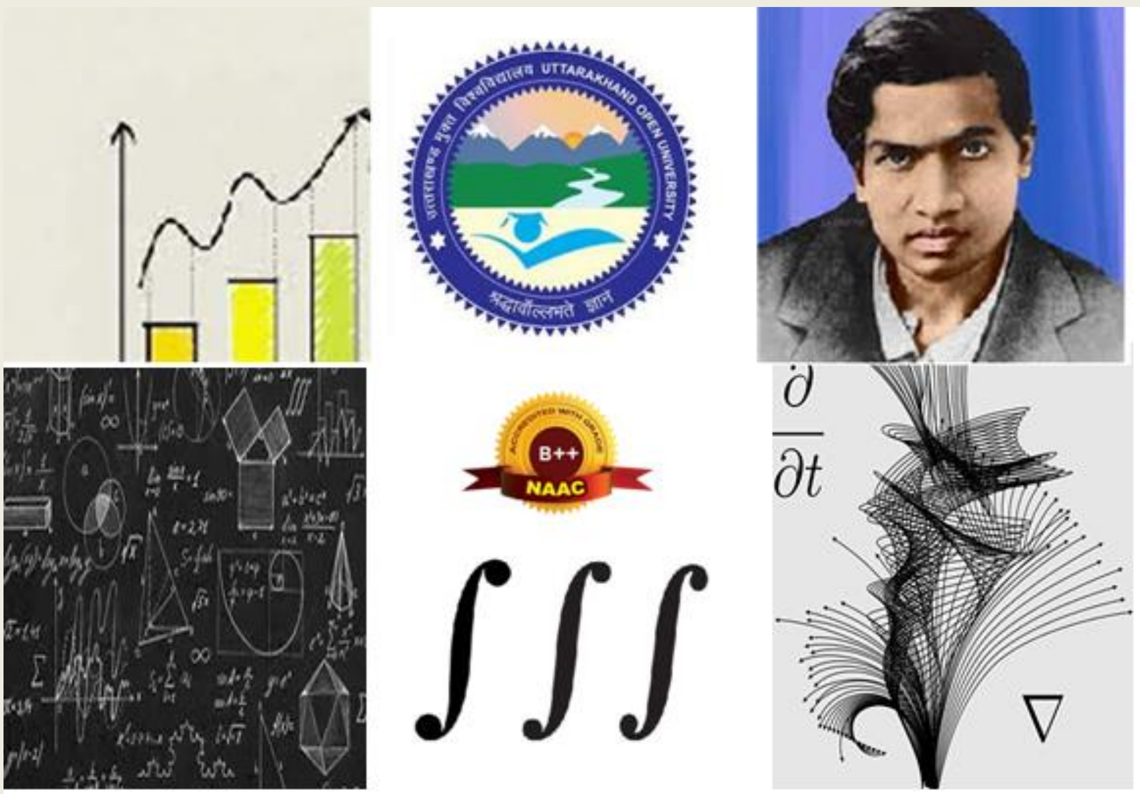
Master of Science

MATHEMATICS

Fourth Semester

MAT 611

GEOMETRY



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COURSE NAME: GEOMETRY

COURSE CODE: MAT 611

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MAT - 611

COURSE INFORMATION

The present self-learning material **"GEOMETRY"** has been designed for **M.Sc. (Fourth Semester)** learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a Mixture of Four Block.

First block is **Space Curves and Its Properties,** in this block Normal and Binormal. Curvature and Torsion. Fundamental Existence Theorem for space curves. Intrinsic Properties of a Surface, Osculating circle. Osculating sphere defined Clearly.

Second block is **Fundamental Forms,** in this block Fundamental form of first and second kind. Angle between Parametric Curves, Orthogonal Trajectories defined clearly.

Third block is **Local Non- Intrinsic Properties Of A Surface,** in this block Normal Curvature, Principal Curvature, Meusnier's theorem, Minimal Surface, Rodrigue Formula, Euler's Theorem are defined.

Fourth block is **Tensor Analysis,** in this Dummy Suffix Real Suffix, Transformation of Coordinate and Contravariant, Covariant, Addition, Subtraction & Multiplication of Tensor. Inner Product. Metric and angle between two vector & Coordinate Curve. Gradient of a Scalar Function, Christoffel Symbols or Christoffel Brackets. Tensor Laws of Transformation of Christoffel Symbols. Divergence and Curl of a Vector are defined.

Adequate number of illustrative examples and exercises have also been included to enable the leaners to grasp the subject easily.

COURSE NAME: GEOMETRY

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BLOCK-I

SPACE CURVE AND ITS PROPERTIES

UNIT 1: *SPACE CURVES*

CONTENTS:

1.1 *INTRODUCTION*

In differential geometry, the study of smooth spaces and shapes, the fundamental theorem of space curves states that the shape, size, and scale of a regular curve in three-dimensional space is completely determined by its curvature and torsion. different space curves are only distinguished by how they bend and twist. Quantitatively, this is measured by the differential-geometric invariants called the curvature and the torsion of a curve. The fundamental theorem of curves asserts that the knowledge of these invariants completely determines the curve.

1.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Space curve

(ii) Class or function of a curve

(iii) Order of contact between curves and surfaces.

(iv) Normal and Binormal

1.3 *SPACE CURVE*

A curve in Euclidean space of three dimension is the locus of a point whose position vector r with respect to origin say O is function of single parameter t. The cartesian coordinates (x, y, z) of point P are called components of r and are the functions of parameter t. Therefore, we can express the equation of curve in terms of a single parameter t.

Thus $r(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} + z(t)\,\boldsymbol{k}$ represents a curve in space.

The curve is known as a plane curve if it lies on a plane, otherwise it is said to be a skew twisted or tortuous.

The parametric equation of the curve are

$$x = x(t),\ \ y = y(t),\ \ z = z(t)$$

Where, x, y, z are real valued functions of a single real parameter t ranging over a set of values $a \leq t \leq b$.

1.4 *CLASS OR A FUNCTION OF A CURVE*

Let I denote a real interval and let m be a positive integer. Then we say that a real valued function f defined on I is of class m if f has a continuous derivative of m^{th} order at every point I.

In case f is differential an infinitely many number of times, it is said to be of class ∞ or a C^{∞} function.

Note: A regular vector valued function of class m is known as a path of class m.

1.5 *ORDER OF CONTACT BETWEEN CURVES AND SURFACES*

Consider a curve C and surface S given by the following equations

x = f(t), y = g(t) , z = h(t) ……… (1)

F(x, y, z) = 0 ……… (2)

The value of t corresponding to the points which are common to C and S are given by the solution of equation obtained from (1) and (2) on eliminating x, y, z i.e. by

F[f(t), g(t), h(t)] = 0 or F(t) = 0 …….. (3)

Let t_0 be one solution of (3), then F$(t_0) = 0$

Now expanding F(t) about t_0 by Taylor's theorem in power of (t - t_0), we get

$$F(t) = F(t_0) + (t - t_0)F'(t_0) + \frac{(t-t_0)^2}{2!}F''(t_0) + \ldots + \frac{(t-t_0)^n}{n!}F^n(t_0) + \cdots$$

Since F$(t_0) = 0$

$$\text{Therefore, } F(t) = (t - t_0)F'(t_0) + \frac{(t-t_0)^2}{2!}F''(t_0) + \ldots + \frac{(t-t_0)^n}{n!}F^n(t_0) + \cdots$$

Now the following cases are arise

1. If $F'(t_0) \neq 0$, then t_0 is **simple zero** of F(t) and in this case C and S said to have **simple intersection**.
2. If $F'(t_0) = 0$ and $F''(t_0) \neq 0$ then t_0 is double zero of F(t) and the curve C and surface S have **two-point contact** or **contact of first order**.
3. If $F'(t_0) = F''(t_0) = 0$ and $F'''(t_0) \neq 0$ then t_0 is Triple zero of F(t) and the curve C and surface S have **three-point contact** or **contact of second order**.

In general if
If $F'(t_0) = F''(t_0) = \cdots = F^r(t_0) = 0$ and $F^{r+1}(t_0) \neq 0$ then the curve C and surface S have **(r + 1) point contact** or **contact of r^{th} order**.

1.6 *OSCULATING PLANE*

If P, Q, R are three consecutive points on the curve then the limiting position of the plane PQR as the point Q and R tend to P, is called the oscillating plane at the point P.

Or

The oscillating plane at a point P of a curve of class ≥ 2 is the limiting position of the plane which contains the tangent line at P and a neighboring point Q on the curve as Q$\longrightarrow P$.

Equation of oscillating plane.

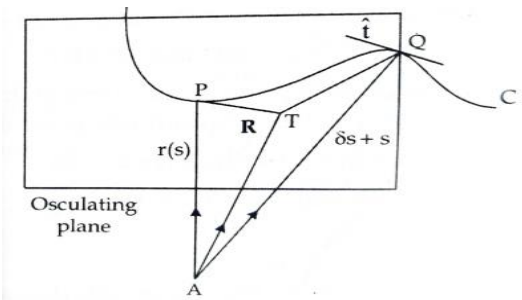


Fig.1.6.1

Let r = r(s) be the given curve C of class ≥ 2 with respect to parameter s, the arc length. Let the arc length be measured from some point say S such that arc AP = s, arc AQ = $s + \delta s$ so that arc PQ = δs. The position vector of P can be taken as r(s). the position vector of the point Q can be taken as r($s + \delta s$). Let R be the position vector of current point T on the plane containing the tangent line at P and the point Q.

The unit tangent vector at P is $\hat{t} = r'(s)$.

$\overrightarrow{PT} = R - r(s), \hat{t} = r'(s)$ and $\overrightarrow{PQ} = r(s + \delta s) - r(s)$ line in the plane TPQ.

Hence their scalar triple product must be zero.

i.e. $[R - r(s)].r'(s) \times [r(s + \delta s) - r(s)] = 0$ **…….. (1)**

equation (1) is equation of the plane TQR. Now expanding r($s + \delta s$) in power of (δs) by Taylor's theorem, we have.

$$r(s + \delta s) = r(s) + \delta s\, r'(s) + \frac{(\delta s)^2}{2!} r''(s) + O(\delta s)^3 \quad \text{……. (2)}$$

Putting the value of $r(s+\delta s)$ from (2) in (1), we get

$$[R-r(s)].r'(s)\times\left[r(s)+\delta s\,r'(s)+\frac{(\delta s)^2}{2!}\,r''(s)+O(\delta s)^3-r(s)\right]=0$$

Therefore, $[R-r(s)].r'(s)\times[\,r''(s)+O(\delta s)^3]=0$

Hence the limiting position of the plane as Q$\longrightarrow P$ i.e. as $\delta s\longrightarrow 0$

$[R-r(s)].r'(s)\times\ r''(s)=0$ **……. (3)**

Provided the vector $r'(s)$ and $r''(s)$ are linearly independent. Equation (3) can be put as $[R-r(s), r'(s),\ r''(s)]=0$ **…….. (4)**

which is the equation of osculating plane at P.

Note: (1) Osculation plane at a point of inflexion.

A point P where $r''=0$ is called a point of inflexion, and tangent line at P is called **inflexional tangent.**

For finding the equation of oscillating plane at a point of inflexion, it will be shown that when a curve is analytic, there exists a definite osculating plane at a point of inflexion P provided that the curve is not a straight line.

Since r' is a vector of constant magnitude unity, it is perpendicular to its derivative

r''so that $r'.\ r''=0$.

Differentiating this we get,

$$r'.\ r'''+r''.\ r''=0 \qquad \textbf{…….. (5)}$$

Again, P is a point of inflexion, $r''=0$. Hence (5) reduced to

$$r'.\ r'''=0$$

This shows that r' is linearly independent of r''' except when $r'''=0$.

Continuing this argument, we shall arrive at the result

$$r'.\ r^{(k)}=0$$

Where, $r^{(k)}$ $(k\geq 2)$ is the first non-zero derivative of r at P. we that have

$$r(s+\delta s)-r(s)=\frac{(\delta s)^k}{k!}\,r^{(k)}(s)+O(\delta s)^{(k+1)}$$

Hence the equation of osculating plane at P is $\boldsymbol{\left[R-r(s), r'(s),\ r^{(k)}(s)\right]=0.}$

Note: (2) Equation of Osculation plane in term of general parameter t.

$$\boldsymbol{[R-r,\dot{r},\ddot{r}]=0.}$$

Note: (3) Equation of Osculation plane in Cartesion form.

Let (X, Y, Z) be the coordinate of the current point T on the osculating plane at P, the coordinates of point P are (x, y, z).

Then $R = Xi + Yj + Zk$ and $r = xi + yj + zk$

Therefore, $R - r = (X - x)i + (Y - y)j + (Z - z)k.$

Again $\dot{r} = \dot{x}i + \dot{y}j + zk$ and $\ddot{r} = \ddot{x}i + \ddot{y}j + \ddot{z}k$

Hence, the equation (5) is equivalent to

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$$

Example 1. Find the equation of oscillating plane at the point 't' on helix

$$r = (a\,cost, a\,sint, ct)$$

Solution. Equation of the helix are

$x = a\,cost, \quad y = a\,sint, \quad z = ct$

Therefore, $\dot{x} = -a\,sint$, $\dot{y} = a\,cost, \quad \dot{z} = c$

And $\ddot{x} = -a\,cost, \quad \ddot{y} = -a\,sint, \quad \ddot{z} = 0$

Therefore, equation of the osculating plane at point t is

$$\begin{vmatrix} X - a\,cost & Y - a\,sint & Z - ct \\ -a\,sint & a\,cost & c \\ -a\,cost & -a\,sint & 0 \end{vmatrix} = 0.$$

Expanding the determinant, we get

$$(Z - ct)[(-asint)(-asint) - (acost)(-cost)] \\ - c[(X - acost)(-asint) - (Y - asint)(-cost)] = 0$$

Or $\quad c[(Xsint - ycost - at] + az = -0.$

1.7 *NORMAL LINES AND NORMAL PLANE, PRINCIPAL NORMAL*

(a) Normal line. The normal line at point P to the given curve is a line perpendicular to the tangent at point to the curve.

For a three-dimensional space curve there will be an infinite number of such normal lines.

(b) Normal plane. The normal plane at point P to the given curve is the plane passing through the point P and perpendicular to the tangent at P.

Thus, we can say that the normal plane at point P on the space curve and **R** be the position vector of any current point on the normal plane at P, thus the vector **(R - r)** lies in the plane. Since the vector $\dot{\mathbf{r}}$ is perpendicular to this plane,

So we have $(\mathbf{R} - \mathbf{r}).\dot{\mathbf{r}} = \mathbf{0}$ .... (1)

Which is the equation of the normal plane at point P.

Again the equation (1) can be put in the form

$$(\mathbf{R} - \mathbf{r}).\boldsymbol{t} = \mathbf{0} \qquad \text{.......} \ (2)$$

Note: Cartesian form:

Let R = Xi + Yi + Zk; r = xi + yj + zk

Therefore, $\dot{\mathrm{r}} = \dot{x}i + \dot{y}j + \dot{z}k$

Putting these values in (1), we get

$$[(X - x)i + (Y - y)j + (Z - z)k].[\dot{x}i + \dot{y}j + \dot{z}k] = 0$$

Or $$(X - x)\dot{x} + (Y - y)\dot{y} + (Z - z)\dot{z} = 0$$

(c) Principal Normal. The normal lying in the osculating plane at a point P on the space curve is called the principal normal at point P.

1.8 *BINORMAL*

The normal perpendicular to the principal normal at point P is called binormal at point P.

Thus, we can say that the binormal at any point P is the line perpendicular to the osculating plane at P. The unit vector along the binormal is denoted by $\boldsymbol{b}$ and we choose the sense of $\boldsymbol{b}$ is such manner that the triad $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ from a right-handed system, i.e. $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$.

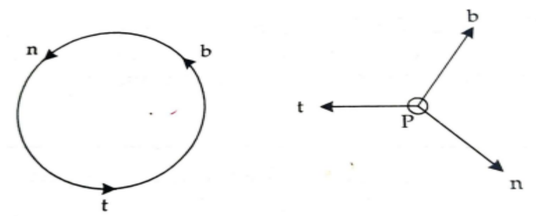


Fig. 1.8.1

Note: Since the binormal is perpendicular to the osculating plane, therefore it must be parallel to the vector $\dot{\mathbf{r}} \times \ddot{\mathbf{r}}$.

1.9 *RECTIFYING PLANE*

The plane containing the tangent and binormal at P is called rectifying plane at P. i.e. it is the plane passing through P and perpendicular to principal normal at P. and equation of this plane is

$$(\mathbf{R}-\mathbf{r}).\mathbf{n}=\mathbf{0}$$

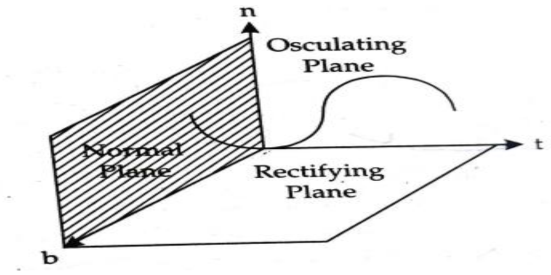


Fig. 1.9.1

Note: Orthonormal triad of Fundamental unit Vectors $\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}$.
We have defined a set of three mutually perpendicular unit vectors associated with each point of a curve. This set of unit orthonormal triad forms a moving trihedral at point P(say) such that

$$\boldsymbol{t}.\boldsymbol{n}=\mathbf{0},\boldsymbol{n}.\boldsymbol{b}=\mathbf{0},\boldsymbol{b}.\boldsymbol{t}=\mathbf{0}$$

And $\boldsymbol{n}\times\boldsymbol{b}=\boldsymbol{t},\boldsymbol{b}\times\boldsymbol{t}=\boldsymbol{n},\boldsymbol{t}\times\boldsymbol{n}=\boldsymbol{b}.$

The vectors $\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}$ are called fundamental unit vectors.

1.10 *FUNDAMENTAL PLANES*

The three planes, osculating plane, normal plane and rectifying plane associated with each point of a curve are called as fundamental planes. These planes are mutually perpendicular and are determined by moving trihedral $\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}$ at the point.

The equations of fundamental planes are:

Osculating plane: it contains $\boldsymbol{t}$ and $\boldsymbol{n}$ is normal to $\boldsymbol{b}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\mathbf{b}=\mathbf{0}.$$

Normal plane: it contains $\boldsymbol{n}$ and $\boldsymbol{b}$ is normal to $\boldsymbol{t}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\mathbf{t}=\mathbf{0}.$$

Rectifying plane: it contains $\boldsymbol{b}$ and $\boldsymbol{t}$ is normal to $\boldsymbol{n}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\mathbf{n}=\mathbf{0}.$$

1.11 *EQUATION OF THE PRINCIPAL NORMAL AND BINORMAL*

Let **r** be the position vector of any point P on the given curve C at which the equation of the principal normal and binormal are to be found. Let $\boldsymbol{R}$ be the position vector of a current point $\boldsymbol{R}$ on the principal, then we have

$\overrightarrow{OP} = r, \overrightarrow{OR} = R$ and $\overrightarrow{PR} = \lambda n$, since n is the unit along the principal normal and λ is some scalar.

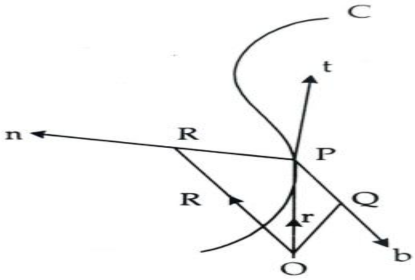


Fig.1.11.1

By triangle law of vectors, we have

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$ or $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{\lambda n}$, which is required equation of the principal normal.

Similarly, if $\boldsymbol{R}$ is the position vector of a current point Q on the binormal, then the equation of binormal is given by

$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{\mu b},$ where μ is a scalar.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Equation of osculating plane is $(\mathrm{R} - \mathrm{r}).\mathrm{b} = 0$.

Problem 2. Equation of Normal plane is $(\mathrm{R} - \mathrm{r}).\mathrm{t} = 0$.

Problem 3. Equation of Rectifying plane is $(\mathrm{R} - \mathrm{r}).\mathrm{t} = 0$.

Problem 4. $b = t \times n$.

Problem 5. Equation of Osculation plane in term of general parameter t is $[R - r, \dot{r}, \ddot{r}] = 0$.

1.12 *SUMMARY*

(i) Osculating Plane: If P, Q, R are three consecutive points on the curve then the limiting position of the plane PQR as the point Q and R tend to P, is called the oscillating plane at the point P.

(ii) Osculating plane: it contains $\boldsymbol{t}$ and $\boldsymbol{n}$ is normal to $\boldsymbol{b}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\,\mathbf{b}=\mathbf{0}.$$

(iii) Normal plane: it contains $\boldsymbol{n}$ and $\boldsymbol{b}$ is normal to $\boldsymbol{t}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\,\mathbf{t}=\mathbf{0}.$$

(iv) Rectifying plane: it contains $\boldsymbol{b}$ and $\boldsymbol{t}$ is normal to $\boldsymbol{n}$**,** its equation is

$$(\mathbf{R}-\mathbf{r}).\,\mathbf{n}=\mathbf{0}.$$

1.13 *GLOSSARY*

(i) Derivatives

(ii) Determinant

(iii) Vector

1.14 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn “Cambridge University Press.”
2. Matrices and Tensors in physics by A.W. Joshi “Wiley Eastern Limited”.
3. Tensors by Ram Bilas Mishra “Hardwari publications Allahabad”.
4. Differential Geometry by Gupta, Malik and Pundir “Pragati Edition’’.

1.15 *TEWRMINAL QUESTIONS*

1. Prove that the necessary and sufficient condition for the curve to be plane is

   $[\dot{r}, \ddot{r},\ \dddot{r}] = 0.$

2. Define rectifying plane, write its equation.
3. Define osculating plane, write its equation.
4. Define normal plane, write its equation.
5. Define equation of principal normal and binormal.

1.16 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. True

CYQ 5. True

UNIT 2: CURVATURE AND TORSION AND FUNDAMENTAL EXISTENCE THEOREM FOR SPACE CURVES

CONTENTS:

2.1 *INTRODUCTION*

In differential geometry, the study of smooth spaces and shapes, the fundamental theorem of space curves states that the shape, size, and scale of a regular curve in three-dimensional space is completely determined by its curvature and torsion. The notion of curvature first began with the discovery and refinement of the principles of geometry by the ancient Greecks circa 800-600 BCE. Curvature was originally defined as a property of the two classical Greek curves, the line and the circle.

In mathematics, **curvature** is any of several strongly related concepts in geometry that intuitively measure the amount by which a curve deviates from being a straight line or by which a surface deviates from being a plane. If a curve or surface is contained in a larger space, curvature can be defined extrinsically relative to the ambient space. Curvature of Riemannian manifolds of dimension at least two can be defined intrinsically without reference to a larger space. First, we show how the notion of torsion emerges in differential geometry. In the context of a Cartan circuit, torsion is related to translations similar as curvature to rotations. Cartan's investigations started by analyzing Einsteins general relativity theory and by taking recourse to the theory of Cosserat continua.

2.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Curvature

(ii) Torsion

(iii) Screw-Curvature

2.3 *CURVATURE*

Definition: The curvature at a point P of a given curve is the arc rate of rotation of tangent at P. its magnitude is denoted by $\boldsymbol{\kappa}$ (Kappa).

To find an expression for the curvature ($\boldsymbol{\kappa}$) at a given point P to a given curve.

Let Q be a point very near to point P on the curve. Arc PQ is δs and let the direction of the tangent at Q makes an angle $\delta\theta$ with the direction of tangent at P.

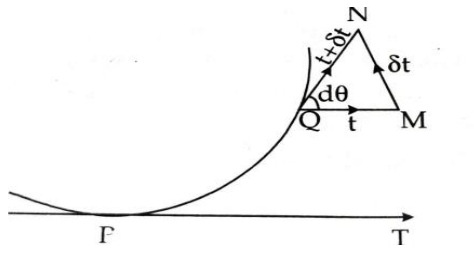


Fig.2.3.1

Again, the unit tangent vector is not unit vector, since its direction changes from point to point. Let $\boldsymbol{t}$ and $\boldsymbol{t} + \boldsymbol{\delta t}$ be its value at P and Q respectively.

If $\overrightarrow{QM} = \boldsymbol{t}$ **and** $\overrightarrow{QN} = \boldsymbol{t} + \boldsymbol{\delta t}$ then we have

$\overrightarrow{MN} = \boldsymbol{\delta t},\ \angle MQN = \delta\theta$ and $|\overrightarrow{QM}| = |\overrightarrow{QN}| = 1.$

From isosceles triangle QMN, we have

MN = 2QM $\sin\frac{1}{2}\delta\theta = 2\sin\frac{1}{2}\delta\theta$

Therefore, $|\delta t| = 2\sin\frac{1}{2}\delta\theta \quad \Rightarrow \left|\frac{\delta t}{\delta\theta}\right| = \frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}$.

taking limits, $\left|\frac{dt}{d\theta}\right| = 1$ ……… (1)

therefore, curvature at P = $\boldsymbol{\kappa} = \lim\limits_{\delta\theta\to 0}\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$, along the direction of the tangent.

$$= \frac{d\theta}{|dt|}\frac{|dt|}{ds} = \left|\frac{d\theta}{dt}\right|\left|\frac{dt}{ds}\right| = \left|\frac{dt}{ds}\right| = \left|\frac{dr'}{ds}\right| = |r''| \quad \text{[using (1)]}$$

Which implies that the curvature is the scalar measure of the arc rate of turning of the unit vector $\boldsymbol{t}$**.** The reciprocal of $\boldsymbol{\kappa}$**,** i.e. $\frac{\mathbf{1}}{\boldsymbol{\kappa}}$ is called radius of curvature and is denoted by ρ.

Deduction: $|r'| = 1 \quad \Rightarrow r'^2 = 1$

Differentiating, we get $2\boldsymbol{r}'.\boldsymbol{r}'' = 0$

i.e. $\boldsymbol{r}''$ is perpendicular to $\boldsymbol{r}'$, i.e. to **t.**

but $\boldsymbol{r}''$ at P lies in the osculating plane at P or $\boldsymbol{r}''$ is a vector in osculating plane perpendicular to **t,** implying that $\boldsymbol{r}''$ is collinear with **n.**

Also $|r''| = \boldsymbol{\kappa}$**,** so we have $\boldsymbol{r}'' = \pm\,\boldsymbol{\kappa}\,\boldsymbol{n}$**.**

we choose the direction of $\boldsymbol{n}$ such that curvature $\boldsymbol{\kappa}$ is always positive**.**

i.e. we take $\boldsymbol{r}'' = \boldsymbol{\kappa}\,\boldsymbol{n}$ **or** $\frac{dt}{ds} = \boldsymbol{\kappa}\,\boldsymbol{n}$ **.**

Theorem 1.12.1 A necessary and sufficient condition for the curve to be a straight line is that curvature $\kappa = 0$ at all points of the curve.

Proof. Vector equation of the straight line can be put as $r = sa + b$ where a and b are constant vectors.

Hence, $t = r' = a$ and $t' = r'' = 0$

Therefore, $\kappa = |r''| = 0$

i.e., if a curve is a straight line, then $\kappa = 0$ i.e. κ is a necessary condition for a curve to be straight line.

Converse. In case $\kappa = 0$ for all points on the curve, then

$$r'' = 0 \qquad \ldots\ldots (1)$$

Integrating (1), we get $\quad r' = a \qquad \ldots\ldots (2)$

Integrating (2), we get $\quad r = as + b \qquad \ldots\ldots (3)$

Where a and b are arbitrary constant vectors. The equation (3) represents a straight line for all values of a and b.

2.4 *TORSION*

Definition: Torsion at point P of a given curve is the arc rate of the change in the direction of the bonormal at P its magnitude is denoted by τ(Tau).

Let Q be a point contiguous to P on the curve. ArcPQ = δs, b and b + δb are the unit binormal vector at P and Q respectively and $\delta\theta$ is the angle between b and b + δb.

If $\overrightarrow{QR} = b$, $\overrightarrow{QS} = b$ then $\overrightarrow{RS} = \delta b$.

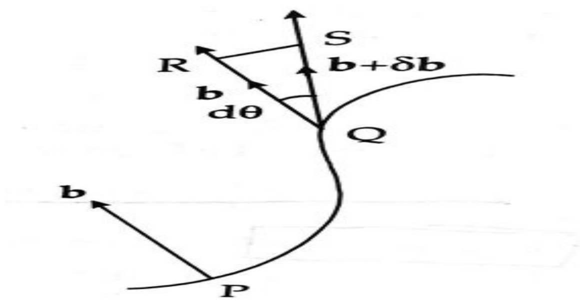


Fig. 2.4.1

Now from then isosceles triangle QRS, we have RS = 2QR sin $\frac{\delta\theta}{2}$

$\Longrightarrow |\overrightarrow{RS}| = 2|\overrightarrow{QR}| \sin\frac{\delta\theta}{2}$

$\Longrightarrow |\delta b| = 2.1. \sin\frac{\delta\theta}{2}$

$\Longrightarrow \left|\frac{\delta b}{\delta\theta}\right| = 2\frac{\sin\frac{\delta\theta}{2}}{\delta\theta} \quad \Longrightarrow \frac{db}{d\theta} = \lim_{\delta\theta\to 0}\frac{\sin\frac{\delta\theta}{2}}{\frac{1}{2}\delta\theta} = 1.$

Thus, by definition, torsion at P

$\tau = \lim_{\delta s\to 0}\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$

$= \left|\frac{d\theta}{db}\right|\left|\frac{db}{ds}\right| = \left|\frac{db}{ds}\right| = |b'|$

$\Longrightarrow \tau$ is the scalar measure of the arc rate of the unit vector b.

The reciprocal of the torsion is called the radius of the torsion and is denoted by σ . thus $\sigma = \frac{1}{\tau}$.

Deduction: We have $t.b = 0$ whence differentiating

$t.b' + t'.b = 0 \quad \Longrightarrow t.b' + \kappa n.b = 0 \quad [\because t' = \kappa n]$

$\Longrightarrow t.b' = 0 \qquad [\because n.b = 0]$

$i.e.$ b' is perpendicular to t.

Further $b.b = 1 \Longrightarrow 2b.b' = 0$

i.e. b' is perpendicular to b.

$\therefore$ b' is normal to the plane containing t and b. $i.e.$ to rectifying plane.

Thus b' is collinear with n.

Thus $b' = \pm\,\tau\, n$

Since b has the opposite direction to n, so negative sign is taken

$\boldsymbol{i.e.}$ $b' = -\,\tau\, n$ or $\frac{d\boldsymbol{b}}{ds} = -\,\tau\, n$

Theorem. A necessary and sufficient condition that a given curve is plane curve is that $\tau = 0$ at all points.

Proof. Let the curve be a plane curve then the tangent and normal at all points of the curve lie in the plane of the curve, i.e. the plane of the curve is the osculating plane at all points of the curve. This implies that the unit vector **b** along the binormal is constant.

$\frac{d\boldsymbol{b}}{ds} = 0$ or $\tau = 0$. Hence the condition is necessary.

Converse. Let $\tau = 0$ at all points of the curve. This implies that $\frac{d\boldsymbol{b}}{ds} = 0$ i.e. **b.**

This implies that $\frac{db}{ds} = 0$ i.e. b is a constant vector.

Again $\frac{\boldsymbol{d}}{\boldsymbol{ds}}(\boldsymbol{r}.\boldsymbol{b}) = \frac{\boldsymbol{dr}}{\boldsymbol{ds}}.\boldsymbol{b} + \boldsymbol{r}.\frac{\boldsymbol{db}}{ds} = \boldsymbol{t}.\boldsymbol{b} + \boldsymbol{r}.\boldsymbol{b}'$

As t and b are orthogonal, we have $t.b = 0$. Also $b' = 0$.

Therefore $\frac{\boldsymbol{d}}{\boldsymbol{ds}}(\boldsymbol{r}.\boldsymbol{b}) = 0$**, i.e.** $\boldsymbol{r}.\boldsymbol{b}$ = constant.

Again $\boldsymbol{b}$ is constant vector of magnitude unity, $\boldsymbol{r}.\boldsymbol{b}$ is the projection of the position vector $\boldsymbol{r}$ on $\boldsymbol{b}$ and is same at all points of the curve by the condition $\boldsymbol{r}.\boldsymbol{b}$ = constant. This implies that the curve must lie in a plane.

2.5 *SCREW-CURVATURE*

The arc rate at which principal normal changes direction $\left(i.e., \frac{d\boldsymbol{n}}{ds}\right)$ is called the screw curvature vector and its magnitude is given by $\sqrt{\{k^2 + \tau^2\}}$.

Note. Serret-Frenet Formulae.

The following set of three relations involving space derivatives of fundamental unit vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ are known as Serret-Frenet Formulae.

1. $\frac{dt}{ds} = k\boldsymbol{n}$ **(2)** $\frac{d\boldsymbol{n}}{ds} = \tau\boldsymbol{b} - k\boldsymbol{t}$ **(3)** $\frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{n}$

Proof: (1) Since $|r'| = 1 \Rightarrow r'^2 = 1$

Differentiating, we get $2\boldsymbol{r}'.\boldsymbol{r}'' = 0$

i.e. $\boldsymbol{r}''$ is perpendicular to $\boldsymbol{r}'$, i.e. to **t.**

but $\boldsymbol{r}''$ at P lies in the osculating plane at P or $\boldsymbol{r}''$ is a vector in osculating plane perpendicular to **t,** implying that $\boldsymbol{r}''$ is collinear with **n.**

Also $|r''| =$ $\boldsymbol{\kappa}$**,** so we have $\boldsymbol{r}'' = \pm\,\boldsymbol{\kappa}\,\boldsymbol{n}$**.**

we choose the direction of $\boldsymbol{n}$ such that curvature $\boldsymbol{\kappa}$ is always positive**.**

i.e. we take $\boldsymbol{r}'' = \boldsymbol{\kappa}\,\boldsymbol{n}$ **or** $\frac{dt}{ds} = \boldsymbol{\kappa}\,\boldsymbol{n}$ **.**

(3) We have $t.b = 0$ whence differentiating

$t.b' + t'.b = 0 \quad \Longrightarrow t.b' + \kappa n.b = 0 \quad [\because t' = \kappa n]$

$\Longrightarrow t.b' = 0 \qquad [\because n.b = 0]$

$i.e.$ b' is perpendicular to t.

Further $b.b = 1 \Rightarrow 2b.b' = 0$

i.e. b' is perpendicular to b.

$\therefore$ b' is normal to the plane containing t and b. $i.e.$ to rectifying plane.

Thus b' is collinear with n.

Thus $b' = \pm\, \tau\, n$

Since b has the opposite direction to n, so negative sign is taken

$\boldsymbol{i.e.}$ $b' = -\,\tau\, n$ or $\frac{d\boldsymbol{b}}{ds} = -\,\tau\, n$

$(\mathbf{2})$ We know that $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}$

Differentiating w.r.t. 's', we get

$$\frac{d\boldsymbol{n}}{ds} = b \times \frac{d\boldsymbol{t}}{ds} + \frac{d\boldsymbol{b}}{ds} \times \boldsymbol{t} = \boldsymbol{b} \times \boldsymbol{kn} + (-\tau\, n) \times \boldsymbol{t}$$

$$= \boldsymbol{k}(\boldsymbol{b} \times \boldsymbol{n}) - \boldsymbol{\tau}(\boldsymbol{n} \times \boldsymbol{t}) = \boldsymbol{k}(-\boldsymbol{t}) - \boldsymbol{\tau}(-\boldsymbol{b})$$

$$= \tau\, \boldsymbol{b} - k\boldsymbol{t}\,.$$

- **Remark.** Serret-Frenet Formulae can be represented in the form of matrix equation as below:

$$\begin{bmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{k} & \mathbf{0} \\ -\boldsymbol{k} & \mathbf{0} & -\boldsymbol{\tau} \\ \mathbf{0} & -\boldsymbol{\tau} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}$$

2.6 *CURVATURE AND TORSION OF ANY CURVE r = r(t) GIVEN BY*

$k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$ and $\tau = \frac{[\,\dot{r},\ \ddot{r}\,,\ \dddot{r}\,]}{|\dot{r} \times \ddot{r}|^3}$

We know that $\dot{r} = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = r'\dot{s} = t\dot{s}$ ……. (1)

$\therefore |\dot{r}\,| = |t\dot{s}| = \dot{s}$ ……. (2)

Now differentiating (1), we get

$\ddot{r} = \frac{d^2r}{dt^2} = \boldsymbol{t}'\dot{s}^2 + \boldsymbol{t}\ddot{s}$ or $\ddot{r} = (kn)\dot{s}^2 + \boldsymbol{t}\ddot{s}$ $[\because \boldsymbol{t}' = kn]$ ……. (3)

Now taking the cross-product of (1) and (3), we get

$$\dot{r} \times \ddot{r} = \dot{s}^3\, kb \quad \Rightarrow |\dot{r} \times \ddot{r}| = k\dot{s}^3 \qquad \ldots\ldots. (4)$$

Differentiating (4), we get

$$\dot{r} \times \dddot{r} + \ddot{r} \times \ddot{r} = \dot{s}^3\, k + \boldsymbol{b}'\dot{s} + \boldsymbol{b}\frac{d}{dt}(\dot{s}^3\, k) \qquad [\because b' = -\tau\, n] \qquad \ldots\ldots. (5)$$

Again, taking the scalar product of (3) and (4), we get

$$[\dot{r}, \ddot{r}, \dddot{r}] = -\dot{s}^6 k^2 \tau \qquad \ldots\ldots.. (6)$$

Also, from (2) and (4), we have

$\dot{s}^3\, k|b| = |\dot{r} \times \ddot{r}|$ or $|\dot{r}|^3 k|b| = |\dot{r} \times \ddot{r}|$

Or $\quad k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} \qquad [\,|b| = 1]$

From (6) and (4), we have $\tau = \frac{[\dot{r},\ \ddot{r},\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^3}$.

An important Result.

$k = |r' \times r''| \quad$ and $\quad \tau = \frac{[r',\ r'',\ r''']}{|r' \times r''|^2}$

Proof: We know that $r' = t$ and $r'' = kn$

$\therefore r' \times r'' = t \times kn = kb \quad$ or $\quad |r' \times r''| = k|b| = k$

Again $\quad \boldsymbol{r}' = \boldsymbol{t} = 1.\boldsymbol{t} + 0.\boldsymbol{n} + 0.\boldsymbol{b} \qquad$ **……. (1)**

$\qquad \boldsymbol{r}'' = kn = 0.\boldsymbol{t} + k\boldsymbol{n} + 0.\boldsymbol{b} \qquad$ **……. (2)**

And $\quad \boldsymbol{r}''' = k\frac{dn}{ds} + \frac{dk}{ds}n = k(\tau\boldsymbol{b} - k\boldsymbol{t}) + k'\boldsymbol{n}$

$\qquad = -k^2\boldsymbol{t} + k'\boldsymbol{n} + k\,\tau\boldsymbol{b} \qquad$ **……. (3)**

From (2), (3) and (4), we have

$$[r',\ r'',\ r'''] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ -k^2 & k' & k\tau \end{vmatrix} = k^2\tau \qquad \ldots\ldots\ldots (4)$$

Or $\qquad \tau = \frac{[r',\ r'',\ r''']}{k^2} = \frac{[r',\ r'',\ r''']}{|r' \times r''|^2}$.

Theorem. The necessary and sufficient condition for the curve to be a plane curve is

$$[r',\ r'',\ r'''] = 0.$$

Proof. From equation (4) above we have

$$[r',\ r'',\ r'''] = k^2\tau$$

In case $[r',\ r'',\ r'''] = 0$ then either $k = 0$ or $\tau = 0$.

Let $\tau \neq 0$ at some point of the curve then in the neighborhood of this point $\tau \neq 0$, therefore $k = 0$ in the neighborhood of this point.

Hence the arc is a straight line and therefore $\tau = 0$ on this line which contradicts our hypothesis. Hence $\tau = 0$ at all points and the curve is a plane.

Conversely. If $\tau = 0$ i.e. the curve is a plane curve then from equation (4), we have

$$[r', \; r'', \; r'''] = 0.$$

Therefore, the condition is necessary as well as sufficient.

Remark: This theorem may also be put as, show that the necessary and sufficient condition for the curve to be plane curve is $[\dot{r}, \; \ddot{r}, \; \dddot{r}] = 0$.

Question 1. For the curve $x = a(3t - t^3), y = 3at^2, z = a(3t + t^3)$, show that

$$k = \tau = \frac{1}{3a(1+t^2)^2}\,.$$

Solution: In this case $\boldsymbol{r}$ in terms of parameter $\boldsymbol{t}$ is given by

$$\boldsymbol{r} = (3at - at^3, 3at^2, 3at + at^3)$$

$\therefore$ $$\dot{r} = (3a - 3at^2, 6at, 3a + 3at^2)$$

$\therefore$ $$\ddot{r} = (-6at, 6a, 6at) \qquad \text{and}$$

$\therefore$ $$\dddot{r} = (-6a, 0, 6a)$$

$\therefore$ $$|\dot{r}| = 3a\sqrt{\{(1-t^2)^2 + 4t^2 + (1+t^2)^2\}} = 3\sqrt{2}a(1+t^2)$$

Again $$\dot{r} \times \ddot{r} = (18a^2t^2 - 18a^2, -36a^2t, \; 18a^2 + 18a^2t^2)$$

$$\therefore\; |\dot{r} \times \ddot{r}| = 18a^2\sqrt{(t^2-1)^2 + 4t^2 + (1+t^2)^2} = 18\sqrt{2}a^2(1+t^2)$$

Therefore, $$k = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} = \frac{18\sqrt{2}a^2(1+t^2)}{54\sqrt{2}a^3(1+t^2)^3} = \frac{1}{3a(1+t^2)^2}$$

Again $$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 3a - 3at^2 & 6at & 3a + 3at^2 \\ -6at & 6a & 6at \\ -6a & 0 & 6a \end{vmatrix}$$

$$= \begin{vmatrix} 6a & 6at & 3a + 3at^2 \\ -0 & 6a & 6at \\ -0 & 0 & 6a \end{vmatrix} \text{ by } c_1 + c_3$$

$$= 6a(36a^2) = 216a^3$$

Therefore, $$\tau = \frac{[\dot{r},\ddot{r},\dddot{r}]}{|\dot{r}\times\ddot{r}|^2} = \frac{216a^3}{\left\{18\sqrt{2}a^2(1+t^2)\right\}^2} = \frac{1}{3a(1+t^2)^2}$$

Hence, $k = \tau = \dfrac{1}{3a(1+t^2)^2}$.

Question 2. For the curve $r = (t, t^2, t^3)$, show that

$$k^2 = \frac{4(9t^4 + 9t^2 + 1)}{(9t^4 + 4t^2 + 1)^2} \text{ and } \tau = \frac{3}{9t^4 + 4t^2 + 1}$$

Solution: here $r = (t, t^2, t^3)$

$\therefore \quad \dot{r} = (1,\ 2t, 3t^2)$

$\therefore \quad \ddot{r} = (0, 2, 6t)$ and

$\therefore \quad \dddot{r} = (0, 0, 6)$

Therefore, $\dot{r} \times \ddot{r} = (6t^2, -6t, 0)$

$\therefore \quad |\dot{r} \times \ddot{r}| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$

Again, $[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2a & 6t \\ 0 & 0 & 6 \end{vmatrix} = 12$

Also, $|\dot{r}|^2 = (9t^4 + 4t^2 + 1)$

Now, $k^2 = \frac{|\dot{r}\times\ddot{r}|^2}{|\dot{r}|^6} = \frac{4(9t^4+9t^2+1)}{(9t^4+4t^2+1)^3}$ and $\tau = \frac{[\dot{r},\ \ddot{r},\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{12}{4(9t^4+9t^2+1)} = \frac{3}{(9t^4+9t^2+1)}$.

Question 3. Find the curvature and torsion for the curve $x = a\,cost, y = a\,sint, z = at\,cot\alpha$.

Solution: Here position vector r in term of parameter t is given by

$$r = (a\,cost, a\,sint, at\,cot\alpha)$$

$\therefore \quad \dot{r} = (-asint,\ acost, acot\alpha)$

$\therefore \quad \ddot{r} = (-acost, -sint, 0)$ and

$\therefore \quad \dddot{r} = (asint, -acost, 0)$

Therefore, $\dot{r} \times \ddot{r} = (a^2 sint\,cot\alpha, a^2 cost\,cot\alpha, a^2)$

$\therefore \quad |\dot{r} \times \ddot{r}| = a^2 cosec\alpha$

And $|\dot{r}| = (a^2 sin^2 t + a^2 cos^2 t + a^2 cot^2 \alpha)^{1/2} = acosec\alpha$

Again, $[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} -asint & acost & acot\alpha \\ -acost & -asint & 0 \\ asint & -acost & 0 \end{vmatrix} = a^2 cot\alpha$

Therefore, $k = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} = \frac{a^2 cosec\alpha}{a^3 cosec^3\alpha} = \frac{1}{a} sin^2\alpha$

And $\tau = \frac{[\dot{r},\ \ddot{r},\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{a^3 cot\alpha}{a^4 cosec^2\alpha} = \frac{1}{a} sin\alpha\, cos\alpha$.

Question 4. If $x = acost, y = asint, z = ct$ a plane curve? Calculate the curvature and torsion

of the above curvature.

Solution: This is exactly above question 3 put a cot $\alpha =$ c in question 3 the we get

$k = \frac{1}{a} sin^2\alpha = \frac{1}{a}.\frac{a^2}{a^2+c^2} = \frac{a}{a^2+c^2}$ and $\tau = \frac{1}{a} sin\alpha\, cos\alpha = \frac{1}{a}.\frac{ac}{a^2+c^2} = \frac{c}{a^2+c^2}$

Since in this case $\tau \neq 0$, hence curve is not a plane curve.

Question 5. Show that the Serret-Frenet formulae can be written in the form $\frac{d\boldsymbol{t}}{ds} = \omega \times \boldsymbol{t}$, $\frac{d\boldsymbol{n}}{ds} = \omega \times \boldsymbol{n}$, $\frac{d\boldsymbol{b}}{ds} = \omega \times \boldsymbol{b}$ and determine ω.

Solution: Since we know that $\frac{d\boldsymbol{t}}{ds} = kn = \tau t \times t + kb \times t \quad [\because t \times t = 0 \text{ and } b \times t = n]$

$$= (\tau t + kb) \times t \qquad \ldots\ldots (1)$$

And $\frac{d\boldsymbol{n}}{ds} = \tau b - kt = \tau(t \times n) + k(b \times n)$

$$= (\tau t + bk) \times n \qquad \ldots\ldots. (2)$$

$$\frac{d\boldsymbol{b}}{ds} = -\tau n = \tau(t \times b) + kb \times b$$

$$= (\tau t + kb) \times b \qquad \ldots\ldots\ldots (3)$$

From equation (1), (2), (3), we have

$\frac{d\boldsymbol{t}}{ds} = \omega \times \boldsymbol{t}$, $\frac{d\boldsymbol{n}}{ds} = \omega \times \boldsymbol{n}$, $\frac{d\boldsymbol{b}}{ds} = \omega \times \boldsymbol{b}$ Where, $\omega = \tau t + kb$

Question 6. Show that $r''' = k'n - k^2 t + k\tau b$ and show that

$$r'''' = (k'' - k^3 - k\tau^2)n - 3k'kt + (2k'\tau + \tau' k)b.$$

Solution: We have $r'' = kn$

$\therefore r''' = kn' + k'n = k(\tau b - kt) + k'n$

$= k'n - k^2 t + k\tau b.$

Hence, $r'''' = k''n + k'n' - 2kk't - k^2 t' + (k\tau' + k'\tau)b + k\tau b'$

$= k''n + k'(\tau b - kt) - 2kk't - k^3 n + (k\tau' + k'\tau)b - k\tau^2 n$

$= (k'' - k^3 - k\tau^2)n - 3kk't + (2k'\tau + \tau' k)b$

Question 7. Prove that $[r', r'', r'''] = k^2 \tau.$

Solution: We have $r' = t$ so that $r'' = t' = kn$

And $r''' = k'n - k^2 t + k\tau b$ (by. Question 6)

$\therefore [r', r'', r'''] = r'.[r'' \times r''']$

$= t.[kn \times (-k^2 t + k'n + k\tau b)]$

$= t.[k^2 b + k^2 \tau t] = k^2 \tau t$ [since t.b = 0 and t.t = 1]

Question 8. If the tangent and binormal at a point of a curve make an angle θ and φ respectively with a fixed direction, show that $\frac{\sin\theta}{\sin\varphi} \cdot \frac{d\theta}{d\varphi} = -\frac{k}{\tau}$.

Solution: Let the tangent t and bonormal b at a point of a curve make angles θ and φ with the fixed direction, say a in space, then

$t.a = a\cos\theta$ where $|a| = a$

$b.a = c\cos\varphi$

Differentiating w.r.to 's' we get

$t'.a = -a\sin\theta\frac{d\theta}{ds}$ (Differentiating of $\boldsymbol{a}$ is zero, since, $\boldsymbol{a}$ is a constant vector)

Or kn. $a = -a\sin\theta\frac{d\theta}{ds}$ ……. (1)

Also $b'.a = -a\sin\varphi\frac{d\varphi}{ds}$ i.e., $-\tau n.a = -a\sin\varphi\frac{d\varphi}{ds}$ …… (2)

Dividing (1) by (2), we get

$\frac{\sin\theta}{\sin\varphi} \cdot \frac{d\theta}{d\varphi} = -\frac{k}{\tau}$.

<u>CHECK YOUR PROGRESS</u>

True or false Questions

Problem 1. If $\tau = 0$ then the curve is a plane curve.

Problem 2. The curvature at a point P of a given curve is the arc rate of rotation of tangent at P.

Problem 3. If for a curve $[r', \ r'', \ r'''] = 1$ then curve is a plane curve.

Problem 4 A necessary and sufficient condition for the curve to be a straight line is that curvature $\kappa = 1$ at all points of the curve.

Problem 5. A necessary and sufficient condition for the curve to be a straight line is that curvature $\kappa = 0$ at all points of the curve.

2.7 *SUMMARY*

(1) Curvature: The curvature at a point P of a given curve is the arc rate of rotation of tangent at P. its magnitude is denoted by $\boldsymbol{\kappa}$ (Kappa).

(2) Serret-Frenet Formulae.

The following set of three relations involving space derivatives of fundamental unit vectors $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ are known as Serret-Frenet Formulae.

$(\boldsymbol{i})$ $\frac{dt}{ds} = k\boldsymbol{n}$ **(ii)** $\frac{dn}{ds} = \tau\boldsymbol{b} - k\boldsymbol{t}$ **(iii)** $\frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{n}$

(3) The necessary and sufficient condition for the curve to be a plane curve is

$$[r', \ r'', \ r'''] = 0.$$

(4) Serret-Frenet Formulae can be represented in the form of matrix equation as below:

$$\begin{bmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{k} & \boldsymbol{0} \\ -\boldsymbol{k} & \boldsymbol{0} & -\boldsymbol{\tau} \\ \boldsymbol{0} & -\boldsymbol{\tau} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}$$

2.8 *GLOSSARY*

(i) Derivatives

(ii) Determinant

(iii) Vector

2.9 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn “Cambridge University Press.”
2. Matrices and Tensors in physics by A.W. Joshi “Wiley Eastern Limited”.
3. Tensors by Ram Bilas Mishra “Hardwari publications Allahabad”.
4. Differential Geometry by Gupta, Malik and Pundir “Pragati Edition’’.

2.10 *TEWRMINAL QUESTIONS*

1. Prove that the necessary and sufficient condition for the curve to be plane is $[r', \ r'', \ r'''] = 0$.
2. Find the osculating plane, curvature and torsion at any point of the curve

   $x = a\ cos2u, y = a\ sin2u, z = 2a\ sinu.$
3. Define curvature, also prove that $\frac{dt}{ds} = \boldsymbol{\kappa\, n}$.
4. Find the curvature and torsion of the curve given by

   $r = (at - asint,\ a - acost, bt).$
5. For the curve $r = (\sqrt{at^3}, a(1+3t^2), \sqrt{6at}\ )$, show that $\tau = k = \frac{1}{a(1+3t^2)^2}$.

2.11 *ANSWERS*

TQ 2. The equation of osculating plane is $3u\ sinu,$ torsion $\frac{3}{a(5secu + 3cosu)}$.

TQ 4. $k = \frac{a(\ b^2 + 4a^2 sin^4\frac{t}{2})^{1/2}}{b^2 + 4a^2 sin^2\frac{t}{2}}$ and $\tau = \frac{-b}{b^2 + 4a^2 sin^4\frac{t}{2}}$.

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. False

CYQ 5. True

UNIT 3: INTRINSIC PROPERTIES OF A SURFACE

CONTENTS:

3.1 *INTRODUCTION*

In geometry, an intrinsic equation is an equation that defines a curve using its intrinsic properties. These properties do not depend on the curve's location or orientation and in differential geometry, the fundamental theorem of space curves states that every regular curve in three-dimensional space, with non-zero curvature, has its shape (and size or scale) completely determined by its curvature and torsion.

3.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Intrinsic Equations

(ii) Osculating Circle

(iii) The Osculating Sphere

3.3 *INTRINSIC EQUATIONS*

We have defined the curve with respect to a set of three mutually orthogonal axis but in case the same curve be referred to a different set of cartesian axes, then its defining equations are altogether different and its is not at all clear that they refer to the same curve. Thus, it is required to describe a curve without reference to a particular set of cartesian axes, this can be done by expressing the curvature and torsion at any of its points as function of arc length s, say $k = f(s)$; $\tau = g(s)$. These are called intrinsic equation of the curve.

3.4 *FUNDAMENTAL THEOREMS FOR SPACE CURVES*

Theorem 1. (Existence Theorem). If $k(s)$ and $\tau(s)$ are continuous functions of a real variable s $(s \geq 0)$ then there exists a space curve for which k is the curvature, τ is the torsion, and s is the arc length measured from some suitable base point.

Proof. From existence theorem on linear differential equation, we know that the differential equations

$$\frac{d\alpha}{ds} = k\beta, \qquad \frac{d\beta}{ds} = \tau\gamma - k\alpha, \qquad \frac{d\gamma}{ds} = -\tau\beta \qquad \ldots\ldots\ldots (1)$$

Admit a unique set of solutions for a given set of values α, β, γ at s = 0.

Therefore, we have unique set α_1 , β_1 , γ_1 for which the values at s = 0 are 1, 0.

Similarly, there exist a unique set α_2 , β_2 , γ_2 with values 0, 1, 0 at s = 0 and α_3 , β_3 , γ_3 is a unique set with values 0, 0, 1 at s = 0.

Now $\frac{d}{ds}\left({\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2\right) = 2\left(\alpha_1 \frac{d\alpha_1}{ds} + \beta_1 \frac{d\beta_1}{ds} + \gamma_1 \frac{d\gamma_1}{ds}\right)$

$$= [\alpha_1(k\beta_1) + \beta_1(\tau\gamma_1 - \mathrm{k}\alpha_1) + \gamma_1(-\tau\beta_1)] = 0 \text{ [From (1)]}$$

Integrating, we get ${\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2 = c_1$ (constant)

Initially at s = 0, $\alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0, \quad \therefore c_1 = 1$

Hence $$\left.\begin{array}{l} {\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2 = 1 \\ {\alpha_2}^2 + {\beta_2}^2 + {\gamma_2}^2 = 1 \\ {\alpha_3}^2 + {\beta_3}^2 + {\gamma_3}^2 = 1 \end{array}\right\} \qquad \ldots\ldots..(2)$$

Again $\frac{d}{ds}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)$

$$= \left(\alpha_1 \frac{d\alpha_2}{ds} + \beta_1 \frac{d\beta_2}{ds} + \gamma_1 \frac{d\gamma_2}{ds}\right) + \left(\alpha_2 \frac{d\alpha_1}{ds} + \beta_2 \frac{d\beta_1}{ds} + \gamma_2 \frac{d\gamma_1}{ds}\right)$$

$$= \alpha_1(k\beta_2) + \beta_1(\tau\gamma_2 - \mathrm{k}\alpha_2) + \gamma_1(-\tau\beta_2) + k\beta_1\alpha_2 + \beta_2(\tau\gamma_1 - \mathrm{k}\alpha_1) + \gamma_2(-\tau\beta_1)$$

$$= 0$$

Thus, on integrating, we have $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = c^2$ (constant)

Initially at s = 0, $\alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0, \alpha_2 = 0, \beta_2 = 1, \gamma_2 = 0$

$\therefore \quad c^2 = 0$

Hence, $$\left.\begin{array}{l} \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 1 \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 = 1 \\ \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 1 \end{array}\right\} \qquad \ldots\ldots..(3)$$

Thus, we have six equations given by (2) and (3) in elements of three sets namely $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$. Hence it follows that there are three mutually orthogonal unit vectors $t = (\alpha_1, \beta_1, \gamma_1), n = (\alpha_2, \beta_2, \gamma_2)$ and $b = (\alpha_3, \beta_3, \gamma_3)$ defined for each curve of s.

Now let the be defined by

$r = r(s) = \int t(s)ds$ …….. (4)

Thus k is curvature of the curve given by (4).

Again $b = t \times n \qquad \Rightarrow b' = t' \times n + t \times n'$

$= k(n \times n) + t \times (-kt + \tau n)$ putting for t' and n'

$= (n \times n) - k(t \times t) + \tau(t \times n)$

$= \tau b$ Where, $|b| = 1$

Thus τ is a torsion of the curve, so there exists a curve given by (4) where t, n, b are unit vectors along the tangent, principal normal and bonormal respectively and k and τ are its curvature and torsion respectively.

Example 1. Show that the intrinsic equations of the curve given by

$$x = ae^u \cos u, y = ae^u \sin u, z = be^u \text{ are}$$

$$k = \frac{a\sqrt{2}}{s\sqrt{\{2a^2+b^2\}}}, \tau = \frac{a\sqrt{2}}{s\sqrt{\{2a^2+b^2\}}}.$$

Solution: Here $r = (ae^u \cos u, ae^u \sin u, be^u)$

Therefore $\dot{r} = [ae^u(cosu - sinu), ae^u(sinu + cosu), be^u]$

$$|\dot{r}| = \dot{s} = e^u\sqrt{[a^2(cosu - sinu)^2 + a^2(cosu + sinu)^2 + b^2]}$$

$$= e^u\sqrt{(2a^2 + b^2)} = \dot{s} \qquad \ldots\ldots\ldots (1)$$

$$r' = \frac{\dot{r}}{\dot{s}} = \frac{[a(cosu - sinu),\ a(sinu + cosu),\ b]}{\sqrt{(2a^2 + b^2)}}$$

$$r'' = kn = \frac{[-a(sinu + cosu),\ a(cosu - sinu),\ 0]}{\sqrt{(2a^2 + b^2)}}.\frac{1}{s} \qquad \ldots\ldots\ldots\ldots (2)$$

Taking module of both sides, we get

$$k = |r''| = \frac{a\sqrt{2}}{\sqrt{(2a^2+b^2)}} \qquad \left[\because \frac{1}{\dot{s}} = \frac{1}{s}\right] \text{ from (1)}$$

Also, from (2)

$$sr'' = \frac{[-a(sinu + cosu),\ a(cosu - sinu),\ 0]}{\sqrt{(2a^2 + b^2)}}$$

Differentiating w.r.t. ‘s’, we get

$$sr''' + r'' = \frac{[-a(cosu - sinu),\ a(cosu + sinu),\ 0]}{\sqrt{(2a^2 + b^2)}}.\frac{1}{s} \quad \text{From (1)}$$

Or $\qquad s^2r''' + sr'' = \dfrac{[-a(cosu - sinu), -a(cosu + sinu),\ 0]}{\sqrt{(2a^2 + b^2)}}$

Now, $[r', sr'', s^2r''' + sr''] =$

$$\frac{1}{(2a^2 + b^2)^{3/2}}\begin{vmatrix} a(cosu - \text{sinu}) & a(sinu + cosu) & b \\ -a(sinu + cosu) & a(cosu - sinu) & 0 \\ -a(cosu - sinu) & -a(sinu + cosu) & 0 \end{vmatrix}$$

Or $s^2[r', r'', r'''] = \frac{1}{(2a^2+b^2)^{3/2}} a^2 b^2 [(sinu + cosu)^2 + (cosu - sinu)^2]$

Or $s^3 k^2 \tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$ or $s^3 \frac{2a^2}{(2a^2+b^2)^2} . \frac{1}{s^2} \tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}$

Or $\tau = \frac{b}{(2a^2+b^2)^{1/2}} \frac{1}{s}$.

Hence, the intrinsic equations of the given curve are

$k = \frac{\sqrt{2}a}{(2a^2+b^2)^{1/2}} \frac{1}{s}, \quad \tau = \frac{b}{(2a^2+b^2)^{1/2}} \frac{1}{s}$.

3.5 *OSCULATING CIRCLE (OR THE CIRCLE OF CURVATURE)*

Definition: Let P, Q, R be three points on any curve then the circle of curvature at point P is the limiting position of the circle through P, Q, R when the points Q, R tend to P.

Alternatively: The osculating circle at point P on any curve is the circle which has three-point contact with the curve at P.

Obviously, the osculating circle at any point of a curve lies in the osculating plane at the point since the osculating plane at P has three-point contact at P with the curve.

The radius and the Centre of circle of curvature:

Let a circle in the osculating plane be given as intersection of the plane and the sphere $|r - c| = a$ i.e. $(r - c)^2 = a^2$ where r is the position vector of the generic point and c is the position vector of Centre C and a is radius of the sphere.

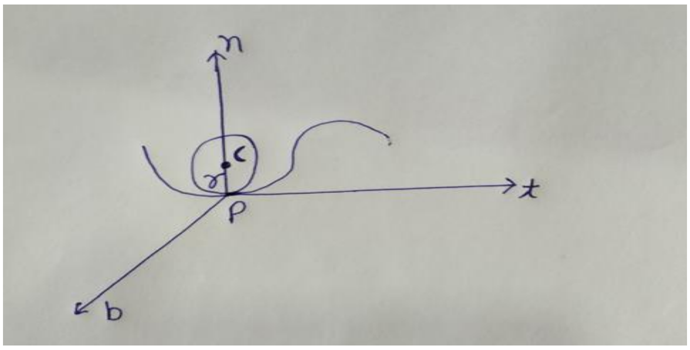


Fig.3.5.1

Let the equation of the curve be $r = r(s)$ (i.e. parametric in s). now the positions of intersection of the curve and sphere are given by

$$F(s) = [r(s) - c]^2 - a^2 = 0$$

For three-point contact, we have

F(s) = $F'(s) = F''(s) = 0$

These conditions give

$(r-c)^2 = a^2$, $(r-c).r' = 0, (r-c).r'' + r'.r' = 0$

Since $r' = t, r'' = t' = kn, r'.r' = t^2 = 1$

These equations may be put as

$(r-c)^2 = a^2$ ……… (1)

$(r-c).t = 0$ ……... (2)

And $(r-c).n = -\rho$ …….. (3)

Equation (2) shows that (r – c) lies in the normal plane at P. but definition it also lies in the osculating plane at P, hence (r – c) must be along the line of intersection of osculating plane and normal plane, thus it must lie in the direction of principal normal at P. thus

$(r-c) = \lambda n$ where λ is any scalar, substitutions in (1) and (3) give $a = \rho$ and $\lambda = -\rho$

Therefore, position vector c of the centre of osculating circle is given by

$c = r - \lambda n = r + \rho n$

It is evident that centre lies on the principal normal and is at a distance ρ from P.

- **Properties of the locus of the centre of curvature:**
  Let C be the original curve and C_1 be the locus of the centre of curvature, then it has following two important properties.
  (i) The tangent to C_1 lies in the normal plane of the original curve C.
  (ii) In case the original curve C has constant curvature k then the curvature of C_1 is also constant and torsion of C_1 varies inversely as that of C.

  Proof: Let the suffix unit be used for quantities belonging to the locus of the centre of curvature i.e. for C_1.

  (i) The position vector c of the curvature of C_1 is given by
  $c = r + \rho n$
  Differentiating this w.r.t. 's', we have
  $C' = t_1 = (r + \rho n)' \frac{ds}{ds_1}$ or $t_1 = (r' + \rho n' + \rho' n)\frac{ds}{ds_1}$
  $t_1 = [t + \rho n' + \rho(\tau b - kt)]\frac{ds}{ds_1}$ $[by\ Fernet's\ formula]$
  $t_1 = (\rho n' + \rho \tau b)\frac{ds}{ds_1}$ $[\because \rho k = 1]$ ……… (1)

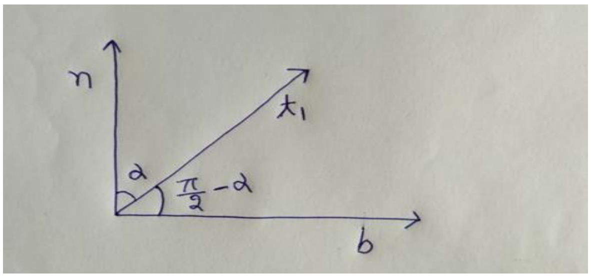


Fig.3.5.2

The relation (1) shows that the tangent to C_1 lies in the plane containing $\boldsymbol{n}$ and $\boldsymbol{b}$ i.e. in normal plane of C if it is inclined at an angle α to the principal normal $\boldsymbol{n}$**.**

Then $\cos\alpha = n.t_1 = n.(\rho n' + \rho\tau \mathrm{b})\frac{ds}{ds_1}$ [from (1)]

$\cos\alpha = \rho'\frac{ds}{ds_1}$ and also $\sin\alpha = \rho\tau\frac{ds}{ds_1}$ …….. (2)

Again squaring (1) we get $1 = \sqrt{(\rho'^2 + \rho^2\tau^2)}\frac{ds}{ds_1}$ …….. (3)

Using (2) and (3), we have

$\cos\alpha = \frac{\rho'}{\sqrt{\rho'^2+\frac{\rho^2}{\sigma^2}}} = \frac{\rho'\sigma}{\sqrt{\rho^2+\rho'^2\sigma^2}}$ or $\tan\alpha = \frac{\rho}{\rho'\sigma}$

Or $\alpha = tan^{-1}\left(\frac{\rho}{\rho'\sigma}\right)$ ……… (4)

(ii) In case k is constant i.e. ρ is constant, we have $\rho' = 0$

Thus from)1), we have $t_1 = \rho\tau \mathrm{b}\frac{ds}{ds_1}$ …….. (5)

Squaring both sides of this equation, we get

$\frac{ds}{ds_1} = \frac{1}{\rho\tau}$ $[\therefore t_1{}^2 = 1 - b^2]$ ………. (6)

From (5) and (6), we have $t_1 = b$

Now differentiating this relation w.r.t. 's', we get

$t_1{}' = b\frac{ds}{ds_1}$ or $t_1{}' = k_1 n_1 = -\tau \mathrm{n}\frac{ds}{ds_1}$

Or $k_1 n_1 = -\tau \mathrm{n}\frac{1}{\rho\tau} = -kn$

This implies that n_1 is parallel to n and choosing the direction of n_1 opposite to that of n such that $n_1 = -n$. Therefore $k_1 = k$.

Again $b_1 = t_1 \times n_1 = b \times (-n) = t$

Differentiating this relation, we get

$-\tau_1 n_1 = \frac{db_2}{ds_1} = t' \frac{ds}{ds_1} \quad = kn\left(\frac{k}{\tau}\right)$

But $n_1 = -b$ therefore $\tau_1 = \frac{k^2}{\tau} = \frac{constant}{\tau}$

i.e. torsion of C_1 varies inversely as that of C.

Example: Show that the principal normal to a curve is normal to the locus of the centre at points where curvature k is stationary.

Solution: The position vector of the centre of curvature is given by

c = $r + \rho n$ [r = r(s)] …….. (1)

let the suffix unity be used for quantities belonging to the locus of c(= r_1), then

$$\frac{dr}{ds} = t_1 \frac{ds}{ds_1} = t + \rho(\tau b - kt) + \rho' n$$

Or $t_1 = (\rho\tau b + \rho' n)\frac{ds}{ds_1}$ $(\because \rho k = 1)$ ………. (2)

Now taking scalar product of (2) with n, we get

$t_1.n = \rho' \frac{ds}{ds_1}$ ……… (3)

In case k is constant, then $\rho' = 0$

Hence from (3), we have

$t_1.n = 0$

i.e. principal normal is normal to the locus of centre of curvature.

3.6 *THE OSCULATING SPHERE (OR THE SPHERE OF CURVATURE)*

Definition: Let P, Q, R, S are four points on a curve then the sphere of curvature at point P is the limiting position of the sphere PQRS when the points Q, R, S tend to coincide with P. its radius and centre are called radius and centre of spherical curvature.

Alternatively: The sphere which has a four-point contact with the curve at a point P is called osculating sphere at P.

The radius and the centre of the sphere of curvature.

Let c be the position vector of the centre and R be the radius of the sphere.

Then its equation is $(\boldsymbol{r} - c)^2 = R^2$ ………. (1)

Where $\boldsymbol{r}$ is the position vector of the generic point.

The point of intersection of the curve r = r(s) with the sphere are given by

F(s) ≡ $(\boldsymbol{r}(\boldsymbol{s}) - c)^2 - R^2 = 0$

Again, for a four-point contact, we have

F(s) = 0, $F'(s) = 0$, $F''(s) = 0$, $F'''(s) = 0$

Now these conditions give rise to following equations

$(\boldsymbol{r} - c)^2 = R^2$; $(r - c).r' = 0$; $(r - c).r'' + r'.r' = 0$

And $(r - c).r''' + r'.r'' + 2r'.r'' = 0$

Again, we know that

$r' = t, r'.r' = t^2 = 1$

$r'' = t' = kn;\ r'.r'' = r'.t' = r'.kn = t.kn = 0$

$r''' = (t')' = (kn)' = kn' + k'n = k(\tau b - kt) + k'n$

Using these relations, above equations reduce to

$(\boldsymbol{r} - c)^2 = R^2$ …….. (1)

$(r - c).t = 0$ ……... (2)

$(r - c).n = -\rho$ ……… (3)

$(r - c).\{k(\tau b - kt) + k'n\} = 0$ ………. (4)

Again equation (4) by making use of (2) and (3) reduces to

$(r - c).b = \frac{k'\rho}{k\tau} = \rho^2\sigma k' = -\rho^2\sigma\frac{\rho'}{\rho^2} = -\sigma\rho'$ ……… (5)

From (2), we observe that $(r - c)$ is perpendicular to t i.e. it lies in the normal plane at P. Thus, we can express $(r - c)$ as linear combination of n and b

i.e. there exists scalar λ and μ such that

$$(r - c) = \lambda n + \mu b$$

Substitution in (3) and (5) we get

$\lambda = -\rho$ and $\mu = -\sigma\rho'$

Whence $r - c = -\rho n - \sigma\rho' b$ or $c = r + \rho n + \sigma\rho' b$ ………. (6)

Again, substitution in (1), gives

$R^2 = \lambda^2 + \mu^2 = \rho^2 + \sigma^2 \rho'^2$ ………. (7)

$R^2 = \frac{\left(k^2\tau^2 + k'^2\right)}{k\tau^2}$ ………. (8)

Remark: if $k = \frac{1}{\rho}$ is constant, then $\rho' = 0$, then ρ is constant, so (7) gives R = ρ and (6) gives $c = r + \rho n$, i.e. centre of osculating sphere coincides with centre of osculating circle.

Example 1. Find the equation of the osculating sphere and osculating circle at (1, 2, 3) on the curve $x = 2t + 1, y = 3t^2 + 2, z = 4t^3 + 3$.

Solution: the equation of the curve can be put as

$r = (2t + 1, 3t^2 + 2, 4t^3 + 3)$ ……… (1)

Evidently t = 0 at point (1, 2, 3) on the curve

Differentiating equation (1) w.r.t. 't' we get

$$\dot{r} = (2, 6t, 12t^2), \ddot{r} = (0, 6, 24t), \dddot{r} = (0, 0, 24)$$

At t = 0, we have

$$\dot{r} = (2, 0, 0), \ddot{r} = (0, 6, 0), \ \dddot{r} = (0, 0, 24)$$

Let the equation of osculating sphere be $(r - c)^2 = R^2$ …….. (2)

Where c is the position vector of the centre of osculating sphere, let

$c = \alpha i + \beta j + \gamma k$

Now, for a four-point contact at r, we have on differentiating equation (2), three times w.r.t. 't'

$(r - c).\dot{r} = 0, (r - c).\ddot{r} + \dot{r}^2 = 0$ and $(r - c).\dddot{r} + 3\dot{r}.\ddot{r} = 0$

At t = 0, these reduce to

$$[(i + 2j + 3k) - (\alpha i + \beta j + \gamma k)].2i = 0$$

i.e. $1 - \alpha = 0$ or $\alpha = 1$

Similarly, from other two equations, we have

$$(2 - \beta)6 + 4 = 0 \ \ i.e. \beta = \frac{8}{3}$$

And

$$(3 - \gamma)24 + 0 = 0 \ \ i.e. \gamma = 3$$

also, osculating sphere (2) passes through $(1, 2, 3)$

therefore, $\left[(i + 2j + 3k) - \left(i + \frac{8}{3}j + 3k\right)\right]^2 = R^2$

or $\frac{4}{9} = R^2$ or $\frac{2}{3} = R$

Hence, the equation of the osculating sphere is

$$\left[(i + 2j + 3k) - \left(i + \frac{8}{3}j + 3k\right)\right]^2 = \frac{4}{9}$$

Or $(x-1)^2 + \left(y - \frac{8}{3}\right)^2 + (z-3)^2 = \frac{4}{9}$

Or $3x^2 + 3y^2 + 3z^2 - 6x - 16y - 18z + 50 = 0$

The osculating circle is the intersection of the osculating plane and osculating sphere and the equation of this plane is

$$[R - r, \quad \dot{r}, \quad \ddot{r}] = 0$$

i.e., at t = 0, it reduces to $\{(x-1)i + (y-2)j + (z-3)k\}.12k = 0$

i.e. $z - 3 = 0$

Hence, the equation of osculating circle is

$3x^2 + 3y^2 + 3z^2 - 6x - 16y - 18z + 50 = 0, z - 3 = 0.$

Example 2. Show that the radius R of the sphere of curvature is given by

$$R^2 = \rho^4\sigma^2 r'''^2 - \sigma^2.$$

Solution: we know that

$$r'' = kn = \frac{1}{\rho}n$$

Therefore, $r''' = \frac{1}{\rho}(\tau b - kt) - \frac{\rho'}{\rho^2}n = -\frac{1}{\rho^2}t - \frac{\rho'}{\rho^2}n + \frac{1}{\sigma\rho}b$

Squaring, we get $r'''^2 = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\sigma^2\rho^2}$

Or $\rho^4\sigma^2 r'''^2 = \sigma^2 + \rho'^2\sigma^2 + \rho^2 = \sigma^2 + R^2$

$R^2 = \rho^4\sigma^2 r'''^2 - \sigma^2.$

Example 3. Show that $x'''^2 + y'''^2 + z'''^2 = \frac{1}{\sigma^2\rho^2} + \frac{1+\rho'^2}{\rho^4} = \frac{1}{\rho^4} + \frac{R^2}{\sigma^4\rho^2}.$

Solution: we have

$$r = xi + yj + zk$$

Therefore, $r''' = x'''i + y'''j + z'''k$ ……… (1)

Squaring we get, $r'''^2 = x'''^2 + y'''^2 + z'''^2$ ……. (2)

Therefore, $r''' = \frac{1}{\rho}(\tau b - kt) - \frac{\rho'}{\rho^2}n = -\frac{1}{\rho^2}t - \frac{\rho'}{\rho^2}n + \frac{1}{\sigma\rho}b$

Squaring, we get $r'''^2 = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\sigma^2\rho^2} = \frac{1}{\sigma^2\rho^2} + \frac{1+\rho'^2}{\rho^4}$ …….. (3)

From (2) and (3), we get

$x'''^2 + y'''^2 + z'''^2 = \frac{1}{\sigma^2\rho^2} + \frac{1+\rho'^2}{\rho^4}$ ……… (4)

Since $R^2 = \rho^2 + \sigma^2\rho'^2$, equation (4) can be put as

$x'''^2 + y'''^2 + z'''^2 = \frac{1}{\sigma^2\rho^2} + \frac{1+\rho'^2}{\rho^4} = \frac{1}{\rho^4} + \frac{R^2}{\sigma^4\rho^2}$.

Example 4. Show that the radius of spherical curvature of a circular helix is equal to the radius of circular curvature.

Solution: the radius R of a spherical curvature is given by

$R^2 = \rho^2 + \sigma^2\rho'^2$ ……… (1)

Again, for a circular helix, we know that $\rho =$ constant $(\rho' = 0)$

Therefore, (1) reduces to $R^2 = \rho^2$ or $R = \rho$.

Example 5. Show that the radius of spherical curvature of a circular helix $x = a\cos\theta, y = a\sin\theta, z = a\theta\cot\alpha$ is equal to the radius of circular curvature.

Solution: Here $r = a(\cos\theta, \sin\theta, \cot\alpha)$

Differentiating w.r.t. 's', we get

$$r' = t = a(-\sin\theta, \cos\theta, \cot\alpha)\frac{d\theta}{ds}$$

Squaring above, we get

$$1 = a^2(\sin^2\theta + \cos^2\theta + \cot^2\alpha)\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2\,\mathrm{cosec}^2\,\alpha}$$

Or $\frac{d\theta}{ds} = \frac{\sin\alpha}{a}$

Therefore, $t = \sin\alpha\,(-\sin\theta\,, \cos\theta\,, \cot\alpha)$

Differentiating w.r.t. 's', we get

$t' = \sin\alpha\,(-\cos\theta\,, -\sin\theta\,, 0)\frac{d\theta}{ds}$

$or \quad kn = \frac{\sin^2\alpha}{a}(-\cos\theta,\ -\sin\theta\,, 0\,)$

Squaring, we get

$k^2 = \frac{\sin^4\alpha}{a^2}[\cos^2\theta + \sin^2\theta] = \frac{\sin^4\alpha}{a^2}$

Therefore, $k = \frac{\sin^2\alpha}{a}$ which is constant.

Therefore, in this case $\rho =$ constant, therefore $\rho' = 0$

Again, radius R of spherical curvature is given by

$R^2 = \rho^2 + \sigma^2\rho'^2 = \rho^2$ (since $\rho' = 0$)

Hence, $R = \rho.$

Example 6. If a curve lie on a sphere, show that ρ and σ are connected by

$$\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0.$$

Or show that the necessary and sufficient condition that a curve lies on a sphere is that $\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$ at every point on the curve.

Solution: Necessary condition. in case curve lies on a sphere then that sphere is the osculating sphere for every point, then the radius R of the osculating sphere is constant. The radius R is given by

$R^2 = \rho^2 + \sigma^2\rho'^2 = \rho^2 + (\sigma\rho')^2$ ………. (1)

Differentiating w.r.t. 's', we get

$0 = 2\rho'\rho + 2\sigma\rho'\frac{d}{ds}(\sigma\rho')$ or $\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$

Sufficient condition. If $\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$, then to show that the curve lies on a sphere.

By reversing the order of steps, we see that the radius of osculating sphere is independent of the point on the curve.

Again, the centre of spherical curvature is given by

$C = r + \rho n + \sigma\rho' b$

Therefore, $\frac{dC}{ds} = t + \rho' n + \rho(\tau \mathrm{b} - \mathrm{kt}) + \sigma'\rho' b + \sigma\rho'' b - \sigma\rho'\tau \mathrm{n}$

$$= \left(\frac{\rho}{\sigma} + \sigma'\rho' + \sigma\rho''\right) b = 0 \qquad \left(\frac{\rho}{\sigma} + \sigma'\rho' + \sigma\rho'' = 0 \; by\ hypothesis\right)$$

Or C is a constant vector.

i.e. the centre of osculating sphere is independent of the point on the curve. Hence the curve lies on the sphere.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. If $\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$, then the curve lies on a sphere.
Problem 2. The radius R of the sphere of curvature is given by
$R^2 = \rho^4\sigma^2 r'''^2 - \sigma^2$.
Problem 3. The radius of spherical curvature of a circular helix is equal to the radius of circular curvature.
Problem 4 For three-point contact, we have
F(s) = $F'(s) = F''(s) = 1$
Problem 5. The radius of spherical curvature of a circular helix is not equal to the radius of circular curvature.

3.7 *SUMMARY*

(1) (Existence Theorem). If $k(s)$ and $\tau(s)$ are continuous functions of a real variable s $(s \geq 0)$ then there exists a space curve for which k is the curvature, τ is the torsion, and s is the arc length measured from some suitable base point.

(2) The sphere which has a four-point contact with the curve at a point P is called osculating sphere at P.

(3) The principal normal to a curve is normal to the locus of the centre at points where curvature k is stationary.

(4) The circle which has a three-point contact with the curve at a point P is called osculating circle at P.

3.8 *GLOSSARY*

(i) Derivatives

(ii) Determinant

(iii) Vector

3.9 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."
2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".
3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".
4. Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.

3.10 *TEWRMINAL QUESTIONS*

1. Prove that the curve given by $r = (a \sin^2 u,\ \ a\, sinu\, cosu\, , a\, cosu)$ lie on a sphere.
2. For spherical curve, prove that $\rho + \frac{d^2\rho}{d\varphi^2} = 0$, where, φ is such that $d\varphi = \tau ds.$
3. Define osculating sphere, find its equation.
4. Define osculating sphere, find its equation.

3.11 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. True

CYQ 4. False

CYQ 5. False

UNIT 4: INVOLUTE AND EVOLUTE

CONTENTS:

4.1 *INTRODUCTION*

In differential geometry, an "involute" is a curve created by tracing the path of a taut string as it unwinds from another curve, while the "evolute" is the original curve from which the string is unwinding; essentially, the evolute is the curve of which the involute is derived by "unwinding a string" from it, making the original curve the "evolute" of its "involute" curve.

4.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Involute and Evolute

(ii) Spherical Indicatrices

(iii) Bertrand Curves

4.3 *INVOLUTE AND EVOLUTE*

Definition: If there be one-one correspondence between points of two curve C and C_1 such that the tangent at any point of C is a normal to the corresponding point of C_1, then C_1 is called involute of C and C is called an evolute of C_1.

(i) Involute of a given space:

Let $\mathrm{r} = \mathrm{r(s)}$ be the given curve C and let $\mathrm{C_1}$ be involute of C. the quantities belonging to the $\mathrm{C_1}$ will be distinguished by using the suffix unity. Then the position vector r of any point $\mathrm{P_1}$ on $\mathrm{C_1}$ is given by

$$\mathrm{r_1} = \mathrm{r} + \lambda \mathrm{t} \qquad \ldots\ldots. (1)$$

Where λ is to be determined.

Differentiating (1) w.r.t. '$\mathrm{s_1}'$ we get

$$\mathrm{t_1} = (\mathrm{t} + \lambda' \mathrm{t} + \lambda \mathrm{kn}) \frac{\mathrm{ds}}{\mathrm{ds_1}} \qquad \ldots\ldots\ldots (2)$$

By definition t is perpendicular to $\mathrm{t_1}$ so taking dot product of both sides of (2) with t and using t. $\mathrm{t_1} = 0$, we get

$$(1 + \lambda') \frac{\mathrm{ds}}{\mathrm{ds_1}} = 0 \quad \text{or} \quad 1 + \lambda' = 0$$

Which on integration gives $s + \lambda = \mathrm{c}$ or $\lambda = \mathrm{c} - \mathrm{s}$, where c is constant of integration.

Therefore, $r_1 = r + (c - s)t$ .... (3)

Which is the required equation of involute C_1 of C.

Again putting the value of λ in (2), the unit tangent vector t_1 is given by

$t_1 = (c - s)k \frac{ds}{ds_1} n$ $(\because \lambda' = -1)$ ...... (4)

From (4), we observe that t_1 is parallel to n. taking the positive direction along the involute such that

$t_1 = n$ thus from (4) $\frac{ds}{ds_1} = k(c - s)$.

(ii) Curvature k_1 and Torsion τ_1 of the Involute:

Differentiating $t_1 = n$ w.r.t. 's_1' we get

$$t_1{}' = k_1 n_1 = (\tau b - kt)\frac{ds}{ds_1} = \frac{\tau b - kt}{k(c - s)}$$

Therefore, on squaring both sides, we get

$k_1{}^2 = \frac{\tau^2 + k^2}{k^2(c-s)^2}$ or $k_1 = \frac{(\tau^2+k^2)^{1/2}}{k(c-s)}$

Obviously, the unit principal normal to involute is

$n_1 = \frac{\tau b - kt}{k k_1 (c-s)} = \frac{\tau b - kt}{(\tau^2+k^2)^{1/2}}$ .... (5)

$b_1 = t_1 \times n_1 = n \times n_1 = \frac{kb + \tau t}{k k_1 (c-s)} = \frac{kb+\tau t}{(\tau^2+k^2)^{1/2}}$ .... (6)

Now differentiating (6) w.r.t. 's' we get

$-\tau_1 n_1 \frac{ds}{ds_1} = \frac{(k^2+\tau^2)(k'b+\tau' t+kb'+\tau t') - (kb+\tau t)(kk'+\tau\tau')}{(\tau^2+k^2)^{3/2}}$ $[\because b' = -\tau n, t' = kn]$

Squaring both sides and putting $\frac{ds_1}{ds} = k(c - s)$, we get

$\tau_1 = \frac{(k\tau' - k'\tau)}{k(c-s)(\tau^2+k^2)}$.

(iii) To find the equation of a given Curve C.

In this case we will find a curve C_1 such that C is involute of C_1 and consequently C_1 will be evolute of C.

Let $r = r(s)$ be equation of curve C. we shall use the suffix unit for quantities belonging to curve C_1. Let r_1 be the position vector of an evolute C_1 and that of corresponding point P on C be r. since the tangents to the curve C_1 are normal to curve C, the vector PQ must lie in the plane to the curve C at P.

$\Rightarrow r_1 - r = \lambda n + \mu b \Rightarrow r_1 = r + \lambda n + \mu b$ .... (1)

Where the values of λ and μ are to be determined.
Now differentiating (1) w.r.t. 's_1' we get

$$\frac{dr_1}{ds_1} = [r' + \lambda' n + \lambda n' + \mu' b + \mu b']\frac{ds}{ds_1} \qquad \ldots\ldots\ldots.\ (1)$$

$$t_1 = [t + \lambda' n + \lambda(\tau b - kt) + \mu' b + \mu(-\tau n)]\frac{ds}{ds_1}$$

$$= [(1 - k\lambda)t + (\tau' - \mu\tau)n + (\mu' + \lambda\tau)b]\frac{ds}{ds_1}$$

Since t_1 lies in the normal plane at P to the curve C, so it must be parallel to $\lambda n + \mu b$ and $t_1 = (\lambda n + \mu b)$ at P.

Therefore, on comparing the coefficients of this with that of relation (2), we get

$$1 - k\lambda = 0 \quad \text{or} \quad \lambda = \frac{1}{k} = \rho \quad \text{and } \lambda' - \mu\tau = P\lambda,\ \mu' + \lambda\tau = P\mu$$

$$\Rightarrow \frac{\lambda' - \mu\tau}{\tau} = \frac{\mu' + \lambda\tau}{\mu} \quad \text{i.e.} \quad \tau = \frac{\lambda'\mu - \lambda\mu'}{\lambda^2 + \mu^2} = \frac{d}{ds}\tan^{-1}\left(\frac{\lambda}{\mu}\right)$$

Or $\quad \tau = \frac{d}{ds}\tan^{-1}\left(\frac{\lambda}{\mu}\right) \qquad \ldots\ldots\ldots.\ (3)$

Integrating (3), we get

$a + \int \tau\, ds = \tan^{-1}\left(\frac{\lambda}{\mu}\right)$, where a is an arbitrary constant.

$or \quad \lambda = \mu \tan^{-1}(\int \tau\, ds + a)$

Or $\quad \mu = \lambda cot(\int \tau\, ds + a)$

Or $\quad \mu = \rho cot(\int \tau\, ds + a)$

Therefore, on putting the value of λ and μ in (1), we get

$$r_1 = r + \rho n + \rho cot(\int \tau\, ds + a)b \qquad \ldots\ldots\ldots.\ (4)$$

Which gives the required equation of evolute C_1 of C. in case we give different values of a, we shall an in case we assume $\int \tau ds = \phi(s)$ and $c - \frac{\pi}{2} = a$

Equation (4) may be put as

$$r_1 = r + \rho n - \rho tan(\phi(s) + c)b \qquad \ldots\ldots..\ (5)$$

Example 1. Find the involute of circular helix is

Solution: The equation of circular of helix is

$$r = [acosu, asinu, bu]$$

$$\therefore \dot{r} = [-asinu, acosu, b]\ ;\ \dot{s} = |\dot{r}| = \sqrt{a^2 + b^2}$$

$$r' = \frac{\dot{r}}{\dot{s}} = \frac{1}{\sqrt{a^2+b^2}} = [-asinu, acosu, b] = t$$

Therefore $s = \int_0^u |\dot{r}|du = \int_0^u \sqrt{a^2+b^2}du$

Again equation of an involute is $r_1 = r + (c-s)t$

$$= [-asinu, acosu, b] + (c-s)\frac{1}{\sqrt{a^2+b^2}}[-asinu, acosu, b]$$

$$r_1 = \left[acosu - \frac{a(c-s)}{\sqrt{a^2+b^2}}sinu, asinu + \frac{(c-s)}{\sqrt{a^2+b^2}}cosu, acosu, b\right]$$

Where $s = \sqrt{a^2+b^2}$u.

Example 2. Prove that the involute of a circular helix are plane curves.

Solution. For a circular helix, $\frac{k}{\tau} = a$ (constant)

$$k' = a\tau'$$

Again, torsion of an involute of a given curve $r = r(s)$ is given by

$\tau_1 = \frac{k\tau' - k'\tau}{k(c-s)(\tau^2+k^2)}$ …….. (1)

On putting the value of k and k' in terms of τ and τ' in (1), we have

$\tau_1 = 0$

i.e. torsion for the involute is zero and hence the involute is a plane curve.

Example 3. The locus of the centre of curvature is an evolute only when the curve is a plane curve.

Solution. The position vector of a current point on the evolute is given by

$r_1 = r + \rho n - \rho tan(\phi(s) + c)b$ ……… (1)

Where c is an arbitrary constant and for its various values we get an infinite system of evolutes.

Also, the locus of the centre of curvature is given by

$c = r + \rho n$ ……… (2)

We observe that equation (1) and (2) will coincide, i.e. the locus of the centre c is an evolute r_1 then we must have

$tan(\phi + c) = 0$

i.e. $\phi + c = m\pi$

or $\frac{d\emptyset}{ds} = 0$ i.e. $\tau = 0$

hence curve must be a plane curve.

4.4 *SPHERICAL INDICATRICES*

When we move all unit tangent vectors $\boldsymbol{t}$ of a curve C to a point, their extremities describe a curve C_1on the unit sphere; this curve $C_1$is called the **Spherical image** (Spherical Indicatrix) of C. we can similarly obtain the spherical image of C when its bonormal or principal normal are moved to a point.

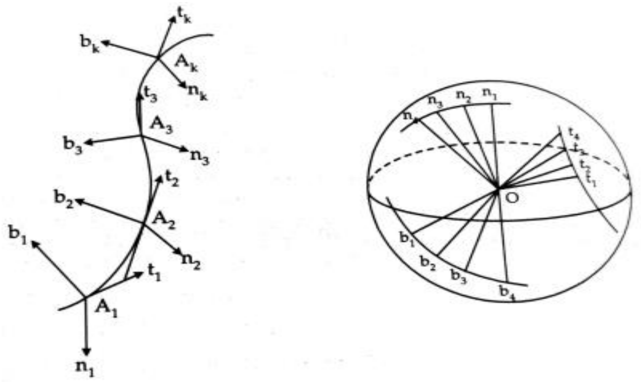


Fig.4.4.1

To construct the spherical indicatrix of the tangent, draw lines parallel to the positive directions of the tangents at the points of the given curve from the centre O of the unit sphere. Let t_1, t_2, t_3, t_4, ... be the points where these lines meets the surface of the sphere, the curve joining these points is spherical indicatrix of the tangent. Similarly, the spherical indicatrices of the principal normal and binormal can be constructed. Below we give the precise definition of various indicatrices.

(i) **The spherical indicatrix of the tangent.**

The locus of a point whose position vector is equal to the unit tangent $\boldsymbol{t}$ at any point of a given curve is called spherical indicatrix of the tangent.

(ii) **The spherical indicatrix of the principal normal.**

The locus of a point whose position vector is equal to the unit principal normal $\boldsymbol{n}$ at any point of a given curve is called spherical indicatrix of the principal normal.

(iii) **The spherical indicatrix of the binormal.**

The locus of a point whose position vector is equal to the unit binormal b at any point of a given curve is called the spherical indicatrix of the binormal.

4.5 *BERTRAND CURVES*

If a pair of curves C and C_1 are such that the principal normal to the C are also principal normal to C_1 , then the curves C and C_1 are conjugate or associate Bertrand curves.

Properties of Bertrand Curves.

Properties (i). The distance between corresponding points of two Bertrand curves is constant. (we shall use the suffix unity for quantities belonging to C_1).

Proof: Consider the principal normal to C and C_1 in the same sense, so that

$$n_1 = n \qquad \text{……. (1)}$$

Let r be the position vector of a point P on the curve C and r_1 be the position vector of the corresponding point P_1 on the associate Bertrand curve C_1 and C with respect to the some fixed origin O, then

$$r_1 = r + \lambda n \qquad \text{…….. (2)}$$

Where λ is a suitable function of s and represents the distance between corresponding points of the two curves.

Now taking the dot product of (1) and (3), we get

$0 = \lambda' \Rightarrow \lambda =$ constant.

$i.e.$ distance PP_1 is constant.

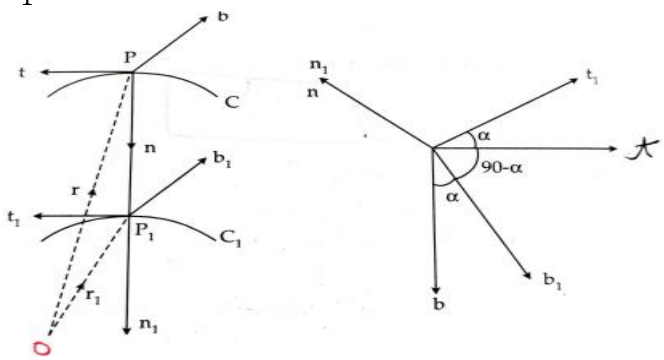


Fig.4.5.1

Properties (ii). The tangents at the corresponding point of the two curves are inclined at a constant angle.

Proof: We have

$$\frac{d}{ds}(t.t_1) = \frac{dt}{ds}.t_1 + t.\frac{dt_1}{ds_1}\frac{ds_1}{dt_1} = k.n.t_1 + t.k_1 n_1 \frac{ds_1}{ds}$$

$$= k.n_1.t_1 + k_1 \frac{ds_1}{ds} t.n \qquad \{since\ n_1 = n\}$$

$$= 0$$

Integrating, we get $t.t_1 =$ constant.

Now if α be the angle between t and t_1, then we have

$\Rightarrow t_1 . t = |t_1||t| \cos\alpha$

Therefore $\cos\alpha = t_1 . t =$ constant.

Since the principal normal coincide and tangents are inclined at a constant angle α and therefore the binormals of the two curves are also inclined at the same constant angle α.

Properties (iii). Curvature and torsion of either curve are connected by a linear relation.

Proof: We have shown above in properties (i) that $\lambda' = 0$

∴ equation (3) reduces to

$t_1 \frac{ds_1}{ds} = (1 - \lambda k)t + \lambda\tau b$ ……….. (4)

Now taking the dot product both sides of (4) with b_1, we have

$0 = (1 - \lambda k)t . b_1 + \lambda\tau b . b_1$ ……… (5)

Again $t . b_1 = \cos(90 - \alpha) = \sin\alpha$

And $b . b_1 = \cos\alpha$

Therefore from (5), we have

$0 = (1 - \lambda k)\sin\alpha + \lambda\tau\cos\alpha$ ……… (6)

The above relation (6) shows that there exists a linear relation with constant coefficients between curvature and torsion of curve C.

We may put relation (6) in the form

$\tau = \left(k - \frac{1}{\lambda}\right)\tan\alpha$ ……… (7)

Again, the relation between the curves C and C_1 is a reciprocal one, thus the point $P(r)$ is a distance $-\lambda$ along the normal at $P_1(r_1)$ and t is inclined at an angle $-\alpha$ with t_1.

Thus, for curvature C_1, we have a relation corresponding to (7) as

$\tau_1 = -\left(k_1 + \frac{1}{\lambda}\right)\tan\alpha.$

Properties (iv). The torsion of the two associate Bertrand curves have the same sign, and their product is constant.

Proof: We have that $t_1 = t\cos\alpha - b\sin\alpha$ …….. (8)

Comparing (4) of properties (iii) and (8) of properties (iv) we have

$$\frac{ds_1}{ds} = \frac{1 - \lambda k}{\cos\alpha} = \frac{\lambda\tau}{-\sin\alpha}$$

$\Rightarrow \cos\alpha = (1 - \lambda k)\frac{ds}{ds_1}$ ……… (9)

$\Rightarrow \sin\alpha = -\lambda\tau\frac{ds}{ds_1}$ .... (10)

Now, for the curve C_1, the relations corresponding to (9) and (10) are obtained by putting $-\lambda$ for λ and $-\alpha$ for α and interchanging s and s_1, we get

$\cos\alpha = (1+\lambda k_1)\frac{ds_1}{ds}$ .... (11) and $\sin\alpha = -\lambda\tau_1\frac{ds_1}{ds}$ ...... (12)

On multiplying (9) by (11) and (100 by (12), we get

$(1-\lambda k)(1+\lambda k_1) = \cos^2\alpha$ .... (13)

And $\tau\tau_1 = \frac{1}{\lambda^2}\sin^2\alpha$ .... (14)

Relation (14) shows that the torsion of the two curve have same sign and their product is constant.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The distance between corresponding points of two Bertrand curves is constant.

Problem 2. The tangent at the corresponding points of the curves are inclined at a constant angle.

Problem 3. The distance between corresponding points of two Bertrand curves is not constant.

Problem 4 The tangent at the corresponding points of the curves are inclined at a variable angle.

4.6 *SUMMARY*

(1) If there be one-one correspondence between points of two curve C and C_1 such that the tangent at any point of C is a normal to the corresponding point of C_1, then C_1 is called involute of C and C is called an evolute of C_1.

(2) If a pair of curves C and C_1 are such that the principal normal to the C are also principal normal to C_1 , then the curves C and C_1 are conjugate or associate Bertrand curves.

(3) $(\boldsymbol{i})$ **The spherical indicatrix of the tangent.**

The locus of a point whose position vector is equal to the unit tangent $\boldsymbol{t}$ at any point of a given curve is called spherical indicatrix of the tangent.

$(\boldsymbol{ii})$ **The spherical indicatrix of the principal normal.**

The locus of a point whose position vector is equal to the unit principal normal $\boldsymbol{n}$ at any point of a given curve is called spherical indicatrix of the principal normal.

(iii) **The spherical indicatrix of the binormal.**
The locus of a point whose position vector is equal to the unit binormal b at any point of a given curve is called the spherical indicatrix of the binormal.

4.7 *GLOSSARY*

(i) Derivatives

(ii) Torsion

(iii) Curvature

4.8 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."

2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".

3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".

4. Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.

4.9 *TEWRMINAL QUESTIONS*

1. Prove that corresponding points on the spherical indicatrix of the tangent to a curve C and on the indicatrix of the binormal to C have parallel tangent lines.
2. Find the involutes and evolutes of the circular helix

   $x = a\cos\theta, y = a\sin\theta, z = a\theta\tan\alpha.$
3. Define Bertrand curves.
4. Define Spherical Indicatrices.

4.10 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. False

COURSE NAME: GEOMETRY

COURSE CODE: MAT 611

BLOCK-II

FUNDAMENTAL FORM

UNIT 5: *FUNDAMENTAL FORMS I*

CONTENTS:

5.1 *INTRODUCTION*

In differential geometry, the study of smooth spaces and shapes, the fundamental theorem of space curves states that the shape, size, and scale of a regular curve in three-dimensional space is completely determined by its curvature and torsion. different space curves are only distinguished by how they bend and twist. Quantitatively, this is measured by the differential-geometric invariants called the curvature and the torsion of a curve. The fundamental theorem of curves asserts that the knowledge of these invariants completely determines the curve.

5.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) First Fundamental form or Metric

(ii) Second fundamental coefficient

5.3 *FIRST FUNDAMENTAL FORM OR METRIC*

Let $r = r(u, v)$ be the equation of a surface and let $E = {r_1}^2 = r_1.r_1$, $F = r_1.r_2$ and $G = {r_2}^2 = r_2.r_2$. the quadratic differential form $Edu^2 + 2Fdudv + Gdv^2$ in du, dv is called metric or first fundamental form of the surface and the quantities E, F, G are called first fundamental coefficients or first order fundamental magnitudes. Since E, F and G are functions of u, v the quantities will generally vary from point to point on the surface. These quantities are of much importance and will hence forth occur very frequently throughout the remainder part of the book.

Geometrically Interpretation on Metric:

Let $= r(u, v)$ be the surface and $u = u(t)$, $v = v(t)$ be a curve on the surface. Let P and Q be two neighboring point on the curve with position vectors r and $r + dr$, corresponding to the parameter values u, v and $u + du, v + dv$ respectively.

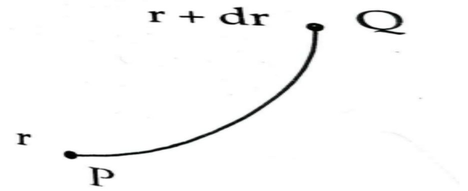


Fig.5.3.1

Then we have $dr = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv = r_1 du + r_2 dv$

Let the arc PQ be ds. Since the points P and Q are adjacent points, therefore

$ds = |dr|$ or $ds^2 = dr^2 = (r_1 du + r_2 dv)^2 dr^2$

$= {r_1}^2\, du^2 + 2r_1.r_2 du dv + {r_2}^2\, dv^2$

$= E\, du^2 + 2F du dv + G\, dv^2$ .... (1)

Thus, we have following interpretation of the first fundamental form.

If ds is the "infinitesimal distance" from the point (u, v) to the point $(u + du, v + dv)$ on the surface, then

$ds^2 = E\, du^2 + 2F du dv + G\, dv^2$

The name metric is assigned to the first fundamental form as it is chiefly used for the calculation of arc lengths on the surface. The arc length s of the curve is related to the parameter t by

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

Special cases: on the parametric curve $u =$ constant, we have $du = 0$ and hence metric (1) reduces to $ds^2 = G\, dv^2$. Similarly on the parametric curve $v =$ constant, we have $dv = 0$, thus the metric (1) reduces to $ds^2 = E\, du^2$.

An important relation between the coefficients E, F, G, and H:

From vector identity

$$(r_1 \times r_2)^2 = {r_1}^2 {r_2}^2 - (r_1 . r_2)^2$$

We have $H^2 = EG - F^2$ where $H = |r_1 \times r_2|$

Since H is positive quantity so $EG - F^2$ is also a positive quantity and H is equal to a positive square root of $EG - F^2$. Again at the ordinary point $r_1 \neq 0, r_2 \neq 0$ thus $E = {r_1}^2 > 0$ and $E = {r_2}^2 > 0$. Hence we have $E > 0, G > 0, EG - F^2 > 0$.

5.4 *IMPORTANT PROPERTIES OF THE METRIC*

Property 1. The metric of first fundamental form is a positive definite quadratic form in du, dv.

Proof. We have from first fundamental form that

E du^2 + 2Fdu dv + G dv^2

$= \frac{1}{E}\left(E^2 du^2 + 2EF du\, dv + EG\, dv^2\right)$

$= \frac{1}{E}\left((E\, du + F\, dv)^2 + \left(EG - F^2\right) dv^2\right) \geq 0,$

for all real values of du and dv

As E > 0 and EG - F^2 > 0.

Also, we have

$\left((E\, du + F\, dv)^2 + \left(EG - F^2\right) dv^2\right) = 0$

$\Rightarrow$ (E du + F dv) =0 and (EG - F^2) dv^2 = 0

$\Rightarrow$ E du + Fdv = 0 and dv = 0. (As EG-$F^2 \neq 0$.)

$\Rightarrow$ E du = 0 and dv = 0
$\Rightarrow$ du = 0 and dv = 0. As E $\neq$ 0.
Hence E du^2 + 2Fdu dv + G dv^2 i.e. metric or first fundamental form is a positive definite quadratic form in du and dv.

Property 2. Invariance property: the metric is invariant under a transformation of parameter.
Proof. Let in the equation of the surface $r = r(u, v)$, the parameters u, v are the transformed to the parameters u', v' by the relation $u' = \emptyset(u, v), v' = \Psi(u, v)$.
Thus $r_1' = \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial u'} = r_1\frac{\partial u}{\partial u'} + r_2\frac{\partial v}{\partial u'}$ …….. (1)
Similarly, $r_2' = \frac{\partial r}{\partial v'} = r_1\frac{\partial u}{\partial v'} + r_2\frac{\partial v}{\partial v'}$ ……… (2)
Again, $du = \frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'$ …….. (3)
And $dv = \frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'$ ……. (4)
Now, $E'du'^2 + 2F'du'dv' + G'dv'^2$

$$= r_1'^2du'^2 + 2r_1'.r_2'du'dv' + r_2'^2dv'^2$$
$$= (r_1'du' + r_2'dv')^2$$
$$= \left[\left(r_1\frac{\partial u}{\partial u'} + r_2\frac{\partial v}{\partial u'}\right)du' + \left(r_1\frac{\partial u}{\partial v'} + r_2\frac{\partial v}{\partial v'}\right)dv'\right]^2 \quad [from\ (1) and\ (2)]$$
$$= \left[r_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + r_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right]^2 \ [from\ (3) and\ (4)]$$
$$= (r_1du + r_2dv)^2 = r_1^2du^2 + 2r_1.r_2dudv + r_2^2dv^2$$
$$= E\,du^2 + 2Fdudv + G\,dv^2$$

Hence the metric is invariant.

Example 1. Calculate first fundamental magnitudes for the surface
$$r = [ucosv, usinv, f(u)].$$
Solution. The given surface is $r = [ucos, vsinv, f(u)]$,
$\therefore$ $r_1 = [cosv, sinv, f(u)'], r_2 = [-usinv, ucosv, 0]$
$$E = r_1^2 = cos^2v + sin^2v + f'^2 = 1 + f'^2$$

$$F = r_1.r_2 = -ucosvsinv + usinvcosv + 0 = 0$$
$$G = r_2^2 = u^2sin^2v + u^2cos^2v + 0 = u^2.$$

Example 2. Calculate E, F, G, H for the paraboloid
$$x = u, y = v, z = u^2 - v^2.$$

Solution. The given surface is $r = (u, v, u^2 - v^2)$
We have $r_1 = (1, 0, 2u), r_1 = (0, 1, -2v)$
$\therefore E = r_1^2 = 1 + 4u^2, F = r_1.r_2 = 0 + 0 - 4uv = -4uv$
$$G = r_2^2 = 1 + 4u^2$$
Also $H = \sqrt{(EG - F^2)} = \sqrt{(1 + 4u^2)(1 + 4v^2) - 16u^2v^2} = \sqrt{1 + 4u^2 + 4v^2)}$.

Example 3. Show that for the surface of revolution $x = ucosv, y = usinv, z = f(u)$ the parameter curves form an orthogonal system and $ds^2 = \left(1 + f'^2\right)du^2 + u^2dv^2$ where dash denotes differentiation with respect to u.

Solution. Here surface of revolution is same as in example 1, we have shows in example 1 that for this surface $E = 1 + f'^2, F = 0$ and $G = u^2$
since $F = 0$, therefore the parametric curves are orthogonal.
Again $ds^2 = Edu^2 + 2Fdudv + Gdv^2 = \left(1 + f'^2\right)du^2 + u^2dv^2$.

Example 4. Calculate the first fundamental coefficients and show that parametric curves are orthogonal, and find the area corresponding to the domain $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ for the anchor ring $[x = (b + acosu)cosv, y = (b + acosu)sinv, z = asinu]$.
Solution 3. The equation of the surface is

$$r = [x = (b + acosu)cosv, y = (b + acosu)sinv, z = asinu]$$

We have
$\mathbf{r_1} = \{$(-a sinu cosv, -a sinu sinv, a cosu $\}$.
$\mathbf{r_2} = \{$-(b + a cosu) sinv, (b + a cosu) cosv, 0 $\}$.
Then as we know that
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^{2,}$ $F = \mathbf{r_1}.\mathbf{r_2} = \mathbf{r_2}.\mathbf{r_1}$ and $G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2$
Therefore, we have for this problem, by taking scalar products suitably
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2 = a^2$
$F = \mathbf{r_1}.\mathbf{r_2} = 0$
$G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2 = (b + a\cos u)^2$
$H = \sqrt{(EG - F^2)} = a(b + a\cos u)$
Now F = 0 implies that parametric curves are orthogonal on the given surface.
Now the area bounded by the limits $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ is given by

$$A = \iint H dudv$$

$$= \int_0^{2\pi}\int_0^{2\pi} a\left(b + a\cos u\right)dudv$$

$$= \int_0^{2\pi} a\left[v\right]_0^{2\pi}\left(b + a\cos u\right)du$$

$$= 2a\pi\left[bu + a\sin u\right]_0^{2\pi}$$

$$= 4ab\pi^2.$$

This is required area.

5.5 *SECOND FUMDAMENTAL FORM AND SECOND ORDER MAGNITUDES*

Let $r = r(u, v)$ be the equation of the surface and N be the unit normal vector to this surface at the point $r(u, v)$ then

$$N = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H}$$

We know that $r_{11} = \frac{\partial^2 r}{\partial u^2}, r_{12} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2 r}{\partial v \partial u} = r_{21}, r_{22} = \frac{\partial^2 r}{\partial v^2}$

If $L = r_{11}.N, M = r_{12}.N = r_{21}.N, \boldsymbol{N} = r_{22}.N$, then the quadratic differential form is $\boldsymbol{L\,du^2 + 2Mdudv + Ndv^2}$ in du, dv is called the second fundamental form of the surface. The quantities L, M, N are called second fundamental coefficient or second order fundamentals magnitudes.

Geometrical interpretation of the second fundamental form:

To show that the length of the perpendicular, as far as terms of the second order, on the tangent plane to a surface at the point (u, v) from a neighboring point $(u + du, v + dv)$ is $\frac{1}{2}(\boldsymbol{L\,du^2 + 2Mdudv + Ndv^2})$

Proof: let $p(r)$ be the point of contact of the tangent plane with the square with parametric values (u, v) and let $Q(r + dr)$ be a neighboring point with parametric values $(u + du, v + dv)$ on the surface.

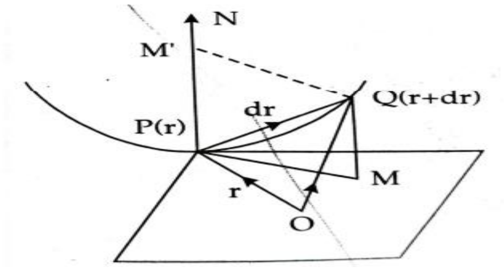


Fig.5.5.1

By Taylor's series we have

$$\mathrm{r + dr = r + (r_1du + r_2dv) + \frac{1}{2}(r_{11}du^2 + 2r_{12}dudv + r_{22}dv^2) + \cdots}$$

[neglecting quantities of order higher than two]

$$\mathrm{dr = (r_1du + r_2dv) + \frac{1}{2}(r_{11}du^2 + 2r_{12}dudv + r_{22}dv^2)}$$

let QM be the length of the perpendicular from Q on the tangent plane at P.

therefore QM = projection of the vector PQ on the normal at P

$$= N.dr = N.\mathrm{(r_1du + r_2dv)} + \frac{1}{2}N.\mathrm{(r_{11}du^2 + 2r_{12}dudv + r_{22}dv^2)}$$

upto terms of second order

$\frac{1}{2}\mathrm{(L\,du^2 + 2Mdudv + Ndv^2)}$.

Note: Some important products.

1. The scalar triple product of $\mathrm{N, r_1}$ and $\mathrm{r_2}$ has the value H;

$$[\boldsymbol{N}, \mathrm{r_1, r_2}] = \boldsymbol{N}.\mathrm{r_1 \times r_2} = N^2H = H \qquad [since\ N^2 = 1]$$

2. Cross product of N with $\mathrm{r_1}$ and $\mathrm{r_2}$

(i) $\mathrm{r_1 \times N = r_1 \times \frac{r_1 \times r_2}{\mathit{H}}} = \frac{1}{H}[\mathrm{(r_1.r_2)r_1 - (r_1.r_1)r_2}] = \frac{1}{H}[F\mathrm{r_1} - E\mathrm{r_2}]$

(ii) $\mathrm{r_2 \times N = r_2 \times \frac{r_1 \times r_2}{\mathit{H}}} = \frac{1}{H}[\mathrm{(r_2.r_2)r_1 - (r_2.r_1)r_2}] = \frac{1}{H}[G\mathrm{r_1} - F\mathrm{r_2}]$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. E du^2 + 2Fdu dv + G dv^2 is called metric.

Problem 2. $\mathrm{L\ du^2 + 2Mdudv + Ndv^2}$ is called Second fundamental form.

Problem 3. The metric is not invariant under a transformation of parameter.

Problem 4 The metric of first fundamental form is a positive definite quadratic form in du, dv.

5.6 *SUMMARY*

(1) E du^2 + 2Fdu dv + G dv^2 i.e. metric or first fundamental form is a positive definite quadratic form in du and dv.

(2) The scalar triple product of N, r_1 and r_2 has the value H;

$$[\boldsymbol{N}, \mathrm{r}_1, \mathrm{r}_2] = \boldsymbol{N}.\mathrm{r}_1 \times \mathrm{r}_2 = N^2 H = H \qquad [since\ N^2 = 1]$$

Cross product of N with r_1 and r_2

$$(ii)\mathrm{r}_1 \times \mathrm{N} = \mathrm{r}_1 \times \frac{\mathrm{r}_1 \times \mathrm{r}_2}{H} = \frac{1}{H}[(\mathrm{r}_1.\mathrm{r}_2)\mathrm{r}_1 - (\mathrm{r}_1.\mathrm{r}_1)\mathrm{r}_2] = \frac{1}{H}[F\mathrm{r}_1 - E\mathrm{r}_2]$$

$$(iii)\mathrm{r}_2 \times \mathrm{N} = \mathrm{r}_2 \times \frac{\mathrm{r}_1 \times \mathrm{r}_2}{H} = \frac{1}{H}[(\mathrm{r}_2.\mathrm{r}_2)\mathrm{r}_1 - (\mathrm{r}_2.\mathrm{r}_1)\mathrm{r}_2] = \frac{1}{H}[G\mathrm{r}_1 - F\mathrm{r}_2]$$

5.7 *GLOSSARY*

(i) Derivatives

(ii) Torsion

5.8 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn “Cambridge University Press.”

2. Matrices and Tensors in physics by A.W. Joshi “Wiley Eastern Limited”.

3. Tensors by Ram Bilas Mishra “Hardwari publications Allahabad”.

4. Differential Geometry by Gupta, Malik and Pundir “Pragati Edition’’.

5.9 *TEWRMINAL QUESTIONS*

1. Prove that E du^2 + 2Fdu dv + G dv^2 is positive definite.
2. Define Second fundamental form.
3. Define First fundamental form.

5.10 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. True

UNIT 6: *FUNDAMENTAL FORMS II*

CONTENTS:

6.1 *INTRODUCTION*

In differential geometry, the study of smooth spaces and shapes, the fundamental theorem of space curves states that the shape, size, and scale of a regular curve in three-dimensional space is completely determined by its curvature and torsion. different space curves are only distinguished by how they bend and twist. Quantitatively, this is measured by the differential-geometric invariants called the curvature and the torsion of a curve. The fundamental theorem of curves asserts that the knowledge of these invariants completely determines the curve.

6.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) First Fundamental form or Metric

(ii) Second fundamental coefficient

6.3 *SURFACE OF REVOLUTION*

Definition: A surface generated by the revolution of a plane curve about and axis in its plane is called a surface of revolution.

Let us take $z-$ axis as the axis of revolution and let the generating curve in z x-plane $(y=0)$ be given by $x=f(u), y=0, z=g(u)$.

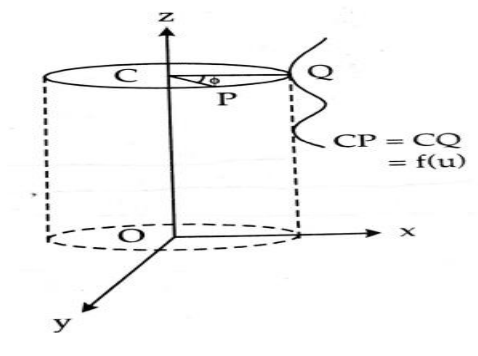


Fig.6.3.1

Let this plane curve be rotated through an angle φ, then the co-ordinates of a point $P=(x,y,z)$ are given by $x=CQ\cos\varphi\,, y=CP\sin\varphi\,, z=ZQ=g(u)$

$\Rightarrow x = f(u)\cos\varphi\,, y = f(u)\sin\varphi\,, z = g(u)$

For convenience, we often take $f(u) = u$

$\Rightarrow x = u\cos\varphi\,, y = u\sin\varphi\,, z = g(u)$

Or in vector notation, position vector of P is $r = (u\cos\varphi\,, u\sin\varphi\,, g(u)\,)$

This is the equation of the surface of revolution.

(a) Sphere:

Sphere can be regarded as a surface of revolution formed by rotating the circle $\mathrm{x}^2 + \mathrm{z}^2 = \mathrm{a}^2, \mathrm{y} = 0$ [in $\mathrm{zx} -$ plane] about $\mathrm{z} -$ axis.

Coordinates of any point, say Q on any point of the circle can be taken as

$\mathrm{x} = \mathrm{asinu}, \mathrm{y} = 0, \mathrm{z} = \mathrm{acosu}$ whence the equation of the sphere can be written as

$$\mathrm{x} = \mathrm{asinu}\cos\varphi\,, \mathrm{y} = \mathrm{asinu}\ \sin\varphi, \mathrm{z} = \mathrm{acosu}$$

Or $$\mathrm{x} = \mathrm{asinu}\cos\mathrm{v}\,, \mathrm{y} = \mathrm{asinu}\ \mathrm{sinv}, \mathrm{z} = \mathrm{acosu}$$

Where φ is replaced by v.

Or in vector form $\mathrm{r} = (\mathrm{asinu\ cosv}, \mathrm{asinu\ sinv}, \mathrm{acosu})$.

Note: The **Co-latitude** u of the point P may be defined as the inclination of the radius OP to the z-axis and **longitude** v as inclination of the plane containing P and the z-axis to $y = 0$ plane. The parametric curve $u =$ constant are the small circles called the **parallels of latitude;** the parametric curve $u =$ constant are the great circles called the meridians of longitude. The poles $u = 0$ and $u = \pi$ are called artificial singularities. The domain of u is $0 < u < \pi$ and that of v as

$0 \leq v < 2\pi$.

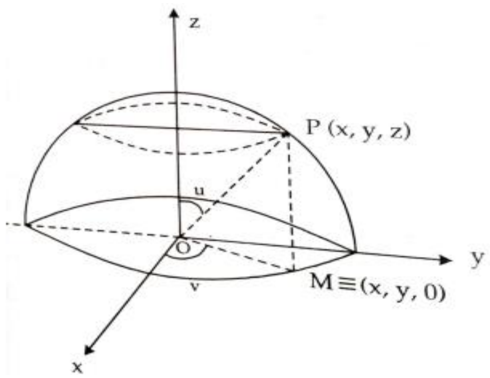


Fig.6.3.2

Here $\mathrm{r} = (\mathrm{asinu\ cosv}, \mathrm{asinu\ sinv}, \mathrm{acosu})$

$\Rightarrow \quad r_1 = (a\cos u \cos v, a\cos u \sin v, -a\sin u)$

$\Rightarrow \quad r_2 = (-\,a\sin u \sin s v, a\sin u \cos v, 0)$

$\Rightarrow \quad r_1.r_2 = -a^2\sin u \sin v \cos u \cos v + a^2\sin u \sin v \cos u \cos v = 0$

$\Rightarrow \quad F = 0$

$\Rightarrow$ Parametric curves are orthogonal

$\Rightarrow$ Latitudes and longitudes intersect orthogonally.

(b) Right circular cone of semi-vertical angles α.

Let the line OZ be taken as axis of the cone and let $P = (x, y, z)$ be any point on the surface of the cone. Further let u be the distance of P from z −axis and let v be the inclination of the plane containing P and the $z -$ axis to zx −plane. Then as shown in the above figure, we have

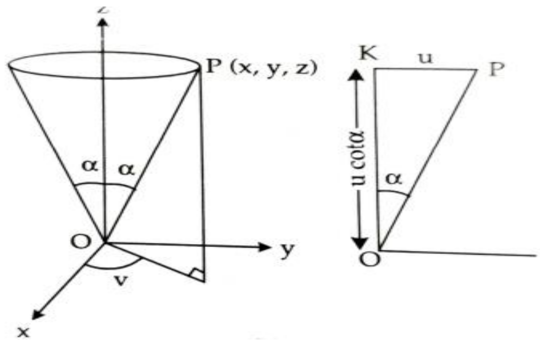


Fig.6.3.3

$$x = u\cos v, y = u\sin v, z = OK = u\cot \propto.$$

$\Rightarrow$ position vector of $P = (u\cos v, u\sin v, u\cot \propto)$

$\Rightarrow \quad r = (u\cos v, u\sin v, u\cot \propto)$

$\Rightarrow \quad r_1 = (\cos v, \sin v, \cot \propto), r_2 = (-u\sin v, u\cos v, 0)$

$\Rightarrow \quad r_1.r_2 = 0 \Rightarrow F = 0, \Rightarrow$ parametric curve are orthogonal.

Here $\quad r_1 \times r_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & \cot \propto \\ -u\sin v & u\cos v & 0 \end{vmatrix}$

$$= (-u\cos v \cot\alpha)\hat{\imath} - (u\sin v\cot\alpha)\hat{\jmath} + \hat{k}(u)$$

$\Rightarrow (r_1 \times r_2)_{u=0}$ i.e. vertex is only singularity.

(c) **Anchor Ring**

A surface obtained by rotating a circle (radius a) about a line in its plane at a distance $b(>a)$ from the centre, is called anchor ring.

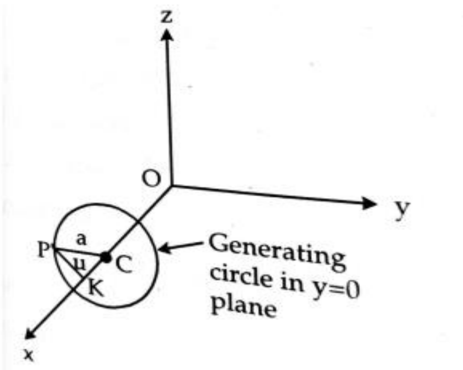


Fig.6.3.4

Let the generating circle lie in $y=0$ plane and let it rotate about $z-axis$. Further let C be the centre of the circle such that $CO=b(>a)$ if P' is any point on the circumference of this circle, then co-ordinates of P' are

$x=b+acosu, y=0, z=asinv$, where u is the angle which the radius CP' makes with positive direction of x −axis. When the generating circle has been rotated through an angle v, let P' take the position P and as such its co-ordinates are

$x=(b+acosu)cosv, y=(b+acosu)sinv, z=asinu$ these are the equations of the anchor ring and domain of u,v is $0<u<2\pi; 0<v<2\pi$.

(d) **Helicoids**

General Helicoid. A surface generated by a curve which is simultaneously rotated about a fixed axis and translate in the direction of the axis with a velocity proportional to the velocity of rotation, is called helicoids.

this kind of the motion of the curve called a screw motion or a helicoidal motion. Different positions of the generating curve can be obtained by first translating it through a distance λ parallel to the axis and then rotating it through an angle $\boldsymbol{v}$

about the axis. The ration (λ/v) is always constant. Let it be c i.e. (λ/v) = c. the constant $2\pi c$ is called the pitch of the helicoids (distance translated in one complete revolution). As a matter of face, pitch is positive or negative according as the helicoids is right of or left -handed (right or left screw) and is zero when the surface is of revolution.

Equations of General Helicoids.

For a general helicoid, the meridians i.e. the sections of the surface by planes containing the axis are congruent plane curves. There is no loss of generality if we assume the generating curve to be a plane curve. The surface be thought as being generating be given by equation $x = f(u), y = 0, z = g(u)$; whence the positive vector of any current point on the helicoids is given by

$r = (f(u)cosv, f(u)sinv, g(u) + cv)$ where c be any constant.

$\Rightarrow \quad r_1 = \left(f'(u)cosv, f'(u)sinv, g'(u)\right)$

$r_2 = (-f(u)sinv, f(u)cosv, c)$

$\Rightarrow \quad r_1.r_2 = cg'(u)$.

If $r_1.r_2 = 0$, we have $cg'(u) = 0 \Rightarrow c = 0$ or $g'(u) = 0 \Rightarrow c = 0$

Or $\quad g(u) =$ constant.

Parametric curves $v =$ constant are the various positions of the generating curves whereas parametric curves $u =$ constant are circular helices. When, $c = 0$, we have

$r = \left(f(u)cosv, f(u)sinv, g(u)\right)$. This represents the equation of the surface of revolution.

Further if $g(u) =$ constant $= k$, say we have

$r = (f(u)cosv, f(u)sinv, k + cv)$, which represents a right helicoid.

Right Helicoid

Definition. The surface generated by the helicoids motion of a straight line meeting the axis in perpendicular direction is called right-helicoid.

Let us take the axis as z-axis, then the position vector of any current point on the right helicoids is given by

$r = (ucosv, usinv, cv)$ where $u =$ distance of a point from z −axis, $v =$ angle of rotation; and the generator is assumed to be x −axis when $v = 0$.

Now $r_1 = (cosv, sinv, cv)$ and $r_2 = (-usinv, ucosv, c)$

$\Rightarrow \quad r_1.r_2 = -usinvcosv + ucosvsinv = 0, \Rightarrow F = 0$

$\Rightarrow$ Parametric curve are orthogonal.

It is to be noted that the curves $v = 0$ constant are the generators whereas $u = 0$ are the circular helices.

(e) Surface generated by tangents to a twisted curve.

Let $r = r(s)$ be the equation of the curve in space. Consider a point $P = r$ on the space curve and let Q be any point (having position vector R) on it,

then $\vec{R} - \vec{r} = u\vec{t}$ where scalar u is the distance of Q from P. this can be further

re-written as $\vec{R} = \vec{r} + u\vec{t}$ ……. (1)

and represents the surface by the tangents to the curve $r = r(s)$. As $\vec{r}$ and $\vec{t}$ are both functions of the arc lengths s of the given curve, therefore in equation (1), $\vec{R}$ is the function of two parameters u and s.

The parametric curves $s =$ constant give the generators of the surface and are the tangents to the given curve, Also the parametric curves u = constant are the curves which cut the tangents at a constant distance from the given curve.

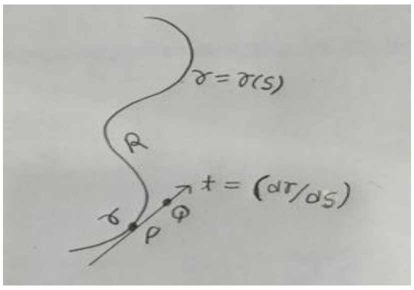


Fig.6.3.5

We know proceed to calculate the fundamental magnitudes for surface $\vec{R} = \vec{r} + u\vec{t}$, Where $\vec{R} = \vec{R}(u, s)$, u is first parameters and s is second parameter

$\vec{R}_1 = \frac{\partial \boldsymbol{R}}{\partial u} =$ t, $\vec{R}_2 = \frac{\partial \boldsymbol{R}}{\partial s} = \frac{d\boldsymbol{R}}{ds} + u\frac{dt}{ds} = t + ukn$

$$E = {R_1}^2 = t^2 = 1, F = R_1.R_2 = t.(t + ukn) = 1,$$

$$G = {R_2}^2 = (t + utn).(t + ukn) = 1 + u^2k^2,$$

$$H^2 = EG - F^2 = u^2k^2$$

Now, $R_1 \times R_2 = t \times (t \times ukn) = ukb.$

$\Longrightarrow N =$ unit normal vector to the surface $= \frac{R_1 \times R_2}{H} = \frac{ukb}{uk} = b$, where $H = |\vec{N}|$

Further, $R_{11} = 0$, $R_{12} = \frac{dt}{ds} = kn,$

$$R_{22} = \frac{dt}{ds} + u\frac{d(kn)}{ds} = kn + u[k'n + k(\tau b - kt)]$$

$$= kn + uk'n + uk\tau b - uk^2 t.$$

$$\Longrightarrow \quad L = R_{11}.N = 0\ ,\ M = R_{12}.N = (kn).b = 0 \text{ and } N = R_{22}.N = uk\tau.$$

Example 1. Find the fundamental magnitudes for some important surfaces.

(a) The general surface of the revolution

In the case surface of revolution, the position, the position vector **r** of a current point is given by

$$\mathbf{r} = [u\cos \mathrm{v}, \mathrm{u}\sin \mathrm{v}, f(\mathrm{u})]$$

$$\mathbf{r}_1 = (\cos \mathrm{v}, \sin \mathrm{v}, f');\ \mathbf{r}_2 = (-u \sin \mathrm{v}\,, \mathrm{u} \cos \mathrm{v}, 0),$$

$$\mathrm{r}_{11} = (0,0, f'');\ \mathrm{r}_{21} = (-\sin \mathrm{v}, \cos \mathrm{v}, 0),$$

$$\mathrm{r}_{22} = (-\mathrm{u} \cos \mathrm{v}, -\mathrm{u} \sin \mathrm{v}, 0)$$

$$\mathrm{E} = \mathrm{r}_1^2 = \cos^2 \mathrm{v} + \sin^2 \mathrm{v} + f'^2 = 1 + f'^2$$

$$F = r_1 \cdot r_2 = -u\sin v\cos v + u\sin v\cos v + 0 = 0,$$

$$\mathrm{G} = \mathbf{r}_2^2 = \mathrm{u}^2\sin^2 \mathrm{v} + \mathrm{u}^2\cos^2 \mathrm{v} + 0 = \mathrm{u}^2$$

$$\mathbf{r}_1 \times \mathbf{r}_2 = (-uf'\cos v, -uf'\sin v, u\cos^2 v + u\sin^2 v)$$

$$= u(-f'\cos v, -f'\sin v, 1)$$

$$\therefore H^2 = EG - F^2 = u^2(1 + f'^2) - 0 = u^2(1 + f'^2)$$

$$\mathrm{H} = \mathrm{u}\sqrt{1 + f'^2}$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{\mathrm{u}(-f'\cos \mathrm{v}, -f'\sin \mathrm{v}, 1)}{\mathrm{u}\sqrt{1 + f^2}} = \frac{(-f'\cos \mathrm{v}, -f'\sin \mathrm{v}, 1)}{\sqrt{1 + f^2}}$$

$$\mathrm{L} = \mathrm{N} \cdot \mathbf{r}_{11} = \frac{(-f\cos \mathrm{v}, -f'\sin \mathrm{v}, 1) \cdot (0,0, f'')}{\sqrt{1 + f'^2}}$$

$$= \frac{f''}{\sqrt{(1 + f'^2)}},$$

$$M = N \cdot \mathbf{r}_{12} = \frac{(-f'\cos \mathrm{v}, -f'\sin \mathrm{v}, 1) \cdot (-\sin \mathrm{v}, \cos \mathrm{v}, 0)}{\sqrt{(1 + f'^2)}}$$

$$= \frac{f'\sin v\cos v - f'\sin v\cos v + 0}{\sqrt{(1 + f'^2)}} = 0$$

$$\mathbf{N} = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{(-f'\cos \mathrm{v}, -f'\sin \mathrm{v}, 1) \cdot (-u\cos \mathrm{v}, -\mathrm{u}\sin \mathrm{v}, 0)}{\sqrt{\{(1 + f'^2)\}}}$$

$$= \frac{uf'}{\sqrt{\{(1+f'^2)\}}}.$$

(b) The Conoidal Surface.

The position vector of a current point on the conoidal surface is given by

$$\mathbf{r} = [u\,cosv, u\,sinv, f(v)]$$
$$\therefore\ \mathbf{r}_1 = (\cos v\,, \sin v\,, 0); r_2 = (-u\sin v, u\cos v, f');$$
$$\mathbf{r}_{11} = (0,0,0); \mathbf{r}_{12} = (-\sin \mathrm{v}, \cos \mathrm{v}, 0);$$
$$\mathbf{r}_{22} = (-u\cos v, -u\sin v, f'')$$
$$\because\ E = \mathbf{r}_1 \cdot \mathbf{r}_1 = \cos^2 v + \sin^2 v = 1$$
$$F = \mathbf{r}_1 \cdot \mathbf{r}_2 = -u \cos v \sin v + u \cos v \sin v + 0 = 0$$
$$G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2 \sin^2 v + u^2 \cos^2 v + f'^2 = u^2 + f'^2;$$

Again $\quad H^2 = EG - F^2 = (u^2 + f'^2) - 0 = u^2 + f'^2;$

$\because \qquad \mathrm{H} = \sqrt{\mathrm{u}^2 + f'^2}$

Now

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{(\cos v, \sin v, 0) \times (-u\sin v, u\cos v, f')}{\sqrt{\{(u^2 + f'^2)\}}}$$
$$= \frac{(\sin v f', -\cos v f', u\cos^2 v + u\sin^2 v)}{\sqrt{\{(u^2 + f'^2)\}}} = \frac{(\sin v f', -\cos v f', u)}{\sqrt{\{(u^2 + f'^2)\}}}.$$

Therefore

$$\mathbf{L} = \mathbf{N} \cdot \mathbf{r}_{11} = \mathbf{N} \cdot (0,0,0) = 0,$$
$$\mathbf{M} = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{(\sin v f'\,, -\cos v f'\,, u)}{\sqrt{(f'^2 + u^2)}} \cdot (-\sin v\,, \cos v\,, 0) = \frac{-f'}{\sqrt{(f' + u^2)}},$$
$$\mathbf{N} = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{(\sin v f'\,, -\cos v f'\,, u) \cdot (-\mathrm{u}\cos v\,, -\mathrm{u}\sin v\,, f'')}{\sqrt{(f'^2 + u^2)}}$$
$$= \frac{-\mathrm{u}\sin \mathrm{v}\cos v f' + \mathrm{u}\sin \mathrm{v}\cos v f' + u f''}{\sqrt{(f'^2 + u^2)}} = \frac{u f''}{\sqrt{(f'^2 + u^2)}}.$$

(c) Monge's Form of the Surface.

The equation of a surface given in the form z = f(x, y) is called Monge’s form. Again from the knowledge differential equation, we know that

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Taking x, y as parameters, we have

$$\mathbf{r} = [x, y, f(x, y)]$$

$$\mathbf{r}_1 = [1,0,p];\ \mathbf{r}_2 = (0,1,q);\ \mathbf{r}_{11} = (0,0,r);\ \mathbf{r}_{22} = (0,0,t);\ \mathbf{r}_{12} = (0,0,s) = \mathbf{r}_{21}$$

Therefore

$$E = \mathbf{r}_1 \bullet \mathbf{r}_1 = 1 + p^2;\ F = \mathbf{r}_1 \bullet \mathbf{r}_2 = pq;\ G = \mathbf{r}_2 \bullet \mathbf{r}_2 = 1 + q^2$$

Again $H^2 = EG - F^2 = (1+p^2)(1+q^2) - p^2q^2 = 1 + p^2 + q^2$

or $$H = \sqrt{\{(1+p^2+q^2)\}}$$

Now $$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{(1,0,p)\times(0,1,q)}{H} = \frac{(-p,-q,1)}{\sqrt{\{(1+p^2+q^2)\}}}$$

$$L = \mathbf{N} \bullet \mathbf{r}_{11} = \frac{(-p,-q,1)\bullet(0,0,r)}{\sqrt{\{(1+p^2+q^2)\}}} = \frac{r}{\sqrt{\{(1+p^2+q^2)\}}}$$

$$M = \mathbf{N} \bullet \mathbf{r}_{12} = \frac{(-p,-q,1)\bullet(0,0,s)}{\sqrt{\{(1+p^2+q^2)\}}} = \frac{s}{\sqrt{\{(1+p^2+q^2)\}}}$$

$$N = \mathbf{N} \bullet \mathbf{r}_{22} = \frac{(-p,-q,1)\bullet(0,0,t)}{\sqrt{\{(1+p^2+q^2)\}}} = \frac{t}{\sqrt{\{(1+p^2+q^2)\}}}$$

(d) Right Helicoid

To find the fundamental magnitudes for the right helicoid given by

$$x = u\cos\phi,\ y = u\sin\phi,\ z = c\phi$$

Let the suffix 1 and 2 represent partial differentiation w.r.t. $'u'$, and 'ϕ'.
Now

$$\mathbf{r} = (u\cos\phi, u\sin\phi, c\phi)$$

$\therefore\ \mathbf{r}_1 = (\cos\phi, \sin\phi, 0); \mathbf{r}_2 = (-u\sin\phi, u\cos\phi, c)$
$\mathbf{r}_{11} = (0,0,0); \mathbf{r}_{12} = (-\sin\phi, \cos\phi, 0);$
$\mathbf{r}_{22} = (-u\cos\phi, -u\sin\phi, 0)$

Therefore $E = \mathbf{r}_1 \cdot \mathbf{r}_1 = \cos^2\phi + \sin^2\phi = 1, F = \mathbf{r}_1 \cdot \mathbf{r}_2 = 0; G = \mathbf{r}_2 \cdot \mathbf{r}_2 = u^2 + c^2$
Again $H^2 = EG - F^2 = u^2 + c^2$ or $H = \sqrt{\{(u^2 + c^2)\}}$

Now $\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{(\mathrm{c}\sin\phi, -\cos\phi, \mathrm{u})}{\sqrt{\{(\mathrm{u}^2+\mathrm{c}^2)\}}}$

$$\therefore \mathrm{L} = \mathbf{N}\cdot\mathbf{r}_{11} = 0; \mathrm{M} = \mathbf{N}\cdot\mathbf{r}_{12} = \frac{-\mathrm{c}}{\sqrt{\{(\mathrm{u}^2+\mathrm{c}^2)\}}}, \mathbf{N} = \mathbf{N}\cdot\mathbf{r}_{22} = 0$$

Example 2. Calculate the fundamental magnitudes and the normal to the surface $2z = ax^2 + 2hxy + b^2$ taking x, y as parameters.

Solution. The position vector of any current point on the surface is given by

$$\mathbf{r} = \left(\mathrm{x}, \mathrm{y}, \frac{\mathrm{a}}{2}\mathrm{x}^2 + \mathrm{hxy} + \frac{\mathrm{b}}{2}\mathrm{y}^2\right)$$

$$\therefore \mathbf{r}_1 = (1,0, ax + hy); \mathbf{r}_2 = (0,1, hx + by)$$

$$\mathbf{r}_{11} = (0,0,a); \mathbf{r}_{12} = (0,0,h); \mathbf{r}_{22} = (0,0,b)$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{(-(ax+hy), -(hx+by), 1)}{\mathrm{H}}$$

$$E = \mathbf{r}_1\cdot\mathbf{r}_1 = 1 + (ax+hy)^2; F = r_1\cdot\mathbf{r}_2 = (ax+hy)(hx+by);$$

$$G = \mathbf{r}_2\cdot\mathbf{r}_2 = 1 + (hx+by)^2$$

$$H^2 = EG - F^2 = 1 + (ax+hy)^2 + (hx+by)^2$$

$$\therefore L = \mathbf{N}\cdot\mathbf{r}_{11} = \frac{a}{H}; M = \mathbf{N}\cdot\mathbf{r}_{12} = \frac{h}{H}; N = \mathbf{N}\cdot\mathbf{r}_{22} = \frac{b}{H}\ .$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Parametric curves are orthogonal

Problem 2. Latitudes and longitudes intersect orthogonally.

Problem 3. $r = (f(u)cosv, f(u)sinv, k + cv)$, represents a right helicoid.

6.4 *SUMMARY*

(1) Definition. The surface generated by the helicoids motion of a straight-line meeting the axis in perpendicular direction is called right-helicoid.

(2) Definition: A surface generated by the revolution of a plane curve about and axis in its plane is called a surface of revolution.

6.5 *GLOSSARY*

(i) Derivatives

(ii) Torsion

6.6 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."

2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".

3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".

4 Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.

6.7 *TEWRMINAL QUESTIONS*

1. Define Right Helicoid, find its equation.
2. Define Monge's Form of the Surface, find its equation.
3. Right circular cone of semi-vertical angles α, find its equation.
4. Sphere, find its equation.

6.8 *ANSWERS*

CYQ 1. True **CYQ 2.** True **CYQ 3.** True

UNIT 7: ANGLE BETWEEN PARAMETRIC CURVES

CONTENTS:

7.1 *INTRODUCTION*

Dear learners, in the previous units, you might have studied and learnt by now that

1. A curve in space is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed the function of a single variable parameter t (for example time), s (for example arc-length) or u (any other notion), say.

Example: of a space curve is circular helix, whose equation is $r(t) = a\, cos\, t\, \hat{\imath} + a \sin t\, \hat{\jmath} + ct\, \hat{k}$, where t lies in $(-\infty, \infty)$.

Equation of a space curve can also be expressed as intersection of two surfaces in space. Various properties of apace curves have been discussed in previous units in detail.

2. A surface is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed the function of two independent variable parameters, u and v (any other notion), say.

Thus x = x (u, v), y = y(u, v), z = z(u, v) are called parametric equations of a surface. The parameters u and v take real values and vary in some region D. This type of representation is an explicit form of surface.

Example of a surface is conicoid whose equation is x = u cos v, y = u sin v, z = f(v).

Equation of a surface can also be written as z = f(x, y). This representation is called Monge's form of the surface. For example, z= x^2 - y^2 represents a hyperboloid. It's parametric equation is x= u cosh v, y= u sinh v, z= u^2. It is obvious that we can write the equation of surface in any form as desired.

We have the following notations for partial differentiation of position vector **r** with respect to the parameter u and v.

$$r_1 = \frac{\partial r}{\partial u}\ ,\ \ r_2 = \frac{\partial r}{\partial v},\ \ r_{11} = \frac{\partial^2 r}{\partial u^2},\ \ r_{21} = r_{21} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u},\ \ r_{22} = \frac{\partial^2 r}{\partial v^2}.$$

That is to say, suffixes 1 and 2 denote the partial differentiation with respect to parameter u and v respectively.

7.2 *OBJECTIVES*

Dear learners, after studying this unit, you should be able to -

(i) Understand the concept of parametric curves.

(ii) Understand the concept of tangential vector.

(iii) To find angle between the parametric curves

(iv) To find angle between any two-space curve.

(v) Understand direction ratios and direction coefficients.

7.3 *CURVES ON A SURFACE*

Dear learners, you might have studied the equation of curves length of curves, radius of curvature, angle between two intersecting curves in a plane, viz. xy plane, in geometry and differential calculus in undergraduate course. Now, we shall have knowledge of curves on surface in three dimensional spaces. Suppose $\mathbf{r} = \mathbf{r}$ (u, v) is the parametric equation of a surface. Any relation between the parameters, say g (u, v) = constant, gives the equation of a curve lying on this surface. For then anyone parameter, say u, can be written in terms of the other parameter v and hence the position vector **r** becomes a function of only one parameter v, and thus its locus is, by definition, a curve. In a similar way if the parameters u and v are expressed as function of a single parameter t, then again, the position vector **r** becomes a function of only one independent parameter t, hence its locus is a curve lying on the given surface. Hence u = u(t) and v = v(t), then $\mathbf{r} = \mathbf{r}$ (u(t), v(t)) is the equation of a curve on the surface $\mathbf{r} = \mathbf{r}$(u, v). Then the we call the equations u = u(t) and v = v(t) as curvilinear equations of the curve.

7.4 *CURVES ON A SURFACE*

We shall now go for the basic knowledge of parametric curves on a given surface. Suppose

$\mathbf{r} = \mathbf{r}$(u, v) is the equation of a surface. Now if we keep either u or v constant, then we obtain curves lying on this surface and these curves are of independent interest and importance. These curves obtained by keeping one of the parameters as constant are called parametric curves.

If we take v = c (constant) and u is allowed to vary, then position vector **r** becomes a function of single parameter u and hence its locus is a curve by definition. Thus $\mathbf{r} = \mathbf{r}$(u, c) is a curve lying on the surface $\mathbf{r} = \mathbf{r}$(u, v). This curve is called parametric curve v = c or the u-curve. There is one such curve for every value of c and if c is an arbitrary constant, then v = c forms a family of parametric curves v = constant.

In a similar way, if we take u = c (a constant) and allow v to vary, then we obtain the family of parametric curves u = constant.

We should know that through each point on the surface $\mathbf{r} = \mathbf{r}$(u, v), one and only curve of each system passes. From this we can conclude that no two parametric curves of same family intersect.

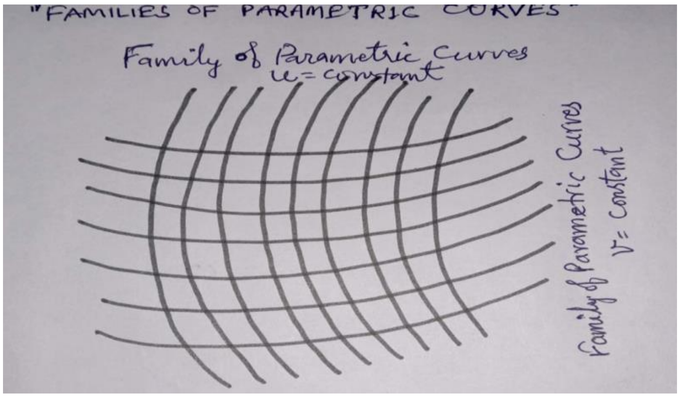


Fig. 7.4.1

7.5 *VECTORS TANGENTIAL TO PARMETRIC CURVES ON A SURFACE*

Dear learners, we are aware of finding tangent and normal in case of plane curves at different points of it using ordinary differential in calculus. Here, we shall find the similar things in parametric versions.

Suppose **r** = **r**(u, v) be the equation of a given surface. We are now clear that the equation of the parametric curve u = a (constant) can be written as **r** = **r**(a, v). For this parametric curve v takes the variable values. Therefore we can differentiate this equation **r** = **r**(a, v) partially with respect to v. Differentiating **r** = **r**(u, v), partially with respect to v, we get a vector $r_2 = \frac{\partial r}{\partial v}$, tangential to curve **r** = **r**(a, v) along the direction of increasing v.

In the same way, equation of the parametric curve v = b (constant) can be written as **r** = **r**(u, b). For this parametric curve u takes the variable values. Therefore we can differentiate this equation **r** = **r**(u, v) partially with respect to u. Differentiating **r** = **r**(u, v), partially with respect to u we get a vector$r_1 = \frac{\partial r}{\partial v}$, tangential to curve **r** = **r**(u, b) along the direction of increasing u.

In this unit, we only consider the surfaces which have no singularities at all of any kind; therefore, we always have $\mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0}$. Hence parametric curves of different systems never touch each other, as for condition of touching is $\mathbf{r}_1 \times \mathbf{r}_2 = \mathbf{0}$.

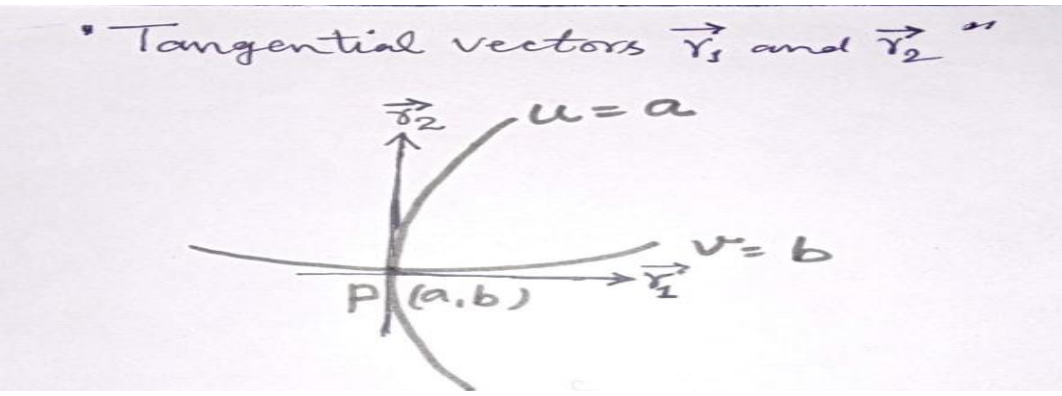


Fig. 6.2

7.6 *NORMAL TO THE SURFACE*

Dear learners, you might have learnt how to find unit normal vector on a given surface in Monge' form. You can recall the process learnt in vector calculus by finding the gradient of surfaces f(x, y, z) = constant or z = z(x, y). But those methods are useful only if surface is not given in parametric forms (Gaussian form) but given in Monge's form. Here shall learn to find normal on surface if equation of surface is given in parametric form i.e. containing two parameters.

Let **r** = **r**(u, v) be the equation of a surface. Then the normal to the surface at any point on it is a line passing through that point and perpendicular to the tangent plane at the concerning point.

If **r** = **r**(u, v) is the given surface then we have learnt that the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ are tangential to the surface. Therefore, normal to the surface at any point is perpendicular to both the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ and hence parallel to the vector $\mathbf{r_1}\times\mathbf{r_2}$ as the vector $\mathbf{r_1}\times\mathbf{r_2}$ itself is perpendicular in direction to both individual vectors $\mathbf{r_1}$ and $\mathbf{r_2}$.

The direction of the normal is taken in such a way that if **N** is the unit normal vector at any point P, then vectors $\mathbf{r_1}$, $\mathbf{r_2}$ and **N** always constitute a right-handed system. Thus the vectors $\mathbf{r_1}\times\mathbf{r_2}$ and **N** are in the same direction. Therefore

$$\mathbf{N} = \frac{\boldsymbol{r_1}\times\boldsymbol{r_2}}{|\boldsymbol{r_1}\times\boldsymbol{r_2}\,|}\,.$$

We, by convention, denote $\boldsymbol{r_1} \times \boldsymbol{r_2}$ by H. since $\mathbf{r_1}\times\mathbf{r_2} \neq \mathbf{0}$, therefore $\mathbf{H} \neq \mathbf{0}$ and H > 0 i.e. H is always positive definite quantity. So we can write

$$\mathbf{N} = \frac{\boldsymbol{r_1}\times\boldsymbol{r_2}}{\boldsymbol{H}}, \quad \text{or} \qquad H\,\mathbf{N} = \mathbf{r_1}\times\mathbf{r_2}$$

7.7 FUNDAMENTAL COEFFICIENTS AND RELATION AMONG THEM

Dear learners, in previous units you have learnt about first and second fundamental coefficients. Let **r** = **r**(u, v) be the equation of a surface, then for brevity, we introduce first fundamental coefficients as-

$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2$, $F = \mathbf{r_1}.\mathbf{r_2} = \mathbf{r_2}.\mathbf{r_1}$ and $G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2$.

Similarly, the second fundamental coefficients are defined as-

$L = \mathbf{r_{11}}.\mathbf{N}$, $M = \mathbf{r_{12}}.\mathbf{N} = \mathbf{r_{21}}.\mathbf{N}$ and $N = \mathbf{r_{22}}.\mathbf{N}$.

Here as we know the symbols

$$r_{11} = \frac{\partial^2 r}{\partial u^2},\ \ r_{21} = r_{21} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u},\ \ r_{22} = \frac{\partial^2 r}{\partial v^2}.$$

The second fundamental coefficients can also be suitably written as-

$L = -\mathbf{r_1}.\mathbf{N_1}$, $M = -\mathbf{r_1}.\mathbf{N_2} = -\mathbf{r_2}.\mathbf{N_1}$ and $N = -\mathbf{r_2}.\mathbf{N_2}$.

If ø is angle between the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$, then from the definition of cross product or vector product of two vectors, we have

$$\mathbf{r_1}\times\mathbf{r_2} = |\,\mathbf{r_1}|\,|\mathbf{r_2}\,|\sin ø\ \mathbf{N}, \qquad \ldots\ldots\ldots \quad (1)$$

where **N** is the unit normal vector to the surface at the that point under consideration.

Squaring or making self-scalar product of both the sides of equation (1), we have

$(\mathbf{r_1} \times \mathbf{r_2})^2 = |\mathbf{r_1}|^2 |\mathbf{r_2}|^2 \sin^2 ø$ **[since $\mathbf{N}^2=1$]**

Or $(H\,\mathbf{N})^2 = |\mathbf{r_1}|^2 |\mathbf{r_2}|^2 (1-\cos^2 ø)$

$H^2 = |\mathbf{r_1}|^2 |\mathbf{r_2}|^2 - |\mathbf{r_1}|^2 |\mathbf{r_2}|^2 \cos^2 ø$

$= |\mathbf{r_1}|^2 |\mathbf{r_2}|^2 - (\mathbf{r_1} \cdot \mathbf{r_2})^2$

$= EG - F^2.$

Or $H^2 = EG - F^2.$

Since H is positive quantity, therefore $EG - F^2$ is also a positive quantity always. H is positive square root of $EG - F^2$.

7.8 ANGLE BETWEEN PARAMETRIC CURVES

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of a surface. Also let at any point P on this surface, two parametric curves u-constant and v = constant intersect each other. Let $\mathbf{r}$ is position vector of the point P. Then vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ are tangential to surface along the directions of tangents of parametric curves v = constant and u = constant respectively. If ø $(0 < ø < \pi)$ is angle between the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ or say between the parametric curves , then

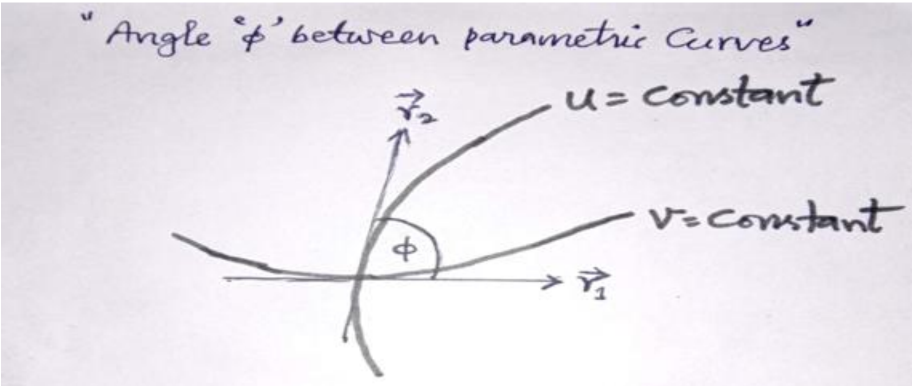


Fig. 6.3

$$\cos ø = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1}|\,|\boldsymbol{r_2}|} = \frac{F}{\sqrt{EG}}$$

and

$$\sin ø = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1} \times \boldsymbol{r_2}|} = \frac{H}{\sqrt{EG}} = \frac{\sqrt{(EG - F2)}}{\sqrt{EG}}$$

and combining these results

$$\tan ø = \frac{\sin ø}{\cos ø} = \frac{H}{F}.$$

In general, the angle between the parametric curves varies from point to point. The parametric curves are said to form an orthogonal system if they cut at right angles at all points of the surface.

From the formula of cos ø or tan ø, it is obvious that if ø is right angle then F=0 and vice-versa. i.e. $\mathbf{r_1.r_2} = 0$.

Thus, the necessary and sufficient condition for parametric or coordinate curves to be orthogonal system is that F = $\mathbf{r_1.r_2} = 0$**,** at each point of the surface.

Now, we shall discuss in detail the orthogonality, angle between any two curves and orthogonal trajectory in coming unit.

7.9 SOLVED EXERCISES

Question1: Find the equation of tangent plane and normal to the surface given by xyz = 4 at the point (i) (1, 2, 2) and (ii) (-1, -1, 4).

Solution: Equation of the surface can be written as

F(x, y, z) = xyz - 4 = 0.

We now have

$$\frac{\partial F}{\partial x} = yz = 4 \ \ at \ \ (1,2,2)$$

$$\frac{\partial F}{\partial y} = xz = 2 \ \ at \ \ (1,2,2)$$

$$\frac{\partial F}{\partial x} = xy = 2 \ \ at \ \ (1,2,2)$$

Therefore, the equation of the tangent plane at (1, 2, 2) by the formula

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0,$$

is (x-1).4 + (y-2).2 + (z-2).2 = 0

Or

4x + 2y + 2z = 12.

i.e. equation of tangent plane is

2x + y + z = 6.

Now the equation of the normal is given by

$$\frac{(X - x)}{\frac{\partial F}{\partial x}} = \frac{(Y - y)}{\frac{\partial F}{\partial y}} = \frac{(Z - z)}{\frac{\partial F}{\partial z}}$$

So, equation of the normal at the point (1, 2, 2) on the surface is

$$\frac{(x-1)}{4} = \frac{(y-2)}{2} = \frac{(z-2)}{2}$$

Or

$$\frac{(x-1)}{2} = \frac{(y-2)}{1} = \frac{(z-2)}{1}.$$

Similarly, following the same lines, the learners can find the required equation of tangent plane and normal at the point (-1, -1, 4).

Question 2: Prove that the metric or the first fundamental form is a positive definite quadratic form in du and dv.

Solution. We have from first fundamental form that

$E\,du^2 + 2F du\,dv + G\,dv^2$

$= \frac{1}{E}\left(E^2 du^2 + 2EF du\,dv + EG\,dv^2\right)$

$= \frac{1}{E}\left((E\,du + F\,dv)^2 + \left(EG - F^2\right)dv^2\right)$

» 0, for all real values of du and dv

As $E > 0$ and $EG - F^2 > 0$.

Also, we have

$\left((E\,du + F\,dv)^2 + \left(EG - F^2\right)dv^2\right) = 0$

$\Rightarrow$ (E du + F dv) =0 and $(EG - F^2)\,dv^2 = 0$

$\Rightarrow$ E du + Fdv = 0 and dv = 0. (As $EG-F^2 \neq 0$.)

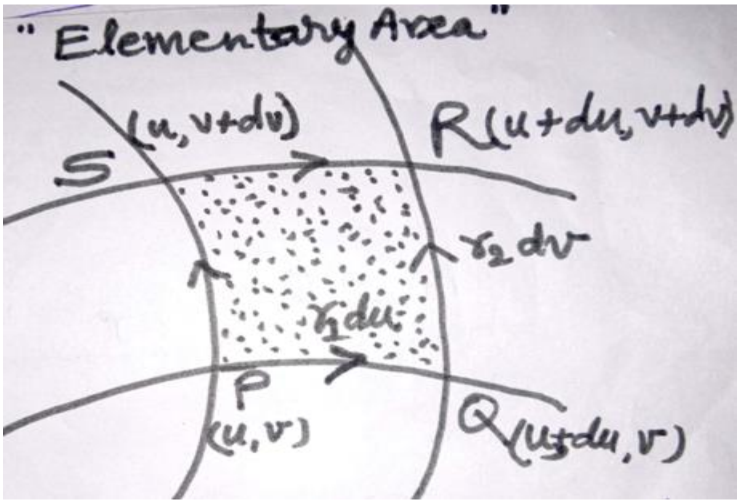
$\Rightarrow$ E du = 0 and dv = 0

$\Rightarrow$ du = 0 and dv = 0. As $E \neq 0$.

Hence $E\,du^2 + 2F du\,dv + G\,dv^2$ i.e. metric or first fundamental form is a positive definite quadratic form in du and dv.

Question 3: Find the expression for the elementary area at a point (u, v) of an arbitrary surface **r =** (u, v)

Solution: Let a very small portion PQRS near the point P(u, v) of an arbitrary surface **r =**(u, v). Let the coordinates of the vertices P, Q, R and S are (u, v), (u + du, v), (u + du, v + dv) and (u, v + dv) respectively. We join P, Q, R, S so that PQRS becomes a parallelogram when du and dv are very small.



Now vector $\overrightarrow{PQ}$ = position vector of Q - position vector of P

= **r** (u + du, v) - **r**(u, v)

= [**r**(u, v) - $\frac{\partial r}{\partial u}$ du] - **r**(u, v)

= $\mathbf{r_1}$ du.

Similarly vector $\overrightarrow{PS}$ = $\overrightarrow{QR}$ = $\mathbf{r_2}$ dv. Therefore, if ds is the area of this elementary parallelogram PQRS, then

ds= $|r_1 du \times r_2 dv|$

= $|r_1 \times r_2|\ du\, dv$

=H du dv.

Thus H du dv is expression for the elementary area on a surface **r = r**(u, v).
If complete area is required between specified limits, then

Area = $\iint H du dv$, such bounded area is covered within given limits.

Question 4: Calculate the first fundamental coefficients and show that parametric curves are orthogonal, and find the area corresponding to the domain $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ for the anchor ring

x = (b + a cosu) cosv, y = (b + a cosu) sinv, z = a sinu.

Solution. The equation of the surface is

r = {(b + a cosu) cosv, (b + a cosu) sinv, a sinu }.

We have

$\mathbf{r_1}$ **= {**(-a sinu cosv, -a sinu sinv, a cosu }.

$\mathbf{r_2}$ **= {-**(b + a cosu) sinv, (b + a cosu) cosv, 0 }.

Then as we know that

E = $\mathbf{r_1.r_1 = r_1}^2$, F= $\mathbf{r_1.r_2 = r_2.r_1}$ and G= $\mathbf{r_2.r_2 = r_2}^2$

Therefore, we have for this problem, by taking scalar products suitably

E = $\mathbf{r_1.r_1 = r_1}^2$ = a^2

$F = \mathbf{r_1}.\mathbf{r_2} = 0$
$G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2 = (b + a \cos u)^2$
$H = \sqrt{(EG - F^2)} = a (b + a \cos u)$
Now F = 0 implies that parametric curves are orthogonal on the given surface.
Now the area bounded by the limits $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ is given by

$$A = \iint H du dv$$
$$= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u) du dv$$
$$= \int_0^{2\pi} a\,[v]_0^{2\pi} (b + a\cos u) du$$
$$= 2a\pi[bu + a\sin u]_0^{2\pi}$$
$$= 4ab\pi^2.$$

This is required area.

Question 5: Show that on the surface given by x = a (u + v), y = b (u - v) and z = uv, the parametric curves are straight lines. Also calculate the fundamental coefficients for this surface and find the condition for the orthogonality of parametric curves on this surface.

Solution. The surface is given as
$\mathbf{r} = (a (u + v), b (u-v), z = uv)$.
On this surface, the parametric curves u = constant = c, are given by
X = a (c + v), y = b (c - v) and z = cv. In these equations now v is the parameter. Eliminating v, we get
X = a (c + z / c), y = b (c – z / c). These are equations of two planes, whose intersection is a straight line. Therefore, the parametric equation u = c are straight lines on the surface.
Again, taking the parametric curves v= c (constant), we get
X = a (u + c), y = b (u - c) and z = uc. In these equations, now u has become the parameter. Eliminating u in these equations, we get
X = a ((z / c) + c), y = b ((z / c) – c).
These are again the equations two planes, whose intersection is a straight line. Thus the parametric curves v = c are also straight lines.
Now the equation of the surface is given as
$\mathbf{r} = (a (u + v), b (u - v), z = uv)$.
So that, we get on differentiating the above equation w.r.t. u and v respectively
$\mathbf{r_1} = (a, b, v)$.

$\mathbf{r_2} = (a, -b, u)$.
$\mathbf{r_{11}} = (0, 0, 0)$.
$\mathbf{r_{21}} = (0, 0, 1)$.
$\mathbf{r_{22}} = (0, 0, 0)$.
Now as we know that
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2$, $F = \mathbf{r_1}.\mathbf{r_2} = \mathbf{r_2}.\mathbf{r_1}$ and $G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2$
Therefore, we have for this problem, by taking scalar products suitably
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2 = a^2 + b^2 + v^2$
$F = \mathbf{r_1}.\mathbf{r_2} = a^2 - b^2 + uv$
$G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2 = a^2 + b^2 + u^2$
$H^2 = EG - F^2 = 4a^2b^2 + a^2 (u - v)^2 + b^2(u + v)^2$.
Also, we know that
$\mathbf{r_1}\times\mathbf{r_2} = (bu + bv, av-au, -2ab)$. So that
$\mathbf{N} = (\mathbf{r_1}\times\mathbf{r_2}) / H = \dfrac{(bu + bv,\ av - au,\ -2ab).}{H}$
So that the second fundamental coefficients are given as
$L = \mathbf{r_{11}}.\mathbf{N} = 0$
$M = \mathbf{r_{12}}.\mathbf{N} = \dfrac{(-2ab)}{H}$
$N = \mathbf{r_{22}}.\mathbf{N} = 0$
Finally, condition of orthogonality of parametric curves is given by
$F = 0$, which means
$a^2-b^2+uv = 0$
or
$uv = b^2- a^2$.

Question 6: Find the metric and elementary area for the paraboloids
$\mathbf{r} = (u, v, u^2 - v^2)$.
Solution: For the given paraboloids $\mathbf{r} = (u, v, u^2 - v^2)$.
$\mathbf{r_1} = (1, 0, 2u)$.
$\mathbf{r_2} = (0, 1, -2v)$.
Now as we know,
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2$, $F = \mathbf{r_1}.\mathbf{r_2} = \mathbf{r_2}.\mathbf{r_1}$ and $G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2$
Therefore, we have for this problem, by taking scalar products suitably
$E = \mathbf{r_1}.\mathbf{r_1} = \mathbf{r_1}^2 = 1 + 4u^2$
$F = \mathbf{r_1}.\mathbf{r_2} = -4uv$
$G = \mathbf{r_2}.\mathbf{r_2} = \mathbf{r_2}^2 = 1 + 4v^2$
Therefore, the metric is given by
$E\,du^2 + 2F du\,dv + G\,dv^2$

$= (1 + 4u^2)\, du^2 - 8dudv + (1 + 4v^2)\, dv^2$

Now $H = \sqrt{(EG - F^2)} = \sqrt{\{(1 + 4u^2)(1+ 4v^2) - 16u^2 v^2\}}$

$= \sqrt{(1 + 4u^2 + 4v^2)}$

Therefore, elementary area is given by

$H.\, du\, dv = \sqrt{(1 + 4u^2 + 4v^2)}\, du\, dv$

Question 7: Show that if L, M, N vanish everywhere on a surface, then the surface is part of a plane.

Solution: We know that

$$L= -\mathbf{r_1}.\mathbf{N_1},\ M = -\mathbf{r_1}.\mathbf{N_2} = -\mathbf{r_2}.\mathbf{N_1} \text{ and } \mathbf{N} = -\mathbf{r_2}.\mathbf{N_2}.$$

Since L = M = N = 0 everywhere on the surface, therefore

$\mathbf{r_1}.\mathbf{N_1} = 0,\ \mathbf{r_2}.\mathbf{N_1} = 0$ ………(1)

And

$\mathbf{r_1}.\mathbf{N_2} = 0,\ \mathbf{r_2}\ \mathbf{N_2} = 0$ ……… (2)

Since $\mathbf{r_1} \neq 0$, $\mathbf{r_2} \neq 0$, therefore from (1), either $\mathbf{N_1} = 0$, **or** $\mathbf{N_1}$ is perpendicular to both $\mathbf{r_1}$ **and** $\mathbf{r_2}$.

This means N1 is parallel to $\mathbf{r_1} \times \mathbf{r_2}$. Which clearly implies that $\mathbf{N_1}$ is parallel to **N.** ($\mathbf{r_1} \times \mathbf{r_2} = H\,\mathbf{N}$). But $\mathbf{N_1}$ is perpendicular to **N,** being a vector of constant modulus. Therefore, $\mathbf{N_1}$ cannot be parallel to **N.**

Hence from (1), we see that $\mathbf{N_1} = 0$, implying that **N** is independent of parameter u.

In the same way, we can show that **N** is independent of parameter v, by taking equation (2).

Therefore, **N** is independent of both the parameters' u and v, thereby a constant vector at every point on the surface. Thus, at every point of the surface the normal to the surface are parallel. Hence the surface is part of a plane.

Thus, the result is proved.

Question 8: State and Weingarten equations. **Or**

Show that in terms of E, F, G, L, M, N and H, the Weingarten equations are

$$H^2\, \mathbf{N_1} = (FM-GL)\, \mathbf{r_1} + (FL-EM)\, \mathbf{r_2}$$

$$H^2\, \mathbf{N_2} = (FN-GM)\, \mathbf{r_1} + (FM-EN)\, \mathbf{r_2}$$

And deduce the formula

$$H\, \mathbf{N_1} \times \mathbf{N_2} = (LN-M^2)\, \mathbf{N}.$$

Solution: Since **N** is a vector of constant modulus, therefore both the vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ are perpendicular to **N.** Thus, both the vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ are tangential to the surface. So, both the vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ lie in the plane of the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ and so we can write

$$\mathbf{N_1} = a\,\mathbf{r_1} + b\,\mathbf{r_2} \qquad \ldots\ldots \quad (1)$$

$$\mathbf{N_2} = c\,\mathbf{r_1} + d\,\mathbf{r_2} \qquad \ldots\ldots \quad (2)$$

For some scalars a, b, c and d.

Taking scalar product of both sides of (1) with $\mathbf{r_1}$ and then with $\mathbf{r_2}$, we have

$$\mathbf{r_1}.\ \mathbf{N_1} = a\,\mathbf{r_1}.\mathbf{r_1} + b\,\mathbf{r_2}.\mathbf{r_1}$$

and $$\mathbf{r_2}.\ \mathbf{N_1} = a\,\mathbf{r_2}.\mathbf{r_1} + b\,\mathbf{r_2}.\mathbf{r_2}$$

These equations become

$$-L = aE + bF \qquad \ldots\ldots \quad (3)$$

$$-M = aF + dG \qquad \ldots\ldots\ldots (4)$$

Solving (3) and (4) for a and b, we get

$$a = \frac{FM - GL}{H^2}$$

$$b = \frac{FL - EM}{H^2}.$$

Putting these values of a and b in equation (1), we get

$$H^2\,\mathbf{N_1} = (FM - GL)\,\mathbf{r_1} + (FL - EM)\,\mathbf{r_2} \quad \ldots\ldots\ldots \quad (5)$$

Now taking scalar product of both sides of (2) with $\mathbf{r_1}$ and then with $\mathbf{r_2}$, we have

$$\mathbf{r_1}.\ \mathbf{N_2} = c\,\mathbf{r_1}.\mathbf{r_1} + d\,\mathbf{r_2}.\mathbf{r_1}$$

and $$\mathbf{r_2}.\ \mathbf{N_2} = c\,\mathbf{r_2}.\mathbf{r_1} + d\,\mathbf{r_2}.\mathbf{r_2}.$$

These equations reduce to

$$-M = cE + dF \qquad \ldots\ldots\ldots\ldots (6)$$

$$-N = cF + dG \qquad \ldots\ldots\ldots\ldots (7)$$

Solving (6) and (7) for c and d, we get

$$c = \frac{FN - GM}{H^2}$$

$$d = \frac{FM - EN}{H^2}.$$

Putting these values of c and d in equation (2), we get

$$H^2\,\mathbf{N_2} = (FN - GM)\,\mathbf{r_1} + (FM - EN)\,\mathbf{r_2} \quad \ldots\ldots\ldots (8)$$

Now taking vector product i.e. cross product of equations (5) and (8), we have

$$H^2\,\mathbf{N_1} \times H^2\,\mathbf{N_2} = \{(FM-GL)\,\mathbf{r_1} + (FL-EM)\,\mathbf{r_2}\} \times \{(FN - GM)\,\mathbf{r_1} + (FM - EN)\,\mathbf{r_2}\}$$

$$H^4\,\mathbf{N_1} \times \mathbf{N_2} = \{(FM - GL) \times (FM - EN)\}\,\mathbf{r_1} \times \mathbf{r_2} + \{(FL - EM) \times (FN - GM)\}\,\mathbf{r_2} \times \mathbf{r_1}$$

$$= [\{(FM - GL) \times (FM - EN)\} - \{(FL - EM) \times (FN - GM)\}]\,\mathbf{r_1} \times \mathbf{r_2}$$

As we know that $\mathbf{r_1} \times \mathbf{r_1} = 0$, $\mathbf{r_2} \times \mathbf{r_1} = -\,\mathbf{r_1} \times \mathbf{r_2}$, $\mathbf{r_2} \times \mathbf{r_2} = 0$.

Or on solving and little manipulation, we get
H^4 $\mathbf{N_1} \times \mathbf{N_2}$ = (LN - M^2) (EG - F^2) H **N**
H^4 $\mathbf{N_1} \times \mathbf{N_2}$ = (LN - M^2) H^2 H **N**
H $\mathbf{N_1} \times \mathbf{N_2}$ = (LN - M^2) **N**.
Which is the desired result.

CHECK YOUR PROGRESS

True or false Questions

Problem 1. The angle ø between the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ is given by

$$\cos ø = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1}|\,|\boldsymbol{r_2}|} = \frac{F}{\sqrt{EG}}\ .$$

.

Problem 2. The necessary and sufficient condition for parametric or coordinate curves to be orthogonal system is that F = $\mathbf{r_1}.\mathbf{r_2}$ = 0 at each point of the surface.

Problem 3. **N** = $\frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{\boldsymbol{H}}$.

Problem 4. H^2 = EG - F^2.
.

7.10 *SUMMARY*

1. A curve in space is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed as the function of a single variable parameter.

2. A surface is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed as the function of two independent variable parameters.

3. Some standard notations are

$$r_1 = \frac{\partial r}{\partial u}\ ,\ \ r_2 = \frac{\partial r}{\partial v},\ \ r_{11} = \frac{\partial^2 r}{\partial u^2},\ \ r_{21} = r_{21} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u}\ ,\ \ r_{22} = \frac{\partial^2 r}{\partial v^2}$$

4. $\mathbf{N} = \dfrac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1} \times \boldsymbol{r_2}|} = \dfrac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{H}$,

5. First fundamental coefficients are as-

$E = \mathbf{r_1.r_1} = \mathbf{r_1}^2$, $F = \mathbf{r_1.r_2} = \mathbf{r_2.r_1}$ and $G = \mathbf{r_2.r_2} = \mathbf{r_2}^2$.

6. $H = \sqrt{(EG - F^2)}$

7. Second fundamental coefficients are as-

$L = \mathbf{r_{11}.N}$, $M = \mathbf{r_{12}.N} = \mathbf{r_{21}.N}$ and $\mathbf{N} = \mathbf{r_{22}.N}$.

8. The second fundamental coefficients can also be written as-

$L = -\mathbf{r_1.N_1}$, $M = -\mathbf{r_1.N_2} = -\mathbf{r_2.N_1}$ and $N = -\mathbf{r_2.N_2}$.

9. The angle ø between the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ is given by

$$\cos ø = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1}||\boldsymbol{r_2}|} = \frac{F}{\sqrt{EG}}$$

and

$$\sin ø = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1} \times \boldsymbol{r_2}|} = \frac{H}{\sqrt{EG}} = \frac{\sqrt{(EG - F2)}}{\sqrt{EG}}$$

and combining these results

$$\tan ø = \frac{\sin ø}{\cos ø} = \frac{H}{F}.$$

10. The necessary and sufficient condition for parametric or coordinate curves
to be orthogonal system is that $F = \mathbf{r_1.r_2} = 0$. At each point of the surface.

11. The equation of the tangent plane at the point (x, y, z) on the surface
F(x, y, z) is given by the formula

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0.$$

12. The equation of the normal at the point (x, y, z) on the surface F(x, y, z) is
given by the formula

$$\frac{(X - x)}{\frac{\partial F}{\partial x}} = \frac{(Y - y)}{\frac{\partial F}{\partial y}} = \frac{(Z - z)}{\frac{\partial F}{\partial z}}$$

7.11 *GLOSSARY*

(i) Orthogonal – mutually perpendicular.

(ii) Curve – Path traversed by point or locus of point in space if point depends on one parameter only.

(iii) Surface- Path traversed by point or locus of point in space if point depends on two parameters only

7.12 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."
2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".
3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".
4. Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.
5. Differential Geometry, An Integrated Approach: Nirmala Prakash, Tata Mc Graw-Hill Publishing Company Limited, New Delhi

7.13 *TEWRMINAL QUESTIONS*

(TQ - 1) Find the equation of tangent plane and normal to the surface given by $z = x^2 + y^2$ at the point (i) (1, -1, 2) and (ii) (2, 2, 4).

(TQ - 2) Find the equation of tangent plane and normal to the surface given by z = xy at the point (i) (2, 3, 6) and (ii) (1, 3, 3).

(TQ - 3) Prove that the metric is invariant under a transformation of parameters.

(TQ - 4) Find expressions for the second fundamental coefficients in terms of differentials of unit normal vector **N** or prove the results $L = -\mathbf{r}_1.\mathbf{N}_1$, $M = -\mathbf{r}_1.\mathbf{N}_2 = -\mathbf{r}_2.\mathbf{N}_1$ and $N = -\mathbf{r}_2.\mathbf{N}_2$.

(TQ - 5) Taking x and y as parameters, calculate the fundamental coefficients and the unit normal vector to surface $2z = ax^2+2hxy+by^2$.

(TQ - 6) Calculate the fundamental coefficients and the unit normal vector to surface (conoid) (u cosv, u sinv, f(v)).

(TQ - 7) Calculate the fundamental coefficients and the unit normal vector to surface (helicoids) (u cosv, u sinv, f(u)+cv).

(TQ - 8) Prove that on the surface of revolution (u cosv, u sinv, f(u)), the parametric curves u= constant reduces into circles lying in the plane parallel to the xy-plane.

(TQ - 9) Prove that if for a surface the condition (E/L) = (F/M) = (G/N) holds at all points, then the surface is either a plane or spherical one.

(TQ - 10) Obtain Weingarten equations and deduce the following results

$H\,[\mathbf{N}, \mathbf{N_1}, \mathbf{r_1}] = EM\text{-}FL$

$H\,[\mathbf{N}, \mathbf{N_1}, \mathbf{r_2}] = FM\text{-}GL$

$H\,[\mathbf{N}, \mathbf{N_2}, \mathbf{r_1}] = EN\text{-}FM$

$H\,[\mathbf{N}, \mathbf{N_2}, \mathbf{r_2}] = FN\text{-}GN$.

7.14 ANSWERS

(TQ -1) (i) $2x - 2y - z = 2$, $(x - 1) / 2 = (y + 1) / (-2) = (z - 2)(-1)$

(ii) $4x + 4y - z = 12$, $(x - 2) / 4 = (y - 2) / 4 = (z - 4) / (-1)$

(TQ -2) (i) $3x + 2y - z = 6$, $(x - 2) / (-3) = (y - 3) / (-2) = (z - 6) / 1$

(ii) $3x + y - z = 3$, $(x - 1) / 3 = (y - 3) / 1 = (z - 3) / (-1)$

(TQ -5) $E = 1 + (ax + hy)^2$, $f = (ax + hy)(hx + by)$, $G = 1 + (hx + by)^2$

$L = a / H$, $M = h / H$, $N = b / H$, Where $H^2 = 1 + (ax + hy)^2 + (hx + by)^2$.

UNIT 8: ORTHOGONAL TRAJECTORIES

CONTENTS:

8.1 *INTRODUCTION*

Dear learners, in the previous units, you should have studied and learnt by now that

1. A curve in space is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed the function of a single variable parameter t (for example time), s (for example arc-length) or u (any other notion), say.
2. A surface is defined as the locus of a point whose Cartesian co-ordinates (x, y, z) can be expressed the function of two independent variable parameters, u, v (any other notion), say.

Thus x = f(u, v), y = (u, v), z = (u, v) are called parametric equations of a surface. The parameters u and v take real values and vary in some region D. This type of representation is an explicit form of surface.

Example of a surface is conicoid whose equation is x= u cos v, y= u sin v, z= f(v).

We have the following notations for partial differentiation of position vector **r** with respect to the parameter u and v.

$$r_1 = \frac{\partial r}{\partial u},\ r_2 = \frac{\partial r}{\partial v},\ r_{11} = \frac{\partial^2 r}{\partial u^2},\ r_{21} = r_{21} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u},\ r_{22} = \frac{\partial^2 r}{\partial v^2}.$$

That is to say, suffixes 1 and 2 denote the partial differentiation with respect to parameter u and v respectively.

3. Angle between two intersecting curves is defined as angle between their tangents at that point of intersection.
4. Two curves are said to intersect orthogonally iff angle between their tangents is right angle at the point of intersection or in other words tangents are mutually perpendicular.
5. Curves of same parameter of family do not ever intersect.
6. expression H= √ (EG- F^2) is a positive definite quantity.
7. The unit normal vector **N** is always perpendicular to surface as well as vector $\mathbf{r}_1$ and $\mathbf{r}_2$.

8.2 *OBJECTIVES*

(i) After completion of this unit learners will be able to:

(ii) Understand the concept of direction coefficients and direction ratios.

(iii) Understand the concept of family of curves and their differential equations.

(iv) To find angle between the different directions

(v) To understand concept of orthogonal trajectories

(vi) To obtain condition of orthogonality.

8.3 *DIRECTION COEFFICIENTS*

Suppose $\mathbf{r} = \mathbf{r}$ (u, v) is the parametric equation of a surface. Let P be any point on this surface and also let u = constant and v = constant are any two parametric curves passing through this point P. Then we have learnt that vectors $\mathbf{r}_2$ and $\mathbf{r}_1$ are the direction vectors of the tangents at P to these curves respectively. Since $\mathbf{r}_2 \times \mathbf{r}_1 \neq 0$, therefore the vectors $\mathbf{r}_2$ and $\mathbf{r}_1$ are independent and the unit normal vector **N** is perpendicular to these vectors. Thus, these three vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and the unit normal vector **N,** can treated as linearly independent vectors on the surface and coordinates of an arbitrary point P can be represented in terms of these three vectors on the surface. Hence, we can conclude that every vector passing through a point on the surface can be uniquely expressed as the linear combination of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and **N.** If **a** is any vector passing through P then there exist unique scalars a_n, λ and μ such that

$$\mathbf{a} = a_n \mathbf{N} + \lambda \mathbf{r}_1 + \mu \mathbf{r}_2 \qquad \ldots\ldots\ldots (1)$$

The scalar a_n is called the normal component of the vector **a.** If vector **a** lie on the surface, then $a_n = 0$ and vice-versa.

The part $\lambda\mathbf{r}_1 + \mu\mathbf{r}_2$ is called the tangential part and the scalars λ, μ are called the tangential components of the vector **a.**

For a tangential vector i.e. vector lying on the given surface, the normal component is zero and so we have for any tangential vector **T** the expression is

$$\mathbf{T} = \lambda\mathbf{r}_1 + \mu\mathbf{r}_2 \qquad \ldots\ldots\ldots\ldots (2)$$

Here λ and μ are called direction ratios or direction components of the vector **T**.

Now taking self-dot product of equation (2), we have

$$\mathbf{T}^2 = \lambda^2\, \mathbf{r}_1 2 + 2\mu\,\lambda\, \mathbf{r}_1 \cdot \mathbf{r}_2 + \mu^2\, \mathbf{r}_2 2$$

$$= \mathbf{E}\,\lambda^2 + 2F\,\mu\,\lambda + G\,\mu^2 \quad \textbf{Or}$$

$$|\mathbf{T}| = (\mathbf{E}\,\lambda^2 + 2F\mu\,\lambda + G\,\mu^2)^{1/2}.$$

If we wish to know the direction in the tangent plane then we can take unit vector **e** in place of arbitrary vector **T**. Hence if there is a unit vector **e** on the tangent plane of the given surface then we can write

$$\mathbf{e} = l\mathbf{r}_1 + m\mathbf{r}_2 \qquad \ldots\ldots\ldots.. (3)$$

then here in this case l and m are called direction coefficients of **e** we write

(l, m) as direction coefficients.

Now taking self-dot product of equation (3), we have

$\mathbf{e}^2 = l^2\, \mathbf{r}_1 2 + 2lm\, \mathbf{r}_1 \mathbf{r}_2 + m^2\, \mathbf{r}_2 2$

$1 = El^2 + 2Flm + Gm^2$ ………(4)

This is the condition for the scalars l and m to the direction coefficients on the surface.

Here it is understood that direction coefficients opposite to (l, m) are given by (-l, -m).

8.4 *ANGLE BETWEEN ANY TWO ARBITRARY DIRECTIONS OR CURVES*

Dear learners, we have the idea of angle between two lines or directions with given direction cosines (*l*, *m*, *n*) or direction ratios (*a*, *b*, *c*) in three -dimensional geometry or plane geometry in undergraduate courses. Let we are given two different curves, whose direction coefficients are given by (l_1, m_1) and (l_2, m_2) through a given point P of intersection of the curves on the surface. We wish to find the angle between these two directions. We proceed as-

If (l_1, m_1) and (l_2, m_2) are given directions then unit vectors along these directions can be taken as

$\mathbf{e}_1 = l_1\, \mathbf{r}_1 + m_1\, \mathbf{r}_2$ and $\mathbf{e}_2 = l_2\, \mathbf{r}_1 + m_2\, \mathbf{r}_2$

Taking dot product of these unit vectors, by definition

$\mathbf{e}_1 \cdot \mathbf{e}_2 = (l_1\, \mathbf{r}_1 + m_1\, \mathbf{r}_2) \cdot (l_2\, \mathbf{r}_1 + m_2\, \mathbf{r}_2)$

where ɵ is the angle between the given directions. Hence

$|\mathbf{e}_1|\, |\mathbf{e}_1| \operatorname{Cos} \theta = l_1 l_2\, \mathbf{r}_1 2 + (l_1 m_2 + l_2 m_1)\mathbf{r}_1 . r_2 + m_1 m_2\, \mathbf{r}_2 2$

or

$\operatorname{Cos} \theta = El_1 l_2 + F(l_1 m_2 + l_2 m_1) + Gm_1 m_2$ ………. (5)

This relation gives the angle between the directions (l_1, m_1) and (l_2, m_2).

Further, we also know that

Sin ϴ = |$\mathbf{e_1} \times \mathbf{e_2}$|

|($l_1\,\mathbf{r}_1 + m_1\,\mathbf{r}_2$) × ($l_2\,\mathbf{r}_1 + m_2\,\mathbf{r}_2$) |

Or

Sin ϴ $\left|l_1 \times m_2 - l_2 m_1 (r_1 \times r_2)\right|$ =

$$\left|H(l_1 m_2 - l_2 m_1)N\right|$$

Sin ϴ = H($l_1\, m_2$-$l_2 m_1$) ………….. (6)

As H is always positive definite and $|N| = 1$, being unit normal vector.

Hence, we have, from (5) and (6)

$$\text{Tan}\theta = \frac{H(l_1 m_2 - l_2 m_1)}{El_1 l_2 + F(l_1 m_2 + l_2 m_1) + Gm_1 m_2}$$

Or

Equation (7) can also be written as, in terms of direction ratios (λ_1, μ_1) and (λ_2, μ_2) as

tanθ =

$$\frac{H(\lambda_1\mu_2 - \lambda_2\mu_1)}{E\lambda_1\lambda_2 + F(\lambda_1\mu_2 + \lambda_2\mu_1) + G\mu_1\mu_2}$$

Corollary 1: If θ =90^0 then Cos ϴ= 0 then the directions with directions coefficients given by (l_1, m_1) and (l_2, m_2) are orthogonal and the condition for the same is given by

E$l_1\, l_2$+ F($l_1\, m_2$+$l_2\, m_1$)+ G$m_1\, m_2$ =0. ………… (8)

If, however we are given direction ratios (λ_1, μ_1) and (λ_2, μ_2) in place of direction coefficients, then above condition of orthogonality becomes

E$\lambda_1\, \lambda_2$+ F($\lambda_1\, \mu_2$+$\lambda_2\, \mu_1$)+ G$\mu_1\, \mu_2$ =0. ………….. (9)

Direction Ratios of a direction: Let us suppose that (l, m) are the direction coefficients of a direction at a point on a given surface **r = r**(u, v), then scalars λ and μ which are proportional to l and m respectively are called direction ratios of the that given direction (l, m). Very often we find it convenient to use direction ratios (λ, μ) in place of direction coefficients (l, m), while solving the problems and finding other relations in our discussions, as proved in previous article for orthogonal condition (8) of directions in terms of direction ratio (9).

8.5 RELATIION BETWEEN GIVEN DIRECTION COEFFICIENTS AND CORRESPONDING DIRCTION RATIOS

Let us suppose that we are given the direction with direction coefficients l and m, and the direction ratios μ and λ proportional to these coefficients respectively. Then therefore we can write

$l/\lambda = m/\mu = c$ (constant)

or

$l = c\lambda$ and $m = c\mu$

so that on applying the condition (4) for direction coefficients, we get

$El^2 + 2Flm + Gm^2 = 1$

i.e.

$Ec^2\lambda^2 + 2Fc^2\lambda\mu + Gc^2\mu^2 = 1$

Or

$$c^2 = \frac{1}{E\lambda^2 + 2F\lambda\mu + G\mu^2}$$

or

$$c = \frac{1}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

so that
$$l = \frac{\lambda}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

and

$$m = \frac{\mu}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

Henceforth,we can write the direction coefficients (l, m) as

$$(l, m) = \frac{(\lambda, \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}. \tag{10}$$

We can see that these relations have close similarity with the relations we have studied in three dimensional geometry in undergraduate classes, for direction cosines and direction ratios. (Learners have a look on those relations to learn here the relations).

Corollary 2: As we already know that the vector $\mathbf{r_1}$ at an arbitrary point P is tangential to the parametric curve v = constant, and therefore we can write the vector $\mathbf{r_1} = 1\, \mathbf{r_1} + 0\, \mathbf{r_2}$, so that the vector $\mathbf{r_1}$ has the components (1, 0). So the direction ratios of the direction of parametric curve v= constant are (1, 0) and therefore the direction coefficients are, by equation (10), as

$$\frac{(1, 0)}{\sqrt{E1^2 + 2F1.0 + G0^2}}$$

Or

$$\left(\frac{1}{\sqrt{E}},\ 0\right).$$

In the same way, we know that the vector $\mathbf{r_2}$ at any arbitrary point P is tangential to the parametric curve u= constant, and therefore we can write the vector $\mathbf{r_2} = 0\, \mathbf{r_1} + 1\, \mathbf{r_2}$, so that the vector $\mathbf{r_2}$ has the components (0, 1). So the direction ratios of the direction of parametric curve u= constant are (0, 1) and therefore the direction coefficients are, by equation (10), as

$$\frac{(0, 1)}{\sqrt{E0^2 + 2F0.1 + G1^2}}$$

$$\left(0,\ \frac{1}{\sqrt{G}}\right).$$

8.6 *FAMILY OF CURVES AND ITS DIFFERENTIAL EQUATION*

Let us suppose that **r** = **r** (u, v) be a surface. Also let ψ (u, v) is a single valued function of u and v. ψ has continuous derivatives ψ_1 and ψ_2 , with respect to u, v respectively. ψ_1 and ψ_2 do not vanish together. Then the equation

$$\psi (u, v) = k \qquad \text{………..} (11)$$

where c is a real parameter, gives a family of curves lying on the given surface **r** = **r** (u, v). For different values of parameter k, we have different members of the family (11). In case k is a fixed constant, equation (11) gives one particular member of the family (11).

Suppose for a point (u_0, v_0) on the surface **r** = **r** (u, v), we have ψ (u_0, v_0) =k_0 is a member of the family of curves (11) passing through the point (u_0, v_0). Thus we have the following proposition.

"i *passes one and only member of the family* (11) *of curves, through every point of on the surface* **r** = **r** (u, v) ".

Dear learners, we now try to find differential equation and direction ratio of family of curves.

Let us suppose ψ (u, v) =k i.e. equation (11) be a family of curves on a given surface **r** = **r** (u, v).

Differentiating equation (11), we have

$$\frac{\partial \psi}{\partial u} du + \frac{\partial \psi}{\partial v} dv = 0$$

Or

$$\Psi_1 \, du + \psi_2 \, dv = 0 \qquad (12)$$

Or

$$\frac{du}{dv} = -\frac{\psi_2}{\psi_1} \quad \Rightarrow \quad \frac{du}{-\psi_2} = \frac{dv}{\psi_1}$$

i.e. ($-\psi_2$, Ψ_1) are direction ratios of the tangent at the point (u, v) to the member of family of curves (11) which passes through (u, v).

If we suppose that integral of equation (12) is (11), then the curves of family ψ (u, v) =constant are the solutions of the differential equation

Ψ_1 du + ψ_2 dv = 0 . Conversely we can say that every first order differential equation of the form

P(u, v) du + Q(u, v) dv= 0 (13)

Where P and Q are functions of class 1 and do not vanish together, always defines a family of curves.

It follows from (13) that at any point (u, v) the tangent to the curve through this point has direction ratios (-Q, P), since these are proportional to (du, dv).

8.7 *ORTHOGONAL TRAJECTORY AND ITS DIFFERENTIAL EQUATION*

Suppose we have a family of curves ψ (u, v) =k ………….. (14)

lying on the given surface **r** = **r** (u, v) .

Again if we have another family of curves as ø (u, v) =k_1 ……… (15) lying on the same surface.

If families (14) and (15) are such that at every point of the surface the two curves, one from each family cut each other orthogonally, then the family of curves (15) is called the orthogonal trajectory of the family of curves (14) and vice-versa.

Dear learners, we now move to find differential equation of orthogonal trajectories of any given family of curves.

Let us suppose that the given surface is **r** = **r** (u, v) and ψ (u, v) =k is a family of curves lying on this surface. Again let ψ has continuous first order derivatives ψ_1 and ψ_2 that do not vanish together. Let us suppose that ψ_1 = P and ψ_2 = Q.

Now from ψ (u, v) =k , we have

$$\frac{\partial \psi}{\partial u}\delta u + \frac{\partial \psi}{\partial v}\delta v = 0$$

Or

$$\Psi_1 \,\delta u + \psi_2 \,\delta v = 0$$

Or

$$\frac{\delta u}{\delta v} = -\frac{\psi_2}{\psi_1} \Rightarrow \frac{\delta u}{\delta v} = \frac{-Q}{P}$$

Thus (- Q, P) are direction ratios of the tangent at (u, v) of a member of the family ψ (u, v) =k .

Let (du, dv) be the direction ratios of tangent at the point (u, v) of a member of family the orthogonal trajectories of ψ (u, v) =k . Thus the direction (du, dv) and (- Q, P) are mutually orthogonal to each other. So from the condition (9) of orthogonally or perpendicularly, viz.

$E\lambda_1 \lambda_2 + F(\lambda_1 \mu_2 + \lambda_2 \mu_1) + G\mu_1 \mu_2 = 0$, we get

$E(-Q)du + F(-Q\, dv + P\, du) + G\, P\, dv = 0$

Or

$$\mathbf{(F\,P - E\,Q)\,du + (G\,P - F\,Q)\,dv = 0} \qquad (16)$$

i.e.

$$\mathbf{(F\,\psi_1 + E\,\psi_2)\,du + (G\,\psi_1 + F\,\psi_2)\,dv = 0} \qquad (17)$$

Equation (16) or (17) is the required differential equation of the orthogonal trajectories of the family of the curves ψ (u, v) =k.

Corollary 3: For a given surface, parameters can always be chosen so that the curves of a family and their orthogonal trajectories become parametric curves.

As we can see that coefficients of du and dv in equation (17) are continuous functions of u and v, coefficients do not vanish together because $EG-F^2 \neq 0$ and P and Q also do not vanish together. Hence we can conclude that equation (17) is completely integrable. Let solution or integration of equation (17) is

$$\varnothing\,(u, v) = k_1 \qquad (18)$$

On differentiating equation (18), we get

$$\frac{\partial\phi}{\partial u}\,du + \frac{\partial\phi}{\partial v}\,dv = 0$$

Or

$$Ø_1 \text{ du} + ø_2 \text{ dv} = 0 \qquad (19)$$

Now equation (17) and (19) must be equivalent, as (18) is obtained by integrating (17) whereas (19) is obtained by differentiating (18).

Therefore comparing equation (17) and (19), we get

$$\frac{FP-EQ}{\phi_1} = \frac{GP-FQ}{\phi_2} = \lambda \neq 0$$

Or

FP-EQ = λ $ø_1$ and GP-FQ = λ $ø_2$, λ≠0.

Again , the jacobian of ψ and ø with respect to u and v is

$$\frac{\partial(\varphi,\phi)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} P & Q \\ \frac{1}{\lambda}(FP-EQ) & \frac{1}{\lambda}(GP-FQ) \end{vmatrix}$$

$$= \frac{1}{\lambda}\left(EQ^2 - 2FPQ + GP^2\right)$$

$$\neq 0$$

Since the quadratic expression inside the bracket is positive definite and P and Q do not vanish together.

Hence we conclude that ψ is indefinite integral of ø , so the transformation

U= ψ (u, v) and V= ø (u, v) is a proper parametric transformation. In the new system of parameters U and V, the given family of curves ψ (u, v) = constant and their orthogonal trajectories given by ø (u, v) = constant become parametric curves U = constant and V= constant. Hence the result.

8.8 *REPRESENTTION OF DOUBLE FAMILY OF CURVES AND ITS DIFFERENTIAL EQUATION*

The quadratic differential equation of the standard for

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0. \qquad \ldots\ldots\ldots (20)$$

Always represents two families of curves on the surface provided $Q^2 - PR > 0$, and here P, Q, R are continuous functions of parameters u and v, as well as functions P, Q, R do not vanish simultaneously.

Equation (20) can also be expressed as in more compact way

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \ . \qquad \ldots\ldots\ldots\ldots (21)$$

Equation (21) being quadratic in (du/dv), always gives, on solving the equation, two separate differential equations of first order are obtained and thus we get two families of curves can be obtained.

8.9 *CONDITION FOR ORTHOGONALITY OF DOUBLE FAMILY OF CURVES*

Dear learners, we now wish to obtain the condition that the two families of curves given by a quadratic differential of the form

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0. \qquad (22)$$

represents orthogonal families of curves or two orthogonal directions on the given surface.

The given differential equation (22) can be rewritten as

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \qquad (23)$$

Let the direction ratios of the curves of the two families obtained by (22)

through a point (u, v) on the given surface be (λ_1 , μ_1) and (λ_2 , μ_2).

Then obviously the corresponding roots of differential equation (23) will be as

(λ_1/μ_1) and (λ_2/μ_2) respectively.

Hence by standard relations on sum and products of roots of quadratic equation and its coefficients, we have

$$(\lambda_1/\mu_1) + (\lambda_2/\mu_2) = -2\,Q/P \qquad \ldots\ldots.. (24)$$

and

$$(\lambda_1/\mu_1)\,.\,(\lambda_2/\mu_2) = R/P \qquad \ldots\ldots (25)$$

Now applying the condition of orthogonality

$$E\lambda_1\lambda_2 + F(\lambda_1\mu_2+\lambda_2\mu_1) + G\mu_1\mu_2 = 0. \qquad \ldots\ldots.. (26)$$

Or

$$E(\lambda_1/\mu_1).(\lambda_2/\mu_2) + F\{(\lambda_1/\mu_1)+(\lambda_2/\mu_2)\} + G = 0.$$

Of two directions (λ_1, μ_1) and (λ_2, μ_2).

Hence two directions (λ_1/μ_1) and (λ_2/μ_2) obtained from equation (23) will be orthogonal if $E(\lambda_1/\mu_1).(\lambda_2/\mu_2) + F\{(\lambda_1/\mu_1)+(\lambda_2/\mu_2)\} + G = 0$.

Or, applying the relations (24) and (25) we have

$E\,(R/P) - 2(Q/P)\,F + G = 0$ Or

$$ER - 2FQ + GP = 0 \qquad \ldots\ldots\ldots.. (27)$$

Thus (27) is the necessary and sufficient condition for the families of curves obtained by solving equation (22) to be orthogonal.

Corollary 4: The necessary and sufficient condition for parametric curves to be orthogonal is that F is zero.

The proof of this result is obvious as we know that combined quadratic equation of parametric curves is

$$du.\,dv = 0 \qquad \ldots\ldots..(28)$$

Therefore, general quadratic equation of families of curves given by

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0. \qquad \ldots\ldots(29)$$

will express parametric curves if and only if $P = 0, Q \neq 0, R = 0$.

Substituting $P = 0, Q \neq 0, R = 0$, in equation (27), of orthogonality, we have

$F = 0$.

Hence $F = 0$ is the necessary and sufficient condition for parametric curves to be orthogonal.

Example 1: The parametric equation of a surface in terms of parameters u and v is given by (x,y,z) = (u Cosv, u Sinv, a log $\sqrt{(u^2-a^2)}$).
Prove that the parametric curves on the sphere always form an orthogonal system. Determine the families of curves (i)v = constant, (ii) u = constant, at angle $\pi/4$ and $3\pi/4$.
Solution. Equation of sphere is given as
r = (x,y,z) = (u Cosv, u Sinv, a log $\sqrt{(u^2-a^2)}$).
Differenciting w.r.t. paramaeters u and v, and using standard notations, we have
$\mathbf{r_1}$ = (Cosv, Sinv, au/(u^2-a^2)).
$\mathbf{r_2}$ = (-u Sinv, u Cosv, 0).
Therefore, taking dot (scalar) product, we have
E = $\mathbf{r_1^2} = \mathbf{r_1} . \mathbf{r_1} = a^2u^2/(u^2-a^2)^2$
F = **r1.r2 =** 0
G = $\mathbf{r_2^2} = \mathbf{r_2} . \mathbf{r_2} = u^2$
Since F= 0, the parametric curves are orthogonal.
$H^2 = EG- F^2 = a^2u^2/(u^2-a^2)^2. u^2$,
Which implies that $H= au^2/(u^2-a^2)$.
As in previous case, we solve the remaining part of the problem.
Example 2: If the parametric curves are orthogonal, show that the differential equation of the curves cutting the curves u = constant, at a constant angle δ is
du/dv = tan δ $\sqrt{(G/E)}$
Solution: Since parametric curves are given orthogonal, therefore F = 0, and
H= $\sqrt{(EG-F^2)}$, gives H= $\sqrt{(EG)}$. Now for the curves u = constant, the direction ratios are (0, 1). If (du, dv) be the direction ratios of the curve which cuts u = constant at an angle δ, then
Using the relation, for the angle between two directions

$$\text{Tan } \theta \quad \frac{H(\lambda_1\mu_2-\lambda_2\mu_1)}{E\lambda_1\lambda_2+F(\lambda_1\mu_2+\lambda_2\mu_1)+G\mu_1\mu_2} =$$

We have,

$$tan\delta = \frac{H(du-0)}{E.0+0+G\,dv}$$

Or

$$tan\delta = \frac{H(du-0)}{E.0+0+G\,dv}$$

$$= \frac{\sqrt{EG}\,du}{G\,dv}$$

$$= \sqrt{\frac{E}{G}}\;\frac{du}{dv}$$

or

(du/dv) =tan δ $\sqrt{(G/E)}$, as desired.

Example 3: Prove that, if ɵ is the angle at the point (u, v) between the two directions given by P du^2 + 2Q du dv + R dv^2 = 0. Then $\tan\theta = \dfrac{2H\left(Q^2 - PR\right)^{1/2}}{ER - 2FQ + GP}$

Solution: The given quadratic equation is P du^2 + 2Q du dv + R dv^2 =0. (1) Or

P (du/dv)^2 + 2Q (du/ dv) + R =0. ………. (2)

Let the direction ratios of the curves of the two families obtained by the above equation (1) through a point (u, v) on the given surface be (λ_1, μ_1) and (λ_2, μ_2).

Then obviously the corresponding roots of differential equation (2) will be as (λ_1/μ_1) and (λ_2/μ_2) respectively.

Hence by standard relations on sum and products of roots of quadratic equation and its coefficients, we have

$$(\lambda_1/\mu_1) + (\lambda_2/\mu_2) = -2\,Q/P \qquad (3)$$

and

$$(\lambda_1/\mu_1)\,.\,(\lambda_2/\mu_2) = R/P \qquad (4)$$

As we know that angle ɵ between two directions is given by

$$\tan ɵ \quad \frac{H(\lambda_1\mu_2 - \lambda_2\mu_1)}{E\lambda_1\lambda_2 + F(\lambda_1\mu_2 + \lambda_2\mu_1) + G\mu_1\mu_2} =$$

$$= \frac{H\left(\dfrac{\lambda_1}{\mu_1} - \dfrac{\lambda_2}{\mu_2}\right)}{E\dfrac{\lambda_1\lambda_2}{\mu_1\mu_2} + F\left(\dfrac{\lambda_1}{\mu_1} + \dfrac{\lambda_2}{\mu_2}\right) + G}$$

$$= \frac{H\left\{\left(\dfrac{\lambda_1}{\mu_1} + \dfrac{\lambda_2}{\mu_2}\right)^2 - 4\left(\dfrac{\lambda_1}{\mu_1}.\dfrac{\lambda_2}{\mu_2}\right)\right\}^{1/2}}{E\dfrac{\lambda_1\lambda_2}{\mu_1\mu_2} + F\left(\dfrac{\lambda_1}{\mu_1} + \dfrac{\lambda_2}{\mu_2}\right) + G}$$

Using relations (3) and (4) we have

$$= \frac{H\left\{\left(\dfrac{-2Q}{P}\right)^2 - 4\left(\dfrac{R}{P}\right)\right\}^{1/2}}{E\dfrac{R}{P} + F\left(\dfrac{-2Q}{P}\right) + G}$$

$$= \frac{2H\left\{Q^2 - PR\right\}^{1/2}}{ER - 2FQ + GP}$$

Thus the desired result.

8.10 *SUMMARY*

1. The necessary and sufficient condition for parametric curves to be orthogonal is that F is zero.

2. Angle between two direction ratios (λ_1,μ_1) and $(\lambda_{2,}\mu_2)$ is given by

$$\text{Tan } \theta) \frac{H(\lambda_1\mu_2 - \lambda_2\mu_1)}{E\lambda_1\lambda_2 + F(\lambda_1\mu_2 + \lambda_2\mu_1) + G\mu_1\mu_2} =$$

5. If there is a unit vector **e** on the tangent plane of the given surface then we can writ**e** = $l\mathbf{r}_1 + m\mathbf{r}_2$.

5. The condition for the scalars l and m to the direction coefficients on the surface.

$El^2 + 2Flm + Gm^2 = 1$.

6. The directions coefficients given by (l_1, m_1) and $(l_{2,} m_2)$ are orthogonal and the condition for the same is given by $El_1 l_2 + F(l_1 m_2 + l_2 m_1) + Gm_1 m_2 = 0$.

7. The direction coefficients (l, m) can be written as

$$(l, m) = \frac{(\lambda, \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}} .$$

Where (λ, μ) are direction ratios of the same direction.

8.11 *GLOSSARY*

1. Orthogonal- mutually perpendicular.
2. Quadratic – of degree two.
3. Transformation – mapping or function.
4. Vanish– to become zero.
5. Trajectory- path.

8.12 *REFERENCES*

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8.13 *TERMINAL QUESTIONS*

(TQ -1) Find the direction which makes an angle $\pi/2$ with the whose direction coefficient are (l, m).

(TQ-2) Find the the equation of the curves bisecting the angles between the parametric curves.

(TQ-3) Show that on a right helicoids, the family of curves orthogonal to the curves u.cosv = constant is the family $(u^2 + a^2) \text{Sin}^2 v$ = constant.

(TQ-4) Show that parametric curves are orthogonal on the surface x= u.cosv, y=u.sinv, $Z = a.\log \{u+\sqrt{(u^2 - a^2)}\}$.

(TQ-5) Find the differential equation of the orthogonal trajectories of the family of the curves given by P du + Q dv = 0.

(TQ-6) Find the orthogonal trajectories of the curves obtained by the section of the planes z = constant on the surface, paraboloids $x^2 - y^2 = z$.

8.14 *ANSWERS TO SELECTD TERMINAL QUESTIONS*

(TQ -1) (l’, m’) = (-(Fl+Fm)/H , (El+Fm)/H)

(TQ-2) $E\, du^2 - G\, dv^2 = 0$.

(TQ-5) (EQ-FP) du + (FQ-GP) dv = 0.

(TQ-6) xy = constant.

COURSE NAME: GEOMETRY

COURSE CODE: MAT 611

BLOCK-III

LOCAL NON-INTRINSIC PROPERTIES OF A SURFACE

UNIT 9: *NORMAL AND PRINCIPAL CURVATURES*

CONTENTS:

9.1 *INTRODUCTION*

In differential geometry, the two **principal curvatures** at a given point of a surface are the maximum and minimum values of the curvature as expressed by the eigenvalues of the shape operator at that point. They measure how the surface bends by different amounts in different directions at that point. The concepts of normal and principal curvatures, fundamental in differential geometry, emerged from the study of curves and surfaces, with key contributions from figures like Euler, Monge, and Gauss, culminating in a systematic analysis by Darboux.

9.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Principal curvatures

(ii) Normal Curvature

9.3 *NORMAL CURVATURE*

Before defining normal curvature firstly, we shall define normal section. A plane P′ drawn through a point P on the surface, cuts the surface in a curve

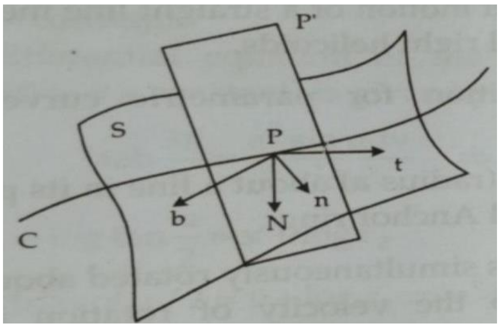

Fig. 9.3.1

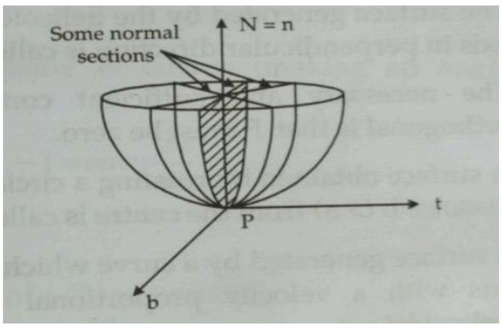

Fig. 9.3.2

which is called a section of the surface. In case the plane **P′** is so drawn that it contains the normal to the surface, then the curve is called Normal section, otherwise the curve is called an Oblique section. We observe that in fig. 2, the principal normal **n** to the normal section is parallel to the surface normal **N**. We shall adopt the convention that vector **n** has the same direction as that of vector **N**, with this convention, we have $\mathbf{n} = \mathbf{N}$.

Formula for normal curvature in terms of fundamental magnitudes.

Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be the equation of surface and P is any point (u, v) on the surface. Let κ_n represents the curvature of the normal section, it will be positive when the curve is concave on the side towards which N points out. Then, we

$$\frac{dt}{ds} = \mathbf{r}'' = \kappa_n \boldsymbol{n} = \kappa_n \boldsymbol{N}$$

$$\therefore \quad \kappa_n = \mathbf{N} \cdot \mathbf{r}''$$

$$[\because \mathbf{r}'' = \kappa_{\mathrm{n}} \mathbf{N}] \qquad \ldots \ldots \ldots \ldots \quad (1)$$

Again, we know that

$$\mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds}$$

$$= \boldsymbol{r}_1 u' + \mathbf{r}_2 v'$$

$$\therefore \mathbf{r}'' = \mathbf{r}_1 u'' + \frac{dr_1}{ds} u' + \mathbf{r}_2 v'' + \frac{dr_2}{ds} v' \qquad \ldots \ldots \ldots. \quad (2)$$

$$= \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \left(\frac{\partial \mathbf{r}_1}{\partial u}\frac{\partial u}{ds} + \frac{\partial \mathbf{r}_1}{\partial v}\frac{dv}{ds}\right) u' + \left(\frac{\partial \mathbf{r}_2}{\partial u}\frac{\partial u}{ds} + \frac{\partial \mathbf{r}_2}{\partial v}\frac{dv}{ds}\right) v'$$

$$= \mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11} u'^2 + \mathbf{r}_{12} u'v' + \mathbf{r}_{21} u'v' + \mathbf{r}_{22} v'^2$$

$$\therefore \kappa_n = \mathbf{r}'' \cdot \mathbf{N} = (\mathbf{r}_1 u'' + \mathbf{r}_2 v'' + \mathbf{r}_{11} u'^2 + 2\mathbf{r}_{12} u'v' + \mathbf{r}_{22} v'^2) \cdot \boldsymbol{N}.$$

Again, $\mathbf{r}_1 \cdot \mathbf{N} = 0, \mathbf{r}_2 \cdot \mathbf{N} = 0, \mathbf{r}_{11} \cdot \mathbf{N} = \mathrm{L}, \mathbf{r}_{12} \cdot \mathbf{N} = \mathrm{M}, \mathbf{r}_{22} \cdot \mathbf{N} = \mathrm{N}$

Therefore

$$\kappa_n = \boldsymbol{N} \cdot \boldsymbol{r}'' = Lu'^2 + 2Mu'v' + Nv'^2$$

$$= L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}$$

$$= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + \mathrm{G}dv^2} \text{ [Using first fundamental form] } \quad \ldots \ldots \ldots (3)$$

Equation (3) gives the curvature of the normal section usually called normal curvature parallel to the direction $(\mathrm{du}, \mathrm{dv})$. Its reciprocal is called the radius of normal curvature and is denoted by ρ_{n}.

We define the normal curvature as follows:

- **Definition.** If P be a point with a position vector $\mathbf{r}(\mathrm{u}, \mathrm{v})$ on the surface $\mathrm{r} = \mathrm{r}(\mathrm{u}, \mathrm{v})$, the normal curvature at P in the direction $(\mathrm{du}, \mathrm{dv})$ is equal to the curvature at P of the normal section at P parallel to the direction $(\mathrm{du}, \mathrm{dv})$.
- **Alternative definition.** Suppose $\mathbf{r} = \mathbf{r}(\mathbf{u}, \mathbf{v})$ is a surface and P is any point with a position vector $\mathbf{r}(\mathrm{u}, \mathrm{v})$ on it. If $\mathbf{r} = \mathbf{r}(\mathrm{s})$ is a curve through P on this surface, then the component of the curvature vector **r** "along the normal to the surface is defined to be the normal curvature of the curve at P and is generally denoted by κ_{n}. Therefore, $\kappa_{\mathrm{n}} = \mathbf{N} \cdot \mathbf{r}''$
- **Equivalence of two definitions.** Let **N** be the unit normal vector to the surface at P, then by alternative definition the normal curvature κ_n is given by $\kappa_n = \mathbf{N} \cdot \mathbf{r}$ ", where **r** " is curvature vector at P

Let κ be the curvature at P of the normal section at P containing the direction (du,dv). Then

$$r'' = \kappa_n = \kappa \mathbf{N} \qquad [\because \mathbf{n} = \mathbf{N}]$$
$$\mathbf{N} \cdot \mathbf{r}'' = \mathbf{N} \cdot (\kappa N) = \kappa \qquad [\because \mathbf{N} \cdot \mathbf{N} = 1]$$

i.e.,

$$\kappa_n = \kappa$$

Thus, we see that curvature at P of normal section at P containing the direction (du, dv) is equal to the normal curvature at P in the same direction.

Remarks

- We know that **r**" is the curvature vector at P of a curve lying on the surface $\mathbf{r}(u, v)$ then we have shown that
$$N \cdot r'' = \frac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}$$
$$= L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$
- We know that all curves having the same direction at P have the same values of their direction coefficients $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ at P. Also, the values of second-order fundamental magnitudes; L, M, N are fixed at P. Therefore, for all curves having the same direction at P, the value of $\mathbf{N} \cdot \mathbf{r}''$ is fixed which is equal to the normal curvature at P of any one of these curves. Therefore, normal curvature is a property of the surface and a direction at a point on the surface.

9.4 *MEUSNIER'S THEOREM*

STATEMENT. If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ be the angle between these sections, then

$$\kappa_{\text{n}} = \kappa \cos\theta$$

Proof. Let P be a point (u, v) on the surface $\mathbf{r} = \mathbf{r}(u, v)$ and r'' be the curvature vector at P of the oblique section through P, containing the direction (du, dv). Then

$$\mathbf{r}'' = \mathbf{kn} \qquad \ldots\ldots\ldots\ldots \qquad (1)$$

where **n** is the unit principal normal vector to the oblique section at P.

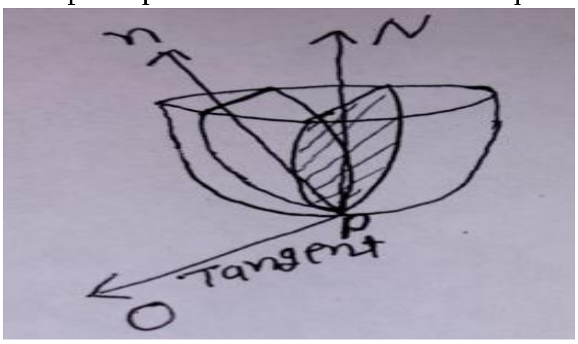

Fig. 9.4.1

Again, the unit normal vector **N** to the surface at P is the unit principal normal vector of the normal section at P parallel to the direction (du, dv). Since θ is the angle between

oblique and normal sections at P through the same tangent line so θ is the angle between oblique and normal sections at P through the same tangent line, so θ is the angle between the vectors **n** and **N**
i.e.

$$\mathbf{n} \cdot \mathbf{N} = \cos\theta \qquad [\because |\mathbf{n}| = 1, |N| = 1]$$

Now taking dot product of both sides of (1) by **N**, we have

$$\mathbf{r}'' \cdot \mathbf{N} = \kappa\mathbf{n} \cdot \mathbf{N} = \kappa\cos\theta$$

Again $\mathbf{r}'' \cdot \mathbf{N} =$ normal curvature at P in the direction (du, dv) = curvature of the normal section at P parallel to be direction $(du, dv) = \kappa_n$
Therefore, $\kappa_{\mathrm{n}} = \kappa\cos\theta$.

9.5 *PRINCIPAL DIRECTIONS*

The normal sections of a surface through a given point having maximum or minimum curvatures at the point are called principal sections of the surface at that point and the tangents to these sections are called principal directions at the point. In general there are two principal directions at every point on a surface and it will be shown that they are mutually orthogonal.

Principal curvature. The curvatures of the principal sections of a surface through a given point, i.e., the maximum and minimum curvatures at that point are called principal curvatures at that point, and their corresponding radius of curvatures are called principal radius of curvatures.

➢ EQUATION GIVING PRINCIPAL CURVATURES

We know that the normal curvature κ_n at point $\mathrm{P}(\mathrm{u}, \mathrm{v})$ in the direction (du, dv) is given by

$$\boldsymbol{\kappa}_{\mathbf{n}} = \frac{\mathrm{Ldu}^{\,2} + 2\mathrm{Mdudv} + \mathrm{Ndv}^{\,2}}{Edu^2 + 2Fdudv + Gdv^2}$$

If (l, m) be actual direction coefficients of the direction (du, dv), then

$$\boldsymbol{\kappa}_{\mathbf{n}} = \frac{\mathrm{L}l^2 + 2\mathrm{M}l\,\mathrm{m} + \mathrm{Nm}^2}{\mathrm{E}l^2 + 2\mathrm{Fm}l + \mathrm{Gm}^2}$$

Where,

$$\mathrm{E}l^2 + 2\,\mathrm{F}l\,\mathrm{m} + \mathrm{Gm}^2 = 1 \qquad \dots\dots\dots.. \quad (1)$$

$$\therefore\ \kappa_{\mathrm{n}} = \mathrm{Ll}^2 + 2\mathrm{Mlm} + \mathrm{Nm}^2 \qquad \dots\dots\dots. \qquad (2)$$

Since L, M, N are fixed at P, so the value of κ_n at P depends upon the values 1, m, at P , i.e., K_{n} is a function of two variables l, m and are connected by relation (2). We shall find the maximum value of κ_n by Lagrange's method of undetermined multipliers. For a maximum or minimum value of k_{n}, we have

$$\mathrm{d}\kappa_{\mathrm{n}} = 0$$

i.e. , $2\,\mathrm{L}l\,\mathrm{d}l + 2\mathrm{M}l\mathrm{dm} + 2\mathrm{Mmd}l + 2\mathrm{Nmdm} = 0$
or

$$(\mathrm{L}l + \mathrm{Mm})\mathrm{d}l + (\mathrm{M}l + \mathrm{Nm})\mathrm{dm} = 0 \qquad \dots\dots\dots\dots. \qquad (3)$$

Differentiating (1), we get

$$2\mathrm{E}l\mathrm{d}l + 2\mathrm{Fldm} + 2\mathrm{Fmd}l + 2\mathrm{Gmdm} = 0$$

or

$$(\mathrm{El} + \mathrm{Fm})\mathrm{d}l + (\mathrm{Fl} + \mathrm{Gm})\mathrm{dm} = 0 \quad \dots\dots\dots\dots \quad (4)$$

Now multiplying (4) by λ and adding to (3) and then equating to zero the coefficients of $\mathrm{d}l$ and dm, we get

$$(\mathrm{L}l + \mathrm{Mm}) + \lambda(\mathrm{El} + \mathrm{Fm}) = 0 \quad \dots\dots\dots \quad (5)$$

$$(\mathrm{M}l + \mathrm{Nm}) + \lambda(\mathrm{Fl} + \mathrm{Gm}) = 0 \quad \dots\dots\dots\dots \quad (6)$$

and

Now multiplying (5) by l and (6) by m and adding, we get

or

$$(\mathrm{Ll}^2 + 2\mathrm{Mlm} + \mathrm{Nm}^2) + \lambda(\mathrm{E}l^2 + 2\mathrm{Flm} + \mathrm{Gm}^2) = 0$$

$$\kappa_\mathrm{n} + \lambda = 0 \quad \text{or} \quad \lambda = -\kappa_\mathrm{n}.$$

Putting $\lambda = -\kappa_\mathrm{n}$ in (5) and (6), we get

$$(\mathrm{L}l + \mathrm{Mm}) - \kappa_\mathrm{n}(\mathrm{E}l + \mathrm{Fm}) = 0 \quad \dots\dots\dots \quad (7)$$

$$(\mathrm{M}l + \mathrm{Nm}) - \kappa_\mathrm{n}(\mathrm{Fl} + \mathrm{Gm}) = 0 \quad \dots\dots\dots \quad (8)$$

and

Now we shall eliminate l, m between (7) and (8). From (7), we have

$$(L - \kappa_n E)l + (M - \kappa_n F)m = 0 \quad \dots\dots\dots\dots\dots \quad (9)$$

From (8),

$$(M - \kappa_n F)l + (N - \kappa_n G)m = 0 \quad \dots\dots\dots\dots\dots \quad (10)$$

From (9) and (10), we have

$$(L - \kappa_n E)(N - \kappa_n G) = (M - \kappa_n F)(M - \kappa_n F)$$

Or

$$\kappa_n^2(EG - F^2) - \kappa_n(En + LG - 2FM + (LN - M^2)$$

This is the required quadratic equation giving the maximum or minimum values of normal curvature at P . Its roots are principal curvatures of the surface at P and are usually denoted by κ_a and κ_b. Thus we have

$$\boldsymbol{k_a} + \boldsymbol{k_b} = \frac{EN + LG - 2FM}{EG - F^2} \text{ and } \boldsymbol{k_a k_b} = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2}$$

➢General Definition.

1. **First curvature.** The sum of the principal curvature κ_a and κ_b is called the first curvature at the point and it is denoted by symbol J,

i.e.,

$$\mathrm{J} = \mathrm{K_a} + \kappa_\mathrm{b} = \frac{\mathrm{EN} + \mathrm{LG} - 2\mathrm{FM}}{\mathrm{EG} - \mathrm{F}^2}$$

2. **Mean curvature or Mean normal curvature.**

The arithmetic mean of the principal curvature k_a and k_b at a point is called the mean curvature at the point and is denoted by symbol μ.

$$\text{i.e.,} \qquad \mu = \frac{1}{2}(\kappa_a + \kappa_b) = \frac{EN + LG - 2FM}{2(EG - F^2)}$$

Some authors denote the mean normal curvature by B

$$\Rightarrow \; B = \frac{1}{2}(\kappa_a + \kappa_b).$$

Also, **amplitude** of normal curvature is defined by

$$A = \frac{1}{2}(\kappa_b + \kappa_a)$$

3. **Gaussian curvature.** The product of the principal curvatures k_a and k_b at point is called Gaussian curvature at the point and is denoted by symbol K.
i.e.,

$$K = \kappa_a \cdot \kappa_b = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2}$$

It is also called, **specific curvature**, **second curvature** or **total curvature**.

➢TO FIND EQUATION GIVING THE PRINCIPAL DIRECTION AT A POINT ON THE GIVEN SURFACE

We know that directions having the maximum and minimum normal curvatures are given by equations (5) and (6), which are

$$(Ll + Mm) + \lambda(El + Fm) = 0 \qquad (1)$$

$$(Ml + Nm) + \lambda(Fl + Gm) = 0 \qquad (2)$$

Eliminating λ form (1) and (2) we get

$$(Ll + Mm)(Fl + Gm) = (Ml + Nm)(El + Fm)$$

Or

$$(EM - Fl)l^2 + (EN - GL)l\,m + (FN - GM)m^2 = 0 \qquad (3)$$

Equation (3) gives principal directions at the given point. Now replacing the actual direction coefficients $(1, m)$ by direction ratios (du, dv), the equation (3) reduces to

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0 \qquad (4)$$

Now (4) is a quadratic equation in $\frac{du}{dv}$, therefore there are in general two principal directions at each point of the surface.

Again, we know that the two directions given by

$$Pdu^2 + 2Qdudv + Rdv^2 = 0$$

Are orthogonal if

$$ER - 2FQ + GP = 0 \qquad (5)$$

On comparing (4) and (5), we have

$$P = EM - FL, Q = \frac{EN - GL}{2}, R = FN - GM$$

Hence in this case

$$ER - 2FQ + GP = E(FN - GM) - F(EN - GL) + G(EM - FL) = 0$$

Therefore, the two directions given by (4) are orthogonal hence the principal direction are orthogonal.

The discriminant of equation (4) is

$$(\mathrm{EN}-\mathrm{GL})^2-4(\mathrm{EM}-\mathrm{FL})(\mathrm{FN}-\mathrm{GM})$$
$$\equiv \frac{4(\mathrm{EG}-\mathrm{F}^2)}{\mathrm{E}^2}\left((\mathrm{EM}-\mathrm{FL})^2+\left\{\mathrm{EN}-\mathrm{GL}-\frac{2\,\mathrm{F}}{\mathrm{E}}(\mathrm{EM}-\mathrm{FL})\right\}\right.$$

But $\mathrm{EG}-\mathrm{F}^2>0$, if follows that the roots of equation (4) are real and distinct, provided that the coefficients E,F,G and L,M,N are not proportional. Thus, if at a point P the coefficients E, F, G are not proportional to L, M, N then we have two real and distinct principal direction at P which are orthogonal. When $\frac{L}{E}=\frac{M}{F}=\frac{N}{G}$, the equation (4) fails to determine principal directions i.e., the principal directions are indeterminate.

Umbilic Definition. A point on a surface is called an umbilic if at that point we have

$$\frac{L}{E}=\frac{M}{F}=\frac{N}{G}$$

The normal curvature k at a point (u, v) in the direction $(\mathrm{du}, \mathrm{dv})$ is given by

$$\kappa=\frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$$

Therefore, at an umbilic, the normal curvature is the same in all direction.

9.6 *MINIMAL SURFACE*

Definition. If mean curvature of a surface is zero at all points, then the surface is called a minimal surface.

Hence the surface will be minimal if, $\mu=0 \Rightarrow \mathrm{K_a}+\mathrm{K_b}=0 \;\; \mathrm{EN}+\mathrm{GL}-2\mathrm{FM}=0$ at every point of the surface.

Theorem. If there is a surface of minimum passing through a closed space curve, it is necessarily a minimal surface.

Proof. Let $\mathbf{r}=\mathbf{r}(\mathrm{u}, \mathrm{v})$ be the equation of a surface S bounded by a closed curve C. Given to S, a small displacement ε in the direction of normal to derive a surface S.

Here ε is a function of u and v. Let $\frac{\partial \varepsilon}{\partial u}=\varepsilon_1$ and $\frac{\partial \varepsilon}{\partial v}=\varepsilon_2$ be both small quantities. More exactly we take $\varepsilon_1=O(\varepsilon), \varepsilon_2=O(\varepsilon)$ as $\varepsilon \rightarrow 0$

Let **R** be the position vector of any point on the surface S^*, then

$$\mathbf{R}=\mathbf{r}+\varepsilon\mathbf{N}$$

where $\mathbf{r}, \varepsilon, \mathbf{N}$ are all functions of u and v.

$$\therefore \quad \mathbf{R}_1=\mathbf{r}_1+\varepsilon_1\mathbf{N}+\varepsilon\mathbf{N}_1,$$
$$\mathbf{R}_2=\mathbf{r}_2+\varepsilon_2\mathbf{N}+\varepsilon\mathbf{N}_2$$

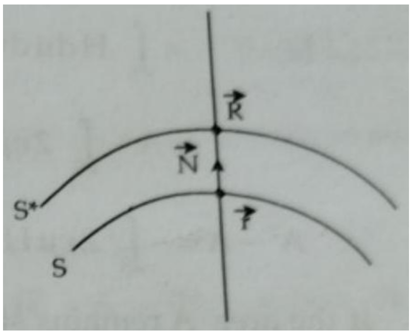

Fig. 9.6.1

Let E^*, F^*, G^* be first order fundamental coefficients of S^*.

$$\therefore E^* = \mathbf{R}_1^2 = (\mathbf{r}_1 + \varepsilon_1\mathbf{N} + \varepsilon\mathbf{N}_1)^2.$$
$$= \mathbf{r}_1^2 + 2\varepsilon\mathbf{r}_1 \cdot \mathbf{N} + 2\varepsilon\mathbf{r}_1 \cdot \mathbf{N}_1 + O(\varepsilon^2)$$
$$= \mathrm{E} - 2\varepsilon\,\mathrm{L} + \mathrm{O}(\varepsilon^2)$$
$$\mathrm{F}^* = \mathbf{R}_1 \cdot \mathbf{R}_2 = (\mathbf{r}_1 + \varepsilon_1\mathbf{N} + \varepsilon\mathbf{N}_1) \cdot (\mathbf{r}_2 + \varepsilon_2\mathbf{N} + \varepsilon\mathbf{N}_2)$$
$$= \mathbf{r}_1 \cdot \mathbf{r}_2 + \varepsilon_2\mathbf{r}_1 \cdot \mathbf{N} + \varepsilon\mathbf{r}_2 \cdot \mathbf{N}_2 + \varepsilon_1\mathbf{N} \cdot \mathbf{r}_2 + \varepsilon\mathbf{N}_1 \cdot \mathbf{r}_2 + \cap\,(\varepsilon^2)$$
$$= F - 2\varepsilon M + O(\varepsilon^2)$$
$$G^* = \mathbf{R}_2^2 = (\mathbf{r}_2 + \varepsilon\mathbf{N} + \varepsilon\mathbf{N}_2)^2 = G - 2\varepsilon\mathbf{N} + O(\varepsilon)^2$$
$$\therefore\; H^{*2} = E^*G^* - F^{*2}$$
$$= \{E - 2\varepsilon L + O(\varepsilon)^2\} \mid G - 2\varepsilon N + O(\varepsilon^2)\} - \{F - 2\varepsilon M + O(\varepsilon^2)\}^2$$
$$= \mathrm{EG} - \mathrm{F}^2 - 2\varepsilon(\mathrm{EN} - 2\mathrm{FM} + \mathrm{GL}) + \mathrm{O}(\varepsilon^2)$$
$$= \mathrm{H}^2 - 4\varepsilon\mathrm{H}^2\left(\frac{\mathrm{EN} - 2\mathrm{FM} + \mathrm{GL}}{2\mathrm{H}^2}\right) + \mathrm{O}(\varepsilon^2)$$

$= \mathrm{H}^2 - 4\varepsilon\mathrm{H}^2\mu + \mathrm{O}(\varepsilon^2)$, where μ is mean curvature of S

$= \mathrm{H}^2[1 - 4\varepsilon\mu + \mathrm{O}(\varepsilon^2)]$.

Example 1. Show that the equation for the principal curvature through a point of the surface $\mathrm{z} = \mathrm{f}(\mathrm{x}, \mathrm{y})$ is

$$\mathrm{H}^4\kappa_\mathrm{n}{}^2 - \mathrm{H}[(1 + \mathrm{p}^2)\mathrm{t} + (1 + \mathrm{q}^2)\mathrm{r} - 2\mathrm{spq}]\kappa_\mathrm{n} + (\mathrm{rt} - \mathrm{s}^2) = 0$$

Solution: The given surface is $\mathrm{z} = \mathrm{f}(\mathrm{x}, \mathrm{y})$, thus we have

$\mathbf{r} = (\mathrm{x}, \mathrm{y}, \mathrm{f}(\mathrm{x}, \mathrm{y}))$

$\therefore\; \mathbf{r}_2 = (1,0,\mathrm{p}); \mathbf{r}_2 = (0,1,\mathrm{q}); \mathbf{r}_{11} = (0,0,\mathrm{r}); \mathbf{r}_{12} = (0,0,\mathrm{s});$

$$\mathbf{r}_{22} = (0,0,\mathrm{t})$$

Therefore, $\mathrm{E} = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + \mathrm{p}^2; \mathrm{F} = \mathbf{r}_1 \cdot \mathbf{r}_2 = \mathrm{pq}; \mathrm{G} = \mathbf{r}_2 \cdot \mathbf{r}_2 = 1 + \mathrm{q}^2$.

Therefore, $\mathrm{H}^2 = \mathrm{EG} - \mathrm{F}^2 = (1 + \mathrm{p}^2)(1 + \mathrm{q}^2) - \mathrm{p}^2\mathrm{q}^2 = 1 + \mathrm{p}^2 + \mathrm{q}^2$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{(-\mathrm{p}, -\mathrm{q}, 1)}{\mathrm{H}}$$
$$\mathrm{L} = \mathbf{N} \cdot \mathbf{r}_{11} = \frac{\mathbf{r}}{\mathrm{H}}; \mathrm{M} = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{\mathrm{s}}{\mathrm{H}}; \mathrm{N} = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{\mathrm{t}}{\mathrm{H}};$$

and

$$\mathrm{T}^2 = \mathrm{LN} - \mathrm{M}^2 = \frac{\mathrm{rt} - \mathrm{s}^2}{\mathrm{H}^2}$$

Equation of principal curvature is given by

$$\mathrm{H}^2\kappa_\mathrm{n}{}^2 - (\mathrm{EN} + \mathrm{GL} - 2\mathrm{FM})\kappa_\mathrm{n} + \mathrm{T}^2 = 0$$

Putting the values of $\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{L}, \mathrm{M}, \mathrm{N}$ in this equation, we get

$$\mathrm{H}^4\kappa_\mathrm{n}^2 - \mathrm{H}[(1 + \mathrm{p}^2)\mathrm{t} + (1 + \mathrm{q}^2)\mathrm{r} - 2\mathrm{spq}]\kappa_\mathrm{n} + (\mathrm{rt} - \mathrm{s}^2) = 0$$

which is the equation of principal curvatures.

Example 2. Show that the principal radii of curvature of the surface $\mathrm{y}\cos\frac{\mathrm{z}}{\mathrm{a}} = \mathrm{x}\sin\frac{\mathrm{z}}{\mathrm{a}}$ are equal to $\pm\frac{\mathrm{x}^2+\mathrm{y}^2+\mathrm{a}^2}{\mathrm{a}}$. Find the lines of curvature.

Solution. Surface is $\mathrm{y}\cos\frac{\mathrm{z}}{\mathrm{a}} = \mathrm{x}\sin\frac{\mathrm{z}}{\mathrm{a}}$ i.e. $\mathrm{z} = \mathrm{a}\tan^{-1}\frac{\mathrm{y}}{\mathrm{x}}$

The parametric equations of the surface are

X=u cos v, y=u sin v, z=av

$$\therefore \mathrm{r} = (\mathrm{u}\cos\mathrm{v}, \mathrm{u}\sin\mathrm{v}, \mathrm{av})$$

$$\mathrm{r}_1 = (\cos\mathrm{v}, \sin\mathrm{v}, 0); \mathrm{r}_2 = (-\mathrm{u}\sin\mathrm{v}, \mathrm{u}\cos\mathrm{v}, \mathrm{a})$$

$$\mathbf{r}_{11} = (0,0,0); \mathrm{r}_{12} = (-\sin\mathrm{v}, \cos\mathrm{v}, 0)$$

$$\mathrm{r}_{22} = (-\mathrm{u}\cos\mathrm{v}, -\mathrm{u}\sin\mathrm{v}, 0);$$

$$\therefore\ \mathrm{E} = \mathrm{r}_1^2 = 1; \mathrm{F} = \mathrm{r}_1 \cdot \mathrm{r}_2 = 0, \mathrm{G} = \mathrm{r}_2^2 = \mathrm{u}^2 + \mathrm{a}^2;$$

$$\mathrm{H}^2 = \mathrm{EG} - \mathrm{F}^2 = \mathrm{u}^2 + \mathrm{a}^2$$

$$\mathbf{N} = \frac{\mathrm{r}_1 \times \mathbf{r}_2}{\mathrm{H}} = \frac{(\mathrm{a}\sin\mathrm{v}, -\mathrm{a}\cos\mathrm{v}, \mathrm{u})}{\mathrm{H}}.$$

$$\mathrm{L} = \mathbf{N} \cdot \mathbf{r}_{11} = 0, \mathbf{M} = \mathbf{N} \cdot \mathbf{r}_{12} = -\frac{\mathrm{a}}{\mathrm{H}}, \mathrm{N} = \mathrm{N} \cdot \mathbf{r}_{22} = 0.$$

The principal curvatures of the surface are given by

$$\mathrm{H}^2{\kappa_\mathrm{n}}^2 - (\mathrm{EN} - 2\mathrm{FM} + \mathrm{GL})\kappa_\mathrm{n} + \mathrm{T}^2 = 0$$

or

$$(\mathrm{EG} - \mathrm{F}^2){\kappa_\mathrm{n}}^2 - (\mathrm{EN} - 2\mathrm{FM} + \mathrm{GL})\kappa_\mathrm{n} + (\mathrm{LN} - \mathrm{M}^2) = 0$$

Putting the values of E, F, G, L, M, N in the equation, we get

$$(\mathrm{u}^2 + \mathrm{a}^2)\kappa^2 + \left(0 - \frac{\mathrm{a}^2}{\mathrm{H}^2}\right) = 0 \text{ or } \rho^2 = \frac{(\mathrm{u}^2 + \mathrm{a}^2)\mathrm{H}^2}{\mathrm{a}^2} = \frac{(\mathrm{u}^2 + \mathrm{a}^2)^2}{\mathrm{a}^2}$$

or $$\rho = \pm\frac{(\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{a}^2)}{\mathrm{a}}$$

which are the principal radii of curvatures.

Again the lines of curvature are given by the equation

$$(\mathrm{EM} - \mathrm{FL})\mathrm{du}^2 + (\mathrm{EN} - \mathrm{GL})\mathrm{dudv} + (\mathrm{FN} - \mathrm{GM})\mathrm{dv}^2 = 0$$

On putting the values of E, F, G, L, M and N in this equation we get

$$-\frac{\mathrm{a}}{\mathrm{H}}\mathrm{du}^2 + (\mathrm{u}^2 + \mathrm{a}^2)\frac{\mathrm{a}}{\mathrm{H}}\mathrm{dv}^2 = 0 \text{ or } \frac{\mathrm{du}}{\pm\sqrt{[(\mathrm{u}^2 + \mathrm{a}^2)]}} = \mathrm{dv}$$

Integrating, $\mathrm{v} = \pm\sinh^{-1}\frac{\mathrm{u}}{\mathrm{a}} + \mathrm{C}$, where C is constant.

or $$\tan^{-1}\frac{\mathrm{y}}{\mathrm{x}} = \pm\sinh^{-1}\frac{[(\mathrm{x}^2+\mathrm{y}^2)]}{\mathrm{a}} + \mathrm{C}$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Equation of First curvature is

$$\mathrm{J} = \mathrm{K}_\mathrm{a} + \kappa_\mathrm{b} = \frac{\mathrm{EN}+\mathrm{LG}-2\mathrm{FM}}{\mathrm{EG}-\mathrm{F}^2}.$$

Problem 2. The amplitude of normal curvature is defined by

$$A = \frac{1}{2}(\kappa_b + \kappa_a).$$

Problem 3. If mean curvature of a surface is one at all points, then the surface is called a minimal surface.

Problem 4. If mean curvature of a surface is zero at all points, then the surface is called a minimal surface.

9.7 *SUMMARY*

i. **Meusnier's theorem:** If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ be the angle between these sections, then $\kappa_{\mathrm{n}} = \kappa \cos\theta$.

ii. **Minimal surface:** If mean curvature of a surface is zero at all points, then the surface is called a minimal surface.

iii. **Principal curvature:** The curvatures of the principal sections of a surface through a given point, i.e., the maximum and minimum curvatures at that point are called principal curvatures at that point, and their corresponding radius of curvatures are called principal radius of curvatures.

9.8 *GLOSSARY*

(i) Derivatives

(ii) Determinant

9.9 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."
2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".
3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".
4. Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.

9.10 *TEWRMINAL QUESTIONS*

1. Define Normal Curvature, write its equation.
2. Define Principal Curvature, write its equation.
3. Define Mean Curvature, find its equation.

4. Define Gaussian Curvature, find its equation.

9.11 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. False

CYQ 4. True

UNIT 10: *RODRIGUE'S FORMULA AND EULER'S THEOREM*

CONTENTS:

10.1 *INTRODUCTION*

In geometry, Rodrigues' formula is used to define orthogonal polynomials, like Legendre polynomials, providing a way to derive them from a differential operator. It's a general formula that expresses a polynomial as a derivative of a product of a polynomial and a weight function. Local non-intrinsic properties in surface differential geometry are those that depend on how a surface is embedded in space, like the normal vector at a point, whereas intrinsic properties depend only on the surface itself, like the Gaussian curvature.

10.2 *OBJECTIVES*

After completion of this unit learners will be able to:

(i) Rodrigue's Formula

(ii) Euler's theorem

(iii) Monge's Theorem

10.3 *RODRIGUE'S FORMULA*

Statement. A necessary and sufficient condition that a curve on a surface be the line of curvature is that

$$\frac{\mathrm{dN}}{\mathrm{ds}} \propto \frac{\mathrm{d}\mathbf{r}}{\mathrm{ds}} \text{ or } \mathrm{dN} + \kappa \mathrm{d}\mathbf{r} = \mathbf{0}$$

at each of its points, where κ denotes the normal curvature.

Proof. Let $(\mathrm{du}, \mathrm{dv})$ be a line of curvature on the surface, then it is a principal direction at the point (u, v) to the surface, so we have from equations

$$(L - \kappa E)du + (M - \kappa F)dv = 0 \qquad (1)$$

and

$$(M - \kappa F)du + (N - \kappa G)dv = 0 \qquad (2)$$

where κ is one of the principal curvatures.

Putting the values of L,M,N,E,F,G in (1) and (2) by their expressions in terms of derivatives of $\mathbf{r}$ and $\mathbf{N}$, viz.

$$\mathrm{L} = -\mathbf{N}_1 \cdot \mathbf{r}_1, \mathrm{M} = -\mathbf{N}_1 \cdot \mathbf{r}_2 = -\mathbf{N}_2 \cdot \mathbf{r}_1, \mathrm{N} = -\mathbf{N}_2 \cdot \mathbf{r}_2$$
$$\mathrm{E} = \mathbf{r}_1^2, \mathrm{F} = \mathbf{r}_1 \cdot \mathbf{r}_2, G = \mathbf{r}_2^2.$$

We have
or

$$(-\mathbf{N}_1 \cdot \mathbf{r}_1 - \boldsymbol{\kappa} \boldsymbol{r}_1^2)du + (-\mathbf{N}_2 \cdot \mathbf{r}_1 - \kappa \mathbf{r}_1 \cdot \mathbf{r}_2)\mathrm{dv} = 0$$

or $$(\mathbf{N}_1 du + \mathbf{N}_2 dv) \cdot \mathbf{r}_1 + \kappa(\mathbf{r}_1 du + \mathbf{r}_2 dv) \cdot \mathbf{r}_1 = 0$$

Or

$$(\mathrm{dN}) \cdot \mathbf{r}_1 + (\kappa d\mathbf{r}) \cdot \mathbf{r}_1 = 0 \text{ or } (\mathrm{d}\mathbf{N} + \kappa d\mathbf{r}) \cdot \mathbf{r}_1 = 0 \qquad (3)$$

Similarly form (2), we have

$$(\mathrm{d}\mathbf{N} + \kappa \mathrm{d}\mathbf{r}) \cdot \mathbf{r}_2 = 0 \qquad (4)$$

Again

$$\mathbf{N} \cdot \mathbf{N} = 1$$

On differentiating it, we get $2\mathbf{N} \cdot \mathrm{d}\mathbf{N} = 0$ i.e., d**N** is normal to **N** or d**N** is a tangent vector. Also $d\mathbf{r}$ is a tangential vector, so the vector $\kappa d\mathbf{r} + \mathrm{d}\mathbf{N}$ is tangential vector to the surface. Also $\mathbf{r}_1$ and $\mathbf{r}_2$ are tangential vectors, therefore in order that equations (3) and (4) are satisfied, we must have

$$\mathrm{dN} + \kappa \mathrm{d}\mathbf{r} = 0$$

or

$$\frac{\mathrm{dN}}{\mathrm{ds}} \propto \frac{\mathrm{d}\mathbf{r}}{\mathrm{ds}}, \text{ i.e., the condition is necessary.}$$

Sufficient condition. Let there be curve on the surface at each point of which

$$\frac{d\mathbf{N}}{ds} \propto \frac{d\mathbf{r}}{ds}, \text{ i.e., } \kappa d\mathbf{r} + \mathrm{d}\mathbf{N} = 0.$$

where κ is any function, now reversing the order of steps, we get the equations (3) and (4) which are
and

$$(L-\kappa E)du+(M-\kappa F)dv=0$$
$$(M-\kappa F)du+(N-\kappa G)dv=0$$

Thus the curve is line of curvature in case K is normal curvature to the surface. Given, $\kappa dr+dN=0 \Rightarrow \kappa dr=-dN$

$$\Rightarrow \qquad \kappa(\mathbf{r}_1 du+\mathbf{r}_2 dv)=-(\boldsymbol{N}_1 du+\boldsymbol{N}_2 dv)$$

Taking dot product of ($\mathbf{r}_1$du + $\mathbf{r}_2$dv) both sides,

$$\Rightarrow \ \boldsymbol{\kappa}(\mathbf{r}_1 du+\mathbf{r}_2 dv)\cdot(\mathbf{r}_1 du+\mathbf{r}_2 dv)=-(\mathbf{N}_1 du+\mathbf{N}_2 dv)\cdot(\mathbf{r}_1 du+\mathbf{r}_2 dv)$$

$$\Rightarrow \ \kappa(Edu^2+2Fdudv+\mathrm{G}\mathrm{dv}^2)=\mathrm{Ldu}^2+2\mathrm{Mdudv}+\mathrm{Ndv}^{\,2}$$

$$\Rightarrow \ \kappa=\frac{\mathrm{L}du^2+2\mathrm{Mdudv}+\mathrm{Ndv}^2}{\mathrm{E}du^2+2\mathrm{Fdudv}+\mathrm{Gdv}^2}$$

$\Rightarrow \kappa$ is a normal curvature at the point (u,v) in the direction (du,dv).

Hence, the direction at each point of the curve is a principal direction and thus the curve is a line of curvature on the surface.

Remark.

σ In Rodrigue's formula, it is not necessary that k is the curvature of the curve under consideration. It is simply a scalar function. In fact, at any point of the curve, κ is the normal curvature of the surface at that point in the direction of the curve.

10.4 *MONGE'S THEOREM*

A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Proof. Let $\mathbf{r}=\mathbf{r}(\mathrm{u},\mathrm{v})$ be a surface and $\mathbf{r}=\mathbf{r}(\mathrm{s})$ be a curve on it. Let N denote a unit vector along normal to the surface $\mathbf{r}=\mathbf{r}(\mathrm{u},\mathrm{v})$ at any point P, $\mathrm{r}(\mathrm{s})$ on the curve $\mathbf{r}=\mathbf{r}(\mathrm{s})$; **N** can be taken as a function of s alone. Let **R** denote the position vector of any point on this normal, then

$$\mathbf{R}=\mathbf{r}(\mathrm{s})+\mathrm{vN}(\mathrm{s})$$

This equation can be taken as the function of the surface generated by the normals to the given surface at points on the curve $\mathbf{r}=\mathbf{r}(\mathrm{s})$. In equation (1) the two parameters are s and v.

$\Rightarrow$

$$\frac{\partial \mathbf{R}}{\partial s}=\mathbf{R}_1=\frac{d\mathbf{r}}{ds}+v\frac{d\mathbf{N}}{ds}=t+v\mathbf{N}'$$
$$\frac{\partial \mathbf{R}}{\partial v}=\mathbf{R}_2=\mathbf{N},\mathbf{R}_{12}=N',\mathbf{R}_{22}=0$$

Then $HM=[\mathbf{R}_{12},\mathbf{R}_1,\mathbf{R}_2]=[\mathbf{N}',\mathbf{t}+\mathbf{vN}',\mathbf{N}]$

$$=[\mathbf{N}',\mathbf{t},\mathbf{N}]+v[\mathbf{N}',\mathbf{N}',\mathbf{N}]$$
$$=[\mathbf{N}',\mathbf{t},\mathbf{N}]\ [\because [\mathbf{N}',\mathbf{N}',\mathbf{N}]=0,\mathbf{R}_{22}=0]$$

and

$$HN=[\mathbf{R}_{22},\mathbf{R}_1,\mathbf{R}_2]=0,$$

Now $HN=0\Rightarrow N=0$, since $H\neq 0$

Further

$$\mathbf{M}=\frac{1}{H}[\mathbf{N}',\mathbf{t},\mathbf{N}]=\frac{1}{H}[\mathbf{t},\mathbf{N},\mathbf{N}']$$

Now the surface (1) is developable if and only if its Gaussian curvature is zero

i.e., if and only if $\frac{LN-M^2}{H^2}=0$

$$\Leftrightarrow \quad M=0 \qquad [\because \mathbf{N}=0 \text{ for } (1)]$$
$$\Leftrightarrow \quad [\mathbf{t},\mathbf{N},\mathbf{N}']=0.$$

Hence, the normals to the surface $\mathbf{r}=\mathbf{r}(u,v)$ along the curve $\mathbf{r}=\mathbf{r}(s)$ form a developable if and only if $[\mathbf{t},\mathbf{N},\mathbf{N}']=0$. Therefore in order to prove the theorem we are now to prove that

$$[\mathbf{t},\mathbf{N},\ \mathbf{N}']=0$$

is a necessary and sufficient condition for $\mathbf{r}=\mathbf{r}(s)$ to be a line of curvature on the surface $\mathbf{r}=\mathbf{r}(u,v)$.

Now

$$[\mathbf{t},\mathbf{N},\mathbf{N}']=0,[\mathbf{t},\mathbf{N}',\mathbf{N}]=0,(\mathbf{t}\times\mathbf{N}')\cdot\mathbf{N}=0$$

But $\mathbf{N}\neq 0$.

Also $\mathbf{N}'$ is perpendicular to $\mathbf{N}$. So $\mathbf{N}'$ lies in the tangent plane to the surface $\mathbf{r}=\mathbf{r}(u,v)$. Now $\mathbf{t}\times\mathbf{N}'$ is perpendicular to both $\mathbf{t}$ and $\mathbf{N}'$. Therefore $\mathbf{t}\times\mathbf{N}'$ is normal to the surface $\mathbf{r}=\mathbf{r}(u,v)$. Thus $\mathbf{t}\times\mathbf{N}'$ is parallel to $\mathbf{N}$. Therefore if $t\times\mathbf{N}'\neq 0$, then $(\mathbf{t}\times\mathbf{N}').\mathbf{N}$ cannot be zero.

$$\Rightarrow \qquad (\mathbf{t}\times\mathbf{N}')\cdot\mathbf{N}=0,\Rightarrow \mathbf{t}\times\mathbf{N}'=0$$

$\Rightarrow$ $N'=-\kappa t$ for some scalar function κ .

$$\Rightarrow\ \kappa\frac{d\mathbf{r}}{ds}+\frac{d\mathbf{N}}{ds}=0$$

$\Rightarrow$ The given curve is a line of curvature by Rodrigue's formula.

Conversely. If the given curve is a line of curvature, then by Rodrigue's formula, we have

$$\boldsymbol{\kappa}\frac{dr}{ds}+\frac{d\mathbf{N}}{ds}=0$$
$$-\kappa t=N'$$
$$\therefore\ [\mathbf{t},\mathbf{N},\mathbf{N}']=[\mathbf{t},\mathbf{N},-\boldsymbol{\kappa}\mathbf{t}]=0$$

Hence the proof of the theorem is complete.

10.5 *LINE OF CURVATURE AS PARAMETRIC CURVE*

The necessary and sufficient condition that parametric curves be lines of curvature $F = 0, M = 0$.

Proof. The parametric curves are $u =$ constant and $v =$ constant therefore combined differential equation of parametric curves is given by

$$dudv = 0 \qquad (1)$$

Again the differential equation of lines of curvatures is

$$[EM - FL]du^2 + [EN - GL]dudv + [FN - GM]dv^2 = 0 \qquad (2)$$

If the lines of curvature are taken as parametric curves, then $F = 0$, since the principal directions are orthogonal.

Comparing (1) and (2), we have

$$EM - FL = 0, FN - GM = 0 \text{ and } EN - GL \neq 0$$

Since $F = 0$, so we have

$$EM = 0, GM = 0 \text{ which gives } M = 0$$

Hence $F = 0, M = 0$ are necessary conditions for the parametric curves to be lines of curvature.

Sufficient condition. If $F = 0, M = 0$ the equation (2) of lines of curvature becomes $(EN - GL)dudv = 0$

But $\qquad EN - GL \neq 0; \therefore dudv = 0$

Which is the differential equation of the parametric curves.

10.6 *EULER'S THEOREM*

Statement. The normal curvature κ_n at a point on a surface is given in terms of principal curvatures κ_a and κ_b by the formula

$$\kappa_n = \kappa_a \cos^2 \psi + \kappa_b \sin^2 \psi$$

(known as Euler's formula) where κ_a and κ_b are the principal curvatures and ψ is the angle at which the direction (du, dv) of the normal section made with the principal direction $dv = 0$

Proof. Let the lines of curvature be taken as parametric curves, then $F = 0$, $M = 0$ and the normal curvature

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

reduce to

$$\kappa_n = \frac{Ldu^2 + Ndv^2}{Edu^2 + Gdv^2} \qquad (1)$$

Since the direction $\mathbf{v}$ = constant [having direction coefficients $\left(\frac{1}{\sqrt{E}}, 0\right)$] and u = constant [having direction coefficient $\left(0, \frac{1}{\sqrt{G}}\right)$] are principal directions, so the curvatures for these directions are principal curvatures and are given by

$$\kappa_a = \frac{L}{E} \text{ [Put } dv = 0 \text{ in (1)]} \qquad (2)$$

And

$$\kappa_b = \frac{N}{G} \text{ [Put } du = 0 \text{ in (1)]} \qquad (3)$$

As the direction du = (du, dv) makes angle ψ with the parametric curve v = constant, so we have

$$\left(\frac{1}{\sqrt{E}}, 0\right)$$

$$\cos\psi = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$$

$$= \sqrt{(E)}\frac{du}{ds}$$

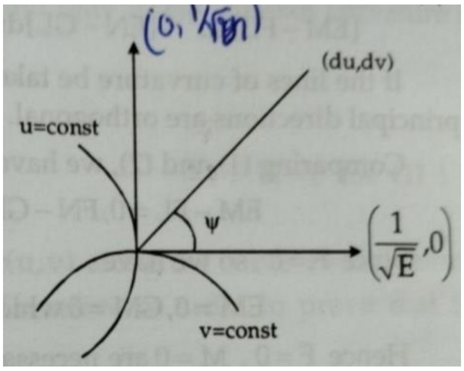

Fig.(10.5.1)

$$\sin\psi = \frac{H}{\sqrt{E}}\frac{dv}{ds} = \frac{\sqrt{(EG - F^2)}dv}{\sqrt{E}ds} = \sqrt{(G)}\frac{dv}{ds} \qquad [\because F = 0]$$

Again, $$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$= Edu^2 + Gdv^2 \qquad [\because F = 0]$$

Again from (1)

$$\kappa_n = \frac{Ldu^2 + Ndv^2}{ds^2} = L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin\psi$$

or

$$\kappa_n = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi \qquad (4)$$

- **Deduction.** The sum of the normal curvature in two orthogonal directions is equal to the sum of the principal curvatures at that point.

Proof. Let κ_{n_1} and κ_{n_2} denote normal curvature in two orthogonal directions on the surface and ψ be the angle between the first direction and the principal direction $dv = 0$; thus, the angle between the second direction and the principal direction $du = 0$ will be $\frac{\pi}{2} + \psi$. Thus, from Euler's formula (4), we have

$$\kappa_{n_1} = \kappa_a \cos^2 \psi + K_b \sin^2 \psi \tag{1}$$

and

$$\begin{aligned}\kappa_{n_2} &= \kappa_a \cos^2\left(\frac{\pi}{2} + \psi\right) + \kappa_b \sin^2\left(\frac{\pi}{2} + \psi\right) \\ &= \kappa_a \sin^2 \psi + \kappa_b \cos^2 \psi \end{aligned} \tag{2}$$

Now on adding (1) and (2), we get

$$\kappa_{n_1} + \kappa_{n_2} = \kappa_a + \kappa_b$$

This is known as **Dupin's theorem**.

- **Elliptic points.** The points on the surface at which the principal curvatures κ_a and κ_b have the same sign i.e., the Gaussian curvature K is positive are called elliptic points.

Again $K = K_a K_b = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{H^2}$.
Thus, we conclude that a point is an elliptic point if

$$LN - M^2 > 0.$$

Note: If κ_a and κ_b have different signs then the indicatrix is one of two conjugate hyperbolas depending on the sign of h. In this case surface in the neighbourhood of Olies on both sides of the tangent plane. Such portions of the surface are called **Anticlastic at that point.**

- **Hyperbolic points**. The points on the surface at which the Gaussian curvature K is negative i.e., $LN - M^2 < 0$ are called hypervolic points. In this case principal curvatures at the points are of opposite signs.

Note: If one of the principal curvatures is zero, i.e., either $k_a = 0$ or $\kappa_b = 0$, then the indicatrix is a pair of parallel straight lines.

- **Parabolic points.** The points on the surface at which the Gaussian curvature $K = 0$ are called parabolic points. In this case $LN - M^2 = 0$.

Example 1. Show that the equation for the principal curvature through a point of the surface $z = f(x, y)$ is

$$H^4 \kappa_n{}^2 - H[(1 + p^2)t + (1 + q^2)r - 2spq]\kappa_n + (rt - s^2) = 0$$

Solution: The given surface is $z = f(x, y)$, thus we have

$\mathbf{r} = (x, y, f(x, y))$
$\therefore\ \mathbf{r}_2 = (1,0, p); \mathbf{r}_2 = (0,1, q); \mathbf{r}_{11} = (0,0, r); \mathbf{r}_{12} = (0,0, s);$

$$\mathbf{r}_{22} = (0,0, t)$$

Therefore, $E = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1 + p^2; F = \mathbf{r}_1 \cdot \mathbf{r}_2 = pq; G = \mathbf{r}_2 \cdot \mathbf{r}_2 = 1 + q^2.$

Therefore, $H^2 = EG - F^2 = (1+p^2)(1+q^2) - p^2q^2 = 1 + p^2 + q^2$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{(-p, -q, 1)}{H}$$

$$L = \mathbf{N} \cdot \mathbf{r}_{11} = \frac{\mathbf{r}}{H}; M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{s}{H}; N = \mathbf{N} \cdot \mathbf{r}_{22} = \frac{t}{H};$$

and

$$T^2 = LN - M^2 = \frac{rt - s^2}{H^2}$$

Equation of principal curvature is given by

$$H^2{\kappa_n}^2 - (EN + GL - 2FM)\kappa_n + T^2 = 0$$

Putting the values of E, F, G, L, M, N in this equation, we get

$$H^4\kappa_n^2 - H[(1+p^2)t + (1+q^2)r - 2spq]\kappa_n + (rt - s^2) = 0$$

which is the equation of principal curvatures.

Example 2. Show that the principal radii of curvature of the surface $y\cos\frac{z}{a} = x\sin\frac{z}{a}$ are equal to $\pm\frac{x^2+y^2+a^2}{a}$. Find the lines of curvature.

Solution. Surface is $y\cos\frac{z}{a} = x\sin\frac{z}{a}$ i.e. $z = a\tan^{-1}\frac{y}{x}$

The parametric equations of the surface are

X= u cos v, y = u sin v, z = av

$\therefore r = (u\cos v, u\sin v, av)$

$r_1 = (\cos v, \sin v, 0); r_2 = (-u\sin v, u\cos v, a)$

$\mathbf{r}_{11} = (0,0,0); r_{12} = (-\sin v, \cos v, 0)$

$r_{22} = (-u\cos v, -u\sin v, 0);$

$\therefore E = r_1^2 = 1; F = r_1 \cdot r_2 = 0, G = r_2^2 = u^2 + a^2;$

$H^2 = EG - F^2 = u^2 + a^2$

$\mathbf{N} = \frac{r_1 \times \mathbf{r}_2}{H} = \frac{(a\sin v, -a\cos v, u)}{H}$.

$L = \mathbf{N} \cdot \mathbf{r}_{11} = 0, \mathbf{M} = \mathbf{N} \cdot \mathbf{r}_{12} = -\frac{a}{H}, N = N \cdot \mathbf{r}_{22} = 0.$

The principal curvatures of the surface are given by

$$H^2{\kappa_n}^2 - (EN - 2FM + GL)\kappa_n + T^2 = 0$$

or

$$(EG - F^2){\kappa_n}^2 - (EN - 2FM + GL)\kappa_n + (LN - M^2) = 0$$

Putting the values of E, F, G, L, M, N in the equation, we get

$$(u^2 + a^2)\kappa^2 + \left(0 - \frac{a^2}{H^2}\right) = 0 \text{ or } \rho^2 = \frac{(u^2+a^2)H^2}{a^2} = \frac{(u^2+a^2)^2}{a^2}$$

or $$\rho = \pm\frac{(x^2 + y^2 + a^2)}{a}$$

which are the principal radii of curvatures.

Again the lines of curvature are given by the equation

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0$$

On putting the values of E, F, G, L, M and N in this equation we get

$$-\frac{a}{H}du^2 + (u^2 + a^2)\frac{a}{H}dv^2 = 0 \text{ or } \frac{du}{\pm\sqrt{[(u^2 + a^2)]}} = dv$$

Integrating, $v = \pm\sinh^{-1}\frac{u}{a} + C$, where C is constant.

or $\tan^{-1}\frac{y}{x} = \pm\sinh^{-1}\frac{[(x^2+y^2)]}{a} + C$

Example 3. For the hyperboloid $2z = 7x^2 + 6xy - y^2$, prove that the principal radii at the are $\frac{1}{8}, -\frac{1}{2}$ and the principal sections are $x = 3y, 3x = -y$.

Solution. The given surface is

$$z = \frac{1}{2}(7x^2 + 6x - y^2) \qquad (1)$$

which is of the form $z = f(x, y)$

In Example 1, we have calculated the fundamental magnitudes of such surface which are given by

$$E = 1 + p^2; F = pq; G = 1 + q^2$$

$$L = \frac{r}{H}; M = \frac{s}{H}; N = \frac{t}{H} \qquad \left[\because H = \sqrt{[(1 + p^2 + q^2)]}\right.$$

In this case

$$p = \frac{\partial z}{\partial x} = 7x + 3y = 0 \qquad \text{[at origin]}$$

$$q = \frac{\partial z}{\partial y} = (3x) - y = 0 \qquad \text{[at origin]}$$

$$r = \frac{\partial^2 z}{\partial x^2} = 7; s = \frac{\partial^2 z}{\partial x\,\partial y} = 3; t = \frac{\partial^2 z}{\partial y^2} = -1$$

Therefore, we have

$$E = 1, F = 0, G = 1, H = 1, L = 1, M = 3, N = -1$$

Equations giving the principal curvatures is

$$(EG - F)\kappa_n^2 - (EN - 2FM + LG)\kappa_n + (LN - M^2) = 0$$

Putting the values of E, F, G, L, M, N and H in this equation, we get $\kappa_n^2 - 6\kappa_n - 16 = 0$, i.e., $\kappa_n = 8, -2$. Hence the principal radii are $\frac{1}{8}, -\frac{1}{2}$.

Again, the equation of line curvature is

$$(EM - FL)dx^2 + (EN - GL)dxdy + (FN - GM)dy^2 = 0$$

Or $3dx^2 - 8dxdy - 3dy^2 = 0$ or $(3dx + dy)(dx - 3dy) = 0$

or $3x + y = c_1, x - 3y = c_2$

At origin, $c_1 = 0 = c_2$

Therefore, principal sections at origin are

$$x = 3y, 3x = -y$$

Example 4. Find the principal radii of the surface $a^2x^2 = z^2(x^2 + y^2)$ at the points where $x = y = z$.

Solution. Parametric equations of the surface are given by

$$x = u\cos\theta, y = u\sin\theta, z = a\cos\theta$$

where u and θ are parameters.

$\therefore\ r = (u\cos\theta, u\sin\theta, a\cos\theta)$
$\mathbf{r}_1 = (\cos\theta, \sin\theta, 0), \quad \mathbf{r}_2 = (-u\sin\theta, u\cos\theta, -a\sin\theta)$
$\mathbf{r}_{11} = (0,0,0); \mathbf{r}_{12} = (-\sin\theta, \cos\theta, 0)$
$\mathbf{r}_{22} = (-u\cos\theta, -u\sin\theta, -a\cos\theta)$
$NH = r_1 \times \mathbf{r}_2 = (-a\sin^2\theta, a\sin\theta\cos\theta, u);$
$\therefore E = 1, F = 0, G = u^2 + a^2\sin^2\theta, H^2 = u^2 + a^2\sin^2\theta$

$$\therefore L = 0, M = \frac{a\sin\theta}{H}, N = -\frac{au\sin\theta}{H}, T^2 = \frac{a^2\sin^2\theta}{H^2}$$

The principal curvatures are given by the equation

$$H^2\kappa_n^2 - (EN + GL - 2FM)\kappa_n + T^2 = 0$$

or Principal radii are given by the equation

$$T^2\rho^2 - (EN + GL - 2FM)\rho + H^2 = 0$$

Substituting values of E, F, G, L, M, N, T and H , we have

$$\frac{-a^2\sin^2\theta}{(u^2 + a^2\sin^2\theta)}\rho^2 - \left[\frac{-au\cos\theta}{\sqrt{(u^2 + a^2\sin^2\theta)}}\right]\rho + (u^2 + a^2\sin\theta) = 0$$

or $\rho^2(a^2\sin^2\theta) - \rho\left[(u^2 + a^2\sin^2\theta)^{1/2}\right]au\cos\theta - (u^2 + a^2\sin\theta)^2 = 0$

Again at $x = y = z, u = a, \theta = \pi/4$,

$$\therefore\ \frac{\rho^2 a^2}{2} - \rho\sqrt{\left(a^2 + \frac{a^2}{2}\right)} \cdot a \cdot a \cdot \frac{1}{\sqrt{2}} - \left(a^2 + \frac{a^2}{2}\right)^2 = 0$$

or $$\rho^2 a^2 - \rho a^3\sqrt{3} - \frac{9a^4}{2} = 0$$

or $$2\rho^2 a^2 - 2a^3\rho\sqrt{3} - 9a^4 = 0$$

or $$2\rho^2 - 2a\rho\sqrt{3} - 9a^2 = 0$$

or $$\rho = \frac{2a\sqrt{3} \pm \sqrt{12a^2 + 72a^2}}{4}$$

or $$\rho = \frac{2a\sqrt{3} \pm a \cdot 2\sqrt{(21)}}{4} = \frac{a\sqrt{3}}{2}(1 \pm \sqrt{7}).$$

Example 5. At a point of the curve of intersection of the paraboloid $xy = cz$ and the hyperboloid $x^2 - y^2 + z^2 + c^2 = 0$ the principal radii of curvatures are $\frac{z^2}{c}(1 \pm \sqrt{2})$.

Solution. The position vector rof any point on the curve is given by

$$\mathbf{r} = \left(u, v, \frac{uv}{c}\right) \tag{1}$$

As this is the point of intersection of the two given surfaces, so we have from $x^2 + y^2 - z^2 + c^2 = 0$, the equation

$$u^2 + v^2 + c^2 = \frac{u^2v^2}{c} \tag{2}$$

Now $r_1 = (1.0.v)\ 289$

Now $r_1 = \left(1,0,\frac{v}{c}\right), r_2 = \left(0,1,\frac{u}{c}\right), r_{11} = (0,0,0),$

$$\mathbf{r}_{12} = \left(0,0,\frac{1}{c}\right), \mathbf{r}_{22} = (0,0,0)$$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\mathbf{H}} = \left(\frac{-v}{c}, \frac{-u}{c}, 1\right)/H$$

$$\therefore\ E = r_1^2 = 1 + \frac{v^2}{c^2}, F = r_1 \cdot r_2 = \frac{uv}{c^2}, G = r_2^2 = 1 + \frac{u^2}{c^2}$$

$$H^2 = EG - F^2 = 1 + \frac{u^2}{c^2} + \frac{v^2}{c^2}$$

$$L = \mathbf{N} \cdot \mathbf{r}_{11} = 0, M = \mathbf{N} \cdot \mathbf{r}_{12} = \frac{1}{cH}, N = \mathbf{N} \cdot \mathbf{r}_{22} = 0$$

$$T^2 = \frac{-1}{c^2H^2}$$

The principal radii of curvature are given by the equation

$$T^2\rho^2 - (EN + GL - 2FM)\rho + H^2 = 0$$

$$-\frac{1}{c^2H^2}\rho^2 + \frac{2uv}{c^2} \cdot \frac{1}{cH}\rho + H^2 = 0$$

or $$\rho - \frac{2uv}{c} \cdot \rho \sqrt{\left[\left(1 + \frac{u^2}{c^2} + \frac{v^2}{c^2}\right)\right]} - c^2\left(1 + \frac{u^2}{c^2} + \frac{v^2}{c^2}\right)^2 = 0$$

or $$\rho - \frac{2uv\rho}{c} \cdot \frac{1}{c} \cdot \frac{uv}{c} - \frac{c^2}{c^4} - \frac{u^4v^4}{c^4} = 0$$

or

$$\rho^2 - \frac{2z^2}{c}\rho - \frac{z^4}{c^4} = 0 \qquad \left(\because z = \frac{uv}{c}\right)$$

or

$$\rho = \frac{\frac{2z^2}{c} \pm \sqrt{\left[\frac{4z^4}{c^2} + \frac{4z^4}{c^2}\right]}}{2} = \frac{z^2}{c}(1 \pm \sqrt{2})$$

Example 6. Show that the points of intersection of the surface $x^m + y^m + z^m = a^m$ and the line $x = y = z$ are umbilics and that the curvature at an umbilic is given by $\rho = \frac{a}{m-1}(3)^{\frac{m-2}{2m}}$.

Solution. We have $x^m + y^m + z^m = a^m$

$$\therefore\ mx^{m-1} + mz^{m-1} \cdot \frac{\partial z}{\partial x} = 0$$

$\therefore$ $$p = -\frac{x^{m-1}}{z^{m-1}} \qquad \left(\because p = \frac{\partial z}{\partial x}\right) \qquad \text{(A)}$$

Similarly, $q = -\frac{y^{m-1}}{z^{m-1}}$

Now, $\log p = -|\ m - 1)\log x - (m-1)\log z\ |$

$$\frac{1}{p}\frac{\partial P}{\partial x} = -(m-1)\left[\frac{1}{x} - \frac{1}{z}\cdot P\right]$$

$$\therefore\ \frac{r}{p} = (m-1)\left(\frac{P}{z} - \frac{1}{x}\right) \qquad (1)$$

Also

$$\frac{s}{p} = \frac{(m-1)q}{z} \qquad (2)$$

And

$$t = (m-1)\left(\frac{q}{z} - \frac{1}{y}\right) \qquad (3)$$

Now for an umbilical,

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{z}{m-1} \qquad [\text{Using}(2)]$$

$$\therefore\ (1+p^2) = \frac{z}{(m-1)}\left[p(m-1)\left(\frac{p}{z} - \frac{1}{x}\right)\right] = p^2 - \frac{pz}{x}$$

$$p = \frac{-z}{x} = -\frac{x^{m-1}}{z^{m-1}} \qquad (\text{Using A})$$

Similarly, $y^{m-2} = z^{m-2}$

$\therefore\ x^{m-2} = y^{m-2} = z^{m-2}$

$$\Rightarrow\ x = y = z$$

$\therefore$ For an umbilic, $x = y = z$

Again if

$$x = y = z, p = 1, q = 1, s = \frac{m-1}{a/(3)^{1/m}}$$

Now

$$H = \sqrt{(1+p^2+q^2)} = \sqrt{1+1+1)} = \sqrt{3}$$

$$\therefore\ \rho = \frac{pqH}{S} = \frac{\sqrt{3}a}{(m-1)3^{1/m}} = \frac{a(3)^{\frac{m-2}{2m}}}{m-1}$$

Example 7. Find the principal directions and the principal curvature on the surface $x = a(u+v), y = b(u-v), z = uv$.

Solution. The position vector r of any point on the surface is given by

$$\mathbf{r} = [a(u+v), b(u-v), uv]$$

$$\therefore\ \mathbf{r}_1 = (a, b, v)\mathbf{r}_2 = (a, -b, u)$$

$$\mathbf{r}_1 \times \mathbf{r}_2 = [b(u+v), a(v-u), -2ab].$$

Also, $\mathbf{r}_{11} = (0,0,0), \mathbf{r}_{12} = (0,0,1), \mathbf{r}_{22} = (0,0,0).$

Nor $E = r_1^2 = a^2 + b^2 + v^2,$

$F = r_1 \cdot r_2 = a^2 - b^2 + uv,$

$G = r_2^2 = a^2 + b^2 + u^2.$

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{H} = \frac{[b(u+v), a(v-u), -2ab]}{H}$$

$\therefore L = \mathbf{N} \cdot \mathbf{r}_{11} = 0, M = \mathbf{N} \cdot \mathbf{r}_2 = \frac{-2ab}{H}, \mathbf{N} = \mathbf{N} \cdot \mathbf{r}_{22} = 0.$

The differential equation of lines of curvature (or principal directions) is given by

$$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0$$

Or

$$-(a^2 + b^2 + v^2)\frac{2ab}{H}du^2 + (a^2 + b^2 + u^2)\frac{2ab}{H}dv^2 = 0$$

Or

$$(a^2 + b^2 + v^2)du^2 - (a^2 + b^2 + u^2)dv^2 = 0$$

or

$$\frac{du}{\sqrt{(a^2 + b^2 + u^2)}} = \pm\frac{dv}{\sqrt{(a^2 + b^2 + v^2)}}$$

Integrating we get

$$\sinh^{-1}\frac{u}{\sqrt{(a^2 + b^2)}} = \pm\sinh^{-1}\frac{v}{\sqrt{a^2 + b^2}} + c$$

where c is a constant.

The equation giving principal curvatures is

$$H^2\kappa^2 - \kappa(EN - 2FM + GL) + (LN - M^2) = 0$$

Or

$$H^2\kappa^2 - \kappa\left\{-2(a^2 - b^2 + uv)\left(-\frac{2ab}{H}\right)\right\} - \frac{4a^2\, b^2}{H^2} = 0$$

or

$$H^4\kappa^2 - 4abH(a^2 - b^2 + uv)\kappa - 4a^2\, b^2 = 0$$

Where,

$$H^2 = EG - F^2 = (a^2 + b^2 + v^2)(a^2 + b^2 + u^2) - (a^2 - b^2 + uv)^2$$

Example 8. Find the Gaussian curvature at the point (u, v) of the anchor ring $x = (b + a\cos u)\cos v, y = (b + a\cos u)\sin v, z = a\sin u$, where the domain of u, v is $0 < u < 2\pi, 0 < v < 2\pi$. Verify that the total curvature of the whole surface is zero.

Solution. Position vector $\mathbf{r}$ of any point on the surface is given by

$$\mathbf{r} = [(b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u]$$
$$\mathbf{r}_1 = (-a\sin u\cos v, -a\sin u\sin v, a\cos u$$
$$\mathbf{r}_2 = [-(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0],$$

$$\begin{aligned}\mathbf{r}_1 \times \mathbf{r}_2 = [&-a(b + a\cos u)\cos u\cos v \\ &- a(b + a\cos u)\cos u\sin v\ , -a(b + a\cos u)\sin u\ ] \\ &= (b + a\cos u)(-a\cos u\cos v\ , -a\cos u\sin v, -a\sin u)\end{aligned}$$

Also

$$r_{11} = (-a\cos v\cos u, -a\cos u\sin v, -a\sin u),$$
$$r_{12} = (a\sin \cdot u\sin v, -a\sin u\cos v, 0),$$
$$r_{22} = [-(b + a\cos u)\cos v, -(b + a\cos u)\sin v, 0]$$

Now,

$$E = r_1^2 = a^2, F = r_1 \cdot r_2 = 0,$$
$$G = r_2^2 = (b + a\cos u)^2,$$
$$H^2 = EG - F^2 = a^2(b + a\cos u)^2,$$

Again,

$$\begin{aligned}
\mathbf{N} &= \frac{r_1 \times \mathbf{r}_1}{H} \\
&= \frac{(b + a\cos u)(-a\cos u\cos v, -a\cos u\sin v, a\sin u)}{H}
\end{aligned}$$

$$\therefore \quad L = N \cdot r_{11} = \frac{a^2(b + a\cos u)}{H} = \frac{a^2(b + a\cos u)}{a(b + a\cos u)} = a$$

$$M = N \cdot r_{12} = u; N = N \cdot r_{22} = (b + a\cos u)\cos u.$$

Now Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{a(b + a\cos u)\cos u}{a^2(b + a\cos u)^2} = \frac{\cos u}{a(b + a\cos u)}$$

Total curvature of the whole surface

$$= \int KdS, \text{ integrated over the whole surface S}$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} KHdudv, \text{ since } dS = Hdudv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} \frac{\cos u}{a(b + a\cos u)} a(b + a\cos u)dudv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos u du dv = 0.$$

Example 9. Prove that the cone $\kappa xy = z \mid \sqrt{[(x^2 + z^2)]} + \sqrt{[(y^2 + z^2)]}$ passes through a line curvature of the paraboloid $xy = az$.

Solution. From the paraboloid $z = \frac{xy}{a}$, we have

$$p = \frac{\partial z}{\partial x} = \frac{y}{a}, q = \frac{\partial z}{\partial y} = \frac{x}{a}, r = 0, s = \frac{1}{a}, t = 0$$

The lines of curvature are given by

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 1 + p^2 & pq & 1 + q^2 \\ r & s & t \end{vmatrix} = 0.$$

CHECK YOUR PROGRESS

True or false Questions

Problem 1. Equation of Euler's equation $\kappa_n = \kappa_a \cos^2\psi + \kappa_b \sin^2\psi$.

Problem 2. The sum of the normal curvature in two orthogonal directions is equal to the sum of the principal curvatures at that point.

Problem 3. A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

10.7 *SUMMARY*

i. **MONGE'S THEOREM:** A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Euler's Theorem: The normal curvature κ_n at a point on a surface is given in terms of principal curvatures κ_a and κ_b by the formula

$$\kappa_n = \kappa_a \cos^2 \psi + \kappa_b \sin^2 \psi$$

(known as Euler's formula) where κ_a and κ_b are the principal curvatures and ψ is the angle at which the direction (du, dv) of the normal section made with the principal direction $dv = 0$

10.8 *GLOSSARY*

(i) Derivatives

(ii) Determinant

(iii) Vector

10.9 *REFERENCES AND SUGGESTED READINGS*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."
2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".
3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".

4. Differential Geometry by Gupta, Malik and Pundir "Pragati Edition''.

10.10 *TERMINAL QUESTIONS*

1. States and prove Euler's theorem.
2. States and Drive Rodrigue's formula.
3. States and prove Monge's theorem.

10.11 *ANSWERS*

CYQ 1. True

CYQ 2. True

CYQ 3. True

COURSE NAME: GEOMETRY

COURSE CODE: MAT 611

BLOCK-IV

TENSOR ANALYSIS

UNIT 11: *n – DIMENSIONAL SPACE*

CONTENTS:

11.1 *INTRODUCTION*

In geometry, **Two Dimensional Space:** In two dimensional space the coordinates of a point are given by the doublets of the form (x, y), where x, y are two numbers.

Three Dimensional Space: In three dimensional space the coordinates of a point are given by the triplets of the form (x, y, z), where x, y, z are three numbers.

Four Dimensional Space: In four dimensional space the coordinates of a point are given by the four touples of the form (x, y, z, u), where x, y, z, u are four numbers.

11.2 *OBJECTIVES*

After studying this unit Learner will be able to

i. Understand the concept of n- dimensional space.

ii. Define a subspace.

iii. Write superscript and subscript.

iv. Understand the Einstein Summation Convention.

11.3 *n – DIMENSIONAL SPACE*

Consider an ordered set of n real variables

$$(x^1, x^2, \dots, x^i, \dots x^n)$$

These variables $x^1, x^2, \dots, x^i, \dots x^n$ are called coordinates. The space generated by all points corresponding to different values of the coordinates is called n- dimensional space and is denoted by V_n. Here $1, 2, \dots n$ are not the powers of x but are the labels only. The suffix i in the coordinate x^i does not have the character of power indices. Usually powers will be denoted by brackets e.g., $\left(x^i\right)^3$ means the cube of x^i.

Note:

- A subspace $V_m (m < n)$ of V_n is defined as the collection of points which satisfy the n-equations $x^i = x^i(u^1, u^2, \dots, u^m)$, (i = 1,2, … , n).
- The variables $u^1, u^2, \dots, u^m$ are the coordinates of V_m . The suffixes $1,2, \dots, n$ serve as labels only and do not possess any significance as power indices.
- A curve in V_n is defined as the collection of points which satisfy the n-equations

  $x^i = x^i(\mathrm{u}), (\mathrm{i} = 1,2, \dots, \mathrm{n})$ u being a parameter and $x^i(\mathrm{u})$ denotes a function of u.
- **Superscript and Subscript:** The suffixes i and j in B^i_j are called superscript and subscript respectively. The upper position always denotes the superscript and the lower position denotes subscript.

11.4 *EINSTEIN SUMMATION CONVENTION*

We know that the expression

$$a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

is represented by $\sum_{i=1}^{n} a_i x^i$.

Dropping the sigma sign and writing the sum $\sum_{i=1}^{n} a_i x^i$ as $a_i x^i$ is called the summation convention.

Thus summation convention means if a suffix occurs twice in a term, once in the lower position and once in the upper position, then that suffix implies sum over defined range. If the range is not given, then we assume that the range is from 1 to n.

Example 1: Write the following by using summation convention

$$A^k_1 B^1 + A^k_2 B^2 + \cdots + A^k_n B^n$$

Solution: By using summation convention we can write

$$A^k_1 B^1 + A^k_2 B^2 + \cdots + A^k_n B^n = A^k_i B^i$$

Example 2: Write the following by using summation convention

$$g^{21} g_{11} + g^{22} g_{21} + g^{23} g_{31} + g^{24} g_{41} + g^{25} g_{51} + g^{26} g_{61} + g^{27} g_{71}$$

Solution: By using summation convention, we can write

$$g^{21} g_{11} + g^{22} g_{21} + g^{23} g_{31} + g^{24} g_{41} + g^{25} g_{51} + g^{26} g_{61} + g^{27} g_{71} = g^{2i} g_{i1}, n = 7$$

Example 3: Write the terms in the following indicated sums

$$A^k_i B^i, \; n = 5$$

Solution: Here the index i is repeated and n= 5 therefore i takes the values 1 to 5

$$A_i^k B^i = \sum_{i=1}^{5} A_i^k B^i = A_1^k B^1 + A_2^k B^2 + A_3^k B^3 + A_4^k B^4 + A_5^k B^5$$

EXAMPLE 4: Write the terms in the following indicated sums

$$a_i x^i x^3$$

SOLUTION: Here the index i is repeated therefore the sum is over the index i. Hence

$$a_i x^i x^3 = \sum_{i=1}^{n} a_i x^i x^3 = a_1 x^1 x^3 + a_2 x^2 x^3 + \cdots + a_n x^n x^3 \quad .$$

11.5 *DUMMY SUFFIX*

A suffix which occurs twice in a term, once in the upper position and once in the lower position, is called dummy suffix. For example q is a dummy suffix in $A_{pq}A^{qr}$. Umbral suffix and dextral index are the other names for dummy suffix.

Theorem: To show that a dummy suffix can be replaced by another dummy suffix not used in that term.

Proof: Let us take $a_i^\mu x^i$ in which i is a dummy suffix. Evidently

$$a_i^\mu x^i = a_1^\mu x^1 + a_2^\mu x^2 + \cdots + a_n^\mu x^n \ldots\ldots\ldots (1)$$

$$a_j^\mu x^j = a_1^\mu x^1 + a_2^\mu x^2 + \cdots + a_n^\mu x^n \ldots\ldots\ldots (2)$$

As R.H.S. of both the equations are same, we can say that $a_i^\mu x^i = a_j^\mu x^j$ which proves that a dummy suffix can be replaced by another dummy suffix not used in that term.

Similarly, it can be proved that two or more than two dummy suffixes can also be interchanged i.e., $a_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} = a_{\beta\alpha} \frac{\partial x^\beta}{\partial x'^i} \frac{\partial x^\alpha}{\partial x'^j}$.

11.6 *REAL SUFFIX*

A suffix which is not repeated is called a real or free suffix. It may be in superscript or in subscript also. For example α is a real suffix in $a_i^\alpha x^i$. A real suffix can not be replaced by another real suffix. Since $a_i^\alpha x^i \neq a_i^\beta x^i$.

Kronecker delta:- It is denoted by δ_j^i and is defined as

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Some Properties of Kronecker delta

(i) $\delta_i^i = N$

Here i is a dummy suffix. So by summation convention,

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_n^n$$
$$= 1 + 1 + \cdots + 1 = n$$

Remark: Due to this property we use the statement $\delta^i_j = 1$, if $i = j$ instead of saying $\delta^i_i = 1$.

(ii) $\delta^i_j A^{jk} = A^{ik}$

Here j is a dummy suffix. So by summation convention,

$$\delta^i_j A^{jk} = \delta^i_1 A^{1k} + \delta^i_2 A^{2k} + \cdots + \delta^i_i A^{ik} + \cdots \delta^i_n A^{nk}$$
$$= 0 + 0 + \cdots + 1.\,A^{ik} + \cdots 0$$
$$= A^{ik}$$

(iii) If $x^1, x^2, \dots, x^N$ be N independent variables, Then

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j$$

Clearly, x^i and x^j are independent variables when $i \neq j$ and dependent when $i = j$.

So $\frac{\partial x^i}{\partial x^j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Hence, $\frac{\partial x^i}{\partial x^j} = \delta^i_j$.

Example 1: Prove that $\delta^i_j \; \delta^j_k = \delta^i_k$.

Solution: Here j is a dummy suffix. So

$$\delta^i_j \; \delta^j_k = \delta^i_1 \; \delta^1_k + \delta^i_2 \; \delta^2_k + \cdots + \delta^i_i \; \delta^i_k \dots + \delta^i_n \; \delta^n_k$$
$$\delta^i_j \; \delta^j_k = 0 + 0 + \cdots \delta^i_k + \cdots + 0 = \delta^i_k$$
$$= \delta^i_k$$

Example 2: Prove that $\delta^i_j \; A^{jl}_m = A^{il}_m$.

Solution: Here j is a dummy suffix. So

$$\delta^i_j \; A^{jl}_m = \delta^i_1 \; A^{1l}_m + \delta^i_2 \; A^{2l}_m + \cdots + \delta^i_i \; A^{il}_m \dots + \delta^i_n \; A^{nl}_m$$

$$\delta^i_j \; A^{jl}_m = 0 + 0 + \cdots + A^{il}_m \dots + 0$$
$$= A^{il}_m$$

11.7 *EINSTEIN'S SUMMATION CONVENTION*

Let *n* quantities be denoted by $x^1, x^2, x^3, \dots x^2$ where the upper indices (subscripts) are in dentification labels and do not indicate powers. Consider the expression

$$\sum_{i=1}^{n} a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

These expressions will be written by introducing the summation convention of **Einstein;** where a index/suffice occurs twice in a term, once in the lower position and once in the upper position; a summation implied, the range of the summation being known from the context.

Example 1. Write the following using the summation convention.

$$ds^2 = g_1(dx^1)^2 + g_{22}(dx^3)^2 + g_{33}(dx^3)^2 + \ldots + g_{nn}(dx^n)^2$$

Solution: Given expression can be expressed as

$$ds^2 = g_{11}dx^1\, dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^{33} + \ldots + g_{nn}dx^ndx^n$$

Here a index occurs twice in each term, once in the lower position and once in the upper position, therefore summation convention in applicable as

$$ds^2 = g_{ij}dx^idx^j \qquad (\because \textbf{Riemannian metric})$$

Example 2. Write value of a determinant using summation convention.

Solution : Let A be a matrix of order n and let a^i_j be its j^{th} element of i^{th} row i.e., symbols I and j denoted the row and column to which the element belongs. Let A^j_i is the **cofactor** of the element a^i_j in the determinant a = det (A).

It is well-know that the sum of the products of the elements of the i^{th} row (or column) by the cofactors of the corresponding elements of the j^{th} row (column) is equal to determinant if $i = j$ and to zero if $i \neq j$ i.e.,

$$a^j_1 A^1_j + a^i_2 a^2_j + a^i_3 A^3_j + \ldots + a^i_n A^n_j = a^i_k A^k_j = \begin{bmatrix} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{bmatrix}$$

Example 3. Write matrix multiplication of two matrices using summation convention.

Solution: Let A and B are two matrices, compatible for matrix multiplication. Let a^i_j and b^j_k are $(i,\, j)^{\text{th}}$ and $(j, k)^{\text{th}}$ element of A and B respectively where i, j, k take integral value from I to m, n, p respectively. Then

$$AB = \left[\sum_{j=1}^{n} a^i_j b^j_k\right]_{\text{mxp}},\ a^i_j b^j_k = c^i_k \text{ i.e., } AB = C$$

11.8 *KRONECKER DELTA*

The symbol δ^i_j introduced by German Mathematician **L. Kronecker**, is called **Kronecker delta**, which is defined

$$\delta^{ij} = \delta_{ij} = \delta^i_j = \begin{bmatrix} 1 \ ; & i = j \\ 0 \ ; & i \neq j \end{bmatrix}$$

Properties: (i) If $x\ , x^2, \ldots x^n$ are independent co-ordinates then by differential calculus, we have

$$\frac{\partial x^i}{\partial x^j} = 0, i \neq j \text{ and } \frac{\partial x^i}{\partial x^j} = 1, i = j$$

Thus $\boxed{\dfrac{\partial x^i}{\partial x^j} = \delta^i_j}$

By Chain rule, we have

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^k}\frac{\partial x^k}{\partial x^j}$$

$$\boxed{\delta^i_j = \delta^i_k\ \delta^k_j}$$

(ii) $\delta^i_i = \delta^1_1 + \delta^2_2 + \ldots + \delta^n_n$

$= 1 + 1 + \ldots + 1$

$\boxed{\delta^i_i = n}$

(iii) $\boxed{a_{ij}\delta^i_k = a_{ik}}$ and $\boxed{a^{ij}\delta^k_j = a^{ik}}$

Since $a_{ij}\delta^l_k = \sum\limits_{j=1}^{n} a_{ij}\delta^j_k$

$$= a_{i1}\ \delta^1_k + a_{i2}\delta^2_k + \ldots a_{ik}\delta^k_k\ \ldots + a_{iA}\delta^n_k$$

$$= a_{i1}\ 0 + a_{i2}\, 0 + \ldots a_{ik}\ 1 + \ldots + a_{in}.\,0 = a_{ik}$$

Therefore, $a_{ij}\delta^j_k = a_{ik}$

Similarly, $a^{ij}\delta^j_k = \sum\limits_{j=1}^{n} a^{ij}\delta^k_j = a^{il}\delta^k_1 + a^{i2}\delta^k_2 + a^{in}\delta^k_n$

$$= a^{i1}\ .\,0 + a^{i\,2}.\,0 + \ldots + a^{ik}\ .1\ + a \ldots + a^{in}.\,0$$

$$a^{ij}\delta^k_j = a^{ik}$$

(iv) If a_{ij} are constant and $a_{ij} = a_{ji}$ then

$$\frac{\partial}{\partial x^k}(a_{ij}x_i x_j) = 2a_{ik}x_i \quad \text{and} \quad \frac{\partial^2(a_{ij}x_i x_j)}{\partial x_k \partial x_l} = 2$$

Since $\dfrac{\partial}{\partial x_k}(a_{ij}x_i x_j) = a_{ij}\dfrac{\partial(x_i x_j)}{\partial x_k}$

$$= a_{ij}\left(x_i\frac{\partial x_j}{\partial x_k} + \frac{\partial x_i}{\partial x_k}x_j\right)$$

$$= a_{ij}\ (x_i\delta^j_k + x_j\delta^i_k)$$

$$= (a_{ij}\delta^j_k)\ x_i + (a_{ij}\delta^l_k)\ x_j$$

$$= a_{ik}x_i + a_{kj}x_j$$

$$= a_{ik}x_i + a_{jk}x_j$$

$$\frac{\partial}{\partial x_k}(a_{ij}x_i x_j) = 2a_{ik}x_i \qquad \because\ j = \text{dummy index}$$

and again differentiation gives

$$\frac{\partial x^2 (a_{ij} x_i x_j)}{\partial x_k \partial x_l} = \frac{\partial}{\partial x_l} \left(\frac{\partial}{\partial x_k} (a_{ij} x_l x_j) \right)$$

$$= \frac{\partial}{\partial x_l} (2a_{ik} x_i) = 2a_{ik} \frac{\partial x_i}{\partial x_l}$$

$$\frac{\partial^2 (a_{ij} x_i x_j)}{\partial x_k \partial x_l} = 2a_{ik} \delta_l^i = 2a_{lk}$$

11.9 *TRANSFORMATION OF CO-ORDINATES*

In a 2-dimensional Euclidean space E_2, let (x^1, x^2) and (y^1, y^2) are coordinate of two points P and Q respectively. Then coordinates of vector $\overrightarrow{\mathbf{PQ}}$ are given by $(y^1 - x^1, y^2 - x^2)$.

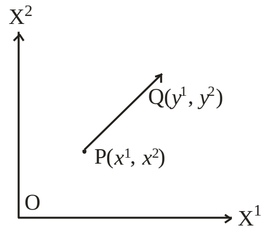

Fig. 1

Consider a simple transformation i.e., shifting origin O to O' whose coordinates are (b^1, b^2) with reference to old coordinate system. Let $o'\overline{x'}.\, o'\overline{x}^2$ are axes parallel to ox^1, ox^2 respectively. Then transformation is given by

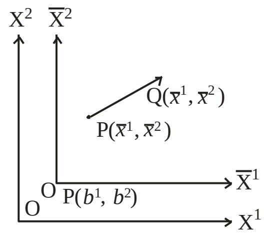

Fig. 2

$$\overline{x}^i = x^i - b^i \quad ; i = 1, 2$$

… (1)

and coordinates of points P and Q with reference to new coordinate system are $P(x^{-1}, x^{-2})$, $Q(y^{-1}, y^{-2})$ where

$$x^{-1} = x^l - b^1, x^{-2} - b^{-2}; y^{-1} - b^1, y^{-2} = y^2 - b^2$$

… (2)

Therefore coordinate of vector $\overrightarrow{\mathbf{PQ}}$ with reference to new coordinate system are given by $(y^{-1} - x^{-1}, y^{-2} - x^{-2}) = (y^1 - x^1, y^2 - x^2)$

Consider other form of transformation i.e., rotation of coordinate axes through an angle α; Which transform the coordinates by the rule

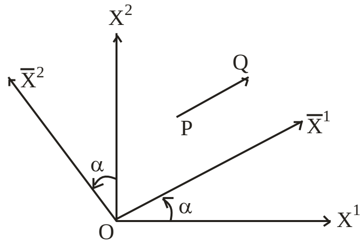

Fig. 3

$$x^{-1} = x^1 \cos\alpha + x^2 \sin\alpha,\ x^{-2} = -x^1 \sin\alpha + x^2 \cos\alpha$$

…. (3)

Thus $\quad P(x^{-1}, x^{-2}) = (x^1 \cos\alpha + x^2 \sin\alpha, -x^1 \sin\alpha + x^2 \cos\alpha)$

$$Q\,(y^{-1}, y^{-2}) = (y^1 \cos\alpha + y^2 \sin\alpha, -y^1 \sin\alpha + y^2 \cos\alpha)$$

And coordinates of $\overrightarrow{\mathbf{PQ}}$ are given

$$\overrightarrow{\mathbf{PQ}} = (y^1 - x^1)\cos\alpha + (y^2 - x^2)\sin\alpha, -(y^1 - x^1)\sin\alpha + (y^2 - x^2)\cos\alpha)$$

… (4)

In view of above discussion we observe the following facts :

(i) Magnitude of $\overrightarrow{\mathbf{PQ}}$ or distance between P and Qin invariant in both the case i.e.,

$$|\overrightarrow{\mathbf{PQ}}| = \sqrt{(y^{-1} - x^{-1}) + (y^{-2} - x^{-2})^2} = \text{ Euclidean distance.}$$

(ii) Both the transformation are reversible i.e.,

Shifting of origin : $x^i = x^{-1} + b^i \ \forall\ i = 1, 2$

Rotation of axes : $x^1 = x^{-1}\cos\alpha - x^{-2}\sin\alpha,\ x^2 = x^{-1}\sin\alpha + x^{-2}\cos\alpha$

(Solving Eqs. (1) and (3) results can easily be verified).

The existence of reversible transformation is not a coincidence instead it is based on well-known theory i.e., transformation is **orthogonal** and it can easily be verified from Eqs. (3) as

$$\begin{bmatrix} x^{-1} \\ x^{-2} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \text{ with det } (A) = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} = 1$$

or
$$AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix} = I$$

Transformation given by (3) can be expressed

$$x^{-1} = a^{11} + x^1 + x^{12}x^2;$$

$$x^{-2} = a^{21}x^1 + x^{22}x^2$$

where $a^{11} = \cos\alpha, a^{12} = \sin\alpha, a^{21} = -\sin\alpha, a^{22} = \cos\alpha$

Both the above transformation can be combined as

$$x^{-1} = a^{11}x^1 + a^{12}x^2 + b^1$$
$$x^{-2} = a^{21}x^1 + a^{22}x^2 + b^2$$

with
$$\begin{vmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{vmatrix} = 1 \neq 0$$

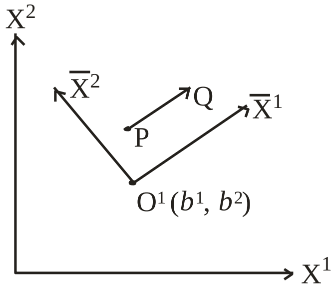

Fig .5

We can writes it as

$$x^{-i} = \sum_{j=1}^{2} a^{ij}x^j + b^i; \, i = 1, 2$$

or $$x^{-i} = a^{ij}x^j + b^i \quad \text{(by summation convention)} \quad \dots (6)$$

The new co-ordinate or $\overrightarrow{\mathbf{PQ}}$ will be $(y^{-1} - x^{-1}, y^{-2} - x^{-2})$

and we write

$$z^{-i} = y^{-1} - x^{-i} = \sum_{j=1}^{2} (a^{ij}y^j + b^i) - \sum_{j=1}^{2} (a^{ij}x^j + b^i)$$

$$z^{-i} = \sum_{j=1}^{2} a^{ij} (y^j - x^j) = \sum_{j=1}^{2} a^{ij}z^j$$

$$z^{-i} = a^{ij}z^j \qquad \text{(By summation convention)}$$

or $$z^{-i} = \frac{\partial x^{-i}}{\partial x^j} z^j \qquad \text{(From (6))} \qquad \dots (7)$$

According to **Klein,** it is law of change of transformation which the coordinates obey. Thus a vector in E^2 can be regarded as an object which is determined by a set of components of obeying the law given by (7).

Let us consider two different n-dimensional frames of reference and let $(x^1, x^2, \dots x^n)$ and $(x^{-1}, x^{-2}, x^{-3}, \dots x^{-n})$ be coordinate of a point with reference to these frames. These two systems have the following n-independent relations.

$$\left.\begin{aligned} x^{-1} &= g^1 (x^1, x^2, x^3, \dots x^n) \\ x^{-2} &= g^2 (x^1, x^2, x^3, \dots, x^n) \\ &\vdots \\ x^n &= g^n (x^1, x^2, x^3, \dots x^n) \end{aligned}\right\} \qquad \dots (8)$$

where $g^1, g^2, \dots g^n$ are the single-value continuous function of $x^1, x^2, x^3 \dots x^n$ and have continuous partial derivatives upto desire order. The necessary and sufficient condition for reversables transformation is automatically fulfilled by choice of n-independent relations given by (8). This condition is also expressed in terms of **"functional determinant", J** formed by the partial derivatives $\frac{\partial x^{-i}}{\partial x^j}$ as

$$J = \frac{\partial(x^{-1}, x^{-2}, \dots x^{-n})}{\partial\,(x^1, x^2, \dots x^n)} = \begin{vmatrix} \frac{\partial x^{-1}}{\partial x^1}, \frac{\partial x^{-1}}{\partial x^2}, \dots \frac{\partial x^{-1}}{\partial x^2} \\ \frac{\partial x^{-2}}{\partial x^1}, \frac{\partial x^{-2}}{\partial x^2}, \dots \frac{\partial x^{-2}}{\partial x^n} \\ \vdots \\ \frac{\partial x^{-n}}{\partial x^1}, \frac{\partial x^{-n}}{\partial x^2}, \dots \frac{\partial x^{-n}}{\partial x^n} \end{vmatrix} \neq 0$$

Under this condition, equation (8) can be solved for x^i as

$$x^i = h^i\,(x^{-1}, x^{-2}, \dots x^{-n});\ i = 1, 2, 3 \dots n \qquad \dots (9)$$

The class of coordinate transformation satisfying these properties as **"admissible transformations".**

- **Spherical Polar-Coordinates**

Let P be point in space and let OP = r. The lines OP, OZ, OX are regarded as radial, polar and equatorial axes respectively, where plane XOY is called equatorial plane.

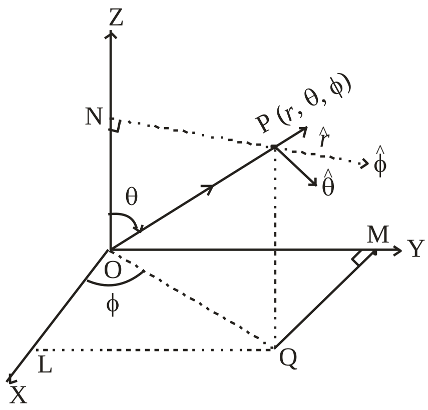

Fig .5

Let OQ is projection of OP an equatorial plane. Suppose θ is angle between OP and polar axis and ϕ is angle between OQ and equatorial angle. This three variable r, θ, ϕ are related to point P and are called **radial, polar** and **azimuthual** coordinates of point P.

Let, L, M, N are points on OX, OY, OZ axes respectively such that they are foot of perpendicular drawn from Qand Prespectively.If (x, y, z) are coordinates of P in rectangular coordinates system then

$$x = OL,\ y = OM,\ z = ON \qquad \dots (1)$$

If $\rho = OQ$ be projections of OP, then

$$OQ = \rho = OP\cos(\angle POQ) \text{ i.e., } \rho = r\cos\left(\frac{\pi}{2} - \theta\right) = r\sin\theta$$

i.e., $\rho = r\sin\theta$

The geometrical correspondence between coordinate system with common origin is

$$\left.\begin{aligned} x &= OQ\cos\phi = r\sin\theta\cos\phi \\ y &= OQ\cos\left(\frac{\pi}{2} - \phi\right) = r\sin\theta\sin\phi \\ z &= OP\cos\theta = r\cos\theta \end{aligned}\right\} \qquad \dots (2)$$

In vector notation, $\bar{r} = |r|\,\hat{r}$

i.e., $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,j + \cos\theta k$

Similarly, $\hat{\theta} = \hat{r}\left(\theta + \dfrac{\pi}{2}, \phi\right) = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,j - \sin\theta k$

$$\phi = \hat{r}\left(\theta = \frac{\pi}{2}, \phi + \frac{\pi}{2}\right) = -\sin\phi\,\hat{i} + \cos\phi\,j$$

The range of r, θ, ϕ is $0 < r < \theta, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$

Remarks: (1) $\hat{r}$ is along $\overrightarrow{\mathbf{OP}}$ i.e., as r increases.

(2) $\hat{\theta}$ is perpendicular to OP i.e., as θ increases.

(3) $\hat{\phi}$ is in a plane parallel to the equatorial plane.

(4) $r =$ Constant, represents the surface of sphere.

(5) $\theta =$ Constant, represents the surface of cone with vertex at the origin and polar axis as its axis.

(6) $\phi =$ Constant, represents a half-plane bounded on one side by the entire polar axis.

- **Cylindrical Coordinate**

In cylindrical coordinate system, z-axis and x-axis i.e., polar and equatorial axes are kept same as in spherical coordinate system. The third axis is take along equatorial projection OP i.e., along OQ and is called cylindrical radial coordinate axis such that $OQ = \rho$. Then cylindrical polar coordinates are (ρ, ϕ, z) with

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z \qquad \ldots (1)$$

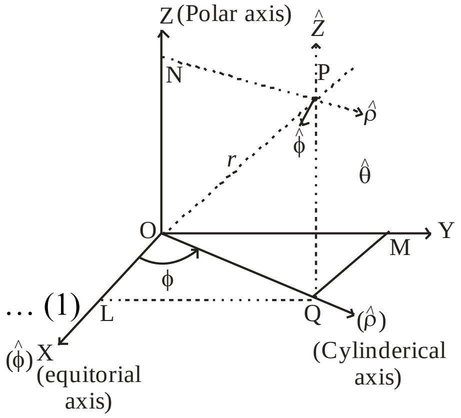

Fig .6

In vector notation,

$$\hat{r} = \rho\cos\phi\,\hat{\mathbf{i}} + \rho\sin\phi\,\hat{\mathbf{j}} + z\hat{k}$$

$$\bar{\rho} = \rho\,\hat{\rho} \quad \text{i.e.,} \quad \hat{\rho} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi j \quad \text{as} \quad \hat{\phi} = \hat{\rho}\left(\phi + \frac{\pi}{2}\right) \qquad \ldots (2)$$

$$\hat{z} = \hat{k}$$

The range of $\rho > 0, 0 \le \phi \le 2\pi; -\infty < z < \infty$

Remarks:

(1) ρ is along $\overrightarrow{\mathbf{OQ}}$ i.e., as ρ increases

(2) ϕ is along perpendicular to $\overrightarrow{\mathbf{OQ}}$ as ϕ increases and lies in plane parallel to equitorial plane.

(3) $\hat{\mathbf{z}}$ is along z-axis.

(4) $\rho =$ constant, represents the surface of infinite cylinder.

(5) $\phi =$ constant, represents vertical plane.

(6) $z =$ constant, represents horizontal plane.

Note : (1) The distinction between the terms spherical polar and cylindrical polar is due to the fact that $r =$ constant represents the surface of sphere but $\rho =$ constant represents the surface of cylinder infinite.

(2) Both are polar due to symmetry about the axes of z.

(3) The plane polar coordinate system is a special case of both the spherical and cylindrical polar coordinates, i.e., (ρ, ϕ)

Example 1. Find the cylindrical coordinate system in terms of the rectangular Cartesian coordinate system.

Solution: Let (x^1, x^2, x^3) and (y^1, y^2, y^3) be coordinates of a point in certesian and cylindrical coordinates. Then

$$x^1 = y^1 \cos y^2, x^2 = y^1 \sin y^2 \; x^3 = y^3$$

… (1)

and
$$J = \begin{vmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \frac{\partial x^1}{\partial y^3} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \frac{\partial x^2}{\partial y^3} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} & \frac{\partial x^3}{\partial y^3} \end{vmatrix} = \begin{vmatrix} \cos y & -y \sin y^2 & 0 \\ \sin y^2 & y^1 \cos y^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = y^1 \neq 0$$

If $y^1 \neq 0$ then

$$(x^1)^2 + (x^2)^2 = (y^1)^2; \; y^2 = \tan^{-1}\left(\frac{x^2}{x^1}\right)$$

and
$$y^1 = [(x^1)^2 + (x^2)^2]^{1/2}, \; y^2 = \tan^{-1}\left(\frac{x^2}{x^1}\right), \; y^3 = x^3 \qquad \dots (2)$$

Example 2. Find the spherical coordinate system in terms of the rectangular cartesian coordinate system.

Solution : Let (x^1, x^2, x^3) and (y^1, y^2, y^3) be coordinates of a point in rectangular cartesian and spherical coordinates respectively. Then

$$x^1 = y^1 \sin y^2 \cos y^3, x^2 = y^1 \sin y^2 \sin y^3, x^3 = y^1 \cos y^2 \qquad \dots (1)$$

and
$$J = \begin{vmatrix} \sin y^2 \cos y^3 & y^1 \cos y^2 \cos y^3 & -y^1 \sin y^2 \sin y^3 \\ \sin y^2 \sin y^3 & y^1 \cos y^2 \sin y^3 & y^1 \sin y^2 \cos y^3 \\ \cos y^2 & -y^1 & 0 \end{vmatrix}$$

$$= (y^1)^2 \sin(y^2) \neq 0 \text{ as } y_1 > 0, 0 < y^2 < \pi$$

Therefore $(x^1)^2+(x^2)^2+(x^3)^2=(y^1)^2$,

$$y^2=\cos^{-1}\left(\frac{x^3}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}}\right)$$

$$y^3=\tan^{-1}\left(\frac{x^2}{x^1}\right)$$

Example 3. If $\left(\frac{6}{\sqrt{2}},\frac{6}{\sqrt{2}},1\right)$ are rectangular coordinates then find the spherical and cylindrical coordinates.

Solution: Here $x^1=\frac{6}{\sqrt{2}}$, $x^2=\frac{6}{\sqrt{2}}$, $x^3=1$. Therefore

(I) $$y^1=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}=\sqrt{\frac{36}{2}+\frac{36}{2}+1}=\sqrt{37}$$

$$y^2=\cos^{-1}\left(\frac{1}{\sqrt{37}}\right),\ y^3=\tan^{-1}\left(\frac{\frac{6}{\sqrt{2}}}{\frac{6}{\sqrt{2}}}\right)=\frac{\pi}{4}$$

Thus, $(y^1,y^2,y^3)=\left(\sqrt{37},\cos^{-1}\left(\frac{1}{37}\right),\frac{\pi}{4}\right)$

(I) $$y^1=\sqrt{(x^1)^2+(x^2)^2}=\sqrt{\frac{36}{2}+\frac{36}{2}}=6$$

$$y^2=\tan^{-1}\left(\frac{x^2}{x^1}\right)=\frac{\pi}{4},\ y^3+x^3=1$$

Thus $(y^1,y^2,y^3)=\left(6,\frac{\pi}{4},1\right)$

Example 4. If $\left(4,\frac{\pi}{3},1\right)$ is cylindrical coordinates then find cartesian coordinates and spherical coordinates.

Solution: Here, $y^1=4$, $y^2=\frac{\pi}{3}$, $y^3=1$. Therefore,

(I) Let (x^1,x^2,x^3) be Cartesian coordinates. Then

$$x^1=y^1\cos y^2,\ x^2=y^1\sin y^2,\ x^3=y^3$$

$$x^1=4\cos\left(\frac{\pi}{3}\right)=2,\ x^2=4\sin\left(\frac{\pi}{3}\right)=2\sqrt{3},\ x^3=1$$

i.e., $$(x^1,x^2,x^3)=(2,2\sqrt{3},1)$$

(II) Let (z^1, z_2^2, z_2^3) be spherical polar coordinates of $(x^1, x^2, x^3) = (2, 2\sqrt{3}, 1)$. Then

$$z^1 = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{4+12+1} = \sqrt{17}$$

$$z^2 = \cos^{-1}\left(\frac{x^3}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{17}}\right)$$

$$z^3 = \tan^{-1}\left(\frac{x^2}{x^1}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Thus $(z^1, z^2, z^3) = \left(\sqrt{17}, \cos^{-1}\left(\frac{1}{\sqrt{17}}\right), \frac{\pi}{3}\right)$.

11.10 *SUMMARY*

1. The symbol δ_j^i introduced by German Mathematician **L. Kronecker**, is called **Kronecker delta**, which is defined

$$\delta^{ij} = \delta_{ij} = \delta_j^i = \begin{bmatrix} 1 \ ; & i = j \\ 0 \ ; & i \neq j \end{bmatrix}$$

2. **Superscript and Subscript:** The suffixes i and j in B_j^i are called superscript and subscript respectively. The upper position always denotes the superscript and the lower position denotes subscript.

11.11 *GLOSSARY*

(i) Derivatives

(ii) Determinant

(iii) Vector

11.12 *REFERENCES AND SUGGESTED READINGS*

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iii. Tensors and Differential geometry, Dr. P. P. Gupta, Dr. H. D. Pandey, Prof. G. S. Malik, 1997 Pragati Prakashan Meerut.

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11.13 *TERMINAL QUESTIONS*

(TQ-1) Write each of the following by using summation convention

(i) $a_1x^1x^3 + a_2x^2x^3 + \cdots + a_nx^nx^3$

(ii) $\frac{d\bar{x}^k}{dt} = \frac{\partial\bar{x}^k}{\partial x^1}dx^1 + \frac{\partial\bar{x}^k}{\partial x^2}dx^2 + \frac{\partial\bar{x}^k}{\partial x^3}dx^3$

(iii) $d\theta = \frac{\partial\theta}{\partial x^1}dx^1 + \frac{\partial\theta}{\partial x^2}dx^2 + \ldots + \frac{\partial\theta}{\partial x^n}dx^n$

(TQ-2) Write the terms in each of the following indicated sums

(i) $a_{ij}x^j$

(ii) $\frac{\partial(\sqrt{g}A^i)}{\partial x^i}$

11.14 *ANSWERS*

(TQ-1(i)) $a_ix^ix^3$

(TQ-1(ii)) $\frac{d\bar{x}^k}{dt} = \frac{\partial\bar{x}^k}{\partial x^p}dx^p, p = 1,2,3 \ or \ n = 3$

(TQ-1(iii)) $d\theta = \frac{\partial\theta}{\partial x^i}dx^i$

(TQ-2(i)) $a_{i1}x^1 + a_{i2}x^2 + \cdots a_{in}x^n$

(TQ-2(ii)) $\frac{\partial(\sqrt{g}A^1)}{\partial x^1} + \frac{\partial(\sqrt{g}A^2)}{\partial x^2} + \cdots \frac{\partial(\sqrt{g}A^n)}{\partial x^n}$

UNIT 12: CHRISTOFFEL SYMBOLS OR CHRISTOFFEL BRACKETS

CONTENTS:

12.1 *INTRODUCTION*

In mathematics and physics, the Christoffel symbols are an array of numbers describing a metric connection. The metric connection is a specialization of the affine connection to surfaces or other manifolds endowed with a metric, allowing distances to be measured on that surface. The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor g_{ik}: As a shorthand notation, the nabla symbol and the partial derivative symbols are frequently dropped, and instead a semicolon and a comma are used to set off the index that is being used for the derivative.

12.2 *OBJECTIVES*

After studying this unit Learner will be able to

i. Understand the relationship between first and second kind Christoffel symbols.
ii. Understand the covariant derivatives of vector are tensor.
iii. Understand the Christoffel symbol are not tensors.
iv. Understand the tensor form of differential operators.
v. Understand the Ricci Tensor and Ricci theorem.

12.3 *CONSTANT VECTOR FIELD*

Let a vector field $\vec{\mathbf{A}}$ is constant in cartesian coordinates as $\overline{A}$. Transforming the curvilinear system x^{-i} to cartesian x^i, we get

$$A^{-i} = \frac{\partial x^{-i}}{\partial x^j} . A^j \qquad \dots (1)$$

The $C = C(t)$ be curve in the space and observe the change in A^{-i}, when we move on the curve by an infinitesimally small step *dt* as

$$\frac{dA^{-i}}{dt} = \frac{\partial^2 x^{-i}}{\partial x^j \partial x^k} \frac{\partial x^k}{dt} A^j + \frac{\partial x^{-i}}{\partial x^j} \frac{\partial A^j}{\partial t} \qquad \dots (2)$$

Since A^{-i} is constant in Cartesian coordinates therefore $\frac{dA^{-i}}{dt} = 0$ and Eq. (2) becomes

$$\frac{\partial^2 x^{-i}}{\partial x^j \partial x^k}\frac{\partial x^k}{dt}A^j + \frac{\partial x^{-i}}{\partial x^j}\frac{\partial A^j}{dt} = 0 \qquad \ldots (3)$$

Which shows that a vector field is expressed in curvilinear coordinates.

12.4 *CHRISTOFFEL SYMBOLS*

Multiplying equation (3) by $g^{rp}\dfrac{\partial x^{-i}}{\partial x^p}$ we get

$$g^{rp}\frac{\partial x^{-i}}{\partial x^p}\frac{\partial^2 x^{-i}}{\partial x^j \partial x^k}\frac{\partial x^k}{dt}A^j + g^{rp}\frac{\partial x^{-i}}{\partial x^p}\frac{\partial x^{-i}}{\partial x^j}\frac{dA^j}{dt} = 0$$

$$g^{rp}\frac{\partial x^{-i}}{\partial x^p}\frac{\partial^2 x^{-i}}{\partial x^j \partial x^k}\frac{\partial x^k}{\partial t}A^j + g^{rp}\,g_{pj}\frac{dA^j}{dt} = 0 \qquad \because g_{ij} = \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}$$

$$g^{rp}\frac{\partial x^{-i}}{\partial x^p}\frac{\partial^2 x^{-i}}{\partial x^j \partial x^k}\frac{\partial x^k}{\partial t}A^j + \delta^r_j\frac{dA^j}{dt} = 0$$

$$g^{rp}\left(\frac{\partial x^{-i}}{\partial x^p}\frac{\partial^2 x^{-i}}{\partial x^j\,\partial x^k}\right)\frac{\partial x^k}{\partial t}A^j + \frac{dA^r}{dt} = 0$$

$$g^{rp}[i,\ j,\ p]\frac{\partial x^k}{\partial t}A^j + \frac{dA^r}{dt} = 0 \qquad \ldots (4)$$

Where $[jk, p]$ is called **Chrisloffel symbol of first kind.**

and
$$[jk,\ p] = \frac{\partial x^{-i}}{\partial x^p}\frac{\partial^2 x^{-i}}{\partial x^j\,\partial x^k}$$

Find partial derivatives of the metric tensor g_{ij} w.r.t. to all three coordinates as

Since
$$g_{ij} = \frac{\partial x^t}{\partial x^i}\frac{\partial x^j}{\partial x^j}$$

therefore,
$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial x^t}{\partial x^i}\frac{\partial^2 x^l}{\partial x^j \partial x^k} + \frac{\partial^2 x^l}{\partial x^i\,\partial x^k}\frac{\partial x^l}{\partial x^j} \qquad \ldots (5)$$

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial x^l}{\partial x^j}\frac{\partial^2 x^l}{\partial x^k \partial x^i} + \frac{\partial^2 x^l}{\partial x^j\,\partial x^i}\frac{\partial x^l}{\partial x^k} \qquad \ldots (6)$$

$$\frac{\partial g_{ki}}{\partial x^j} = \frac{\partial x^l}{\partial x^k}\frac{\partial^2 x^l}{\partial x^i\,\partial x^j} + \frac{\partial^2 x^l}{\partial x^k \partial x^j}\frac{\partial x^l}{\partial x^i} \qquad \ldots (7)$$

Adding (6) and (7) then subtracting (5) we get

$$\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] = 2\frac{\partial x^l}{\partial x^k}\frac{\partial^2 x^l}{\partial x^i\,\partial x^j}$$

Thus,
$$[ij, k] = \frac{\partial x^l}{\partial x^k}\frac{\partial^2 x^l}{\partial x^i \partial x^j} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad \ldots (8)$$

It is event that in rectangular Cartesian coordinates

$$g_{ij} = \delta_{ij} \text{ and } \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \delta_{ij}}{\partial x^k} = 0$$

Therefore $[ij, k]$ is always zero and equation (4) reduces to

$$\frac{dA^r}{dt} = 0$$

It is expected result i.e., A^r is constant Cartesian coordiantes.

Introduing $\begin{Bmatrix} l \\ ij \end{Bmatrix} = g^{lk}\,[ij, k]$, know as the Christoffel symbol of second kind, the equation (4) can be simplified to

$$\frac{dA^r}{dt} + \begin{Bmatrix} r \\ jk \end{Bmatrix} \frac{\partial x^k}{dt} A^j = 0 \qquad \ldots (9)$$

or

$$\frac{\partial A^r}{\partial x^k}\frac{dx^k}{dt} + \begin{Bmatrix} r \\ jk \end{Bmatrix} \frac{dx^k}{dt} A^j = 0$$

$$\left(\frac{\partial A^r}{\partial x^k} + \begin{Bmatrix} r \\ jk \end{Bmatrix} A^j\right)\frac{dx^k}{dt} = 0 \qquad \ldots (10)$$

Since $\frac{dx^k}{dt} \neq 0$ along a general curve $c(t)$ therefore equation (10) can be written as

$$\frac{dA^r}{dx^k} + \begin{Bmatrix} r \\ jk \end{Bmatrix} A^j = 0 \qquad \ldots (11)$$

This is the differential equation of constant vector field in any Riemannian space. In case of covariant vector

$$\frac{dA_r}{\partial x^k} - \begin{Bmatrix} j \\ rk \end{Bmatrix} A_j = 0 \qquad \ldots (12)$$

Finally, the Christoffel symbols are defined as

$$\left.\begin{aligned} [ij, k] &= \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \\ \begin{Bmatrix} k \\ ij \end{Bmatrix} &= \Gamma^k_{ik} = g^{kl}\,[ij, l] \end{aligned}\right\}$$

and equations for parallel transmission are

$$\frac{\partial A^i}{\partial x^j} + \Gamma^k_{ik} A^k = 0$$

$$\frac{\partial A^i}{\partial x^j} - \Gamma^k_{ik} A^k = 0$$

Theorem: The Christoffel is symbol $[ij, k]$ and $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ are symmetric with respect to the indices i and j.

Proof: The christoffel symbol of first kind is

$$[ij,k]=\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x_k}\right] \qquad \dots (1)$$

Interchanging *i* and *j* we get

$$[ij,k]=\frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ji}}{\partial x^k}\right]$$

$$=\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right] \qquad \text{as} \quad g_{ij}=g_{ji}$$

$$[ij,k]=[ji,k] \qquad \dots (2)$$

The Christoffel symbol of second kind is

$$\begin{Bmatrix} k \\ ij \end{Bmatrix}=g^{kh}\,[ij,h] \qquad \dots (3)$$

Interchanging *i* and *j*

$$\begin{Bmatrix} k \\ ji \end{Bmatrix}=g^{kh}\,[ji,h]=g^{kh}\,[ij,h] \qquad \text{(From (2))}$$

$$\begin{Bmatrix} k \\ ji \end{Bmatrix}=\begin{Bmatrix} k \\ ij \end{Bmatrix} \qquad \dots (4)$$

Theorem: Prove that

(i) $[ij,m]=g_{km}\begin{Bmatrix} k \\ ij \end{Bmatrix}$

(ii) $[ik,\ j]+[jk,i]=\dfrac{\partial g_{ij}}{\partial x_k}$

(iii) $\dfrac{\partial g_{ij}}{\partial x^k}=-\,g^{jl}\begin{Bmatrix} i \\ lk \end{Bmatrix}-g^{im}\begin{Bmatrix} j \\ mk \end{Bmatrix}$

Proof: (i) Multiplying Christoffel symbol of second by g_{mk} we get

$$g_{mk}\begin{Bmatrix} k \\ ij \end{Bmatrix}=g_{km}g^{kl}\,[ij,l]$$

$$=\delta^l_m[ij,l]$$

$$g_{km}\begin{Bmatrix} k \\ ij \end{Bmatrix}=[ij,m]$$

(ii) By def. of Chrestoffel symbol of second kind, we directly obtain

$$[ij,k]+[jk,l]=\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{ji}}{\partial x^k}\right)=\frac{\partial g_{ij}}{\partial x_k} \qquad \because g_{ji}=g_{ji}$$

(iii) Differentiation of $g^{ij}g_{lj}=\delta^i_l$ w.r.t. x^k gives

$$g^{ij}\frac{\partial g_{lk}}{\partial g^{k}}+g_{lj}\frac{\partial g^{ij}}{\partial x^{k}}=0$$

Multiplying by g^{lm} we get

$$g^{lm}g^{ij}\frac{\partial x_{lj}}{\partial x^{k}}+g^{lm}g_{lj}\frac{\partial g^{ij}}{\partial x^{k}}=0$$

$$g^{lm}g^{ij}\frac{\partial g_{lj}}{\partial x^{k}}+\delta^{m}_{j}\frac{\partial g^{ij}}{\partial x^{k}}=0$$

$$g^{lm}g^{ij}\frac{\partial x_{lj}}{\partial g^{k}}+\frac{\partial g^{im}}{\partial g^{k}}=0$$

i.e., $$\frac{\partial g^{lm}}{\partial x^{k}}=-\,g^{lm}[g^{ij}([lk,\ j]+[\,jk,l\,])] \qquad \text{(From (ii))}$$

$$=-\,g^{lm}\begin{Bmatrix} i \\ lk \end{Bmatrix}=g^{ij}\begin{Bmatrix} m \\ jk \end{Bmatrix}$$

$$\frac{\partial g^{ij}}{\partial x^{k}}=-\,g^{lj}\begin{Bmatrix} i \\ lk \end{Bmatrix}-g^{lm}\begin{Bmatrix} m \\ mk \end{Bmatrix}\ \text{ as } g^{ij}=g^{ji}$$

Theorem : Prove that $\begin{Bmatrix} i \\ ij \end{Bmatrix}=\dfrac{\partial(\log\sqrt{g})}{\partial x^{j}}$

Proof : As we have already prove that

$$\frac{\partial g}{\partial x^{k}}=\text{ cofactor of } g_{ij}\frac{\partial g_{ij}}{\partial x^{k}}$$

$$\frac{1}{g}\frac{\partial g}{\partial x^{k}}=\frac{\text{Cofactor of } g_{ij}}{g}\frac{\partial g_{ij}}{\partial x^{k}}$$

$$\frac{\partial}{\partial x^{k}}(\log g)=g^{ij}\frac{\partial g_{ij}}{\partial x^{k}}$$

$$=g^{ij}\ [[ik,\ j]+[lk.\,i]] \qquad \text{(From previous theorem)}$$

$$=\begin{Bmatrix} i \\ ik \end{Bmatrix}+\begin{Bmatrix} j \\ kj \end{Bmatrix}$$

$$=2\begin{Bmatrix} i \\ ij \end{Bmatrix}\ \text{ as } k \text{ is dummy suffices.}$$

Example 1. If $g_{ij}=0$ for $i\neq j$ then

(i) $\begin{Bmatrix} k \\ ij \end{Bmatrix}=0$ for $i\neq j\neq k$

(ii) $\begin{Bmatrix} i \\ ii \end{Bmatrix}=\dfrac{1}{2}\dfrac{\log g_{ii}}{\partial x^{i}}$

(iii) $\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{1}{2} \frac{\partial \log g_{ii}}{\partial x^j}$

(iv) $\begin{Bmatrix} i \\ jj \end{Bmatrix} = -\frac{1}{2} \frac{1}{g_i} \frac{\partial g_{jj}}{\partial x^i}$

Solution : We have

$$[ij, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad \ldots (1)$$

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kh} [ij, h] \qquad \ldots (2)$$

and $\quad g_{ij} = 0$ when $i \neq j$ and $g^{ii} = \frac{1}{g_{it}}$; $g^{ij} = 0$ if $i \neq j \qquad \ldots (3)$

Then (i) For $i \neq j \neq k$; $g_{ij} = g_{jk} = g_{ik} = 0$ and by (1) $[ij, k] = 0 = [ij, h]$ (From (3))

Using in (2) we have

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kh}\, 0 = 0$$

(ii) $$\begin{Bmatrix} i \\ ii \end{Bmatrix} = g^{ii} [ii, i]$$

From (2))

$$= g^{ii} \frac{1}{2}\left[\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right] \qquad \text{(From (1))}$$

$$= g^{ii} \frac{1}{2} \frac{\partial g_{ii}}{\partial x^i}$$

$$= \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x^i}$$

$$= \frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^i} \qquad \because g^{ii} = \frac{1}{g_{ii}} \text{ as } g_{ij} = 0 \text{ if } i \neq j$$

$$\begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{1}{2} \frac{\partial (\log g_{ii})}{\partial x^i}$$

(iii) $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = g^{ii} [ij, i]$$

$$= \frac{1}{2} g^{ii} \left[\frac{\partial g_{ji}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^i}\right]$$

$$= \frac{1}{2} \frac{1}{g^{ii}} \left[\frac{\partial g_{ii}}{\partial x^j}\right] = \frac{1}{2} \frac{\partial}{\partial x^j} (\log g_{ii}) \qquad \because g^{ii} = \frac{1}{g_{ii}}$$

(iv) $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = g^{ii} \{jj, i\}$$

$$= \frac{1}{2} g^{ii} \left[\frac{\partial g_{ji}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right] \text{as } g_{ij} = 0 \text{ if } i \neq j$$

$$\left\{ \begin{matrix} i \\ jj \end{matrix} \right\} = -\frac{1}{2} \frac{1}{g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

Example 2. If $ds^2 = dr^2 + r^2 d\theta + r^2 \sin\theta \, d\phi^2$, find

$$[22, 1], [13, 3], \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}, \left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\}$$

Solution : We have

$$ds^2 = (dr)^2 + (r\, d\theta)^2 + (r \sin\theta)^2 \, d\phi$$

i.e., $g_{11} = 1, g_{22} = r^2, g_{33} = r^2 \sin^2\theta, g_{ij} = 0, \text{ when } i \neq j$

Then by result, $g^{ii} = \dfrac{1}{g_{ii}}$ and $g^{ij} = 0 \ \forall \ i \neq j$, we get

$$g^{11} = 1, g^{22} = \frac{1}{r^2}, g^{33} = \frac{1}{r^2 \sin^2\theta}; g^{12} = g^{13} = g^{23} = 0 \qquad \ldots (2)$$

(i) By definition of Christoffel symbol first kind

$$[ij, k] = \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial x_{ij}}{\partial x^k} \right]$$

$$[22, 1] = \frac{1}{2} \left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right]$$

$$= \frac{1}{2} \left[0 + 0 - \frac{\partial r^2}{\partial r} \right] = -r \qquad \text{(From (1))}$$

$$[13, 3] = \frac{1}{2} \left[\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3} \right]$$

$$= \frac{1}{2} \left[\frac{\partial (r^2 \sin^2\theta)}{\partial r} + 0 - 0 \right]$$

$$= r \sin^2\theta$$

(ii) By definition of Christoffel symbol of second kind

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = g^{kh} \, [ij, h]$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = g^{1h} \, [22, h]$$

$$= g^{11} \, [22, 1] + g^{12} \, [22, 2] + g^{13} \, [22, 3]$$

$$= 1 \, [22, 1] + 0 + 0 \qquad \text{(From (2))}$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -r \qquad \text{(From (1))}$$

and $$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = g^{3h}\,[13, h]$$

$$= g^{31}\,[13,1] + g^{32}\,[13,2] + g^{33}\,[13,3]$$

$$= 0 + 0 + \frac{1}{r^2 \sin^2 \theta}\,[13,3] \qquad \text{(From (2))}$$

$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{r}$$

(From (1))

12.5 *LAW OF TRANSFORMATION OF CHRISTOFFEL SYMBOL OF FIRST KIND*

From the law of transformation of covariant tensor of type (0, 2) we have

$$\bar{g}_{lm} = g_{ij} \frac{\partial x^i}{\partial x^{-l}} \frac{\partial x^j}{\partial x^{-m}} \qquad \dots (1)$$

Different w.r.t. x^{-n} we get

$$\frac{\partial x_{lm}}{\partial x^{-n}} = \frac{\partial g_{ij}}{\partial x^k} \cdot \frac{\partial x^k}{\partial x^{-n}} \cdot \frac{\partial x^i}{\partial x^{-l}} \cdot \frac{\partial x^j}{\partial x^{-m}} + g_{ij}\left(\frac{\partial^2 x^i}{\partial x^{-n} \partial x^{-l}} \frac{\partial x^j}{\partial x^{-m}} + \frac{\partial x^i}{\partial x^{-l}} \frac{\partial^2 x^j}{\partial x^{-n} \partial x^{-m}}\right)$$

... (2)

Similarly,

$$\frac{\partial g_{\ln}}{\partial x^{-m}} = \frac{\partial x_{ij}}{\partial x^k} \frac{\partial x^k}{\partial x^{-m}} \frac{\partial x^i}{\partial x^{-l}} \frac{\partial x^j}{\partial x^{-l}} + g_{ij}\left(\frac{\partial^2 x^i}{\partial x^{-m} \partial x^{-l}} \frac{\partial x^j}{\partial x^{-n}} + \frac{\partial x^i}{\partial x^{-l}} \frac{\partial x^2 x^j}{\partial x^{-m} \partial x^{-n}}\right) \qquad \dots (3)$$

$$\frac{\partial g_{mn}}{\partial x^{-n}} = \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^k}{\partial x^{-l}} \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} + g_{ij}\left(\frac{\partial^2 x^i}{\partial x^{-l} \partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} + \frac{\partial x^i}{\partial x^{-m}} \frac{\partial^2 x^j}{\partial x^{-l} \partial x^{-n}}\right)$$

... (4)

Using (2) and (4) in

$$\overline{[lm, n]} = \frac{1}{2}\left[\frac{\partial g_{\ln}}{\partial x^{-m}} + \frac{\partial g_{mn}}{\partial x^{-l}} - \frac{\partial g_{lm}}{\partial x^{-n}}\right]$$

We obtain

$$\overline{[lm, n]} = [ik,\ j] \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} \frac{\partial x^k}{\partial x^{-l}} + g_{ij} \frac{\partial^2 x^i}{\partial x^{-l} \partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} \qquad \dots (5)$$

The exitance of 2nd term in R.H.S. of (5) shows that the Chirisloffel symbol of first kind is not a tensor.

12.6 LAW OF TRANSFORMATION OF CHRISTOFFEL SYMBOL FOR SECOND KIND

Taking inner product of (5) with $g^{-ns} = g^{pq} \dfrac{\partial x^{-n}}{\partial x^p} \dfrac{\partial x^{-s}}{\partial x^q}$

we get

$$g^{-ns}\,\overline{[lm, n]} = [ik,\ j] \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} \frac{\partial x^k}{\partial x^{-l}} g^{pq} \frac{\partial x^{-n}}{\partial x^p} \frac{\partial x^{-s}}{\partial x^q}$$

$$+ g_{ij} g^{pq} \frac{\partial^2 x^i}{\partial x^{-l} \partial x^{-m}} \frac{\partial x^j}{\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^p} \frac{\partial x^{-s}}{\partial x^q}$$

$$\overline{\begin{Bmatrix} s \\ lm \end{Bmatrix}} = g^{pq}\,[ik,\ j] \frac{\partial x^i}{\partial x^{-m}} \delta_p^j \frac{\partial x^k}{\partial x^{-l}} \frac{\partial x^{-s}}{\partial x^q} + g_{ij} g^{pq} \frac{\partial^2 x^i}{\partial x^{-l} \partial x^{-m}} . \delta_p^j \frac{\partial x^{-s}}{\partial x^q}$$

$$= \left(g^{jq}\,[ik,\ j] \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^k}{\partial x^{-l}} + g_{ij} g^{js} \frac{\partial^2 x^i}{\partial x^{-l} \partial x^{-m}} \right) \frac{\partial x^{-s}}{\partial x^q}$$

$$\overline{\begin{Bmatrix} s \\ lm \end{Bmatrix}} = \begin{Bmatrix} q \\ ik \end{Bmatrix} \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^k}{\partial x^{-l}} \frac{\partial x^{-s}}{\partial x^q} + \frac{\partial^2 x^q}{\partial x^{-l} \partial x^{-m}} \frac{\partial x^{-s}}{\partial x^q} \qquad \dots (1)$$

The existance of 2nd term in R.H.S. of above equation shows that the Christoffel symbol of second kind is not a tensor.

This result can be expressed as

$$\frac{\partial^2 x^q}{\partial x^{-l} \partial x^{-m}} = \overline{\begin{Bmatrix} s \\ lm \end{Bmatrix}} \frac{\partial x^q}{\partial x^{-s}} - \begin{Bmatrix} q \\ ik \end{Bmatrix} \frac{\partial x^i}{\partial x^{-m}} \frac{\partial x^k}{\partial x^{-l}}$$

$$\frac{\partial^2 x^q}{\partial x^{-l} \partial x^{-m}} = \overline{\begin{Bmatrix} s \\ lm \end{Bmatrix}} \frac{\partial x^q}{\partial x^{-s}} - \begin{Bmatrix} q \\ ik \end{Bmatrix} \frac{\partial x^k}{\partial x^{-l}} \frac{\partial x^i}{\partial x^{-m}} \qquad \dots (2)$$

Note : Similarly result can easily be obtained

$$\frac{\partial^2 x^q}{\partial x^{-l} \partial x^{-m}} = \overline{\begin{Bmatrix} s \\ lm \end{Bmatrix}} \frac{\partial x^q}{\partial x^{-s}} - \overline{\begin{Bmatrix} q \\ ik \end{Bmatrix}} \frac{\partial x^{-k}}{\partial x^{-l}} \frac{\partial x^{-i}}{\partial x^{-m}}$$

12.7 *COVARIANT DIFFERENTIATION OF VECTORS*

We have already studied the algebraic operations on tensors and a natural question arises, whether differentiation of tensors, can produces a tensor or not. But it has been shown that partial differentiation of an invariant is a tensor and partial differentiation of a covariant vector of order ≥ 1 is not a tensor in general. Thus there is an urgent need to introduce a new kind of differentiation when applied to tensors will produce again tensors.

12.7.1 Covariant Differentiation of a Covariant Vector

From the transformation of a covariant vector

$$\overline{A}_k = A_i \frac{\partial x^i}{\partial x^{-k}} \qquad \ldots (1)$$

Differentiation w.r.t. x^{-j} we get

$$\frac{\partial \overline{A}_k}{\partial x^{-j}} = \frac{\partial A_i}{\partial x^m}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^i}{\partial x^{-k}} + A_i \frac{\partial^2 x^i}{\partial x^{-j}\partial x^{-k}}$$

Using result

$$\frac{\partial^2 x^i}{\partial x^{-j}\partial x^{-k}} = \overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}}\frac{\partial x^i}{\partial x^{-x}} - \begin{Bmatrix} i \\ pq \end{Bmatrix}\frac{\partial x^p}{\partial x^{-j}}\frac{\partial x^q}{\partial x^{-k}}$$

In above equation we get

$$\frac{\partial \overline{A}_k}{\partial x^{-j}} = \frac{\partial A_i}{\partial x^m}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^i}{\partial x^{-k}} + A_i\left(\overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}}\frac{\partial x^i}{\partial x^{-s}} - \begin{Bmatrix} i \\ pq \end{Bmatrix}\frac{\partial x^p}{\partial x^{-j}}\cdot\frac{\partial x^q}{\partial x^{-k}}\right)$$

$$= \frac{\partial A_i}{\partial x^m}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^i}{\partial x^{-k}} + \overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}} A_i \frac{\partial x^i}{\partial x^{-s}} - \begin{Bmatrix} i \\ pq \end{Bmatrix} A_i \frac{\partial x^p}{\partial x^{-j}}\cdot\frac{\partial x^q}{\partial x^{-k}}$$

$$= \frac{\partial A_i}{\partial x^m}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^i}{\partial x^{-k}} + \overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}}\overline{A}_s - A_i\begin{Bmatrix} i \\ pq \end{Bmatrix}\frac{\partial x^p}{\partial x^{-j}}\frac{\partial x^q}{\partial x^{-k}}$$

Changing the dummy index $i \to q$ in first term and $p \to m$ in the third term on the right hand side of the above equation we get

$$\frac{\partial \overline{A}_k}{\partial x^{-j}} - \overline{A}_s\overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}} = \frac{\partial A_q}{\partial x^m}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^q}{\partial x^{-k}} - A_i\begin{Bmatrix} i \\ m,q \end{Bmatrix}\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^q}{\partial x^{-k}}$$

$$= \left(\frac{\partial A_q}{\partial x^m} - A_i\begin{Bmatrix} i \\ mq \end{Bmatrix}\right)\frac{\partial x^m}{\partial x^{-j}}\frac{\partial x^q}{\partial x^{-k}}$$

Define $\overline{A}_{k,j} = \frac{\partial \overline{A}_k}{\partial x^{-j}} - \overline{A}_s\overline{\begin{Bmatrix} s \\ kj \end{Bmatrix}}$ to obtain

$$\overline{A}_{k,j} = A_{q,m}\frac{\partial x^i}{\partial x^{-k}}\frac{\partial x^q}{\partial x^{-k}}$$

$$\overline{A}_{k,j} = A_{i,m}\frac{\partial x^i}{\partial x^{-k}}\frac{\partial x^m}{\partial x^{-j}} \qquad \ldots (2)$$

Which shows that $A_{i,m}$ is a covariant tensor of type (0, 2) and $A_{i,m}$ is covariant derivative of covariant vectors A_k w.r.t. x^j.

12.7.2 Covariant Differentiation of a Contravariant Vectors

From the law of transformation for a contravariant vector,

$$A^k = \overline{A}^{-i}\frac{\partial x^k}{\partial x^{-i}}$$

Different w.r.t. x^j

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial A^{-i}}{\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-i}} + A^{-i} \frac{\partial^2 x^k}{\partial x^{-i}\,\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^j}$$

Using result

$$\frac{\partial^2 x^k}{\partial x^{-i}\partial x^{-n}} = \overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \frac{\partial x^k}{\partial x^{-s}} - \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial x^{-i}} \cdot \frac{\partial x^q}{\partial x^{-n}}$$

We get

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial A^{-i}}{\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^{-j}} + A^{-i} \left(\overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \frac{\partial x^k}{\partial x^{-s}} - \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial x^{-i}} \frac{\partial x^q}{\partial x^{-n}} \right) \frac{\partial x^{-n}}{\partial x^j}$$

$$= \frac{\partial A^{-i}}{\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-i}} + A^{-i} \overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \frac{\partial x^k}{\partial x^{-s}} \frac{\partial x^{-n}}{\partial x^j} - \begin{Bmatrix} k \\ pq \end{Bmatrix} \frac{\partial x^p}{\partial x^{-i}} \delta^q_n A^{-i}$$

$$= \frac{\partial A^{-i}}{\partial x^{-n}} \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-i}} + A^{-i} \overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \frac{\partial x^k}{\partial x^{-s}} \frac{\partial x^{-n}}{\partial x^j} - \begin{Bmatrix} k \\ pn \end{Bmatrix} \frac{\partial x^p}{\partial x^{-i}} A^{-i}$$

Interchanging $i \to s$ in first term of R.H.S.

$$\frac{\partial A^k}{\partial x^j} = \left(\frac{\partial A^{-i}}{\partial x^{-n}} + A^{-i} \overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \right) \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-s}} - A^p \begin{bmatrix} k \\ pn \end{bmatrix} \because A^{-i} = \frac{\partial x^{-i}}{\partial x^p} A^p$$

$$\left(\frac{\partial A^k}{\partial x^j} + A^p \begin{Bmatrix} k \\ pn \end{Bmatrix} \right) = \left(\frac{\partial A^{-i}}{\partial x^{-n}} + A^{-i} \overline{\begin{Bmatrix} s \\ in \end{Bmatrix}} \right) \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-s}}$$

Define $A^k_j = \dfrac{\partial A^k}{\partial x^j} + A^p \begin{Bmatrix} k \\ pn \end{Bmatrix}$. Then

$$A^k_{,j} = A^{-i}_{,n} \frac{\partial x^{-n}}{\partial x^j} \frac{\partial x^k}{\partial x^{-s}}$$

Which shows that $A^k_{,j}$ is a tensor of type (0, 2) and is called covariant derivative of a contravariant vector.

12.7.3 Covariant Differentiation of Tensors

From the transformation of tensors of type (0, 2),

$$\overline{A}_{pq} = A_{ij} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-q}} \qquad \ldots (1)$$

Differentiation w.r.t. x^{-r} we get

$$\frac{\partial \overline{A}_{pq}}{\partial x^{-r}} = \frac{\partial A_{ij}}{\partial x^l} \frac{\partial x^l}{\partial x^{-r}} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-q}} + A_{ij} \frac{\partial^2 x^i}{\partial x^{-p}\partial x^{-r}} \frac{\partial x^j}{\partial x^{-q}} + A_{ij} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial^2 x^j}{\partial x^{-q}\partial x^{-r}}$$

Using result

$$\frac{\partial^2 x^a}{\partial x^{-b}\partial x^{-c}} = \overline{\begin{Bmatrix} d \\ bc \end{Bmatrix}} \frac{\partial x^a}{\partial x^{-d}} - \begin{Bmatrix} a \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\alpha}{\partial x^{-b}} \frac{\partial x^\beta}{\partial x^{-c}} \qquad \ldots (3)$$

We get

$$\frac{\partial \bar{A}_{pq}}{\partial x^{-r}} = \frac{\partial A_{ij}}{\partial x^l} \frac{\partial x^i}{\partial x^{-r}} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-q}} + A_{ij} \frac{\partial x^j}{\partial x^{-q}} \left(\overline{\begin{Bmatrix} s \\ pr \end{Bmatrix}} \frac{\partial x^i}{\partial x^{-x}} - \begin{Bmatrix} i \\ uv \end{Bmatrix} \frac{\partial x^u}{\partial x^{-p}} \frac{\partial x^v}{\partial x^{-r}} \right)$$

$$+ A_{ij} \frac{\partial x^i}{\partial x^{-p}} \left(\overline{\begin{Bmatrix} s \\ qr \end{Bmatrix}} \frac{\partial x^j}{\partial x^{-s}} - \begin{Bmatrix} j \\ m, n \end{Bmatrix} \frac{\partial x^m}{\partial x^{-q}} \cdot \frac{\partial x^n}{\partial x^{-r}} \right)$$

$$= \frac{\partial A_{ij}}{\partial x^l} \frac{\partial x^l}{\partial x^{-r}} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-q}} + \bar{A}_{sq} \overline{\begin{Bmatrix} s \\ pr \end{Bmatrix}} - A_{ij} \begin{Bmatrix} i \\ uv \end{Bmatrix} \frac{\partial x^u}{\partial x^{-p}} \frac{\partial x^v}{\partial x^{-r}} \frac{\partial x^j}{\partial x^{-q}}$$

$$+ \bar{A}_{ps} \overline{\begin{Bmatrix} s \\ qr \end{Bmatrix}} - A_{ij} \begin{Bmatrix} j \\ mn \end{Bmatrix} \frac{\partial x^m}{\partial x^{-q}} \frac{\partial x^n}{\partial x^{-r}} \frac{\partial x^i}{\partial x^{-p}}$$

Changing the dummy indices $u \to i$, $v \to l$, $i \to u$ in 3rd term and $j \to u$, $m \to j$, $n \to l$ in 5th term, we get

$$\frac{\partial \bar{A}_{pq}}{\partial x^{-r}} - A_{sq} \overline{\begin{Bmatrix} s \\ pr \end{Bmatrix}} - \bar{A}_{ps} \overline{\begin{Bmatrix} s \\ qr \end{Bmatrix}} = \left(\frac{\partial A_{ij}}{\partial x^i} - A_{uj} \begin{Bmatrix} u \\ il \end{Bmatrix} - A_{iu} \begin{Bmatrix} u \\ jl \end{Bmatrix} \right) \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-q}} \frac{\partial x^l}{\partial x^{-r}}$$

$$\bar{A}_{pq,\, r} = A_{ij,\, l} \frac{\partial x^i}{\partial x^{-p}} \frac{\partial x^j}{\partial x^{-p}} \frac{\partial x^l}{\partial x^{-r}} \qquad \ldots (4)$$

Where $$A_{ij,\, l} = \frac{\partial A_{ij}}{\partial x^i} - A_{ul} \begin{Bmatrix} u \\ il \end{Bmatrix} - A_{iu} \begin{Bmatrix} u \\ jl \end{Bmatrix} \qquad \ldots (5)$$

This shows that the covariant derivative of a tensors of type (0, 2) is a covariant of type (0, 3).

Note: Similarly, the covariant derivative of A^{ij} w.r.t. x^k

i.e., $$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + A^{lj} \begin{Bmatrix} i \\ lk \end{Bmatrix} + A^{il} \begin{Bmatrix} j \\ lk \end{Bmatrix}$$

To covariant derivative of A^i_j w.r.t. x^k is

$$A^i_{j,\, k} = \frac{\partial A^i_j}{\partial x^k} + A^i_j \begin{Bmatrix} i \\ lk \end{Bmatrix} - A^i_l \begin{Bmatrix} l \\ jk \end{Bmatrix}$$

12.8 *RICCI'S THEOREM*

Theorem: The covariant derivative of the Kronecker delta and the fundamental tensor, $g_{ij,}\ g^{ij}$ is zero.

Proof: The covariant derivative δ^i_j w.r.t. x^k is

$$\delta^i_{j,k} = \frac{\partial \delta^i_j}{\partial x^k} + \delta^l_j \begin{Bmatrix} i \\ lk \end{Bmatrix} - \delta^i_l \begin{Bmatrix} l \\ j,k \end{Bmatrix}$$

$$= 0 + \begin{Bmatrix} i \\ jk \end{Bmatrix} - \begin{Bmatrix} i \\ jk \end{Bmatrix} \text{ as } \delta^i_j = \begin{bmatrix} 1 \;; & i = j \\ 0 \;; & i \neq j \end{bmatrix} \text{ i.e., constant}$$

$$\delta^l_{j,k} = 0$$

The covariant derivative of g_{ij} w.r.t. x^k is

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k} - g_{mj} \begin{Bmatrix} m \\ ik \end{Bmatrix} - g_{im} \begin{Bmatrix} m \\ jk \end{Bmatrix}$$

$$= [ik, j] + [jk, i] - [ik, j] - [jk, i]$$

$$g_{ij,k} = 0 \qquad \because [ij, m] = g_{km} \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

We know $g^{im} g_{mj} = \delta^i_j$. Differentiation w.r.t. x^k

$$g^{im}_{,k} g_{mj} + g^{im} g_{mj,k} = \delta^i_{j,k}$$

$$g^{im}_{,k} g_{mj} + g^{im}\, 0 = 0 \qquad \text{(From (1) and (2))}$$

i.e., $$g^{im}_{,k} = 0 \qquad \dots (3)$$

Example. If at a specified point, the derivatives of g_{ij} w.r.t. x^k are all zero, prove the components of covariant derivatives at that point are the same as ordinary derivatives.

Solution: Given that

$$\frac{\partial g_{ij}}{\partial x^k} = 0 \; \forall \; i, j, k \text{ at point } P_0 \qquad \dots (1)$$

Let A^i_j be tensor. Then covariant derivative w.r.t. x^k is.

$$A^i_{j,k} = \frac{\partial A^i_j}{\partial x^k} + A^l_k \begin{Bmatrix} i \\ lk \end{Bmatrix} - A^i_l \begin{Bmatrix} l \\ jk \end{Bmatrix} \qquad \dots (2)$$

Since $\begin{Bmatrix} i \\ lk \end{Bmatrix} = g^{ij}\,[lk, j] = \frac{1}{2} g^{ij} \left[\frac{\partial g_{ki}}{\partial x^l} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{lk}}{\partial x^j} \right]$, therefore in view of (1), $\begin{Bmatrix} i \\ lk \end{Bmatrix}$ and $\begin{Bmatrix} l \\ jk \end{Bmatrix}$ are zero. Thus from (2), we observe that

$$A^i_{j,k} = \frac{\partial A^i_j}{\partial x^k} \text{ at } P_0.$$

12.9 *SUMMARY*

1. The covariant derivative of the Kronecker delta and the fundamental tensor, $g_{ij},\ g^{ij}$ is zero.
2. **Superscript and Subscript:** The suffixes i and j in B_j^i are called superscript and subscript respectively. The upper position always denotes the superscript and the lower position denotes subscript.

12.10 *GLOSSARY*

(i) Derivatives

(ii) Determinant

12.11 *REFERENCES AND SUGGESTED READINGS*

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iii. Tensors and Differential geometry, Dr. P. P. Gupta, Dr. H. D. Pandey, Prof. G. S. Malik, 1997 Pragati Prakashan Meerut.

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12.12 *TERMINAL QUESTIONS*

1. Prove that $\dfrac{\partial^2 x^{-r}}{\partial x^k \partial x^l} = \begin{Bmatrix} i \\ kl \end{Bmatrix} \dfrac{\partial x^{-r}}{\partial x^i} - \overline{\begin{Bmatrix} r \\ st \end{Bmatrix}} \dfrac{\partial x^{-s}}{\partial x^k} \dfrac{\partial x^{-i}}{\partial x^i}$

2. $A_{ij} = B_{i,\,j} - B_{j,\,i}$ prove that $A_{ij,\,k} + A_{jk,\,i} + A_{ki,\,j} = 0$

3. Prove that $\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^i} = [jk,\, i] = [ij,\, k]$

4. Prove that $\dfrac{\partial}{\partial x^k}(g_{ij} A^i B^j) = A_{i,\,k} B^i + A^i B_{i,\,k}$

UNIT 13: TENSOR LAW OF TRANSFORMATION OF CHRISTOFFEL SYMBOLS

CONTENTS:

13.1 *INTRODUCTION*

The subject matter of this block is presented to explain the origin of the concept of tensor and in what sense tensor is the extension of the vector. Beside it, a mathematically and geometrically systematic approach is developed to explain the distinction between Contravariant and Covariant vectors. For the purpose of further use and application, various operations in tensors are also discussed.

13.2 *OBJECTIVES*

After studying this block, you should be able to

1. Understand the distinction between Contravariant and Covariant vectors (Tensors).
2. Understand the relationship between scalars, vectors and tensors.
3. Understand whether the given mathematical entities are tensors or not.
4. Understand the transformation of operators from one co-ordinate system to other coordinate system.

13.3 *DEFINITIONS*

Tensor Calculus is a generalization of Differential Geometry of Gauss and Riemann. Its systematic exposition was elaborated by mathematician **Ricci** and **Levi-Civita.** Later on Einstein observed that it to be a most suitable tool for General Relativity. The reason is that a physicist wants a formulation of the Laws of physics which remains invariant with respect to observers and view of Mathematician, the main preoccupation of Tensor Calculus is the study of the behavior of an expression under Co-ordinates transformation.

Elementary physical laws such as that the acceleration of a body is proportional to the force acting on it, which can be stated mathematically as

$$a = \frac{F}{m} \quad \text{i.e.,} \quad F = ma$$

where *a, F, m* are the acceleration, force and mass of the body respectively. It should be kept in mind that, however that the law is a special case and apply strictly only to **isotopic media** or to media of high symmetry. In real life, many media are **anisotropic** and as a result, the acceleration "*a*" is not necessarily parallel to the applied force. In such situation, this equation can be generalized as

$$a_x = \frac{1}{m_{xx}} F_x + \frac{1}{m_{xy}} F_y + \frac{1}{m_{xx}} F_z$$

$$a_y = \frac{1}{m_{yx}} F_x + \frac{1}{m_{yy}} F_y + \frac{1}{m_{yz}} F_z$$

$$a_z = \frac{1}{m_{zx}} + \frac{1}{m_{zy}} F_y + \frac{1}{m_{zz}} F_z$$

where a_x, a_y, a_z and F_x, F_y, F_z are Cartesian components of "*a*" and "*F*" respectively and $\frac{1}{m_{ij}}; i, j = x, y, z$ are components of the mass tensor.

13.3.1 Invariants

A function or equation in invariant if it preserves its value or form in a transformation of co-ordinates.

For example if $\phi = AX^2 + 2ABXY + B^2Y^2$, then it will be invariant if it is transformed $\phi = \overline{A}\,\overline{X}^2 + 2\overline{A}\overline{B}\overline{X}\overline{Y} + \overline{B}\overline{Y}^2$. But contrary to it, the temperature *T* of a fluid is not same in terms of Cartesian and spherical co-ordinates as the function *T*. Thus *T* is **"physical invariant"** is not **"tensorial invariant".**

13.3.2 Notation and Conventions

Let V_N be and N-dimensional space and let $x^1, x^2, x^3, \ldots X^N$ be any set of co-ordinates in V_N. It is important to note that x^i does not denote the i^{th} power of x and $(x^i)^r$ denotes that r^{th} power of x^i.

let $x^{-\alpha}; \alpha = 1, 2, 3, \ldots N$ is another set of co-ordinate in the space V_N. Then it is clear that each co-ordinates x^i will be function of the *N*-co-ordinates $x^{-\alpha}$ and conversely.

In symbol,
$$\left.\begin{aligned} x^i &\equiv x^i\,(x^{-1}, x^{-2}, \ldots x^{-N}); 1 \le i \le N \\ x^{-\alpha} &\equiv (x^1, x^2, \ldots, x^N); 1 \le \alpha \le N \end{aligned}\right\} \quad \ldots (1)$$

Differentiating, we find that

and
$$\left.\begin{aligned} dx^i &= \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial x^{-\alpha}} dx^{-\alpha}; 1 \le i \le N \\ dx^{-\alpha} &= \sum_{i=1}^{N} \frac{\partial x^{-\alpha}}{\partial x^i} dx^i; 1 \le \alpha \le N \end{aligned}\right\} \quad \ldots (2)$$

Due to Einstein, summations appear in RHS of (2) are expressed as

$$dx^i = \frac{\partial x^i}{\partial x^{-\alpha}} dx^{-\alpha}; 1 \le i \le N \quad \ldots (3)$$

and
$$dx^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i} dx^i; 1 \le \alpha \le N \quad \ldots (4)$$

In equation (3), α appears twice in RHS however *i* appears twice in RHS of equation (4). The index appears only once in any term and let has a definite value between 1 and *N* is called **"free index".** In equations (3) and (4); i and α are free indices respectively. These equations can be expressed as

$$dx^i = \frac{\partial x^i}{\partial x^{-\alpha}} dx^{-\alpha} \qquad \ldots (5)$$

and
$$dx^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i} dx^i \qquad \ldots (6)$$

These are set of N equations for each value of i and α respectively.

On the other hand, an index which is repeated and over which summation is implied, is called a **"dummy index"** and it is important to note that this **"dummy index"** can be replaced by an other index other than free and dummy indices.

The equation (5) and (6) can be expressed as

$$dx^1 = \frac{\partial x^i}{\partial x^{-\beta}} dx^{-\beta}$$

… (7)

and
$$dx^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^j} dx^j$$

13.3.3 Kroneker Delta

Since the co-ordinates x^i are independent of each other, therefore by differential calculus it is clear that

$$\frac{dx^i}{dx^j} = \begin{bmatrix} 1 \; ; & i = j \\ 0 \; ; & 1 \neq j \end{bmatrix}$$

and we can define it as

$$\frac{dx^i}{dx^j} = \delta^i_j \quad \text{where} \quad \delta^i_j = \begin{bmatrix} 1 \; ; & i = j \\ 0 \; ; & 1 \neq j \end{bmatrix}$$

Where δ^i_j is called Kronecker Delta

Similarly,
$$\frac{dx^{-\alpha}}{dx^{-\beta}} = \delta^\alpha_\beta$$

13.4 *CONTRAVARIANT AND COVARIANT VECTORS*

Let $x^i = (x^1, x^2, \ldots x^N)$ be co-ordinates of a point in a co-ordinate system and let $x^{-\alpha} = (x^{-1}, x^{-2}, \ldots, x^N)$ be co-ordinates of the same point in another co-ordinate system.

Let $A^i = i = 1, 2, 3, \ldots N$ be N-functions of co-ordinate x^i. If A^i are transformed to A^{-i} in another co-ordinate system as

$$A^{-i} = \frac{\partial x^{-i}}{\partial x^\alpha} A^\alpha \quad \text{or} \quad A^{-\alpha} = A^i \frac{\partial x^{-\alpha}}{\partial x^i}$$

Then A^i is called components of contravariant vector of order 1 or of type (1, 0).

Similarly, let $A_i \; ; i = 1, 2, 3, \ldots N$ be N-functions of the co-ordinates x^i in a co-ordinates system. If the quantities A_i are transformed to $\overline{A}_i$ in another co-ordinate system as

$$\overline{A}_i = \frac{\partial x^{\alpha}}{\partial x^{-i}} A_{\alpha} \qquad \text{or} \qquad \overline{A}_a = \frac{\partial x^{i}}{\partial x^{-\alpha}}$$

Then A_i is called components of covariant vector of order 1 or of type (0, 1).

Note: A superscript /subscript is always used to indicate contravariant/covariant component or character.

13.4.1 Geometrical Interpretation

Let $r = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ be the positions vector of a point P in E_3. In curvilinear co-ordinates position vector of point P is $r = r\,(u_1, u_2, u_3)$. Define unit tangent and unit normal vector as

$$t_1 = \frac{\dfrac{\partial r}{\partial u_1}}{\left|\dfrac{\partial r}{\partial u_1}\right|};\ i = 1, 2, 3 \text{ to the curve } u_1 \text{ at point } P.$$

and $N_i = \dfrac{\nabla u_1}{|\nabla u_1|};\ i = 1, 2, 3$ to the surface $u_1 =$ Constant at P.

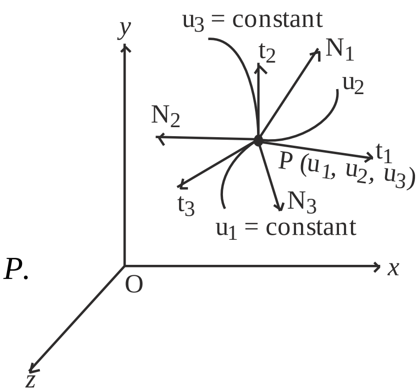

Fig. 1

Thus at each point P of a curvilinear system there exists two sets of unit vectors (t_1, t_2, t_3) tangential to the co-ordinate curves and (N_1, N_2, N_3) normal to the co-ordinate surface. Therefore any vector A can be represented in-terms of unit vector t_1 or N_i as

$$A = a_1 t_1 + a_2 t_2 + a_3 t_3 = a^1 N_1 + a^2 N_2 + a^3 N_3$$

$$A = a_1 \left(a_1 \frac{\dfrac{\partial r}{\partial u_1}}{\left|\dfrac{\partial r}{\partial u_1}\right|}\, a_2 + \frac{\dfrac{\partial r}{\partial u_2}}{\left|\dfrac{\partial r}{\partial u_2}\right|} + a_3 \frac{\dfrac{\partial r}{\partial u_3}}{\left|\dfrac{\partial r}{\partial u_3}\right|} \right)$$

$$A = A_1 \frac{\partial r}{\partial u_1} + A_2 \frac{\partial r}{\partial u_2} + A_3 \frac{\partial r}{\partial u_3};\ A_i = \frac{a_1}{\left|\dfrac{\partial r}{\partial u_1}\right|}$$

and

$$A = a^1 \frac{\nabla u_1}{|\nabla u_1|} + a^2 \frac{\nabla u_2}{|\nabla u_2|} + a^{23} \frac{\nabla u_3}{|\nabla u_3|};\ A^i = \frac{a^t}{|\nabla u_1|}$$

$$A = A^1 \nabla u_1 + A^2 \nabla u_2 + A^3 \nabla u_3$$

where A_1, A_2, A_3 and A^1, A^2, A^3 are contravariant and covariant components of vector A.

or $$A = A_i \frac{\partial r}{\partial u_i} \quad \text{and} \quad A = A^i \nabla u_3$$

13.4.2 Transformation law for Contravariant and Covariant Vectors

Let a given vector A is define in two general curvilinear co-ordinate system (x^1, x^2, x^3) and (x^{-1}, x^{-2}, x^{-3}) as

$$A = A^t \frac{\partial r}{\partial x^i} \quad \text{and} \quad A = A^{-i} \frac{\partial r}{\partial x^{-i}} = A^{-\alpha} \frac{\partial r}{\partial x^{-\alpha}} \qquad \dots (1)$$

By ordinary partial differentiation,

$$dr = \frac{\partial r}{\partial x^i} dx^i \quad \text{and} \quad dr = \frac{\partial r}{\partial x^{-\alpha}} dx^{-\alpha} \qquad \dots (2)$$

Hence $$\frac{dr}{dx^i} dx^i = \frac{\partial r}{\partial x^{-a}} \partial x^{-\alpha} \qquad \dots (3)$$

Since x^i and x^{-1} are two co-ordinate system therefore

$$x^i = x^i \, (x^{-1}, x^{-2}, x^{-3}) \quad \text{i.e.,} \quad x^i = x^i \, (x^{-\alpha})$$

and by differentiation, we have

$$dx^i = \frac{\partial x^i}{\partial x^{-\alpha}} dx^{-\alpha} \qquad \dots (4)$$

Using in (3) we get

$$\frac{\partial r}{\partial x^i} \frac{\partial x^i}{\partial x^{-\alpha}} = \frac{\partial r}{\partial x^{-\alpha}} \qquad \dots (5)$$

and by (1) we have

$$A^i \frac{\partial r}{\partial x^i} = A^{-\alpha} \frac{\partial r}{\partial x^i} \frac{\partial x^i}{\partial x^{-\alpha}}$$

$$A^i = A^{-\alpha} \frac{\partial x^i}{\partial x^{-\alpha}}$$

i.e., $$\boxed{\overline{A}^{\alpha} = A^i} \frac{\partial x^{-\alpha}}{\partial x^i} \qquad \dots (6)$$

or $$\overline{A}^i = A^{\alpha} \frac{\partial x^{-i}}{\partial x^{\alpha}}$$

Similarly, $$A = A_i \, (\nabla x^i) \quad \text{and} \quad A = \overline{A}_{\alpha} \, (\nabla x^{-\alpha}) \qquad \dots (7)$$

where $$\nabla \equiv i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

By ordinary partial differential, we have

$$\frac{\partial x^{-\alpha}}{\partial x} = \frac{\partial x^{-\alpha}}{\partial x^i} \frac{\partial x^i}{\partial x} \quad \text{as} \quad x^{-\alpha} = x^{-\alpha} \, (x^i) \qquad \dots (8)$$

and $$\nabla x^{-\alpha} = i \frac{\partial x^{-\alpha}}{\partial x} + j \frac{\partial x^{-\alpha}}{\partial y} + k \frac{\partial x^{-\alpha}}{\partial z}$$

$$= \frac{\partial x^{-\alpha}}{\partial x^i}\left(i\,\frac{\partial x^i}{\partial x} + j\,\frac{\partial x^i}{\partial y} + k\,\frac{\partial x^i}{\partial z}\right)$$

$$\nabla x^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i}\,\nabla x^i \qquad \ldots (9)$$

Using in (7), we have

$$A_i\,(\nabla x^i) = \overline{A}_\alpha \left(\frac{\partial x^{-\alpha}}{\partial x^i}\,\nabla x^i \right)$$

$$A_i = \overline{A}_\alpha\,\frac{\partial x^{-\alpha}}{\partial x^i}$$

$$\boxed{\overline{A}_\alpha = A_i\,\frac{\partial x^i}{\partial x^{-\alpha}} \quad \text{or} \quad \overline{A}_i = A_\alpha}\,\frac{\partial x^\alpha}{\partial x^{-i}} \qquad \ldots (10)$$

Remarks: 1. Contravariant vector involve derivatives of the new co-ordinates w.r.t. the old co-ordinates.

2. Covariant vector involve derivative of the old co-ordinates w.r.t. the new co-ordinates.

3. In rectangular (orthogonal) system, both curvilinear co-ordinates are identical. Hence contravariant and covariant vector are identical.

Example 1. If x^i be the co-ordinate of a point in N-dimensional space V_N, then show that dx^i is component of a contravariant vector. Also show that velocity and acceleration are contravariant vectors.

Solution : Let x^i and $x^{-\alpha}; i, \alpha = 1, 2, 3, \ldots N$ be two co-ordinates in V_N Then each co-ordinates $x^{-\alpha}$ is function of the N-co-ordinates x^i and conversely i.e.,

$$x^{-\alpha} = x^{-\alpha}\,(x^1, x^2, \ldots x^N);\, 1 \le \alpha \le N$$

Differentiating x^i we have

$$dx^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i}\,dx^i \qquad \ldots (1)$$

Which show that dx^i component of a contravariant vector. If $x^i = x^i(t)$ where t is time $\forall\, i = 1, 2, 3, \ldots N$ then by differential calculus, we have

$$\frac{dx^{-\alpha}}{dt} = \frac{dx^{-\alpha}}{dx^i}\,\frac{dx^i}{dt} \qquad \ldots (2)$$

If we define the component of the velocity in both co-ordinate system by

$$v^{-\alpha} = \frac{dx^{-\alpha}}{dt},\; v^i = \frac{dx^i}{dt} \qquad \ldots (3)$$

Equation (2) becomes

$$v^{-\alpha} = \frac{\partial x^{-\alpha}}{dx^i}\,v^i \qquad \ldots (4)$$

which shows that the velocity v^i is a contravariant vector. Taking time-derivatives of (4), we obtain

$$\frac{dv^{-\alpha}}{dt} = \frac{dx^{-\alpha}}{dx^i}\frac{dv^t}{dt} \qquad \ldots (5)$$

If we define the components of acceleration in both co-ordinate system by

$$a^{-\alpha} = \frac{dv^{-a}}{dt}, a^i = \frac{dv^i}{dt} \qquad \ldots (6)$$

Equation (5) becomes

$$a^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i} a^i$$

which shows that the acceleration a^i is a contravariant vector.

It should be noted that the coefficient $\frac{\partial x^{-\alpha}}{\partial x^i}$ are independent of time for fixed co-ordinate systems because the co-ordinates x^i in $\frac{dx^i}{dt}$ are co-ordinate of a particles in motion while $\frac{\partial x^{-\alpha}}{\partial x^i}$ is a relation between two co-ordinate system, which is independent of time.

If each $x^i = x^i\,(s)$ where s is arc-length parameter, then by Eq. (2) we have

$$\frac{\partial x^{-\alpha}}{ds} = \frac{\partial x^{-\alpha}}{\partial x^i}\frac{dx^i}{ds} \qquad \ldots (8)$$

Which shown that $\frac{dx^{-\alpha}}{ds}$ (tangent) is a contravariant vector.

Example 2. Show that the gradient of sealar function is a covariant vector. *OR* show that $\frac{\partial\phi}{\partial x^i}$ is a covariant vector where ϕ is a scalar function.

Solution : Let $\phi = \phi(x^i)$ be a field. Being a scalar field, its functional form remains unchanged under coordinate transformations, so that

$$\phi(x^i) = \bar{\phi}\,(x^{-\alpha}) = \phi\,(x^{-\alpha}) \quad \ldots (1)$$

On using partial derivatives, it is clear that

$$\frac{\partial\phi}{\partial x^i} = \frac{\partial\phi}{\partial x^{-\alpha}}\frac{\partial x^{-\alpha}}{\partial x^i}$$

i.e., $$(\nabla\phi)_i \equiv (\nabla\bar{\phi})_\alpha \frac{\partial x^{-\alpha}}{\partial x^i} \quad \text{or} \quad A_i = \bar{A}_\alpha \frac{\partial x^{-\alpha}}{\partial x^i} \quad \ldots (2)$$

Which shws that the gradient of a scalar function/field is a covariant vector.

Example 3. Show that the co-ordinates x^i do not form a contravariant vector A^i.

Solution : Let A^i is a contravariant vector. Then by law of transformation from co-ordinates x^i to $x^{-\alpha}$, we have

$$A^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i} A^i \qquad \ldots (1)$$

If we put $A^i = x^i$ then

$$A^{-\alpha} = \frac{\partial x^{-\alpha}}{\partial x^i} x^i$$

In general, it does not reduce to $A^{-\alpha} = x^{-\alpha}$. Hence x^i is not a contravariant vector.

Example 4. Show that the second derivatives of a scalar field ϕ i.e., $A_{ij} = \dfrac{\partial^2 \phi}{\partial x^i \partial x^i}$ are not the components of a second rank tensor.

Solution : Let $\phi = \phi\,(x^i)$ be a scalar function of x^i and let x^{-i} be another co-ordinate system such that

$$x^{-\alpha} = x^{-\alpha}\,(x^i)$$

Then,
$$\frac{\partial \phi}{\partial x^{-\alpha}} = \frac{\partial \phi}{\partial x^i}\,\frac{\partial x^i}{\partial x^{-\alpha}} \qquad \ldots (1)$$

Differentiating (1) partially w.r.t. $x^{-\beta}$

$$\frac{\partial^2 \phi}{\partial x^{-\beta}\partial x^{-\alpha}} = \frac{\partial \phi}{\partial x^i}\,\frac{\partial^2 x^i}{\partial x^{-\beta}\partial x^{-\alpha}} + \frac{\partial^2 \phi}{\partial x^{-\beta}\partial x^i}\,\frac{\partial x^i}{\partial x^{-\alpha}}$$

$$= \frac{\partial \phi}{\partial x^i}\,\frac{\partial^2 x^i}{\partial x^{-\beta}\partial x^{-\alpha}} + \frac{\partial^2 \phi}{\partial x^i \partial x^i}\,\frac{\partial x^i}{\partial x^{-\beta}}\cdot\frac{\partial x^i}{\partial x^{-\alpha}}$$

$$\frac{\partial^2 x^i}{\partial x^{-\beta}\partial x^{-\alpha}} = \frac{\partial^2 \phi}{\partial x^i \partial x^i}\,\frac{\partial x^i}{\partial x^{-\beta}}\cdot\frac{\partial x^i}{\partial x^{-\alpha}} + \frac{\partial \phi}{\partial x^i}\,\frac{\partial^2 x^i}{\partial x^{-\beta}\partial x^{-\alpha}} \qquad \ldots (2)$$

which shows that Eq. (2) does not represent tensor law of transformation due to presence of second term in RHS of Eq. (2).

Example 5. There is no distinction between contravariant and covariant vector under rectangular cartesian co-ordinates transformations.

Solution : Let $P(x, y)$ be a point w.r.t. the rectangular Cartesian co-ordinates axes X and Y and let $P(\bar{x}, \bar{y})$ be the same point w.r.t. the rectangular coordinates axes $\bar{X}$ and $\bar{Y}$, which is obtained by rotating $X - Y$ systems about OZ axis. Let (l_1, m_1) and (l_2, m_2) be the direction cosines of the axes $\bar{X}$ and $\bar{Y}$ respectively. Then the transformation relations are given by.

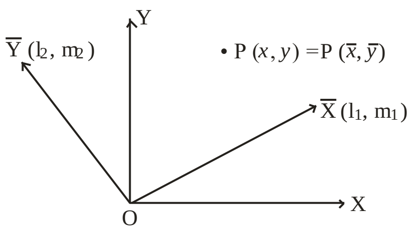

Fig. 2

$$\bar{y} = l_1 x + m_1 y \qquad \ldots (1)$$

$$\bar{y} = l_2 x + m_2 y \qquad \ldots (2)$$

and $$x = l_1 \bar{x} + l_2 \bar{y} \qquad \ldots (3)$$

$$y = m_1 \bar{x} + m_2 \bar{y} \qquad \ldots (4)$$

Also $$\frac{\partial \bar{x}}{\partial x} = l_1, \frac{\partial \bar{x}}{\partial y} = m_1, \frac{\partial \bar{y}}{\partial x} = l_2, \frac{\partial \bar{y}}{\partial y} = m_2 \qquad \ldots (5)$$

Suppose $x^1 = x, x^2 = y; x^{-1} = \bar{x}, x^{-2} = \bar{y}$. Then by contravariant transformation for A with components A^1, A^2 we have

$$A^{-i} = A^{\alpha} \frac{\partial x^{-i}}{\partial x^{\alpha}}; \alpha = 1, 2 \text{ and } i = 1, 2 \qquad \ldots (6)$$

Thus, $$A^{-1} = \frac{\partial x^{-i}}{\partial x^1} + A^2 \frac{\partial x^{-1}}{\partial x^2}$$

$$A^{-2} = A^2 \frac{\partial x^{-2}}{\partial x^1} + A^2 \frac{\partial x^{-2}}{\partial x^2}$$

Using (2) we have

$$A^{-1} = \boxed{A^1 l_1 + A^2 m_1 \quad ;} \quad A^{-2} = \boxed{A^1 l_2 + A^2 m_2} \qquad \ldots (7)$$

Consider covariant transformation for A with components A_1, A_2 we have

$$\bar{A}_1 = A_{\alpha} \frac{\partial x^{\alpha}}{\partial x^{-i}}; \alpha = 1, 2 \text{ and } i = 1, 2$$

Thus, $$\bar{A}_1 = A_1 \frac{\partial x^1}{\partial x^{-1}} + A_2 \frac{\partial x^2}{\partial x^{-1}}$$

$$\bar{A}_2 = A_1 \frac{\partial x^1}{\partial x^{-2}} + A_2 \frac{\partial x^2}{\partial x^{-2}}$$

Using (3) we have

$$\boxed{A_1 = A_1 l_2} + A_2 m \quad \boxed{; \quad A_2 = A_1 l_2} + A_2 m_2 \qquad \ldots (8)$$

Result (7) and (8) show that

$$\bar{A}_1 = A^{-1}; \bar{A}_2 = A^{-2}$$

Hence contravariant and covariant vectors are identically same in rectangular co-ordinate system.

Example 6. If a vector has components $\dot{x}.\dot{y}$ in rectangular Cartesian co-ordinates, then they are $\dot{r}.\dot{\theta}$, in polar co-ordinates and if vector has components $\ddot{x}, \ddot{y}$ in rectangular Cartesian co-ordinates then they are $\ddot{r} - r\dot{\theta}^2, \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}$ in polar co-ordinates.

Solution: Let (x, y) and (r, θ) be position of a point in rectangular Cartesian and polar co-ordinates respectively. Then relations between these two co-ordinates system are

$$x = r\cos\theta, y = r\sin\theta \qquad \ldots (1)$$

with $$r^2 = x^2 + y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots (2)$$

By simple application of differential calculus,

$$\frac{\partial r}{\partial x} = \frac{x}{y}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial \theta}{\partial x} = \frac{-y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \quad \dots (3)$$

$$\dot{x} = \dot{r}\cos\theta - r\,\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \quad \dots (4)$$

and $$\dot{x}^2 + \dot{y}^2 = \dot{r} + r^2\dot{\theta}^2 \quad \text{(squaring and adding (4))} \quad \dots (5)$$

Also from (2), we have

$$r\dot{r} = x\dot{x} + y\dot{y}, \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2}\ \ddot{\theta} = \frac{x\ddot{y} - y\ddot{x}}{r^2} - \frac{2\dot{r}}{r^3}(x\dot{y} - \dot{x}y) \quad \dots (6)$$

$$\dot{r}^2 + r\ddot{r} = \dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y} \quad \text{i.e.,} \quad x\ddot{x} + y\ddot{y} = r^2 + r\dot{r} - (\dot{x}^2 + \dot{y}^2) \quad \dots (7)$$

Let $x^1 = x, x^2 = y$ and $x^{-1} = r, x^{-2} = \theta$. Then by contravariant transformation we have

$$A^{-i} = A^{\alpha}\frac{\partial x^{-i}}{\partial x^{\alpha}};\ \alpha = 1, 2 \quad \text{and} \quad i = 1, 2 \quad \dots (8)$$

$$A^{-1} = A^1\frac{\partial x^{-1}}{\partial x^1} + A^2\frac{\partial x^{-1}}{\partial x^2} \quad \dots (9)$$

$$A^{-2} = A^1\frac{\partial x^{-2}}{\partial x^1} + A^2\frac{\partial x^{-2}}{\partial x^2} \quad \dots (10)$$

Case I : Let $A^1 = \dot{x}, A^2 = \dot{y}$. Then by (9) and (10), we have

$$A^{-1} = \dot{x}\frac{x}{r} + \dot{y}\frac{y}{r} = \frac{x\dot{x} + \dot{y}y}{r} \quad \text{(From (3))}$$

Using (6), we have

$$A^{-1} = \dot{r}$$

and $$A^{-2} = \dot{x}\left(\frac{-y}{r^2}\right) + \dot{y}\left(\frac{x}{r^2}\right) \quad \text{(From (3))}$$

Using (4) we have

$$= \frac{x\dot{y} - \dot{x}y}{r^2} = \frac{r\cos(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}) - r\sin\theta\,(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)}{r^2}$$

$$A^{-2} = \frac{r^2\dot{\theta}}{r^2} = \dot{\theta}$$

Case II : $A^1 = \ddot{x}, A^2 = \ddot{y}$. Then by (9) and (10) we have

$$A^{-1} = \ddot{x}\frac{x}{r} + \ddot{y}\frac{y}{r}$$

$$= \frac{x\ddot{x} + y\ddot{y}}{r}$$

$$= \frac{\dot{r}^2 + r\ddot{r} - (\dot{x}^2 + \dot{y}^2)}{r} \quad \text{(From (7))}$$

$$= \frac{\dot{r}^2 + r\dot{r} - \dot{r}^2 - r^2\dot{\theta}^2}{r} \quad \text{(From (5))}$$

$$A^{-1} = r - r\dot{\theta}^2$$

And $$A^{-2} = \ddot{x}\left(\frac{-y}{r^2}\right) + \ddot{y}\left(\frac{x}{r^2}\right)$$
$$= \frac{x\ddot{y} - \ddot{x}y}{r^2}$$
$$= \left[\ddot{\theta} + \frac{2\dot{r}}{r^3}(x\dot{y} - \dot{x}y)\right]$$
$$A^{-2} = \ddot{\theta} + \frac{2\dot{r}}{r}(\dot{\theta}) \qquad \text{(From (6))}$$

Note : 1. In case (i) and (ii); A^{-1} represent radial velocity and radial acceleration respectively.

2. In both cases, $A^{-2}r$ is transverse velocity and transverse accelerations respectively.

Example 7. If $a_x = \frac{d^2x}{dt^2}, a_y = \frac{d^2y}{dt^2}, a_z = \frac{d^2z}{dt^2}$ be Cartesian components of the acceleration vector then find the component of the acceleration vector in the spherical polar co-ordinates.

Solution : Since acceleration is a contravariant vector therefore
$$a^{-\alpha} = a^i \frac{\partial x^{-\alpha}}{\partial x^i} \qquad \ldots (1)$$
let $x^1 = x,\ x^2 = y,\ x^3 = z;\ \ a^1 \equiv a_x,\ a^2 \equiv a_y,\ a^3 \equiv a_z$

and $x^{-1} = r, x^{-2} = \theta, x^{-3} = \phi; a^{-1} \equiv a_{r,} a^{-2} \equiv a_\theta, a^{-3} \equiv a_\phi$

The relationship connecting and spherical polar co-ordinates is given by
$$x = r \sin \cos \phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta \qquad \ldots (2)$$
and $$r = (x^2 + y^2 + z^2)^{1/2}, \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \phi = \tan^{-1}\left(\frac{y}{x}\right) \quad \ldots (3)$$

By partial differentiation, we have
$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \frac{\partial r}{\partial y} = \sin\phi\sin\phi, \frac{\partial r}{\partial z} = \cos\theta$$
$$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}, \frac{\partial r}{\partial y} = \frac{\cos\theta\sin\theta}{r}, \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r} \qquad \ldots (4)$$
$$\frac{\partial\phi}{\partial x} = -\frac{\sin\theta}{r\sin\theta}, \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}, \frac{\partial\phi}{\partial z} = 0$$

For $\alpha = 1;$ Eq. (1) gives
$$a_r = a^{-1} = a^1\frac{\partial x^{-1}}{\partial x^1} + a^2\frac{\partial x^{-1}}{\partial x^2} + a^3\frac{\partial x^{-1}}{\partial x^3}$$
$$= a_x\frac{\partial r}{\partial x} + a_y\frac{\partial r}{\partial y} + a_z\frac{\partial r}{\partial z}$$

$$= \frac{\partial^2 x}{\partial t^2}\frac{\partial r}{\partial x} + \frac{\partial y^2}{dt}\frac{\partial r}{\partial y} + \frac{d^2 z}{dt}\frac{\partial r}{\partial z}$$

Using (2) and first Eq. of (4) we get

$$a_r = \sin\cos\phi\,\frac{d^2}{dt^2}(r\sin\cos\phi) + \sin\theta\sin\phi\,\frac{d^2}{dt^2}(r\sin\theta\sin\phi)$$

$$+ \cos\frac{d^2}{dt^2}(r\cos\theta)$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 \qquad \ldots (5)$$

Similarly $\quad a_\theta = \ddot{\theta} - \sin\theta\cos\phi\,\dot{\phi}^2 \qquad \ldots (6)$

$$a_\phi = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}^2 \qquad \ldots (7)$$

These are required components of acceleration in spherical co-ordiantes.

Remarks : 1. The only component a_r has the dimensions of acceleration.

2. The components a_0, a_ϕ have the dimensions $(\text{times})^{-2}$.

3. If $\theta = \frac{\pi}{2}$ then spherical polar co-ordiantes reduce to polar co-ordinates and components of acceleration become

$$a_r = \ddot{r}\,r\,\dot{\phi}^2,\ a_\phi = \ddot{\phi} + \frac{2}{r}\dot{r}\,\dot{\phi}$$

Which is exactly same as in example (5).

Example 8. Find the components of acceleration in cylindrical co-ordinates r, θ, z which are related to the Cartesian co-ordinates $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

Solution : Since the acceleration is a contravariant vector therefore,

$$a^{-\alpha} = a^i\,\frac{\partial x^{-\alpha}}{\partial x^i}$$

let $\quad x^1 = x,\ x^2 = y,\ x^2 = z;\ a^1 \equiv a_x,\ a^2 \equiv a_y,\ a^3 \equiv a_z$

and $\quad x^{-1} = r,\ x^{-2} = \theta,\ x^{-3} = z;\ a^{-1} \equiv a_r,\ a^{-2} \equiv a_\theta,\ a^{-3} \equiv a_z$

The relationship connecting carteisan and cylindrical coordinates is given by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \qquad \ldots (2)$$

and $\quad r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots (3)$

By partial differentiation, we have

$$\frac{\partial r}{\partial x} = \cos\theta,\ \frac{\partial r}{\partial y} - \sin\theta;\ \frac{\partial r}{\partial z} = 0$$

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r},\ \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r},\ \frac{\partial\theta}{\partial z} = 0 \qquad \ldots (4)$$

$$\frac{\partial z}{\partial x} = 0,\ \frac{\partial z}{\partial y} = 0,\ \frac{\partial z}{\partial z} = 1$$

For $\alpha = 1$; Eq. (1) gives

$$a_r = a^{-1} = a^1 \frac{\partial x^{-1}}{\partial x^1} + a^2 \frac{\partial x^{-1}}{\partial x^2} + a^3 \frac{\partial x^{-1}}{\partial x^3}$$

$$= a_x \frac{\partial x}{\partial x} + a_x \frac{\partial r}{\partial y} + a_z \frac{\partial r}{\partial z} = \frac{d^2x}{dt^2}\frac{\partial r}{\partial x} + \frac{d^2y}{dt^2}\frac{\partial r}{\partial y} + \frac{d^2z}{dt^2}\frac{\partial r}{\partial z}$$

Using Eq. (2) and first equation of (4) we have

$$a_r = \cos\theta \frac{d^2}{dt^2}(r\cos\theta) + \sin\theta \frac{d^2}{dt^2}(r\sin\theta)$$

$$= \cos\theta \frac{d}{dt}(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}) + \sin\theta \frac{d}{dt}(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})$$

$$= \cos\theta\,(\ddot{r}\cos\theta - 2\dot{r}\,\dot{\theta}\sin\theta - \dot{\theta}^2 - r\sin\dot{\theta}\,\ddot{\theta})$$

$$+\sin\theta\,(\ddot{r}\sin\theta + 2\dot{r}\,\dot{\theta}\cos\theta - r\,\dot{\theta}^2\sin\theta + r\cos\dot{\theta}\,\ddot{\theta})$$

$$a_r = \ddot{r} - r\,\theta^2 \qquad \text{... (5)}$$

For $\alpha = 2;$ Eq. (1) gives

$$a_\theta = \frac{d^2x}{dt^2}\frac{\partial\theta}{\partial x} + \frac{d^2y}{dt^2}\frac{\partial\theta}{\partial y} + \frac{d^2z}{dt^2}\frac{\partial\theta}{\partial z}$$

$$= -\frac{\sin\theta}{r^2}(\ddot{r}\cos\theta - 2\dot{r}\,\dot{\theta}\sin\theta - r\,\dot{\theta}^2\cos\theta - r\,\ddot{\theta}\sin\theta)$$

$$+\frac{\cos\theta}{r}(\ddot{r}\sin\theta + 2\dot{r}\,\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\,\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta)$$

$$a_0 = \frac{2\dot{r}\dot{\theta}}{r} + \ddot{\theta} \qquad \text{... (6)}$$

From $\alpha = 3;$ Eq. (1) gives

$$a_z = \frac{d^2z}{dt^2}\frac{dz}{dx} + \frac{d^2y}{dt^2}\frac{\partial z}{\partial y} + \frac{d^2z}{dt^2}\frac{\partial z}{\partial z}$$

$$= 0 + 0 + \frac{d^2z}{dt^2}.1$$

$$a_z = \ddot{z} = \frac{d^2z}{dt^2} \qquad \text{... (7)}$$

Example 9. Find $div\ A^i$, $div\ A_i$ and $\nabla^2\phi$ in cylindrical co-ordinates where A^i and A_i are vectors and ϕ is a scalar.

Solution : In Cartesian co-ordinates, we have

$$div\ A^i = \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z} \qquad \text{... (1)}$$

The cylindrical co-ordinates r, θ, z are related to the Cartesian co-ordinates given by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \qquad \text{... (2)}$$

If the components of A^i in cylindrical co-ordiantes, are A^r, A^θ, A^z then

$$A^x = \frac{\partial x}{\partial r} A^r + \frac{\partial x}{\partial \theta} A^\theta + \frac{\partial x}{\partial z} A^z$$

$$= \cos\theta\, A^r - r \sin\theta\, A^\theta \qquad \ldots (3)$$

$$A^y = \frac{\partial y}{\partial r} A^r + \frac{\partial y}{\partial \theta} A^\theta + \frac{\partial y}{\partial z} A^z$$

$$= \sin\theta\, A^r + r\cos\theta\, A^\theta \qquad \ldots (4)$$

$$A^z = \frac{\partial z}{\partial r} A^r + \frac{\partial z}{\partial \theta} A^\theta + \frac{\partial z}{\partial z} A^z = A^z \qquad \ldots (5)$$

Also,
$$\frac{\partial A^x}{\partial x} = \frac{\partial A^x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial A^x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial A^x}{\partial z}\frac{\partial z}{\partial x}$$

$$= \cos\theta \frac{\partial A^x}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A^x}{\partial \theta} \qquad \ldots (6)$$

$$\left[\begin{array}{ll} \because & r^2 = x^2 + y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \text{and} & \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} \end{array}\right]$$

$$\frac{\partial A^y}{\partial y} = \frac{\partial A^y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial A^y}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial A^y}{\partial z}\frac{\partial z}{\partial y}$$

$$= \sin\theta \frac{\partial A^y}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A^y}{\partial \theta} \qquad \ldots (7)$$

$$\frac{\partial A^z}{\partial z} = \frac{\partial A^z}{\partial z} \qquad \ldots (8)$$

Using (3) in (6) we get

$$\frac{\partial A^x}{\partial x} = \cos\theta\left(\cos\theta \frac{\partial A^y}{\partial r} - \sin\theta\, A^\theta - r\sin\theta \frac{\partial A^\theta}{\partial r}\right)$$

$$- \frac{\sin\theta}{r}\left(-\sin\theta\, A^r + \cos\theta \frac{\partial A^r}{\partial \theta} - r\cos\theta\, A^\theta - r\sin\theta \frac{\partial A^\theta}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial A^r}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial A^r}{\partial \theta} - r\sin\theta\cos\theta \frac{\partial A^\theta}{\partial r} + \sin^2\theta \frac{\partial A^\theta}{\partial \theta}$$

$$+ \frac{\sin^2\theta A^r}{r} \qquad \ldots (9)$$

Using (4) in (7) we get

$$\frac{\partial A^y}{\partial y} = \sin^2\theta \frac{\partial A^r}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial A^r}{\partial \theta} + r\sin\theta\cos\theta \frac{\partial A^\theta}{\partial r}$$

$$+ \cos^2\theta \frac{\partial A^\theta}{\partial \theta} + \frac{\cos^2\theta}{r} A^r \qquad \ldots (10)$$

$$\frac{\partial A^z}{\partial z} = \frac{\partial A^z}{\partial z} \qquad \ldots (11)$$

Using (9) to (10) in (1) we get

$$div\ A^i = \frac{\partial A^r}{\partial r} + \frac{\partial A^\theta}{\partial \theta} + \frac{A^r}{r}$$

(II) For a covariant vlector A_i,

$$div\ A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \ldots (13)$$

If the components of A_i in cylindrical coordinates are A_r, A_θ, A_z then

$$A_x = A_r \frac{\partial r}{\partial x} + A_\theta \frac{\partial \theta}{\partial x} + A_z \frac{\partial z}{\partial x}$$

$$A_x = A_y \cos\theta - \frac{\sin\theta}{r} A_\theta \qquad \ldots (14)$$

$$A_y = A_r \frac{\partial z}{\partial y} + A_\theta \frac{\partial \theta}{\partial y} + A_z \frac{\partial z}{\partial z}$$

$$A_y = A_r \sin\theta + \frac{\cos\theta}{r} A_\theta \qquad \ldots (15)$$

$$A_z = A_r \frac{\partial r}{\partial z} + A_\theta \frac{\partial \theta}{\partial z} + A_z \frac{\partial z}{\partial z}$$

$$A_z = A_z \qquad \ldots (16)$$

Also,

$$\frac{\partial A_x}{\partial x} = \frac{\partial A_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial A_x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial A_x}{\partial z}\frac{\partial z}{\partial x}$$

$$= \cos\theta \frac{\partial A_x}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A_x}{\partial \theta} \qquad \ldots (17)$$

$$\frac{\partial A_y}{\partial y} = \frac{\partial A_y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial A_y}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial A_y}{\partial \theta}\frac{\partial z}{\partial y}$$

$$= \sin\theta \frac{\partial A_y}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A_y}{\partial \theta} \qquad \ldots (18)$$

$$\frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial z} \qquad \ldots (19)$$

Using (14) in (17) we get

$$\frac{\partial A_x}{\partial x} = \cos\theta\left(\cos\theta \frac{\partial A_r}{\partial r} + \frac{\sin\theta}{r^2} A_\theta - \frac{\sin\theta}{r}\ \frac{\partial A_\theta}{\partial r}\right)$$

$$\frac{-\sin\theta}{r}\left(-\sin\theta\ A_r + \cos\theta \frac{\partial A_r}{\partial \theta} - \frac{\cos\theta}{r} A_\theta - \frac{\sin\theta}{r}\frac{\partial A_\theta}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial A_z}{\partial r} + \frac{\sin^2\theta}{r} A_r + \frac{2\sin\theta\cos\theta}{r} A_\theta - \frac{\sin\theta\cos\theta}{r}\frac{\partial A_\theta}{\partial r}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial A_\theta}{\partial \theta} \qquad \ldots (20)$$

Using (15) in (18) we get

$$\frac{\partial A_y}{\partial y} = \sin\theta \left(\sin\theta \frac{\partial A_y}{\partial r} - \frac{\cos\theta}{r^2} A_\theta + \frac{\cos\theta}{r} \frac{\partial A_\theta}{\partial r} \right)$$

$$+ \frac{\cos\theta}{r} \left(\sin\theta \frac{\partial A_r}{\partial \theta} + \cos\theta\, A_r - \frac{\sin\theta}{r} A_\theta + \frac{\cos\theta}{r} \frac{\partial A_\theta}{\partial \theta} \right)$$

$$= \sin^2\theta \frac{\partial A_r}{\partial r} + \frac{\cos^2\theta}{r} A_r - 2\frac{\sin\theta\cos\theta}{r} A_\theta + \frac{\sin\theta\cos\theta}{r} \frac{\partial A_\theta}{\partial r}$$

$$+ \frac{\cos^2\theta}{r^2} \frac{\partial A_\theta}{\partial \theta} \qquad \dots (21)$$

$$\frac{\partial A_z}{\partial z} = \frac{\partial A_z}{\partial z} \qquad \dots (22)$$

Using (20) – (22) in (13) we get

$$div\ A_i = \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{1}{r^2} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \quad \dots (23)$$

(III) Since ϕ is a scalar field, therefore

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \dots (24)$$

By Calculus

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial \phi}{\partial r} \cos\theta - \frac{\sin\theta}{r} \frac{\partial \phi}{\partial \theta} \qquad \dots (25)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)\frac{\partial z}{\partial x}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial x}\right)\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial x}\right)\frac{\sin}{r}$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial \phi}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \cos\theta\left(\cos\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)$$

$$\frac{-\sin\theta}{r}\left(-\sin\theta \frac{\partial \phi}{\partial r} + \cos\theta \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$

$$= \cos^2\theta \frac{\partial^2 \phi}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial \phi}{\partial r}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \dots (26)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \sin^2\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \phi}{\partial r\,\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial \phi}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \ldots (27)$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial z^2} \qquad \ldots (28)$$

Using (26) and (28) in (24) we get

$$\nabla^2\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \ldots (29)$$

13.5 *TENSOR*

Mathematically, physical quantities are represented by as either scalar or vectors depending on their transformation properties under rotation of the coordinate axes. A physical quantity which requires magnitude only for its complete specification is called **"scalar"**. For instance, mass, length, temperature are scalar quantities. In similar way, a physical quantity which requires a direction beside magnitude for its complete specification is called **"vector"**. For instance, displacement, velocity, acceleration are vector quantities.

In fact, there are many physical quantities which requires multi-directions along with magnitude for their complete specification are called **"tensors"**. For example, Stress, Strain, Conductivity, moment of inertia, dielectric susceptibility etc are tensors. Particularly, in case of stress, magnitude, direction of force and direction of normal on which the component acts are required i.e., two directions, one for direction of force and other is direction of normal to plane. In symbol, σ_{ij} is a stress tensor at a point in Euclidean space E_3 having nine components.

In view of above facts, "Tensors" are a natural and logical generalization of vectors, which are of great use in general relativity theory, differential geometry, mechaincs, elasticity, electromegatic theory etc. According to German mathematician F. Klein, how term tensor is generalization of the vector, can easily be understood with the help of theory of group of transformations.

13.5.1 Contravariant Tensor of Rank Two or Order (2, 0)

A set of n^2 functions A_{ij} of the n coordinates x^i $i = 1, \ldots n$ are said to be the component of a contravariant tensor of rank 2 or order (2, 0) if they transform according to the law

$$A^{-\alpha\beta} = \frac{\partial x^{-\alpha}}{\partial x^i}\frac{\partial x^{-\beta}}{\partial x^j} A_{ij}$$

On changing coordinates x^i to x^{-i}; $i = 1, 2, 3, \ldots n$.

13.5.2 Covariant Tensor of Rank Two or Order (0, 2)

A set of n^2 functions A_{ij} of the n coordinates x^i; $i = 1, 2, \ldots n$ are said to be the components of a covariant tensor of rank 2 or order (0, 2) if they transform according to the law

$$\overline{A}_{\alpha\beta} = \frac{\partial x^i}{\partial x^{-\alpha}}\frac{\partial^j}{\partial x^{-\beta}} A^{ij}$$

On changing coordinates x^i to x^{-i} ; i = 1, 2, 3, ... n.

13.5.3 Mixed Tensor of Rank Two Or Order (1, 1)

A set of n^2 functions A^i_j of the n coordinates x^i ; 1, 2, ... n are said to be the components of a mixed tensor of rank 2 or order (1, 1) if they transform according to the law

$$A^{-\alpha}_{\beta} = \frac{\partial x^{-\alpha}}{\partial x^i} \frac{\partial x^j}{\partial x^{-\beta}} A^i_j$$

On changing coordinates x^i to x^{-i} ; i = 1, 2, ... n.

13.5.4 Rank or Order of Tensor

The number of indices (excluding the dummy indices) of a tensor is called the rank of the tensor i.e., total number of indices per component.

Physically, the rank of tensor means, number of directions besides magnitude are required for its complete specification and it is not an indicator of the nature of tensor i.e., it does not characterise that whether a tensor is contravariant or covariant or mixed. Therefore it is better to use form "order" instead of "rank" because order (r, s) of a tensor shows that r is contravariant order and s is covariant order of given tensor.

Note : 1. Scalars and vectors are tensors of zero and one rank respectively.

2. The number of components of a tensor is n^r where n is the dimension of the space and r is the rank of tensor.

13.5.5 Tensors Field :

In a region of a space, if a tensor is defined for each point of the region, there is a tensor field defined in the region.

13.6 *TENSORS OF HIGHER ORDER*

13.6.1 Contravariant Tensor of Rank *r* or Order (*r*, 0)

A set of n^r functions $A^{i_1, i_2, i_3, \dots i_r}$ of the coordinates x^i ; i = 1, 2, ...n is said to be the components of a contravariant tensor of order (r, 0) if they are transform according to law

$$A^{-\alpha_1, \alpha_2, \dots \alpha_r} = \frac{\partial x^{-\alpha_1}}{\partial x^{i_1}} . \frac{\partial x^{-\alpha_2}}{\partial x^{i_2}} \dots \frac{\partial x^{-\alpha_r}}{\partial x^{i_r}} A^{i_1, j_2, \dots i_r}$$

A changing of coordinates x^i to x^{-i} .

13.6.2 Covariant Tensor of Rank 's' or Order (*o*, *s*)

A set of n^s funtions $A_{i_1, i_2, \dots i_s}$ of the n coordinates $i = 1, 2, \dots n$ is said to be the components of a covariant tensor of order (0, s) if they transform according to law

$$\overline{A}_{\alpha_1, \alpha_2, \dots a_x} = \frac{\partial x^{i_1}}{\partial x^{-\alpha_1}} \frac{\partial x^{i_2}}{\partial x^{-\alpha_2}} \dots \frac{\partial x^{i_x}}{\partial x^{-\alpha_2}} A_{i_2, i_2, \dots i_s}$$

On changing coordinates x_i to x^{-i} .

13.6.3 Mixed Tensor of Rank (*r* + *s*) or order (*r*, *s*)

A set of n^{r+s} functinons $A^{i_1, i_2, i_r}_{j_1, j_2, \dots j_z}$ of the n coordinates $x^i ; i = 1, 2, 3, \dots n$ is said be the components of a mixed tensor of order *(r, s)* if they transform according to law

$$\bar{A}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_1} = \frac{\partial \bar{x}^{\alpha}}{\partial \bar{x}^{i}} \cdots \frac{\partial \bar{x}^{\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{\beta_s}} A^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$$

on changing of coordinates x^i to $\bar{x}^i$.

13.7 *ALGEBRA OF TENSORS*

13.7.1 Summation of Tensors

The sum of two tensors that have the same number of covariant and contravariant indices and the same dimension in all indices is again a tensor of the same number of covariant and contravariant indices :

$$C^{ij}_k = A^{ij}_k + B^{ij}_k \text{ (summation convention is not applied)}$$

where elements are all sums of the corresponding elements of the two summed tensors.

Theorem : The sum (and difference) of two tensors which have same number of covariant and the same contravariant indices is again a tensor of the same rank and type as the given tensors.

Proof : Let $A^{i_1,i_2,\dots i_r}_{j_1,j_2,}$ and $B^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$ be two tensors of type *(r, s)*. Then by law of transformation

$$\bar{A}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_1} = \frac{\partial \bar{x}^{\alpha}}{\partial \bar{x}^{i}} \cdots \frac{\partial \bar{x}^{\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{\beta_s}} A^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} \quad \dots (1)$$

$$\bar{B}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_s} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{j_5}}{\partial \bar{x}^{\beta_1}} B^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} \quad \dots (2)$$

Then $$\bar{C}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_s} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{j_5}}{\partial \bar{x}^{\beta_s}} C^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} \quad \dots (3)$$

where $$\bar{C}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots b_s} = \bar{A}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_s} \pm \bar{C}^{\alpha_1,\alpha_2,\dots\alpha_r}_{\beta_1,\beta_2,\dots\beta_s}$$

and $$C^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} = A^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} \pm B^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$$

This is law of transformation of a mixed tensor of rank $(r + s)$. Therefore $C^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$ is a tensor of type (r, s).

Note: The summation is commutative and associative i.e., $A^i_j + B^i_j = B^i_j + A^i_j$

$$A_{ij} + (B_{ij} + C_{ij}) = (A_{ij} + B_{ij} + C_{ij})$$

13.7.2 Multiplication of a Tensor by a Constant

The product of a constant scalar and a tensor is again a tensor of the same rank or order whose elements are equal to the corresponding elements (of the multiplicand tensor) multiplied by the constant

If be a scalar and $A^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$ be a tensor, then

$$B^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s} = \phi\, A^{i_1,i_2,\dots i_r}_{j_1,j_2,\dots j_s}$$

Note that the multiplication is associative and commutative

i.e., $\phi\,(A^{ij}) = A^{ij}\,\phi$ and $\phi\,(\psi\, A^{ij}) = (\phi\,\psi)\, A^{ij}$

13.7.3 Opposite Tensor

Opposite tensor B^{ij} to a tensor A^{ij} is again a tensor of the same rank whose elements/ components are equal to the negatively taken corresponding elements/components of A^{ij}. It is denoted by □□ A^{ij} and we have

$$A^{ij} + B^{ij} = 0 \quad \text{or} \quad B^{ij} = (-1)\, A^{ij}$$

Its tensorial properties follows immediately from multipication by a constant if □□□□□□1.

13.7.4 Multiplication of Tensors

The product of two tensors is a tensor of a rank that is a sum of the ranks of the two constituent tensors and its components are products of the corresponding components of the constituent tensors. This product is called **outer product** of tensors.

Theorem: The outer product of two tensors is a tensor whose rank is the sum of the rank of two tensors.

Proofs : Let $A^{i_1, i_2, \ldots i_r}_{j_1, j_2, \ldots j_s}$ and $B^{k_1, k_2, \ldots k_m}_{l_1, l_2, \ldots l_n}$ be two tensors of type (r, s) and (m, n) respectively.

Then by law of transformation.

$$A^{-\alpha_1, \alpha_2, \ldots \alpha t_2}_{\beta_1, \beta_2, \ldots \beta_s} = \frac{\partial x^{-\alpha_1}}{\partial x^{i}} \cdots \frac{\partial x^{-\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial x^{-\beta_1}} \cdots \frac{\partial x^{j_s}}{\partial x^{-\beta_s}} A^{i_1, i_2, \ldots i_r}_{j_1, j_2, \ldots j_s}$$

and

$$B^{-a_1, a_2, \ldots a_m}_{b_1, b_2, \ldots b_n} = \frac{\partial x^{-\alpha_1}}{\partial x^{k_1}} \cdots \frac{\partial x^{-\alpha_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial x^{-b_1}} \cdots \frac{\partial x_{l_n}}{\partial x^{-b_n}} B^{k_1, k_2, \ldots k_m}_{l_1, l_2, \ldots l_n}$$

Then

$$C^{-\alpha_1, \alpha_2 \ldots d_r, a_1, a_2, a_m}_{\beta_1, \beta_2, \ldots \beta_s, b_1, b_2, \ldots b_n} = \frac{\partial x^{-\alpha_1}}{\partial x^{i}} \cdots \frac{\partial x^{-\alpha_r}}{\partial x^{i_r}} \frac{\partial x^{-\alpha_1}}{\partial x^{k_1}} \cdots \frac{\partial x^{-a_m}}{\partial x^{m}}$$

$$\frac{\partial x^{j_1}}{\partial x^{-\beta_1}} \cdots \frac{\partial x^{j_s}}{\partial x^{-\beta_s}} \frac{\partial x^{b_1}}{\partial x^{-b_1}} \cdots \frac{\partial x^{l_n}}{\partial x^{-b_n}} \cdot C^{l_1, l_2 \ldots i_r\, k_1\, k_2 \ldots k_m}_{j_1 j_2 \ldots j_s\, l_1, l_2 \ldots l_n}$$

where

$$C^{l_1, l_2 \ldots i_r\, k_1\, k_2 \ldots k_m}_{j_1 j_2 \ldots j_s\, l_1, l_2 \ldots l_n} = A^{l_1, l_2 \ldots i_r}_{j_1, j_2 \ldots j_s} \; B^{k_1, k_2 \ldots k_m}_{l_1, l_2 \ldots l_m}$$

This is law of transformation of a mixed tensor of rank ($r + m$, $s + n$)

Note : 1. In general outer product is not commutative i.e.,

$$A^{ij}_{k} B^{l}_{mn} \neq B^{l}_{mk} A^{ij}_{n};\; A^{ij} B^{l}_{k} = B^{l}_{k} A^{ij}$$

2. Outer product is associative and distributive i.e.,

$$A^{ij} (B^{l}_{m} + C^{l}_{m}) = A^{ij} B^{l}_{in} + A^{ij} C^{l}_{m}$$

Theorem : If A^{i} and B_{j} are two vectors then $A^{i} B_{j}$ is tensor of type (1, 1).

Proof : Same as previous theorem.

Note: The outer product of two contravariant (covariant) vectors is a contrvariant (covariant) tensor rank 2. But a contravariant (covariant) tensor of rank 2 is not necessarily the outer product of two vectors.

13.7.5 Contraction

Contraction is an operation by which we reduce the rank of the tensor by two. If we set one contravarriant and one covariant indices to be equal to each other i.e., this index becomes dummy index.

or

Contraction is an operation by which the rank of a mixed tensor is reduced by 2 when one contravariant and one covariant indices are kept same and performing the summation process. For instance

$$A^{il}_{ij} = B^{l}_{j},\ A^{jl}_{ij} = C^{l}_{j},\ A^{kj}_{ij} = D^{k}_{i},\ A^{ji}_{ik} = E^{j}_{k}$$

In order to visualize, let $[a^i_j]$ is matrix of order n i.e.,

$$[a^i_j] = \begin{bmatrix} a^1_1 & a^1_2 & \dots a^1_n \\ a^2_1 & a^2_2 & \dots a^2_n \\ a^n_1 & a^n_2 & \dots a^n_n \end{bmatrix}$$

Then $\lambda = \sum_{i=1}^{n} a^i_i =$ trace $(A) =$ scalar. In that case $A = [a^i_j]$ is a tensor of rank 2, then contraction of a^i_j is a^i_i = trace (A) = scalar.

In case of tensor of type (r, s), tensor by applying contraction is of type (r □□1, s □□1)

13.7.6 Inner Product of Two Tensors

The inner product of two tensor is outer product of the tensors followed by a contraction.

Example: Then inner product of tensors A^i_j and B^{kj}_m is tensor of type (2, 1).

Solution: Let A^i_j and B^{kl}_m be two tensors of type (1, 1) and (2, 1) respectively. Then

$$A^{-\alpha}_{\beta} = \frac{\partial x^{-\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial x^{-\beta}} A^i_j$$

and
$$B^{-ab}_{c} = \frac{\partial x^{-a}}{\partial x^{k}} \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x^{-c}} B^{kl}_m$$

Then
$$A^{-\alpha}_{\beta} B^{-ab}_{c} = \frac{\partial x^{-\alpha}}{\partial x^{i}} \frac{\partial x^{-a}}{\partial x^{k}} \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{j}}{\partial x^{-\beta}} \frac{\partial x^{m}}{\partial x^{-c}} A^i_j B^{kl}_m$$

Set $j = k$,

$$A^{-\alpha}_{\beta} B^{-ab}_{c} = \frac{\partial x^{-\alpha}}{\partial x^{i}} \frac{\partial x^{-a}}{\partial x^{k}} \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{j}}{\partial x^{-\beta}} \frac{\partial x^{m}}{\partial x^{-c}} A^i_j B^{kl}_m$$

$$= \frac{\partial x^{-\alpha}}{\partial x^{i}} \left(\frac{\partial x^{-a}}{\partial x^{k}} \frac{\partial x^{k}}{\partial x^{-\beta}} \right) \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x^{-c}} A^i_k B^{kl}_m$$

$$= \frac{\partial x^{-\alpha}}{\partial x^{i}} \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x^{-c}} A^i_k B^{kl}_m \left(\text{for } a = \beta \text{ and } \frac{\partial x^{-\alpha}}{\partial x^{-\beta}} = 1 \right)$$

$$(A^{-a}_{a} B^{-ab}_{c}) = \frac{\partial x^{-\alpha}}{\partial x^{i}} \frac{\partial x^{-b}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x^{-c}} (A^i_k B^{kl}_m)$$

This is the law of transformation of a mixed tensor of rank three i.e., of order (2, 1).

Note: 1. Inner and outer product of vectors (tensors) are same as scalar and vector product of vectors in vector calculus.

2. Inner product is also called contracted product. For instance

$$A_j^i B_{ik}^l = \delta_m^i (A_j^m B_{ik}^l)$$

13.8 *TENSOR CHARACTER AND QUOTIENT LAW*

It is noticed that the three basic tensorial operations i.e. summation, multiplication and contraction, produce always again a tensor. They are said to preserve the tensor character. This property serves as another means of distinguishing a tensor and called tensor algebra on
Riemannian space.

This property is known as **"Quotient law"** which is stated as "if the inner product of set of function with an arbitrary tensor is a tensor then these set of functions are component of a tensor".

Example: Show that the expression *A (I, j, k)* is a convariant tensor of rank three if *A (I, j, k)* B^k is a covariant tensor of rank 2 and B^k is a cotravariant vector.

Solution: Since $A(i, j, k)\, B^k$ is a covariant tensor of rank 2 therefore by tensor law of transformation, we have

$$\overline{A}(i, j, k)\, B^{-k} = \frac{\partial x^a}{\partial x^{-i}} \frac{\partial x^b}{\partial x^{-j}} A(a, b, c)\, B^c$$

Using tensor law of transformation for vector B^k i.e.,

$$B^{-k} = \frac{\partial x^{-k}}{\partial x^c} B^c \quad \text{or} \quad B^c = \frac{\partial x^c}{\partial x^{-k}} B^{-k}$$

we get

$$\overline{A}(i, j, k)\, B^{-k} = \frac{\partial x^a}{\partial x^{-i}} \frac{\partial x^b}{\partial x^{-j}} \frac{\partial x^c}{\partial x^{-k}} A(a, b, c) \frac{\partial x^c}{\partial x^{-k}} B^{-k}$$

$$\left(\overline{A}(i, j, k) - \frac{\partial x^a}{\partial x^{-i}} \frac{\partial x^b}{\partial x^{-j}} \frac{\partial x^c}{\partial x^{-k}} A(a, b, c)\right) B^{-k} = 0$$

Since B^{-k} is arbitrary vector therefore

$$\overline{A}(i, j, k)\, B^{-k} = \frac{\partial x^a}{\partial x^{-i}} \frac{\partial x^b}{\partial x^{-j}} \frac{\partial x^c}{\partial x^{-k}} A(a, b, c)$$

This is law of transformation of a tensor of type (0, 3) i.e., *A* (*i*, *j*, *k*) are component of a covariant tensor of rank 3.

13.9 *SYMMETRIC AND SKEW-SYMMETRIC TENSORS*

The order of the index in a tensor is important. The tensor A^{ij} (or A_{ij}) is not necessarily the same as that of the tensor A^{ji} (or A_{ij}). In case of matrices, A^{ji} is the transpose of A^{ij}.

If two contravariant or covariant indices of a tensor can be interchanged without altering the tensor, then it is called "**symmetric**" in every pair of such indices.

In symbols, A_{ij} is symmetric iff $A_{ij}^k = A_{ji}^k$ and A_{ij} is skew-symmetric iff $A_{ij}^k = -A_{ji}^k$.

Following facts should be noted:

- In general, symmetry and skew-symmetry of a tensor can not be defined for a tensor with respect to two indices of which one is contravariant and the other is covariant.
- The kronecker delta δ^i_j is symmetric in i and j i.e., symmetry is reversed under coordinate transformation i.e., $\delta^i_j = \delta^j_i$ and $\delta^{-}_{j} = \delta^{-j}_{i}$.
    - In skew symmetric tensor $A_{ii} = 0 \ \forall \ i = 1, 2, 3, \dots n$
- No of independent distinct component of symmetric and skew-symmetric tensor of rank 2 is $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$ respectively.

Theorem: A covariant or contravariant tensor of rank two can always be expressed as the sum of a symmetric and skew-symmetric tensors.

Proof: Let A_{ij} be a covariant tensor. Then

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$A_{ij} = S_{ij} + T_{ij} \qquad \dots (1)$$

where
$$S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}); T_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$$

Now
$$S_{ji} = \frac{1}{2}(A_{ji} + A_{ij}) = \frac{1}{2}(A_{ij} + A_{ji}) = S_{ij}$$

i.e., S_{ij} is symmetric tensor.

and
$$T_{ij} = \frac{1}{2}(A_{ij} - A_{ji}) = -\frac{1}{2}(A_{ji} - A_{ij}) = -T_{ji}$$

i.e., T_{ij} is skew-symmetric tensor.

Thus A is sum of a symmetric and a skew - symmetric tensors.

Theorem: If T_i be the component of a covariant vector than $\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right)$ are components of a skew-symmetric covariant tensor of rank two.

Proof: Since T_i is a covariant vector therefore by law of transformation,

$$\overline{T}_i = \frac{\partial x^\alpha}{\partial x^{-i}} T_\alpha \qquad \dots (1)$$

Differentiating w.r.t. x^{-j} partially

$$\frac{\partial \overline{T}}{\partial x^{-j}} = \frac{\partial x^\alpha}{\partial x^{-i}} \frac{\partial T_\alpha}{\partial x^{-j}} + \frac{\partial^2 x^2}{\partial x^{-j} \partial x^{-i}} T_\alpha$$

i.e.,
$$\frac{\partial \overline{T}}{\partial x^{-j}} = \frac{\partial x^\alpha}{\partial x^{-i}} \frac{\partial T^\beta}{\partial x^{-j}} \frac{\partial T_\alpha}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial x^{-j} \partial x^{-i}} T_\alpha \qquad \dots (2)$$

Similarly,
$$\frac{\partial \overline{T}}{\partial x^{-i}} = \frac{\partial x^\beta}{\partial x^{-j}} \cdot \frac{\partial x^\alpha}{\partial x^{-i}} \frac{\partial T^\beta}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial x^{-i} \partial x^{-j}} T_\alpha \qquad \dots (3)$$

Subtracting (3) from (2) we get

$$\left(\frac{\partial \overline{T}_i}{\partial x^{-j}} - \frac{\partial \overline{T}_j}{\partial x^{-i}}\right) = \frac{\partial x^{\alpha}}{\partial x^{-i}} \frac{\partial x^{\beta}}{\partial x^{-j}} \left(\frac{\partial T_{\alpha}}{\partial x^{\beta}} - \frac{\partial T_{\beta}}{\partial x^{\alpha}}\right) \quad \dots (4)$$

This law of transformation of a covariant tensor of rank 2.

let
$$A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$$

Then
$$A_{ji} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j} = -\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right) = -A_{ij}$$

Which shows that $A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ are components of a skew-symmetric tensor of rank 2. **Note:** Due to presence of 2nd form in Eq. (2), $\frac{\partial T_i}{\partial x^j}$ are not components of a Tensor.

Theorem: Show that kronecker delta is a mixed tensor of order (1, 1) and it is invariant.

Proof: Let x^{-i} □□i □□1, 2,□n be coordinates of a point in V_n .Then

$$\frac{\partial x^{-i}}{\partial x^{-j}} = \frac{\partial x^{-i}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{-j}}$$

$$= \frac{\partial x^{-i}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{b}} \frac{\partial x^{b}}{\partial x^{-j}}$$

$$\delta_j^{-i} = \frac{\partial x^{-i}}{\partial x^{-j}} = \frac{\partial x^{-i}}{\partial x^{a}} \cdot \frac{\partial x^{b}}{\partial x^{-j}} \cdot \frac{\partial x^{a}}{\partial x^{b}} = \frac{\partial x^{-i}}{\partial x^{a}} \frac{\partial x^{b}}{\partial x^{-j}} \delta_b^a \quad \dots (1)$$

Which shows that δ_j^i i.e., Kronecker delta is a mixed tensor of rank 2.

Now from (1)

$$\delta_j^{-i} = \frac{\partial x^{-i}}{\partial x^{a}} \left(\frac{\partial x^{b}}{\partial x^{-j}} \delta_b^a\right) = \frac{\partial x^{-i}}{\partial x^{a}} \left(\frac{\partial x^{a}}{\partial x^{-j}}\right)$$

$$\delta_j^{-i} = \frac{\delta x^{-i}}{\delta x^{-j}}$$

i.e., δ_j^i is invariant.

Theorem: Prove that the transformation of a contravariant (covariant) tensor is transitive. Prove that the transformation of a contravariant (covariant) tensor form a group.

Proof: Let A^{ij} be a contravariant vector in a coordinate system $x^i = (i = 1, 2, \dots n)$. Then by law of transformation we have

$$A^{-ij} = \frac{\partial x^{-i}}{\partial x^{\alpha}} \frac{\partial x^{-j}}{\partial x^{\beta}} A^{\alpha} B^{\beta} \quad (\text{From } x^i \text{ to } x^{-i}) \quad \dots (1)$$

Applying law of transformation from x^{-i} to $x^{=i}$ we have

$$A^{=ab} = \frac{\partial x^{=a}}{\partial x^{-i}} \frac{\partial x^{=b}}{\partial x^{-j}} A^{-ij} \quad \dots (2)$$

Using (1) in (2) we get

$$\bar{\bar{A}}^{ab} = \frac{\partial \bar{\bar{x}}^{a}}{\partial \bar{x}^{i}} \cdot \frac{\partial \bar{\bar{x}}^{b}}{\partial \bar{x}^{j}} \frac{\partial \bar{x}^{i}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{j}}{\partial x^{\beta}} A^{\alpha} B^{\alpha}$$

$$= \left(\frac{\partial \bar{\bar{x}}^{a}}{\partial x^{i}} \frac{\partial \bar{x}^{i}}{\partial x^{\alpha}} \right) \left(\frac{\partial \bar{\bar{x}}^{b}}{\partial \bar{x}^{j}} \frac{\partial \bar{x}^{j}}{\partial x^{\beta}} \right) A^{\alpha} B^{\beta}$$

$$\bar{\bar{A}}^{ab} = \frac{\partial \bar{\bar{x}}^{a}}{\partial x^{\alpha}} \frac{\partial \bar{\bar{x}}^{b}}{\partial x^{\beta}} A^{\alpha} B^{\beta} \qquad \ldots (3)$$

This is law of transformation of a contravariant from x^i to $\bar{\bar{x}}^i$. This property is called that transformation of contravariant is transitive or form a group.

Note: 1. Proceeding on the same lines as above, theorem for covariant tensor/vector can easily be proved.

2. This theorem can easily be proved for mixed tensor.

Theorem: If all components of tensor in one coordinate system are zero at a point, then they are all zero at this point in every coordinate system.

Proof: Let $A^{\alpha\beta} = 0$ in x^{α} $\forall\ \alpha, \beta = 1, 2, \ldots n$. Then on using in (1) and (2) of previous theorem we get $\bar{A}^{ij} = 0 = \bar{\bar{A}}^{ab}$. Hence proved.

Example: If a_{ij} and g_{ij} are symmetric and u^i, v^i are components of contravariant vector such that

$$(a_{ij} - kg_{ij})\, u^i = 0; (a_{ij} - k'g_{ij})\, v' = 0\ i,\ j = 1, 2, 3 \ldots n \text{ and } k \neq k'$$

then
$$g_{ij} u^i v^j = 0 = a_{ij} u^i v^j$$

Solution : We have

$$(a_{ij} - kg_{ij})\, u^i = 0; (a_{ij} - k'g_{ij})\, v^i = 0 \qquad \ldots (1) \text{ and } (2)$$

Multiplying (1) and (2) v^j and u^j and subtracting we get

$$a_{ij} u^i v^j - kg_{ij} u^i v^j - a_{ij} v^j u^j + k'g_{ij} v^i u^j = 0 \qquad \ldots (3)$$

Interchanging i and j we get

$$a_{ji} u^j v^i - kg_{ij} u^j v^i - a_{ji} u^i v^j + k'g_{ji} u^i v^j = 0$$

Using $a_{ij} = a_{ji}$ and $g_{ij} = g_{ji}$ we have

$$a_{ij} u^j v^i - kg_{ij} u^j v^i - a_{ij} u^j v^j + k'g_{ij} u^i v^j = 0 \qquad \ldots (4)$$

Adding (3) and (4) we have

$$(k' - k)\, g_{ij} u^j v^i + (k' - k)\, g_{ij} u^i v^j = 0$$

$$g_{ji} u^i v^j + g_{ij} v^i u^j = 0 \ \because\ k \neq k^i$$

$$2g_{ij} u^i v^j = 0 \text{ i.e., } g_{ij} u^i v^j = 0 \qquad \ldots (5)$$

Multiplying u^j in (1) and using (5) we get

$$a_{ij} u^i v^j = 0 \qquad \ldots (6)$$

13.10 *IMPORTANT FACTS*

There are same simple rules for checking the correctness of the indices in a tensor equation:

1. A free index should match in all terms throughout the equation at the same level.
2. A dummy index should match in each term of the equation separately i.e., twice at opposite levels.
3. No index should occur more than twice in any term.
4. If a tensor equation is true in one coordinate system then it is true in all other co-ordinate system i.e., they are in one to one correspondence.
5. The rank/order of each term of tensor equation is same.

13.11 *SUMMARY*

In this block, we have learned about the various type of tensors and their addition, substation multiplication (inner and outer product) along with symmetric property, contraction of a tensor and quotient law.

13.12 *REFERENCE*

1. An introduction to Riemannian Geometry and the Tensor calculus by C.E. Weatherburn "Cambridge University Press."
2. Matrices and Tensors in physics by A.W. Joshi "Wiley Eastern Limited".
3. Tensors by Ram Bilas Mishra "Hardwari publications Allahabad".

13.13 TERMINAL QUESTIONS

1. The components of a contravariant vector A^i in x-co-ordinate system are $A^i = f;\ A = 0;\ j = 2, 3, 4, \ldots n$. Find its components in $\bar{x}$-coordiante system.

2. Let A^1, A^2 are the contravariant component of a vector in (x^1, x^2) coordinate system. Find the contravariant and covariant components of A in (x^{-1}, x^{-2}) if

$x^{-1} = x^1 - x^2 \cot \alpha, x^{-2} = x^2 \operatorname{cosec} \alpha$ and A_1, A_2 are covariant components of A in (x^1, x^2)

3. The components of a contravariant vector in the *x*-coordinate system are 2 and 3. Find its components in the $\bar{x}$ coordinate system if

$$x^{-1} = 3\,(x^1)^2,\ x^{-2} = 5\,(x^1)^2 + 3(x^2)^2$$

4. If $\frac{x^1}{x^2}, \frac{x^2}{x^1}$ be covariant components of a vector in rectangular coordinates x^1, x^2, find its components in polar coordinates r, θ.

5. Show that the tensor equation $a_n^m \theta_m = \beta \theta_n$ where β is an invariant and θ_m are arbitrary vector, demands that $a_n^m = \delta_n^m \beta$.

6. If the components of a contravariant tensor of type (2, 0) in $V^2 = \{x^1, x^2) \mid x^1, x^2 \in R\}$ are $T^{11} = 1, T^{12} = 0, T^{21} = 0, T^{22} = 1$ find T^{-ij} in $V^{-2} = \{(x^{-1}, x^{-2}) \in R\}$ where functional relation between the two coordinate systems are $x^{-1} = (x^1)^2, x^{-2} = (x^2)^2$

7. Show that the contracted tensor A_n^m is scalar.

8. Applying contraction on δ_j^i find is value.

9. If $u_{ij} \neq 0$ are components of a tensor of type (0, 2) and if the equation

$$fu_{ij} + gu_{ji} = 0$$

holds, then prove that either $f = g$ and u_{ij} is skew-symmetric or $f = -g$ and u_{ij} is symmetric.

10. If A_{ij} is a skew-symmetric tensor, prove that

$$(\delta_j^i \delta_l^k + \delta_l^i \delta_j^k)\, A_{ik} = 0$$

11. If A_{ij} is a skew-symmetric tensor and B^i is a contravariant vector then show that $A_{ij} B^i B^j = 0$. Is the converse true?

12. If $X(i, j)\, B^j = C_i$, B^j is an arbitrary contravariant vector and C_i is a covariant vector, show that $X(i, j)$ is a tensor. What is its type?

13.14 ANSWERS

1. $A^{-k} = \dfrac{\partial x^{-k}}{\partial x^1} f$

2. $A^{-1} = A^1 - A^2 \cot \alpha,\ A^{-2} = a^2 \operatorname{cosec} \alpha$, $\overline{A}_1 = A_1, \overline{A}_2 = A_2 \cos \alpha + A_2 \sin \alpha$

3. $12x^1, 20x^1 + 18x^2$

4. $\dfrac{\cos^3 \theta + \sin^3 \theta}{\sin \theta \cos \theta}, r\,(\sin \theta - \cos \theta)$

6. $T^{-11} = 4x^{-1}, T^{-12} = 0 = T^{-21}, T^{-22} = 4x^{-2}$

8. n

12. (0, 2).

UNIT 14: *DIVERGENCE AND CURL OF A VECTOR*

CONTENTS:

14.1 *INTRODUCTION*

In tensor geometry, the divergence of a vector field is a scalar field representing the "outward flow" from a point, while the curl is a vector field measuring the rotational tendency or "swirling" at a point. Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl.

14.2 *OBJECTIVES*

After studying this block, you should be able to

1. Gradient
2. Divergence
3. Curl

14.3 *GRADIENT*

Gradient

Let $f(x^i)$ be a scalar function. As in Euclidean space, the gradient of $f(x^i)$ in Riemannian space V_n is defined as

$$\text{grade } (f) = \frac{\partial f}{\partial x^i} = \partial_i f = \nabla_i f \qquad \ldots (1)$$

It is earlier verified that gradient of a scalar function is a covariant vector and is normal to the surface represented by $f(x^i) =$ constant.

In view of the definition of magnitude of a covariant vector, the square of the magnitude of the gradient vector is defined as

$$\nabla_1 f \equiv g^{ij} (\nabla_i f)(\nabla_j f) \qquad \ldots (2)$$

It is also known as **Beltrami first order** differential order operator.

Another Belitrami first order different operator $\Delta_1(f, \phi)$ is defined as

$$\Delta_1(f, \phi) = g^{ij} (\nabla_i f)(\nabla_j \phi)$$

In view of above $\qquad \Delta_1(f) = \Delta_1(f, f)$

14.4 *DIVERGENCE*

Divergence

Let A^i be a contravariant vector, function of coordinates $x^i; i = 1, 2, 3, \ldots n$. As Euclidean space, the divergence of A^i Riemannian space V_n is defined as

$$\text{div}\, A^i = \nabla_i A^i = A^i_{,i} \qquad \ldots (1)$$

But covariant differentiation of contrvariant vector is defined as

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + A^k \begin{Bmatrix} i \\ kj \end{Bmatrix} \qquad \ldots (2)$$

Therefore (1) can be expressed as

$$\text{div}\, A^i = \nabla_i A^i = \frac{\partial A^i}{\partial x^i} + A^k \begin{Bmatrix} i \\ ki \end{Bmatrix}$$

or
$$\text{div}\, A^i = \frac{\partial A^i}{\partial x^i} + A^j \begin{Bmatrix} i \\ ij \end{Bmatrix} \qquad \ldots (3)$$

The divergence of a covariant vector A^i is defined as

$$\text{div}\, A_i = \nabla_i A^i = g^{ij} \nabla_j A_i \qquad \ldots (4)$$

Theorem: Prove that div $A_i = \text{div}\, A^i$ and div $A^i = \dfrac{1}{\sqrt{g}} \dfrac{\partial(\sqrt{g}\, A^k)}{\partial x^k}$

Proof: We have

$$\text{div}\, A^i = g^{ik} A_{i,k}$$

$$\text{div}\, A^i = (g^{ik} A_i)_{,k} = (A^k)_{,k} = A^k_{,k} = \text{div}\, A^i$$

and
$$\text{div}\, A^i = A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \begin{Bmatrix} i \\ ik \end{Bmatrix} A^k$$

$$= \frac{\partial A^i}{\partial x^i} + \frac{\partial}{\partial x^k}(\log \sqrt{g})\, A^k \because \begin{Bmatrix} l \\ lk \end{Bmatrix} = \frac{\partial}{\partial x^k}(\log \sqrt{g}) \text{ and}$$

$g = |g_{ij}|$

$$= \frac{\partial A^i}{\partial x^i} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} A^k$$

$$\text{div}\, A^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, A^i)$$

.

Theorem: The div A^i is a scalar function.

Proof: By law transformation

$$A^i = A^{-a} \frac{\partial x^i}{\partial x^{-a}} \quad \text{or} \quad A^{-a} = A^i \frac{\partial x^{-a}}{\partial x^i} \qquad \dots (1)$$

Different w.r.t. x^k is

$$\frac{\partial A^i}{\partial x^k} = \frac{\partial A^{-a}}{\partial x^{-b}} \frac{\partial x^{-b}}{\partial x^k} \frac{\partial x^i}{\partial x^{-a}} + A^{-a} \frac{\partial^2 x^i}{\partial x^{-b} \partial x^{-a}} \cdot \frac{\partial x^{-b}}{\partial x^k} \qquad \dots (2)$$

Using $\dfrac{\partial^2 x^i}{\partial x^{-b} \partial x^{-a}} = \overline{\begin{Bmatrix} c \\ ab \end{Bmatrix}} \dfrac{\partial x^i}{\partial x^c} - \begin{Bmatrix} i \\ jl \end{Bmatrix} \dfrac{\partial x^j}{\partial x^{-a}} \dfrac{\partial x^l}{\partial x^{-b}}$

We get

$$\frac{\partial A^i}{\partial x^k} = \frac{\partial A^{-a}}{\partial x^{-b}} \frac{\partial x^{-b}}{\partial x^k} \frac{\partial x^i}{\partial x^{-a}} + A^{-a} \left[\overline{\begin{Bmatrix} c \\ ab \end{Bmatrix}} \frac{\partial x^i}{\partial x^c} - \begin{Bmatrix} i \\ jl \end{Bmatrix} \frac{\partial x^j}{\partial x^{-a}} \frac{\partial x^l}{\partial x^{-b}} \right] \cdot \frac{\partial x^{-b}}{\partial x^k}$$

$$\frac{\partial A^i}{\partial x^k} + A^j \begin{Bmatrix} i \\ jk \end{Bmatrix} = \left(\frac{\partial x^{-a}}{\partial x^{-b}} + \overline{A}^a \overline{\begin{Bmatrix} a \\ cb \end{Bmatrix}} \right) \frac{\partial x^{-b}}{\partial x^k} \frac{\partial x^i}{\partial x^{-a}} \qquad \dots (3)$$

let $\nabla_k A^i = \dfrac{\partial A^i}{\partial x^k} + A^j \begin{Bmatrix} i \\ jk \end{Bmatrix}$ $\qquad \dots (4)$

Then $\nabla_k A^i = \overline{\nabla}_b \overline{A}_a \dfrac{\partial x^{-b}}{\partial x^k} \dfrac{\partial x^i}{\partial x^{-a}}$

Therefore

$$\text{div } A^i = \nabla_i A^i = \overline{\nabla}_b A^{-a} \frac{\partial x^{-b}}{\partial x^k} \frac{\partial x^i}{\partial x^{-a}} \because i = k$$

$$= \overline{\nabla}_b A^{-a} \delta^b_a$$

$$\text{div } A^i = \nabla_i A^i = \overline{\nabla}_a A^{-a}$$

Which shows that div A^i is a scalar function.

Note: $\text{div}(A^{ij}) = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^j} (\sqrt{g}\ A^{ij}) + A^{aj} \begin{Bmatrix} i \\ aj \end{Bmatrix}$

$$\text{div}(A^i_j) = \nabla_i A^i_j = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\ A^i_j) - A^i_a \begin{Bmatrix} a \\ ij \end{Bmatrix}$$

14.5 *CURL*

Curl of a Covariant Vector

Let A_i be covariant vector. The skew-symmetric part of the covariant derivative of A_i w.r.t. the indices *i* and *j* is a covariant tensor of order (0, 2) :

$$\text{Curl } (A_i) = \nabla_i A_j - \nabla_j A_i$$

is called the **curl (or rotation)** of A_l i.e.,

$$\text{Curl } (A_i) = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x^j}$$

It should be noted that Curl (A_i) is skew-symmetric tensor and can have at most $\frac{n\,(n-1)}{2}$ linearly independent components in a V_n.

Theorem: Show that if the covariant derivative of a covariant vector is symmetric then vector is gradient.

Proof: If the covariant derivative of a covariant vector is symmetric then

$$A_{i,\,j} = A_{j,\,i}$$

where $$A_{i,\,j} = \frac{\partial A_i}{\partial x^j} - A_m \begin{Bmatrix} m \\ ij \end{Bmatrix}$$

Thus $$\frac{\partial A_i}{\partial x^j} - A_m \begin{Bmatrix} m \\ ij \end{Bmatrix} = \frac{\partial A_j}{\partial x^i} - A_m \begin{Bmatrix} m \\ ji \end{Bmatrix}$$

$$\frac{\partial A_i}{\partial x^j} = \frac{\partial A_j}{\partial x^i} \text{ as } \begin{Bmatrix} a \\ bc \end{Bmatrix} \text{ is symmetric w.r.t. } b \text{ and } c$$

$$\frac{\partial A_i}{\partial x^j} .\, dx^j = \frac{\partial A_j}{\partial x^i} .\, dx^j$$

$$dA^i = \frac{\partial}{\partial x^i} (A_j dx^i) \because \frac{\partial x^j}{\partial x^i} = 1 \text{ iff } i = j$$

Integration gives

$$A_i = \int \frac{\partial}{\partial x^i} (A_j \partial x^j) = \frac{\partial}{\partial x^i} \int A_j dx^j$$

Hence $A_i = \frac{\partial \phi}{\partial x^i} = \phi_{,\,i} = \text{grad } \phi$, because $\int A_j dx^i$ is scalar quantity.

Theorem: A necessary and sufficient condition that the curl of a vector vanishes is that the vector field be gradient.

Proof: Let the curl of a vector A_i vanishes i.e.,

$$\text{curl}(A_i) = A_{i,\,j} - A_{j,\,i} = 0$$

In view of above theorem it can easily be prove that $A_i = \text{grad } \phi$,

Let $A_i = \nabla\phi$. Then

$$A_i = \nabla\phi = \frac{\partial \phi}{\partial x^i}$$

and $$\frac{\partial A_i}{\partial x^j} = \frac{\partial^2 \phi}{\partial x^j \partial x^i} ; \frac{\partial A_j}{\partial x^i} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$$

Therefore $$\frac{\partial A^i}{\partial x^j} = \frac{\partial A_j}{\partial x^i} \text{ as } \frac{\partial^2 \phi}{\partial x^j \partial x^i} = \frac{\partial^2 \phi}{\partial x^i\, \partial x^j}$$

i.e., curl $(A_i) = 0$

Theorem: If ϕ is a scalar function of x^i then

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\, g^{kr}\frac{\partial\phi}{\partial x^r}\right)$$

Proof: Since $\nabla^2\phi =$ div (grad ϕ), therefore

and $\quad$ grad $(A^i) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\, A^k)$

Also, $\quad$ grad $\phi = \frac{\partial\phi}{\partial x^r}$ and $g^{kr}\frac{\partial\phi}{\partial x^r} = A^k$

Thus $\quad \nabla^2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\, g^{kr}\frac{\partial\phi}{\partial x^r}\right)$

Example 4. Divergence and Laplacian operators in cylindrical coordinates.
Solution: In cylindrical coordinates,

$$ds^2 = (dx^1)^2 + (x^1 dx^2)^2 + (dx^3)^2;\ x^i \geq 0$$

where $\quad x^1 = r,\ x^2 = \theta,\ x^3 = z$

Thus, $\quad g_{11} = 1,\ g_{22} = (x^1)^2,\ g_{33} = 1;\ g_{12} = g_{23} = g_{13} = 0;\ g = (x^1)^2$

And $\quad g^{11} = 1,\ g^{22} = \frac{1}{(x^1)^2},\ g^{33} = 1$

We know that div $(A^i) = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}A^k)}{\partial x^k}$

Therefore $\quad$ div $(A^i) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x}(\sqrt{g}A^1) + \frac{\partial}{\partial x^2}(\sqrt{g}\, A^2) + \frac{\partial}{\partial x^3}(\sqrt{g}\, A^3)\right]$

$$= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}(x^1A^1) + \frac{\partial}{\partial x^2}(x^1A^2) + \frac{\partial}{\partial x^3}(x^1A^3)\right]$$

$$= \frac{1}{x^1}\left[A_1^1 + x^1\frac{\partial A^1}{\partial x^1} + x^1\frac{\partial A^2}{\partial x^2} + x^1\frac{\partial A^3}{\partial x^3}\right]$$

$$= \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} + \frac{A^1}{x^1}$$

or $\quad$ div $(A^1) = \frac{\partial A^1}{\partial r} + \frac{\partial A^2}{\partial\theta} + \frac{\partial A^3}{\partial z} + \frac{A_1}{r} \quad \ldots (1)$

Since div $(A^i) =$ div (A_1) therefore

$$\text{div}(A_1) = \frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3} + \frac{A_1}{x_1} \quad \ldots (2)$$

Since $A_i = g_{ij}A^j$ therefore

$$A_1 = g_{11}A^1 + g_{12}A^2 + g_{13}A^3 = A^1$$

$$A_2 = g_{22}A^2 = (x^1)^2 A^2 = A^2 r^2$$
$$A_3 = g_{33}A^3 = A^3$$

Using in div $(A^i) = \text{div}(A_i)$ we get (From (1))

$$\text{div}(A_i) = \frac{\partial}{\partial r}(A_1) + \frac{\partial}{\partial \theta}\left(\frac{A_2}{r^2}\right) + \frac{\partial}{\partial z}(A_3) + \frac{A_1}{r}$$
$$\text{div}(A_i) = \frac{\partial A_1}{\partial r} + \frac{1}{r^2}\frac{\partial A_2}{\partial \theta} + \frac{\partial A_3}{\partial z} + \frac{A_1}{r} \qquad \dots (3)$$

We know that

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, g^{kr} \frac{\partial \phi}{\partial x^r}\right)$$

Therefore, $$\nabla^2 \phi = \frac{1}{x^1} \frac{\partial}{\partial x^k}\left(x^1 g^{kr} \frac{\partial \phi}{\partial x^r}\right)$$

$$= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1 g^{1r} \frac{\partial \phi}{\partial x^r}\right) + \frac{\partial}{\partial x^2}\left(x^1 g^{2r} \frac{\partial \phi}{\partial^{rr}}\right) + \frac{\partial}{\partial x^3}\left(x^1 g^{3r} \frac{\partial \phi}{\partial x^r}\right)\right]$$

$$= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1 g^{11} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(x^1 g^{22} \frac{\partial \phi}{\partial^2}\right) + \frac{\partial}{\partial x^3}\left(x^1 g^{33} \frac{\partial \phi}{\partial x^3}\right)\right]$$

$$= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1 \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(x^1 \frac{1}{(x^1)^2} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(x_1 \frac{\partial \phi}{\partial x^3}\right)\right]$$

$$\because g^{22} = \frac{1}{g_{22}} = \frac{1}{(x^1)^2}$$

$$= \frac{1}{x^1}\left[\frac{\partial \phi}{\partial x^1} + x^1 \frac{\partial^2 \phi}{\partial (x^1)^2} + \frac{1}{x^1}\frac{\partial^2 \phi}{\partial (x^2)^2} + x_1 \frac{\partial^2 \phi}{\partial (x^3)^2}\right]$$

$$= \frac{\partial^2 \phi}{(dx^1)^2} + \frac{1}{(x^1)^2}\frac{\partial^2 \phi}{\partial (x^2)^2} + \frac{\partial^2 \phi}{\partial (x^3)^2} + \frac{1}{x^1}\frac{\partial \phi}{\partial x^1}$$

$$\boxed{\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}} + \frac{\partial^2 \phi}{\partial z^2}$$

Example 5. Find div and ∇^2 in spherical polar coordinates.

Solution: In spherical polar coordinates,

$$ds^2 = (dx^1)^2 + (x^1 dx^2)^2 + (x^1 \sin x^2)^2\,(dx^3)^2$$

Therefore, $$g_{11} = 1,\ g_{22} = (x^1)^2,\ g_{33} = (x^1 \sin x^2)^2\ ;\ g_{12} = g_{13} = g_{22} = 0$$

and $$g^{11} = 1,\ g^{22} = \frac{1}{(x^1)^2},\ g^{33} = \frac{1}{(x^1 \sin x^3)^2};\ g = (x^1)^4 \sin^2(x^2)$$

we know $$\text{div}\,(A^i) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\, A^k)$$

$$= \frac{1}{(x^1)^2 \sin x^2} \left[\frac{\partial}{\partial x^1} ((x^1)^2 \sin x^2 A^1) + \frac{\partial}{\partial x^2} ((x^1)^2 \sin x^2 A^2) + \frac{\partial}{\partial x^3} ((x^1)^2 \sin x^2 A^2 \right]$$

$$= \frac{1}{(x^1)^2 \sin x^2} \left[\begin{array}{l} 2x^1 \sin x^2 A^1 + (x^1)^2 \sin x^2 \dfrac{\partial A^1}{\partial x^1} \\ + (x^1)^2 \sin x^2 \dfrac{\partial A^2}{\partial x^2} + (x^1)^2 \cos x^2 A^2 \end{array} \right.$$

$$\left. + (x^1)^2 \sin x^2 \frac{\partial A^3}{\partial x^3} \right]$$

$$div\,(A^i) = \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} + \frac{2}{x^1} A^1 + \cot x^2 A^2 \qquad \ldots (1)$$

$$div\,(A^i) = \frac{\partial A^1}{\partial r} + \frac{\partial A^2}{\partial \theta} + \frac{\partial A^3}{\partial w} + \frac{2A^1}{r} + \cot^2 \theta\, A^2$$

We also know that div $(A^i) = \text{div}\,(A_1)$ and $A^j = g^{ij} A_j$ i.e.,

$$A^1 = A_1,\ A^2 = \frac{1}{(x^1)^2} A_2,\ A^3 = \frac{A_3}{(x^1)^2 (\sin x^2)^2}$$

Therefore by (i) we have

$$\text{div}\,(A_i) = \frac{\partial A_1}{\partial x^1} + \frac{1}{(x^1)^2} \frac{\partial A_2}{\partial x^2} + \frac{\text{cosec}^2\, x^2}{(x^1)^2} \frac{\partial}{\partial x^3} (A_3) + \frac{2A_1}{x^1} + \frac{\cot x^2}{(x^1)^2} A_2$$

$$\text{div}\,(A_1) = \frac{\partial A_1}{\partial r} + \frac{1}{(r^2)} \frac{\partial A_2}{\partial \theta} + \frac{1}{(r^2)(\sin \theta)} \frac{\partial A_3}{\partial \phi} + \frac{2A_1}{r} + \frac{\cot (\theta)}{r^2} A_2 \qquad \ldots (2)$$

We know that

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g}\, g^{kr} \frac{\partial \phi}{\partial x^r} \right)$$

$$= \frac{1}{(x^1)^2 \sin x^2} \left[\frac{\partial}{\partial x^1} \left((x^1)^2 \sin x^2 . \frac{\partial \phi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\frac{(x^1)^2 \sin x^2}{(x^1)^2} . \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left\{ \frac{(x^1)^2 \sin x^2}{(x^1)^2 \sin x^2} \frac{\partial \phi}{\partial x^3} \right\} \right]$$

$$= \frac{1}{(x^1)^2 \sin x^2} \left[2x^1 \sin x^2 \frac{\partial \phi}{\partial x^1} + (x^1)^2 \sin x^2 \frac{\partial^2 \phi}{\partial (x^1)^2} + \cos x^2 \frac{\partial \phi}{\partial x^2} + \sin x^2 \frac{\partial^2 \phi}{\partial (x^2)^2} + \frac{\partial^2 \phi}{\partial (x^3)^2} \right]$$

$$\nabla^2 \phi = \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial w^2} \quad \ldots (3)$$

14.6 SUMMARY

In this block, we have learned how tensor calculus is distinct from classical calculus and in which sense both are identically same. The conversion of differential operators from Cartesian coordinates to cylindrical polar coordinates and spherical polar coordinates is studied in easiest way with the help of tensor Calculus.

14.7 REFERENCES

1. Differential Geometry by Majumdar and Bhattacharya Books and Allied (P) Ltd.
2. Tensors by R.B. Mishra "Hardwari Publication Allahabad."

14.8 TERMINAL QUESTIONS

1. For the metric $ds^2 = (dx^1)^2 + [(x^2)^2 - (x^1)^2]\ (dx^2)^2$, find $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}$
2. Prove that the necessary and sufficient condition that all the Christoffel symbols Vanish at a point is that g_{ij} are constant.
3. Prove that $\dfrac{\partial^2 \bar{x}^r}{\partial x^k \partial x^l} = \begin{Bmatrix} i \\ kl \end{Bmatrix} \dfrac{\partial \bar{x}^r}{\partial x^i} - \overline{\begin{Bmatrix} r \\ st \end{Bmatrix}} \dfrac{\partial \bar{x}^s}{\partial x^k} \dfrac{\partial \bar{x}^i}{\partial x^i}$
4. $A_{ij} = B_{i,j} - B_{j,i}$ prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$
5. If A_{ij} is the curl of a covariant vector, prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$
6. Prove that $\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^i} = [jk, i] = [ij, k]$
7. Prove that $\dfrac{\partial}{\partial x^k}(g_{ij} A^i B^j) = A_{i,k} B^i + A^i B_{i,k}$
8. Show that only non-zero Christoffel symbols, of the second kind for a space Where $ds^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2$ are
$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\sin x^1 \cos x^1, \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \cot x^1$.
9. If A^{ij} is a skew-symmetric tensor, show that $A^{jk} \begin{Bmatrix} i \\ jk \end{Bmatrix} = 0$
10. Find the Christoffel symbols corresponding to the metric

$ds^2 = (dx^1)^2 + G\,(x^1, x^2)\,(dx^2)^2$ where G is function of x^1, x^2

14.9 ANSWERS

1. x^1

10. $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\frac{1}{2}\frac{\partial\phi}{\partial x^1}(x^1, x^2)$, $\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{2}\frac{\partial}{\partial x^1}G(x^1, x^2)$

$\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{1}{2}\log G(x^1, x^2)$.

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