$= \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \ldots \quad \text{...... (3)} \quad \left[b_n = \frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right]$$

Substituting the values of values of both integrals from (2) and (3) in (1), we get

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots. + b_1(z-a)^{-1} + b_2(z-a)^{-2} + \ldots$$

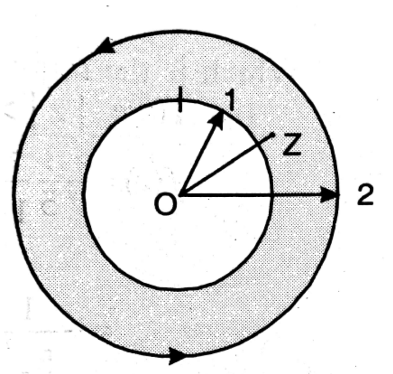
$$f(z) = \sum_{n=0}^{n=\infty} a_n (z-a)^n + \sum_{n=1}^{n=\infty} \frac{b_n}{(z-a)^n}$$

This is Laurent Theorem.

Que: Expand $f(z) = \frac{1}{(z-1)(z-2)}$ for $1<|z|<2$

Sol:- $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

In first bracket $|z| < 2$ we take out 2 as common and from second bracket z is taken out common as $1 < |z|$.



$$f(z) = -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+...\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right]$$

$$= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - .... - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} ....$$

This is the required expansion.

Que: Find the Laurent series expansion of

$$f(z) = \frac{z}{(z-1)(z-2)} \text{ valid for } |z-1| > 1.$$

Sol. $f(z) = \frac{z}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{2}{z-2} = -\frac{1}{z-1} + \frac{2}{z-1-1}$

$$= -\frac{1}{z-1} + \frac{2}{z-1}\frac{1}{1-\frac{1}{z-1}} = -\frac{1}{z-1} + \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= -\frac{1}{z-1} + \frac{2}{z-1}\left(1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+...\right)$$

$$= -\frac{1}{z-1} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^4} + ... \qquad \cdots \textbf{Ans}$$

3.16 SINGULARITY: A singular point or singularity of a function is the point of a function at which the function ceases to be analytic.

For example:- If $f(z) = \frac{1}{z-2}$

Then z=2 is a singularity of *f*(z).

3.17 Types of singularities:

3.17.1 Isolated singularity-If the function f (z) has a singularity at z = a and in a neighborhood of 'a' (i.e. a region of the complex plane which contains a) there are no other singularities then 'a' is an isolated singularity of *f* (z).

For example: If *f*(z)= 1/z then z=0 is an isolated singularity of *f*(z).

3.17.2 Removable singularity- In this type, if f(z) has a singularity at z=a then we can remove this singularity.

For example, consider the function f (z) = $\frac{\sin z}{z}$, f (z) has an isolated singularity at z=0.

$$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....$$

Since no negative power of z occurs in the expansion. Hence z=0 is a removable singularity.

3.17.3 Poles-Poles and Zeros of a function are the values for which the value of the denominator and numerator of function becomes zero respectively. If the number of terms are 'm' then z = a is said to be a pole of order m. A pole of order 1 is called a simple pole.

If $f(z) = \frac{1}{z(z-5)^2(z-4)^3}$ then z=0 is a simple pole, z=5 is a pole of order 2 and z=4 is a pole of order 3.

3.17.4 Essential singularity-

If $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2.2!} + \frac{1}{z^3.3!} + ...$

Since in the expansion there is an infinite series of negative powers of z thus z=0 is the essential singularity of f(z).

3.18 Residue:

For a function f(z), the Laurent expansion is-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

If this function f (z) has a pole of order m at z=a then its principal part is given by-

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Where, $a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz, b_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{-n+1}} dz$

Evidently, $b_1 = \frac{1}{2\pi i} \int_c f(z) dz$

The coefficient b_1 is called residue of f(z) at the pole z=a.

Res (z = a) = $b_1 = \frac{1}{2\pi i} \int_c f(z) dz$

When z = a is a simple pole then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

$$(z-a) f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$$\lim_{z \to a} (z-a) f(z) = b_1$$

1. Hence, for simple pole

Res (z=a) = $\lim_{z \to a} (z-a) f(z)$

2. For pole of order m , $\operatorname{Re}s(z=a)=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$

3. Residue at infinity (a) $\operatorname{Re}s(z=\infty)=\lim_{z\to\infty}-zf(z)$ if limit exists.

(b) Res (z = ∞) = negative of the coefficient of 1/z in the expansion of f(z)

3.19 CAUCHY RESIDUES THEOREM: If f(z) is analytic within and on a closed contour c, except at a finite number of poles $z_1, z_2, z_3, ..., z_n$ within c, then,

$$\int_c f(z)dz = 2\pi i\sum_{r=1}^{n}\operatorname{Re}s(z=z_r) = 2\pi i\sum R^+$$

Where $\sum R^+$ = sum of residues of f(z).

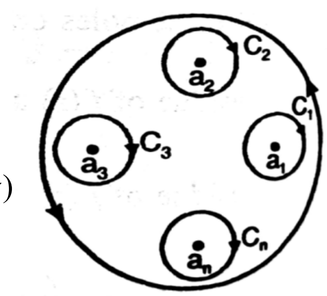
Proof-

Consider $c_1, c_2, c_3, ..., c_n$ are the circles with centre $a_1, a_2, a_3, ..., a_n$ respectively and radii so small that they lie within closed contour c and do not overlap.

Since *f*(z) is analytic within the annulus bounded between these circles and the contour c, then we know-

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + ... + \int_{c_n} f(z)dz$$

Dividing by 2πi



$$\frac{1}{2\pi i}\int_c f(z)dz = \frac{1}{2\pi i}\int_{c_1} f(z)dz + \frac{1}{2\pi i}\int_{c_2} f(z)dz + ... + \frac{1}{2\pi i}\int_{c_n} f(z)$$ **.......(1)**

By the definition of residue,

Residue of $f(z) = \frac{1}{2\pi i}\int_c f(z)dz$

$$\operatorname{Re}s(z=z_1) = \frac{1}{2\pi i}\int_{c_1} f(z)dz$$

Hence, from equation (1), we get

$$\frac{1}{2\pi i}\int_c f(z)dz = \text{Res}(z=a_1) + \text{Res}(z=a_2) + \ldots\ldots\ldots\ldots + \text{Res}(z=a_n) = \sum_{r=1}^{n}\text{Res}(z=a_r)$$

$$\int_c f(z)dz = 2\pi i\sum_{r=1}^{n}\text{Res}(z=a_r)$$

This is Cauchy residues theorem.

3.20 EVALUATION OF RESIDUES

Question: Find the order of each pole and residue of $\frac{1-2z}{z(z-1)(z-2)}$.

Ans:-

Let $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

The poles of *f*(z) are given by z(z-1)(z-2)=0
z=0, 1, 2 all are simple poles.

Residue of f (z) at (z=0) = $\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{z(1-2z)}{z(z-1)(z-2)}$

$$= \lim_{z\to 0}\frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$$

Residue of f (z) at (z=1)= $\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{(z-1)(1-2z)}{z(z-1)(z-2)} = \lim_{z\to 1}\frac{1-2z}{z(z-2)} = 1$

Residue of f (z) at (z=2)= $\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{(z-2)(1-2z)}{z(z-1)(z-2)} = \lim_{z\to 2}\frac{1-2z}{z(z-1)} = -\frac{3}{2}$

Hence, the residue of f (z) at z=0, 1 and z=2 are ½, 1 and -3/2 respectively.

Question: Evaluate the residue of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at 1,2,3 and infinity and show that their sum is zero.

Ans: Let $f(z)=\frac{z^2}{(z-1)(z-2)(z-3)}$

$$\operatorname{Re}s(z=1)=\lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}(z-1)\frac{z^2}{(z-1)(z-2)(z-3)}=\frac{1}{2}$$

$$\operatorname{Re}s(z=2)=\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}\frac{z^2}{(z-1)(z-3)}=-4$$

$$\operatorname{Re}s(z=3)=\lim_{z\to 3}\frac{z^2}{(z-1)(z-2)}=\frac{9}{2}$$

$$\operatorname{Re}s(z=\infty)=\lim_{z\to\infty}-zf(z)$$

$$=\lim_{z\to\infty}\frac{-z^3}{(z-1)(z-2)(z-3)}$$

$$=\lim_{z\to\infty}\frac{-z^3}{z^3(1-\frac{1}{z})(1-\frac{2}{z})(1-\frac{3}{z})}=-1$$

Sum of residues = $\frac{1}{2}-4+\frac{9}{2}-1=0$

Question:-Evaluate the residue of $f(z)=\frac{z+3}{z^2-2z}$

Ans:

$$f(z)=\frac{z+3}{z(z-2)}$$

Poles are z=0 and z=2

$$\operatorname{Re}s(z=0)=\lim_{z\to 0}\frac{(z-0)(z+3)}{z(z-2)}=\lim_{z\to 0}\frac{z+3}{z-2}=-\frac{3}{2}$$

$$\operatorname{Re}s(z=2)=\lim_{z\to 2}\frac{(z-2)(z+3)}{z(z-2)}=\lim_{z\to 2}\frac{z+3}{z}=\frac{5}{2}$$

Question:-Find the residue of $f(z)=\frac{1}{(z^2+a^2)^2}$ at z=ia.

Ans: $$f(z)=\frac{1}{(z^2+a^2)^2}$$
$$=\frac{1}{(z+ia)^2(z-ia)^2}$$

Poles are z=ia, -ia of order 2.

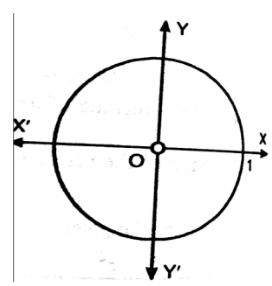
$$\operatorname{Re}s(z=ia)=\lim_{z\to ia}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-ia)^2\frac{1}{(z+ia)^2(z-ia)^2}\right]$$
$$=\lim_{z\to ia}\frac{d}{dz}\left[\frac{1}{(z+ia)^2}\right]$$
$$=\lim_{z\to ia}-\frac{2}{(z+ia)^3}=\frac{1}{4ia^3}$$

Question:-Find the residue of $\frac{z^3}{z^2-1}$ at z=∞.

Ans:-Let $f(z)=\frac{z^3}{z^2-1}$

$$f(z)=\frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)}=z\left(1-\frac{1}{z^2}\right)^{-1}$$
$$=z\left(1+\frac{1}{z^2}-........\right)$$
$$=z+\frac{1}{z}-.............$$

Res (z=∞) = negative coefficient of 1/z = -1



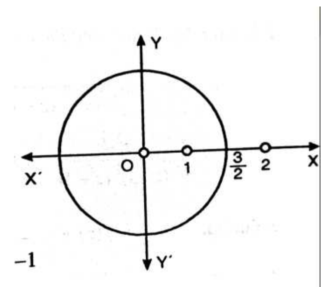
3.21 EVALUATION OF INTEGRALS

(a) If f (z) is analytic in a closed curve C, except at a finite number of poles within C, then $\int_c f(z)dz=2\pi i$ i (sum of residues at the poles within C).

Question:-Evaluate the following integral using residue theorem $\int_c \frac{1+z}{z(2-z)} dz$, where c is the circle $|z| = 1$.

Ans. Let f(z)= $\frac{1+z}{z(2-z)}$

Poles are z=0, 2.
The integrand is analytic on $|z| = 1$ and all points inside except z=0, as a pole at z=0 is inside the circle $|z| = 1$.
Hence by residue theorem,



$$\int_c \frac{1+z}{z(2-z)} dz = 2\pi i[resf(0)]$$

Residue f(0)=$\lim_{z\to 0} \frac{z(1+z)}{2-z} = \frac{1}{2}$

Putting the value of Residue f (0) in eq.(1), we get

$$\int_c \frac{1+z}{z(2-z)} dz = 2\pi i[\frac{1}{2}] = \pi i$$

Question: Evaluate the following integral using residue theorem $\int_c \frac{4-3z}{z(z-1)(z-2)} dz$

Where, c is the circle $|z| = \frac{3}{2}$.
Ans: The poles of the function *f*(z) are given by
z=0, 1, 2

The function has poles at z=0, 1, 2 of which the given circle encloses the pole at z=0 and z=1.

Residue of f (z) at the simple pole z=0 is

$$= \lim_{z\to 0} \frac{z(4-3z)}{z(z-1)(z-2)} = \lim_{z\to 0} \frac{(4-3z)}{(z-1)(z-2)} = \frac{4-0}{(-1)*(-2)} = 2$$

Residue of f(z) at the simple pole z=1 is

$$= \lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)} = \lim_{z\to 1}\frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{1(-1)} = -1$$

By Cauchy's integral formula

$$\int_c f(z)dz = 2\pi i \text{ (sum of the residue within c)}$$

$$= 2\pi i\,(2-1) = 2\pi i.$$

(b) Evaluation of $\int_{-\infty}^{\infty} f(x)dx$

Let f (z) be a function such that-

(i) f (z) is analytic throughout the upper half plane except at certain points which are its poles.

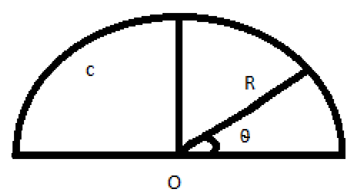
(ii) f (z) has no poles on the real axis i.e., if $R\to\infty$ (R being the radius of semi-circle), then it will cover entire upper half plane.

(iii) $zf(z)\to 0$, uniformly as $z\to\infty$ for $0\le \arg z\le \pi$.

(iv) $\int_0^{\infty} f(x)dx$ and $\int_{-\infty}^{0} f(x)dx$ both converges then

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$$

Where $\sum R^+$ denotes the sum of the residues of *f*(z) at its pole in the upper half plane.



Question: prove that $\int_0^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}$

Ans: let $f(z) = \frac{1}{1+z^2} = \pi/2$

Only z=i lies inside the contour c.

Res (z=i) = $\lim_{z\to i}(z-i)f(z)$

$$= \lim_{z\to i}(z-i)\frac{1}{(z+i)(z-i)} = \frac{1}{2i}$$

Hence, $\int_{-\infty}^{\infty} f(z)dz = 2\pi i\sum R^+ = 2\pi i\frac{1}{2i}$

$$\int_{-\infty}^{\infty} f(x)dx = \pi$$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \pi$$

$$\int_0^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}$$

Question:-Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)(x^2+4)}$

Ans: Consider $\int_c \frac{z^2 dx}{(1+z^2)(z^2+4)} = \int_c f(z)dz$

Where, C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from –R to +R.

The integral has simple poles at
$z = \pm i, z = \pm 2i$, of which z=i, 2i only lie inside C.

The residue (at z = i) = $\lim_{z \to i} \frac{(z-i)z^2}{(z+i)(z-i)(z^2+4)}$

$\lim_{z \to i} \frac{z^2}{(z+i)(z^2+4)} = -\frac{1}{2i(-1+4)} = -\frac{1}{6i}$

The residue (at z=2i) = $\lim_{z \to 2i} \frac{(z-2i)z^2}{(z^2+1)(z+2i)(z-2i)}$

$\lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{(2i)^2}{(-4+1)(2i+2i)} = 1/3i$

By residue theorem,

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum R^+ = 2\pi i \left[-\frac{1}{6i} + \frac{1}{3i} \right] = \frac{\pi}{3}$$

Question: Prove that $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log z$

Ans: Let f (z) = $\frac{\log(1+z^2)}{1+z^2}$

Poles of f (z) are given by

$1+z^2 = 0$, only z=i lies inside the contour c.

Res (z=i) = $\lim_{z \to i} (z-i) f(z) = \lim_{z \to i} (z-i) \frac{\log(z+i)}{(z+i)(z-i)} = \log(2i)/2i$

$$=\frac{\log(2e^{i\pi/2})}{2i}=(\log 2+\log e^{i\pi/2})/2i$$

$$=\frac{\log 2+i\pi/2}{2i}$$

Hence $\int_{-\infty}^{\infty}\frac{\log(z+i)}{1+z^2}dz=2\pi i\frac{\log 2+i\pi/2}{2i}=\pi\log 2+i\pi^2/2$

$$2\int_{0}^{\infty}\frac{\log(x+i)}{1+x^2}dx=\pi\log 2+i\pi^2/2$$

Using formula log (α+iβ) $=\frac{1}{2}\log(\alpha^2+\beta^2)+i\tan^{-1}(\beta/\alpha)$

$$\int_{0}^{\infty}\left[\frac{\log(1+x^2)}{1+x^2}+\frac{2i\tan^{-1}(\beta/\alpha)}{1+x^2}\right]dx=\pi\log 2+i\pi^2/2$$

Equating real parts, we get

$$\int_{0}^{\infty}\left[\frac{\log(1+x^2)}{1+x^2}\right]dx=\pi\log 2$$

(c) Integration round unit circle of the type $\int_{0}^{2\pi}f(\cos\theta,\sin\theta)d\theta$

Here $\int_{0}^{2\pi}f(\cos\theta,\sin\theta)d\theta$ is a rational function of cosθ and sinθ.
Convert cosθ, sin θ into z.
Consider a circle of unit radius with centre at origin, as contour.

$$\cos\theta=\frac{e^{i\theta}+e-^{i\theta}}{2i}=\frac{1}{2i}\left[z-\frac{1}{z}\right]$$

$$\sin\theta=\frac{e^{i\theta}-e-^{i\theta}}{2}=\frac{1}{2i}\left[z+\frac{1}{z}\right]$$

As $z=re^{i\theta}=1*e^{i\theta}=e^{i\theta}$

As we know, $z=e^{i\theta}\to dz=e^{i\theta}id\theta=izd\theta$

The integrand is converted into a function of z.
Then apply Cauchy's residue theorem to evaluate the integral.

Question: Evaluate the integral $\int_{0}^{2\pi}\frac{d\theta}{5-3\cos\theta}$

Ans. $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-3\left(\frac{e^{i\theta}+e^{-i\theta}}{2!}\right)}$ **[**Let $e^{i\theta} = z, d\theta = \frac{dz}{iz}$]

[c is the unit circle $|z| = 1$]

$$= -\frac{2}{i}\int_C \frac{dz}{(3z-1)(z-3)} = 2i\int_C \frac{dz}{(3z-1)(z-3)}$$

$$I = \int_0^{2\pi} \frac{d\theta}{10-3e^{i\theta}-3e^{-i\theta}} = \int_C \frac{2}{10-3z-\frac{3}{z}}\frac{dz}{iz} = \frac{1}{i}\int_C \frac{2dz}{10z-3z^2-3}$$

Poles of the integrand are given by $(3z-1)(z-3)=0$
i.e., z=1/3, 3.There is only one pole at z=1/3 inside the unit circle c.

Residue at z=1/3

$$\mathrm{Res}\left(z=\frac{1}{3}\right) = \lim_{Z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)f(z)$$

$$= \lim_{Z\to\frac{1}{3}} \frac{\left(z-\frac{1}{3}\right)2i}{(3z-1)(z-3)} = \lim_{Z\to\frac{1}{3}} \frac{2i}{3(z-3)} = \frac{2i}{3\left(\frac{1}{3}-3\right)} = -\frac{i}{4}$$

Hence by Cauchy residues theorem
I = 2πi (sum of the residues within contour) = 2πi (-i/4) = π/2 **Ans.**

Question:-Evaluate contour integration of the real integral $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$.

Ans: $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$ = Real part of $\int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}d\theta$

= Real part of $\int_0^{2\pi} \frac{e^{3i\theta}}{5-2(e^{i\theta}+e^{-i\theta})}d\theta$ Let $z = e^{i\theta}$ and $d\theta = \frac{dz}{iz}$

= Real part of $\int_C \frac{z^3}{5-2\left(z+\frac{1}{z}\right)}\left(\frac{dz}{iz}\right)$ c is the unit circle.

$$= \text{Real part of } -\frac{1}{i}\int_0^{2\pi} \frac{z^3}{2z^2 - 5z + 2} dz$$

$$= \text{Real part of } \ i\int \frac{z^3}{(2z-1)(z-2)} dz$$

Poles are given by (2z-1) (z-2) =0 i.e. z=1/2, z=2
z=1/2 is the only pole inside the unit circle.

$$\text{Residue (at z=1/2)} = \lim_{Z\to\frac{1}{2}} \frac{i\left(z-\frac{1}{2}\right)z^3}{(2z-1)(z-2)}$$

$$= \lim_{Z\to\frac{1}{2}} \frac{iz^3}{2(z-2)} = -\frac{i}{24}$$

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \text{Real part of } 2\pi\text{i}\left(-\frac{i}{24}\right) = \frac{\pi}{12}.$$ Ans.

Question: Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Ans. Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{d\theta}{2 + \frac{e^{i\theta}+e^{-i\theta}}{2i}} = \int_0^{2\pi} \frac{2d\theta}{4 + e^{i\theta} - e^{-i\theta}}$

Put $e^{i\theta} = z$ so that $e^{i\theta}(id\theta) = dz, d\theta = \frac{dz}{i\theta}$

$$I = \int_C \frac{2\, dz/iz}{4 + z + \frac{1}{z}} = \frac{1}{i}\int_C 2\frac{dz}{z^2 + 4z + 1}$$

The poles are given by

$$z^2 + 4z + 1 = 0 \text{ or } z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

The pole within the unit circle C is a simple pole at z=$-2+\sqrt{3}$. Now we calculate the residue at this pole.

Residue at (z=$-2+\sqrt{3}$) =

$$\lim_{Z\to(-2+\sqrt{3})}\frac{1}{i}\frac{(z+2-\sqrt{3})2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}=\lim_{Z\to(-2+\sqrt{3})}\frac{2}{i(z+2+\sqrt{3})}=\frac{2}{i(-2+\sqrt{3}+2+\sqrt{3})}=\frac{1}{\sqrt{3}i}$$

Hence by Cauchy's residues theorem, we have

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = 2\pi i \text{ (sum of the residues within the contour)}$$

$$=2\pi i\frac{1}{i\sqrt{3}}=\frac{2\pi}{\sqrt{3}} \text{ .} \qquad \text{Ans}$$

3.22 Summary: This chapter introduces imaginary and complex numbers. Complex numbers are numbers of the form a + ib, where i = $\sqrt{-1}$ and a and b are real numbers. They are used in a variety of computations and situations. Complex numbers are useful for our purposes because they allow us to take the square root of a negative number and to calculate imaginary roots.

In the beginning of this chapter, we have discussed the complex plane, along with the algebra and geometry of complex numbers, and then we have made our way via differentiation, integration, complex dynamics, power series representation and Laurent series into territories at the edge of what is known today. Complex Integration now includes a new and simpler proof of the general form of Cauchy's theorem. There is a short section on the concept of Singularities, residues, poles and Evaluation of Integrals by using Cauchy residue theorem.

3.23 References:

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3. Joseph Bak, Donald J. Newman, *Complex Analysis, Springer*.
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3.24 EXERCISE

1. If z=a cosθ+ ia sinθ, prove that $\left(\frac{z}{\bar{z}}+\frac{\bar{z}}{z}=2\cos\theta\right)$.
2. Prove that $\left|\frac{z-1}{\bar{z}-1}\right|=1$.
3. Test the analyticity of the function w= sin z.

4. $\int_c \frac{e^z}{z-1} dz$ Where c is the circle $|z| = 2$. **Ans:** $2\pi ie$

5. $\int_c \frac{2z^2 + z}{z^2 - 1} dz$, where c is the circle $|z-1| = 1$ **Ans:** $3\pi i$

6. Expand $\frac{z}{(z^2-1)(z^2+4)}$ in $1 < |z| < 2$

   Ans: $\frac{1}{10}\left[\left(\frac{2}{z} + \frac{2}{z^3} + \frac{2}{z^5} + ...\right) - \left(\frac{z}{2} + \frac{z^3}{8} + ...\right)\right]$

7. Find Taylor Expansion of $f(z) = \frac{2z^3 + 1}{z^2 + z}$ about the point z=i.

   Ans: $\left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty}(-1)^n \left\{\frac{1}{(1+i)^{n+1}} + \frac{1}{(i)^{n+1}}\right\}(z-i)^n$

8. Find the poles or singularity of the following function

   $\frac{1}{(\sin z - \cos z)}$

   Ans: Simple pole at z=π/4

9. Evaluate $\int_0^{\infty} \frac{1}{1+x^2} dx$ **Ans:** π/2

10. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$ **Ans:** $\pi/\sqrt{2}$

Choose the correct alternative:

1. If $z = r(\cos\theta + i\sin\theta)$ then $|z|^3$ is equal to:

   (i) $(\cos\theta + i\sin\theta)^3$ (ii) $r^3(\cos\theta + i\sin\theta)^3$ (iii) $\frac{r^3}{2}$ (iv) r^3.

2. If $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{\alpha x}{y}$ be an analytic function if α is equal to:

   (i) 1 (ii) -1 (iii) 2 (iv) -2 .

3. The value of $\int_c \frac{z^2 - z + 1}{z-1} dz$, c being $|z| = \frac{1}{2}$ is:

(i) 2πi (ii) $\frac{1}{2\pi i}$ (iii) 0 (iv) πi.

4. Let $f(z) = \frac{1}{(z-2)^4 (z+3)^6}$, then z=2 and z=-3 are the poles of order:

   (i) 6 and 4 (ii) 2 and 3 (iii) 3 and 4 (iv) 4 and 6.

5. What is the value of $\int_c \frac{e^{2z}}{(z+1)^4}$, where C is circle $|z| = 1$?

   (i) Zero (ii) $4\pi i e^{-2}$ (iii) $\frac{4\pi i}{3} e^{-2}$ (iv) $\frac{8\pi i e^{-2}}{3}$.

6. The value of integral $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$ is :

   (i) π (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{3}$.

7. Find the sum of residues at all poles of function $\frac{z}{\cos z}$.

   (i) π (ii) –π (iii) zero (iv) π/2.

8. Evaluate $\int_c |z| dz$ where the contour C is straight line from z= -i to z=+i

   (i)zero (ii)1 (iii) –i (iv) i.

9. Evaluate $\int_{-\infty}^{\infty} e^{-x^2} \cos 2x dx$:

   (i) $\sqrt{\pi}$ (ii) √π/e (iii) $\sqrt{\pi e}$ (iv) none of these.

10. What is the residue at all poles of the function $\frac{e^{ikz}}{a^2 + z^2}$?

    (i) $\frac{\sinh(ka)}{a}$ (ii) $\frac{i\sinh(ka)}{a}$ (iii) $-\frac{i\sinh(ka)}{a}$ (iv) zero.

Ans: **(1)**(iv) **(2)** (i) **(3)** (iii) **(4)** (iv) **(5)**(iv) **(6)** (iii) **(7)** (iii) **(8)**(iv) **(9)** (ii) **(10)**(ii).

UNIT 4: TENSOR

STRUCTURE

4.22.2 Long Answer type

4.22.3 Numerical Answer type

4.1 OBJECTIVES

After studying this unit, you should be able to-

- Define rank of tensor
- Define Covariant and Conravariant vectors
- Define Covariant and Conravariant tensors
- Define mixed tensor
- Define contraction of tensor
- Define summation and subtraction of tensor
- Define inner and outer product
- Define geodesics
- Define Christofell's symbol

4.2 INTRODUCTION

In three dimensional space a point is determined by a set of three numbers called the co-ordinates of that point in particular system. Tensor analysis is intimately connected with the subject of co-ordinate transformations. Number of indices present in a physical quantity is called its rank. A Tensor of rank zero is said to be scalar or invariant. A Tensor of rank one is said to be vector. A Tensor having indices in superscript is said to be contravariant while Tensor having indices in subscript is said to be covariant. A Tensor having indices both in subscript and superscript is called mixed Tensor. If two contravariant or covarint indices can be interchanged without altering the tensor, then the tensor is said to be symmetric with respect to these two indicas. A tensor, whose each component alters in sign but not in magnitude when two contravariant or covariant indices are interchanged, is said to be skew symmetric or anti- symmetric with respect to these two indices. The tensor which has the same components in all co-ordinate systems are said to be invariant tensors. The Levi-Civita symbol is defined as a quantity ε_{ijk} in three dimensional space which is antisymmetric in all its indices. The sum or difference of two tensors of the same rank and same type is also a tensor of the same rank and same type. Two tensors of the same rank and same type are said to be equal if their components are one to one equal. The algebraic operation by which the rank of a mixed tonsor is lowered by 2 is known as contraction. An expression which are express the distance between two adjacent point is called a metric or line element. The path of extremum (maximum or minimum) distance between any two points in

Riemannian space is called the geodesic. The quadratic differential form $g_{jk}dx^j dx^k$ is independent of the coordinates system and is called the Riemannian metric for n dimensional space. The space which is characterised by Riemannian metric is called Riemannian space.

4.3 TENSORS: Tensors are important in physics as they provide a concise mathematical framework for formulating and solving physics problems in areas such as mechanics (stress, elasticity, fluid mechanics, moment of inertia etc.) and in electrodynamics (electromagnetic tensor, Maxwell tensor, permittivity, magnetic susceptibility etc.) or general relativity (curvature tensor, stress- energy tensor etc.).

In applications, it is common to study situations in which a different tensor can occur at each point of an object; for example the stress within an object may vary from one location to another. This leads to the concept of a tensor field. In some areas, tensor fields are so ubiquitous that they are often simply called "tensors".

Number of indices present in a physical quantity is called its rank. A Tensor of rank zero is said to be scalar or invariant. A Tensor of rank one is said to be vector.

4.4 CO-ORDINATE TRANSFORMATIONS

Tensor analysis is connected with the subject of co-ordinate transformations.

Consider two sets of variables $(x^1, x^2, x^3, \ldots, x^n)$ and $\left(\overline{x^1}, \overline{x^2}, \overline{x^3}, \ldots\ \overline{x^n}\right)$ in two different frames of reference which determine the co-ordinates of point in an n-dimensional space. Let the two sets of variables be related to each other by the transformation equations

$$\overline{x^1} = P^1(x^1, x^2, x^3, \ldots x^n)$$

$$\overline{x^2} = P^2(x^1, x^2, x^3, \ldots x^n)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\overline{x^n} = P^n(x^1, x^2, x^3, \ldots x^n)$$

or briefly $\overline{x^\mu} = P^\mu(x^1, x^2, x^3, \ldots, x^i, \ldots, x^n)$ …(4.1)

(i = 1, 2, 3, …, n)

where function P^μ are single valued, continuous differentiable functions of co-ordinates. It is essential that the n-function P^μ be independent. Equations (4.1) can be solved for co-ordinates x^i as functions of $\overline{x^\mu}$ to yield

$$x^i = A^i(\overline{x^1}, \overline{x^2}, \overline{x^3}, \ldots, \overline{x^\mu}, \ldots, \overline{x^n}) \qquad \ldots(4.2)$$

Equations (4.1) and (4.2) are said to define co-ordinate transformations.

From equations (4.1) the differentials $\overline{dx^\mu}$ are transformed as

$$\overline{dx^\mu} = \frac{\partial \bar{x}^\mu}{\partial x^1} dx^1 + \frac{\partial \bar{x}^\mu}{\partial x^2} dx^2 + \cdots + \frac{\partial \bar{x}^\mu}{\partial x^n} dx^n$$
$$= \sum_{i=1}^{n} \frac{\partial \bar{x}^\mu}{\partial x^i} dx^i, \ (\mu = 1, 2, 3, \ldots, n). \qquad \ldots(4.3)$$

4.5 INDICAL AND SUMMATION CONVENTIONS

The summation convention implies the sum of the term for the index appearing twice in that term over defined range. An index repeated as sub and superscript in a product represents summation over the range of the index.

We can define two types of convention:

4.5.1 Indicial convention-Any index, used either as subscript or superscript will take all values from 1 to n unless the contrary is specified. Thus, equations (4.1) can be written as

$$\bar{x}^\mu = P^\mu(x^i). \qquad \ldots(4.4)$$

The convention reminds us that there are n equations with μ = 1, 2, …n and A^μ are the functions of n-co-ordinates with (i = 1, 2, …, n).

4.5.2 Einstein's summation convention-If any index is repeated in a term then a summation with respected to that index over the range 1, 2, 3, …, n is implied. This convention is called Einstein's summation convention.

According to this conversation instead of expression $\sum_{i=1}^{n} a_j x^j$

we write $a_i x^i$.

Using above tow conversation eqn. (4.3) is written as

$$\overline{dx}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^i} dx^i. \qquad \ldots(4.5a)$$

Thus, the summation convention means the drop of sigma sign for the index appearing twice in a given term.

4.6 DUMMY AND REAL INDICES

Any index which is repeated in a given term, so that the summation convention implies, is called a dummy index and it may be replaced freely by any other index not already used in the term. For example i is a dummy index in $a_i^\mu x^i$.

$$\overline{dx}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^k} dx^k = \frac{\partial \bar{x}^\mu}{\partial x^\lambda} dx^\lambda. \qquad \text{...(4.5b)}$$

Also two or more dummy indices can be interchanged. Any index which is not repeated in a given term is called a real index. For example μ is a real index in $p_i^\mu x^i$. A real index cannot be replaced by another real index, e.g.

$$p_i^\mu x^i \neq p_i^\nu x^i.$$

SAQ 1: What is difference between real and dummy indices?

4.7 KRONECKER DELTA SYMBOL

In mathematics, the **Kronecker delta** (named after Leopold Kronecker) is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal and 0 otherwise:

The symbol Kronecker delta $\delta_k^j = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$ ... (4.6)

The Kronecker delta δ_{ij} is a piecewise function of variables i and j. For example, $\delta_{1\,2} = 0$, whereas $\delta_{3\,3} = 1$.

4.7.1 Some properties of Kronecker delta

(i) If $x^1, x^2, x^3, \ldots x^n$ are independent variables, as

$$\frac{\partial x^j}{\partial x^k} = \delta_k^j. \qquad \text{... (4.7)}$$

(ii) An another property of Kronecker delta symbol is

$$\delta_k^j P^j = P^k. \qquad \text{... (4.8)}$$

Since by summation convention in the left hand side of this equation the summation is with respect to j and by definition of kronecker delta, the only surviving term is that for which j = k.

(iii) If we are dealing with n dimensions, then

$$\delta_j^j = \delta_k^k = n. \qquad \text{...(4.9)}$$

By summation convention

$$\delta_j^j = \delta_1^1 + \delta_2^2 + \delta_3^3 + \cdots + \delta_n^n$$

$$= 1 + 1 + 1 + \cdots + 1 = n.$$

(iv) $$\delta_j^i \delta_k^j = \delta_k^i. \quad \ldots(4.10)$$

By summation convention

$$\delta_j^i \delta_k^j = \delta_1^i \delta_k^1 + \delta_2^i \delta_k^2 + \delta_3^i \delta_k^3 + \cdots + \delta_i^i \delta_k^i + \cdots \delta_n^i \delta_k^n$$

$$= 0 + 0 + 0 + \cdots + 1.\, \delta_k^i + \cdots + 0$$

$$= \delta_k^i.$$

(v) $$\frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^k} = \frac{\partial x^j}{\partial x^k} = \delta_k^j. \quad \ldots(4.11)$$

4.7.2 Generalised Kronecker Delta: The generalised Kronecker delta is symbolized as

$$\delta^{j_1\ j_2\ \ldots\ j_m}_{k_1 k_2 \ldots k_m}$$

and defined as follows:

(i) The subscripts and superscripts can have any value from 1 to n.

(ii) If either at least two superscripts or at least two subscripts have the same value or the subscribts are not the same set as super-scripts, then the generalised Kronecker delta is zero. For example

$$\delta^i_{jkl} kk = \delta^{ijk}_{lmm} = \delta^{ijk}_{klm} = 0.$$

(iii) If all the subscripts are separately different and the subscripts are the same set of numbers as the superscripts, then the generalised Kronecker delta has value +1 or -1 according to whether it requires as even or odd number of permutations to arrange the superscripts in the same order as the subscripts.

For example

$$\delta^{123}_{123} = \delta^{123}_{231} = \delta^{1452}_{4125} = +1$$

and $$\delta^{123}_{213} = \delta^{123}_{132} = \delta^{1452}_{4152} = -1.$$

It should be noted that

$$\delta_1^{i_1}, \delta_2^{i_2}, \delta_{3\dots n}^{i_3\dots i_n} \delta_{j_1}^1, \delta_{j_2}^2, \delta_{j_3\dots j_n}^{3\dots n} = \delta_{j_1}^{i_1}, \delta_{j_2}^{i_2}, \delta_{j_3\dots j_n}^{i_3\dots i_n}.$$

4.8 SCALARS, CONTRAVARIANT VECTORS and COVARIANT VECTORS

4.8.1 SCALARS: Consider a function ϕ in a co-ordinate system of variables x^i and let his function have the value $\overline{\phi}$ in another system of variables $\overline{x}^\mu$. If

$$\overline{\phi} = \phi.$$

Then the function ϕ is said to be scalar or invariant or a tensor of the order zero.

The quantity

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 + \cdots + \delta_n^n = n.$$

Is a scalar or an invariant.

4.8.2 CONTRAVARIANT VECTORS: Consider a set of n quantities $P^1, P^2, P^3, \dots P^n$ in a system of variables x^i and let these quantities have values $\overline{P}^1, \overline{P}^2, \overline{P}^3, \dots \overline{P}^n$ in another co-ordinate system of variables$\overline{x}^\mu$. If these quantities obey the transformation relation

$$\overline{P}^\mu = \frac{\partial \overline{x}^\mu}{\partial x^i} = P^i \qquad \dots(4.12)$$

where the quantities P^i are said to be the components of a contravariant vector or a contravariant tensor of first tank.

Any n functions can be chosen as the components of a contravariant vector in a system of variables $\overline{x}^\mu$.

Multiplying equation (4.12) by $\frac{\partial x^j}{\partial \overline{x}^\mu}$ and taking the sum over the index μ from 1 to n, we get

$$\frac{\partial x^j}{\partial \overline{x}^\mu} \overline{P}^\mu = \frac{\partial x^j}{\partial \overline{x}^\mu} \frac{\partial \overline{x}^\mu}{\partial x^i} P^i = \frac{\partial x^j}{\partial x^i} P^i = P^j$$

or

$$P^j = \frac{\partial x^j}{\partial \overline{x}^\mu} \overline{P}^\mu. \qquad \dots(4.13)$$

Equations (4.13) represent the solution of equations (4.12).

The transformation of differentials dx^i and $d\overline{x}^\mu$ in the systems of variables x^i and $\overline{x}^\mu$ respectively, from eqn. (8.5a), is given by

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{i}}\, dx^{i}. \qquad \ldots(4.14)$$

As equations (4.12) and (4.14) are similar transformation equations, we can say that the differentials dx^i form the components of contravariant vector, whose components in any other system are the differentials $d\bar{x}^{\mu}$ of that system. Also we conclude that the components of a contravariant vector are actually the components of a contravariant tensor of rank one.

Let us now consider a further change of variables from $\bar{x}^{\mu}$ to x'^{q}, then the new components P'^{q} must be given by

$$P'^{q} = \frac{\partial x'^{q}}{\partial \bar{x}^{\mu}} P^{\mu} = \frac{\partial x'^{q}}{\partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \,.\, P^{i} \qquad \text{(using 4.12)}$$

$$= \frac{\partial x'^{q}}{\partial x^{i}} P^{i}. \qquad \ldots(4.15)$$

This equation has the same from as eqn. (4.12). This indicates that the transformations of contravariant vectors form a group.

4.8.3 COVARIANT VECTORS: Consider a set of n quantities P_1, P_2, P_3, …P_n in a system of variables x^i and let these quantities have values $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots \bar{P}_n$in another system of variables $\bar{x}^{\mu}$. If these quantities obey the transformation equations

$$P = \frac{\partial x^{i}}{\partial \bar{x}^{\mu}} P_{i.} \qquad \ldots(4.16)$$

where the quantities P_j are said to be a covariant tensor of rank one.

Any n functions can be chosen as the components of a covariant vector in a system of variables x^i and equations (4.16) determine the n-components in the new system of variables$\bar{x}^{\mu}$.

Multiplying equation (4.16) by $\frac{\partial \bar{x}^{\mu}}{\partial x^{i}}$ and taking the sum over the index μ from 1 to n, we get

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{j}} \bar{P}_{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{j}} \frac{\partial x^{i}}{\partial \bar{x}^{\mu}} P_{i} = \frac{\partial x^{i}}{\partial x^{j}} P_{i} = P_{j}$$

Thus,

$$P_{j} = \frac{\partial \bar{x}^{\mu}}{\partial x^{j}} \bar{P}_{\mu}. \qquad \ldots(4.17)$$

Let us now consider a further change of variables from$\bar{x}^{\mu}$ to x'^{q}. Then the new components P'_q must be given by

$$P'_q = \frac{\partial \overline{x}^{\mu}}{\partial x'^{q}} \overline{P}_{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x'^{q}} \frac{\partial x^{i}}{\partial \overline{x}^{\mu}} P$$

$$= \frac{\partial x^{i}}{\partial x'^{q}} A_l. \qquad \ldots(4.18)$$

This equation has the same form as eqn. (4.16). This indicates that the transformation of contravariant vectors form a group.

As $$\frac{\partial \psi}{\partial \overline{x}^{\mu}} = \frac{\partial \psi}{\partial x^{i}} \frac{\partial x^{i}}{\partial \overline{x}^{\mu}} = \frac{\partial x^{i}}{\partial \overline{x}^{\mu}} \frac{\partial \psi}{\partial x^{i}}.$$

It follows from (4.16) that $\frac{\partial \psi}{\partial x^{i}}$ form the components of a contravariant vector, whose components in any other system are the corresponding partial derivatives$\frac{\partial \psi}{\partial \overline{x}^{\mu}}$. This convariant vector is called grad ψ.

4.9 TENSORS OF HIGHER RANKS

The laws of transformation of vectors are given by following formulas

$$\text{Contravariant} \ldots \overline{P}^{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} P^{i} \qquad \ldots (4.12)$$

$$\text{Covariant} \ldots \overline{P}^{\mu} = \frac{\partial x^{i}}{\partial \overline{x}^{\mu}} P_i. \qquad \ldots (4.16)$$

4.9.1 CONTRAVARIANT TENSORS OF SECOND RANK-

Let us consider $(n)^2$ quantities P^{ij} (here i and j take the values from 1 to n independently) in a system of variables x^i and let these quantities have values $\overline{P}^{\mu v}$ in another system of variables$\overline{x}^{\mu}$. If these quantities obey the transformation equations

$$\overline{P}^{\mu v} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{v}}{\partial x^{j}} P^{ij} \qquad \ldots(4.19)$$

where the quantities P^{ij} are said to be the components of a contravariant tensor of second rank.

4.9.2 COVARIANT TENSOR OF SECOND RANK- If $(n)^2$ quantities A^{ij} in a system of variables x^i are related to another $(n)^2$ quantities $\overline{A}_{\mu v}$ in another system of variables $\overline{x}^{\mu}$ by the transformation equations

$$\overline{P}_{\mu v} = \frac{\partial x^i}{\partial \overline{x}^\mu}\frac{\partial x^j}{\partial \overline{x}^v} P_{ij} \quad \ldots(4.20)$$

where the quantities P_{ij} are said to be the components of a covariant tensor of second rank.

4.9.3 MIXED TENSOR OF SECOND RANK: If $(n)^2$ quantities P_j^i in a system of variables x^i are related to another $(n)^2$ quantities P_v^μ in another system of variables $\overline{x}^\mu$ by the transformation equations

$$\overline{P}_v^\mu = \frac{\partial \overline{x}^\mu}{\partial x^j}\frac{\partial x^j}{\partial \overline{x}^v} P_j^i \quad \ldots(4.21)$$

where the quantities P_j^i are said to be component of a mixed tensor of second rank.

SAQ 2: Show that Kronecker deltamixed tensor of rank 2?

4.9.4 TENSOR OF HIGHER RANKS, RANK OF A TENSOR-The tensors having ranks more than two are called tensor of higher rank. Tensor of higher ranks are defined by similar laws. The rank of a tensor only indicates the number of indices attached to its per component. For example P_{lm}^{ijk} are the components of a mixed tensor of rank 5; contravariant of rank 3 and covariant of rank 2.

They transform according to the equation

$$\overline{P}_q^{\mu v \sigma} = \frac{\partial \overline{x}^\mu}{\partial x^i}\frac{\partial \overline{x}^v}{\partial x^j}\frac{\partial \overline{x}^\sigma}{\partial x^k}\frac{\partial \overline{x}^l}{\partial x^q} P_l^{ijk} \quad \ldots(4.22)$$

Where $\overline{P}_q^{\mu v \sigma}$ and P_l^{ijk} are tensors of rank 4.

4.10 SYMMETRIC AND ANTISYMMETRIC TENSORS

4.10.1 SYMMETRIC TENSORS- If two contravariant or covarint indices can be interchanged without changing the tensor, then the tensor is said to be symmetric with respect to these two indices.

For example if

or

$$\left.\begin{matrix} P^{ij} = P^{ji} \\ P_{ij} = P_{ji} \end{matrix}\right\} \quad \ldots(4.23)$$

then the contravariant tensor of second rank P^{ij} or covariant tensor P_{ij} is said to be symmetric.

For a tensor of higher rank P_l^{ijk} if

$$P_l^{ijk} = P_l^{jik}$$

where the tensor P_l^{ijk} is said to be symmetric with respect to indices i and j.

So if a tensor is symmetric with respect to two indices in any co-ordinate system, it remains symmetric with respect to these two indices in any other co-ordinate system.

This can be seen as follows:

If tensor P_l^{ijk} is symmetric with respect to first indices i and j, we have

$$P_l^{ijk} = P. \qquad \ldots(4.24)$$

We have $$\overline{P}_p^{\mu\nu\sigma} = \frac{\partial\overline{x}^{\mu}}{\partial x^i}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial\overline{x}^{l}}{\partial x^p}P_l^{ijk}$$

$$= \frac{\partial\overline{x}^{\mu}}{\partial x^i}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}\text{ P jik.} \qquad \text{(using 4.24)}$$

Now interchanging the dummy indices i and j, we get

$$\overline{P}_\rho^{\mu\nu\sigma} = \frac{\partial\overline{x}^{\mu}}{\partial x^j}\frac{\partial\overline{x}^{\nu}}{\partial x^i}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}\text{ P jik} = \overline{P}_\rho^{\nu\mu\sigma}$$

i.e., given tensor is gain symmetric with respect to first two indices in new co-ordinate system. Thus, the symmetry property of a tensor is independent of coordinate system.

Let P_l^{ijk} be symmetric with respect to two indices, one contravarient i and the other covariant l, then we have

$$P_l^{ijk} = P_i^{ljk} \qquad \ldots(4.25)$$

We have $$\overline{P}_\rho^{\mu\nu\sigma} = \frac{\partial\overline{x}^{\mu}}{\partial x^i}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}P_l^{ijk}$$

$$= \frac{\partial\overline{x}^{\mu}}{\partial x^i}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}P_i^{ljk}. \qquad \text{[(using 4.25)]}$$

Now interchanging dummy indices i and l, we have

$$\overline{P}_\rho^{\mu\nu\sigma} = \frac{\partial\overline{x}^{\mu}}{\partial x^i}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}P_l^{ijk}$$

$$= \frac{\partial x^{i}}{\partial\overline{x}^{\rho}}\frac{\partial\overline{x}^{\nu}}{\partial x^j}\frac{\partial\overline{x}^{\sigma}}{\partial x^k}\frac{\partial\overline{x}^{\mu}}{\partial x^l}P_l^{ijk}. \qquad \ldots(4.26)$$

According to tensor transformation law,

$$\overline{P}_{\mu}^{\rho\nu\sigma} = \frac{\partial\overline{x}^{\rho}}{\partial x^{i}}\frac{\partial\overline{x}^{\nu}}{\partial x^{j}}\frac{\partial\overline{x}^{\sigma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial\overline{x}^{\mu}}P_{l}^{ijk}. \qquad \ldots(4.27)$$

Comparing (4.26) and (4.27), we see that

$$\overline{P}_{\rho}^{\mu\nu\sigma} \neq \overline{P}_{\mu}^{\rho\nu\sigma}$$

i.e., symmetry is not preserved after a change of co-ordinate system.

A symmetric tensor of rank 2 in n-dimensional space has at most $\frac{n(n+1)}{2}$ independent components.

SAQ 3: Show that the symmetry property of a tensor in independent of co-ordinate system used?

4.10.2 ANTISYMMETRIC TENSORS OR SKEW SYMMETRIC TENSORS-

A tensor, whose each component alters in sign but not in magnitude when two contravariant or covariant indices are interchanged, is said to be skew symmetric or anti-symmetric with respect to these two indices.

For example if

$$\left.\begin{aligned} P^{ij} &= -P^{ji} \\ \text{or} \qquad P_{ij} &= -P_{ji} \end{aligned}\right\} \qquad \ldots(4.28)$$

then contravariant tensor P^{ij} or covariant tensor P_{ij} of second rank is antisymmetric or for a tensor of higher rank P_{l}^{ijk} if

$$P_{l}^{ijk} = -P_{l}^{ikj}$$

where tensor P_{l}^{ijk} is antisymmetric with respect to indices j and k.

If tensor P_{l}^{ijk} is antisymmetric with respect to first two indices i and j.

We have

$$P_{l}^{ijk} = -P_{l}^{jik} \qquad \ldots(4.29)$$

and

$$\overline{P}_{\rho}^{\mu\nu\sigma} = \frac{\partial\overline{x}^{\mu}}{\partial x^{i}}\frac{\partial\overline{x}^{\nu}}{\partial x^{j}}\frac{\partial\overline{x}^{\sigma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}P_{l}^{ijk}$$

$$= -\frac{\partial\overline{x}^{\mu}}{\partial x^{i}}\frac{\partial\overline{x}^{\nu}}{\partial x^{j}}\frac{\partial\overline{x}^{\sigma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial\overline{x}^{\rho}}P_{l}^{jik}. \qquad \text{[using (4.29)]}$$

Now interchanging the dummy indices i and j, we get

$$\overline{P}_{\rho}^{\mu\nu\sigma} = \frac{\partial \overline{x}^{\mu}}{\partial x^{j}} \frac{\partial \overline{x}^{\nu}}{\partial x^{i}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} P_{l}^{ijk} = -\overline{P}_{\rho}^{\nu\mu\sigma}$$

i.e., given tensor is again antisymmetric with respect to first two indices in new co-ordinate system. Thus, antisymmetry property is retained under co-ordinate transformation.

An antisymmetric tensor of rank 2 in n-dimensional space has $\frac{n(n-1)}{2}$ independent components. Any tensor having either two contravariant or two covariant indices can be expressed as a sum parts, one symmetric and the other antisymmetric.

Thus, $$P^{ij} = \frac{1}{2}\left(P^{ij} + P^{ji}\right) + \frac{1}{2}\left(P^{ij} + P^{ji}\right) \qquad \ldots(4.30)$$

the first term on the right is the symmetric part and the second is the antisymmetric part. The symmetric part is a symmetric tensor and the antisymmetric part is an antisymmetric tensor.

The process of writing a tensor as a sum of symmetric and antisymmetric parts not only holds for tensors of rank 2 but is quite general. For example a tensor P_{kl}^{ij} can be written as

$$P_{kl}^{ij} = \underbrace{\frac{1}{2}\left[P_{kl}^{ij} + P_{kl}^{ji}\right]}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left[P_{kl}^{ij} + P_{kl}^{ji}\right]}_{\text{Antisymmetric part}}$$

or $$P_{kl}^{ij} = \underbrace{\frac{1}{2}\left[P_{kl}^{ij} + P_{kl}^{ji}\right]}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left[P_{kl}^{ij} - P_{lk}^{ij}\right]}_{\text{Antisymmetric part}}$$

SAQ 4: Show that the skew-symmetry property of a tensor of is independent of co-ordinate system used?

4.11 INVARIANT TENSORS

The tensor which has the same components in all co-ordinate systems are said to be invariant tensors.

Kronecker delta symbol and Levi Civita symbol (Epsilon tensor) are the important examples of such tensors.

Kronecker delta: The kronecker delta symbol; is defined as

$$\delta_j^i = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}.$$

We shall now prove that **Kronecker delta is an invariant tensor.**

Let δ_j^i be the components of Kronecker delta in a system of variables x^i and $\overline{\delta}_\nu^\mu$ as the corresponding components in another system of variables $\overline{x}^\mu$.

If Kronecker delta is a mixed tensor of rank two then it must transform according to the rule.

$$\bar{\delta}^{\mu}_{v} = \frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{v}} \delta^{i}_{j} \qquad \ldots(4.31)$$

$$= \frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{v}} \frac{\partial x^{i}}{\partial x^{j}} = \frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \bar{x}^{v}} . \qquad \ldots(4.32)$$

Since new variables $\bar{x}^{\mu}$ are the functions of old variables x^{i} which in turn are the functions of new variables$\bar{x}^{v}$, we have by chain rule

$$= \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{v}} = \frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \bar{x}^{v}} .$$

This gives the changes $\delta\bar{x}^{\mu}$ consequent upon change $\delta\bar{x}^{v}$.

Since $\bar{x}^{\mu}$ and $\bar{x}^{v}$ are the coordinates of the same system, hence their variations are independent of each other unless $\mu = v$ in which case

$$\delta\bar{x}^{\mu} = \delta\bar{x}^{v}$$

$$\therefore \qquad \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{v}} = \begin{cases} 1 \text{ for } \mu = v \\ 0 \text{ for } \mu \neq v \end{cases} .$$

Therefore, by definition of Kronecker delta, we have

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{i}}{\partial \bar{x}^{v}} = \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{v}} \delta^{\mu}_{v} . \qquad \ldots(4.33)$$

4.11.1 LEVI-CIVITA SYMBOL (OR EPSILON TENSOR OR ALTERNATING TENSOR OR PERMUTATION TENSOR): Levi-Civita symbol in three dimensional space in a tensor of rank 3 and is denoted by ε_{ijk} while in four dimensional space it is a tensor of rank four and denoted by ε_{ijkl}. The Levi-Civita symbol is defined as a quantity ε_{ijk} in three dimensional space which is antisymmetric in all its indices. Thus, the only non-vanishing components of ε_{ijk} are those for which all the indices are different and they are equal to +1 or -1 according as (i, j, k) is an even or odd permutation of (1, 2, 3), i.e.,

$$\varepsilon_{ijk} = \begin{bmatrix} +1 \text{ if } (i,j,k) \text{is an even permutation of } (1,2,3) \\ -1 \text{ if} (i,j,k) \text{is an odd permutation of } (1,2,3) \\ 0 \text{ otherwise (contain two or more repeated indices).} \end{bmatrix} \qquad \ldots(4.34)$$

4.12 ALGEBRAIC OPERATIONS ON TENSORS

4.12.1 ADDITION AND SUBTRACTION: The tensors are added and subtracted only if the tensors have some rank and same type. Same type means the same number of contravarient and covariant indices. To add or subtract two tensors the corresponding elements are added or subtracted.

The sum or difference of two tensors of the same rank and same type is also a tensor of the same rank and same type.

If there are two tenors $P_k^{ij}$and Q_k^{ij} of the same rank (3) and same type (mixed with two indices in contravariant and one in covariant), then the laws of addition and subtraction are given by

$$P_k^{ij} + Q_k^{ij} = R_k^{ij} \text{ (Addition)} \qquad \ldots(4.35)$$

$$P_k^{ij} - Q_k^{ij} = S_k^{ij} \text{ (Subtraction)} \qquad \ldots(4.36)$$

where R_k^{ij} and S_k^{ij} are the tensors of the same rank (3) and same type (mixed with two indices in contravariant and one in covariant) as the given tensors.

The transformation laws for the given tensors are

$$\overline{P}_\sigma^{\mu v} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^v}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} P_k^{ij} \qquad \ldots(4.37)$$

and

$$\overline{Q}_\sigma^{\mu v} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^v}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} Q_k^{ij} \qquad \ldots(4.38)$$

Adding (4.37) and (4.38), we get

$$\overline{S}_\sigma^{\mu v} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^v}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} S_k^{ij} \qquad \ldots(4.39)$$

where is a transformation law for the sum and is similar to transformation laws for P_k^{ij} and Q_k^{ij} given nby (4.37) and (4.38). Hence the sum $R_k^{ij}(= P_k^{ij} + Q_k^{ij})$ is itself a tensor of the same rank and same type as the given tensors.

Subtracting eqn. (4.38) from (4.37), we get

$$\overline{P}_\sigma^{\mu v} - \overline{Q}_\sigma^{\mu v} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^v}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} (P_k^{ij} - Q_k^{ij})$$

or

$$\overline{S}_\sigma^{\mu v} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^v}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} S_k^{ij}. \qquad \ldots(4.40)$$

which is a transformation law for the difference and is again similar to the transformation law for P_k^{ij} and Q_k^{ij}. Hence the difference $S_k^{ij}\left(= P_k^{ij} - Q_k^{ij}\right)$ is itself a tensor of the same rank and same type as the given tensors.

SAQ 5: Show that sum and difference of tensor of same rank and same type is also a tensor of the same rank and same type.

4.12.2 EQUITY of TENSORS: Two tensors of the same rank and same type are said to be equal if their components are one to one equal, i.e., if

$$P_k^{ij} = Q_k^{ij} \text{ for all values of the indices.}$$

If two tensors are equal in one co-ordinate system, they will be equal in any other co-ordinate system.

SAQ 6: Show that two tensors are equal in one co-ordinate system, they will be equal in any other co-ordinate system.

4.12.3 OUTER PRODUCT: The outer product of two tensors is a tensor whose rank is the sum of the ranks of given tensors.

Thus, if t and t' are the ranks of two tensors, then rank of their outer product will be (t + t').

For example if P_k^{ij} and Q_m^{l} are two tensors of ranks 3 and 2 respectively, then

$$P_k^{ij} Q_m^{l} = R \text{ (say)} \qquad \ldots(4.41)$$

is a tensor of rank 5 (= 3 + 2).

For proof of this statement we write the transformation equations of the given tensors as

$$\overline{P}_\sigma^{\mu\nu} - Q_\sigma^{\mu\nu} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^\nu}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\sigma} P_k^{ij} \qquad \ldots(4.42)$$

$$\overline{Q}_\lambda^{\rho} = \frac{\partial \overline{x}^\rho}{\partial x^l} \frac{\partial x^m}{\partial \overline{x}^\lambda} Q_m^{l}. \qquad \ldots(4.43)$$

Multiplying (8.42) and (8.43), we get

$$\overline{P}_\sigma^{\mu\nu} - \overline{Q}_\lambda^{\rho} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^\nu}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^\sigma} \frac{\partial \overline{x}^\rho}{\partial x^l} \frac{\partial x^m}{\partial \overline{x}^\lambda} P_k^{ij} Q_m^{l}$$

or

$$\overline{R}_{\sigma\lambda}^{\mu\nu\rho} = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial \overline{x}^\nu}{\partial x^j} \frac{\partial \overline{x}^\rho}{\partial x^l} \frac{\partial x^k}{\partial \overline{x}^\sigma} \frac{\partial x^m}{\partial \overline{x}^\lambda} R_{km}^{ijl} \qquad \ldots(4.44)$$

which is a transformation law for tensor of rank 5. Hence the outer product of two tensors P_k^{ij} and Q_m^l is a tensor R_{km}^{ijl} of rank (3 + 2 =) 5.

The outer product of tensors is commutative and associative.

4.12.4 CONTRACTION OF TENSORS: The algebraic operation by which the rank of a mixed tensor is lowered by 2 is known as contraction.

For example consider a mixed tensor P_{lm}^{ijk} of rank 5 with contravariant indices i, j, k and covariant indices l, m.

The transformation law of the given tensor is

$$\overline{P}_{\sigma\lambda}^{\mu\nu\rho} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{\nu}}{\partial x^{j}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} \frac{\partial x^{m}}{\partial \overline{x}^{\lambda}} P_{lm}^{ijk}. \qquad \ldots(4.45)$$

To apply the process of contraction, we put $\lambda = \sigma$ and obtain

$$\overline{P}_{\rho\sigma}^{\mu\nu\sigma} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{\nu}}{\partial x^{j}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} \frac{\partial x^{m}}{\partial \overline{x}^{\sigma}} P_{lm}^{ijk}$$

$$= \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{\nu}}{\partial x^{j}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} \frac{\partial \overline{x}^{\sigma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \overline{x}^{\sigma}} P_{lm}^{ijk}$$

$$= \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{\nu}}{\partial x^{j}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} \delta_k^m P_{lm}^{ijk}$$

$$\left(\text{since substitution operator} \frac{\partial \overline{x}^{\sigma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \overline{x}^{\sigma}} = \delta_k^m\right)$$

i.e., $$\overline{P}_{\rho\sigma}^{\mu\nu\sigma} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial \overline{x}^{\nu}}{\partial x^{j}} \frac{\partial x^{l}}{\partial \overline{x}^{\rho}} P_{lk}^{ijk} \qquad \ldots(4.46)$$

which is a transformation law for a mixed tensor of rank 3.

SAQ 7: Show that contraction of a tensor of rank 7 is tensor of rank 5?

4.12.5 INNER PRODUCT: The outer product of two tensors followed by a contraction results a new tensor called and **inner product** of the two tensors and the process is called the **inner multiplication** of two tensors.

Example (a) Consider two tensors P_k^{ij} and Q_m^l

The outer product of these two tensors is

$$P_k^{ij} Q_m^l = R_{km}^{ijl} \text{ (say)}$$

Applying contraction process by setting m = i, we obtain

$$P_k^{ij} Q_m^l = R_{km}^{ijl} = S_k^{jl} \text{ (a new tensor)}$$

The new tensor S_k^{jl} is the inner product of the two tensors P_k^{ij} and Q_m^l.

(b) An another example consider two tensors of rank 1 as P^i and Q_j. The outer product of P^i and Q_j is

$$P^i Q_j = R_j^i.$$

Applying contraction process by setting i = j, we get

$$P^i Q_j = R_j^j \text{ (a scalar or a tensor of rank zero).}$$

Thus, the inner product of two tensors of rank one is a tensor of rank zero. (i.e., invariant).

4.12.6 QUOTIENT LAW: Quotient law provided a direct method to find out if the given entity is a tensor or not. Quotient law states that:

An entity whose inner product with an arbitrary tensor (contravariant or covariant) is a tensor, is itself a tensor.

Example: Let P (i, j, k) be the given entity to be tested whether it is a tensor or not. To apply quotient law let us consider an arbitrary tensor Q_{jk}^l whose inner product with P (i, j, k) is a tensor. i.e.,

$$P(i, j, k) Q_{jk}^l = R_i^l$$

We have to show that P (i, j, k) is a tensor. In the other system of variables $\overline{x}^\mu$, we must have

$$\overline{P}(\mu, \nu, \sigma) \overline{Q}_{\nu\sigma}^\rho = \overline{R}_\mu^\rho.$$

Now $$\overline{Q}_{\nu\sigma}^\rho = \frac{\partial \overline{x}^\rho}{\partial x^l} \frac{\partial x^j}{\partial \overline{x}^\nu} \frac{\partial x^k}{\partial \overline{x}^\sigma} Q_{jk}^l$$

and $$R_\mu^\rho = \frac{\partial \overline{x}^\rho}{\partial x^l} \frac{\partial x^i}{\partial \overline{x}^\mu} R_j^j .$$

4.12.7.EXTENSION OF RANK:

So that $$\overline{R}_\nu^\mu = \overline{P}^\mu \overline{Q}_\nu = \frac{\partial \overline{x}^\mu}{\partial x^i} P^i \frac{\partial x^j}{\partial \overline{x}^\nu} Q_j$$

$$= \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^\nu} P^i Q_j = \frac{\partial \overline{x}^\mu}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^\nu} R_j^i .$$

The rank of a tensor can be extended by differentiating its each component with respect to variables x^i.

As an example consider a simple case in which the original tensor is of rank zero, i.e., a scalar S (x^i) whose, derivatives relative to the variables x^i are $\frac{\partial S}{\partial x^i}$. In other system of variables $\bar{x}^\mu$ the scalar is $\bar{S}(\bar{x}^\mu)$, such that

$$\frac{\partial \bar{S}}{\partial \bar{x}^\mu} = \frac{\partial S}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^\mu} = \frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial S}{\partial x^i}. \quad \ldots(4.49)$$

This shows that $\frac{\partial S}{\partial x^i}$, transforms like the components of a tensor of rank one. Thus, the differentiation of a tensor of rank zero gives a tensor of rank one. In general we may say that the differentiation of a tensor with respect to variables x^i yields a new tensor of rank one greater than the original tensor.

The rank of a tensor can also be extended when a tensor depends upon another tensor and the differentiation with respect to that tensor is performed. As an example consider a tensor S of rank zero (i.e., a scalar) depending upon another tensor P_{ij}, then

$$\frac{\partial S}{\partial A_{ij}} = Q_{ij} = \text{a tensor of rank 2.} \quad \ldots(4.50)$$

Thus, the rank of the tensor of rank zero has been extended by 2.

4.13 RIEMANNIAN SPACE: METRIC TENSOR

An expression which are express as the distance between two adjacent point is called a metric or line element. In three dimensional space the line element, i.e., the distance between two adjacent points (x, y, z) and (x + dx, y + dy, z + dz) in Cartesian coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

In terms of general curvilinear coordinates, the line element becomes

$$ds^2 = \sum_{j=1}^{3}\sum_{k=1}^{3} g_{jk}du_j du_k = g_{jk}du_j du_k \quad \text{(Using summation convention).}$$

This idea was generalised by Riemann to n-dimensional space.

The distance between two neighbouring points with coordinates x^j and $x^j + dx^j$ is given by

$$ds^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} g_{jk}dx^j dx^k = g_{jk}dx^j dx^k \quad \ldots(4.54)$$

(Using summation convention)

where the coefficients g_{jk} are the functions of coordinates x^j, subject to the restriction g = determinant of g_{jk}, i.e, $|g_{jk}| \neq 0$.

The quadratic differential form $g_{jk}dx^jdx^k$ is independent of the coordinates system and is called the Riemannian metric for n dimensional space. The space which is characterised by Riemannian metric is called Riemannian space. Hence the quantities g_{jk} are the components of a covariant symmetric tensor of rank two, called the metric tensor or fundamental tensor.

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \dots + (dx^n)^2 \text{ or } dx^jdx^k,$$

the space is called n-dimensional Euclidean space. It is now obvious that Euclidean spaces are the particular cases of Riemannian space.

In general theory of relativity (four dimensional space), the line element is given by

$$Ds^2 = g_{jk}dx^jdx^k \ (j, k = 1, 2, 3, 4).$$

In special theory of relativity, the line element is given by

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \dots + (dx^n)^2 \text{ or } dx^jdx^k.$$

the space is called n-dimensional Euclidean space. The Euclidean spaces are the particular cases of Riemannian space.

In general theory of relativity (four dimensional space), the line element is given by

$$ds^2 = g_{jk}dx^idx^jdx^k \ (j, k = 1, 2, 3, 4).$$

In special theory of relativity, the line element is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad [\text{with } x^4 = ict, i = \sqrt{(-1)}]$$

$$= dx^jdx^j \ (j = 1, 2, 3, 4).$$

As $ds^2 = g_{jk}dx^jdx^k$ has been defined in general space, it is independent of the coordinate system, i.e., $dx^2 = g_{jk}dx^jdx^k$ is an invariant.

4.14 FUNDAMENTAL TENSORS g_{jk}, g^{jk} AND δ^j_k

4.14.1 COVARIANT FUNDAMENTAL TENSOR g_{jk}: The line element or interval ds in Riemannian space is given by

$$ds^2 = g_{jk}dx^jdx^k. \qquad \dots(4.55)$$

As $dx^j dx^k$ are contravariant vectors and ds^2 is invariant for arbitrary choice of vectors dx^j and dx^k, it follows from quotient law that g_{jk} is a covariant tensor, we have

$$ds^2 = g_{jk}dx^j dx^k \text{ in system of variables } x^j$$

$$= \bar{g}_{\mu v}\overline{dx}^{\mu}\overline{dx}^{v} \qquad \text{in system of variables } \bar{x}^{\mu}$$

i.e., $$= \bar{g}_{\mu v}\overline{dx}^{\mu}\overline{dx}^{v} = g_{jk}dx^j dx^k. \qquad \ldots(4.56)$$

Now applying inverse transformation law to dx^j and dx^j, i.e.,

$$dx^j = \frac{\partial x^j}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu} \text{etc.}$$

$$\bar{g}_{\mu v}\overline{dx}^{\mu}\overline{dx}^{v} = g_{jk}\frac{\partial x^j}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu}\frac{\partial x^k}{\partial \bar{x}^{v}} d\bar{x}^{v}$$

$$= g_{jk}\frac{\partial x^j}{\partial \bar{x}^{\mu}}\frac{\partial x^k}{\partial \bar{x}^{v}} d\bar{x}^{\mu} d\bar{x}^{v}$$

i.e., $$\left\{\bar{g}_{\mu v} - g_{jk}\frac{\partial x^j}{\partial \bar{x}^{\mu}}\frac{\partial x^k}{\partial \bar{x}^{v}}\right\} d\bar{x}^{\mu} d\bar{x}^{v} = 0. \qquad \ldots(4.57)$$

As $d\bar{x}^{\mu}$ and $d\bar{x}^{v}$ are arbitrary contravarient vectors, we must have

$$\bar{g}_{\mu v} - c\frac{\partial x^j}{\partial \bar{x}^{\mu}}\frac{\partial x^k}{\partial \bar{x}^{v}} = 0$$

$$\bar{g}_{\mu v} = \frac{\partial x^j}{\partial \bar{x}^{\mu}}\frac{\partial x^k}{\partial \bar{x}^{v}} g_{jk}\,.$$

Hence g_{jk} is a covariant tensor of rank 2.

g_{jk} may be expressed as

$$g_{jk} = \frac{1}{2}\left(g_{jk} + g_{kj}\right) + \frac{1}{2}\left(g_{jk} - g_{kj}\right)$$

$$= P_{jk} + Q_{jk} \qquad \ldots(4.58)$$

where and
$$\left.\begin{aligned} A_{jk} &= \tfrac{1}{2}\left(g_{jk} + g_{kj}\right) \text{ is symmetric tensor} \\ B_{jk} &= \tfrac{1}{2}\left(g_{jk} - g_{kj}\right) \text{ is symmetric tensor} \end{aligned}\right\} \qquad \ldots(4.59)$$

$\therefore$ $$ds^2 = g_{jk}dx^j dx^k = (A_{jk} + B_{jk})\, dx^j dx^k. \qquad \ldots(4.60)$$

We have

$$B_{jk}dx^j dx^k = B_{kj}dx^k dx^j \quad \text{(interchanging dummy indices j and k)}$$

$$= -B_{jk}dx^j dx^k$$

(since B_{jk} is antisymmetric i.e., $B_{jk} = -B_{kj}$)

i.e., $$2B_{jk}\, dx^j dx^k = 0.$$

As dx^j and dx^k are arbitrary vectors, we have

$$B_{jk} = 0$$

i.e., $$\frac{1}{2}\left(g_{jk} + g_{kj}\right) = 0$$

i.e., $$g_{jk} + g_{kj} = 0$$

i.e., g_{jk} is symmetric.

So, we can write g_{jk} as

$$g_{\mu\nu}. = \frac{1}{2}\left(g_{\mu\nu} + g_{\nu\mu}\right).$$

Thus, we have proved that the metric tensor g_{jk} is covariant symmetric tensor of rank 2. This is called covariant fundamental tensor of rank 2.

4.14.2 CONTRAVARIANT FUNDAMENTAL TENSOR g^{jk}

Let us define g^{jk} as

$$g^{jk} \frac{\text{cofactor of } g_{jk} \text{ in } g}{g} \quad \ldots(4.61)$$

where g is the determinant of g_{jk}, i.e.,

$$g = \left|g_{jk}\right| = \begin{vmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} \end{vmatrix}.$$

Since g_{jk} is symmetric, g is symmetric which implies cofactor of g_{jk} in g is symmetric and so g^{jk} is symmetric.

Let P^j be an arbitrary contravariant vector, then by quotient law,

$$P_k = g_{jk}P^j \quad \ldots(4.62)$$

is an arbitrary covariant vector.

Now multiplying eqn. (4.62) by g^{kl}, we get

$$g^{kl}P_k = g_{jk}g^{kl}P^j. \qquad \ldots(4.63)$$

But $g_{jk}g^{kl} = g_{jk}\frac{\text{cofactor of } g_{kl} \text{ in } g}{g}$ (using 4.61)

$$= \delta^l_j \text{ (by theory of determinants).} \qquad \ldots(4.64)$$

Therefore, equation (4.63) yields

$$g^{kl}A_k = \delta^l_j\, P^j = P^l \qquad \ldots(4.65)$$

i.e., the inner product of g^{kl} with an arbitrary covariant vector P_k yields a contravariant vector. Hence by quotient law g^{kl} is a contravariant tensor of rank 2.

4.14.3 MIXED FUNDAMENTAL TENSOR g^j_k or δ^l_j

From equation (4.64), we have

$$g_{jk}g^{kl} = \delta^l_j\,. \qquad \ldots(4.66)$$

As g_{jk} and g^{kl} are covariant and contravariant tensors of rank 2 respectively, therefore, from quotient law δ^l_j is also a tensor of rank 2; it is a mixed tensor, contravariant in l and covarian in j and is known as mixed fundamental tensor.

4.15 CHRISTOFELL'S 3-INDEX SYMBOLS

We now introduce two expressions formed of the fundamental tensors, known as Christofell's symbols of first and second kind.

Christofell's symbol of first kind

$$[jk, l] = \Gamma_{l,jk} = \frac{1}{2}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \qquad \ldots(4.67)$$

Christofell's symbol of second kind.

$$\begin{Bmatrix} l \\ jk \end{Bmatrix} = \Gamma^l{}_{.jk} = \frac{1}{2} g^{lm}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right). \qquad \ldots(4.68)$$

From the symmetry property of g_{jk} it follows that

$$[jk, l] = [kj, l] \text{ or } \Gamma_{l,jk} = \Gamma_{l.kj} \qquad \ldots(4.69)$$

and $$\begin{Bmatrix} l \\ jk \end{Bmatrix} = \begin{Bmatrix} l \\ kj \end{Bmatrix} \text{ or } \Gamma^{l}{}_{.jk} = \Gamma^{l}{}_{kj} \qquad \ldots(4.70)$$

there by indicating that Christofell's symbols $\Gamma_{l.jk}$ and $\Gamma^{l}{}_{jk}$ are symmetrical with respect to indices j and k.

Relations between Christofell's symbols of first and second kind

(i) Replacing l by m in eqn. (4.67), we get

$$\Gamma_{m,jk} = \frac{1}{2}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial x_{jk}}{\partial x^m}\right).$$

Multiplying both sides of above equation by g^{lm}, we get

$$g^{lm}\Gamma_{m,jk} = \frac{1}{2} g^{lm}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \qquad (\text{since } g_{jm} = g_{mj})$$

$$= \Gamma^{l}_{jk} \qquad [\text{Using } (4.68)]$$

i.e., $$\Gamma^{l}_{jk} = g^{lm}\Gamma_{m,jk}. \qquad \ldots(4.70)$$

(ii) Interchanging l and m in eqn. (4.68), we get

$$\Gamma^{m}_{jk} = \frac{1}{2} g^{lm}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

Multiplying above equation by g_{lm}, we get

$$g_{lm}\Gamma^{m}_{jk} = \frac{1}{2} g_{lm}\, g^{lm}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \qquad (\text{since } g_{lm}g^{ml} = \delta^{l}_{l} = 1)$$

$$= \Gamma_{l,jk}.$$

4.16 GEODESICS

In Euclidean three dimensional space the path of shortest distance between two fixed points is a straight line. Here we shall generalise this fundamental concept to Riemannian space.

The path of extremum (maximum or minimum) distance between any two points in Riemannian space is called the geodesic. Thus, a geodesic is determined by the condition that the path between two fixed points A and B given by be extremum, i.e.,

$$\int_A^B ds$$

$$\int_A^B ds \quad \text{extremum (or stationary),} \qquad \ldots(4.71)$$

i.e., $$\delta\int_A^B ds = 0 \qquad \ldots(4.72)$$

Where δ represents the variation symbol.

In Riemannian space, we have

$$ds^2 = g_{jk}dx^j dx^k \qquad \ldots(4.73)$$

$$2\, ds\, \delta\,(ds) = \delta\,(g_{jk})\, dx^j dx^k + g_{jk}\delta\,(dx^j)\, dx^k + g_{jk}dx^j\delta\,(dx^k)$$

$$= dx^j dx^k \frac{\partial g_{jk}}{\partial x^m}\delta x^m + g_{jk}dx^k\delta\,(dx^j) + g_{jk}dx^j\delta\,(dx^k).$$

Dividing both sides by 2 ds and using the relation

$$\delta\left(\frac{\partial x^j}{ds}\right) = \frac{d}{ds}\left(\delta x^j\right).$$

We get$(ds) = \frac{1}{2}\left\{\frac{dx^j}{ds} + \frac{dx^k}{ds} - \frac{\partial g_{jk}}{\partial x^m}\delta x^m + g_{jk}\frac{dx^k}{ds}\frac{d}{ds}\left(\delta x^j\right) + g_{jk}\frac{dx^j}{ds}\frac{d}{ds}\left(\delta x^k\right)\right\} ds.$

Substituting the value of $\delta(ds)$ from (4.73) in (4.72), we get

$$\frac{1}{2}\int_A^B\left\{\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m}\delta x^m + g_{jk}\frac{dx^j}{ds}\frac{d}{ds}\left(\delta x^k\right) + g_{jk}\frac{dx^k}{ds}\frac{d}{ds}\left(\delta x^j\right)\right\}ds = 0$$

On changing the dummy indices in the last two terms, we get

$$\frac{1}{2}\int_A^B\left\{\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m}\delta x^m + \left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds}\right)\frac{d}{ds}.\left(\delta x^m\right)\right\}ds = 0.$$

Integrating the second term by parts and remembering that the variation δ is zero at the fixed end points A and B,

$$\frac{1}{2}\int_A^B\left\{\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m} - \frac{d}{ds}\left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds}\right)\right\}\delta x^m ds = 0.$$

As the infinitesimal displacements δx^m are arbitrary, therefore for the integral to be stationary the coefficient of δx^m in the integrand must vanish at all points on the path, i.e.,

$$\frac{1}{2}\left[\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m} - \frac{d}{ds}\left(g_{jm}\frac{dx^j}{ds} + g_{mk}\frac{dx^k}{ds}\right)\right] = 0$$

i.e., $$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{\partial g_{jk}}{\partial x^m}-\frac{1}{2}\frac{dg_{jm}}{ds}\frac{dx^j}{ds}-\frac{1}{2}g_{mk}\frac{d^2x^k}{ds^2}=0.$$

$$-\frac{1}{2}\frac{dg_{mk}}{ds}\frac{dx^k}{ds}-\frac{1}{2}g_{mk}\frac{d^2x^k}{ds^2}=0. \qquad \dots(4.74)$$

But we have

$$\frac{dg_{jm}}{ds}=\frac{\partial g_{jm}}{\partial x^k}\frac{dx^k}{ds}\, and\, \frac{dg_{mk}}{ds}=\frac{\partial g_{mk}}{\partial x^j}\frac{dx^j}{ds}$$

With these substitutions, equation (4.74) becomes

i.e., $$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{jk}}{\partial x^m}-\frac{\partial g_{jm}}{\partial x^k}-\frac{\partial g_{mk}}{\partial x^j}\right)-\frac{1}{2}\left(g_{jm}\frac{d^2x^j}{ds^2}+g_{mk}\frac{d^2x^k}{ds^2}\right)=0.$$

Replacing the dummy indices j and k and l in the second bracketed terms, we get

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{jk}}{\partial x^m}-\frac{\partial g_{jm}}{\partial x^k}-\frac{\partial g_{mk}}{\partial x^j}\right)-\frac{1}{2}\left(g_{jm}\frac{d^2x^l}{ds^2}+g_{mk}\frac{d^2x^l}{ds^2}\right)=0.$$

Using symmetry property of g_{lm} (i.e., $g_{lm} = g_{ml}$) above equation mau be written as

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{\partial g_{mj}}{\partial x^k}-\frac{\partial g_{km}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^m}\right)+g_{lm}\frac{d^2x^l}{ds^2}=0.$$

Now multiplying throughout by g^{mp}, we get

$$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\left(\frac{\partial g_{mj}}{\partial x^k}+\frac{\partial g_{km}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^m}\right)+g_{lm}g^{mp}\frac{d^2x^l}{ds^2}=0$$

or $$\frac{1}{2}\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\Gamma_{m.jk}+\delta_l^p\frac{d^2x^l}{ds^2}=0$$

i.e., $$\frac{d^2x^p}{ds^2}+\frac{dx^j}{ds}\frac{dx^k}{ds}g^{mp}\Gamma_{m.jk}=0 \qquad \dots(4.75)$$

or $$\frac{d^2x^p}{ds^2}+\frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma^p_{jk}=0. \qquad \dots(4.76)$$

4.17 COVARIANT DERIVATIVE OF A CONTRAVARIANT VECTOR

Let A^j be a contravariant vector, then by tensor transformation law

$$\overline{A}^{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x^j} A^j. \qquad \ldots(4.77)$$

Differentiating above equation with to $\overline{x}^{v}$, we get

$$\frac{\partial \overline{A}^{\mu}}{\partial \overline{A}^{v}} = \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial A^j}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^{v}} + \frac{\partial^2 \overline{x}^{\mu}}{\partial x^j \partial x^k}\frac{\partial x^k}{\partial \overline{x}^{v}} A^j$$

$$\frac{\partial \overline{A}^{\mu}}{\partial \overline{A}^{v}} = \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^{v}}\frac{\partial A^j}{\partial x^k} + \frac{\partial^2 \overline{x}^{\mu}}{\partial x^j \partial x^k}\frac{\partial x^k}{\partial \overline{x}^{v}} A^j \qquad \ldots(4.78)$$

The presence of the last term on the R.H.S. of eqn. (4.77) shows that the partial derivatives $\frac{\partial A^j}{\partial A^k}$ or $\frac{\partial \overline{A}^{\mu}}{\partial \overline{x}^{v}}$ do not transform like the components of tensor.

From eqn. (4.77) interchanging x and $\overline{x}$ co-ordinates, we get

$$\frac{\partial^2 \overline{x}^{\mu}}{\partial x^j \partial x^k} = \Gamma^{p}_{jk}\frac{\partial \overline{x}^{\mu}}{\partial x^p} - \frac{\partial \overline{x}^{\sigma}}{\partial x^j}\frac{\partial \overline{x}^{\lambda}}{\partial x^k}\overline{\Gamma}^{\mu}_{\sigma\lambda} \qquad \ldots(4.79)$$

$$\frac{\partial \overline{A}^{\mu}}{\partial \overline{A}^{v}} = \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^{v}}\frac{\partial A^j}{\partial x^k} + \left(\Gamma^{p}_{jk}\frac{\partial \overline{x}^{\mu}}{\partial x^p} - \frac{\partial \overline{x}^{\sigma}}{\partial x^j}\frac{\partial \overline{x}^{\lambda}}{\partial x^k}\overline{\Gamma}^{\mu}_{\sigma\lambda}\right)\frac{\partial x^k}{\partial \overline{x}^{v}} A^j$$

$$= \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^{v}}\frac{\partial A^j}{\partial x^k} + \Gamma^{p}_{jk}\frac{\partial \overline{x}^{\mu}}{\partial x^p}\frac{\partial x^k}{\partial \overline{x}^{v}} A^j - \delta^{\lambda}_{v}\overline{\Gamma}^{\mu}_{\sigma\lambda}\frac{\partial \overline{x}^{\sigma}}{\partial x^j} A^j$$

$$= \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^{v}}\frac{\partial A^j}{\partial x^k} + \Gamma^{p}_{jk}\frac{\partial \overline{x}^{\mu}}{\partial x^p}\frac{\partial x^k}{\partial \overline{x}^{v}} A^j - \overline{\Gamma}^{\mu}_{\sigma v} A^{\sigma}$$

Interchanging dummy indices j and p in the second term on R.H.S. of above equation, we obtain

$$\frac{\partial \overline{A}^{\mu}}{\partial \overline{A}^{v}} + \overline{\Gamma}^{\mu}_{\sigma v}\overline{A} = \frac{\partial \overline{x}^{\mu}}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^{v}}\left(\frac{\partial A^j}{\partial x^k} + \Gamma^{j}_{pk} A^p\right) \qquad \ldots(4.80)$$

Introducing the comma notation

$$\partial \overline{A}^{\mu}_{;k} = \frac{\partial A^j}{\partial x^k} + \Gamma^{j}_{pk} A^p \qquad \ldots(4.81)$$

Eqn. (8.108) can be written in the form

$$\partial \overline{A}^{\mu}_{;k} = \frac{\partial \overline{x}^{\mu}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \overline{x}^{v}} A^{j}_{;k} \qquad \ldots(4.82)$$

This equation shows that $A^{j}_{;k}$ defined by (4.82) is a mixed of rank two, called the covariant derivative of A^{j} with respect to x^{k}.

4.18 SUMMARY

In this unit you have learned about tensor analysis, rank of tensor, types of tensor etc. You have learnt coordinate transformation in terms of covariant and contravariant vector and tensor. You have also learnt reduction of rank of a tensor and algebra rule like addition, subtraction, multiplication etc. You have also learnt symmetric and skew symmetric tensor. You have also learnt Christofell's three index symbol and their relationship. In this unit you have studied fundamental tensor, Remanian metric, Geodesic. Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

4.19 GLOSSARY

Dummy indices – which can be changed without altering meaning

Covariant – set of quantities remain unchanged

Contravariant - not comparable

Geodesic- the shortest line between two points that lies in a given surface.

Fundamental - basic

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2. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd

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4.22 TERMINAL QUESTIONS

(Should be divided into Short Answer type, Long Answer type, Numerical, Objective type)

4.22.1 Short Answer type

1. What do you understand by dummy and real indices?
2. Explain Kronecker delta. Discuss some properties of kronecker delta.
3. Explain contravariant vectors.
4. Explain covariant vectors.
5. Discuss contravariant tensors of second rank.
6. Discuss Covariant tensor of second rank.
7. What do you understand by mixed tensor of second rank?
8. Explain symmetric and antisymmetric tensors.

4.22.2 Long Answer type

1. Show that the sum or difference of two tensors of the same rank and same type is also a tensor of the same rank and same type?
2. What do you understand by Contraction of tensors? Discuss it with an example.
3. Explain Metric tensor in Riemannian space.
4. Explain Christofell's 3-index symbols. Establish Relations between Christofell's symbols of first and second kind.

UNIT - 5 LINEAR DIFFERENTIAL EQUATIONS OF FIRST and SECOND ORDER

STRUCTURE

UNIT 5:

Differential equations: Linear ordinary differential equations of first and second order

5.1 Objectives

The Learning objectives of this unit are
1. To know the difference between linear and non-linear differential equations.
2. To classify the differential equations according to their order.
3. To find the solution of linear first and second order differential equations using different approaches.

5.2 Introduction

A great many number of problems in nature, either scientific or non-scientific, involves rate of change of one quantity with respect to another, this referred to as derivative in mathematics.
A differential equation is an equation expressing a relation between a function and its derivatives it contains derivatives either ordinary or partial. Most common example of differential equation, which every one might have come across with, is the Newton's second law. The equation is F = ma, where F is the force applied on a particle of mass m and 'a' is the acceleration which results because of that force. We may also write the equation as: $F = m\frac{d^2x}{dt^2}$we can see that the force on the body is expressed as a double differential of the position w.r.t time. Other frequently encountered differential equations are; the Laplace's equation, the Poisson's equation and many others.

5.3 Ordinary and Partial Differential equations

The function which is being described by a differential equation decides whether it is an ordinary differential equation or partial differential equation. If the function has single independent variable it is ordinary differential equation, whereas if the function has more than one independent variable then it is partial differential equation. Ordinary differential equations are expressed as complete derivatives $\left(\frac{d}{dx}\right)$, whereas the partial differential equations are expressed as $(\frac{\partial}{\partial x})$.
As an example consider the function y = 2x or any higher powers of x, now the differential equation expressing the different derivatives of the function would be called an ordinary differential equation since the function has a single independent variable x.
On the other hand for the function of type $y = xz + xz^2$, the differential equation should contain terms representing rate of change of y with respect to both the variables x and z, thus this differential equation is termed as partial differential equation.

Example of ordinary differential equation isNewton's second law, rate of change equations etc
Examples of Partial differential equations are Schrodinger equation, Maxwell's equations etc

Order of a differential equation:

The order of a differential equation is the order of the highest derivative of the unknown function involved in the equation, for example a first order differential equation contains only first order derivatives such as the expression for slope of a line; $m = \frac{dy}{dx}$, whereas a second order differential equation contains at least one second order derivative such as Newton's second law. The order of a differential equation does not depend on whether the equation is ordinary or partial. Some examples of differential equations ca be summarized as

$$ay'' + by' + cy = g(t) \tag{1}$$

$$Sin\,(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2e^{-5y}. \tag{2}$$

5.4 Linear Differential equations

The differential equations are further classified as linear differential equations and nonlinear differential equations. The linear differential equations are those differential equations in which the dependent variable and their derivatives do not occur in product form or in powers other than single power. For example a linear differential equation ca be written as

$$a_n(t)\,y^{(n)}(t) + a_{(n-1)}(t)\,y^{(n-1)}(t) + \text{- - - - - - - - - -} + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t). \tag{3}$$

The coefficients $a_0(t)$, ….. $a_n(t)$ and g(t) can be zero or non - zero functions, constant or non - constant functions, linear or non – linear functions. Only the function y (t) and its derivatives are considered to determine whether a differential equation is linear or not.If a differential equation cannot be written in the form of (3) it is a non - linear differential equation as is equation (2). The fundamental equations of atmospheric physics are non-linear.

For the present unit we will restrict ourselves to linear differential equations of first and second orders only as these are more frequently involved in studying different physical phenomenon.

We start here with first order ordinary linear differential equation.

5.4.1 Linear Differential equation of First order

A first order ordinary linear differential equation is one which involves only first derivative of the independent variable.
The standard form of first order ordinary differential equation can be expressed as:

$$\mathbf{y' + P(x)\,y = Q(x).} \tag{4}$$

where P and Q are functions of x.
As defined this equation is a first order differential equation as the derivative of the unknown variable y is of first order only, the equation is also linear because dependent variable y is neither in product form nor in any power other than single power.

A first order linear differential equation can be encountered in different fields of study such as scientific research, engineering, and economics etc.

An example of first order linear differential equation we may study the radioactive decay equation.

According to law of radioactivity, a unstable nuclei decays to more stable nuclei and the rate of this decay is proportional to the initial number of the unstable nuclei, we may express this as

$$\frac{dN}{dt} \alpha N$$

$$\frac{dN}{dt} = -\lambda N$$

$$\frac{dN}{N} = -\lambda \, dt$$

This is of the form of eq 4, a linear differential equation of first order

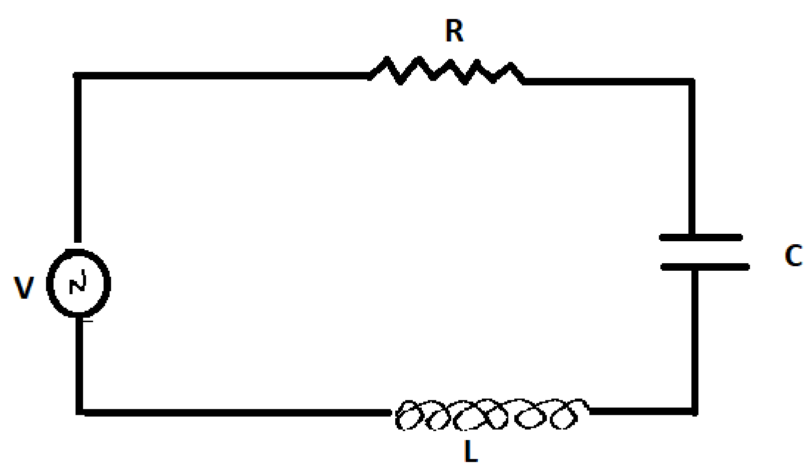
The solution of this differential equation can be done as

Integrating both sides we get

$$\ln N = -\lambda dt + Constant.$$

Using initial conditions we can solve this to get $N = N_0 e^{-\lambda t}$.

Further example of differential equation of this type may be encountered in electronic circuits such as a simple series circuit as shown below



For the given series circuit above containing resistor R, capacitor C, and inductor L and a source of emf V, we can study this circuit using a first order linear differential equation as follows
If at time t the current flowing through the circuit is I(t) and the charge on the capacitor is q(t) then I = dq/dt, The voltage across R is RI, the voltage across capacitor is q/V and the voltage across inductor is L(dI/dt). At any time t we must have

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V.$$

This is the required equation of which we have to find the solution.

Now we move on to study the techniques to solve a first order linear differential equation.

5.4.2 Solution of a First order linear differential equation

A Solution of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity.

There are three main techniques for solving linear differential equations depending on the form.

5.4.2.1. Separation of variables method: If the first order differential equation is of the form

$$\frac{dy}{dx} = F(x, y).$$

F(x,y) can be expressed as F(x,y) = X(x)Y(y) (separation of variable form).
Then the equation can be expressed as

$$\frac{dy}{dx} = X(x)Y(y)$$

$$\frac{dy}{Y(y)} = X(x)dx.$$

The equation can thus be solved by integrating both sides as

$$\int \frac{dy}{Y} = \int X(x)dx.$$

Examples

1. Solve the following differential equation using separation of variable method
   i) $y' = \frac{4y}{t}$.
   ii) $x^2yy' = e^y$, $x \neq 0$.

Sol. i) The equation can be solved by separation of variables method we separate the variables as

$$\frac{dy}{dt} = \frac{4y}{t}$$

$$\frac{dy}{4y} = \frac{dt}{t}.$$

Integrating this we get

$$\int \frac{dy}{4Y} = \int \frac{dt}{t} + C_1$$

$$\frac{\ln|y|}{4} = ln|t| + C_1$$

$$\frac{\ln|y|}{4} - ln|t| = C_1$$

$$l\,n\left|y^{\frac{1}{4}}\right| - ln|t| = C_1$$

$$\ln\left|\frac{y^{\frac{1}{4}}}{t}\right| = C_1$$

$$\frac{y^{\frac{1}{4}}}{t} = e^{C_1}$$

$$y^{\frac{1}{4}} = te^{C_1}$$

$$y = Ct^4.$$

This is the general solution of the given differential equation which defines a family of solution curves corresponding to various initial conditions.

Sol. ii) $x^2yy' = e^y$.

This equation can be written as

$$ye^{-y}dy = x^{-2}dx.$$

Now the variables are separated, now we integrate it

$$\int ye^{-y}dy = \int x^{-2}dx$$

$$-(1+y)e^{-y} + C = -x^{-1}$$

where C is a constant of integration.
The solution can thus, be expressed implicitly in the form as

$$x(y+1) = (1+Cx)e^y.$$

5.4.2.2 Using Integrating factor: This method is used for those first order linear differential equations, which are in standard for as

$$y' + p(t)y = q(t).$$

The ODE can be solved using an integrating factor as $I = e^{\int p(t)dt}$.

Next step is to multiply both sides by this integrating factor

$$e^{\int p(t)dt}[y' + p(t)y] = e^{\int p(t)dt}q(t)$$

$$[e^{\int p(t)dt}y]' = e^{\int p(t)dt}q(t)$$

$$e^{\int p(t)dt}y = \int e^{\int p(t)dt}q(t)dt + C.$$

Now solving the integration on R.H.S and dividing both sides by the integrating factor gives the general solution of the equation.

Examples

Q.2. Solve the following equations using Integrating factor method

i) $y' - 2ty = t$

ii) $y' + \frac{3y}{x} = \frac{e^x}{x^3}$.

Sol. i) $y' - 2ty = t$

We first find the integrating factor as I. F. = $e^{\int -2tdt}$

Multiplying both sides by $e^{\int -2tdt} = e^{-t^2}$

We get

$$e^{\int -2tdt}\frac{dy}{dt} - e^{\int -2tdt}2ty = e^{\int -2tdt}t$$

$$\frac{d}{dt}\left[e^{\int -2tdt}y\right] = e^{\int -2tdt}t$$

$$e^{\int -2tdt}y = \int e^{\int -2tdt}tdt$$

$$e^{-t^2}y = \int e^{-t^2}tdt$$

$$y = Ce^{t^2} - \frac{1}{2}.$$

Where C is constant

Sol. ii) $y' + \frac{3y}{x} = \frac{e^x}{x^3}$

The integrating factor I.F = $e^{\int \frac{3}{x} dx} = e^{\ln x^3} = x^3$.

Now we multiply both sides of the differential equation by this integrating factor

$$x^3[y' + \frac{3y}{x}] = x^3 \frac{e^x}{x^3}$$

$$x^3 \frac{dy}{dx} + 3x^2 y = e^x$$

$$\frac{d}{dx}(x^3 y) = e^x.$$

Integrating both sides we get

$$x^3 y = \int e^x \, dx + C.$$

Which gives the solution as

$$y = \frac{e^x + C}{x^3}.$$

5.4.2.3. **Change of Variable method:**

This method is applied for those differential equations which gets convertible to integrable forms under proper substitution.

Examples

i) $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol. Multiplying both sides by e^y we get

$$e^y \frac{dy}{dx} = (e^{2x} - e^x e^y)$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}.$$

Put $e^y = v$, which gives $e^y \frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dv}{dx} + v e^x = e^{2x}.$$

This equation is linear in v and x.

Put P = e^x and Q = e^{2x}

Integrating factor I. F = $I.F = e^{\int p dx} = e^{e^x}$.

Multiplying both sides of the differential equation by I.F, we get

$$e^{e^x}[\frac{dv}{dx} + ve^x] = e^{e^x}e^{2x}$$

$$\frac{d}{dx}\left[ve^{e^x}\right] = e^{e^x}e^{2x}.$$

Integrating both sides we have

$$ve^{e^x} = \int e^{e^x}e^{2x}dx + C$$

Put $e^x = t$

$$ve^{e^x} = \int te^t dt + c$$

$$= (t-1)e^t + C = (e^x - 1)e^{e^x} + C$$
$$v = e^x - 1 + Ce^{-e^x}$$

$$e^y = e^x - 1 + Ce^{-e^x}.$$

5.5 Second order linear differential equation

A second order linear differential equation is one in which the highest order derivative occurring in the equation is 2 and the coefficients are functions of only x. The general form of the second order linear differential equation can be written as

$$y'' + a(x)y' + b(x)y = Q(x)$$

This is an example of inhomogeneous differential equation of second order; the homogeneous differential equation of second order is obtained if Q(x) becomes zero.

$$y'' + a(x)y' + b(x)y = 0$$

The second order linear differential equations are used to model many situations in physics and engineering. The behavior of simple models such as spring mass - system and an LCR circuit can be studied using the differential equations involving both single as double derivatives, these can then be used to approximate other more complicated situations such as the bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

5.5.1 The solution of the differential equations of second order

The two forms of the second order differential equations as described above i.e. non-homogeneous and homogeneous differential equations are related to each other and there is an important connection between the solution of a nonhomogeneous linear equation and the solution of its corresponding homogeneous equation. The two principal results of this relationship are as follows:

Theorem 1:
If y_1, y_2 _ _ _ _ _ _ y_n are n linearly independent solutions of the differential equation $y^n + a_1y^{n-1} + a_2y^{n-2}$ _ _ _ _ + $a_ny = 0$, then $y = c_1y_1 + c_2y_2 +$ _ _ _ _ _ + c_ny_n is also its solution, where c_1, c_2, _ _ _ _ _ _c_n are arbitrary constants.

Theorem 2:
If y(x) is any particular equation of the linear non homogeneous equation, and if $y_n(x)$ is the general solution of the corresponding homogeneous equation, then the general solution of the linear non homogeneous equation is the linear sum of $y_n(x)$ and the particular solution of given non- homogeneous equation.

The general solution of the homogeneous linear differential equation part is called the complementary function (C.F) of the non - homogeneous equation whereas the other part is the particular integral (P.I) of the equation.

5.5.2 Complementary function:

For a non-homogeneous differential equation,the complementary function is the solution of the differential equation with the right hand side term replaced by zero. To find C.F we need to first find the auxiliary equation.

The Auxiliary equation (Characteristic equation): The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation.

Steps for finding Auxiliary equation

1. Replace y by 1.
2. Replace $\frac{dy}{dx}$ by m
3. Replace $\frac{d^2y}{dx^2}$ by m^2 and so on replace $\frac{d^ny}{dx^n}$
4. By doing so we have an equation in m of degree n called auxiliary equation.

5.5.3 Rules for finding the complementary function

1. Write the corresponding characteristic equation for the given differential equation.
2. The auxiliary equation would be an equation in m of degree n, so it will give n values of m on solving.
3. If m_1, m_2 , …….. m_n are the roots of the auxiliary equation, then the complimentary function depends upon the nature of the roots of the auxiliary equation. The different cases which arise are discussed as follows.

CASE I

When the roots of the Auxiliary equation are real and distinct: In this case the general solution of the homogeneous differential equation comprises of only complementary function.
The general solution is given as

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

Example:

Solve the equation $$y'' + y' - 6y = 0.$$

The auxiliary equation is $m^2 + m - 6 = 0.$

The roots of this equation can be obtained as

$$m^2 + 3m - 2m - 6 = 0\,.$$

Which gives $m_1 = 2$ and $m_2 = -3$.
Since the roots are real and distinct the solution of the differential equation would be

$$y\,(x) = C_1 e^{2x} + C_2 e^{-3x}.$$

This solution can be verified by differentiating and substituting in the original differential equation to get zero.

CASE II

When the roots of the auxiliary equation are equal: In this case the solution of the differential equation is written as $y(x) = C_1 e^{mx} + C_2 x e^{mx}$, where m is the common root.

Example:
Solve the equation $4y'' + 12y' + 9y = 0.$
The auxiliary equation of this differential equation can be written as

$$4m^2 + 12m + 9 = 0.$$

Which can be factored as $$(2m + 3)^2 = 0.$$

The two roots of this equation are same and thus m = -3/2.

Thus the solution of the given differential equation would be

$$y(x) = C_1 e^{-\frac{3}{2}x} + C_2 x e^{-\frac{3}{2}x}.$$

CASE III

When two roots of the auxiliary equation are imaginary: In this case the solution of the differential equation involves sine and cosine function, if the roots of the auxiliary equation are $m_1 = \alpha + i\beta \ and \ m_2 = \alpha - i\beta$ then the solution of the differential equation would be

$$y(x) = e^{\alpha x}(C_1 cos\beta x + C_2 sin\beta x).$$

Example:
Solve the equation $y'' - 6y' + 13y = 0.$
The auxiliary equation of the differential equation would be

$$m^2 - 6m + 13 = 0.$$

We can obtain the roots of this equation as

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Which gives

$$m = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

The solution of the differential equation would be

$$y(x) = e^{3x}(C_1 cos2x + C_2 sin2x).$$

CSAE IV

When roots of the auxiliary equation are repeated imaginary: If the two roots of the differential equation are $m_1 = m_2 = \alpha + i\beta$and $m_3 = m_4 = \alpha - i\beta$, then the complementary function will be

$$y(x) = e^{\alpha x}(C_1 + C_2 x)cos\beta x + (C_3 + C_4 x)sin\beta x.$$

This would be the case for fourth order linear differential equation. Since we are discussing second order linear differential equation this case is not elaborated.

5.5.4 Rules for finding the Particular Integral

For a non -homogeneous second order linear differential equation the solution comprises of both the complimentary function and the Particular integral part. Depending on the nature of the term Q(x) on the R.H.S of the differential equation we have the following cases for the solution.

Case I: When $Q = e^{ax}$

First find the P I as : $P.I. = \frac{1}{f(D)} Q$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, provided\ f(a) \neq 0.$$

Case II: When Q = sin (ax + b) or cos (ax +b)

Replace D^2 by $-a^2$.

If denominator reduces to a constant, it will be the final step in finding P.I.
If denominator reduces to D only, we are then only to integrate the given function Q once.
If denominator reduces to a factor of the form αD + β then operate by its conjugate αD – β on both numerator and denominator from left hand side such as

$$\frac{\alpha D - \beta}{\alpha D - \beta}\left[\frac{1}{\alpha D + \beta}\sin(ax + b)\right].$$

By doing so, denominator will become $\alpha^2 D^2 - \beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.
Now operating sin (ax + b) by αD – β we can find the required particular integral.

CASE III: When Q = x^m, m being a positive integer.

Here $P.I = \frac{1}{f(D)}x^m$.

Take out the lowest degree term from f(D) to make the first term unity (so that Binomial theorem for a negative index is applicable). The remaining factor will be of the form

Examples:

Q.1. Solve the following differential equations

i) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$.

ii) $\frac{d^2y}{dx^2} + 4iy = 0$.

Sol.i) First we write the corresponding characteristic equation

$$k^2 - 6k + 5y = 0.$$

The roots of this equation are $k_1 = 1$ and $k_2 = 5$.

Since the roots are real and distinct, the solution has the form

$$y(x) = c_1e^x + c_2e^{5x}.$$

Sol. ii) The characteristic equation is

$$k^2 + 4i = 0.$$

The roots of the equation are k_1 , $k_2 = \pm 2i\sqrt{i}$.

These can be expressed in trigonometric form as

$$i = \sin \pi/2 + i \cos \pi/2 = e^{i\pi/2} \text{ which implies } \sqrt{i} = e^{i\pi/4} = \sin \pi/4 + i \cos \pi/4.$$

The roots of the equation can thus, be written as

$$k_1 = 2i (\sin \pi/4 + i \cos \pi/4) \text{ and } k_2 = -2i (\sin \pi/4 + i \cos \pi/4)$$

$$k_1 = 2i (1/\sqrt{2} + i/\sqrt{2}) \text{ and } k_2 = -2i (1/\sqrt{2} + i/\sqrt{2})$$

$$k_1 = -2/\sqrt{2} + 2i/\sqrt{2} \text{ and } k_2 = -2i/\sqrt{2} + 2/\sqrt{2}$$

$$k_1 = -\sqrt{2} + \sqrt{2}\, i \text{ and } k_2 = \sqrt{2} - \sqrt{2}i\ .$$

The general solution of the given differential equation would thus be

$$y (x) = C_1 e^{(-\sqrt{2} + \sqrt{2}\, i)\, x} + C_2 e^{(\sqrt{2} - \sqrt{2}i)\, x}$$

where C_1 and C_2 are arbitrary constants.

5.6 Summary

In the present unit, we studied about forms of the differential equations. The different forms of the differential equations were introduced, such as linear, non-linear, ordinary and partial differential equations. The order of the differential equations was defined and difference between first order and second order differential equation was elaborated. We also presented some first and second order linear differential equations and studied different approached to determine their solution.

5.7 Glossary

Equation: relationship between dependent and independent variable.

Differential equation: equations involving dependent variable and their derivatives with respect to independent variables.

Linear differential equation: no multiplication among dependent variables.

Ordinary differential equation: Equations involving only one independent variable.

Auxiliary equation: An equation obtained from the standard form of a linear differential equation by replacing the right members by zero.

Complementary Function:general solution of auxiliary equation of linear differentialequation.

Particular Integral: Any solution to a differential equation

5.8 References:

1. Arfken, G. "A Second Solution." §8.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 467-480, 1985.

2. Boyce, W. E. and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems, 4th ed.* New York: Wiley, 1986.

3. Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 667-674, 1953.

4. Boyce, W. E. and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems, 5th ed.* New York: Wiley, 1992

5.9 Suggested Readings

1. An Introduction to Ordinary Differential equations by Earl A Coddington**,** Dover Publications Inc.; New edition (1 March 1989).

2. Differential Equations with Applications and Historical Notes by George Simmons, McGraw Hill Education; 2nd edition (1 July 2017).

5.10 Terminal Questions

Short Answer type questions

Q.1. What is the order of the differential equation $\frac{dy}{dx} + 4y = Sinx$?

Q.2. The process of formation of the differential equation is given in the wrong order,
1) Eliminate the arbitrary constants.
2) Differential equation which involves x,y,dydx.
3) Differentiating the given equation w.r.t x as many times as the number of arbitrary constants.
Write the correct order.

Q.3. Consider the differential equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, if x =0 at t=0, and x=1 at t=1, determine the value of x at t=2.

Q.4. Find the Particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, y(1) = 1.

Q.5. Solution of the second order differential equation $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + ky = 0$ is $y = e^{2t}$, the value of k is.

Long Answer type questions

Q.1. Solve the Differential equation $x\frac{dy}{dx} = x^2 + 3y$.

Q.2. Find the Solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$

Q.3. Find the Solution of $\frac{d^2y}{dx^2} + 7y = 0$

Q.4. Find the Solution of $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 2x - 3$.

Q.5. Solve the initial value problem $\frac{dy}{dx} + xy = x, y(0) = -6$.

5.11 Answers

Answers to Short Answers type questions
1. First Order
2. 3,1,2
3. $e^2 + e$
4. y = x log |x| + x
5. k = 6

Answer to Long Answer type questions

1. $y = -x^2 + Cx^3$
2. $y = Ae^x + Bxe^x - \frac{1}{2}e^x \ln(1 + x^2) + xe^x arctanx$
3. $y = A\, e^{i\sqrt{7}x} + Be^{-i\sqrt{7}x}$
4. $y = A\, e^x + Be^{-\frac{1}{3}x} - 2x + 7$
5. $y = 1 - 7e^{-\frac{x^2}{2}}$

Unit 6: Partial Differential equations

Structure

6.1 Objectives

The learning objectives of this unit are

1. To introduce students with Partial Differential Equations
2. To derive Heat and wave equations
3. To find solutions of PDE using boundary conditions.

6.2 Introduction: Partial differential equations describe the behavior of ,any engineering phenomena, such as wave propagation, fluid flow (air or liquid), vibration, mechanics of solids, heat flow, electric field, diffusion of chemicals etc. Many of the problems of mathematical physics involves the solution of partial differential equations. In fact, a single partial differential equation may apply to a variety of physical problems.

A partial differential equation is an equation involving functions and their partial derivatives, such as the heat equation, the wave equation and the Laplace's equation.

These are termed, as partial differential equations since these involve partial derivatives, which are derivatives of the functions having more than one variable.

To start with the partial differential equations we first describe the partial derivative and then try to study some important partial differential equations such as the heat equation, the wave equation and the Laplace's equation.

6.3 Partial Derivative: For a function dependent on more than one variables, the change in the function with respect to one variable keeping the other variables constant is represented by partial derivative. For example let *f* be a function of *x,y,z* and *t,* represented as *f(x,y,z,t)*, the change in function *f* with respect to change in variable *x*, keeping the other variables constant is represented by, $\frac{\partial f}{\partial x}$.

6.4 Examples of Partial differential equations:

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = 0$$

$$\frac{\partial^2 U}{\partial y^2} + c^2 \frac{\partial^2}{\partial x^2} = 0$$

$$(x^2 + y^2) \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \eth y} - 3u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \frac{\partial^2 x}{\partial y^2} = x^2 + y^2.$$

The partial differential equations are classified based on their order, form and nature as first, second, third or higher orders, linear, non-linear, homogeneous and non-homogeneous PDE's respectively.

6.5 Order of a Partial differential equation:

The order of the highest derivative term in any partial differential equation is the order of the PDE. All the above equations given above are second order partial differential equations. Depending on the nature of the problem, first or second order PDE's are used to describe it. For example, the gas flow problem, the traffic flow problem, the phenomena of shock waves, motion of wave fronts, Hamilton Jacobi theory, nonlinear continuum mechanics and quantum mechanics etc. can be studied using first order PDE's, whereas the problems of fluid mechanics, heat transfer, rigid body dynamics and elasticity are modelled by second order PDE's.

Some of the examples of first and second order PDE's are

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \ ; \qquad transport\ equation$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ ; \qquad Laplace\ equation\ in\ 2D$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \ ; \qquad Heat\ equation$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \ ; \qquad Wave\ equation.$$

Linear PDE: If the dependent variable and all its partial derivatives occur linearly, i.e. degree at most one, in any PDE then such an equation is called linear partial differential equation, otherwise a non-linear PDE, in above equations all except the last are linear PDE.

Quasi – Linear PDE: A partial differential equation in which all the terms of the highest order derivatives of dependent variable occurs linearly. The coefficients of such terms are functions of only lower order derivatives of the dependent variables.

Homogeneous PDE:A linear differential equation is termed homogeneous if the dependent variable and its partial derivatives appear in terms with degree exactly one, or it has no,non-differential terms. If it has one or more non-differential term, it is non-homogeneous or heterogeneous PDE.

6.6 Solution of a partial differential equation

The solution of Ordinary differential equation contains arbitrary constants, whereas the solution to partial differential equations contains arbitrary functions. While an ODE of order m has m

linearly independent solutions, a PDE has infinitely many solutions. These are consequences of the fact that a function of two variables contains immensely more of information than a function of only one variable. Some of the method for solving PDE are:
Separation of variables method
Integral solutions employing a green function
Use of Integral Transforms
Numerical calculations

We will now discuss some important partial differential equations.

6.7 The Laplace Equation: The steady state of a field that depends on two or more independent variables, which are typically spatial. The Laplace equation arises as a steady state problem for the heat or wave equations that do not vary with time so that $\frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial t^2}$
Laplace equation in 2D.

The physical problems in which the Laplace equation arises are
i) 2D steady state heat conduction
ii) Static deflection of a membrane
iii) Electrostatic potential.

The general form of a 2D Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We can relate this equation to the steady state heat equation, which do not vary with time. Secondly, we can also relate it to the equation of continuity for incompressible potential flow.

The Laplacian represents the flux density of the gradient flow of a function. For instance, the net rate at which a chemical dissolved in a fluid moves toward or away from some point is proportional to the Laplacian of the chemical concentration at that point.

The three-dimensional Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Since there is no time dependence in the Laplace's equation, there is no initial condition to satisfy by their solutions, However there should be certain boundary condition on the bounded curve in which the differential equation is to be solved.

Typically, there are three types of boundary conditions given as

6.7.1 The Dirichlet Boundary Condition:
This boundary condition when imposed on an ODE or PDE specifies the values that a solution needs to take along the boundary of the domain.

The solution u (x,y) of the Laplace equation in a domain D is specified by boundary dS as

$$u(x, y) = f(x, y) \text{ on } dS$$

where f (x, y) is a given function. The question of finding solutions to such equations is known as Dirichlet problem and is expressed as

$$\nabla^2 u(x,y) = 0 \; in \; D; u(x,y) = f(x,y) on \; dS.$$

6.7.2 The Neumann or second type boundary condition:

It is the boundary condition which when imposed on an ODE or PDE, specifies the values in which the derivative of a solution is applied within the boundary of the domain.

$$\frac{\partial u(x,y)}{\partial n} = g(x,y), \qquad for \;\; x,y \in dS\,.$$

In physical terms, the normal component of the solution gradient is known on the boundary. In steady state heat flow problem, Neumann boundary condition means the rate of heat loss or gain through the boundary point is prescribed.
The Laplace equation together with Neumann BC are called the Neumann BVP or the Neumann problem and is written as

$$\nabla^2 u(x,y) = 0 \; in \; D; \;\; \frac{\partial u(x,y)}{\partial n} = g(x,y) \; for \; (x,y) \in dS.$$

The Neumann problem have no solution unless the average value of the function g on dS is assumes zero. This assumption is known as the compatibility condition.

6.7.3 Robin's type condition: This boundary condition also called as the third type boundary condition, when imposed on an ordinary or a partial differential equation, is a specification of a linear combination of the values of a function and the value of its derivative on the boundary of the domain.

$$\frac{\partial u(x,y)}{\partial n} + c(u - g) = \; 0.$$

C is a constant and g is a function, which can vary over the boundary. The Laplace equation together with Robin's conditionis known as Robin's boundary value problem or mixed problem.

6.8 Solution of 2D Laplace equation with Boundary Condition

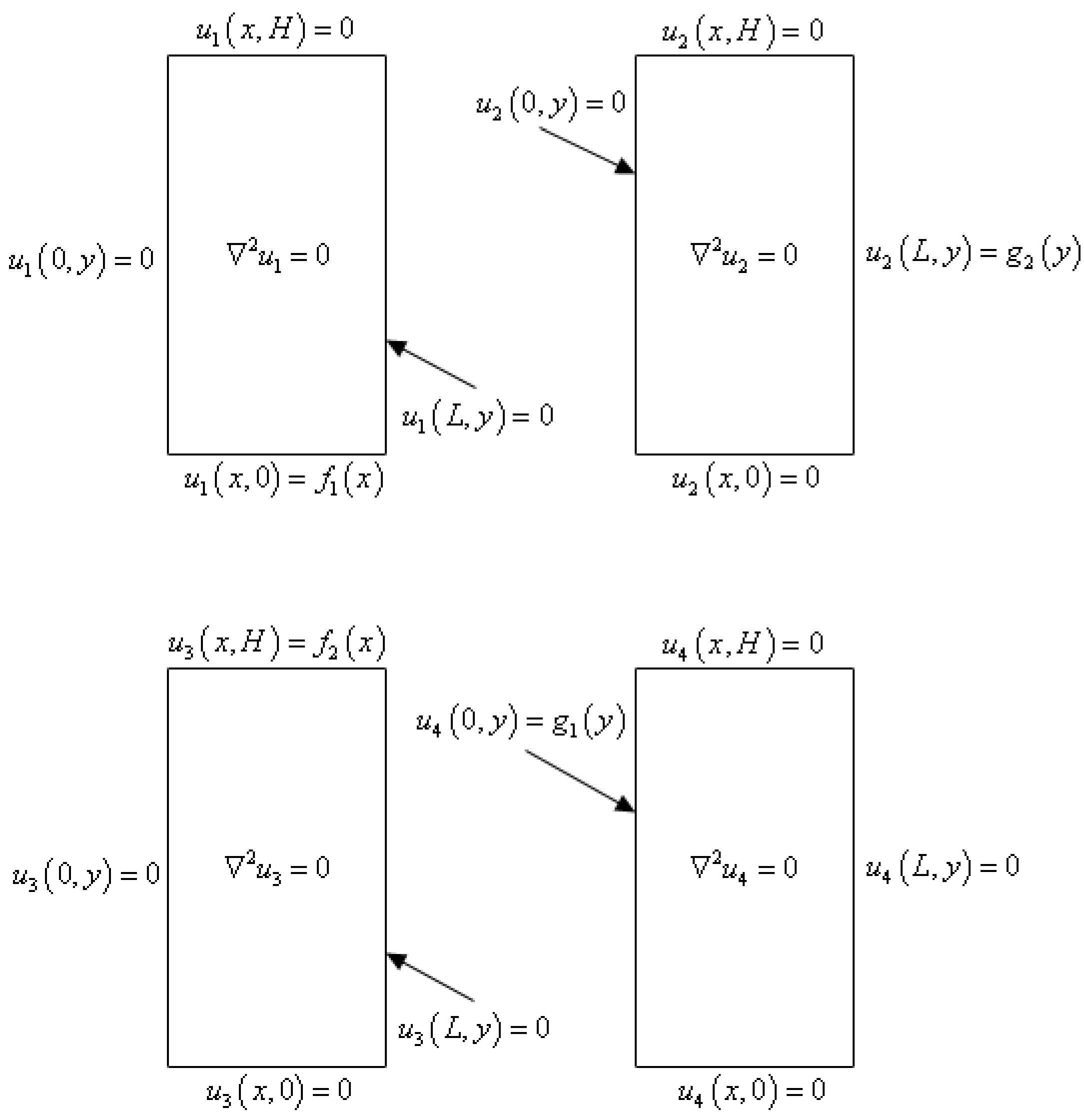
Consider the geometry of a rectangle given by $0 \leq x \leq L$, $0 \leq y \leq H$. For this geometry the Laplace equation along with the four boundary conditions is

$$\nabla^2 u = \frac{d^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(0.y) = g_1(y)$;

$u(L, y) = g_2(y)$
$u(x, 0) = f_1(x)$
$u(x, H) = f_2(x)$.

There is no initial condition here and both the variables are spatial variable and occur in a 2nd order derivative, so we need two boundary conditions for each variable.
The PDE is both linear and homogeneous but the boundary conditions are only linear but not homogeneous. To solve the Laplace equation completely it need to be solved four times. Each time the equation is solved, one of the boundary condition can be non homogeneous while the remaining three will be homogeneous.
The four conditions are as represented by the below figure



These four equations can be solved by separation of variable method. We proceed by solving for $u_1$as follows.

$$u_1(x,y) = X(x)Y(y).$$

The Laplace equation is

$$\frac{d^2u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Substituting the solution in the equation, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

On separating the variables we get

$$\frac{X''}{X} = -\frac{Y''}{Y} = k.$$

Which gives

$$X'' - kX = 0$$

$$Y'' + kY = 0.$$

Substituting $\lambda^2 = -k$.
The solution u(x,y) can be written as

$$u(x,y) = (A\,Cos\,\lambda x + BSin\,\lambda x)(CCosh\,\lambda y + DSinh\,\lambda y).$$

With the boundary conditions
u(0,y) = 0
u(L, y) = 0

we get

A= 0 and $\lambda_n = n\pi/L$, n= 1,2,3

So the solution becomes

$$u_n(x,y) = Sin\frac{n\pi x}{L}\left(A_n Cosh\frac{n\pi y}{L} + B_n Sinh\,\frac{n\pi y}{L}\right).$$

The boundary condition u(x, H) = 0 gives

$$B_n = -\frac{COsh\,\frac{n\pi H}{L}}{SInhn\frac{n\pi H}{L}}A_n.$$

So the solution can now be written as

$$u_n(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} Sin\frac{n\pi x}{L}\left(A_n Cosh\frac{n\pi y}{L} + B_n Sinh\,\frac{n\pi y}{L}\right)$$

$$u_n(x,y) = \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} A_n Sin\frac{n\pi x}{L}\frac{Sinh\,\frac{n\pi(H-y)}{L}}{SInhn\frac{n\pi H}{L}}.$$

The remaining boundary condition u(x, 0) = f_1(x) can be used to find the value of A_n

$$f(x) = \sum_{n=1}^{\infty} A_n Sin\,\frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x)Sin\frac{n\pi x}{L}dx.$$

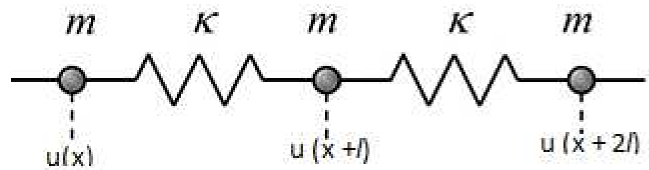
Similarly other solutions u_2, u_3, u_4 can be determined and the overall solution can be written as

$$u = u_1 + u_2 + u_3 + u_4\,.$$

6.9 The Wave equation: The Vibrating string

An important second order partial differential equation for description of waves.

Consider a vertical string of length L, that has been tightly stretched between two points at x = 0 and x = L.Since, the string is tightly stretched, we can assume that the slope$\frac{\partial y}{\partial x}$ of the displaced string at any point is small. The string never getsfar away from its equilibrium position
Consider a point x on the string in this equilibrium position i.e. the location of the point at t = 0. As the string vibrates this point will be displaced both vertically and horizontally, however if we assume that at any point the slope of the string is small then the horizontal displacement will be very small in relation to vertical displacement.
So at any point x on the string the displacement will be purely vertical, let this displacement be u(x,t). we now write the 1D wave equation as
The wave equation in one dimension case can be derived from Hook’s law as follows.
Imagine an array of little weights of mass m interconnected with massless springs of length *l* and spring constant k.



Here the dependent variable u(x) measures the distance from the equilibrium of the mass situated at x, So that u(x) essentially measures the magnitude of a disturbance, travelling in an elastic material.

The forces exerted by the mass m at the location x + l are

$$F_{Newton} = m.a(t) = m.\frac{\partial^2}{\partial t^2} u(x+l,t)$$

$$F_{Hook} = F_{x+2i} - F_x = k[u(x+2l,t) - u(x+l,t)] - k[u(x+l,t) - u(x,t)]$$

$$F_{Hook} = k[u(x+2l,t) - 2u(x+l,t) - u(x,t)].$$

Equating the two forces, we get

$$m.\frac{\partial^2}{\partial t^2} u(x+l,t) = k[u(x+2l,t) - 2u(x+l,t) - u(x,t)]$$

$$\frac{\partial^2}{\partial t^2} u(x+l,t) = \frac{k}{m}[u(x+2l,t) - 2u(x+l,t) - u(x,t)].$$

If the array of weights consists of N weights spaced evenly over the length L = Nh of total mass M = Nm, and the total spring constant of the array K = k/N we can write the above equation as

$$\frac{\partial^2}{\partial t^2} u(x+l,t) = \frac{KL^2}{M}\frac{[u(x+2l,t) - 2u(x+l,t) - u(x,t)]}{h^2}.$$

Under the limits N → ∞ and h → 0, the equation attains the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial x^2}.$$

Where $v = \sqrt{\frac{m}{KL^2}}$ is the propagation speed of the wave.

The generalized 1D wave equation can thus be written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial x^2}.$$

For the wave equation, the only boundary condition is that of the prescribed location of the boundaries.

The 2 D and 3D version of the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

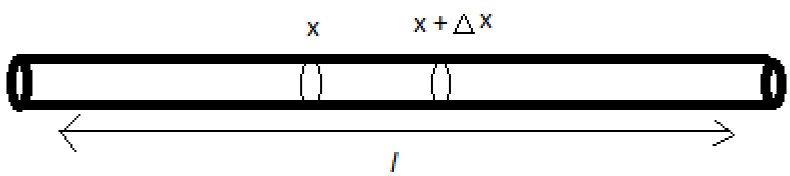
6.10 The Heat Equation:

It is the equation, which governs the temperature distribution in an object. There are in fact several forms of the heat equation; we will focus on one of the forms in 1D and 3D. The heat equation is, derived using the principle of conservation of energy and the fact that heat flows from hot regions to cold regions.

The 1D heat equation: Temperature distribution in a rod of length L

Consider a uniform rod of length *l* with non uniform temperature lying on the x axis from x = 0 to x = *l*.

Let u(x,t)denotes the temperature at x at a time t, and is assumed constant throughout the rod at each time t.



By the principle of energy conservation the net change of heat inside the segment between x and x + Δ x is equal to the net heat flux (influx at x and out flux at x + Δx) across the boundaries and the total heat generated between x and x + Δ x.

If 's' is the, specific heat capacity of the rod, 'k' is the thermal conductivity of the rod, 'ρ' the density of the rod, 'A' the cross sectional area of the rod and *f*(x,t) is the external heat source, then we can have

Total amount of heat inside the segment between x and x + Δ x at time t $=\int_x^{x+\Delta x} s\rho A u(x,t)dx.$

Net change of heat inside the segment $= \frac{d}{dt}\int_x^{x+\Delta x} s\rho A u(x,t)dx = \int_x^{x+\Delta x} s\rho A\, u_t(x,t)dx$.

Net heat flux across the boundaries $= kA\frac{d}{dx}[u(x+\Delta x\,,t) - u(x,t)]$

$$= kA\left[\left(u_x(x+\Delta x,t) - u_x(x,t)\right)\right].$$

Where $u_x = \frac{du}{dx}$.

Heat generated due to external heat source between x and x + Δx = $A \int_x^{x+\Delta x} f(x,t)dx$.

By principle of conservation of energy we now have

$\int_x^{x+\Delta x} s\rho A\, u_t(x,t)dx = kA\left[\left(u_x(x+\Delta x,t) - u_x(x,t)\right)\right] + A\int_x^{x+\Delta x} f(x,t)dx.$

Applying mean value theorem for integral we have

$$spAu_t(\xi_1,t)\Delta x = \left[\left(u_x(x+\Delta x,t) - u_x(x,t)\right)\right] + Af(\xi_2,t)\Delta x.$$

Where $\xi_1, \xi_2 \in (x, x+\Delta x)$ hence

$$u_t(\xi_1,t) = \frac{k}{sp}\frac{\left[\left(u_x(x+\Delta x,t) - u_x(x,t)\right)\right]}{\Delta x} + \frac{f(\xi_2,t)}{sp}.$$

In the limit Δ x → 0, we arrive at

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) + F(x,t).$$

Where $\alpha^2 = \frac{k}{sp}$ is called the thermal diffusivity of the rod and $F(x,t) = \frac{1}{sp} f(x,t)$is called the heat source density.

Now there arises three cases

1. The case when the lateral boundary is not insulated as above, and heat is allowed to flow in and out across the lateral boundary at a rate proportional to the difference between the temperature of the rod u(x,t) and the surrounding medium u_0, the conservation of heat principle yields

$$u_t = \alpha^2 u_{xx} - \beta(u - u_o), \quad \beta > 0.$$

The heat loss or gain is proportional to the difference between the temperatures of the rod and surrounding medium, β is the constant of proportionality.

2. If, the material of the rod is uniform then k is independent of x. for some materials the value of k depends on the temperature u and hence the resulting heat equation is nonlinear and given as:

$$u_t = \frac{1}{c\rho}\frac{\partial}{\partial x}\left\{k(u)\frac{\partial u}{\partial x}\right\}.$$

3. If, the material is non-homogeneous i.e one-half the rod is made of one material and the other half of different materials, the diffusion within the rod depends on x. The heat equation is written as

$$u_t = \alpha^2(x)u_{xx}, \qquad 0 < x < l\,.$$

With

$$\boldsymbol{\alpha_x} = \begin{cases} \boldsymbol{\alpha_1}, & \boldsymbol{0} < x < \dfrac{\boldsymbol{l}}{\boldsymbol{2}}, \\ \boldsymbol{\alpha_2}, & \boldsymbol{l/2} < x < l, \end{cases}$$

Where α_1 and α_2 are the thermal diffusivity of the two materials respectively.

6.11 Summary

In the present unit we studied the partial differential equation and its different forms. We also studied different boundary value problems and methods to solve the partial differential equations. The Laplace equation, the Heat equation and the wave equation for one dimensional and two dimensional cases were discussed in detail.

6.12 Glossary

Partial derivatives: Derivative of a multivariable function w.r.t any one, keeping others fixed.

Linear PDE: Dependent variable and all its derivatives occur in first order.

Quasi-Linear PDE: All terms of highest order derivative of dependent variable occurs linearly.

Homogeneous PDE: A differential equation is homogeneous if it is a homogeneous function of the unknown function and its derivatives

6.13 References

- R. Courant and D. Hilbert, *Methods of Mathematical Physics. Volume 2. Partial Differential Equations*, Wiley-VCH, 1989.
- L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.
- D. Bleecker and G. Csordas, Basic Partial Differential Equations, Van Nostrand Reinhold, New York, 1992.
- E. Kreyszig, Advanced Engineering Mathematics, Wiley, 2011.

6.14 Suggested Readings

- I.N. Sneddon, Elements of Partial Differential Equations, Dover Publications, New York, 2006.
- Peter J. Olver, Introduction to Partial Differential equations, Springer, New York 2014.

6.15 Terminal Questions

Short Answer type questions

1. The nature of the partial differential equations $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$ is

i) elliptic

ii) parabolic

iii) hyperbolic

iv) none of the above.

2. The equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is known as

i) one dimensional heat equation

ii) One dimensional wave equation

iii) Laplace equation

iv) Poisson's equation.

3. Using substitution which of the following are solutions of the PDE $\frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 u}{\partial y^2}$

i) Cos (3x-y)

ii) $x^2 + y^2$

iii) Sin (3x-3y)

iv) $e^{-3\pi y}$Sin(πy).

4. The three dimensional heat equation among following is

i) $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

ii) $\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

iii) $\frac{\partial u}{\partial t} = k\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

iv) $\frac{\partial u}{\partial t} = -k\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x}\right)$.

5. The boundary condition which include direct boundary value is

i) Dirichlet boundary condition

ii) Neumann boundary equation

iii) Forced boundary equation

iv) Discrete boundary equation.

Long Answer type questions

1. Solve $\frac{\partial u}{\partial x} = 6\frac{\partial u}{\partial t} + u$ using the method of separation of variables, if u(x,0) = 10 e^{-x}.

2. Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, 0 \le x \le \pi, 0 < t < \infty$$

u(0,t) = u(π,t) = 0, $0 < t < \infty$

u(x,0) = 3Sin 2x – 6Sin 5x, 0 ≤ x ≤ π.

3. Consider the initial value problem $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, -∞ < x, t >∞ , IC u(x,0) = 0, $u_t(x,0)$ = Sin (x)

6.16 Answers

Answer to short answer type questions

1. iii

2.i

3. iv

4. i

5. i.

Solution to Long Answer type questions

1. 10 e^{-x} $e^{-t/3}$

2.$u(x,t) = 3e^{-12t}$ Sin(2x) – 6 e^{-75t} Sin(5x)

4. $u(x,t) = -\frac{1}{2c}[Cos(x+ct) - Cos\,(x-ct)]$.

UNIT 7: LEGENDRE'S AND BESSEL'S DIFFERENTIAL EQUATIONS

Structure

7.1 Objective

The plan of this unit is as follows:

- First, I am going to briefly review linear second order differential equations which have already been introduced to you in unit 5(Ordinary differential equations of First and Second order).
- Second, I am going to introduce power series methods to solve differential equations.
- Third, we will apply the power series methods to solve above two special differential equations.
- We will learn other methods to generate polynomial like generating function, Rodrigues formula.
- We will discuss various properties of the polynomials such as orthogonality, recurrence relations.

7.2 Introduction

You have already studied the methods to solve first and second order differential equations. The solution of which are continuous i.e. exists for all the values over the real line. However there are certain differential equations whose solution exists only in a defined range. In this unit we are going to learn how to solve linear and homogeneous second order differential equations using power series method. Our focus will be to seek solution of two of the four *special forms of second order differential equations* which are linear and homogeneous in nature. The differential equations are:

a) **Legendre Differential Equation**:

$$(1-x^2)\,y'' - 2x\,y' + n(n+1)y = 0.$$

b) **Bessel's differential equation**:

$$x^2y'' + x\,y' + (x^2-\nu^2)y = 0.$$

The solution of the differential equation turns out to be a polynomial which plays an important role as a part of solution to various important problems of different fields. We see the existence of Legendre polynomials in numerical analysis as a solution of Gaussian quadrature integration method. For weight function as 1the Gauss quadrature method turns out to be Gauss-Legendre quadrature. Bessel's polynomial occurs as one of the solution of partial differential equation of a circular membrane. We also see presence of Bessel polynomial in the intensity distribution of very interesting phenomena well known as Airy's pattern (It is diffraction pattern of a star light when the star is seen through a circular lens/aperture). So let's proceed to learn detail description of these complex but beautiful differential equations.

7.3 Definitions

In this section we will review and learn important definitions which will be used in understanding the mathematical development of the differential equations.

7.3.1Differential equations of second order

A second order linear differential equation is a differential equation that can be written in the following form:

$$f(x)y'' + g(x)y' + h(x)\, y \ = J(x). \qquad (1)$$

Where$y'' = \frac{d^2y}{dx^2}$, $y' = \frac{dy}{dx}$ $and\ f(x) \neq 0$. In any linear differential equations there is no products of the function $y(x)$, and its derivatives and no terms of y and its derivatives with power other than first power, for example: terms like y^2, $y\frac{dy}{dx}$, $\left(\frac{dy}{dx}\right)^2$ $or\ y\frac{d^2y}{dx^2}$ $etc.$. are not part of linear differential equations. The coefficients $f(x),\ g(x), h(x)\ and\ J(x)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions**. Only $\boldsymbol{y(x)}$ and its derivatives are used to determine whether equation is linear or not.**

We can write equation (1) as follows

$$y'' + \frac{g(x)}{f(x)}y' + \frac{h(x)}{f(x)}\ y \ = \frac{J(x)}{f(x)}$$

$$y'' + p(x)y' + q(x)y \ = r(x). \qquad (2)$$

If $r(x) = 0$ then equation (2) is ***homogeneous equation.***

7.3.2 Real and analytic function

A real function $f(x)$ is called ANALYTIC at a point $x = x_0$ if it can be represented by a power series in powers of $(x - x_0)$ with radius of convergence $R > 0$, the convergence interval is $(x_0 - R, x_0 + R)$.

7.3.3 Regular and Singular points

A regular point of $y'' + p(x)\, y + q(x)y \ = 0$ is a point x_0 at which the coefficients $p(x)$ and $q(x)$ are *analytic* i.e. converges to a finite value as $x \to x_0)$. Then the power series method can be applied. If x_0 is not regular, it is singular point (i.e.$\lim_{x\to x_0} p(x)$and $\lim_{x\to x_0} q(x)$**do not converge** to finite value).

Again consider a homogeneous second order differential equation of the form

$$y'' + p(x)y' + q(x)y = 0.$$

If at $x = x_0$ the coefficients $p(x)$ and $q(x)$ are not analytic i.e. $x = x_0$ is a singular point, however the condition for weak singularity is satisfied, which is as follows

$$\lim_{x\to x_0}(x - x_0)\, p(x) \quad \to \textit{finite value}$$

$\lim_{x\to x_0}(x - x_0)^2\, q(x) \quad \to \textit{finite value}$

then$x = x_0$ is called as *regular singular point*.

> **SAQ1**: Find the regular singular point of the differential equation $\mathbf{2\,(x-2)^2\,xy'' + 3x\,y' + (x-2)y = 0}$

7.3.4 Leibniz's rule

Leibniz's rule (named after Gottfried Wilhelm **Leibniz**) is a generalization of the product rule in differential calculus. It expresses the derivative of order$n$ of the product of two functions. Suppose that the functions $u(x)$ and $v(x)$ have the derivatives up to order$n$. Then the derivative of the product of these functions (D is the derivative operator)

The first order derivative will be

$$D(uv) = (Du)v + u\,(Dv).$$

The second order derivative will be

$$D^2(uv) = (D^2u)v + 2(Du)(Dv) + u\,(D^2v).$$

The third order derivative will be

$$D^3(uv) = (D^3u)v + 3(D^2u)(Dv) + 3(Du)(D^2v) + u\,(D^3v).$$

If we keep differentiating with increasing the order of derivative, it is easily visualized that the terms on the right side is similar to binomial expansion with appropriate exponent on the derivative operator D. So in general forn^{th} **order derivative of the product of two functions** can be expanded as (with $D^0u = u$ and $D^0v = v$, i.e. no derivative)

$$\boldsymbol{D^n(uv) = (D^nu)v + {}^nc_1(D^{n-1}u)(Dv) + {}^nc_2(D^{n-2}u)(D^2v) + {}^nc_3(D^{n-3}u)(D^3v) + \cdots + {}^nc_{n-1}(Du)(D^{n-1}v) + {}^nc_n(u)(D^nv)}\,.$$

7.3.5 Generating function

A generating function is a continuous function associated with a given sequence. Consider a sequence of function be $\{f_n\}_{n=0}^{\infty}$. The generating function of this sequence of function is defined as

$$g(z) = \sum_{n=0}^{\infty} f_n z^n$$

for$|x| < R$ and R is radius of convergence of the series. The sequence of function f_n appears as coefficient in the series $g(z)$. It is important that the series has a non-zero radius of convergence, otherwise $g(z)$ would be undefined for all $x \neq 0$.

7.4 Series solution of second order differential equation

There are several methods to find the solution of second order differential equation depending on their form. Series solution is one of the method applied to normally special kind of second order differential equation.

7.4.1 Power series method

Power series method is the standard basic method for solving linear differential equations with variable coefficients. It gives solution in the form of Power series.

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

If $x_0 = 0$ then a power series will have powers of x only i.e.

$$\sum_{n=0}^{\infty} a_n\, x \;= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

For a given homogeneous differential equation of the form of equation (2):

$\boldsymbol{y'' + p(x)y' + q(x)y = 0}$ with $p(x)$and $q(x)$are analytic at $x = 0$, the solution can be found using power series method as follows:

1. Assume the solution in the form of power series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3}$$

and the series obtained by term wise differentiations

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \tag{3.1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \tag{3.2}$$

2. Replace $y, y' and\ y''$ in the given differential equation.
3. Collect the like powers of x and equate the sum of the coefficients of each occurring power of x to zero. We get equations from which the constants coefficients $a_0, a_1, a_2, ...$ can be determined.
4. Once the coefficients are known put it in assumed power series solution (in step 1)to get final solution and simplify it if possible.

Let's take a very simple example of a differential equation whose solution can be obtained from methods explained in unit 5.

SAQ2:Using power series method Solve the differential equation

$$\boldsymbol{y'' + y = 0}$$

7.4.2 Theorem

Several second order differential equations of great practical importance (e.g. Bessel equation, etc.) have coefficients that are not analytic but singular and are such that following theorem holds:

Any differential equation of the form

$$\boldsymbol{y'' + \frac{b(x)}{x} y + \frac{c(x)}{x^2} y = 0.} \tag{4}$$

Or

$$\boldsymbol{x^2y'' + xb(x)y + c(x)y = 0.} \tag{5}$$

With coefficients $p(x) = \frac{b(x)}{x}$ *and* $(x) = \frac{c(x)}{x^2}$*, such that the function* $x\,p(x)$ *and* $x^2q(x)$ *are analytic at* $x = 0$*,(*$x = 0$ *is a regular singular point) has at least one solution that can be represented in the form*

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = x^r(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) \qquad with\ a_0 \neq 0. \tag{6}$$

Where the exponent r *may be any (real or complex) number (**r is chosen so that $a_0 \neq 0$**).*

This provides an extension of power series method called as ***Frobenius Method.*** The method was known for German mathematician F.G. Frobenius (1849 – 1917).

7.4.3 Frobenius Method

To find the solution of equation (4) or (5) first expand $b(x)$ and $c(x)$ in power series:

$b(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots$

$c(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$

and consider the solution is given by equation (6)

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Differentiate this term by term

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} = x^{r-1}[a_0r + a_1(r+1)x + a_2(r+2)x^2 + a_3(r+3)x^3 + \cdots] \tag{7}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

$$= x^{r-2}[a_0r(r-1) + a_1(r+1)r\,x + a_2(r+2)(r+1)x^2 + a_3(r+3)(r+2)x^3 + \cdots] \tag{8}$$

Insert this in equation (5) we get

$$\begin{aligned} &x^r[a_0r(r-1) + a_1(r+1)r\,x + a_2(r+2)(r+1)x^2 + a_3(r+3)(r+2)x^3 + \cdots] \\ &+(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots)x^r[a_0r + a_1(r+1)x + a_2(r+2)x^2 + a_3(r+3)x^3 + \cdots] \\ &+(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots)x^r(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0. \end{aligned} \tag{9}$$

Equate the sum of the coefficients of each power of x i.e. $x^r, x^{r+1}, x^{r+2}, x^{r+3}, etc.$ to zero. This will give system of equations involving unknown coefficients a_i. Here **we are interested in only equation with lowest or smallest power of x**. The equation is:

(Coefficient of smallest power of x $i.e. x^r$)

$$[r(r-1) + b_0 r + c_0] a_0 = 0. \quad (10)$$

From equation (10) it is possible that either $a_0$is zero or the other term in square bracket is zero. We **choose r such that $a_0 \neq 0$.** Therefore, we get a quadratic equation in r

$$[r(r-1) + b_0 r + c_0] = 0. \quad (11)$$

called as ***Indicial equation*** of differential equation (4).

Let $r_1 and r_2$be the roots of the indicial equation thenbasis of the general solution depends on three cases:

- CASE I: distinct root not differing by an integer 1,2,3,…
  The basis is:
  $y_1 = x^{r_1}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$ and
  $y_2 = x^{r_2}(A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots)$.

- CASE II: A double root : $r_1 = r_2 = r = \frac{1-b_0}{2}$.
  The basis is:
  $y_1 = x^r(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$and
  $y_2 = y_1(x)\ ln(x) + x^r(A_1 x + A_2 x^2 + A_3 x^3 + \cdots)$for $x > 0$.

- CASE III: Roots differing by an integer 1,2,3,… , let $r_1 > r_2$
  $y_1 = x^{r_1}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$correspond to the larger root and
  $y_2 = k\ y_1(x)\ ln(x) + x^{r_2}(A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots)$.

Note: In case II and case III the second solution or basis $y_2$is obtained by Reduction of order method which is applied when one solution is known.

The general solution is $= c_1\, y_1(x) + c_2 y_2(x)$.$k$is some constant, may turn out to be zero.

SAQ3: Can we use Frobenius method to solve following differential equation

$$(x^2 - x)\, y'' + (3x - 1)\, y' + y = 0$$

7.5 Legendre differential equation

Any solution of Legendre Differential Equation:

$$(1-x^2)\,y'' - 2x\,y' + m(m+1)y = 0 \;\;; m\ is\ a\ constant\ integer \qquad (\mathbf{12})$$

is called a Legendre Function (Polynomial). Rewrite above equation (12) in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{m(m+1)}{1-x^2}y = 0. \qquad (\mathbf{13})$$

Compare equation (13) with equation (2) we can write

- $r(x) = 0$so Legendre equation is a **homogeneous** equation.
- $p(x) = -\frac{2x}{1-x^2}$ and $q(x) = \frac{m(m+1)}{1-x^2}$ turns out to be **singular** at $\boldsymbol{x = \pm 1}$and **analytic** at $\boldsymbol{x = 0}$.

Therefore solution lies in the interval $-1 < x < 1$, i.e. solution should be bounded (have finite terms) and not divergent.

7.5.1 Solution of Legendre equation

We can apply power series solution to the differential equation with $x_0 = 0$. The expansion of power series solution given by equation(3) i.e.$y = \sum_{n=0}^{\infty} a_n x^n$and its derivatives, which is given by equations (3.1 and 3.2), put this in equation (12) we get

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n\,a_n x^{n-1} + m(m+1)\sum_{n=0}^{\infty} a_n x^n = 0 \qquad (\mathbf{14})$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + m(m+1)\sum_{n=0}^{\infty} a_n x^n = 0 \quad (\mathbf{15})$$

Writing each series and arranging each power of x :

$$2\times 1\,a_2 + 3\times 2\,a_3 x + 4\times 3\,a_4 x^2 + 5\times 4\,a_5 x^3 + \cdots + (s+2)(s+1)a_{s+2}x^s + \cdots$$

$$+\,2\times 1\,a_2 x^2 + 3\times 2\,a_3 x^3 + 4\times 3\,a_4 x^4 + 5\times 4\,a_5 x^5 + \cdots + s(s-1)a_s x^s + \cdots$$

$$-2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots + s a_s x^s + \cdots)$$

$$+m(m+1)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_s x^s + \cdots)(\mathbf{16})$$

Collect the coefficients of same power of x:

Coefficient of x^0 : $2a_2 + m(m+1)a_0 = 0$

Coefficient of x : $3\times 2\,a_3 + [-2 + m(m+1)]a_1 = 0$

Coefficient of x^2 : $4\times 3\,a_4 + [2 - 4 + m(m+1)]a_2 = 0.$

So we can write a general term as

Coefficient of x^s : $(s+2)(s+1)a_{s+2} - s(s-1)a_s - 2sa_s + m(m+1)a_s = 0$ or,

$$\boldsymbol{(s+2)(s+1)a_{s+2} - [s(s-1) + 2s - m(m+1)]a_s = 0}. \qquad \mathbf{(17)}$$

The coefficient of second term of above equation can be written as

$$[s(s-1) + 2s - m(m+1)] = s^2 - s + 2s - m^2 - m = (s-m)(s+m) + (s-m)$$
$$= (s-m)(s+m+1) = -(m-s)(m+s+1).$$

So from equation (17) we get

$$\boldsymbol{a_{s+2} = \frac{-(m-s)(m+s+1)}{(s+2)(s+1)} a_s} \qquad \mathbf{(18)}$$

called as **recurrence relation or recurrence formula.** With $s = 0,1,2,3,4, ...$ we can get all the coefficients in terms of a_0 and a_1 which is an arbitrary constants.

- Put the value of $s = 0,1,2,3,4, ...$ in equation(18) we get

$$s = 0: \quad a_2 = -\frac{m(m+1)}{2!} a_0$$
$$s = 1: \quad a_3 = -\frac{(m-1)(m+2)}{3!} a_1$$
$$s = 2: \quad a_4 = \frac{(m-2)m(m+1)(m+3)}{4!} a_0$$
$$s = 3: \quad a_5 = \frac{(m-3)(m-1)(m+2)(m+4)}{5!} a_1$$
$$s = 4: \quad a_6 = -\frac{(m-4)(m-2)m(m+1)(m+3)(m+5)}{6!} a_0$$
$$s = 5: \quad a_7 = -\frac{(m-5)(m-3)(m-1)(m+2)(m+4)(m+6)}{7!} a_1$$

and so on so that we can write even coefficients (a_i for $i = 2,4,6, ...$) in terms of a_0 and odd coefficients (a_i for $i = 3,5,7, ...$) in terms of a_1.

Put these coefficients in power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

We get

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 - \frac{m(m+1)}{2!} a_0 x^2 + \frac{(m-2)m(m+1)(m+3)}{4!} a_0 x^4 + \cdots$$

$$+a_1x - \frac{(m-1)(m+2)}{3!}a_1x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!}a_1x^5 + \cdots \qquad (19)$$

Or the solution of differential equation can be expressed as

$$\boldsymbol{y = a_0y_1(x) + a_1\, y_2(x)} \ ; \quad \forall\ |x| < 1. \qquad (20)$$

Called **Legendre function** with

$$y_1 = 1 - \frac{m(m+1)}{2!}x^2 + \frac{(m-2)m(m+1)(m+3)}{4!}x^4 - \frac{(m-4)(m-2)m(m+1)(m+3)(m+5)}{6!}x^6 + \cdots \qquad (20.a)$$

$$y_1 = 1 + \sum_{j=1}^{\infty}(-1)^j \frac{[m(m-2)(m-4)\ldots(m-2j+2)][(m+1)(m+3)(m+5)\ldots(m+2j-1)]}{(2j)!}x^{2j} \qquad (20.a)$$

$$y_2 = x - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!}x^5 - \frac{(m-5)(m-3)(m-1)(m+2)(m+4)(m+6)}{7!}x^7 + \cdots \qquad (20.b)$$

$$y_2 = x + \sum_{j=1}^{\infty}(-1)^j \frac{[(m-1)(m-3)\ldots(m-2j+1)][(m+2)(m+4)(m+6)\ldots(m+2j)]}{(2j+1)!}x^{2j+1} \qquad (20.b)$$

The series converges for $|x| < 1$, therefore, the *radius of convergence is unity*. The two basis of the solution y_1 and y_2 are linearly independent as $y_1$consists even power of x and $y_2$consists odd power of x, so the ratio $\frac{y_1}{y_2}$ is not constant.

- If put $s = m$ in equation (18) we see that $a_{m+2} = 0$ which implies

$$a_{m+4} = 0, \quad a_{m+6} = 0, \quad a_{m+8} = 0, \quad a_{m+10} = 0, \ldots \ \textit{and so on}$$

<u>If m is even</u>: then from equation (20.a and 20.b) we see that $y_1$reduces to polynomial of degree m andy_2 diverges. For example

- Take $m = 0$, then Legendre functions

$\boldsymbol{y_1(x) = 1}$which is a polynomial of order zero,

and $y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ which diverges.

So the bounded solution of differential equation for $m = 0$ is $\boldsymbol{y = y_1(x) = 1.}$

- Take $m = 2$, then Legendre functions

  $\boldsymbol{y_1(x) = 1 - 3\,x^2}$a polynomial of order two

  and $y_2(x) = 2x - \frac{x^3}{3} - \frac{3x^7}{7} - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ which diverges.

  So the bounded solution of differential equation for $m = 2$ is $= \boldsymbol{y_1(x) = 1 - 3\,x^2}$ **.**

<u>If m is odd:</u> then from equation (20.a and 20.b) we see that y_2 reduces to polynomial of degree m and y_1 diverges. For example

- Take $m = 1$, then Legendre functions

$y_1(x) = 1 - \frac{1}{2}\, x \ln\left(\frac{1+x}{1-x}\right)$ which diverges

and$\boldsymbol{y_2(x) = x}$ which is a polynomial of order one.

So the bounded solution of differential equation for $m = 1$ is $\boldsymbol{y = y_2(x) = x}$.

- Take $m = 3$, then Legendre functions

  $y_1(x)$ diverges

  and $\boldsymbol{y_2(x) = x - \frac{5}{3}x^3}$which is a polynomial of order three.

  So the bounded solution of differential equation for $m = 3$ is $\boldsymbol{y = y_2(x) = x - \frac{5}{3}x^3}$.

7.5.2 Legendre Polynomials $\boldsymbol{P_m(x)}$

Solution to Legendre's equation are the polynomials $y_1(x)$ and $y_2(x)$ for various values of m gives an independent solution.

If $y_1(x)$ or $y_2(x)$ is the solution of differential equation then $c\, y_1(x)$ or $c\, y_2(x)$ (c is a constant) is also a solution. We shall choose the normalization constant c such that the polynomials (for various m) are normalized to have value unity at $x = 1$ $\; i.e.$ $P_m(1) = 1$ $(or\ cy(1) = 1)$. Such normalized polynomials are called as Legendre Polynomials.

So the few normalized polynomials are

$m = 0:\quad P_0(x) = 1$ which is $y_1(x) = 1\ and\ c = 1$

$m = 1:\quad P_1(x) = x$ which is $y_2(x) = x\ and\ c = 1$

$m = 2:\quad P_2(x) = \frac{1}{2}(3x^2 - 1)$ which is $y_1(x) = 1 - 3\,x^2\ and\ c = -\frac{1}{2}$

$m = 3:\quad P_3(x) = \frac{1}{2}\,(5x^3 - 3x)$ which is $y_2(x) = x - \frac{5}{3}x^3\ and\ c = -\frac{3}{2}$

$m = 4:\quad P_4(x) = \frac{1}{8}\,(35\,x^4 - 30\,x^2 + 3)$which is $y_1(x) = 1 - 10x^2 + 35x^4 and\ c = \frac{3}{8}$

$$m = 5:\quad P_5(x) = \frac{1}{8}(63\,x^5 - 70\,x + 15\,x) \qquad \mathbf{(21)}$$

One canformulate compact or general representation for polynomials in equation (21). To do that rewrite recurrence relation as

$$\boldsymbol{a_s = \frac{-(s+2)(s+1)}{(m-s)(m+s+1)} a_{s+2}} \quad ; \qquad [\boldsymbol{s \le m-2}]. \qquad \mathbf{(22)}$$

Then all non vanishing coefficients may be expressed in terms of coefficient $\boldsymbol{a_m}$of highest power of $x$of the polynomial. The coefficient a_m is then arbitrary. Choose $a_m = 1$ when $m = 0$ and in general

$$\boldsymbol{a_m = \frac{(2m)!}{2^m (m!)^2} = \frac{1 \times 3 \times 5 \times \dots (2m-1)}{m!}}\,. \qquad \mathbf{(23)}$$

For this choice of $\boldsymbol{a_m}$ all those polynomials will have the value 1at$x = 1$.

Now put $s = m - 2$ $(m \ge 2)$ in equation (22)

$$a_{m-2} = \frac{-(m)(m-1)}{(2)(2m-1)} a_m$$

put $\boldsymbol{a_m}$ from equation (23) we get

$$a_{m-2} = \frac{-(m)(m-1)}{(2)(2m-1)} \frac{(2m)!}{2^m (m!)^2} = -\frac{(2m-2)!}{2^m (m-1)!\ (m-2)!}$$

Similarly if put $s = m - 4$ $(m \geq 4)$ in equation (22) and with a_{m-2} we can find

$$a_{m-4} = \frac{-(m-2)(m-3)}{(4)(2m-3)} a_{m-2} = (-1)^2 \frac{(2m-4)!}{2^m\ 2!\ (m-2)!\ (m-4)!}$$

So in general when $\boldsymbol{m - 2l \geq 0}$ we can write for $\boldsymbol{s = m - 2l}$

$$\boldsymbol{a_{m-2l} = (-1)^l \frac{(2m-2l)!}{2^m\ l!\ (m-l)!\ (m-2l)!}} \quad (24)$$

so the Legendre polynomials of degree $\boldsymbol{m}$ denoted by $\boldsymbol{P_m(x)}$ can be written as

$$\boldsymbol{P_m(x) = \sum_{l=0}^{N} (-1)^l \frac{(2m-2l)!}{2^m\ l!\,(m-l)!\ (m-2l)!} x^{m-2l}} \quad (25)$$

where $\boldsymbol{N = \frac{m}{2}}$ if $\boldsymbol{m}$ is **even** and $\boldsymbol{N = \frac{m-1}{2}}$ if $\boldsymbol{m}$ is **odd** i.e. $\boldsymbol{N}$ is an **integer**= $\boldsymbol{\frac{m}{2}}$.

from equation (25) for integer m :

$m = 0 \Rightarrow N = 0:\quad P_0(x) = 1$

$m = 1 \Rightarrow N = 0:\quad P_1(x) = x$

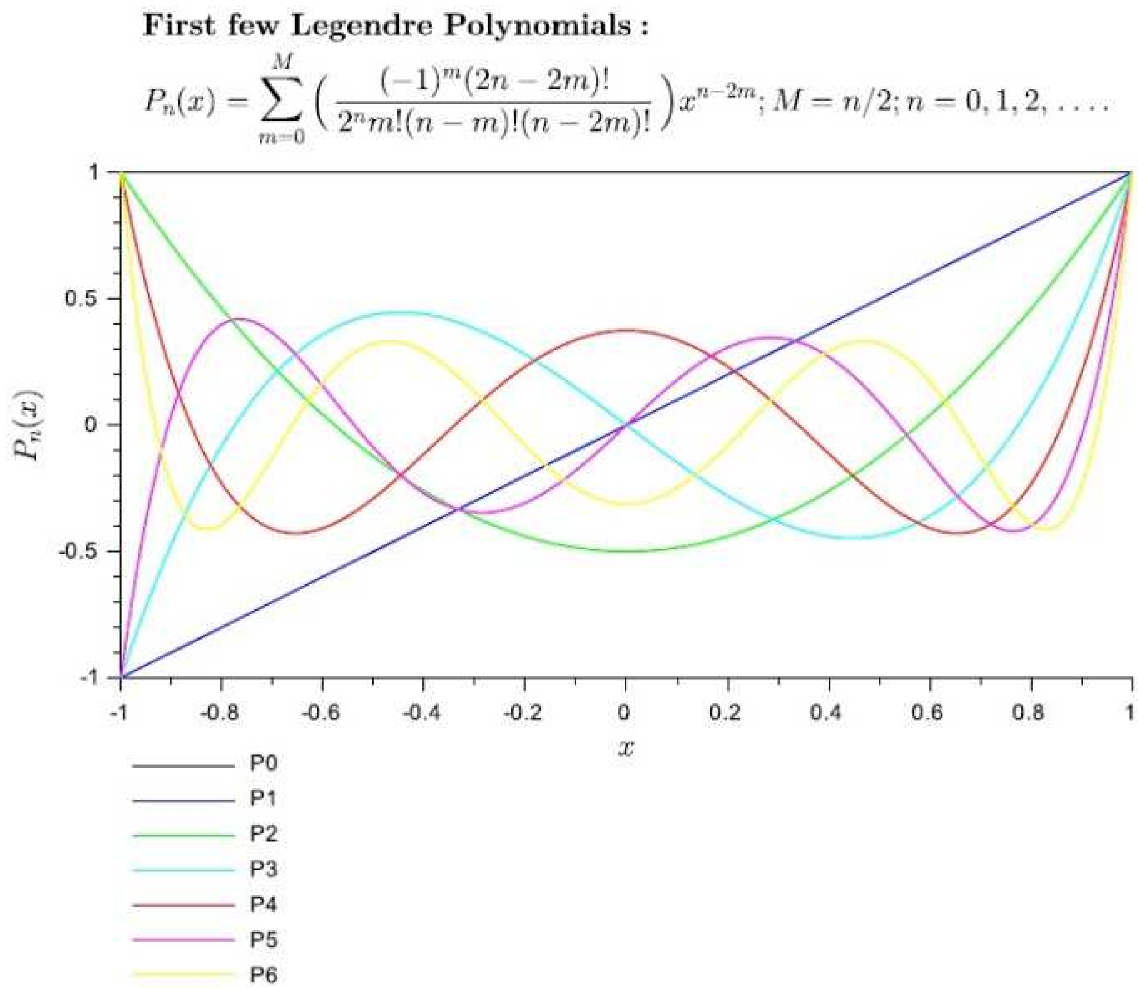
$m = 2 \Rightarrow N = 1:\quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$m = 3 \Rightarrow N = 1:\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$m = 4 \Rightarrow N = 2:\quad P_4(x) = \frac{1}{8}(35\,x^4 - 30\,x^2 + 3)$$

$m = 5 \Rightarrow N = 2:\quad P_5(x) = \frac{1}{8}(63\,x^5 - 70\,x + 15\,x)$

we can cross check that equation (21) is reproduced. We can plot the polynomials given by equation (25), first few have been plotted in the figure below



7.5.3 Rodrigue's Formula

We will try to find yet another solution of Legendre differential equation in terms m^{th} order derivative of some function and from this we will obtain Legendre polynomial.

Consider a function

$$y = (x^2 - 1)^m. \quad \mathbf{(26)}$$

Take derivative of equation (26)

$$\frac{dy}{dx} = 2\, m\, x\, (x^2 - 1)^{m-1}. \quad \mathbf{(27)}$$

Multiply both side of equation (27) by $(x^2 - 1)$

$$\frac{dy}{dx}(x^2 - 1) = 2\, m\, x\, (x^2 - 1)^m$$

$$= y\, 2\, m\, x. \quad \mathbf{(28)}$$

Differentiate both side of equation (28) $(m + 1)$ times by Leibniz's rule taking $u = \frac{dy}{dx}$, $v = (x^2 - 1)$ on L.H.S. and $u = y$, $v = 2mx$ on R.H.S..

$$\frac{d^{m+1}}{dx^{m+1}}\left[\frac{dy}{dx}(x^2-1)\right] = \frac{d^{m+1}}{dx^{m+1}}[y \quad 2\,m\,x\,]$$

$$\frac{d^{m+2}y}{dx^{m+2}}(x^2-1) + \ {}^{m+1}c_1\frac{d^{m+1}y}{dx^{m+1}}(2x) + \ {}^{m+1}c_2\frac{d^m y}{dx^m} \ 2$$
$$= \frac{d^{m+1}y}{dx^{m+1}}\,(2mx) + \ {}^{m+1}c_1\frac{d^m y}{dx^m}(2m)$$

$$(x^2-1)\frac{d^{m+2}y}{dx^{m+2}} + (2x)\frac{d^{m+1}y}{dx^{m+1}} - m(m+1)\frac{d^m y}{dx^m} = 0$$

or,

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} - (2x)\frac{d^{m+1}y}{dx^{m+1}} + m(m+1)\frac{d^m y}{dx^m} = 0$$

Let $\boldsymbol{V} = \boldsymbol{\frac{d^m y}{dx^m}}$ then we can write above equation as

$$(1-x^2)\frac{d^2V}{dx^2} - (2x)\frac{dV}{dx} + m(m+1)\,V = 0. \qquad \mathbf{(29)}$$

Equation (29) is Legendre equation and V is the solution this equation with

$$\boldsymbol{V = \frac{d^m y}{dx^m} = \frac{d^m(x^2-1)^m}{dx^m}}.$$

Now let the Legendre polynomial be

$$\boldsymbol{P_m(x) = CV = C\,\frac{d^m y}{dx^m} = C\frac{d^m(x^2-1)^m}{dx^m}}. \qquad \mathbf{(30)}$$

To find C, put $x = 1$ in equation (30) so that from the definition of Legendre polynomial we have

$$\boldsymbol{P_m(1) = 1 = C\left(\frac{d^m(x^2-1)^m}{dx^m}\right)_{x=1}}.$$

Now as we had considered

$y = (x^2-1)^m = (x-1)^m(x+1)^m.$

Differentiate both sides m times and use Leibniz rule (on R.H.S.) to get

$$\frac{d^m y}{dx^m} = \frac{d^m(x^2-1)^m}{dx^m}$$
$$= \frac{d^m(x+1)^m}{dx^m}(x-1)^m + m\frac{d^{m-1}(x+1)^m}{dx^{m-1}}\{m(x-1)^{m-1}\}$$
$$+ \ m(m+1)\frac{d^{m-2}(x+1)^m}{dx^{m-2}}\{m(m-1)(x-1)^{m-2}\} + \cdots$$
$$+ (x+1)^m\frac{d^m(x-1)^m}{dx^m}$$

$$\left(\frac{d^m y}{dx^m}\right)_{x=1} = \left(\frac{d^m (x^2-1)^m}{dx^m}\right)_{x=1} = 0+0+0+\cdots+2^m (m!)\left\{ as \ \ \frac{d^m (x-1)^m}{dx^m} = m!\right\}$$

hence, $P_m(1) = 1 = C\left(\frac{d^m (x^2-1)^m}{dx^m}\right)_{x=1} = C\, 2^m\ (m!)$ so we get $\boldsymbol{C} = \frac{\mathbf{1}}{\mathbf{2^m(m!)}}$therefore the Legendre polynomials are also obtained from

$$\boldsymbol{P_m(x) = \frac{1}{2^m(m!)}\frac{d^m(x^2-1)^m}{dx^m}} \qquad \mathbf{(31)}$$

well known as ***Rodrigue's Formula.***

7.6 Properties of Legendre Polynomial

7.6.1 Generating function

Since $\boldsymbol{P_m(x)}$is a sequence of polynomials, it may appear as coefficient of some particular series. We observe that on the expansion of a function

$$\boldsymbol{G(z,x) = \frac{1}{\sqrt{1-2xz+z^2}}} \qquad \mathbf{(32)}$$

as power series $\boldsymbol{P_m(x)}$is generated as a coefficient of z^n. Hence, equation (32) is expressed as generating function for Legendre Polynomials.

Proof: Let $w = 2xz - z^2$then $(z,x) = \frac{1}{\sqrt{1-w}}$, Now binomial expansion of this function is

$$\frac{1}{\sqrt{1-w}} = (1-w)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)(-w) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-w)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-w)^3 + \cdots$$

$$= 1 + \frac{1}{2}w + \frac{1\times 3}{2\times 4}w^2 + \frac{1\times 3\times 5}{2\times 4\times 6}w^3 + \cdots + \frac{1\times 3\times 5\times \ldots\times (2n-1)}{2\times 4\times 6\times \ldots \times 2n}w^n + \cdots$$

$$= 1 + \frac{1}{2}(2xz - z^2) + \frac{1\times 3}{2\times 4}(2xz-z^2)^2 + \frac{1\times 3\times 5}{2\times 4\times 6}(2xz-z^2)^3 + \cdots$$

$$= 1 + xz - \frac{z^2}{2} + \frac{3}{8}(4x^2 z^2 + z^4 - 4xz^3) + \frac{5}{16}(8x^3z^3 + z^6 - 8x^2z^4 - 4xz^5) + \cdots$$

$$= 1 + xz + \frac{1}{2}(3x^2-1)z^2 + \frac{1}{8}\left(-12xz^3 + \frac{5}{2}\times 8x^3z^3 - \frac{5}{2}4xz^5\right) + \cdots$$

$$= 1 + xz + \frac{1}{2}(3x^2-1)z^2 + \frac{1}{2}(5x^3-3x)z^3 + \cdots$$

$$= P_0(x) + P_1(x)z + P_2(x)z^2 + P_3(x)z^3 + \cdots$$

$$\boldsymbol{G(x,z) = (1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n.} \qquad \mathbf{(33)}$$

7.6.2 Orthogonality

If $\boldsymbol{P_m(x)}$and $\boldsymbol{P_n(x)}$are two Legendre polynomials then the orthogonality property is defined as

$$\int_{-1}^{1} \boldsymbol{P_m(x) P_n(x)\, dx} = \begin{cases} \boldsymbol{0} & \boldsymbol{if \quad n \neq m} \\ \boldsymbol{\dfrac{2}{2n+1}} & \boldsymbol{if \quad n = m} \end{cases}$$

Proof:

The Legendre equation is

$$(1 - x^2)\, y'' - 2x\, y' + n(n+1)y = 0$$

or

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0.$$

Since $\boldsymbol{P_m(x)}$and $\boldsymbol{P_n(x)}$are solutions of the Legendre equation, we can write

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \tag{34}$$

and

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \tag{35}$$

multiply equation (34) by $\boldsymbol{P_n(x)}$and equation (35) by $\boldsymbol{P_m(x)}$and subtract

$$P_m \frac{d}{dx}\left\{(1 - x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1 - x^2)\frac{dP_m}{dx}\right\} + P_n P_m[n(n+1) - m(m+1)] = 0.$$

Integrating above equation w.r.t. x from -1 to 1

$$\int_{-1}^{1} P_m \frac{d}{dx}\left\{(1 - x^2)\frac{dP_n}{dx}\right\} dx - \int_{-1}^{1} P_n \frac{d}{dx}\left\{(1 - x^2)\frac{dP_m}{dx}\right\} dx + (n - m)(n + m + 1)\int_{-1}^{1} P_n P_m dx = 0.$$

The first and second term will cancel each other so we get

$$(n - m)(n + m + 1)\int_{-1}^{1} P_n P_m dx = 0. \tag{36}$$

Case I: if $\boldsymbol{m \neq n}$then term outside the integration in equation (36) is not zero hence

$$\int_{-1}^{1} \boldsymbol{P_n P_m dx = 0} \qquad (for\ \boldsymbol{m \neq n}). \tag{37}$$

Case II: if $\boldsymbol{m = n}$then equation (36) is zero as term outside the integration is zero, however the integration may not be zero for this case. To find the value of integration $\boldsymbol{\int_{-1}^{1} P_n^2\, dx}$**,** start with the generating function

$$(1 - 2\,x\,z + z^2)^{-\frac{1}{2}} = \sum P_n z^n.$$

Squaring both side we get

$$(1-2\,x\,z\;+z^2)^{-1}=\left(\sum_n P_n z^n\right)\left(\sum_m P_m z^m\right)=\sum_n P_n^2 z^{2n}+\;2\sum_n\sum_{m\neq n} P_n P_m z^{n+m}\;(\mathbf{38})$$

The expression on R.H.S. has been separated into two terms, first for $m=n$ and second for $m\neq n$. The factor 2 with the second is due to repeated terms for each m and n.

For example: Let's look how it would appear, we will evaluate R.H.S. of equation (38) for $m=1{,}2$ and $n=1{,}2$ so R.H.S. is

$$\sum_{n=1}^{2} P_n z^n \sum_{m=1}^{2} P_m z^m \;=(P_1 z+P_2 z^2)\;(P_1 z+P_2 z^2)$$

$$=\;[P_{1\;1}^2 z^2+P_2^2 z^4]+[P_1 z\;P_2 z^2+P_2 z^2 P_1 z]$$

$$=\;[P_{1\;1}^2 z^2+P_2^2 z^4]+\;2\;[P_1 P_2 z^3]$$

$$=\sum_{n=1}^{2} P_n^2 z^{2n}\;+\;2\sum_{n=1}^{2}\;\sum_{m=1,m\neq n}^{2} P_n P_m z^{n+m}$$

Integrating equation (38) both side w.r.t. x from -1 to 1 we get

$$\int_{-1}^{1}\frac{1}{(1-2\,x\,z\;+z^2)}\;dx$$

$$=\int_{-1}^{1}\left(\sum_n P_n z^n\right)\left(\sum_m P_m z^m\right)dx$$

$$=\;\sum_n z^{2n}\int_{-1}^{1} P_n^2\;dx+\;2\sum_n\sum_{m\neq n}\;z^{n+m}\int_{-1}^{1} P_n P_m\;dx.$$

Using orthonality property second term on RHS is zero, so we get

$$-\frac{1}{2z}[ln(1-2\,x\,z\;+z^2)]_{-1}^{1}\;=\;\sum_n z^{2n}\int_{-1}^{1} P_n^2\;dx$$

$$\sum_n z^{2n}\int_{-1}^{1} P_n^2\;dx=\;-\frac{1}{2z}[\ln(1-z)^2-\ln(1+z)^2]=\frac{1}{z}[ln(1+z)-ln(1-z)]$$

$$=\frac{1}{z}\,[\ln\left(\frac{1+z}{1-z}\right)]$$

$$\sum_n z^{2n} \int_{-1}^{1} P_n^2 \, dx \; = \; \frac{2}{z}\left[z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots + \frac{z^{2n+1}}{2n+1} + \cdots \right]$$

$$= 2 \left[1 + \frac{z^2}{3} + \frac{z^4}{5} + \frac{z^6}{7} + \cdots + \frac{z^{2n}}{2n+1} + \cdots \right] = 2\left(\sum_n z^{2n} \left(\frac{1}{2n+1} \right) \right).$$

On comparing the coefficient of $\sum_n z^{2n}$ we get

$$\int_{-1}^{1} P_n^2 \, dx \; = \; \frac{2}{2n+1}$$

Or we can write

$$\int_{-1}^{1} \boldsymbol{P_n \, P_m \, dx} \; = \; \frac{\mathbf{2}}{\mathbf{2n+1}} \qquad\qquad for \;\; \boldsymbol{m = n}. \qquad\qquad (\mathbf{39})$$

Combining equation (37) and equation (39) we get orthoganility property

$$\int_{-1}^{1} \boldsymbol{P_m(x) P_n(x) \, dx} \; = \begin{cases} \mathbf{0} & \boldsymbol{if \quad n \neq m} \\ \dfrac{\mathbf{2}}{\mathbf{2n+1}} & \boldsymbol{if \quad n = m} \end{cases}.$$

7.6.3 Recurrence relations

Differentiating generating function $\boldsymbol{(1 - 2xz + z^2)^{-\frac{1}{2}} \; = \; \sum_{n=0}^{\infty} P_n(x) \, z^n}$both side w.r.t.$\boldsymbol{Z}$, we get

$$-\frac{1}{2}(1 - 2xz + z^2)^{-\frac{3}{2}} (-2x + 2z) = \sum_{n=1}^{\infty} n \, P_n(x) \, z^{n-1}$$

$$(x - z)(1 - 2xz + z^2)^{-\frac{1}{2}} \; = (1 - 2xz + z^2) \sum_{n=1}^{\infty} n \, P_n(x) \, z^{n-1}.$$

Use generating function to replace 2nd factor on LHS

$$(x - z) \sum_{n=0}^{\infty} P_n(x) \, z^n \; = (1 - 2xz + z^2) \sum_{n=1}^{\infty} n \, P_n(x) \, z^{n-1}$$

$$x \sum_{n=0}^{\infty} P_n(x) z^n - \sum_{n=0}^{\infty} P_n(x) z^{n+1}$$

$$= \sum_{n=1}^{\infty} n P_n(x) z^{n-1} - 2x \sum_{n=1}^{\infty} n P_n(x) z^n + \sum_{n=1}^{\infty} n P_n(x) z^{n+1} \qquad (\mathbf{40})$$

Expanding equation (40) both sides

$$[x\left(P_0(x)+P_1(x)z+\cdots+P_{i-1}(x)z^{i-1}+P_i(x)z^i+\cdots\right)]-[(P_0(x)z+\cdots+P_{i-2}(x)z^{i-1}$$
$$+P_{i-1}(x)z^i+\cdots)]$$
$$=[P_1(x)+2P_2(x)z+\cdots+iP_i(x)\,z^{i-1}+(i+1)P_{i+1}(x)\,z^i+\cdots]$$
$$-[2x\,(P_1(x)\,z+\,2P_2(x)\,z^2+\cdots+(i-1)P_{i-1}(x)\,z^{i-1}+(i)P_i(x)\,z^i+\cdots)]$$
$$+[P_1(x)\,z^2+2P_2(x)\,z^3+\cdots+(i-2)P_{i-2}(x)\,z^{i-1}+(i-1)P_{i-1}(x)\,z^i+\cdots]$$

and equating **coefficient of z^{i-1}**

$$xP_{i-1}(x)-P_{i-2}(x)=\,iP_i(x)-2x(i-1)P_{i-1}(x)+(i-2)P_{i-2}(x)$$

and replacing i by n

$$x\,P_{n-1}(x)-P_{n-2}(x)=\,nP_n(x)-2x\,(n-1)P_{n-1}(x)+(n-2)P_{n-2}(x)$$

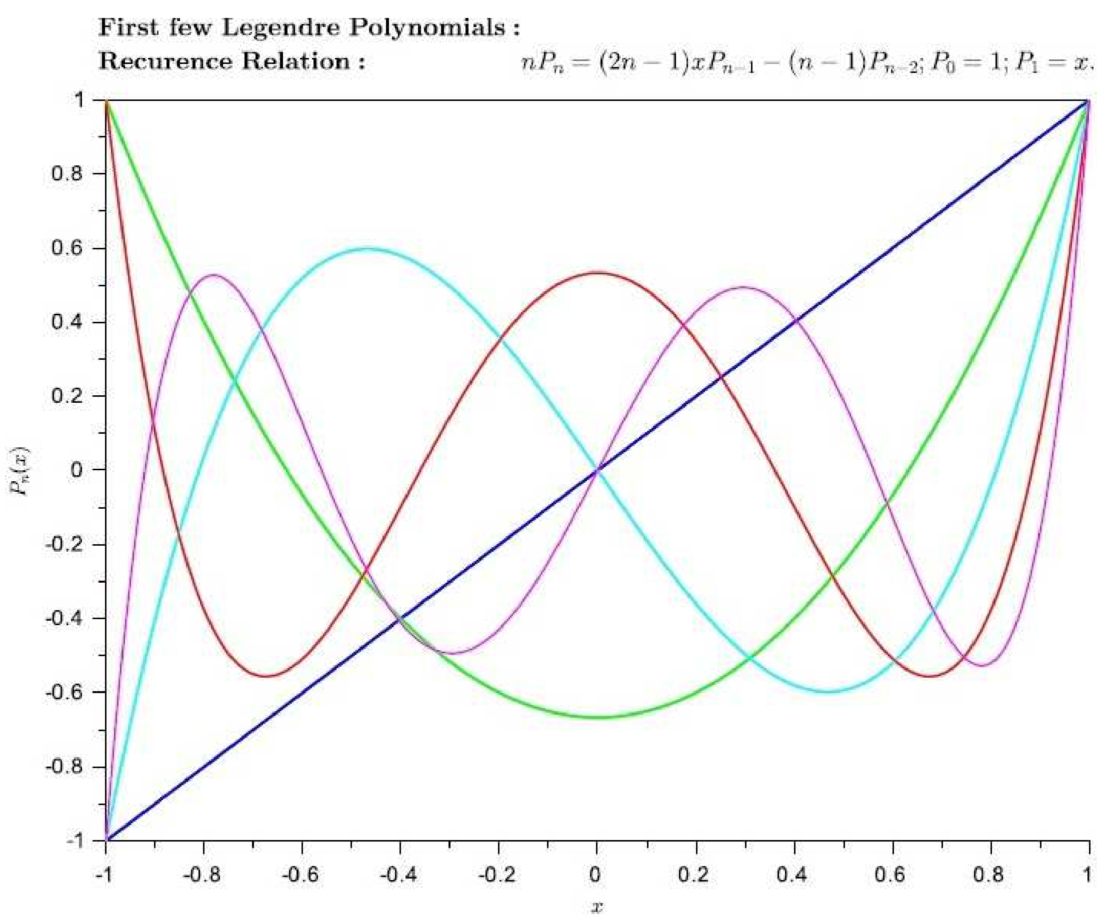
We get one of the recurrence relations from ***coefficient of z^{n-1}*** as

$$\boldsymbol{nP_n(x)=\,2(n-1)\,x\,P_{n-1}(x)-(n-1)P_{n-2}}\,for\ n=2,3,4,\ldots\ \mathbf{(41)}$$

if we equate ***coefficient of z^i*** in the expansion of equation (40) and replacing i by $n$we get other recurrence relation

$$\boldsymbol{(n+1)P_{n+1}(x)=\,2(n+1)\,x\,P_n(x)-nP_{n-1}}\,for\ n=1,2,3,4,\ldots\ \mathbf{(42)}$$

Again first few Legendre polynomialshave been shown in the figurebelow but this time we used recurence relation (41) to draw the figure. The figure has been generated using Sci-Lab package.



SAQ4: Show that $\boldsymbol{nP_n(x) = xP'_n(x) - P'_{n-1}(x)}$ where $\boldsymbol{P'_n(x) = \frac{dP_n(x)}{dx}}$

7.7 Bessel's differential equation

The Bessel's differential equation of order ν(a real and non-negative number) is

$$\boldsymbol{x^2y'' + x\,y' + (x^2 - \nu^2)y = 0}$$

can be rewritten in the form of equation (2) as

$$\boldsymbol{y'' + \frac{1}{x}y' + \frac{x^2 - \nu^2}{x^2}y = 0.}$$

Compare above equation with equation (2) we can write

- $r(x) = 0$so Bessel's equation is a **homogeneous** equation.
- $p(x) = \frac{1}{x}$ and $q(x) = \frac{x^2-\nu^2}{x^2}$ turns out to be **singular** at $\boldsymbol{x = 0}$.
- However in the $\lim_{x\to 0} x\,p(x)$ and $\lim_{x\to 0} x^2q(x)$ are finite at $\boldsymbol{x = 0}$.

Therefore at $x = 0$ the equation has regular singularity, hencesolution can be obtained using Frobenious method. The solution will lie in the interval ($0, \infty$).

7.7.1 Solution of Bessel's equation

Solution: Using Frobenious method we can find the solution of the Bassel's function, let the solution be

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}\ .$$

The first and second derivative of y is

$$y' = \sum_{m=0}^{\infty} (m + r)\ a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2}.$$

Replace $y, y'and\ y''$ in Bessel's differential equation

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)\,a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r} + (x^2-\nu^2)\sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)\,a_m x^{m+r-2} + \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r+2} - \nu^2\sum_{m=0}^{\infty}a_m x^{m+r} = 0. \quad (\mathbf{43})$$

- To find indicial equation, equate the coefficient of lowest power of x i.e. x^r for $m = 0$, we get

$$r\,(r-1)a_0 + r\,a_0 - \nu^2 a_0 = 0. \quad (\mathbf{44})$$

The lowest power of x gives the **indicial equation** so we get

$$[\boldsymbol{r}\,(\boldsymbol{r}-\mathbf{1}) + \boldsymbol{r} - \boldsymbol{\nu}^{\mathbf{2}}]\boldsymbol{a}_{\mathbf{0}} = \mathbf{0} \quad (\mathbf{45})$$

with choice $a_0 \neq 0$

$$[\boldsymbol{r}\,(\boldsymbol{r}-\mathbf{1}) + \boldsymbol{r} - \boldsymbol{\nu}^{\mathbf{2}}] = \boldsymbol{r}^{\mathbf{2}} - \boldsymbol{\nu}^{\mathbf{2}} = (\boldsymbol{r}+\boldsymbol{\nu})(\boldsymbol{r}-\boldsymbol{\nu}) = \mathbf{0}.$$

We get the solution for r as

$$\boldsymbol{r}_{\mathbf{1}} = \boldsymbol{\nu} \quad \boldsymbol{and} \quad \boldsymbol{r}_{\mathbf{2}} = -\boldsymbol{\nu}.$$

- To find $\boldsymbol{a}_{\mathbf{1}}$, equate the coefficient of next lowest power of x i.e. x^{r+1} for $m = 1$, to get

$$(r+1)\,r\,a_1 + (r+1)\,a_1 - \nu^2 a_1 = 0 \quad (\mathbf{46})$$

$$[(r+1)\,r + (r+1) - \nu^2]\,a_1 = 0. \quad (\mathbf{47})$$

The term in square bracket is not zero for the solution of $\boldsymbol{r}$ obtained from indicial equation therefore we get

$$\boldsymbol{a}_{\mathbf{1}} = \mathbf{0}.$$

- To find recursion relation, equate the coefficients of x^{s+r} to zero

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0. \quad (\mathbf{48})$$

Rearrange the terms to get recursion relation as

$$\boldsymbol{a}_{\boldsymbol{s}} = -\frac{\mathbf{1}}{((\boldsymbol{s}+\boldsymbol{r})^{\mathbf{2}} - \boldsymbol{\nu}^{\mathbf{2}})}\boldsymbol{a}_{\boldsymbol{s}-\mathbf{2}}\ for\ s = 2,3,4,\ldots \quad (\mathbf{49})$$

Now we will find first Bessel function for first solution of indicial equation**i.e.** $\boldsymbol{r = r_1 = \nu}$**,** put this in equation (49) to get

$$\boldsymbol{a_s = -\frac{1}{((s+\nu)^2 - \nu^2)} a_{s-2} = -\frac{1}{(s^2 + 2\,s\,\nu)} a_{s-2}} \quad for\ s = 2,3,4, \ldots \qquad (\mathbf{50})$$

Since $a_1 = 0$ we get $a_3 = 0, a_5 = 0, a_7 = 0, \ldots \ldots$, so we get all the odd coefficients zero as all the odd coefficients can be written in terms of a_1 from equation (50). So only even coefficients are non-zero i.e. for $= 2m\ \ with\ \ m = 1,2,2,3$ .... . From eq. (50) **for** $\boldsymbol{s = 2m}$ we get

$$\boldsymbol{a_{2m} = -\frac{1}{4m^2 + 4\,\nu\,m} a_{2m-2} = -\frac{1}{2^2 m(\nu + m)} a_{2m-2}} \qquad (\mathbf{51})$$

m =1 :$a_2 = -\frac{a_0}{2^2(\nu+1)}$

m =2 :$a_4 = -\frac{a_2}{2^2\ 2\ (\nu+2)} = \frac{a_0}{2^4(2!)(\nu+1)(\nu+2)}$

m =3 :$a_6 = -\frac{a_4}{2^2\ 3\ (\nu+3)} = -\frac{a_0}{(2^2 2^4)\ (3\times2!)(\nu+1)(\nu+2)(\nu+3)} = -\frac{a_0}{2^6(3!)(\nu+1)(\nu+2)(\nu+3)}$

in general we can write

$$\boldsymbol{a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)(\nu+1)(\nu+2)(\nu+3) \ldots\ldots\ldots (\nu+m)}}. \qquad for\ \ m = 1,2,3 \ldots \qquad (\mathbf{52})$$

Now $\boldsymbol{a_0}$is still arbitrary so for simplicity take $\boldsymbol{a_0} = \frac{\mathbf{1}}{(2^\nu\ \nu!)}$put in eq.(52) to get

$$\boldsymbol{a_{2m} = \frac{(-1)^m}{2^{2m+\nu}(m!)(\nu+m)!}} for\ m = 1,2,3, \ldots \qquad (\mathbf{53})$$

The solution for$\boldsymbol{r = r_1 = \nu}$is denoted by $\boldsymbol{J_\nu(x)}$**,** we have assumed solution to be $y = \sum_{m=0}^{\infty} a_m x^{m+r}$ from Frobenious method. Replace the coefficient a_m by a_{2m} (as the odd coefficients are zero) to get the Bessel function

$$\boldsymbol{J_\nu(x) = x^\nu \sum_{m=0} a_{2m} x^{2m}}$$

$$\boldsymbol{J_\nu(x) = \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}}. \qquad (\mathbf{54})$$

This is called BESSEL FUNCTION of First Kind of order $\boldsymbol{\nu}$**.**

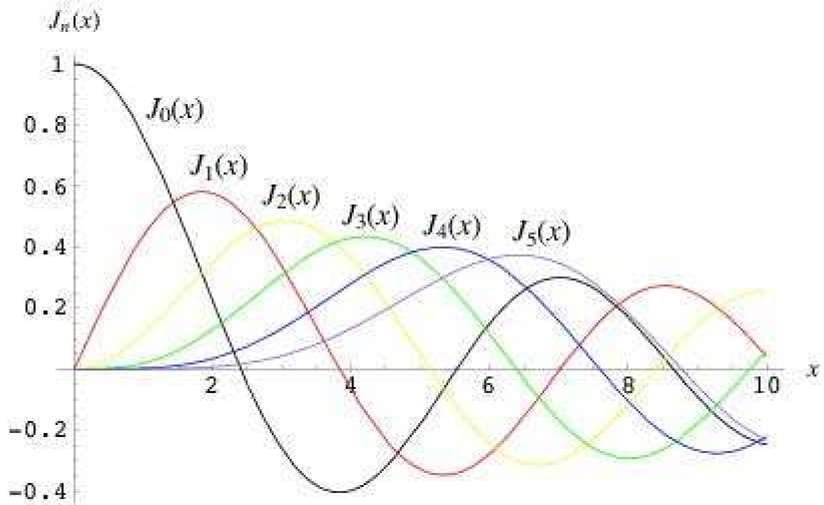


Figure 1: Graphical representation of several Bessel's function. (Source: http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html)

Second Bessel function for other solution of indicial equation **i.e.** $r = r_2 = -\nu$, which can be obtained by replacing ν by $-\nu$ in Eq. (54).

$$J_{-\nu}(x) = \sum_{m=0} \frac{(-1)^m x^{2m-\nu}}{2^{2m-\nu}(m!)(-\nu+m)!} \qquad (for\ \nu\ is\ \mathbf{not\ an\ integer}) \qquad (55)$$

This is called BESSEL FUNCTION of second Kind of order $-\nu$.

So $for\ \nu\ is\ \mathbf{not\ an\ integer}$ the general solution of Bessel differential equation is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \qquad (56)$$

If ν is an integer then Eq. (56) is not a general solution as J_ν and $J_{-\nu}$ are not linearly independent which can be shown as follows

- ***Linear dependence of J_ν and $J_{-\nu}$ when ν is an integer*:**
  Put $k = -\nu + m$ then $m = k + \nu$, put this in Eq.(55) for $J_{-\nu}(x)$ **(with ν to be integer**, summation on **m will start from ν** and not from zero as for any value of $m = 0\ to\ \nu$ the term $(m-\nu)! = \Gamma(m-\nu+1)$ **becomes infinite,** hence summation over **k will start from 0).**

$$J_{-\nu}(x) = \sum_{k=0} \frac{(-1)^{k+\nu} x^{2k+\nu}}{2^{2k+\nu}(m!)(k+\nu)!} = (-1)^\nu \sum_{k=0} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu}(m!)(k+\nu)!}$$

$J_{-\nu}(x) = (-1)^\nu J_\nu(x)$ therefore, If ν is an integer J_ν and $J_{-\nu}$ are linearly dependent.

When ν in an integer i.e. the roots of indicial equation differ by an integer then the

- First solution is $y_1(x) = J_\nu(x)$ and
- second linearly independent solution is given as CASE III of solution of indicial equation when solution is obtained using Frobenous method

$$y_2(x) = A_\nu \ln(x) J_\nu(x) + x^{-\nu} \sum_{k=0} A_k x^{k+\nu}$$

The second solution or basis $y_2$is obtained by Reduction of order method which is applied when one solution is known.

For many purposes, it is convenient to take the linear combination

$$y_2 = Y_\nu(x) = \frac{\left(\cos(\nu\pi) J_{-\nu}(x) - J_{-\nu}(x)\right)}{\sin(\nu\pi)}$$

as the second independent solution instead of $J_{-\nu}(x)$. This is known as the Bessel function of second kind of order ν. Hence the general solution is given as

$$\boldsymbol{y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)} \qquad (\mathbf{57})$$

Bessel function of order zero ($\boldsymbol{\nu = 0}$), $\boldsymbol{J_0(x)}$:

- From indicial equation Eq. (45) we find the solution to be $r^2 = 0$, hence double root $r_1 = r_2 = 0$.
- From recursion relation Eq. (49)

$$\boldsymbol{a_s = -\frac{1}{(s+r)^2} a_{s-2}} \; for\; s = 2,3,4, \ldots \qquad (\mathbf{58})$$

Hence from Eq. (52) one can write even coefficients as

$$\boldsymbol{a_{2m} = \frac{(-1)^m a_0}{(r+2)^2 (r+4)^2 (r+6)^2 \ldots \ldots (r+2m)^2}} \; for\; m \geq 1$$

$$\boldsymbol{a_{2m} = \frac{(-1)^m a_0}{2^{2m} (m!)^2}} \qquad for\; r = 0 \;\; and \;\; m = 1,2,3 \ldots \qquad (\mathbf{59})$$

Therefore, the solution is

$$\boldsymbol{y_r = x^r a_0 \left[1 + \sum_{m=1} \frac{(-1)^m x^{2m}}{(r+2)^2 (r+4)^2 (r+6)^2 \ldots \ldots (r+2m)^2}\right]}$$

and for $r = 0.$,

$$y_{r=0} = a_0 \left[1 + \sum_{m=1} \frac{(-1)^m x^{2m}}{(2)^2(4)^2(6)^2 \dots \dots (2m)^2} \right]$$

$$or, \qquad J_0(x) = a_0 \left[1 + \sum_{m=1} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \right]. \qquad (60)$$

This is Bessel Function of the first kind of order zero.

7.7.2Recurrence relations

Following relations are the recurrence formulae and their proof for Bessel's functions:

1. $\left(x^\nu J_\nu(x)\right)' = x^\nu J_{\nu-1}(x)$

Proof:

$$\frac{d\left(x^\nu J_\nu(x)\right)}{dx} = \frac{d}{dx}\left\{x^\nu \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}\right\} = \frac{d}{dx}\left\{\sum_{m=0} \frac{(-1)^m x^{2(m+\nu)}}{2^{2m+\nu}(m!)(\nu+m)!}\right\}$$

$$\left(x^\nu J_\nu(x)\right)' = \sum_{m=0} \frac{2(m+\nu)(-1)^m x^{2(m+\nu)-1}}{2^{2m+\nu}(m!)(\nu+m)!} = x^\nu \sum_{m=0} \frac{(m+\nu)(-1)^m x^{2m+\nu-1}}{2^{2m+\nu-1}(m!)(\nu+m)(m+\nu-1)!}$$

$$= x^\nu \sum_{m=0} \frac{(-1)^m x^{2m+\nu-1}}{2^{2m+\nu-1}(m!)\ (m+\nu-1)!} = x^\nu J_{\nu-1}(x).$$

2. $\left(x^{-\nu} J_\nu(x)\right)' = -x^{-\nu} J_{\nu+1}(x)$

Proof:

$$\frac{d\left(x^{-\nu} J_\nu(x)\right)}{dx} = \frac{d}{dx}\left\{x^{-\nu} \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}\right\} = \frac{d}{dx}\left\{\sum_{m=0} \frac{(-1)^m x^{2(m)}}{2^{2m+\nu}(m!)(\nu+m)!}\right\}$$

$$\left(x^{-\nu} J_\nu(x)\right)' = \sum_{m=0} \frac{2m\,(-1)^m x^{2(m)-1}}{2^{2m+\nu}(m!)(\nu+m)!} = x^{-\nu} \sum_{m=0} \frac{(-1)^m x^{2(m)+\nu-1}}{2^{2m+\nu-1}(m-1)!\ (\nu+m)!}$$

for $m - 1 = k\ i.e. m = k + 1$

$$\left(x^{-\nu} J_\nu(x)\right)' = x^{-\nu} \sum_{k=0} \frac{(-1)^{k+1} x^{2k+\nu+1}}{2^{2k+\nu+1}(k)!\ (\nu+k+1)!} = -x^{-\nu} \sum_{k=0} \frac{(-1)^k x^{2k+\nu+1}}{2^{2k+\nu+1}(k)!\ (k+\nu+1)!}$$

$$= -x^{-\nu} J_{\nu+1}(x).$$

3. $\boldsymbol{J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)}$ **or** $\boldsymbol{x\,J'_\nu(x) = \nu\, J_\nu(x) - x\,J_{\nu+1}(x)}$

Proof:

$$\frac{d(J_\nu(x))}{dx} = J'_\nu(x) = \frac{d}{dx}\left\{\sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}\right\} = \sum_{m=0} \frac{(2m+\nu)(-1)^m x^{2m+\nu-1}}{2^{2m+\nu}(m!)(\nu+m)!}.$$

Multiplying both side by x

$$x\,J'_\nu(x) = \sum_{m=0} \frac{(2m+\nu)(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}$$

$$= \sum_{m=0} \frac{2m\,(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!} + \nu \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}$$

$$= \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu-1}(m-1)!\;\;(\nu+m)!} + \nu\,J_\nu(x)$$

for $m-1 = k\; i.e. m = k+1$

$$x\,J'_\nu(x) = \sum_{k=0} \frac{(-1)^{k+1} x^{2k+\nu+2}}{2^{2k+\nu+1}(k)!\;\;(\nu+1+k)!} + \nu\,J_\nu(x)$$

$$= -x\sum_{k=0} \frac{(-1)^k x^{2k+\nu+1}}{2^{2k+\nu+1}(k)!\;\;(\nu+1+k)!} + \nu\,J_\nu(x) = -xJ_{\nu+1} + \nu\,J_\nu(x).$$

4. $\boldsymbol{J'_\nu(x) = \frac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)}$.

SAQ 5: Show that

$$\boldsymbol{x\,J'_\nu(x) = -\nu\, J_\nu(x) + x\,J_{\nu-1}(x)}.$$

5. $\boldsymbol{J_\nu(x) = \frac{x}{2\nu}\left(J_{\nu-1}(x) + J_{\nu+1}(x)\right)}$.

7.7.3 Generating function

The Bessel polynomials $J_n(x)$ can be expressed as**coefficients of** $\boldsymbol{t^n}$ in the series expansion of a function $\boldsymbol{\exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right)}$ called "Generating function'.

$$\mathbf{exp}\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \qquad (61)$$

Proof:

$$\exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right) = e^{\frac{xt}{2}}e^{-\frac{x}{2t}} = \left[1+\frac{xt}{2}+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\cdots\right]\times\left[1-\frac{x}{2t}+\frac{1}{2!}\left(\frac{x}{2t}\right)^2-\cdots\right]$$

the coefficient of t^n in this product is

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+1)!}\left(\frac{x}{2}\right)^{n+4} - \cdots = J_n(x)$$

as all the integral powers of t , both positive and negative occurs, we have

$$\exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right) = J_0(x) + t\,J_1(x) + t^2 J_2(x) + \cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \cdots$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$$

Thus, the coefficients of different powers of t in the expansion of $\exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right)$ give Bessel's functions of various orders, hence it is said to be the generating function of Bessel's functions.

7.7.4 Orthogonality

Ifλ andμ are the roots of the equation $J_n(\alpha) = 0$ then condition of orthogonality of Bessel's function over the interval $(0,1)$ with weight function is$x$ is

$$\int_0^1 x\, J_n(\lambda x) J_n(\mu x) dx = \begin{cases} 0 & if \quad \lambda \neq \mu \\ \frac{1}{2}J_{n+1}^2(x) & if \quad \lambda = \mu \end{cases} \qquad (62)$$

Or both the cases can be written in terms of delta function

$$\int_0^1 x\, [J_n(\mu x)]^2\, dx = \frac{1}{2}J_{n+1}^2(x)\delta_{\lambda\mu}$$

Proof: The second order Bessel's differential equation is

$$x^2 y'' + x\, y' + (x^2 - n^2)y = 0$$

Let us change the independent variable$x$to λx, the resulting equation is

$$x^2y'' + x\,y' + (\lambda^2x^2 - n^2)y = 0$$

With the general solution from equation (57) be $C_1J_n(\lambda x) + C_2Y_n(\lambda x)$.

Let $y_1(x) = J_n(\lambda x)$ and $y_2(x) = J_n(\mu x)$ are the solutions of the equation then we have

$$x^2y_1'' + x\,y_1' + (\lambda^2x^2 - n^2)y_1 = 0 \tag{63}$$

and

$$x^2y_2'' + x\,y_2' + (\mu^2x^2 - n^2)y_2 = 0 \tag{64}$$

multiply equation (63) by y_2 and eq.(64) by y_1 and subtract to get

$$(\mu^2 - \lambda^2)x\,y_1y_2 = x\frac{d}{dx}\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right] + \left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right] = \frac{d}{dx}\left(x\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right]\right)$$

integrating on x with limit $0\ to\ 1$ we get

$$(\mu^2 - \lambda^2)\int_0^1 x\,y_1y_2\,dx = \int_0^1 \frac{d}{dx}\left(x\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right]\right)dx = \left(x\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right]\right)_0^1$$

Since $y_1(x) = J_n(\lambda x)$and $y_2(x) = J_n(\mu x)$, replace these in above equation

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,dx = \frac{\left(x\left[J_n(\mu x)\frac{d\,J_n(\lambda x)}{dx} - J_n(\lambda x)\frac{d\,J_n(\mu x)}{dx}\right]\right)_0^1}{(\mu^2 - \lambda^2)}$$

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,dx = \frac{\left(\lambda\,J_n(\mu x)J_n'(\lambda x) - \mu\,J_n'(\mu x)J_n(\lambda x)\right)}{\mu^2 - \lambda^2} \qquad for\ \lambda \neq \mu \tag{65}$$

Therefore, in order to ensure orthogonality we must have $\lambda\ and\ \mu$ be zeros of $J_n(x)$, i.e.$J_n(\lambda) = J_n(\mu) = 0$ then for $\lambda \neq \mu$ in eq. (65) we have

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,dx = 0.$$

For $\boldsymbol{\lambda = \mu}$ if we take the limit then we get RHS of eq. (65) of the form $\frac{0}{0}$. So we apply L'Hospital rule (differentiate w.r.t. μ keeping λ constatnt) and find

$$\int_0^1 x\,J_n^2(\lambda x)\,dx = \lim_{\mu\to\lambda}\frac{\left(\lambda\,J_n'(\lambda)J_n'(\mu) - J_n(\lambda)J_n'(\mu) - \mu J_n(\lambda)J_n''(\mu)\right)}{2\mu}$$

$$= \frac{\left(\lambda\,J_n'(\lambda)J_n'(\lambda) - J_n(\lambda)J_n'(\lambda) - \lambda J_n(\lambda)J_n''(\lambda)\right)}{2\lambda}$$

Since $J_n(\lambda) = 0$ we have

$$\int_0^1 x\,J_n^2(\lambda x)\,dx = \lambda\frac{J_n'(\lambda)J_n'(\lambda)}{2\lambda} = \frac{1}{2}J_n'(\lambda)J_n'(\lambda) = \frac{1}{2}J_{n+1}^2(\lambda).$$

Since $J_{n+1}(\lambda) \neq 0$, thus we have the orthogonality condition.

Steps to show $J_n'(\lambda) = -J_{n+1}(\lambda)$

With the recurrence relation

$$x\, J_n'(x) = n\, J_n(x) - x\, J_{n+1}(x)$$

change the independent variable $x \; to \; \lambda x$

$$\lambda x\, J_n'(\lambda x) = n\, J_n(\lambda x) - \lambda x\, J_{n+1}(\lambda x)$$

For $x = 1$we have

$$\lambda\, J_n'(\lambda) = n\, J_n(\lambda) - \lambda\, J_{n+1}(\lambda) = -\lambda\, J_{n+1}(\lambda) \quad since \;\; J_n(\lambda) = 0.$$

7.8 Summary

- **Legendre differential equation**

$$(1 - x^2)\, y'' - 2x\, y' + m(m+1)y = 0 \;\; ; \; m \; is \; a \; constant \; integer$$

- **Legendre polynomials** of degree $\boldsymbol{m}$ denoted by $\boldsymbol{P_m(x)}$can be written as

$$\boldsymbol{P_m(x) = \sum_{l=0}^{N} (-1)^l \frac{(2m-2l)!}{2^m\, l!\,(m-l)!\;(m-2l)!} x^{m-2l}}$$

where$\boldsymbol{N} = \frac{\boldsymbol{m}}{2}if\boldsymbol{m}$is **even**and $\boldsymbol{N} = \frac{\boldsymbol{m-1}}{2}$ if$\boldsymbol{m}$is **odd** i.e. $\boldsymbol{N}$is an **integer**$= \frac{\boldsymbol{m}}{2}$ **.**

- **Properties of Legendre polynomials**
  - **Rodrigue's Formula**

$$\boldsymbol{P_m(x) = \frac{1}{2^m(m!)} \frac{d^m(x^2-1)^m}{dx^m}}.$$

  - **Generating function**

$$\boldsymbol{G(x,z) = (1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n}.$$

  - **Orthogonality**

$$\boldsymbol{\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{2}{2n+1} & if \quad n = m \end{cases}}.$$

  - **Recurrence relations**
    - $\boldsymbol{nP_n(x) = 2(n-1)\, x\, P_{n-1}(x) - (n-1)P_{n-2}} for \; n = 2,3,4,\ldots$
    - $\boldsymbol{(n+1)P_{n+1}(x) = 2(n+1)\, x\, P_n(x) - nP_{n-1}} for \; n = 1,2,3,4,\ldots$
    - $\boldsymbol{nP_n(x) = xP_n'(x) - P_{n-1}'(x)}$

- $(n+1)\, P_{n+1}(x) = (2n+1)xP_n(x) - n\, P_{n-1}(x)$.

- **Bessel's differential equation**
$$x^2y'' + x\,y' + (x^2 - \nu^2)y = 0.$$
- **Bessel function of First Kind of order ν :**
$$J_\nu(x) = \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}.$$
- **Bessel function of Second Kind of order**
$-\nu J_{-\nu}(x) = \sum_{m=0} \frac{(-1)^m x^{2m-\nu}}{2^{2m-\nu}(m!)(-\nu+m)!}$ $\quad (for\ \nu\ is\ \textbf{not an integer})$
- **Bessel Function of the first kind of order zero:**
$$J_0(x) = a_0\left[1 + \sum_{m=1} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right].$$
- **Properties of Bessel function**
  - **Recurrence Formula**
    - $\left(x^\nu J_\nu(x)\right)' = x^\nu J_{\nu-1}(x)$
    - $\left(x^{-\nu} J_\nu(x)\right)' = -x^{-\nu} J_{\nu+1}(x)$
    - $J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x)$
    - $x\,J'_\nu(x) = -\nu\, J_\nu(x) + x\,J_{\nu-1}(x)$
    - $J'_\nu(x) = \frac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$
    - $J_\nu(x) = \frac{x}{2\nu}\left(J_{\nu-1}(x) + J_{\nu+1}(x)\right)$.
  - **Generating function**
$$\exp\left(\frac{1}{2}x\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$$
  - **Orthogonality:**
$$\int_0^1 x\, J_n(\lambda x) J_n(\mu x)\,dx = \begin{cases} 0 & if \quad \lambda \neq \mu \\ \frac{1}{2} J_{n+1}^2(x) & if \quad \lambda = \mu \end{cases}$$

7.9 Answer to SAQs:

SAQ1: From equation we have $p(x) = \frac{3x}{2x(x-2)^2} = \frac{3}{2(x-2)^2}$ and $(x) = \frac{1}{2x(x-2)}$, so there are two singular points, $x = 0, x = 2$ at which $p(x), q(x)$ are not finite.
To check for regular singular point, evaluate weak singularity condition

$$\lim_{x\to 2}(x-2)\,p(x) = \lim_{x\to 2}(x-2)\,\frac{3}{2(x-2)^2} = \lim_{x\to 2}\frac{3}{2(x-2)} \to \infty \quad i.e. not\ \ a\, finite\ value$$

So $x = 2$ is not a regular singular point. Now check for $x = 0$,

$$\lim_{x\to 0}(x-0)\,p(x) = \lim_{x\to 0} x\,\frac{3}{2(x-2)^2} = \lim_{x\to 0}\frac{3\,x}{2(x-2)^2} \to 0 \qquad i.e. a\, finite\ value$$

$$\lim_{x\to 0}(x-0)^2\,q(x) = \lim_{x\to 0} x^2\frac{1}{2x(x-2)} = \lim_{x\to 0}\frac{x}{2(x-2)} \to 0. \qquad i.e. a\, finite\ value$$

Hence $x = 0$ is a regular singular point or the differential equation has regular singularity at $x = 0$.

SAQ2: **Step1**: Assume the power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Then find derivatives

$$y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} = 2a_2 + 3\times 2\,a_3\,x + 4\times 3\,a_4 x^2 + 5\times 4\,a_5 x^3 + \cdots$$

Step 2: Replace $y\ and\ y''$ in the given differential equation:

$$\therefore \quad y'' + y = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Or,

$$2a_2 + 3\times 2\,a_3\,x + 4\times 3\,a_4 x^2 + 5\times 4\,a_5 x^3 + \cdots + a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 0.$$

Step 3: Collect the like powers of x

$$(2a_2 + a_0) + (3\times 2\,a_3 + a_1)x + (4\times 3\,a_4 + a_2)\,x^2 + (5\times 4\,a_5 + a_3)x^3 + \cdots = 0$$

Now, equate the sum of the coefficient of each occurring power of x to zero and find $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$

Coefficient of x^0 : $2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}$

Coefficient of x : $3\times 2\,a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{3\times 2} = -\frac{a_1}{3!}$

Coefficient of x^2 : $4\times 3\,a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4\times 3} = \frac{a_0}{4\times 3\times 2} = \frac{a_0}{4!}$

Coefficient of x^3 : $5 \times 4\, a_5 + a_3 = 0 \quad \Rightarrow \quad a_5 = -\frac{a_3}{5\times4} = -\frac{a_1}{5\times4\times3\times2} = \frac{a_1}{5!}$

Coefficient of x^4 : $6 \times 5\, a_6 + a_4 = 0 \quad \Rightarrow \quad a_6 = -\frac{a_4}{6\times5} = -\frac{a_0}{6\times5\times4\times3\times2} = -\frac{a_0}{6!}$

Similarly we can find $a_7 = -\frac{a_1}{7!}$, $a_8 = \frac{a_0}{8!}$, $a_9 = \frac{a_1}{9!}$ and so on. Thus in power series of y when n is odd the coefficients are expressed in term of a_1 and when n is even the coefficients are expressed in term of a_0 with alternate signs.

Step 4: Put the value of coefficients in power series of y and simplify the solution by collecting terms of a_0 and a_1.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \frac{a_0}{6!}x^6 - \frac{a_1}{7!}x^7 + \frac{a_0}{8!}x^8 + \frac{a_1}{9!}x^9 - \cdots$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$$

$$\boldsymbol{y = a_0 \cos(x) + a_1 \sin(x)} = c_1 y_1(x) + c_2 y_2(x).$$

This is a well-known solution, and can also be obtained directly from standard methods to solve the second order homogeneous ordinary differential equation. ($y_1(x)$ *and* $y_2(x)$ are two basis of the general solution y).

The example is just to convince you that we can apply power series method to find solution of any differential equation although we apply the method to solve generally special differential equations.

SAQ3: Comparing differential equation with general form, we have

$$p(x) = \frac{3x-1}{x^2-x} = \frac{3x-1}{x(x-1)} \ and\ q(x) = \frac{1}{x(x-1)}$$

so there are two singular points $x = 1\ and\ x = 0$.

Check for regular singularity:

$$\lim_{x\to1}(x-1)p(x) = \lim_{x\to1}(x-1)\frac{3x-1}{x(x-1)} = \lim_{x\to1}\frac{3x-1}{x} \to 2$$

$$\lim_{x\to1}(x-1)^2 q(x) = \lim_{x\to1}(x-1)^2\frac{1}{x(x-1)} = \lim_{x\to1}\frac{x-1}{x} \to 0$$

$$\lim_{x\to 0}(x-0)p(x) = \lim_{x\to 0} x\ \frac{3x-1}{x(x-1)} = \lim_{x\to 0}\frac{3x-1}{(x-1)} \to 1$$

$$\lim_{x\to 0}(x-0)^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{x(x-1)} = \lim_{x\to 0}\frac{x}{(x-1)} \to 0$$

Hence both $x = 0,1$ are regular singular points So solution of the differential equation can be obtained using Frobeneous method.

SAQ4: Step 1: Differentiating generating function given by equation (33)

$$\mathbf{(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\ z^n}$$

w.r.t. z we get

$$(1-2xz+z^2)^{-\frac{3}{2}}(x-z) = \sum n\ z^{n-1}\ P_n(x).$$

Step2: Now differentiating generating function w.r.t. x we get

$$z\,(1-2xz+z^2)^{-\frac{3}{2}} = \sum z^n P_n'(x).$$

Step 3: put result of step 2 in step 1 to get

$$\frac{(x-z)}{z}\sum z^n P_n'(x) = \sum n\ z^{n-1}\ P_n(x)$$

Or

$$(x-z)\sum z^n P_n'(x) = \sum n\ z^n\ P_n(x).$$

Step4: now compare the coefficient of z^n and rearrange terms to get

$$\boldsymbol{nP_n(x) = xP_n'(x) - P_{n-1}'(x).}$$

SAQ 5: Proof:

$$\frac{d(J_\nu(x))}{dx} = J'_\nu(x) = \frac{d}{dx}\left\{\sum_{m=0}\frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}\right\}$$

$$= \sum_{m=0}\frac{(2m+\nu)(-1)^m x^{2m+\nu-1}}{2^{2m+\nu}(m!)(\nu+m)!}.$$

Multiplying both side by x and write $2m+\nu = 2m+2\nu-\nu$

$$x\,J'_{\nu}(x) = \sum_{m=0} \frac{(2m+2\nu-\nu)(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}$$
$$= \sum_{m=0} \frac{2(m+\nu)\,(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!} - \nu \sum_{m=0} \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu}(m!)(\nu+m)!}$$
$$= \sum_{m=0} \frac{(-1)^m x^{2m+\nu-1+1}}{2^{2m+\nu-1}(m)!\ \ (\nu-1+m)!} - \nu\, J_\nu(x)$$
$$= -\nu\, J_\nu(x) + x \sum_{m=0} \frac{(-1)^m x^{2m+\nu-1}}{2^{2m+\nu-1}(m)!\ \ (\nu-1+m)!}$$
$$= -\nu\, J_\nu(x) + xJ_{\nu-1}(x)\,.$$

7.10 References / Bibliography:

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7.11 Terminal and model Questions :

Q1: Show that $\boldsymbol{(n+1)\,P_{n+1}(x) = (2n+1)xP_n(x) - n\,P_{n-1}(x).}$

Q2: Proof the following recurrence relations:

1. $\boldsymbol{J'_{\nu}(x) = \frac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)}$
2. $\boldsymbol{J_\nu(x) = \frac{x}{2\nu}\left(J_{\nu-1}(x) + J_{\nu+1}(x)\right)}$.

Q3: Find a power series solution in powers of $x$of the following differential equation:

1. $y'' + x\,y' + y = 0$
2. $y'' + x\,y = 0$
3. $(1-x^2)y'' - x\,y' + 2\,y = 0.$

Q4: Find the singularity point and solution of following differential equation:

1. $2x\,y'' + 2\,y' + y = 0$
2. $(x+2)^2 y'' + (x+2)y' - y = 0.$

UNIT 8: Hermite and Laguerre differential equations

Structure

8.1 Objective

We will apply the mathematical techniques of power series solution and Frobenious method learned in the previous unit to find the solution of differential equation which is of the form ofHermite and Laguerre. We will also learn the properties, like generating function, orthogonality, recurrence relations and Rodrigue's relation of the Hermite and Laguerre polynomials and their proofs.

8.2 Introduction

In Unit 7 you have already learned to solve two special differential equations whose solution exists only in a defined range. In this unit we will seek solution of rest two of the four *special forms of second order differential equations* which are linear and homogeneous in nature. The differential equations are:

c) **Hermite Differential Equation**:

$$y'' - 2x\, y' + 2\,\lambda\, y = 0 \quad ; \qquad \lambda > 0 \; and \; constant.$$

d) **Laguerre differential equation**:

$$x\, y'' + (1 - x)y' + n\, y = 0.$$

The solution of the differential equation is a polynomial which arises as a part of solution of important fields. In numerical analysis we can observe presence of Hermite and Laguerre polynomials as a solution of Gaussian quadrature integration method. Depending on the weight function the methods are called as Gauss-Hermite or Gauss-Laguerre quadrature. In physics, Quantum mechanics, we can see Hermite polynomials arises as eigen states of the quantum harmonic oscillator, whereas Laguerre polynomials are seen as solution of radial Schrödinger equation for a one-electron atom like Hydrogen. So it is really beautiful to learn the behavior and properties of the above differential equations.

8.3 Hermite Differential equation and polynomial

Any solution of Hermite Differential Equation:

$$y'' - 2x\, y' + 2\,\lambda\, y = 0. \qquad \mathbf{(1)}$$

(*$\lambda$is positive constant*) is called a Hermite Function or Polynomial $H_n(x)$. In the equation $p(x) = -2x$ and $q(x) = 2\lambda$, which is finite for all x so one can obtain solution of the equation using power series method.

8.3.1 Solution of Hermite equation

We can apply power series solution to the differential equation with $x_0 = 0$. Put the expansion of power series given by $y = \sum_{n=0}^{\infty} a_n x^n$and its derivatives $y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2}$to Eq. (1) we get,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n\, a_n x^{n-1} + 2\,\lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \; - \; 2\sum_{n=1}^{\infty} n\, a_n x^n \; + \; 2\,\lambda \sum_{n=0}^{\infty} a_n x^n \; = \; 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \; - 2\left(\sum_{n=1}^{\infty} n\, a_n \; - \; \lambda \sum_{n=0}^{\infty} a_n\right) x^n \; = \; 0. \qquad (\mathbf{2})$$

Expand the series in Eq.(2) and collect the coefficients of same power of x:

Coefficient of x^0 : $\quad 2a_2 + 2\lambda a_0 = 0 \quad$ or $\quad a_2 = \frac{-2\lambda}{2} a_0$

Coefficient of x : $\quad 3 \times 2\, a_3 - 2(1-\lambda)a_1 = 0 \quad$ or $\quad a_3 = \frac{2(1-\lambda)}{3\times 2} a_1$

Coefficient of x^2 : $\quad 4 \times 3\, a_4 - 2\,(2-\lambda)a_2 = 0 \quad$ or $\quad a_4 = \frac{2(2-\lambda)}{4\times 3} a_2.$

So we can write a general term as

Coefficient of $\boldsymbol{x^s}$: $\quad (s+2) \times (s+1)\, a_{s+2} - 2\,(s-\lambda)a_s = 0 \quad$ or $\quad \boldsymbol{a_{s+2} = \frac{2(s-\lambda)}{(s+2)\times(s+1)} a_s} \; (\mathbf{3})$

So we obtain the **recurrence relation or recurrence formula**in the form of Eq.(3) and with $s = 0,1,2,3,4, \ldots$ we can get all the coefficients in terms of a_0 and a_1 which is an arbitrary constants.

- Put the value of $s = 0,1,2,3,4, \ldots$ in Eq. (3) we get

$$s = 0: \quad a_2 = \frac{-2\lambda}{2} a_0$$

$$s = 1: \quad a_3 = \frac{2(1-\lambda)}{3!} a_1$$

$$s = 2: \quad a_4 = -\frac{2^2(2-\lambda)\,\lambda}{4!} a_0$$

$$s = 3: \quad a_5 = \frac{2^2(1-\lambda)(3-\lambda)}{5!} a_1$$

$$s = 4: \quad a_6 = -\frac{2^3\;\lambda\,(2-\lambda)(4-\lambda)}{6!} a_0$$

$$s = 5: \quad a_7 = \frac{2^3(1-\lambda)(3-\lambda)(5-\lambda)}{7!} a_1$$

and so on so that we can write even coefficients (a_{2k}) in terms of a_0 and odd coefficients (a_{2k+1}) in terms of $a_1 for\ k = 1,2,3,4,5,$ .... Such that

$$a_{2k} = -\frac{2^k\ \lambda\,(2-\lambda)(4-\lambda)\ldots\ldots(2k-2-\lambda)}{(2k)!}a_0$$

$$a_{2k+1} = \frac{2^k(1-\lambda)(3-\ \lambda)(5-\ \lambda)\ldots\ldots(2k-1-\lambda)}{(2k+1)!}a_1$$

the solution of differential equation can be expressed as

$$\boldsymbol{y = a_0 y_1(x) + a_1\, y_2(x)}$$

with

$$y_1 = 1 + \sum_{k=1}^{\infty} - \frac{2^k\ \lambda\,(2-\lambda)(4-\lambda)\ldots\ldots(2k-2-\lambda)}{(2k)!}x^{2k}$$

$$y_2 = x + \sum_{k=1}^{\infty}\frac{2^k(1-\lambda)(3-\ \lambda)(5-\ \lambda)\ldots\ldots(2k-1-\lambda)}{(2k+1)!}x^{2k+1}.$$

Polynomials of solution when $\lambda = 0, 1, 2,3$ ... (reduce the complete solution to a finite polynomial)

$\lambda = 0:\ \ y_1(x) = 1$

$\lambda = 1:\ \ y_2(x) = x$

$\lambda = 2:\ \ y_1(x) = (1-2x^2)$

$\lambda = 3:\ \ y_2(x) = (x - \frac{2}{3}x^3)\lambda = 4:\ \ y_1(x) = (1 - 4x^2 + \frac{4}{3}x^4\ )$

$\lambda = 5:\ \ y_2(x) = \left(x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right).$

Since $\boldsymbol{a_{s+2}} = \frac{\boldsymbol{2(s-\lambda)}}{\boldsymbol{(s+2)\times(s+1)}}\boldsymbol{a_s}$for $\boldsymbol{\lambda = s}$we have $\boldsymbol{a_{s+2} = 0}$and hence $\boldsymbol{a_{s+4} = a_{s+6} = \cdots = 0.}$

If $\boldsymbol{s}$is even: solution is the polynomial with even power of x.

If $\boldsymbol{s}$is odd: solution is the polynomial with odd power of x.

We arbitrarily choose coefficient $a_n = \frac{(-1)^n\, n!}{\frac{n}{2}!}$ a multiplicative constant so that coefficient of the term $x^n$is 2^n . The resulting solution is ***Hermite polynomial,*** $\boldsymbol{H_n(x)}$in general expressed as

$$\boldsymbol{H_n(x) = \sum_{k=0}^{N}\frac{n!\,(-1)^k}{k!\,(n-2k)!}(2x)^{n-2k}}. \tag{4}$$

Where the integer $\boldsymbol{N}$ is $\frac{\boldsymbol{n}}{\mathbf{2}}$ when $\boldsymbol{n}$ **is even** and $\frac{\boldsymbol{n-1}}{\mathbf{2}}$ when $\boldsymbol{n}$ **is odd.**

First few Hermite polynomials are

1. $H_0(x) = 1.$
2. $H_1(x) = 2x.$
3. $H_2(x) = 4x^2 - 2.$
4. $H_3(x) = 8x^3 - 12x.$
5. $H_4(x) = 16x^4 - 48x^2 + 12.$

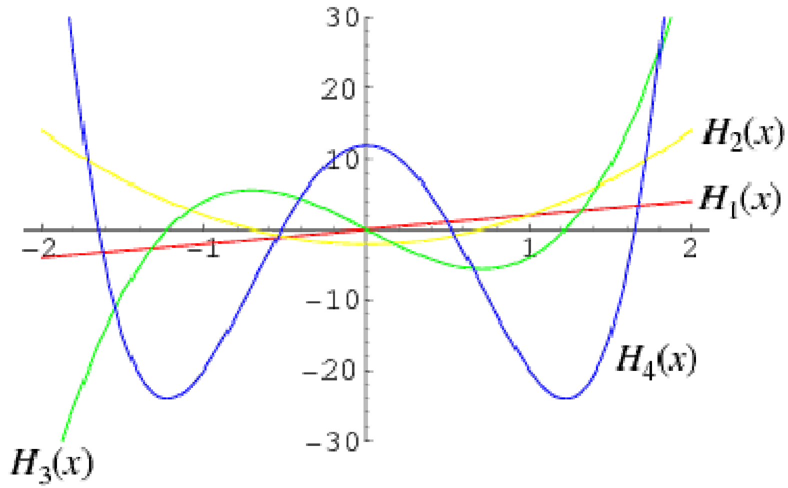

Figure 2: Graph of Hermite Polynomialsover the domain $(-\infty, \infty)$ for n = 1,2,3 and 4.
(Source :https://mathworld.wolfram.com/images/eps-gif/HermiteH_1000.gif)

8.3.2 Generating function

The generating function of Hermite Polynomial is given as

$$\boldsymbol{e^{2xt-t^2} = \sum_n \frac{H_n(x)}{n!} t^n}. \qquad (5)$$

Let's work out how we can write Eq. (5)

$$e^{2xt} = \sum_{r=0}^{\infty} (2x)^r t^r \frac{1}{r!} \qquad and \qquad e^{-t^2} = \sum_{s=0}^{\infty} (-1)^s t^{2s} \frac{1}{s!}.$$

Therefore,

$$e^{2xt}e^{-t^2} = \sum_{r=0}^{\infty}(2x)^r t^r \frac{1}{r!}\sum_{s=0}^{\infty}(-1)^s t^{2s}\frac{1}{s!} = \sum_{r}\sum_{s}(-1)^s\,(2x)^r t^{r+2s}\frac{1}{r!\,s!}$$

put $r = n - 2s$ in above equation,

$$e^{2xt-t^2} = \sum_{n=2s}\sum_{s=0}(-1)^s\,(2x)^{n-2s}\frac{1}{(n-2s)!\,s!}t^n$$

multiply and divide by $n!$ on R.H.S. in above equation, and change the upper limit of summation over s from ∞ to $N = \frac{n}{2}$ to avoid the negative factorial term in factor $(n-2s)!$.

$$\boldsymbol{e^{2xt-t^2}} = \sum_{n}\frac{1}{n!}\left\{\sum_{s}^{N}\frac{(-1)^s\;n!\;\;(2x)^{n-2s}}{(n-2s)!\,s!}\right\}t^n = \sum_{\boldsymbol{n}}\frac{\boldsymbol{H_n(x)}}{\boldsymbol{n!}}\boldsymbol{t^n}.$$

8.3.3 Orthogonality

The orthogonality property of the Hermite polynomial is given as

$$\int_{-\infty}^{\infty}\boldsymbol{e^{-x^2}H_n(x)H_m(x)}\;\;\boldsymbol{dx} = \sqrt{\boldsymbol{\pi}}\,\boldsymbol{2^n}\,\boldsymbol{n!}\;\boldsymbol{\delta_{mn}}. \qquad (6)$$

SAQ 1:Show that orthogonality relation for Hermite polynomial is given by Eq. (6).

8.3.4 Rodrigue's formula of Hermite function

The Rodrigue's formula for Hermite polynomial is given as

$$\boldsymbol{H_n(x)} = (-\boldsymbol{1})^{\boldsymbol{n}}\boldsymbol{e^{x^2}}\frac{\boldsymbol{d^n}}{\boldsymbol{dx^n}}\left(\boldsymbol{e^{-x^2}}\right). \qquad (7)$$

SAQ2: Proof the Rodrigue's relation Eq.(7) of Hermite polynomial.

8.3.5 Recurrence relation for Hermite polynomial ($\boldsymbol{H_n(x)}$)

Some of the recurrence relations for Hermite polynomial are

I. $H_n'(x) = 2\,n\,H_{n-1}(x)\,;\quad n \geq 1.$

II. $2\,x\,H_n(x) = 2\,n\,H_{n-1}(x) + H_{n+1}(x)$.
III. $H_n'(x) = 2\,x\,H_n(x) - H_{n+1}(x)$.

Proof:

(I) from generating function we have

$\sum_{n=0} \frac{H_n(x)}{n!} t^n = e^{2xt-t^2}$.

Differentiate both side with respect to x

$$\sum_{n=0} \frac{H_n'(x)}{n!} t^n = 2\,t\,e^{2xt-t^2} = 2\,t \sum_{n=0} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=0} \frac{H_n(x)}{n!} t^{n+1}$$

comparing both side the coefficient of t^n

$$\frac{H_n'(x)}{n!} = 2\,\frac{H_{n-1}(x)}{(n-1)!} \;; \quad n \geq 1$$

$$= 2\,\frac{n\,H_{n-1}(x)}{n\,(n-1)!} == 2\,n\,\frac{H_{n-1}(x)}{(n)!}.$$

Therefore,

$$H_n'(x) = 2\,n\,H_{n-1}(x)\,; \quad n \geq 1.$$

(II) from generating function we have

$$\sum_{n=0} \frac{H_n(x)}{n!} t^n = e^{2xt-t^2}$$

differentiating both side with respect to t

$$\sum_{n=1} n\,\frac{H_n(x)}{n!} t^{n-1} = 2\,x\,e^{2xt-t^2} - 2\,t\,e^{2xt-t^2} = 2x \sum_{n=0} \frac{H_n(x)}{n!} t^n - 2t \sum_{n=0} \frac{H_n(x)}{n!} t^n$$

$$\sum_{n=1} \frac{H_n(x)}{(n-1)!} t^{n-1} = 2x \sum_{n=0} \frac{H_n(x)}{n!} t^n - 2 \sum_{n=0} \frac{H_n(x)}{n!} t^{n+1}$$

$$2x \sum_{n=0} \frac{H_n(x)}{n!} t^n = \sum_{n=1} \frac{H_n(x)}{(n-1)!} t^{n-1} + 2 \sum_{n=0} \frac{H_n(x)}{n!} t^{n+1}$$

comparing both side the coefficient of t^n

$$2\,x\,\frac{H_n(x)}{n!} = \frac{H_{n+1}(x)}{(n)!} + 2\,\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{(n)!} + 2\,n\,\frac{H_{n-1}(x)}{(n)!}.$$

Hence, we have

$$2\,x\,H_n(x) = 2\,n\,H_{n-1}(x) + H_{n+1}(x).$$

(III) from the first recurrence relation we have

$$H_n'(x) = 2\,n\,H_{n-1}(x)$$

from second recurrence relation we have

$$2\,x\,H_n(x) = 2\,n\,H_{n-1}(x) + H_{n+1}(x)$$

or, $2\,n\,H_{n-1}(x) = 2\,x\,H_n(x) - H_{n+1}(x)$

put this in first recurrence relation on R.H.S., we get

$$H_n'(x) = 2\,x\,H_n(x) - H_{n+1}(x).$$

8.4 Laguerre differential equation and polynomial

Laguerre differential equations is

$$\boldsymbol{xy'' + (1-x)y' + ny = 0} \qquad (\mathbf{11})$$

or

$$\boldsymbol{y'' + \frac{(1-x)}{x}y' + \frac{n}{x}y = 0}.$$

8.4.1 Solution of Laguerre's equation

Thus, $p(x) = \frac{1-x}{x}$and$q(x) = \frac{n}{x}$, implies $x = 0$ is a regural singular point. So we get the solution of the differential equation using Frobeneous method. Assuming solution of the form

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}.$$

Replace above expressions in Laguerresdifferential equation (11) we get

$$x\left(\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}\right) + (1-x)\left(\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}\right) + n\left(\sum_{m=0}^{\infty} a_m x^{m+r}\right) = 0$$

$$\left\{\sum_{m=0}^{\infty}(m+r)(m+r-1) + \sum_{m=0}^{\infty}(m+r)\right\}a_m x^{m+r-1} + n\left(\sum_{m=0}^{\infty} a_m x^{m+r}\right) - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} = 0. \qquad (12)$$

To find the indicial Equation put the coefficient of lowest power of x $i.e.$ x^{r-1} for $m = 0$ to zero,

Indicial equation :$[r\,(r-1) + r]a_0 = 0$; $\qquad a_0 \neq 0.$

Therefore, $[r\,(r-1) + r] = 0$ and the solution of this equation is $[r^2 = 0]$ $\quad or \quad$ $r = 0$ is a double root.

To find recursion relation: equate coefficient of x^{s+r} to zero in Eq.(12)

$$(s+r+1)(s+r)a_{s+1} + (s+r+1)a_{s+1} + n\,a_s - (s+r)a_s = 0$$

$$(s+r+1)^2 a_{s+1} + (n-s-r)a_s = 0$$

$$a_{s+1} = \frac{(s+r-n)}{(s+r+1)^2}a_s\ ; \qquad s = 0,1,2,3,\ldots \qquad (13)$$

Now roots are $r = 0$, put this in recursion relation

$$a_{s+1} = \frac{s-n}{(s+1)^2}a_s \qquad (14)$$

$s = 0$: $\qquad a_1 = -na_0 \qquad = (-1)n\,a_0$

$s = 1$: $\qquad a_2 = \frac{1-n}{2^2}a_1 \qquad = (-1)^2\frac{n\,(n-1)}{2^2}a_0$

$s = 2$: $\qquad a_3 = \frac{2-n}{3^2} a_2 \quad = (-1)^3 \frac{n\,(n-1)(n-2)}{3^2 \times 2^2} a_0$

$s = 3$: $\qquad a_4 = \frac{3-n}{4^2} a_3 \quad = (-1)^4 \frac{n\,(n-1)(n-2)(n-3)}{4^2 \times 3^2 \times 2^2} a_0.$

In general we can write

$$a_s = (-1)^s \frac{n\,(n-1)(n-2) \dots \dots (n-s+1)}{(s!)^2} a_0$$

multiplying and divide by $(n-s)!$ on right hand side

$$a_s = (-1)^s \frac{n\,(n-1)(n-2) \dots \dots (n-s+1) \times (n-s)!}{(s!)^2 (n-s)!} a_0$$

$$a_s = (-1)^s \frac{n!}{(s!)^2 (n-s)!} a_0 . \tag{15}$$

Now put $r = 0$ in the assumed solution y and replace a_m by recursion relation Eq.(15) for $s = m$

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 \sum_{m=0}^{n} (-1)^m \frac{n!}{(m!)^2 (n-m)!} x^m.$$

If we take $a_0 = n!$ then solution y is known as Laguerre polynomial ($L_n(x)$)

$$\boldsymbol{L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{(n!)^2}{(m!)^2 (n-m)!} x^m.} \tag{16}$$

First few Laguerre polynomials are

1. $L_0(x) = 1.$
2. $L_1(x) = -x + 1.$
3. $L_2(x) = \frac{1}{2}(x^2 - 4x + 2).$
4. $L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6).$

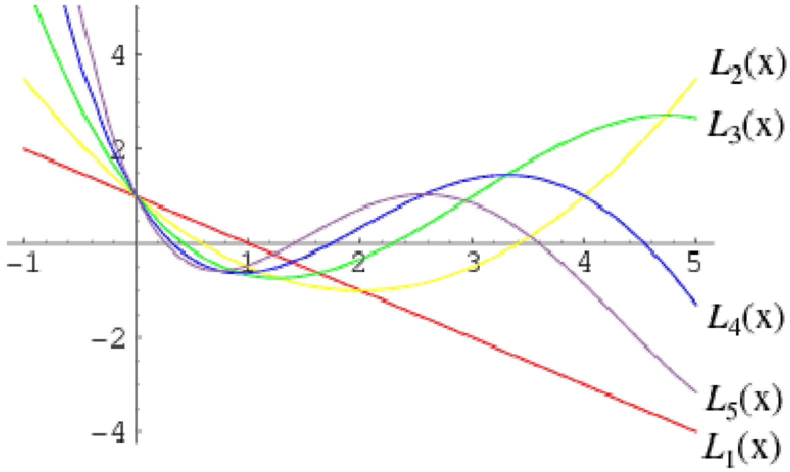

Figure 3: Graph of few Laguerre polynomials for $x \in [-1, 5]$ and $n = 1, 2, \ldots, 5$.
(Source: https://mathworld.wolfram.com/images/eps-gif/LaguerreL_1000.gif)

8.4.2 Generating function

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n = \frac{1}{1-z} \exp\left(-\frac{xz}{1-z}\right). \tag{17}$$

SAQ3: Show that generating function for Laguerre's polynomial is given by Eq.(17)

8.4.3Recurrence relations of Laguerre's Polynomial $L_n(x)$

1. $L_{n+1}(x) + (x - 2n - 1)L_n(x) + n^2 L_{n-1}(x) = 0.$

Proof: use generating function

$$\frac{1}{1-z} \exp\left(-\frac{xz}{1-z}\right) = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n$$

Or, $\exp\left(-\frac{xz}{1-z}\right) = (1-z)\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n$ differentiating w.r.t. z

$$\exp\left(-\frac{xz}{1-z}\right)\left\{-\frac{x}{1-z} - \frac{xz}{(1-z)^2}\right\} = (1-z)\sum n \frac{L_n(x)}{n!} z^{n-1} - \sum \frac{L_n(x)}{n!} z^n$$

$$-\frac{x}{(1-z)^2} \exp\left(-\frac{xz}{1-z}\right) = -\frac{x}{1-z}\sum \frac{L_n(x)}{n!} z^n = (1-z)\sum n \frac{L_n(x)}{n!} z^{n-1} - \sum \frac{L_n(x)}{n!} z^n$$

$$-x\sum\frac{L_n(x)}{n!}z^n \;=\; (1-z)^2\sum n\frac{L_n(x)}{n!}z^{n-1}-(1-z)\sum\frac{L_n(x)}{n!}z^n.$$

Equate the coefficients of z^n

$$-x\frac{L_n}{n!}\;=\frac{L_{n+1}}{n!}-2\frac{L_n}{(n-1)!}+\frac{L_{n-1}}{(n-2)!}-\frac{L_n}{n!}+\frac{L_{n-1}}{(n-1)!}$$

$$xL_n+L_{n+1}-2\frac{n!\;L_n}{(n-1)!}+\frac{n!\;L_{n-1}}{(n-2)!}-L_n+\frac{n!\;L_{n-1}}{(n-1)!}=0$$

So we get

$$L_{n+1}(x)+(x-2n-1)L_n(x)+n^2L_{n-1}(x)=\;0.$$

2. $\boldsymbol{L_n'(x)+nL_{n-1}(x)-nL_{n-1}'(x)=\;0}$.

Proof: use generating function

$$\exp\left(-\frac{xz}{1-z}\right)=(1-z)\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n.$$

differentiating w.r.t. x

$$-\frac{z}{1-z}\exp\left(-\frac{xz}{1-z}\right)=(1-z)\sum_{n=0}^{\infty}\frac{L_n'(x)}{n!}z^n$$

$$-z\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n=(1-z)\sum_{n=0}^{\infty}\frac{L_n'(x)}{n!}z^n.$$

Equate the coefficient of z^n

$$-\frac{L_{n-1}}{(n-1)!}\;=\frac{L_n'}{n!}-\frac{L_{n-1}'}{(n-1)!}$$

$$\frac{n!\;L_{n-1}}{(n-1)!}+L_n'-\frac{n!\;L_{n-1}'}{(n-1)!}=0\,.$$

So we get

$$L_n'(x)+nL_{n-1}(x)-nL_{n-1}'(x)=\;0\,.$$

Problem: show that $xL_n''(x)+\;(1-x)L_n'(x)\;+n\,L_n(x)=0$.

8.4.4 Orthogonality

$$\int_0^\infty \boldsymbol{e^{-x}} \frac{\boldsymbol{L_m(x)}}{\boldsymbol{m!}} \frac{\boldsymbol{L_n(x)}}{\boldsymbol{n!}} \; \boldsymbol{dx} = \boldsymbol{\delta_{mn}}. \qquad (\mathbf{18})$$

Proof: From generating function we have

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n = (1-z)^{-1} \exp\left(-\frac{xz}{1-z}\right) = (1-z)^{-1} \exp\left(x - \frac{x}{1-z}\right) \qquad \ldots (O.1)$$

and

$$\sum_{m=0}^{\infty} \frac{L_m(x)}{m!} t^m = (1-t)^{-1} \exp\left(-\frac{xt}{1-t}\right) = (1-t)^{-1} \exp\left(x - \frac{x}{1-t}\right) \qquad \ldots (O.2)$$

multiply Eq. $(O.1)$and Eq. $(O.2)$ we get

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{L_n(x)}{n!} \frac{L_m(x)}{m!} t^m z^n = \frac{\exp\left(x - \frac{x}{1-z}\right)}{1-z} \frac{\exp\left(x - \frac{x}{1-t}\right)}{1-t} \qquad \ldots (O.3)$$

multiplying both side of Eq. $(O.3)$ by e^{-x} and integrate on x from 0 to ∞

$$\sum_n \sum_m \left(\int_0^\infty e^{-x} \frac{L_n(x)}{n!} \frac{L_m(x)}{m!} \; dx \right) z^n t^m = \int_0^\infty e^{-x} \frac{\exp\left(x - \frac{x}{1-z}\right)}{1-z} \frac{\exp\left(x - \frac{x}{1-t}\right)}{1-t} \, dx \qquad \ldots (O.4)$$

$$\text{RHS} = \int_0^\infty e^{-x} \frac{\exp\left(2x - \frac{x}{1-z} - \frac{x}{1-t}\right)}{(1-z)(1-t)} \; dx = \int_0^\infty \frac{\exp\left(x - \frac{x}{1-z} - \frac{x}{1-t}\right)}{(1-z)(1-t)} \; dx$$

$$= \frac{1}{(1-z)(1-t)} \int_0^\infty \exp\left(-x\left(\frac{1}{1-z} - \frac{1}{1-t} - 1\right)\right) \; dx$$

now we can write $\frac{1}{(1-p)} = 1 + \frac{p}{1-p}$

$$\text{RHS} = \frac{1}{(1-z)(1-t)}\int_0^\infty \exp\left(-x\left(1+\frac{z}{1-z}-\frac{t}{1-t}\right)\right)\ dx$$

$$= \frac{1}{(1-z)(1-t)}\left\{-\frac{1}{\left(1+\frac{z}{1-z}-\frac{t}{1-t}\right)}\exp\left(-x\left(1+\frac{z}{1-z}-\frac{t}{1-t}\right)\right)\right\}_0^\infty$$

$$= \frac{1}{(1-z)(1-t)}\left\{0+\frac{(1-z)(1-t)}{(1-z)(1-t)+z(1-t)+t(1-z)}\right\} = \frac{1}{(1-z\,t)}$$

$$= \sum_{n=0} z^n t^n$$

Therefore, we have

$$\sum_n\sum_m\left(\int_0^\infty e^{-x}\frac{L_n(x)}{n!}\frac{L_m(x)}{m!}\ dx\right)z^n t^m \ = \sum_{n=0} z^n t^n \ = \ \sum_n\sum_m \delta_{mn}\quad z^n t^m$$

when $m = n$ LHS is non zero and the integral is equal to one as the coefficient of $z^n t^m$, otherwise for $m \neq n$ integral is zero. Therefore the orthogonality relation for Laguerre polynomial is

$$\int_0^\infty \boldsymbol{e^{-x}\frac{L_m(x)}{m!}\frac{L_n(x)}{n!}\ dx = \ \delta_{mn}}.$$

8.4.5 Rodrigues Formula

$$\boldsymbol{L_n(x) = \ e^x\frac{d^n}{dx^n}(x^n e^{-x})}. \tag{19}$$

Proof: Generating function is given as

$$(1-z)^{-1}\exp\left(-\frac{xz}{1-z}\right) = \ (1-z)^{-1}\exp\left[\left(1-\frac{1}{1-z}\right)x\right] \ = \sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n \qquad ...(\boldsymbol{R.1})$$

differentiate Eq.$(R.1)$ n times with respect to z (according to Leibnitz theorem) we get

$$e^x\frac{d^n}{dz^n}\left[(1-\text{z})^{-1}\exp\left(-\frac{x}{1-\ \text{z}}\right)\right] = \frac{d^n}{dz^n}\left[L_0+L_1z+\frac{L_2}{2!}z^2+\cdots+\frac{L_n}{n!}z^n+\frac{L_{n+1}}{(n+1)!}z^{n+1}+\cdots\right].(\boldsymbol{R.2})$$

- **To evaluate differential factor of LHS of Eq.**$(R.2)$ $i.e.\boldsymbol{\frac{d^n}{dz^n}\left[(1-\text{z})^{-1}\exp\left(-\frac{x}{1-\text{z}}\right)\right]}$ **we proceed as follows**:

  evaluate derivative for $\boldsymbol{n=1}$ in the limit $\boldsymbol{z\to 0}$

$$\frac{d}{dz}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right]=\frac{1-x-z}{(1-z)^{3}}\exp\left(-\frac{x}{1-\mathrm{z}}\right)$$

$$\lim_{z\to 0}\frac{d}{dz}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right]=(1-x)\mathrm{e}^{-x}=\frac{d}{dx}(x\,e^{-x})$$

evaluate derivative for $n=2$ in the limit $z\to 0$

$$\lim_{z\to 0}\frac{d^2}{dz^2}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right]=(x^2-4x+2)\mathrm{e}^{-x}=\frac{d}{dx}(x^2e^{-x})$$

evaluate derivative for $n=3$ in the limit $z\to 0$

$$\lim_{z\to 0}\frac{d^3}{dz^3}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right]=(6-18x+9x^2-x^3)\mathrm{e}^{-x}=\frac{d}{dx}(x^3e^{-x})$$

similarlily, we can show and express that $\mathbf{n^{th}}$ **orderderivative in the limit $z\to 0$**

$$\mathbf{LHS}=\boldsymbol{e^x}\lim_{z\to 0}\frac{\boldsymbol{d^n}}{\boldsymbol{dz^n}}\left[(\mathbf{1}-\mathbf{z})^{-\mathbf{1}}\,\mathbf{exp}\left(-\frac{\boldsymbol{x}}{\mathbf{1}-\mathbf{z}}\right)\right]=\boldsymbol{e^x}\frac{\boldsymbol{d^n}}{\boldsymbol{dx^n}}(\boldsymbol{x^n e^{-x}}).$$

- **To evaluate RHS of Eq.**$(R.2)$

$$RHS=\frac{d^n}{dz^n}\left[L_0+L_1z+\frac{L_2}{2!}z^2+\cdots+\frac{L_n}{n!}z^n+\frac{L_{n+1}}{(n+1)!}z^{n+1}+\cdots\right]=\frac{d^n}{dz^n}\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n$$

proceed as follows:

$$\frac{d}{dz}z^n=n\,z^{n-1};\quad \frac{d^2}{dz^2}z^n=n\,(n-1)z^{n-2}\;;\quad \frac{d^3}{dz^3}z^n=n\,(n-1)(n-2)z^{n-3}\;;$$

$$\frac{d^k}{dz^k}z^n=n\,(n-1)(n-2)\ldots(n-k+1)z^{n-k}.$$

Therefore, we get for $k=n$

$$\frac{\boldsymbol{d^n}}{\boldsymbol{dz^n}}\boldsymbol{z^n}=\boldsymbol{n}!$$

similarly we can write

$$\frac{d^k}{dz^k}z^{n+r}=(n+r)(n+r-1)(n+r-2)\ldots(n+r-k+1)z^{n+r-k}.$$

Therefore, again for $k=n$ we get

$$\frac{\boldsymbol{d^n}}{\boldsymbol{dz^n}}\boldsymbol{z^{n+r}}=(\boldsymbol{n}+\boldsymbol{r})(\boldsymbol{n}+\boldsymbol{r}-\mathbf{1})(\boldsymbol{n}+\boldsymbol{r}-\mathbf{2})\ldots(\boldsymbol{r}+\mathbf{1})\boldsymbol{z^r}=\frac{(\boldsymbol{n}+\boldsymbol{r})!}{\boldsymbol{r}!}\boldsymbol{z^r}.$$

Hence, using above relations and with the fact that L_n is funtion of x we can evaluate RHS in the limt$\boldsymbol{z}\to\mathbf{0}$

$$RHS = \frac{d^n}{dz^n}\left[L_0 + L_1 z + \frac{L_2}{2!}z^2 + \cdots + \frac{L_n}{n!}z^n + \frac{L_{n+1}}{(n+1)!}z^{n+1} + \frac{L_{n+2}}{(n+2)!}z^{n+2} + \cdots\right]$$

$$= [0 + 0 + 0 + \cdots + n!\,\frac{L_n}{n!} + \frac{(n+1)!}{1!}\frac{L_{n+1}}{(n+1)!}\,z + \frac{(n+2)!}{2!}\frac{L_{n+2}}{(n+2)!}z^2 + \cdots$$

$$\lim_{z\to 0} RHS \;=\; \lim_{z\to 0}\frac{d^n}{dz^n}\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n = L_n(x).$$

Hence, in the limt $z \to 0$ **we get Rodrigue's formula**

$$\mathbf{LHS} \;= \boldsymbol{e^x \frac{d^n}{dx^n}(x^n e^{-x})} \;= \boldsymbol{L_n(x) = RHS}.$$

8.5 Summary

- **Hermite differential equation and polynomial**

$$y'' - 2x\,y' + 2\,\lambda\,y = 0 \;\;;\quad \lambda > 0 \; and\; constant\,.$$

- **Hermite Polynomial**

$$\boldsymbol{H_n(x) = \sum_{k=0}^{N}\frac{n!\,(-1)^k}{k!\,(n-2k)!}(2x)^{n-2k}}.$$

- **Properties of Hermite function**
  - **Generating function**

$$\boldsymbol{e^{2xt-t^2} \;=\; \sum_n \frac{H_n(x)}{n!}t^n}.$$

  - **Orthogonality**

$$\boldsymbol{\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\;\; dx \;\;=\;\; \sqrt{\pi}\,2^n\,n!\;\delta_{mn}}.$$

  - **Rodrigue's formula of Hermite function**

$$\boldsymbol{H_n(x) = \;(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)}.$$

  - **Recurrence relations of Hermite Polynomial**
    1. $\boldsymbol{H_n'(x) = \;2\,n\,H_{n-1}(x)\,;\quad n \geq 1}.$
    2. $\boldsymbol{2\,x\,H_n(x) = 2\,n\,H_{n-1}(x) \;+\; H_{n+1}(x)}.$
    3. $\boldsymbol{H_n'(x) = \;2\,x\,H_n(x) \;-\; H_{n+1}(x)}.$
    4. $\boldsymbol{H_n''(x) \;=\; 2\,x\,H_n'(x) - 2\,n\,H_n(x)}.$

- **Laguerredifferential equation and polynomial**

$$\boldsymbol{xy'' + \;(1 - x)y' + ny = 0}.$$

- **Laguerre’s Polynomial**

$$L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{(n!)^2}{(m!)^2(n-m)!} x^m.$$

- **Properties of Laguerre function**
  - **Generating function**

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n = \frac{1}{1-z} \exp\left(-\frac{xz}{1-z}\right).$$

  - **Recurrence relations of Laguerre's Polynomial $L_n(x)$**
    1. $L_{n+1}(x) + (x - 2n - 1)L_n(x) + n^2 L_{n-1}(x) = 0.$
    2. $L_n'(x) + nL_{n-1}(x) - nL_{n-1}'(x) = 0.$

  - **Orthogonality**

$$\int_0^{\infty} e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} \, dx = \delta_{mn}.$$

  - **Rodrigues Formula**

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}).$$

8.6 Answer to SAQs

SAQ1: Proof:

We will use generating function to show the orthogonality relation as follows

$$e^{2xt-t^2} = \sum_n \frac{H_n(x)}{n!} t^n \quad \ldots\ldots..(a) \qquad and \qquad e^{2xt-t^2} = \sum_m \frac{H_m(x)}{m!} t^m \quad \ldots\ldots..(b)$$

multiply Eq. (a) and $Eq.(b)$

$$e^{4xt-2t^2} = \sum_n \sum_m \frac{H_n(x)H_m(x)}{n!\,m!} t^{n+m}. \qquad \ldots\ldots\ldots(c)$$

Multiply both side of Eq. (c)bye^{-x^2}

$$e^{-x^2+4xt-2t^2} = e^{-x^2+4xt-4t^2+2t^2} = e^{-(x-2t)^2+2t^2} = \sum_n \sum_m \frac{e^{-x^2} H_n(x)H_m(x)}{n!\,m!} t^{n+m}. \quad \ldots(d)$$

Integrating Eq.(d) from $-\infty \; to \; \infty$ on x, and put $y = x - 2t, \quad hence \; dx = dy$ then use definition of Gamma function

L.H.S. = $\int_{-\infty}^{\infty} e^{-(x-2t)^2+2t^2} dx = e^{2t^2} \int_{-\infty}^{\infty} e^{-(y)^2} \; dy = \; e^{2t^2}\sqrt{\pi} = \; \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2t^2)^n}{n!}$

$$\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2t^2)^n}{n!} = \int_{-\infty}^{\infty}\sum_{n}\sum_{m}\frac{e^{-x^2}H_n(x)H_m(x)}{n!\,m!}t^{n+m}\ dx \qquad \dots(e)$$

when $\boldsymbol{m = n}$; on comparing the coefficient of $\boldsymbol{t^{2n}}$in Eq.(e)

$$\int_{-\infty}^{\infty} \boldsymbol{e^{-x^2}\left(H_n(x)\right)^2\, dx \ = \ \sqrt{\pi}\, 2^n\, n!}$$

and $\boldsymbol{m \neq n}$; then comparing the coefficient of $\boldsymbol{t^{m+n}}$in Eq.(e)

$$\int_{-\infty}^{\infty} \boldsymbol{e^{-x^2}H_n(x)H_m(x)\ \ dx \ = \ 0}.$$

SAQ2: Generating function for Hermite polynomial is

$$e^{2xt-t^2} \ = \sum_{n}\frac{H_n(x)}{n!}t^n$$

$$e^{x^2-\,x^2+\,2xt-t^2} \ = \ e^{\{x^2-\,(x-t)^2\}} \ = \sum_{n}\frac{H_n(x)}{n!}t^n$$

differentiating both side $\boldsymbol{n}$ **times** partially with respect to t, and then put $t = 0$.

$$\frac{\partial^n}{\partial t^n}(e^{\{x^2-\,(x-t)^2\}})\bigg|_{t=0} \ = \frac{\partial^n}{\partial t^n}\left(\sum_{n}\frac{H_n(x)}{n!}t^n\right)_{t=0}$$

$$= \frac{\partial^n}{\partial t^n}\left(H_0t^0 + \frac{H_1}{1!}t \ + \frac{H_2}{2!}t^2 + \cdots + \frac{H_n}{n!}t^n + \ \dots..\right)_{t=0}$$

$$= \ H_n(x).$$

Therefore,

$$H_n(x) \ = \ e^{x^2}\left(\frac{\partial^n}{\partial t^n}(e^{-\,(x-t)^2})\ \right)_{t=0}$$

let $z = \ t - x, \quad so,\ dz \ = \ dt$

$$H_n(x) \ = \ e^{x^2}\left(\frac{\partial^n}{\partial z^n}(e^{-z^2})\ \right)_{z=-x} \ = \ e^{x^2}\left((-1)^n\frac{\partial^n}{\partial x^n}(e^{-x^2})\right).$$

Hence, we get Eq. (7)

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

SAQ3: R.H.S. (of Eq. (17))

$$= \frac{1}{1-z}\exp\left(-\frac{xz}{1-z}\right) = \frac{1}{1-z}\left(1 - \frac{xz}{1-z} + \frac{(xz)^2}{2!\ (1-z)^2} - \cdots + \frac{(-1)^k (xz)^k}{k!\,(1-z)^k} + \cdots\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (xz)^k}{k!\,(1-z)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (xz)^k}{k!}(1-z)^{-k-1}.$$

Use binomial expansion

$$\text{RHS} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^k}{k!} z^k \left[1 + (k+1)z + \frac{(k+1)(k+2)}{2!} z^2 + \cdots + \frac{(k+1)(k+2)\ldots(k+l)}{l!} z^l + \cdots\right].$$

Multiply and divide the term in summation by $k!$

$$\text{RHS} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^k}{k!} z^k \left[1 + (k+1)z + \frac{(k+1)(k+2)}{2!} z^2 + \cdots + \frac{(k+1)(k+2)\ldots(k+l)}{l!} z^l + \cdots\right] \times \frac{k!}{k!}$$

$$\text{RHS} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^k (k+l)(k+l-1)\ldots(k+2)(k+1)k!\ \ (x)^k}{k!\ \ l!\ \ \ k!} z^{k+l}$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^k (k+l)!\ \ (x)^k}{(k!)^2\ \ l!} z^{k+l}.$$

Put $k + l = n$ or $l = n - k$, so for $l = 0,$ n will start from $n = k$ in the summation, since summation over k starts from $k = 0$ so e can take summation over n from $n = 0$ with the condition that n is always greater than k to restrict the appearance of any negative factorial term because of replacement of $l! = (n-k)!$. Hence we can take summation of k from 0 to n and summation of n from 0 to ∞.

$$\text{RHS} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (n)!\ \ (x)^k}{(k!)^2\ \ (n-k)!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!}\sum_{k=0}^{n} \frac{(-1)^k (n)!\ \ (x)^k}{(k!)^2\ \ (n-k)!} n!\, z^n = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} z^n = \text{L.H.S.}$$

8.7 References / Bibliography

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8.8 Terminal and model questions

Q1. Show that

$$\lim_{z\to 0}\frac{d^2}{dz^2}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right] = (x^2-4x+2)\mathrm{e}^{-x}.$$

Q2. Show that

$$\lim_{z\to 0}\frac{d^3}{dz^3}\left[(1-\mathrm{z})^{-1}\exp\left(-\frac{x}{1-\mathrm{z}}\right)\right] = (6-18x+9x^2-x^3)\mathrm{e}^{-x}.$$

Q3. Show that

$$\lim_{z\to 0}\frac{d^n}{dz^n}\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}z^n = L_n(x).$$

Q4. Show that

$$H_n''(x) = 2\,x\,H_n'(x) - 2\,n\,H_n(x).$$

Q5. Prove that

$$H_n''(x) = 4\,n\,(n-1)H_{n-2.}$$

Q6. Evaluate

$$\int_{-\infty}^{\infty} x\,e^{-x^2}H_n(x)H_m(x)dx.$$

UNIT 9: FOURIER INTEGRAL AND TRANSFORMS

Structure

9.1 OBJECTIVES

After studying this chapter we will learn about how Fourier transforms is useful many physical applications, such as partial differential equations and heat transfer equations. With the use of different properties of Fourier transform along with Fourier sine transform and Fourier cosine transform, one can solve many important problems of physics with very simple way. Thus we will learn from this unit to use the Fourier transform for solving many physical application related partial differential equations.

9.2 INTRODUCTION

The central starting point of Fourier analysis is **Fourier series**. They are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. This trigonometric system is ***orthogonal***, allowing the computation of the coefficients of the Fourier series by use of the well-known Euler formulas, as shown. Fourier series are very important to the engineer and physicist because they allow the solution of linear differential equations and partial differential. Fourier series are, in a certain sense, more universal than the familiar Taylor series in calculus because many *discontinuous* periodic functions that come up in applications can be developed in Fourier series but do not have Taylor series expansions.

The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions.

The Fourier transform is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The result produced by the Fourier transform is a complex valued function of frequency. The absolute value of the Fourier transform represents the frequency value present in the original function and its complex argument represents the phase offset of the basic sinusoidal in that frequency.

The Fourier transform is also called a generalization of the Fourier series. This term can also be applied to both the frequency domain representation and the mathematical function used. The Fourier transform helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple sinusoids.

So for detailed knowledge of Fourier transform one should know about the Fourier series and Fourier Integral. So we will start the brief review of Fourier series and then I will explain the Fourier Integral and transforms in detailed.

9.3 FOURIER SERIES:

Fourier series are infinite series that represent periodic functions in terms of cosines and sines. As such, Fourier series are of greatest importance to the engineer and applied mathematician. To define Fourier series, we first need some background material. A function *f* (*x*) is called a **periodic function** if *f* (*x*) is defined for all real *x*, except possibly at some points, and if there is some positive number *p*, called a **period** of *f* (*x*) such that

$$f(x+p) = f(x) \qquad\qquad for\ all\ x.$$

Familiar periodic functions are the *cosine, sine, tangent*, and *cotangent*. Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cos hx\ etc.$ to mention just a few.
If *f(x)* has a period of *p* then it has also a period of *2p*

$$f(x+2p) = f\{(x+p)+p\} = f(x+p) = f(x).$$

Or in general we can write

$$f(x+np) = f(x).$$

A Fourier series is defined as an expansion of a real function or representation of a real function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_0, a_n, and\ b_n$ are constants, called the **Fourier coefficients** of the series. We see that each term has the period of 2π Hence *if the coefficients are such that the series converges*, *its sum will be a function of period* 2π.
The **Fourier coefficients** of *f(x)*, given by the **Euler formulas**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad\qquad n = 1, 2, 3, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \qquad\qquad n = 1, 2, 3, ...$$

The above Fourier series is given for period 2π. The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $= 2L$. Then we can introduce a new variable *v* such that, $f(x)$ as a function of *v*, has period 2π.
If we set

$$x = \frac{p}{2\pi}v \ \Rightarrow v = \frac{2\pi}{p}x \ \Rightarrow v = \frac{\pi}{L}x.$$

This means $v = \pm\pi$ corresponds to $x = \pm L$. This represent *f* , as function of *v* has a period of 2π. Hence the Fourier series is

$$f(v) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nv.$$

Now using $v = \frac{\pi}{L}x$ Fourier series for the period of *(-L, L)* is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin n\frac{\pi}{L}x.$$

This is Fourier series we obtain for a function of *f(x)* period 2*L* the Fourier series.
The coefficient is given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

9.4 SOME IMPORTANT RESULTS

1. $\int e^{\,ax} \, sin\, bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\, sin\, bx - bcos\; bx).$
2. $\int e^{\,ax} \, cos\, bx\, dx = \frac{e^{\,ax}}{a^2+b^2}(a\, cos\, bx + bsin\; bx).$
3. $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}.$
4. $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$
5. $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-b)^2+a^2}\, dx = \frac{\pi}{a}\, e^{-am}\, sin\, bm. \quad [m > 0]$

9.5 FOURIER INTEGRAL

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Since, of course, many problems involve functions that are ***non-periodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

9.6 FOURIER INTEGRAL THEOREM

Fourier integral theorem states that $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)dt\, du$

Proof. We know that Fourier series of a function f (x) in (-c, c) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}.$$

Where a_0 , a_n and b_n are given by

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt,$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt,$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt.$$

Substituting the values of a_0 , a_n and b_n in above equation, we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt\cos\frac{n\pi x}{c}$$
$$+ \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt,\sin\frac{n\pi x}{c}$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\left(\frac{n\pi t}{c} - \frac{n\pi x}{c}\right)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi}{c}(t-x)\right]dt$$

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt.$$

Since cosine functions are even functions i.e., $\cos(-\theta) = \cos\theta$ the expression

$$\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\} = \sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x).$$

Hence, we have

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\ \left\{\sum_{n=-\infty}^{\infty} \cos\frac{n\pi}{c}(t-x)\right\} dt$$

$$f(x) = \frac{1}{2\pi}\int_{-c}^{c} f(t)\ \left\{\frac{\pi}{c}\sum_{n=-\infty}^{\infty} \cos\frac{n\pi}{c}(t-x)\right\} dt.$$

We now let the parameter *c* approach infinity, transforming the finite interval [-*c, c*] into the infinite interval (-∞ to +∞). We set

$$\frac{n\pi}{c} = \omega, and\ \frac{\pi}{c} = d\omega \qquad with\ c \to \infty.$$

Then, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\ \left\{\int_{-\infty}^{\infty} d\omega \cos\omega(t-x)\right\} dt$$

On simplifying

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) cos\ \omega(t-x)\, d\omega\, dt. \qquad\qquad Proved$$

9.7. FOURIER SINE AND COSINE INTEGRALS

$$f(x) = \frac{2}{\pi}\int_0^\infty sin\ \omega x\ du\ \int_0^\infty f(t)\ sin\ \omega t\ dt. \qquad \text{(Fourier Sine Integrals)}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\ \omega x\ du\ \int_0^\infty f(t)\ cos\ \omega t\ dt. \qquad \text{(Fourier Cosine Integrals)}$$

Proof: We can write

$$\cos\omega(t-x) = \cos(\omega t - \omega x) = \cos\omega t\cos\omega x + \sin\omega t\sin\omega x.$$

Using this expansion in Fourier integral theorem, we have

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} \cos\omega(t-x)\, d\omega\, dt$$

$$\Rightarrow\ f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)(cos\ \omega t\cos\omega x + \sin\omega t\sin\omega x) d\omega\, dt$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)(cos\ \omega t\cos\omega x\, d\omega\, dt + \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\sin\omega x\, d\omega\, dt.$$

Now to solve the above equation, we have two different cases, using the following conditions

$$\int_{-a}^{a} f(x)dx = 0. \quad for\ odd\ function$$

And

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\, dx. \quad for\ even\ function$$

Case I: when *f(t)* is even function: this means

$$\Rightarrow f(t)\sin\omega t \quad is\ odd\ function\ and$$

$$f(t)\cos\omega t. \quad is\ even\ function$$

Hence

$$\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\sin\omega x\, d\omega\, dt\ = 0.$$

And

$$\Rightarrow f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)(cos\,\omega t\cos\omega x\, d\omega\, dt = \frac{2}{\pi}\int_{0}^{\infty}\cos\omega x\ d\omega\int_{-\infty}^{\infty} f(t)cos\,\omega t\ dt$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} cos\,\omega x\, du\int_{0}^{\infty} f(t)\, cos\,\omega t\, dt.$$

This is known as Fourier cosine integral.

Case II: If *f(t)* is odd function: this means

$$\Rightarrow f(t)\sin\omega t \quad is\ even\ function\ and$$

$$f(t)\cos\omega t. \quad is\ odd\ function$$

Hence

$$\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega t\cos\omega x\, d\omega\, dt\ = 0.$$

And

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) sin\, \omega t\, sin\, \omega x\, d\omega\, dt = \frac{2}{\pi}\int_0^\infty sin\, \omega x\ d\omega \int_{-\infty}^\infty f(t) sin\, \omega t\ dt$$

$$f(x) = \frac{2}{\pi}\int_0^\infty sin\, \omega x\, du \int_0^\infty f(t)\, sin\, \omega t\, dt.$$

This is known as Fourier sine integral.

9.8. FOURIER'S COMPLEX INTEGRALS

We know from Fourier integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) cos\, \omega(t-x)\, d\omega\, dt.$$

Now adding

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^\infty f(t)dt \int_{-\infty}^\infty sin\, \omega(t-x)\, d\omega = 0.$$

Since

$$\int_{-\infty}^\infty sin\ \ \omega(t-x) d\omega = 0. \qquad \textit{because of odd function}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) cos\, \omega(t-x)\, d\omega\, dt + \frac{i}{2\pi}\int_{-\infty}^\infty f(t)dt \int_{-\infty}^\infty sin\, \omega(t-x)\, d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\, dt \left[\int_{-\infty}^\infty cos\, \omega(t-x) + i\, sin\, \omega(t-x)\right]\ d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\, dt \left[\int_{-\infty}^\infty e^{i\omega(t-x)}\right]\ d\omega$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega x} d\omega \int_{-\infty}^\infty f(t)\, e^{i\omega t} dt.}$$

This relation is known as Fourier's complex Integral.

Example 1. Express the following function

$$f(x) = \begin{cases} 1 & when\ \ x \leq 1 \\ 0 & when\ \ x > 1 \end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin u \cos ux}{u} du.$$

Solution: we know the Fourier Integral theorem, the Fourier Integral of a function $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)cos\ \omega(t-x)\, d\omega\, dt.$$

Using $\omega = u$ we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)cos\ u(t-x)\, du\, dt$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^1 cos\ u(t-x)\, dt\, du. \qquad since\ f(t) = 1$$

Now integrating w.r.t. *t* we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin u(t-x)}{u}\right]_{-1}^1 du$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin u(1-x) + \sin u(1+x)}{u}\right] du.$$

Now using $\sin C + sinD = 2\ sin\frac{C+D}{2} cos\frac{C-D}{2}$ and solving it we will get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin u \cos ux}{u} du.$$

We can rewrite this

$$\int_0^\infty \frac{\sin u \cos ux}{u} du = \frac{\pi}{2} f(x)$$

$$\int_0^\infty \frac{\sin u \cos ux}{u} du = \begin{cases} \frac{\pi}{2} \times 1 = \frac{\pi}{2}, & for\ \ x < 1 \\ \frac{\pi}{2} \times 0 = 0, & for\ x > 1. \end{cases}$$

For *x=1,* which is a point of discontinuity of *f(x)*, value of integral $= \frac{\frac{\pi}{2}+0}{2} = \frac{\pi}{4}$ Ans.

Self Assessment Question (SAQ) 1: Find the Fourier cosine integral of

$$f(x) = \begin{cases} \sin x, & for \quad 0 \le x \le \pi \\ 0 & for \quad x > \pi. \end{cases}$$

Self Assessment Question (SAQ) 2: Find the Fourier sine integral of

$$f(x) = e^{-\alpha x}.$$

Hence prove that $\int_0^\infty \frac{\omega \sin \omega x}{\alpha^2+\omega^2} d\omega = \frac{\pi}{2} e^{-\alpha x}.$

9.9. FOURIER TRANSFORMS:

From the Fourier complex integral we know that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt.$$

We can rewrite the above expression as follows using $\omega = s$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} ds \int_{-\infty}^{\infty} f(t)\, e^{ist} dt = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} ds\right]\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist} dt\right].$$

Now using $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist} dt = F(s)$ in above equation, we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\, F(s) ds.$$

Where $F(s)$ is called the Fourier Transform of $f(x)$.

And $f(x)$ is called the Inverse Fourier transform of $F(s)$.

Thus, we obtain the definition of Fourier transform is

$$\boldsymbol{F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{ist} dt}$$

$$\boldsymbol{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}. F(s) ds.}$$

9.10. FOURIER SINE TRANSFORMS

We know that from Fourier sine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx\, ds \int_0^\infty f(t)\sin st\, dt = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, ds \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\, dt\right].$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}\int_0^\infty f(t)\sin st\, dt}.$

We have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\, ds\, F(s).$$

In above equation $F(s)$ is called Fourier Sine transform of $f(x)$

$$\boldsymbol{F(s) = F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\, dt}.$$

And, $f(x)$ given below is known as inverse Fourier Sine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\sin sx\, ds}.$$

9.11 FOURIER COSINE TRANSFORM

From Fourier cosine integral we know that

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\, \omega x\, du \int_0^\infty f(t)\, cos\, \omega t\, dt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, ds\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt\right].$$

Now putting $F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, ds\, F(s).$$

In above equation $F(s)$ is called Fourier cosine transform of $f(x)$

$$\boldsymbol{F(s) = F_c\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt.}$$

And, $f(x)$ given below is known as inverse Fourier cosine transform of $F(s)$

$$\boldsymbol{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, F(s)ds.}$$

Example 2: Find the Fourier transform of e^{-ax^2}, where a>0.

Solution : The Fourier transform of f(x):

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx.$$

Hence

$$F\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2+isx}dx$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-\frac{s^2}{4a}+isx+\frac{s^2}{4a}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2-\frac{s^2}{4a}}\,dx$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{is}{2\sqrt{a}}\right)^2}\,dx.$$

$Putting\ x\sqrt{a} - \frac{is}{2\sqrt{a}} = u \Rightarrow dx = \frac{du}{\sqrt{a}}$ in above expression we get,

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{-u^2}\,du \qquad \left[since\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\right]$$

$$\Rightarrow F\{e^{-ax^2}\} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi a}}\sqrt{\pi} = \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}} \qquad Ans.$$

Example 3: Find the Fourier transform of

$$f(x) = \begin{cases} 2 & for\ \ |x| < a \\ 0 & for\ \ |x| > a \end{cases}$$

Solution: We know that the Fourier transform of a function is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx.$$

Using the given value of *f(x)* we get,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 2e^{isx}dx = \frac{2}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}dx =$$

$$F\{f(x)\} = \frac{2}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{2}{\sqrt{2\pi}\, is}\left[e^{ias} - e^{-ias}\right] = \frac{4}{\sqrt{2\pi}\, s}\frac{\left[e^{ias} - e^{-ias}\right]}{2i}$$

$$F\{f(x)\} = \frac{4}{\sqrt{2\pi}\, s}\sin as = 2\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s} \qquad Ans.$$

Example 4: Find Fourier Sine transform of $\frac{1}{x}$.

Solution: We have to find the Fourier sine transform of $f(x) = \frac{1}{x}$.

We know that from Fourier sine transform

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx.$$

Now using the value of $f(x) = \frac{1}{x}$, we get,

$$F_s\,[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{x}\sin sx\,dx$$

$$now\ using\ sx = t \Rightarrow dx = \frac{dt}{s}.$$

We get $$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin t}{t}\,dt = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right). \qquad \Rightarrow since\ \int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Hence $$F_s\,[f(x)] = \sqrt{\frac{\pi}{2}} \qquad Ans.$$

Example 5: Find the Fourier Sine Transform of e^{-ax}.

Solution: Here, $f(x) = e^{-ax}$.

The Fourier sine transform of $f(x)$:

$$F_s\ [f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx.$$

On putting the value of $f(x)$ in (1), we get

$$F_s\,[e^{-\,ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\,ax}\sin sx\,dx.$$

On Integrating by parts, we get

$$F_s\,[e^{-\,ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-\,ax}}{a^2+s^2}[-a\,sinsx - s\,cos\,sx]\right]_0^\infty$$

$$using\left[\int_0^\infty e^{\,ax}\sin bx\,dx = \frac{e^{\,ax}}{a^2+b^2}(a\sin bx - bcos\,bx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+s^2}\,(-s)\right] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right) \qquad Ans.$$

Example 6: Find the Fourier Cosine Transform of $f(x) = 5e^{-2x} + 2e^{-5x}$.

Solution: The Fourier Cosine Transform of $f(x)$ is given by

$$F_c\,\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.$$

Putting the value of $f(x)$, we get

$$F_c\,\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty (5e^{-2x} + 2e^{-5x})\cos sx\,dx$$

$$= 5\int_0^\infty e^{-2x}\cos sx\,dx + 2\int_0^\infty e^{-5x}\cos sx\,dx$$

$$using\left[\int_0^\infty e^{\,ax}\,cos\,bx\,dx = \frac{e^{\,ax}}{a^2+b^2}(a\,cos\,bx + b\sin bx)\right]$$

$$= 5\left[\frac{e^{-2x}}{(-2)^2+s^2}\,(-2\cos sx + s\sin sx)\right]_0^\infty + 2\left[\frac{e^{-5x}}{(-5)^2+s^2}\,(-5\cos sx + s\sin sx)\right]_0^\infty$$

$$= 5\left[0 - \frac{1}{4+s^2}(-2)\right] + 2\left[0 - \frac{1}{25+s^2}(-5)\right] = 5\left(\frac{2}{s^2+4}\right) + 2\left(\frac{5}{s^2+25}\right)$$

$$= 10\left(\frac{1}{s^2+4} + \frac{1}{s^2+25}\right) \qquad Ans.$$

Self Assessment Question (SAQ) 3: Find Fourier sine transform of $f(x) = e^{-x}$

Assessment Question (SAQ) 4: Find Fourier transform of $f(x)$

$$f(x) = \begin{cases} 1 & for \ |x| < 1 \\ 0 & otherwise \end{cases}$$

Assessment Question (SAQ) 5: Find Fourier cosine transform of $f(x)$

$$f(x) = \begin{cases} x & for\ 0 < x < 1 \\ 2 - x & for\ 1 < x < 2 \\ 0 & for \quad x > 2 \end{cases}$$

9.12 PROPERTIES OF FOURIER TRANSFORMS

9.12.1 LINEAR PROPERTY: If $F_1(s)$ and $F_2(s)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively then

$$F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s). \qquad where\ a\ and\ b\ are\ constants.$$

Proof: we know from the definition of Fourier transform

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

We can write

$$F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx} dx.$$

And

$$F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{isx} dx.$$

Now

$$F[af_1(x) + b\, f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af_1(x) + b\, f_2(x)] e^{isx} dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x)] e^{isx} dx$$

$$\Rightarrow F[af_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s). \qquad Proved$$

9.12.2 CHANGE OF SCALE PROPERTY:

We know that Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

Then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Proof: we know

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx. \qquad \left[now\ put\ ax = t \Rightarrow dx = \frac{dt}{a}\right]$$

We have

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\frac{s}{a}t} \frac{dt}{a} \quad = \quad \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt$$

$$\Rightarrow F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right). \qquad Proved$$

9.12.3 SHIFTING PROPERTY:

The Fourier transform equation is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

Then,

$$F\{f(x-a)\} = e^{isa} F(s).$$

Proof: Given

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

Then,

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$Put\ (x-a) = u \ \Rightarrow x = u + a \ \ and\ dx = du.$$

We have

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{is(u+a)}du = e^{isa}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{isu}du$$

$$\Rightarrow F\{f(x-a)\} = e^{isa}\,F(s). \qquad\qquad Proved$$

Following are few more properties of Fourier transform which can be proved as same manner as above properties. Students are advice to proof these properties by themselves.

Assessment Question (SAQ) 6: Prove following properties of Fourier Transform

1. $F\{e^{iax}\, f(x)\} = F(s+a).$
2. $F\{f(x)\cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)].$
3. $F\{x^n\, f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s).$
4. $F\{f'(x)\} = i\, s\, F(s).$
5. $F\{f^n(x)\} = (-i\, s)^n\, F(s).$
6. $F\{\int_a^x f\,(x)\,dx\} = \frac{F(s)}{i\,s}.$

9.13 FOURIER TRANSFORM OF DERIVATIVES

As we know from the properties of Fourier Transform

$$F\{f^n(x)\} = (-i\,s)^n\,F(s).$$

I. $F\left(\frac{\partial^2 f}{dx^2}\right) = (-i\,s)^2\,F\{f(x)\} = -s^2\bar{f}.$ $\left[where\ \bar{f}\ \ is\ Forier\ Transform\ of\ f\right]$

II. $If\ F_c\ and\ F_s\ are\ cosine\ and\ sine\ Forier\ tranform\ f(x)\ then$

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}\,f(0) + sF_s(s).$$

Proof: From cosine Fourier transform we know that

$$F_c\,[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos sx\,d\{f(x)\}.$$

Now integrating by parts, we get

$$= \sqrt{\frac{2}{\pi}}\,[\cos sx\ \ f(x)]_0^{\infty} - \sqrt{\frac{2}{\pi}}\left[-s\int_0^{\infty} sin\,sx\,f(x)\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\,[0 - f(0)] + s\sqrt{\frac{2}{\pi}}\left[\int_0^{\infty} \sin sx\, f(x)\, dx\right]. \quad \{assumimg\ f(x) \to 0\ as\ x \to \infty\}$$

Hence,

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}\, f(0) + sF_s(s) \qquad where\ \sqrt{\frac{2}{\pi}}\left[\int_0^{\infty} \sin sx\, f(x)\, dx\right] = F_s(s).$$

Proved

Similarly we can prove the following relations

III. $F_s\{f'(x)\} = -sF_c(s).$

IV. $F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}}\, f'^{(0)} - s^2 F_c(s).$

V. $F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\, sf(0) - s^2 F_s(s).$

9.14 FOURIER TRANSFORM OF PARTIAL DERIVATIVE OF A FUNCTION

The Fourier transform of the partial derivatives is given by

$$F\left[\frac{\partial^2 u}{\partial^2 x}\right] = -s^2\, F(u).$$

Where $F(u)$ is the Fourier transform of u.

The Fourier sine transform of the partial derivatives is given by

$$F_s\left[\frac{\partial^2 u}{\partial^2 x}\right] = s(u)_{x=0} - s^2\, F_s(u).$$

Where $F_s(u)$ is the Fourier sine transform of u

The Fourier cosine transform of the partial derivatives is given by

$$F_c\left[\frac{\partial^2 u}{\partial^2 x}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2\, F_c(u).$$

Where $F_c(u)$ is the Fourier cosine transform of u.

Note: From the above formula, it is clear that if $u\ at\ x = 0$ is given then we apply sine Fourier transform and if $\frac{\partial u}{\partial t}\ at\ x = 0$ is given the, we apply Fourier cosine transform.

9.15 APPLICATION TO SIMPLE HEAT TRANSFER EQUATIONS

One of the important applications of Fourier transforms is to solve the simple heat transfer equations. Example of this is given below.

Example 7: Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}.$$

Subject to the conditions

(i) $u = 0$ when $x = 0$. $t > 0$ (ii) $u = \begin{cases} 1 & for\ 0 < x < 1 \\ 0 & for\ x \geq 1 \end{cases}$ $when\ t = 0$

(iii) $u(x\ ,t) is\ bounded.$

Solution: In view of the initial conditions we know that if $u\ at\ x = 0$ is given then we apply sine Fourier transform and if $\frac{\partial u}{\partial t}\ at\ x = 0$ is given the, we apply Fourier cosine transform.

So here we apply Fourier *sine* transform in given equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}.$$

We get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial^2 x}\sin sx\, dx$$

$$\frac{\partial}{\partial t}\int_0^\infty u\sin sx\, dx = \int_0^\infty \frac{\partial^2 u}{\partial^2 x}\sin sx\, dx.$$

Now using the Fourier sine transform partial derivative given below

$$F_s\left[\frac{\partial^2 u}{\partial^2 x}\right] = s(u)_{x=0} - s^2\, F_s(u).$$

Where $F_s(u)$ is the Fourier sine transform of u

We will get

$$\frac{\partial F_s(u)}{\partial t} = s(u)_{x=0} - s^2\, F_s(u) \quad \{ given\, u = 0\, when\, x = 0\}$$

$$\frac{\partial F_s(u)}{\partial t} = -s^2\, F_s(u) \;\Rightarrow \frac{\partial F_s(u)}{\partial t} + s^2\, F_s(u) = 0 \Rightarrow (D + s^2)F_s(u) = 0.$$

Now the auxiliary equation (A.E) is

$$m + s^2 = 0 \;\Rightarrow m = -s^2.$$

Hence, the solution is given by

$$F_s(u) = Ae^{-s^2 t} = \; F_s(u,t)$$

$$F_s(u,t) = \int_0^{\infty} u(x,t) \sin sx\; dx.$$

Now using the given condition

$$u = \begin{cases} 1 & for\; 0 < x < 1 \\ 0 & for\; x \geq 1 \end{cases} \qquad when\; t = 0$$

We get,

$$F_s(u,0) = \int_0^1 u(x,0) \sin sx\; dx = \int_0^1 1. \sin sx\; dx = \left[\frac{-\cos sx}{s}\right]_0^1 = \frac{1-\cos s}{s}.$$

But we have the solution

$$F_s(u,t) = Ae^{-s^2 t} \Rightarrow \; F_s(u,0) = A.$$

Hence, from above equation we get

$$A = \frac{1-\cos s}{s}.$$

Hence solution is

$$F_s(u,t) = \frac{1-\cos s}{s} e^{-s^2 t}.$$

And, finally the complete solution of given equation using inverse Fourier sine transform

$$u = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos s}{s} e^{-s^2 t} ds \qquad Ans.$$

Solution of partial differential Equation by Fourier Transform

Example 8: Solve $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $\quad -\infty < x < \infty, \quad t \geq 0$

With conditions *u(x,0)=f(x)*,

$\frac{\partial u}{\partial t}\,(x,0) = g(x)$ and assuming u, u and $\frac{\partial u}{\partial t} \to 0$ as $x \to \pm\infty$.

Solution: We have given

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

Taking Fourier transform on both sides of the differential equations,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha^2 \frac{\partial^2 u}{\partial x^2} e^{isx} dx.$$

Now using the Fourier transform of derivative

$$\left\{F\left(\frac{\partial^2 f}{dx^2}\right) = (-i\,s)^2\, F\{f(x)\} = -s^2 F\{f(x)\}\right\}.$$

We have

$$\frac{\partial^2 F(u)}{dt^2} = -\alpha^2 s^2 F(u)$$

$$\frac{\partial^2 F(u)}{dt^2} + \alpha^2 s^2 F(u) = 0 \;\Rightarrow D^2 F(u) + \alpha^2 s^2 F(u) = 0.$$

Now Auxiliary equation corresponding to this equation is

$m^2 + \alpha^2 s^2 = 0 \Rightarrow m = \pm i\alpha s.$

Hence, solution corresponding this equation is

$$F\,(s,t) = Ae^{i\alpha st} + Be^{-i\alpha st} \;\ldots\ldots 1$$

Using given condition

$u(x,0) = f(x) and\ \frac{\partial u}{\partial t}(x,0) = g(x)$

$F(s,0) = F(s)$ and $\frac{dF(u)}{dt}(s,0) = G(s)$ on taking transform.

Using these conditions in above equation, we get

$$F(s,0) = A + B = F(s) \ldots\ldots\ldots\ldots\ldots\ldots 2$$

$$\frac{dF(u)}{dt}(s,0) = i\alpha s(A - B) = G(s) \ldots\ldots\ldots 3$$

Solving (2) and (3), we get

$$A = \frac{1}{2}\left[F(s) + \frac{G(s)}{i\alpha s}\right]$$

$$\mathrm{B} = \frac{1}{2}\left[F(s) - \frac{G(s)}{i\alpha s}\right].$$

Using the value of A and B in (1), we get

$$F(s,t) = \frac{1}{2}\left[F(s) + \frac{G(s)}{i\alpha s}\right]e^{i\alpha st} + \frac{1}{2}\left[F(s) - \frac{G(s)}{i\alpha s}\right]e^{-i\alpha st}. \qquad \ldots..(4)$$

By inversion theorem, (4) reduces to

$$\mathrm{u}\,(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left[f(x-\alpha t) - \frac{1}{\alpha}\int_{\alpha}^{x-\alpha t} g(\theta)d\theta\right] + \frac{1}{2}\left[f(x+\alpha t) + \frac{1}{\alpha}\int_{\alpha}^{x+\alpha t} g(\theta)d\theta\right].$$

Using the result

$$F\left(\int_{\alpha}^{x} f(t)dt\right) = \frac{F(s)}{(-is)}.$$

Self Assessment Question (SAQ) 7: Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}. \qquad for\ o \le x < \infty\ and\ t > 0$$

The given conditions are

(i) $u(x,0) = 0,\quad for\ x \ge 0$ $\qquad (ii)$ $\frac{\partial u}{\partial t}(0,t) = -a\ (constant)$

(iii) $u(x,t) is\ bounded.$

9.16 SUMMARY

The main aim of to study the Fourier transforms is the solution of partial differential equations and systems of such equations. Firstly we have learned about brief review of Fourier series, which very important to learn the Fourier transform. After that Fourier Integral along with Fourier sine Integral and Fourier cosine Integral have been explained. The knowledge of Fourier Integral is very important for Fourier transform. After this we have explained Fourier transform and their important properties in detailed. We also have explained Fourier sine transform and

Fourier cosine transform, which is equally important to Fourier transform. To understand this unit more clearly various solved examples are includes almost in each section. For students assessment self assessment question is also incorporated throughout the chapter.

9.17 GLOSSARY

Periodic Function: The function which repeat itself after a fix time.

Computation: Calculations such as addition, subtraction etc.

9.18 TERMINAL QUESTIONS

1. Find the Fourier Transform of $f(x)$ if
$$f(x) = \begin{cases} x, |x| \leq a \\ 0, |x| > a \end{cases}$$
2. Show that the Fourier Transform of
$$f(x) = \begin{cases} a - |x| & for\ |x| < a \\ 0 & for\ |x| > a > 0 \end{cases}$$
Is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$.
Hence show that $\int_0^\infty \left(\frac{sint}{t}\right)^2 dt = \frac{\pi}{2}$.
3. Show that the Fourier Transform of
$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2a} & for\ |x| \leq a \\ 0 & for\ |x| > a \end{cases}$$
Is $\frac{\sin sa}{sa}$.
4. Find the Fourier cosine Transform of e^{-ax}.
5. Find Fourier transform of
$$F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$$
6. Find Fourier Sine Transform of
$$f(x) = \frac{1}{x(x^2 + a^2)}.$$
7. Find the Fourier Sine and Cosine Transform of $ae^{-\alpha x} + be^{-\beta x}$. $\alpha, \beta > 0$
8. Find f(x) if its Fourier Sine transform is $\frac{s}{1+s^2}$.
9. Find f(x) if its Fourier Sine Transform is $(2\pi s)^{\frac{1}{2}}$.
10. Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}$ $for\ x \geq 0\ and\ t \geq 0$ under the given condition $u = u_0\ at\ x = 0, and\ t > 0$ with initial condition $u(x, 0) = 0, x \geq 0$.

9.19 ANSWERS

Self Assessment Question (SAQ)

1. $f(x) = -\frac{2}{\pi}\int_0^{\infty}\left(\frac{1+\cos ux}{u^2-1}\right)\cos ux\, du.$
3. $f(x) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right).$
4. $f(x) = \sqrt{\frac{2}{\pi}}\frac{\sin s}{s}.$
5. $f(x) = \frac{2\sin s(1-\cos s)}{s^2}.$
7. $f(x) = \frac{2}{\pi}.a\int_0^{\infty}\frac{1-e^{ks^2t}}{s^2}\cos sx\, ds.$

Terminal Questions:

1. $\frac{1}{\sqrt{2\pi}}\frac{2i}{s^2}.$
4. $F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right).$
5. $\left(\frac{2a^2}{s} - \frac{4}{s^2}\right)\sin\ as + \frac{4a}{s^2}\cos as.$
6. $\frac{\pi}{2a^2}(1 - e^{-ax}).$
7. $\frac{as}{s^2+\alpha^2} + \frac{bs}{s^2+\beta^2}\ ,\ \frac{a\alpha}{s^2+\alpha^2} + \frac{b\beta}{s^2+\beta^2}.$
8. $\frac{2sin^2\, ax}{\pi^2x^2}.$
9. $\frac{1}{x\sqrt{x}}.$
10. $u(x,t) = \frac{2u_0}{\pi}\int_0^{\infty}\left(\frac{1-e^{ks^2t}}{s}\right)sin\, sx\, ds.$

9.20 REFERENCES

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9.21 SUGGESTED READINGS

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UNIT 10: LAPLACE TRANSFORM

Structure

10.1 OBJECTIVES

After studying this chapter we will learn about how Laplace transforms is useful for solving differential equations with boundary values without finding the general solution. With the use of different properties of Laplace transform and Inverse Laplace transform one can solve many important problem of physics with very simple way. Thus we will learn from this unit to use the Laplace transform for solving the differential equations.

10.2 INTRODUCTION

The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782. Laplace transforms is an integral transform. It helps in solving the differential equations with boundary values without finding the general solution and values of the arbitrary constants. The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving linear differential equations and related initial value problems, as well as systems of linear differential equations, much easier. Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics. The process of solving differential equations using the Laplace transform method consists of three steps:

Step 1. The given differential equations is transformed into an algebraic equation, called the subsidiary equation.

Step 2. The subsidiary equation is solved by purely algebraic manipulations.

Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.

10.3 LAPLACE TRANSFORM

Definition: The Laplace transform of a function *f(t)* is defined as follows

$$\mathrm{F}\ (\mathrm{s}) = \int_0^\infty \mathrm{e}^{-\mathrm{st}} f(t)dt.$$

For all positive values of *t* and integral should exits. The Laplace transform is denoted by

$$\boldsymbol{L[f(t)] = F\ (s) = \int_0^\infty e^{-st} f(t)dt.}$$

The Laplace transform is an operation that transforms a function of *t* (i.e., a function of *time domain*), defined on [0, ∞), to a function of *s* (i.e., of *frequency domain*). *F*(*s*) is the Laplace *transform*, or simply *transform*, of *f* (*t*). Together the two functions *f* (*t*) and *F*(*s*) are called a *Laplace transform pair*.

10.4 LINEARITY OF THE LAPLACE TRANSFORM

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\boldsymbol{L[af(t) + bg(t)] = aL[f(t)] + b[g(t)].}$$

Proof:

$$L[af(t) + bg(t)] = \int_0^\infty e^{-st}[af(t) + bg(t)]dt.$$

As we know that integration is a linear operation. So we can use the linearity property of integration in above equation

$$L[af(t) + bg(t)] = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt$$

$$L[af(t) + bg(t)] = aL[\,f(t)] + bL[g(t)].$$

Proved

10.5 CHANGE OF SCALE PROPERTY

If the Laplace transform *of* $f(t)$ is $F(s)$ then

$$\boldsymbol{L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right).}$$

Proof: From the definition of Laplace transform

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

$$put\ at = r \Longrightarrow dt = \frac{dr}{a}\ and\ also\ t = \frac{r}{a}$$

$$\Rightarrow L[f(at)] = \int_0^\infty e^{-\frac{sr}{a}} f(r)\frac{dr}{a} = \frac{1}{a}\int_0^\infty e^{-Sr} f(r)dr \quad \left[where\ S = \frac{s}{a}\right]$$

$$= \frac{1}{a}F(S) = \frac{1}{a}F\left(\frac{s}{a}\right). \qquad Proved$$

10.6 FIRST SHIFTING THEOREM:

If $F(s)$ has the Laplace transform *of* $f(t)$ then

$$\boldsymbol{L[e^{at}f(t)] = F(s-a)}.$$

Proof: Using the definition of Laplace transform

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-st+at} f(t)dt = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-st}\{e^{at}f(t)\}dt = L[e^{at}f(t)]. \qquad \textbf{Proved}$$

Alternative Method:

$$L[e^{at}f(t)] = \int_0^\infty e^{at}e^{-st}f(t)dt = \int_0^\infty e^{-st+at}f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-ut}f(t)dt$$

Using $(s-a) = u$

$$= F(u) = F(s-a). \qquad \textbf{Proved}$$

10.7 SECOND SHIFTING THEOREM (HEAVISIDE'S SHIFTING THEOREM):

If $L[f(t)] = F(s)$ and $g(t) = \begin{Bmatrix} f(t-a), & for\ t > a \\ 0, & for\ 0 < t < a \end{Bmatrix}$.

Then, $\quad L[g(t)] = e^{-as}F(s)$.

Proof: As per the definition of Laplace transform

$$L[g(t)] = \int_0^\infty e^{-st}\, g(t)dt$$

$$L[g(t)] = \int_0^a e^{-st}\, g(t)dt + \int_a^\infty e^{-st} g(t)dt.$$

Using the given condition $g(t) = 0\ for\ 0 < t < a\ and\ g(t) = \ f(t-a)\ for\ t > a$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)dt.$$

$Now\ using\ (t-a) = \ r \Rightarrow dt = dr\ and\ t = (r+a)$ we get

$$L[g(t)] = \int_0^\infty e^{-s(r+a)} f(r)dr = e^{-sa} \int_0^\infty e^{-sr} f(r)dr = e^{-sa} F(s).$$

Hence, $\quad L[g(t)] = e^{-as}F(s) \qquad Proved$

10.8 LAPLACE TRANSFORM OF THE DERIVATIVE OF $f(t)$

If $L[f(t)] = F(s)$ and $f'(t)$ is the derivative of $f'(t)$ then

$$\boldsymbol{L[f'(t)] = \ sL[f(t)] - f(0)}.$$

Proof: As we know

$$L[f'(t)] = \int_a^\infty e^{-st} f'^{(t)} dt.$$

Solving above equation using integration by parts we get

$$L[f'(t)] = [e^{-st} f(t)]_0^\infty - \int_0^\infty (-se^{-st}) f(t)dt.$$

As we know that

$$e^{-\infty} = 0\ and\ e^0 = 1 \Rightarrow e^{-st} f(t) = 0\ when\ t = \infty\ and\ e^{-st} f(t) = f(0) when\ t = 0$$

$$\Rightarrow L[f'(t)] = -f(0) + s\int_0^\infty e^{-st} f(t)dt = \ -f(0) + sL[f(t)$$

$$\Rightarrow \boldsymbol{L[f'(t)] \ = \ sL[f(t)] - f(0)}. \qquad \boldsymbol{Proved}$$

10.9 LAPLACE TRANSFORM OF THE DERIVATIVE OF ORDER *n*

$$L[f^n(t)] \ = \ s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0).$$

Proof: As we know that the Laplace transform of derivative is given by

$$L[f'(t)] = \ sL[f(t)] - f(0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots.\ 1$$

Using this equation we can find the Laplace transform of $[f''(t)]$

$$L[f''(t)] = sL[f'(t)] - f'(0).$$

Using equation 1 we get

$$L[f''(t)] = s\{sL[f(t)] - f(0)\} - f'(0)$$

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0) \qquad \dots\dots\dots\dots 2$$

Similarly we can find the value of $L[f'''(t)]$ by using equation 1 &2

$$L[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0) \quad \dots\dots\dots\dots 3$$

Similarly, using above method we get

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0).$$

Proved

10.10 LAPLACE TRANSFORM OF THE INTEGRAL OF $f(t)$

If $L[f(t)] = F(s)$ and $f'(t)$ is the derivative of $f'(t)$ then

$$\boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s).}$$

Proof: Let $g(t) = \int_0^t f(t)$ $\quad and\ g(0) = 0$ then $g'(t) = f(t)$.

As we know that $\quad L[g'(t)] = sL[g(t)] - g(0)$

$$\Rightarrow L[g'(t)] = sL[g(t)] \qquad as\ g(0) = 0$$

$$\Rightarrow \qquad L[g(t)] = \frac{1}{s}L[g'(t)].$$

Using the value of $g(t) = \int_0^t f(t)$ $\quad and\ g'(t) = f(t)$ we will get

$$\Rightarrow L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

$$\Rightarrow \boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s).} \qquad \boldsymbol{Proved}$$

10.11 LAPLACE TRANSFORM OF SOME IMPORTANT FUNCTIONS

1. $\boldsymbol{L(1) = \frac{1}{s}}$.

Proof: From the definition of Laplace transform, the Laplace transform of *L(1)* can be written as

$$L(1) = \int_0^\infty 1.e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}.$$

$$[As\ e^{-\infty} = 0\ and\ e^0 = 1]$$

2. $\boldsymbol{L(e^{at}) = \frac{1}{s-a}}$. $\quad where\ s > a$

Proof: As per the definition of Laplace transform

$$L(e^{at}) = \int_0^\infty e^{at}.e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}. \qquad [As\ e^{-\infty} = 0\ and\ e^0 = 1]$$

3. $\boldsymbol{L(\sin at) = \frac{a}{s^2+a^2}}$.

Proof:

$$L(\sin at) = L\left(\frac{e^{iat} - e^{-iat}}{2i}\right) = \frac{1}{2i}\left[L\left(e^{iat}\right) - L\left(e^{-iat}\right)\right].$$

$$\text{Since}\ \sin\theta = \frac{1}{2i}\left[e^{i\theta} - e^{-i\theta}\right]$$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\sin at) = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{(s+ia)-(s-ia)}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2}.$$

4. $\boldsymbol{L(\cos at) = \frac{s}{s^2+a^2}}$.

Proof: $L(\cos at) = L\left(\frac{e^{iat} + e^{-iat}}{2}\right) = \frac{1}{2}\left[L\left(e^{iat}\right) + L\left(e^{-iat}\right)\right].$

$$\text{Since}\ \cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\cos at) = \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2+a^2}\right] = \frac{s}{s^2+a^2}. \quad [as\ we\ know\ i^2 = -1]$$

Proved

5. $\boldsymbol{L(\sinh at) = \frac{a}{s^2 - a^2}}$ **.**

Proof:

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2i}\right) = \frac{1}{2}[L(e^{at}) - L(e^{-at})].$$

Since $\sinh\theta = \frac{1}{2}\left[e^{\theta} - e^{-\theta}\right]$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\sin at) = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}. \qquad Proved$$

6. $\boldsymbol{L(\cosh at) = \frac{s}{s^2 - a^2}}$ **.**

Proof:

$$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2}[L(e^{at}) + L(e^{-at})].$$

Since $\cos\theta = \frac{1}{2}\left[e^{\theta} + e^{-\theta}\right]$

Using Laplace transform $L(e^{at}) = \frac{1}{s-a}$ *we will get*

$$L(\cosh at) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}.$$

Proved

7. $\boldsymbol{L(t^n) = \frac{n!}{s^{n+1}}}$ $\boldsymbol{where\ n\ and\ s\ are\ positive}$

Proof: $\boldsymbol{L(t^n)} = \int_0^\infty t^n . e^{-st} dt$

$$now\ using \quad st = u \Rightarrow t = \frac{u}{s} \ \Rightarrow dt = \frac{du}{s}.$$

We will get

$$L(t^n) = \int_0^\infty \frac{u^n}{s^n} . e^{-u} \frac{du}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-u}\, u^n\, du$$

$$we\ know\ that \int_0^\infty e^{-u}\, u^n\, du = \Gamma(n+1) = n!.$$

Hence we have

$$L(t^n) = \frac{n!}{s^{n+1}}. \qquad Proved$$

Example 1: Find the Laplace transform of $sin^3\, 2t$.

Solution: we have given $f(t) = sin^3\, 2t$

And, we also know that $\sin 3\theta = 3 \sin\theta - 4 sin^3\, \theta$.

From above equation $sin^3\, 2t = \frac{1}{4}[3 \sin 2t - \sin 6t]$.

Hence, $L[sin^3\, 2t] = \frac{1}{4}[3\, \mathrm{L}(\sin 2t) - \mathrm{L}(\sin 6t)]$

$$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right].$$

$$As\ we\ know\ that \quad L(\sin at) = \frac{a}{s^2+a^2}$$

$$= \frac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}.$$

Ans

Example 2: Find the Laplace transform of $sin\, 2t\, sin 3t$

Solution: we have given $f(t) = sin\, 2t\, sin 3t$

$$Using\ relation\ \rightarrow sin\ A\ sinB = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\gg sin\ 2t\ sin3t = \frac{1}{2}[\cos t - \cos 5t].$$

So $L(sin\ 2t\ sin3t) = \frac{1}{2}[L(\cos t) - \mathrm{L}(\cos 5t)]$.

Now using relation $L(\cos at) = \frac{s}{s^2+a^2}$.

We have $L(sin\ 2t\ sin3t) = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right] = \frac{12s}{(s^2+1)(s^2+25)}$.

Ans

Example 3: Show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$. $Given\ that$ $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$

Solution: Suppose $F(t) = \left(2\sqrt{\frac{t}{\pi}}\right)$ $then\ F'(t) = \frac{1}{\sqrt{\pi t}}$ $and\ also\ we\ can\ see\ that\ F(0) = 0$

Now we know that $L[F'(t)] = sL[F(t)] - F(0)$.

Hence $L\left(\frac{1}{\sqrt{\pi t}}\right) = sL\left(2\sqrt{\frac{t}{\pi}}\right) - 0 = s.\frac{1}{s^{3/2}} - 0$

$$\Rightarrow L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}. \qquad Hence\ Proved$$

Example 4: Find the Laplace transform of $t + t^2 + t^3$.

Solution: we have given $f(t) = t + t^2 + t^3$

Now using the relation $L(t^n) = \frac{n!}{s^{n+1}}$

We have$L[\,f(t)] = L(t) + L(t^2) + L(t^3) = \frac{1}{s^2} + \frac{2}{s^3} + \frac{6}{s^4}$ $Ans.$

Example 5: Find the Laplace transform of $t\ cosh\ at$

Solution: we have given $f(t) = t\ cosh\ at$

We know that $L(cosh\ at) = \frac{s}{s^2-a^2}$.

Now using the relation $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

We will get $L(t\, cosh\, at) = -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right) = -\frac{(s^2-a^2).1-s.2s}{(s^2-a^2)^2} = -\frac{(s^2-a^2-2s^2}{(s^2-a^2)^2} = \frac{(s^2+a^2)}{(s^2-a^2)^2}$. *Ans.*

Self Assessment Question (SAQ) 1: Find the Laplace transform of $2sin\, 2t \cos 4\, t$.

Self Assessment Question (SAQ) 2: Find the Laplace transform of $t^{\frac{1}{2}}$.

Self Assessment Question (SAQ) 3: Find the Laplace transform of $F(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & 2 \le t < \infty \end{cases}$

Self Assessment Question (SAQ) 4: Find the Laplace transform of $1 + sin\, 2t$.

Self Assessment Question (SAQ) 5: Find the Laplace transform of $sinh^3\, t$.

10.12 LAPLACE TRANSFORM OF $\frac{1}{t}f(t)$

If $L[f(t)] = F(s)$ then If $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s)\, ds$.

Proof: As per the Laplace transform

$$L[f(t)] = F\,(s) = \int_0^\infty e^{-st} f(t)dt.$$

Integrating with respect to *s,* we get

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= -\int_0^\infty \frac{f(t)}{t}[e^{-st}]_0^\infty dt = -\int_0^\infty \frac{f(t)}{t}[e^{-\infty} - e^{-st}]dt$$

$$= -\int_0^\infty \frac{f(t)}{t}[0 - e^{-st}]dt = \int_0^\infty e^{-st}\left[\frac{1}{t}f(t)\right]dt = L\left[\frac{1}{t}f(t)\right]$$

$$\Rightarrow \boldsymbol{L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)ds.} \qquad \textit{Proved}$$

10.13 LAPLACE TRANSFORM OF $t.f(t)$

$$\boldsymbol{L[t^n.f(t)] = (-1)^n \frac{d^n}{ds^n}F(s).}$$

Example 6: Find the Laplace transform of $\frac{\sin 2t}{t}$.

Solution: $L(\sin 2t) = \frac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty = \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$=\cot^{-1}\frac{s}{2}.$$ Ans

Example 7: Find the Laplace transform of the function

$$f(t) = te^{-t}\sin 2t\,.$$

Solution: $L[\sin 2t] = \frac{2}{s^2+4}$

$$L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4} = F(s) \qquad say$$

$$L[e^{-t}\sin 2t] = -F'(s) = \frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right] = \frac{2.2(s+1)}{[(s+1)^2+4]^2} = \frac{4(s+1)}{[(s+1)^2+4]^2}.$$ Ans.

10.14 UNIT STEP FUNCTION

The unit step function is defined as follows:

$$\boldsymbol{u(t-a)} = \begin{cases} \mathbf{0}, & \boldsymbol{when\ t < a} \\ \mathbf{1}, & \boldsymbol{when\ t \geq a} \end{cases} \qquad \boldsymbol{where\ a \geq 0}.$$

10.15 LAPLACE TRANSFORM OF UNIT STEP FUNCTION

$$\boldsymbol{L[u(t-a)] = \frac{e^{-as}}{s}}.$$

Proof: Using the definition of Laplace transform, we have

$$\boldsymbol{L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)dt}.$$

Now using the condition of unit step function

$$L[u(t-a)] = \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= \int_0^a e^{-st}0.dt + \int_a^\infty e^{-st}1.dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)] = \frac{e^{-as}}{a}. \qquad Proved$$

Example 8: Convert the following function in terms of unit step function and then find the Laplace Transform

$$f(t) = \begin{cases} 6, & when\ t < 2 \\ 4, & when\ t \geq 2 \end{cases}$$

Solution: Given that

$$f(t) = \begin{cases} 6, & when\ t < 2 \\ 4, & when\ t \geq 2 \end{cases}$$

This further can be written as

$$f(t) = \begin{cases} 6+0, & when\ t < 2 \\ 6-2, & when\ t \geq 2 \end{cases} = 6 + \begin{cases} 0, & when\ t < 2 \\ -2, & when\ t \geq 2 \end{cases}$$

$$= 6 + (-2)\begin{cases} 0, & when\ t < 2 \\ 1, & when\ t \geq 2 \end{cases} = 6 - 2u(t-2)$$

[Using the condition of unit step function]

$$L[f(t)] = 6L(1) - 2L[u(t-2)] = \frac{6}{s} - 2\frac{e^{-2s}}{s}. \qquad Ans$$

10.16 PERIODIC FUNCTIONS:

If $f(t)$ be a periodic function with period $T, \Rightarrow f(t+T) = f(t)$ then

$$\boldsymbol{L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}}.$$

Proof: As we know

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

This can be written as in the following manner

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \cdots$$

Now substituting $t = u + T,\ t = u + 2T,\ \ldots and\ dt = du$ in second integral, third integral, and so on respectively, we will get

$$L[f(t)] = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u+T)du + e^{-2sT}\int_0^T e^{-su} f(u+2T)du + \cdots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \cdots$$

As $f(u)$ be a periodic function with period $T, \Rightarrow f(u+T) = f(u+2T) = \cdots = f(u)$.

Now we can write

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-st} f(t)du + e^{-2sT} \int_0^T e^{-st} f(u)dt + \cdots$$

$$= \int_0^T e^{-st} f(t)dt[1 + e^{-sT} + e^{-2sT} + \cdots].$$

$Now\ using\ the\ condition\ [1 + x + x^2 + x^3 +\ \ldots = \frac{1}{1-x}$ we have

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}. \qquad Proved$$

Example 8: Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3.$$

Solution: $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)dt$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} \left(\frac{2}{3}t\right) dt = \frac{1}{1-e^{-3s}} \frac{2}{3} \left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s(1-e^{-3s})} + \frac{2}{3s^2}. \qquad \text{Ans.}$$

Self Assessment Question (SAQ) 6: Find the Laplace transform of $t\ cost$.

Self Assessment Question (SAQ) 7: Find the Laplace transform of $\frac{1}{t} sin^2 t$.

Self Assessment Question (SAQ) 8: Find the Laplace transform of

$$f(t) = \begin{cases} t - 1, & 1 < t < 2 \\ 0 & elsewehre \end{cases}$$

Self Assessment Question (SAQ) 9: Find the Laplace transform of the periodic function

$$f(t) = e^t \quad for\ 0 < t < 2\pi.$$

10.17 SOME IMPORTANT FORMULAE OF LAPLACE TRANSFORM

S.No.	$f(t)$	$F(s)$
1	e^{at}	$\frac{1}{s-a}$
2	$\sin at$	$\frac{a}{s^2+a^2}$
3	$\cos at$	$\frac{s}{s^2+a^2}$
4	$\sinh at$	$\frac{a}{s^2-a^2}$
5	$\cosh at$	$\frac{s}{s^2-a^2}$
6	t^n	$\frac{n!}{s^{n+1}}$
7	$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$
8	$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$
9	$\frac{t}{2a}\sin at$	$\frac{s}{(s^2+a^2)^2}$
10	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

10.18 INVERSE LAPLACE TRANSFORM

If $F(s)$ is the Laplace Transform of a function $f(t)$, then $f(t)$ is known as Inverse Laplace Transform.

$$f(t) = L^{-1}[F(s)].$$

The Inverse Laplace Transform is very useful to solving the differential equations without finding the general solution and arbitrary constants.

10.18 SOME IMPORTANT FORMULAE OF INVERSE LAPLACE TRANSFORM

S.No.	$F(s)$	$f(t) = L^{-1}[F(s)]$
1	$\frac{1}{s-a}$	e^{at}
2	$\frac{a}{s^2+a^2}$	$\sin at$
3	$\frac{s}{s^2+a^2}$	$\cos at$

4	$\frac{a}{s^2-a^2}$	$\sinh at$
5	$\frac{s}{s^2-a^2}$	$\cosh at$
6	$\frac{n!}{s^{n+1}}$	t^n
7	$\frac{a}{(s-b)^2+a^2}$	$e^{bt}\sin at$
8	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
9	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$
10	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
11	$\frac{1}{s}$	1

Example 9: Prove that $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Solution: we know that $L^{-1}\left[\frac{1}{s^n}\right] = \left[\frac{t^{n-1}}{(n-1)!}\right] = \left[\frac{t^{n-1}}{\Gamma n}\right]$.

Using above relation we can write $L^{-1}\left[\frac{1}{s^{1/2}}\right] = \left[\frac{t^{\frac{1}{2}-1}}{\Gamma\frac{1}{2}}\right] = \left[\frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}\right]$ $\quad Since\ \Gamma\frac{1}{2} = \sqrt{\pi}$

$$\Rightarrow L^{-1}\left[\frac{1}{s^{1/2}}\right] = \left[\frac{1}{\sqrt{\pi t}}\right] \quad \Rightarrow \left[\frac{1}{s^{1/2}}\right] = L\left[\frac{1}{\sqrt{\pi t}}\right]. \quad Proved$$

Example 10: Find the inverse Laplace Transform of the following:

(i) $\frac{1}{s-3}$ (ii) $\frac{1}{s^2-25}$ (iii) $\frac{s}{s^2+16}$

(iv) $\frac{1}{s^2+9}$ (v) $\frac{1}{(s-2)^2+1}$ (vi) $\frac{s-1}{(s-1)^2+4}$.

Solution.

(i) $L^{-1}\frac{1}{s-3} = e^{3t}$. $\quad [since\ L^{-1}\frac{1}{s-a} = e^{at}]$

(ii) $L^{-1}\frac{1}{s^2-25} = L^{-1}\frac{1}{5}\left\{\frac{5}{s^2-(5)^2}\right\} = \frac{1}{5}sinh5t.$ $\quad [since \quad L^{-1}\frac{a}{s^2-a^2} = sinh\ at]$

(iii) $L^{-1}\frac{s}{s^2+16} = L^{-1}\frac{s}{s^2+(4)^2} = \cos 4t.$ $\quad [since\ since \quad L^{-1}\frac{s}{s^2+a^2} = cos\ at\ ]$

(iv) $L^{-1}\frac{1}{s^2+9} = L^{-1}\frac{1}{s^2+(3)^2} = \frac{1}{3}\sin 3t.$ $\quad [since\ since \quad L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a} sin\ at\ ]$

(v) $L^{-1}\frac{1}{(s-2)^2+1} = e^{2t}\sin t.$ $\quad \left[since\ since \quad L^{-1}\frac{1}{(s-b)^2+a^2} = e^{bt} sin\ at\right]$

(vi) $L^{-1}\frac{s-1}{(s-1)^2+4} = L^{-1}\frac{s-1}{(s-1)^2+(2)^2} = e^t\cos 2t.$ $\quad \left[since\ since \quad L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at} cos\ bt\right]$

Example 11: Find $L^{-1}\frac{s^2+3s+8}{s^3}$.

Solution: Here, we have

$$L^{-1}\frac{s^2+3s+8}{s^3} = L^{-1}\left[\frac{1}{s} + \frac{3}{s^2} + \frac{8}{s^3}\right] = 1 + \frac{3t}{1!} + \frac{8}{2!}t^2 \qquad [\ since \quad L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}]$$

$$= 1 + 3t + 4\,t^2. \qquad Ans$$

Self Assessment Question (SAQ) 10: Find the Inverse Laplace transform of $\frac{1}{s-5}$.

Self Assessment Question (SAQ) 11: Find the Inverse Laplace transform of $\frac{2s-5}{9s^2-25}$.

10.20 MULTIPLICATION BY S

$$\boldsymbol{L^{-1}[sF(s)\ \ ] = \frac{d}{dt}f(t) + f(0)\delta(t).}$$

Example 12: Find the Inverse Laplace Transform of $\frac{s}{s^2+4}$.

Solution: we know that $L^{-1}\frac{1}{s^2+a^2} = \sin at.$

Hence $\quad L^{-1}\frac{1}{s^2+4} = \sin 2t$

And $\quad L^{-1}\frac{s}{s^2+4} = \frac{d}{dt}(sin\ 2t) + \sin(0)\ \delta(t) = 2\cos 2t. \qquad Ans.$

10.21 DIVISION BY s (MULTIPLICATION BY $\frac{1}{s}$)

$$L^{-1}\left[\ \frac{F\,(s)}{s}\ \right] = \int_0^t [\quad L^{-1}\ \ |F\,(s)|]\ dt\ = \int_0^t f(t)dt.$$

Example 13: Find the Inverse Laplace Transform of

(i) $\frac{1}{s\,(s+a)}$ (ii) $\frac{1}{s(s^2+1)}$ (iii) $\frac{s^2+3}{s(s^2+9)}$.

Solution:

(i) $since \;\; L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$$L^{-1}\left[\;\frac{1}{s(s+a)}\;\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right) dt$$

$$= \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t = \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a}\,[1 - e^{-at}]. \qquad Ans.$$

(ii) $we\ know\ that\;\; L^{-1}\frac{1}{(s^2+1)} = \sin t.$

$$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1)}\right) = \int_0^t L^{-1}\;\left(\frac{1}{s^2+1}\right) = \;\int_0^t \sin t\;dt =$$

$$= [-\cos t]_0^t = [-\cos t + 1] = [1 - \cos t]. \qquad Ans.$$

Self Assessment Question (SAQ) 12: Find the Inverse Laplace transform of $\frac{s^2}{s^2+a^2}$.

Self Assessment Question (SAQ) 13: Find the Inverse Laplace transform of $\frac{1}{s(s^2+a^2)}$.

10.22 FIRST SHIFTING PROPERTY

If the inverse Laplace transform of $F(s)\ is\ f(t)$ such that

$$L^{-1}[F(s)] = \; f(t)$$

$$\text{Then,} \quad L^{-1}F(s+a) = e^{-at}\,L^{-1}[F(s)].$$

Example 14: Find the Inverse Laplace Transform of

(i) $\frac{1}{(s+4)^4}$ (ii) $\frac{s}{s^2+4s+13}$ (iii) $\frac{1}{9s^2+6s+1}$.

Solution:

(i) $we\ know\ that \qquad L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!}$

Then, $L^{-1}\left[\frac{1}{(s+4\,)^4}\right] = \; e^{-4t}L^{-1}\left[\frac{1}{s^4}\right]$ $\qquad using\ first\ shifiting\ property$

$$= e^{-4t}\,\frac{t^3}{3!} = \frac{1}{6}\,e^{-4t}t^3. \qquad Ans.$$

Solution (ii) $\qquad L^{-1}\left(\;\frac{s}{s^2+4s+13}\right) = L^{-1}\frac{s+2-2}{(s+2)^2+(3)^2}$

$$= L^{-1}\frac{s+2}{(s+2)^2+(3)^2} - L^{-1}\frac{2}{(s+2)^2+(3)^2}.$$

Using First shifting property $\Rightarrow L^{-1}F(s+a) = e^{-at}\,L^{-1}[F(s)]$

$$= e^{-2t}L^{-1}\frac{s}{(s)^2+(3)^2} - e^{-2t}L^{-1}\frac{2}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t. \qquad Ans.$$

Solution (iii) $\quad L^{-1}\frac{1}{9s^2+6s+1} = L^{-1}\frac{1}{(3s+1)^2}$

$$= \frac{1}{9}L^{-1}\frac{1}{(s+\frac{1}{3})^2} = \frac{1}{9}e^{-t/3}L^{-1}\frac{1}{s^2} \qquad [\,Using\ First\ shifting\ property]$$

$$= \frac{1}{9}e^{-t/3}t = \frac{te^{-t/3}}{9}\ . \qquad Ans.$$

10.23 SECOND SHIFTING PROPERTY

$$L^{-1}[\,e^{-as}\ F\,(s)\,] = f(t-a)u(t-a).$$

Example 15: Obtain Inverse Laplace Transform of

$(i)\quad \frac{e^{-\pi s}}{(s+3\,)} \qquad (ii)\quad \frac{e^{-s}}{(\,s+1\,)^3}.$

Solution:

(i) As we know that

$$L^{-1}\ \frac{1}{s+3} = e^{-3t}$$

Now using second shifting theorem we can find the inverse Laplace transform of

$$L^{-1}\frac{e^{-\pi s}}{(s+3)} = e^{-3(\,t-\pi)}u(t-\pi). \qquad since\ [L^{-1}[\,e^{-as}\ F\,(s)] = f(t-a)u(t-a)]$$

Ans.

(ii) *As we know that* $\quad L^{-1}\frac{1}{s^3} = \frac{t^2}{2!}.$

Then $\quad L^{-1}\frac{1}{(\,s+1\,)^3}e^{-t}\frac{t^2}{2!}\ . \qquad [using\ first\ shifiting\ property]$

Hence

$$L^{-1}\ \frac{e^{-s}}{(\,s+1\,)^3} = e^{-(\,t-1)}\frac{(\,t-1\,)^2}{2!}\ \ u(t-1). \qquad [using\ second\ shifiting\ property]$$

Ans.

10.24 INVERSE LAPLACE TRANSFORMS OF DERIVATIVES:

$L^{-1}\left[\frac{d}{ds}\ F\ (s)\right] = -tL^{-1}[F(s)] = -tf(t)$

$\Rightarrow L^{-1}[F(s)] = -\ \frac{1}{t}L^{-1}\left[\frac{d}{ds}\ F\ (s)\right].$

Example 16: Find $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$.

Solution: $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(\frac{s+1}{s-1})\right]$

$using\ inverse\ laplace\ transform\ of\ derivatives$

$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\ \log(s-1)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$= -\frac{1}{t}[e^{-t} - e^{t}] = \frac{1}{t}[e^{t} - e^{-t}].$ $\qquad Ans.$

10.25 INVERSE LAPLACE TRANSFORM OF INTEGRALS

$L^{-1}\left[\int_s^\infty F(s)ds\right] = \frac{f(t)}{t} = \frac{1}{t}L^{-1}[F(s)]$

$\Rightarrow L^{-1}[F(s)] = L^{-1}\left[\int_s^\infty F(s)ds\right].$

Example17: Find the Inverse Laplace Transform of $\frac{2s}{(s^2+1)^2}$.

Solution: we have to find $L^{-1}\left(\frac{2s}{(s^2+1)^2}\right)$.

We will solve this using inverse Laplace transform of integrals

$$L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = tL^{-1}\int_s^\infty \frac{2sds}{(s^2+1)^2}$$

$$tL^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty = tL^{-1}\left[-0 + \frac{1}{s^2+1}\right] = tL^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= t\ sin\ t. \qquad Ans.$$

10.26 INVERSE LAPLACE TRANSFORM BY PARTIAL FRACTION METHOD

Example 18: Find the Inverse Laplace Transform of $\frac{1}{s^2-5s+6}$.

Solution: Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}. \qquad Ans.$$

Example 19: Find the Inverse Laplace Transform of $\frac{s+1}{s^2-6s+25}$.

Solution:

$$L^{-1}\left[\frac{1}{s^2-6s+25}\right] = L^{-1}\left[\frac{1}{(s-3)^2+(4)^2}\right] = L^{-1}\left[\frac{s-3+4}{(s-3)^2+(4)^2}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right] + L^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$

$$= e^{3t}\cos 4t + e^{3t}\sin 4t. \qquad [Using\ first\ shifiting\ property] \qquad Ans.$$

Self Assessment Question (SAQ) 14: Find the Inverse Laplace transform of $\frac{s}{(s+7)^4}$

Self Assessment Question (SAQ) 15: Find the Inverse Laplace transform of $\frac{e^{-s}}{(s+2)^3}$.

Self Assessment Question (SAQ) 16: Find the Inverse Laplace transform by partial fraction method of $\frac{1}{s^2-7s+12}$.

10.27 SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants. The method wil be clear from the following examples:

Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t). \qquad y(0) = K_0, \qquad and\ y'(0) = K_1$$

where *a* and *b* are constant. Here is the given **input** (*driving force*) applied to the mechanical or electrical system and is the **output** (*response to the input*) to be obtained.

In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation.

This is an algebraic equation for the transform $Y = L(y)$ obtained by transforming the given differential equation using the Laplace transform of derivatives,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s).$$

Where $R(s) = L(r)$.

Now collecting the Y-terms, we have the subsidiary equation as follows

$$(s^2 + as + b)Y = (s + 1)y(0) + y'(0) + R(\,s).$$

Step 2. Solution of the subsidiary equation by algebra.

We divide by and use the so-called **transfer function**

$$Q(s) = \frac{1}{(s^2 + as + b)}.$$

This gives the solution

$$Y = (s + 1)y(0)Q(s) + y'(0)Q(s) + R(\,s)Q(s).$$

Note that ***Q depends neither on r(t) nor on the initial conditions*** (but only on *a* and *b*).

Step 3. Inversion of Y to obtain $\boldsymbol{y = L^{-1}Y}$.

Now take the inverse Laplace transform to get the solution of differential equations.

Example 20: Solve the following equation by Laplace transform

$$y'' - y = t\,; \qquad y(0) = 1 \; and \; y'(0) = 1.$$

Solution: The given equation is

$$y'' - y = t\,; \qquad y(0) = 1 \; and \; y'(0) = 1$$

Step 1: Taking the Laplace transform of the given equation, we get the subsidiary equation

$$L(y'') - L(y) = L(t).$$

Now using the condition of Laplace transform of derivatives with $L(y) = Y$

$$L[f^n(t)] \;=\; s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0).$$

We get

$$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2}$$

$$(s^2 - 1)Y = sy(0) + y'(0) + \frac{1}{s^2} = s + 1 + \frac{1}{s^2}. \qquad As\; given y(0) = 1 \; and \; y'(0) = 1$$

Step 2: Transfer function is given by

$$Q(s) = \frac{1}{(s^2-1)} \qquad \text{and hence}$$

$$Y = (s+1)Q(s) + \frac{Q(s)}{s^2} = \frac{(s+1)}{(s^2-1)} + \frac{1}{s^2(s^2-1)}.$$

On simplifying

$$Y = \frac{1}{(s-1)} + \left\{\frac{1}{(s^2-1)} - \frac{1}{s^2}\right\}.$$

Step 3: Now taking the inverse Laplace transform to get the solution of differential equation

$$y(t) = L^{-1}Y = L^{-1}\left\{\frac{1}{(s-1)}\right\} + L^{-1}\left\{\frac{1}{(s^2-1)} - \frac{1}{s^2}\right\}$$

$$y(t) = L^{-1}\left\{\frac{1}{(s-1)}\right\} + L^{-1}\left\{\frac{1}{(s^2-1)}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$y(t) = e^t + \sinh t - t. \qquad Ans.$$

Self Assessment Question (SAQ) 17: Solve the differential equation using Laplace transform method

$$\frac{d^2y}{dx^2} + y = 0, \quad where\ y = 1\ and\ \frac{dy}{dx} = -1\ at\ x = 0.$$

Self Assessment Question (SAQ) 18: Solve the differential equation using Laplace transform method

$$y'' + 4y' + 4y = 6e^{-t}, \quad where\ y(0) = -2\ and\ y'(0) = 8.$$

10.28 SUMMARY

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. Firstly we have learned about Laplace transform and their various properties and theorems. Using these properties we have find the Laplace transform of some very important function which were used to solve many important problems. Later we have learned about Inverse Laplace transform and their various properties. And finally we used these properties of Laplace transform and inverse Laplace transform to solve the differential equation and many other important problems which is in the form differential equations. To understand this unit more clearly various solved examples are includes almost in each section. For students assessment self assessment question is also incorporated throughout the chapter.

10.29 GLOSSARY

Domain: one system, one type of region.

Subsidiary equation: contributory equation, secondary equation.

Arbitrary constants: random constant.

Cumbersome method: unmanageable method, bulky method.

10.30. TERMINAL QUESTIONS

Find the Laplace transform of the following:

1. $\sin t \cos t$.
2. $t\, e^{at}$.
3. $t \sinh a\, t$.
4. Prove that $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.
5. $\frac{1}{t}(1 - e^t)$.

Obtain the Inverse Laplace Transform of the following:

6. $\frac{s+8}{(s^2+4s+5)}$.
7. $\frac{s}{(s+3)^2+4}$.
8. $\frac{s}{(s+7)^4}$.
9. $\frac{s+2}{s^2-2s-8}$.
10. $\frac{s}{s^2+6s+25}$.

Solve the differential equation using Laplace transform method

11. $y'' + 9y = 6\cos 3t$. $\quad where\ y(0) = 2\ and\ y'(0) = 0$
12. $y'' - 3y' + 2y = 4e^{2x}$. $\quad where\ y(0) = -3\ and\ y'(0) = 5$
13. $y'' + 2y' + y = te^{-t}$, $\quad where\ y(0) = 1\ and\ y'(0) = -2$

10.31 ANSWERS

Self Assessment Question (SAQ)

1. $\frac{3}{s^2+9} - \frac{1}{s^2+1}$.

2. $\sqrt{\frac{\pi}{s}}$.
3. $\frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} + \frac{3}{s^2}e^{-2s} + \frac{2}{s^3}e^{-2s}$.
4. $\frac{1}{s} + \frac{2}{s^2+4}$.
5. $\frac{6}{(s^2-1)(s^2-9)}$.
6. $\frac{(s^2-1)}{(s^2+1)^2}$.
7. $\frac{1}{4}log\frac{s^2+4}{s^2}$.
8. $\frac{e^{-s}-e^{-2s}}{s^2} - \frac{e^{-2s}}{s}$.
9. $\frac{e^{2(1-s)\pi}-1}{(1-s)(1-e^{-2\pi s})}$.
10. e^{5t}.
11. $\frac{2}{9}cosh\frac{5t}{3} - \frac{1}{3}sinh\frac{5t}{3}$.
12. $-a\,sin\,at + 1$.
13. $\frac{1-\cos at}{a^2}$.
14. $e^{-7t}\frac{t^2}{6}(3-7t)$.
15. $e^{-(t-2)}\frac{(t-2)^2}{2}u(t-2)$.
16. $e^{4t} - e^{3t}$.
17. $y = \cos x - \sin x$.
18. $y = 6e^{-t} - 8e^{-2t} - 2te^{-2t}$.

Terminal Questions:

1. $\frac{1}{s^2+4}$.
2. $\frac{1}{(s-a)^2}$.
3. $\frac{2as}{(s^2-a^2)^2}$.
5. $log\frac{s-1}{s}$.
6. $e^{-2t}(\cos t + 6\sin t)$.
7. $e^{-3t}(\cos 2t - 1.5\, sin\, t)$.
8. $e^{-7t}\frac{t^2}{6}(3 - 7t)$.
9. $e^{-t}(cosh\,3t + sinh\,3t)$.

10. $e^{-3t}\left[\cos 4t - \frac{3}{4}\sin 4t\right]$.
11. $y = 2\cos 3t - t\,sin\,3t$.
12. $y = -7e^{x} + 4e^{2x} + 4xe^{2x}$.
13. $y = \left(1 - t + \frac{t^3}{6}\right)e^{-t}$.

10.32. REFERENCES

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10.33. SUGGESTED READINGS

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