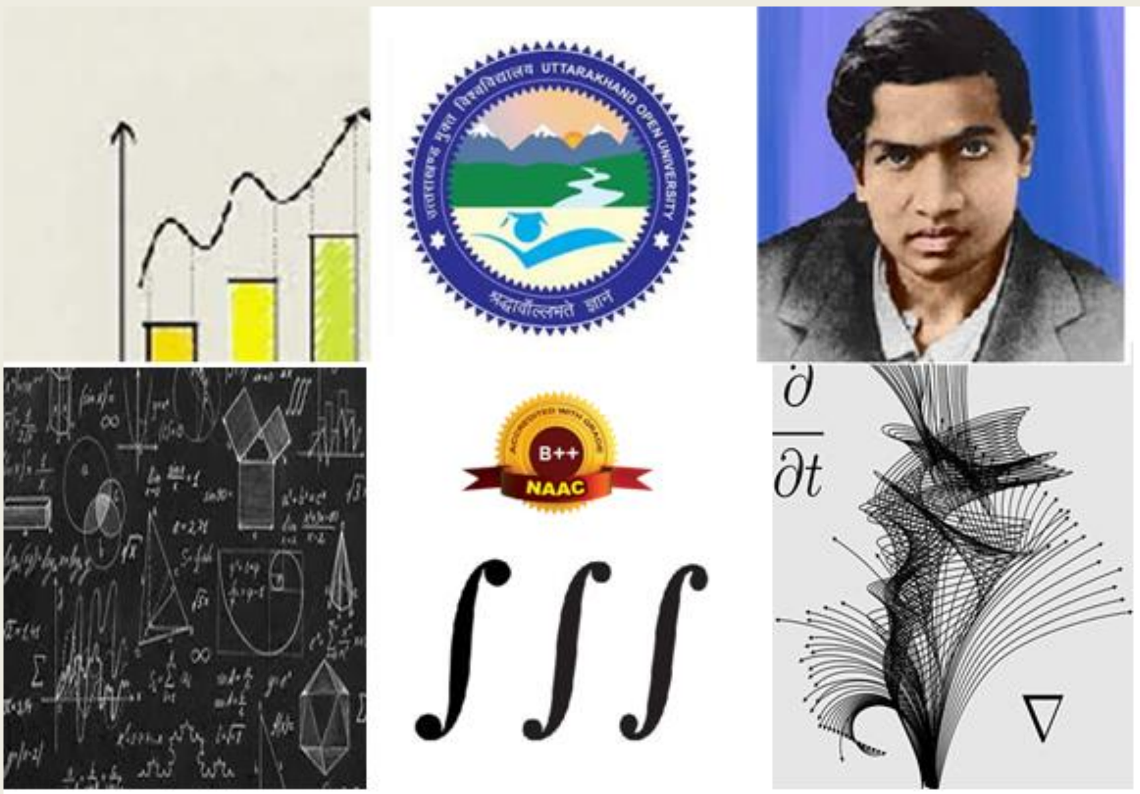
Master of Science

Mathematics

Fourth Semester

MAT - 608
CLASSICAL MECHNAICS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: CLASSICAL MECHANICS

COURSE CODE:MAT-608

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CONTENTS

CLASSICAL MECHANICS

COURSE INFORMATION

The present self-learning material **"Classical Mechanics"** has been designed for M.Sc. (Fourth Semester) learners of Uttarakhand Open University, Haldwani. This course is divided into 14 units of study. This Self Learning Material is a mixture of Three Blocks.

The **first block** is **Foundations of Mechanics**. This block begins with the basic principles of classical mechanics, including the scope and fundamental concepts of mechanics, Newton's laws of motion, systems of units, and frames of reference. It further introduces the kinematics of particles, covering velocity, acceleration, and relative motion. The block concludes with the concept of moment of inertia and its calculation for various standard geometrical bodies, providing the basis for the study of rotational motion.

The **second block** is **Dynamics of Rigid Bodies**. This block focuses on the motion of rigid bodies undergoing translation and rotation. It introduces angular velocity, angular momentum, and the equations of rotational motion. The block further explains D'Alembert's principle as a method for transforming dynamic problems into equilibrium problems and concludes with the principle of virtual work and its applications to the equilibrium and dynamics of constrained mechanical systems.

The **third block** is **Analytical Mechanics**. This block deals with Hamilton's principle and the formulation of mechanics using variational methods. It introduces Hamilton's equations of motion and the Hamiltonian approach to dynamical systems. The block also studies the motion of particles under central forces, including planetary motion and orbital dynamics, and concludes with the theory of small oscillations, normal coordinates, and normal modes of vibration.

The **fourth and final block** is **Advanced Analytical Mechanics**. This block explores canonical (special) transformations and their role in simplifying Hamiltonian systems. It introduces Lagrange and Poisson brackets as important mathematical tools in analytical mechanics and concludes with the study of the motion of a top, including Euler's equations, Lagrangian and Hamiltonian formulations, steady precession, nutation, and the dynamics of symmetric tops. This block provides advanced analytical techniques that form the foundation for modern theoretical mechanics and its applications.

COURSE NAME: CLASSICAL MECHANICS COURSE CODE: MAT-608
CREDIT: 04

SYLLABUS

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Mechanical Systems: The Mechanical System, Angular Velocity, Definition of angular Acceleration, Rate of Change of a Unit Vector in a Plane, Relation Between Angular and Liner Velocity, Component of velocity and acceleration along the coordinate axes in two dimensions, Equations of Motion, Newton's Laws of Rotational Motion, General Theorems on Moment of Inertia, Formulation and Derivation of Moment of Inertia

Motion of rigid body: Motion of rigid body, Theory of small oscillations, Problems on Hamilton-Jacobi Method, Separability. D'Alembert Principle, Angular Momentum of a system of Particle, General equation of a motion, Motion of the center of inertia, Scalar equation, Motion relative the center of inertia, Virtual Work,

Hamilton's equations: Hamilton's Principle, Hamilton's Equation, The principle of least Action, Derivation of Hamilton's Equations from a Variational Principle and Cyclic Coordinates. Hamilton-Jacobi Equation for Hamilton Principal function., Centre Force, Centre Orbit, Differentiable Equation of Centre Orbit, Rate of Description of the Sectorial Area, Elliptic Orbit , Hyperbolic and Parabolic orbits, Velocity in Circle, Given Central orbit, to find the law of force, Apse and apsidal Distance, Property of Apse Line, Areal Velocity, Small Oscillations, 3 Equilibrium of a Mechanical System, Stable Equilibrium, Small Oscillations about Equilibrium.

Canonical form: Special Transformations, Lagrange and Poisson Brackets. Equation of motion of top (deduce LaGrange equations), steady motion, Range of theta.

BLOCK-I
MECHANICAL SYSTEM

UNIT 1- INTRODUCTION OF MECHANICS

CONTENTS:

1.1 INTRODUCTION

This course builds upon the concepts learned in the course “Mechanics of Materials” also known as “Strength of Materials”. In the “Mechanics of Materials” course one would have learnt two new concepts “stress” and “strain” in addition to revisiting the concept of a “force” and “displacement” that one would have mastered in a first course in mechanics, namely “Engineering Mechanics”. Also one might have been exposed to four equations connecting these four concepts, namely strain-displacement equation, constitutive equation, equilibrium equation and compatibility equation. Figure 1.1 pictorially depicts the concepts that these equations relate. Thus, the strain displacement relation allows one to compute the strain given a displacement; constitutive relation gives the value of stress for a known value of the strain or vice versa; equilibrium equation, crudely, relates the stresses developed in the body to the forces and moment applied on it; and finally, compatibility equation places restrictions on how the strains can vary over the body so that a continuous displacement field could be found for the assumed strain field.

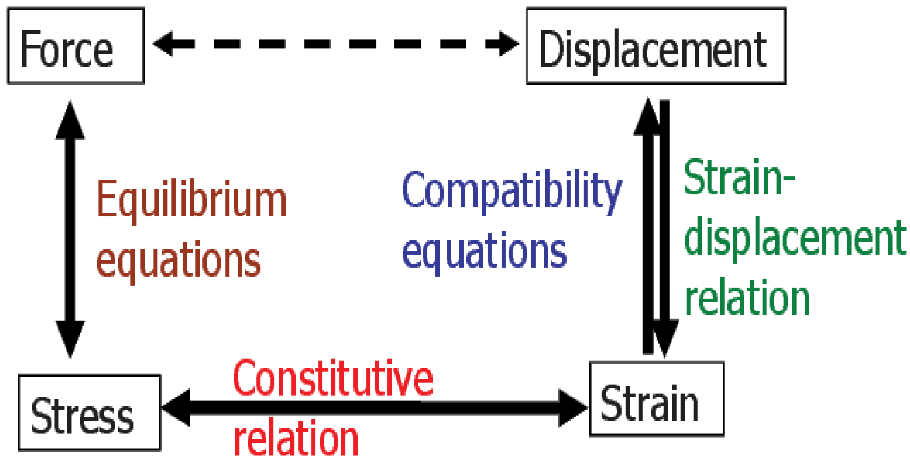


Figure 1.1: Basic concepts and equations in mechanics

In this course too we shall be studying the same four concepts and four equations. While in the "mechanics of materials" course, one was introduced to the various components of the stress and strain, namely the normal and shear, in the problems that was solved not more than one component of the stress or strain occurred simultaneously. Here we shall be studying these problems in which more than one component of the stress or strain occurs simultaneously. Thus, in this course we shall be generalizing these concepts and equations to facilitate three-dimensional analysis of structures.

Before venturing into the generalization of these concepts and equations, a few drawbacks of the definitions and ideas that one might have acquired from the previous course needs to be highlighted and clarified. This we shall do in sections 1.1 and 1.3 Specifically, in section 1.1 we look at the four concepts in mechanics and in section 1.3

1.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of Mechanics as a branch of applied mathematics dealing with motion and forces.
2. To Distinguish between different branches of mechanics, such as: statics, Dynamics, Kinematics.

3. To Explain the physical quantities used in mechanics, including: displacement, velocity, acceleration, Force, mass, momentum
4. To Apply vector algebra to represent forces, velocities, and accelerations.
5. To Understand Newton's laws of motion and their mathematical formulation.

1.3 BASIC CONCEPT OF MECHANICS

1.3.1 What is force?

Force is a mathematical idea to study the motion of bodies. It is not "real" as many think it to be. However, it can be associated with the twitching of the muscle, feeling of the burden of mass, linear translation of the motor, so on and so forth. Despite seeing only displacements we relate it to its cause the force, as the concept of force has now been ingrained.

Let us see why force is an idea that arises from mathematical need. Say, the position[1] ($\mathbf{x}_o$) and velocity ($\mathbf{v}_o$) of the body is known at some time, $t = t_o$, then one is interested in knowing where this body would be at a later time, $t = t_1$. It turns out that mathematically, if the acceleration ($\mathbf{a}$) of the body at any later instant in time is specified then the position of the body can be determined through Taylor's series. That is if

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = \mathbf{f}_a(t),$$

then from Taylor's series

$$\mathbf{x}_1 = \mathbf{x}(t_1) = \mathbf{x}(t_o) + \left.\frac{d\mathbf{x}}{dt}\right|_{t=t_o}(t_1 - t_o) + \left.\frac{d^2\mathbf{x}}{dt^2}\right|_{t=t_o}\frac{(t_1 - t_o)^2}{2!} + \left.\frac{d^3\mathbf{x}}{dt^3}\right|_{t=t_o}\frac{(t_1 - t_o)^3}{3!} + \left.\frac{d^4\mathbf{x}}{dt^4}\right|_{t=t_o}\frac{(t_1 - t_o)^4}{4!} + \dots,$$

which when written in terms of $\mathbf{x}_o$, $\mathbf{v}_o$ and $\mathbf{a}$ reduces to[2]

$$\mathbf{x}_1 = \mathbf{x}_o + \mathbf{v}_o(t_1 - t_o) + \mathbf{f}_a(t_o)\frac{(t_1 - t_o)^2}{2!} + \left.\frac{d\mathbf{f}_a}{dt}\right|_{t=t_o}\frac{(t_1 - t_o)^3}{3!} + \left.\frac{d^2\mathbf{f}_a}{dt^2}\right|_{t=t_o}\frac{(t_1 - t_o)^4}{4!} + \dots.$$

Thus, if the function $\mathbf{f}_a$ is known then the position of the body at any other instant in time can be determined. This function is nothing but force per unit mass[3] , as per Newton's second law which gives a definition for the force. This shows that force is a function that one defines mathematically so that the position of the body at any later instance can be obtained from knowing its current position and velocity.

It is pertinent to point out that this function $\mathbf{f}_a$ could also be prescribed using the position, $\mathbf{x}$ and velocity, $\mathbf{v}$ of the body which are themselves function of time, t and hence $\mathbf{f}_a$ would still be a function of time. Thus, $\mathbf{f}_a = \mathbf{g}(\mathbf{x}(t),\mathbf{v}(t),t)$. However, $\mathbf{f}_a$ could not arbitrarily depend on t, $\mathbf{x}$ and $\mathbf{v}$. At this point it suffices to say that the other two laws of Newton and certain objectivity requirements have to be met by this function. We shall see what these objectivity requirements are and how to prescribe functions that meet this requirement subsequently in chapter - 6.

Next, let us understand what kind of quantity is force. In other words is force a scalar or vector and why? Since, position is a vector and acceleration is second time derivative of position, it is also a vector. Then, it follows from equation (1.3) that $\mathbf{f}_a$ also has to be a vector. Therefore, force is a vector quantity. Numerous experiments also show that addition of forces follow vector addition law (or the parallelogram law of addition). In chapter 2 we shall see how the vector addition differs from scalar addition. In fact it is this addition rule that distinguishes a vector from a scalar and hence confirms that force is a vector.

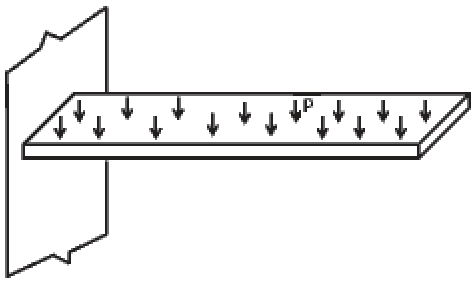
As a summary, we showed that force is a mathematical construct which is used to mathematically describe the motion of bodies.

1.3.2 What is stress?

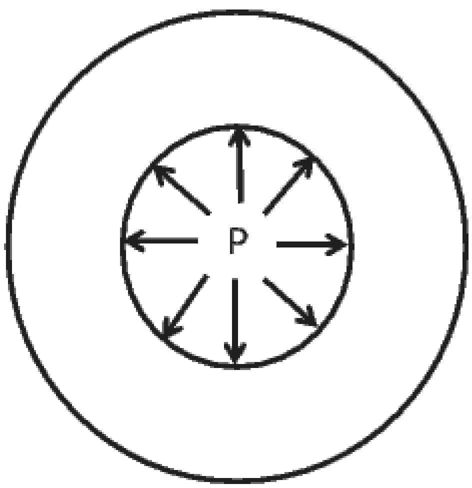
As is evident from figure 1.3, stress is a quantity derived from force. The commonly stated definitions in an introductory course in mechanics for stress are:

- Stress is the force acting per unit area
- Stress is the resistance offered by the body to a force acting on it

While the first definition tells how to compute the stress from the force, this definition holds only for simple loading case. One can construct a number of examples where definition 1 does not hold. The following two cases are presented just as an example. Case -1: A cantilever beam of rectangular cross section with a uniform pressure, p, applied on the top surface, as shown in figure 1.3a. According to the definition 1 the stress in the beam should be p, but it is not. Case -2: An annular cylinder subjected to a pressure, p at its inner surface, as shown in figure 3b. The net force acting on the cylinder is zero but the stresses are not zero at any location. Also, the stress is not p, anywhere in the interior of the cylinder. This being the state of the first definition, the second definition is of little use as it does not tell how to compute the stress. These definitions does not tell that there are various components of the stress nor whether the area over which the force is considered to be distributed is the deformed or the undeformed. They do not distinguish between traction (or stress vector), $\mathbf{t}_{(\mathbf{n})}$ and stress tensor, $\boldsymbol{\sigma}$.



(a) A cantilever beam with uniform pressure applied on its top surface.



(b) An annular cylinder subjected to internal pressure.

Figure 1.3: Structures subjected to pressure loading

Traction is the distributed force acting per unit area of a cut surface or boundary of the body. This traction apart from varying spatially and temporally also depends on the plane of cut characterized by its normal. This quantity integrated over the cut surface gives the net force acting on that surface. Consequently, since force is a vector quantity this traction is also a vector quantity. The component of the traction along the normal direction[4] , **n** is called as the normal stress ($\sigma_{(\mathbf{n})}$). The magnitude of the component of the traction[5] acting parallel to the plane is called as the shear stress ($\tau_{(\mathbf{n})}$).

If the force is distributed over the deformed area then the corresponding traction is called as the Cauchy traction ($\mathbf{t}_{(\mathbf{n})}$) and if the force is distributed over the undeformed or original area that traction is called as the Piola traction ($\mathbf{p}_{(\mathbf{n})}$). If the deformed area does not change significantly from the original area, then both these tractions would have nearly the same magnitude and direction. More details about this traction is presented in chapter 4.

The stress tensor, is a linear function (crudely, a matrix) that relates the normal vector, **n** to the traction acting on that plane whose normal is **n**. The stress tensor could vary spatially and temporally but does not change with the plane of cut. Just like there is Cauchy and Piola traction, depending on over which area the force is distributed, there are two stress tensors. The Cauchy (or true) stress tensor, $\boldsymbol{\sigma}$ and the Piola-Kirchhoff stress tensor (**P**). While these two tensors may nearly be the same when the deformed area is not significantly different from the original area, qualitatively these tensors are different. To satisfy the moment equilibrium in the absence of body couples, Cauchy stress tensor has to be symmetric tensor (crudely, symmetric matrix) and Piola-Kirchhoff stress tensor cannot be symmetric. In fact the transpose of the Piola-Kirchhoff stress tensor is called as the engineering stress or nominal stress. Moreover, there are many other stress measures obtained from the Cauchy stress and the gradient of the displacement which shall be studied in chapter 4.

1.3.3 What is displacement?

The difference between the position vectors of a material particle at two different instances of time is called as displacement. In general, the displacement of the material particle would depend on time; the instances between which the displacement is sought. It is also possible that different particles get displaced differently between the same two instances of time. Thus, displacement in general varies spatially and temporally. Displacement is what can be observed and measured. Forces, traction and stress tensors are introduced to explain (or mathematically capture) this displacement.

The displacement field is at least differentiable twice temporally so that acceleration could be computed. This stems from the observations that the location or velocity of the body does not change abruptly. Similarly, the basic tenant of continuum mechanics is that the displacement field is continuous spatially and is piecewise differentiable spatially at least twice. That is while the displacement field is required to be continuous over the entire body it is required to be twice differentiable not necessarily over the entire body but only on subsets of the body. Thus, in continuum mechanics interpenetration of two surfaces or separation and formation of new surfaces is precluded. The validity of the theory stops just before the body fractures. Notwithstanding this many attempts to use continuum mechanics concepts to understand the process of fracture.

A body is said to undergo rigid body displacement if the distance between any two particles that belongs to the body remains unchanged. That is in a rigid body displacement the particles that belong to a body do not move relative to each other. A body is said to be rigid if it always undergoes only rigid body displacement under action of any force. On the other hand, a body is said to be deformable if it allows relative displacement of its particles under the action of some force. Though, all real bodies are deformable, at times one could idealize a given body as rigid under the action of certain forces.

1.3.4 What is strain?

One observes that rigid body displacements of the body does not give raise to any stresses. Further, stresses are induced only when there is relative displacement of the material particles. Consequently, one requires a measure (or metric) for this relative displacement so that it can be related to the stress. The unique measure of relative displacement is the stretch ratio, $\lambda_{(\mathbf{A})}$, defined as the ratio of the deformed length to the original length of a material fiber along a given direction, **A**. (Note that here **A** is a unit vector.) However, this measure has the drawback that when the body is not deformed the stretch ratio is 1 (by virtue of the deformed length being same as the original length) and hence inconvenient to write the constitutive relation of the form

$$\sigma_{(\mathbf{A})} = f(\lambda_{(\mathbf{A})}),$$

where $\sigma_{(\mathbf{A})}$ denotes the normal stress on a plane who's normal is **A**. Since the stress is zero when the body is not deformed, the function f should be such that $f(1) = 0$. Mathematical implementation of this condition that $f(1) = 0$ and that f be a one to one function is thought to be difficult when f is a nonlinear function of $\lambda_{(\mathbf{A})}$. Consequently, another measure of relative displacement is sought which would be 0 when the body is not deformed and less than zero when compressed and greater than zero when stretched. This measure is called as the strain, $\epsilon_{(\mathbf{A})}$. There is no unique way of obtaining the strain from the stretch ratio. The following functions satisfy the requirement of the strain:

$$\epsilon_{(\mathbf{A})} = \frac{\lambda_{(\mathbf{A})}^{m} - 1}{m}, \qquad \epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})}),$$

where m is some real number and ln stands for natural logarithm. Thus, if $m = 1$ then the resulting strain is called as the engineering strain, if $m = -1$, it is called as the true strain, if $m = 2$ it is Cauchy-Green strain. The second function wherein $\epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})})$, is called as the Hencky strain or the logarithmic strain.

Just like the traction and hence the normal stress changes with the orientation of the plane, the stretch ratio also changes with the orientation along which it is measured. We shall see in chapter 3 that a tensor called the Cauchy-Green deformation tensor carries all the information required to compute the stretch ratio along any direction. This is akin to the stress tensor which when known we could compute the traction or the normal stress in any plane.

1.4 BASIC EQUATION IN MECHANICS

Having gained a superficial understanding of the four concepts in mechanics namely the force, stress, displacement and strain, let us look at the four equations that connect these concepts and the reasoning used to obtain them.

1.4.1 Equilibrium equations

Equilibrium equations are Newton's second law which states that the rate of change of linear momentum would be equal in magnitude and direction to the net applied force. Deformable bodies are subjected to two kinds of forces, namely, contact force and body force. As the name suggest the contact force arises by virtue of the body being in contact with its surroundings. Traction arises only due to these contact force and hence so does the stress tensor. The magnitude of the contact force depends on the contact area between the body and its surroundings. On the other hand, the body forces are action at a distance forces. Examples of body force are gravitational force, electromagnetic force. The magnitude of these body forces depends on the mass of the body and hence are generally expressed as per unit mass of the body and denoted by **b**.

On further assuming that the Newton's second law holds for any subpart of the body and that the stress field is continuously differentiable within the body the equilibrium equations can be written as:

$$div(\boldsymbol{\sigma}) + \rho\mathbf{b} = \rho\mathbf{a},$$

where ρ is the density, **a** is the acceleration and the mass is assumed to be conserved. Detail derivation of the above equation is given in chapter 5. The meaning of the operator $div(\cdot)$ can be found in chapter 2.

Also, the rate of change of angular momentum must be equal to the net applied moment on the body. Assuming that the moment is generated only by the contact forces and body forces, this condition requires that the Cauchy stress tensor to be symmetric. That is in the absence of body couples, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$, where the superscript $(\cdot)^t$ denotes the transpose. Here again the assumptions made to obtain the force equilibrium equation should hold. See chapter 5 for detailed derivation.

1.4.2 Strain-Displacement relation

The relationship that connects the displacement field with the strain is called as the strain displacement relationship. As pointed out before there is no unique definition of the strain and hence there are various strain tensors. However, all these strains are some functions of the gradient of the deformation field, **F**; commonly called as the deformation gradient. The deformation field is a function that gives the position vector of any material particle that belongs to the body at any instance in time with the material particle identified by its location at some time t_o. Then, we show that, the stretch ratio along a given direction **A** is,

$$\lambda_{(\mathbf{A})} = \sqrt{\mathbf{CA} \cdot \mathbf{A}},$$

where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$, is called as the right Cauchy-Green deformation tensor. When the body is undeformed, $\mathbf{F} = \mathbf{1}$ and hence, $\mathbf{C} = \mathbf{1}$ and $\lambda_{(\mathbf{A})} = 1$. Instead of looking at the deformation field, one can develop the expression for the stretch ratio, looking at the displacement field too. Now, the displacement field can be a function of the coordinates of the material particles in the reference or undeformed state or the coordinates in the current or deformed state. If the displacement is a function of the coordinates of the material particles in the reference configuration it is called as Lagrangian representation of the displacement field and the gradient of this Lagrangian displacement field is called as the Lagrangian displacement gradient and is denoted by **H**. On the other hand if the displacement is a function of the coordinates of the material particle in the deformed state, such a representation of the displacement field is said to be Eulerian and the gradient of this Eulerian displacement field is called as the Eulerian displacement gradient and is denoted by **h**. Then it can be shown that (see chapter 3),

$$\mathbf{F} = \mathbf{H} + \mathbf{1}, \qquad \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h},$$

where, **1** stand for identity tensor (see chapter 2 for its definition). Now, the right Cauchy-Green deformation tensor can be written in terms of the Lagrangian displacement gradient as,

$$\mathbf{C} = \mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t\mathbf{H}.$$

Note that the if the body is undeformed then **H** = **0**. Hence, if one cannot see the displacement of the body then it is likely that the components of the Lagrangian displacement gradient are going to be small, say of order 10^{-3}. Then, the components of the tensor $\mathbf{H}^t\mathbf{H}$ are going to be of the order 10^{-6}. Hence, the above equation for this case when the components of the Lagrangian displacement gradient is small can be approximately calculated as,

$$\mathbf{C} = \mathbf{1} + 2\boldsymbol{\epsilon}_L,$$

where

$$\boldsymbol{\epsilon}_L = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^t\right],$$

is called as the linearized Lagrangian strain. We shall see in chapter that when the components of the Lagrangian displacement gradient is small, the stretch ratio (1.7) reduces to

$$\lambda_{\mathbf{A}} = 1 + \boldsymbol{\epsilon}_L\mathbf{A} \cdot \mathbf{A}.$$

Thus we find that $\boldsymbol{\epsilon}_L$ contains information about changes in length along any given direction, **A** when the components of the Lagrangian displacement gradient are small. Hence, it is called as the linearized Lagrangian strain. We shall in chapter 3 derive the various strain tensors corresponding to the various definition of strains given in equation.

Further, since $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$, it follows from that

$$\mathbf{H} = \mathbf{h} + \mathbf{H}\mathbf{h},$$

which when the components of both the Lagrangian and Eulerian displacement gradient are small can be approximated as $\mathbf{H} = \mathbf{h}$. Thus, when the components of the Lagrangian and Eulerian displacement gradients are small these displacement gradients are the same. Hence, the Eulerian linearized strain defined as,

$$\epsilon_E = \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^t\right],$$

and the Lagrangian linearized strain, ϵ_L would be the same when the components of the displacement gradients are small.

Equation (1.14) is the strain displacement relationship that we would use to solve boundary value problems in this course, as we limit ourselves to cases where the components of the Lagrangian and Eulerian displacement gradient is small.

1.4.3 Compatibility equation

It is evident from the definition of the linearized Lagrangian strain, that it is a symmetric tensor. Hence, it has only 6 independent components. Now, one cannot prescribe arbitrarily these six components since a smooth differentiable displacement field should be obtainable from these six prescribed components. The restrictions placed on how these six components of the strain could vary spatially so that a smooth differentiable displacement field is obtainable is called as compatibility equation. Thus, the compatibility condition is

$$curl(curl(\epsilon)) = \mathbf{0}.$$

The derivation of this equation as well as the components of the *curl*$(\cdot)$ operator in Cartesian coordinates is presented in chapter 3.

It should also be mentioned that the compatibility condition in case of large deformations is yet to be obtained. That is if the components of the right Cauchy-Green deformation tensor, $\mathbf{C}$ is prescribed, the restrictions that have to be placed on these prescribed components so that a smooth differentiable deformation field could be obtained is unknown, except for some special cases.

1.4.4 Constitutive relation

Broadly constitutive relation is the equation that relates the stress (and stress rates) with the displacement gradient (and rate of displacement gradient). While the above three equations - Equilibrium equations, strain-displacement relation, compatibility equations - are independent of the material that the body is made up of and/or the process that the body is subjected to, the constitutive relation is dependent on the material and the process. Constitutive relation is required to bring in the dependance of the material in the response of the body and to have as many equations as there are unknowns, as will be shown in chapter 6.

The fidelity of the predictions, namely the likely displacement or stress for a given force depends only on the constitutive relation. This is so because the other three equations are the same irrespective of the material that the body is made up of. Consequently, a lot of research is being undertaken to arrive at better constitutive relations for materials.

It is difficult to have a constitutive relation that could describe the response of a material subjected to any process. Hence, usually constitutive relations are prescribed for a particular process that the material undergoes. The variables in the constitutive relation depends on the process that is being studied. The same material could undergo different processes depending on the stimuli; for example, the same material could respond elastically or plastically depending on say, the magnitude of the load or temperature. Hence it is only apt to qualify the process and not the material. However, it is customary to qualify the material instead of the process too. This we shall desist.

Traditionally, the constitutive relation is said to depend on whether the given material behaves like a solid or fluid and one elaborates on how to classify a given material as a solid or a fluid. A material that is not a solid is defined as a fluid. This means one has to define what a solid is. A couple of definitions of a solid are listed below:

- Solid is one which can resist sustained shear forces without continuously deforming
- Solid is one which does not take the shape of the container

Though these definitions are intuitive they are ambiguous. A class of materials called “viscoelastic solids”, neither take the shape of the container nor resist shear forces without continuously deforming. Also, the same material would behave like a solid, like a mixture of a solid and a fluid or like a fluid depending on say, the temperature and the mechanical stress it is being subjected to. These prompts us to say that a given material behaves in a solid-like or fluid-like manner. However, as we shall see, this classification of a given material as solid or fluid is immaterial. If one appeals to thermodynamics for the classification of the processes, the response of materials could be classified based on (1) Whether there is conversion of energy from one form to another during the process, and (2) Whether the process is thermodynamically equilibrated. Though, in the following section, we classify the response of materials based on thermodynamics, we also give the commonly stated definitions and discuss their shortcomings. In this course, as well as in all these classifications, it is assumed that there are no chemical changes occurring in the body and hence the composition of the body remains a constant.

1.5 SUMMARY

Thus in this unit we introduced the four concepts in mechanics, the four equations connecting these concepts as well as the methodologies used to solve boundary value problems. In the following chapters we elaborate on the same topics. It is not intended that in a first reading of this chapter, one would understand all the details. However, reading the same chapter at the end of this course, one should appreciate the details. This chapter summarizes the concepts that should be assimilated and digested during this course.

1.6 REFERENCES

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1.7 SUGGESTED READING

1. Vector Mechanics for Engineers: Statics – F. P. Beer & E. R. Johnston
A foundational text covering concepts of forces, moments, and equilibrium with clear illustrations.

2. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige
Excellent for understanding theoretical principles and solving analytical problems.

*3.***Engineering Mechanics** – R. K. Bansal
Useful for Indian students; provides numerous solved examples and practice problems.

4. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
Provides strong conceptual explanation with real-life engineering applications.

5. **A Textbook of Engineering Mechanics** – S. S. Bhavikatti
Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6

UNIT 2: KINEMATICS IN TWO DIMENSIONS

CONTENTS:

2.1 INTRODUCTION

Motion is a fundamental concept in physics and mathematics, and while the study of motion in a straight line (linear motion) helps us understand basic kinematics, many real-life motions such as the rotation of wheels, motion of planets, ceiling fans, and spinning tops involve rotational motion. To describe such motion accurately, we introduce the important concepts of angular displacement, angular velocity, and angular acceleration**.**

Angular velocit**y** describes how fast an object rotates, while angular acceleration explains how the angular velocity changes with time. These quantities play a vital role in understanding circular motion, rotational dynamics, and engineering applications.

In this unit, we also study the rate of change of unit vectors in a plan**e**, which is essential for describing motion in two dimensions. This concept helps us express velocity and acceleration in vector form when direction continuously changes, as in circular motion.

Further, a very important relationship between angular velocity and linear (tangential) velocity is developed, which allows us to connect rotational motion with linear motion. This relation is extremely useful in practical problems involving wheels, gears, pulleys, and rotating bodies.

Finally, we discuss the components of velocity and acceleration along coordinate axes in two dimensions. Resolving vectors into components simplifies complex motion into manageable parts and helps in solving a wide variety of problems related to projectile motion, circular motion, and plane motion.

Thus, this unit builds a strong mathematical foundation for understanding rotational motion and prepares students for advanced topics in mechanics.

2.2 OBJECTIVES

After studying this unit learners will be able

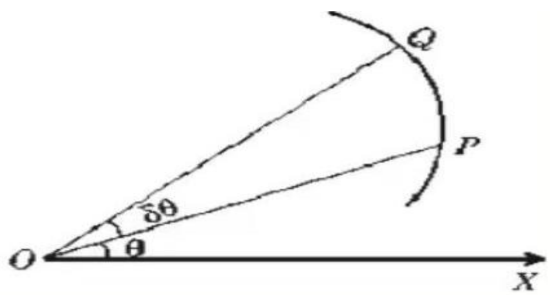
1. To Understand the concept of angular displacement, angular velocity, and angular acceleration and their physical significance.
2. Define and explain angular velocity and angular acceleration in mathematical form.
3. Derive and use expressions for angular velocity and angular acceleration in rotational motion.
4. Understand the rate of change of a unit vector in a plane and apply it in two-dimensional motion.

2.3 ANGULAR VELOCITY

Angular Velocity (Definition): Let P be a point moving in a plane. If O is the fixed point and OX a fixed line through O in the plane of motion, then the angular velocity of the moving point P about O (or the line OP in the plane XOP) is the rate of change of the angle XOP. Let P and Q be the positions of a moving particle at times t and $t + \delta t$ respectively such that $\angle POX = \theta$ and $\angle QOX = \theta + \delta\theta$.

Then the angle turned by the particle in time δt is $\delta\theta$.
$\therefore$ Average rate of change of the angle of P about $O = \frac{\delta\theta}{\delta t}$.
$\therefore$ The angular velocity of the point P about O

$$= \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta},$$



where the dot placed over θ denotes differentiation with respect to the time t.
Since the angular velocity has magnitude as well as direction, it is a vector quantity represented by the vector $\vec{\omega}$. The magnitude of the angular velocity is $\frac{d\theta}{dt}(= \dot{\theta} = \omega)$ and its direction is perpendicular to the plane POQ.
Since the angle θ is measured in radians, the unit of angular velocity is radians/sec.

2.4 DEFINITION OF ANGULAR ACCELERATION

The rate of change of the angular velocity is called angular acceleration.
$\therefore$ Angular acceleration $= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}$.
The unit of angular acceleration is radians/ $\sec^2$.

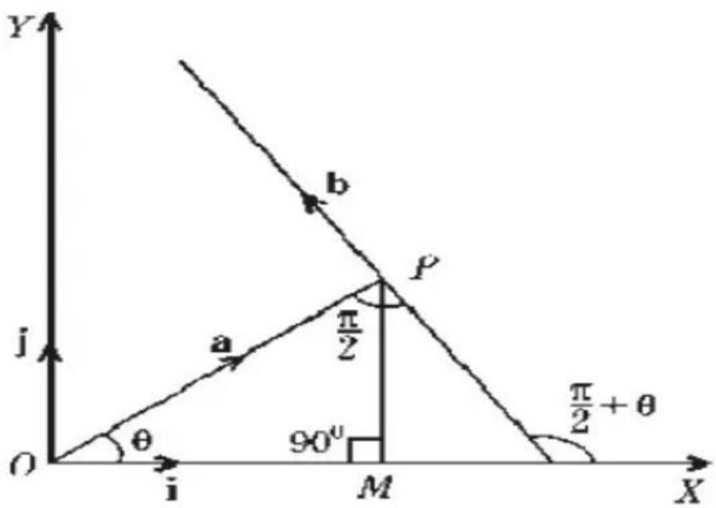
2.5 RATE OF CHANGE OF UNIT VECTOR

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors along two mutually perpendicular fixed lines (say the coordinate axes in the plane). Let a denote a unit vector $\overrightarrow{OP}$ such that $OP = 1$ and $\angle POX = \theta$.

Then $\mathbf{a} = \overrightarrow{OM} + \overrightarrow{MP} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

The vector $\mathbf{a}$ is a function of θ, where θ is a function of the time t. Differentiating (1) w.r.t. t, we have

$$\frac{d\mathbf{a}}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$



[Note that i and j are constant vectors]

$$= \frac{d\theta}{dt}\left[\cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}\right]$$

where $\mathbf{b} = \cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}$ is a unit vector inclined at an angle $\frac{1}{2}\pi + \theta$ with OX. Therefore, $\mathbf{b}$ is a unit vector perpendicular to OP in the sense in which θ increases. Thus remember that if $\mathbf{a}$ is a unit vector which makes a variable angle θ with OX, then

$$\frac{d\mathbf{a}}{dt} = \frac{d\theta}{dt}\mathbf{b} \tag{2}$$

where **b** is a unit vector perpendicular to **a** in the direction of θ increasing.

Particular case : If **t** and **n** are the unit vectors along the tangent and normal respectively at any point P of a plane curve (as shown in the figure), then

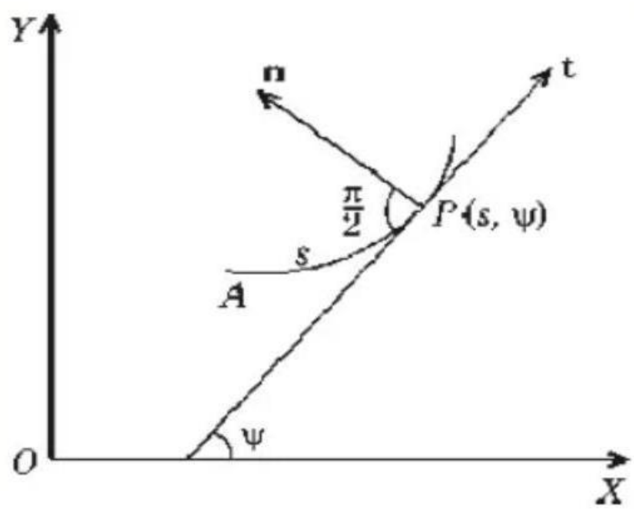
$$\frac{d\mathbf{t}}{dt} = \frac{d\psi}{dt}\mathbf{n} = \dot{\psi}\mathbf{n}$$

where ψ is the angle which the tangent at the point P makes with OX.

Also

$$\frac{d\mathbf{n}}{dt} = -\frac{d\Psi}{dt}\mathbf{t} = -\dot{\psi}\mathbf{t}$$

Here **t** is in the direction of s increasing and **n** is in



2.6 RELATION BETWEEN ANGULAR AND LINER VELOCITY

Now the component of a vector **a** in the direction of a unit vector **b** is given by $\mathbf{a} \cdot \mathbf{b}$. If v_θ is the component of the velocity **v** in the direction perpendicular to OP, then

$$
\begin{aligned}
v_\theta &= \mathbf{v}\cdot\mathbf{e}_\theta = \left(\frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta\right)\cdot\mathbf{e}_\theta \\
&= r\frac{d\theta}{dt} = r\omega, \qquad [\because \mathbf{e}_r \perp \mathbf{e}_\theta \text{ and } |\mathbf{e}_\theta| = l]
\end{aligned}
$$

or

$$\omega = \frac{v_\theta}{r} = \frac{\text{component of the velocity } v \text{ at } P \text{ perpendicular to } OP}{OP}.$$

(Remember)

Since the angle between $\mathbf{v}$ and $\mathbf{e}_\theta$ is $\frac{1}{2}\pi - \phi$, therefore

$$v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta = v\cdot 1\cdot\cos\left(\frac{1}{2}\pi - \phi\right) = v\sin\phi.$$

$$\therefore \quad \omega = \frac{v\sin\phi}{r} = \frac{vr\sin\phi}{r^2}$$

$$\text{or} \quad \omega = \frac{d\theta}{dt} = \frac{vp}{r^2}. \qquad [\because p = r\sin\phi]$$

Remark 1: The angular velocity of P about O

$$= \frac{\text{the resolved part of the velocity of } P \perp \text{ to } OP}{OP}.$$

Remark 2: If A and B are both in motion, then the angular velocity of B relative to A

$$= \frac{\text{the resolved part of the velocity of } B \text{ relative to } A \perp \text{ to } AB}{AB}.$$

Alternative method for finding the relation between angular and linear velocities.

Theorem: If v be the velocity of a point P moving in a plane curve and (r, θ) its coordinates referred to the fixed point O in the plane, then the angular velocity of P about O is equal to vp $/r^2$, where p is the perpendicular from O drawn to the tangent at P.

Proof: The angular velocity of P about O

$$= \frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = v\frac{d\theta}{ds} \qquad \left[\because v = \frac{ds}{dt}\right]$$

$$= \frac{v}{r} \cdot \left(r\frac{d\theta}{ds}\right)$$

$$= \frac{v\sin\phi}{r} \qquad \left[\because \sin\phi = r\frac{d\theta}{ds}, \text{ from differential calculus}\right]$$

$$= \frac{v}{r} \cdot \frac{p}{r}$$

$$= vp/r^2 \qquad [\because p = r\sin\phi]$$

Example 6: Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.

Solution: We know that the path of a projectile is a parabola whose pedal equation referred to its focus S as the pole is given by

$$p^2 = ar. \tag{1}$$

Let v be the velocity of the projectile at the point $P(r, \theta)$.
Then $MP = PS = r$.
Since the velocity of the projectile at a point of its path is equal to the velocity acquired in falling freely from the directrix to that point, therefore,

$$v = \sqrt{(2g \cdot MP)} = \sqrt{(2gr)} \tag{2}$$

$\therefore$ The angular velocity ω of P about the focus S (i.e., about the pole) is given by

$$\omega = \frac{vp}{r^2} = \frac{\sqrt{(2gr)} \cdot \sqrt{(ar)}}{r^2}$$

$$= \frac{\sqrt{(2ag)}}{r} = \frac{\sqrt{(2ag)}}{SP}.$$

$$\therefore \quad \omega \propto \frac{1}{SP}.$$

2.7 COMPONENT OF VELOCITY AND ACCELERATION ALONG THE COORDINATE AXES IN TWO DIMENSIONS

Let us suppose that P(x, y) be the position of a particle moving in a plane at any time t. If $O\overline{P}$ = r, we have

r = $O\overline{P} = O\overline{M} + M\overline{P}$ = xi + yj.

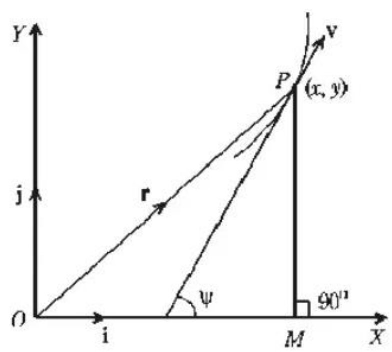
Let v be the vector representing the velocity of the particle at P. Then

v = dr/dt = d/dt (xi + yj)
= (dx/dt)i + (dy/dt)j.

Thus the velocity vector v has been expressed as a linear combination of the vectors i and j.

∴ The x-component of the velocity of P = dx/dt = $\dot{x}$, positive in the direction of vector i, i.e., positive in the direction of x increasing,

and the y-component of the velocity of P = dy/dt = $\dot{y}$, positive in the direction of y increasing.



If v is the resultant velocity of P, we have

$$v = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \frac{ds}{dt}$$

Also, the angle which the direction of v makes with OX

$$= \tan^{-1}\frac{dy/dt}{dx/dt} = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\tan\psi = \psi$$

showing that the resultant velocity at P is along the tangent at P.
If **a** be the acceleration vector of the particle at P, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

$\therefore$ the x-component of the acceleration of $P = \frac{d^2x}{dt^2} = \ddot{x}$, positive in the direction of x increasing, and the y-component of the acceleration of $P = \frac{d^2y}{dt^2} = \ddot{y}$,positive in the direction of y increasing.
The resultant acceleration of

$$P = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

Example 2: The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}$$

If the velocity and displacement are zero at $t = 0$, find the velocity and displacement at any time.
Solution: Here, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}$.
Integrating w.r.t. ' t ', we have

$\mathbf{v} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} + \mathbf{c}_1$, where $\mathbf{c}_1$ is a constant vector.

But at $t = 0, \mathbf{v} = \mathbf{0}$;
$\therefore$ $\mathbf{0} = 4\mathbf{j} + \mathbf{c}_1$ or $\mathbf{c}_1 = -4\mathbf{j}$.
$\therefore$ Velocity $\mathbf{v} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} - 4\mathbf{j}$.
Again $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + 8t^2\mathbf{k} - 4\mathbf{j}$.
Integrating, w.r.t. ' t ', we have

$$\mathbf{r} = -3\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j} + \frac{8}{3}t^3\mathbf{k} - 4t\mathbf{j} + \mathbf{c}_2$$

where $\mathbf{c}_2$ is a constant vector.
But at $t = 0, \mathbf{r} = \mathbf{0}$; $\therefore$ $\mathbf{0} = -3\mathbf{i} + \mathbf{c}_2$ or $\mathbf{c}_2 = 3\mathbf{i}$.
$\therefore$ Displacement from the origin is given by

$$\mathbf{r} = -3\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j} + \frac{8}{3}t^3\mathbf{k} - 4t\mathbf{j} + 3\mathbf{i}$$

or

$$\mathbf{r} = 3(1 - \cos 2t)\mathbf{i} + 2(\sin 2t - 2t)\mathbf{j} + \frac{8}{3}t^3\mathbf{k}$$

Example 4: A particle moves in the curve $y = a\log\sec\left(\frac{x}{a}\right)$ in such a way that the tangent to the curve rotates uniformly; prove that the resultant acceleration of the particle varies as the square of the radius of curvature.

Solution: Since the tangent to the curve rotates uniformly,

$$\therefore\ \frac{d\psi}{dt} = c \text{ (constant)} \tag{1}$$

Equation of the path is

$$\therefore\quad y = a\log\sec(x/a).$$

$$\text{OR}\quad \frac{dy}{dx} = \frac{a}{\sec(x/a)} \cdot \sec\frac{x}{a}\tan\frac{x}{a} \cdot \frac{1}{a}$$

$$\therefore\quad \tan\psi = \frac{dy}{dx} = \tan\frac{x}{a}.$$

$$\therefore\quad \Psi = x/a \text{ or } x = a\psi.$$

$$\text{Now}\quad \frac{dx}{dt} = a\frac{d\psi}{dt} = ac \text{ and } \frac{d^2x}{dt^2} = 0.$$

$$\therefore\quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \left(\tan\frac{x}{a}\right) \cdot ac.$$

$$\therefore\quad \frac{d^2y}{dt^2} = ac\sec^2\frac{x}{a} \cdot \frac{1}{a} \cdot \frac{dx}{dt} \tag{2}$$

$$= ac\sec^2\frac{x}{a} \cdot \frac{1}{a} \cdot ac = ac^2\sec^2\frac{x}{a}.$$

$$\therefore\quad \text{Resultant acceleration } = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

$$= \sqrt{\left[(0)^2 + \left(ac^2\sec^2\frac{x}{a}\right)^2\right]} = ac^2\sec^2\frac{x}{a}.$$

Also, the radius of curvature $\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$.

But $\frac{dy}{dx} = \tan\frac{x}{a}$ implies that

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\frac{x}{a}.$$

$$\therefore \quad \rho = \frac{[1+\tan^2(x/a)]^{3/2}}{(1/a)\sec^2(x/a)} = \frac{a\sec^3(x/a)}{\sec^2(x/a)} = a\sec(x/a)$$

or $\sec(x/a) = \rho/a$.

$\therefore$ from (2), the resultant acceleration $= ac^2(\rho/a)^2 = (c^2/a)\rho^2$.

CHECK YOUR PROGRESS

Multiple Choice Question

1. Angular velocity is defined as:

A. Rate of change of displacement
B. Rate of change of angular displacement
C. Rate of change of linear velocity
D. Rate of change of radial distance

2. The SI unit of angular velocity is:

A. rad
B. rad/s
C. m/s
D. m/s^2

3. If a particle moves in a circle of radius *r* with linear speed *v*, then angular velocity ω is:

A. $\omega = rv$
B. $\omega = v/r$
C. $\omega = r/v$
D. $\omega = v^2r$

4. Angular acceleration is:

A. Rate of change of angular velocity
B. Rate of change of linear velocity
C. Product of ω and r
D. None of these

5. The SI unit of angular acceleration is:

A. rad/s
B. m/s²
C. rad/s²
D. rad/m²

6. When angular velocity is constant, angular acceleration is:

A. Zero
B. Infinite
C. Maximum
D. Constant but non-zero

2.8 SUMMARY

Angular Velocity**:** Angular velocity (ω) measures how fast a body rotates about a fixed point or axis. It is defined as the rate of change of angular displacement.

Angular acceleration: Angular acceleration (α) gives **the** rate at which angular velocity changes**.**

If angular velocity is constant, then angular acceleration is zero**.**

Tangential acceleration in circular motion is given by:at=rα

Rate of Change of a Unit Vector in a Plane: In polar coordinates, unit vectors **change direction** even if the magnitude is fixed.

2.9 GLOSSARY

Angular displacement: The angle through which a body rotates about a fixed point or axis. Measured in radians.

Unit Vector: A vector of magnitude 1, used to represent direction only.

Radial Unit Vector (êr): A unit vector that points outward from the origin toward the particle’s position.

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2.11 SUGGESTED READING

1. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige
Excellent for understanding theoretical principles and solving analytical problems.

*2.***Engineering Mechanics** – R. K. Bansal
Useful for Indian students; provides numerous solved examples and practice problems.

3. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler
Provides strong conceptual explanation with real-life engineering applications.

4 . **A Textbook of Engineering Mechanics** – S. S. Bhavikatti
Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6

2.12 TERMINAL QUESTIONS

1. Define angular velocity**.** Derive the relation between linear velocity and angular velocity of a particle moving along a circular path. Illustrate the physical meaning with a diagram.

2. What is angular acceleration? Obtain the expression for angular acceleration and explain how it affects the tangential acceleration of a particle in circular motion.

2.13 ANSWERS

MCQ 1 – B
MCQ 2 – B
MCQ 3 – B
MCQ 4 – A
MCQ 5 – C
MCQ 6 – A

UNIT 3: MOMENT OF INERTIA

CONTENTS:

3.1 *INTRODUCTION: -*

In the previous units have studied about angular momentum and its conservation, moment of inertia and its physical significance, radius of gyration, equations of angular motion and rotational kinetic energy. In this unit, we shall study some more important concepts of motion and laws of rotational motion. We know that, to find the moment of inertia of a body about a given axis, all that we have to do is to find the sum $\Sigma\ mr^2$ for all particles making up the body by integration or other means. The calculations to find the moment of inertia can be made shorter by the help of some important theorems. In this unit, we shall also study those theorems (theorems of parallel and perpendicular axes).

3.2 OBJECTIVES: -

After studying this unit, you should be able to-

- Solve problems based on equations of motion
- apply equations of motion
- understand laws of rotational motion
- apply theorems of parallel and perpendicular axes.

3.3 EQUATION OF MOTION :-

If an object is moving in a straight line under a constant acceleration, then relations among its velocity, displacement, time and acceleration can be represented by equations. These equations are called 'equations of motion'.

Let us consider that a body starts with an initial velocity 'u' and has a constant acceleration 'a'. Suppose it covers a distance's' in time't' and its velocity becomes 'v'. Then the relations among u, a, t, s and v can be represented by three equations.

First Equation: We know that linear acceleration a = $\frac{dv}{dt}$

or $dv = a\, dt$

Integrating both sides, we get-

$\int_u^v dv = \int_0^t a\, dt$

or $(v - u) = a\,(t - 0)$

or $v = u + at$
.....(1)

This is known as first equation of motion.

Second Equation: We know that linear velocity v = $\frac{ds}{dt}$

But $v = u + at$

Therefore, $u + at = \frac{ds}{dt}$

or $\frac{ds}{dt}$ = u + at

or ds = (u + at) dt

Integrating both sides, we get-

$\int_0^s ds = \int_0^t (u + at)dt$

or s = u (t – 0) + a ($\frac{t^2}{2}$ - 0)

or s = ut + $\frac{1}{2}$ a t^2
.....(2)

This is called second equation of motion.

Third Equation: By first equation, v = u + at

Squaring both sides-

$v^2 = (u + at)^2$

or $v^2 = u^2 + a^2t^2 + 2uat$

$= u^2 + 2a\,(ut + \frac{1}{2} a\,t^2)$

or $v^2 = u^2 + 2as$
…..(3)

(using equation 2)

The above equation (3) is known as third equation of motion.

Example 1: A train starting from rest is accelerated by 0.5 m/sec^2 for 10 sec. Calculate its final velocity after 10 sec. Also calculate the distance travelled by train in 10 sec.

Solution: Here, u = 0, a = 0.5 m/sec^2, t = 10 sec

Using first equation of motion v = u + at

v = 0 + 0.5 × 10 = 5 m/sec

Using second equation of motion s = ut + $\frac{1}{2}$ a t^2

$s = 0\ (10) + \frac{1}{2} \times 0.5\ (10)^2$

$= \frac{1}{2} \times 0.5 \times 100 = 25$ m

Thus the velocity of train after 10 sec is 5 m/sec and the distance travelled is 25 m.

Example 2: A car is accelerated from 8 m/sec to 14 m/sec in 3 sec. What is the acceleration of car ?

Solution: Here, u = 8 m/sec, v = 14 m/sec, t = 3 sec

Using $v = u + at$

$14 = 8 + a \times 3$

$3a = 14 - 8$

$= 6$

$a = 2$ m/sec^2

Self Assessment Question (SAQ) 1: A car is moving with a constant speed of 30 Km/hr. Calculate the distance travelled by car in 1 hr.

Self Assessment Question (SAQ) 2: A particle is shot with constant speed 6×10^6 m/sec in an electric field which produces an acceleration of 1.26×10^{14} m/sec^2 directed opposite to the initial velocity. How far does the particle travel before coming to rest?

Self Assessment Question (SAQ) 3: The initial velocity of a particle is 'u' (at t = 0) and the acceleration is given by 'at^2'. Which of the following relations is valid?

(a) $v = u + at$ (b) $v = u + \frac{at^3}{3}$ (c) $v^2 = u + at^3$ (d) $v = u + \frac{at^3}{2}$

3.4 NEWTON'S LAWS OF ROTATIONAL MOTION : -

As we know, there are Newton's three laws of translational motion, similarly we have the following three laws of rotational motion-

First Law: Unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered.

Second Law: The rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque.

Third Law: When a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation.

3.5 GENERAL THEORUMS OF MOMENT OF INERTIA:-

There are two important theorems on moment of inertia which, in some cases, facilitate the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis be known. These theorems are theorem of parallel axes and theorem of perpendicular axes. Let us discuss these theorems.

3.5.1 Theorem of Parallel Axes

It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two axes.

If 'I_{cm}' be the moment of inertia of a body about a parallel axis through its centre of mass, 'M' be the mass of the body and 'r' be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$

This is the "theorem of parallel axes".

Proof: Let us consider a plane lamina with 'C' as centre of mass. Let 'I' be its moment of inertia about an axis PQ in its plane and I_{cm} the moment of inertia about a parallel axis RS passing through C. Let the distance between RS and PQ be 'r'.

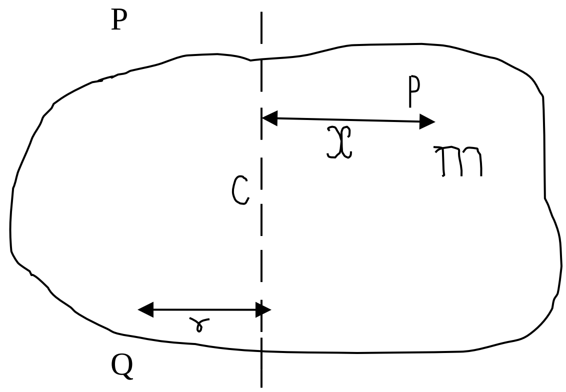


Figure 1

Let us consider a particle P of mass m at a distance x from RS. Its distance from PQ is (r + x) and its moment of inertia about it is $m(r + x)^2$. Therefore, the moment of inertia of the lamina about PQ is given by-

$I = \Sigma\, m(r + x)^2$

$= \Sigma\, m(r^2 + x^2 + 2rx)$

$= \Sigma\, mr^2 + \Sigma\, mx^2 + \Sigma\, 2mrx$

or $I = r^2\, \Sigma\, m + \Sigma\, mx^2 + 2r\, \Sigma\, mx$ …..(4)

(since r is constant)

But $\Sigma\, mx^2 = I_{cm}$, where I_{cm} is the moment of inertia of the lamina about RS, $r^2\, \Sigma\, m = r^2 M$ where M is the total mass of the lamina and $\Sigma\, mx = 0$ because the sum of the moments of all the mass particles of a body about an axis through the centre of mass of the body is zero. Hence, the equation (4) becomes

$I = r^2 M + I_{cm} + 0$

or $I = I_{cm} + M r^2$ …..(5)

It may be seen clearly from equation (5) that the moment of inertia of a body about an axis through the centre of mass is the least. The moment of inertia of the body about an axis not passing through the centre of mass is always greater than its moment of inertia about a parallel axis passing through the centre of mass of the body.

3.5.2 Theorem of Perpendicular Axes

According to this theorem, the moment of inertia of a uniform plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its plane intersecting on the first axis.

If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then

$I_z = I_x + I_y$

This is the "theorem of perpendicular axes".

Proof: Let OZ be the axis perpendicular to the plane of the lamina about which the moment of inertia is to be taken. Let OX and OY be two mutually perpendicular axes in the plane of the lamina and intersecting on OZ.

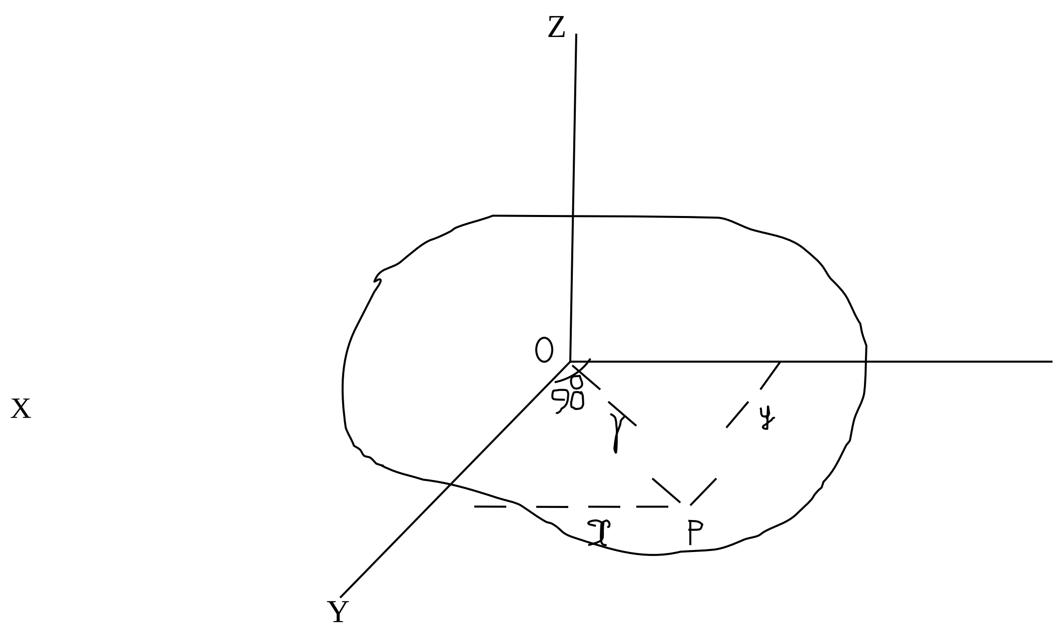


Figure 2

Let us consider a particle P of mass 'm' at a distance of 'r' from OZ. The moment of inertia of this particle about OZ is mr^2. Therefore, the moment of inertia I_z of the whole lamina about OZ is $I_z = \Sigma\, mr^2$

But $r^2 = x^2 + y^2$, where x and y are the distances of P from OY and OX respectively.

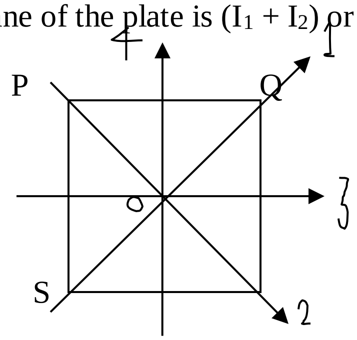
Therefore, $I_z = \Sigma\, m(x^2 + y^2) = \Sigma\, mx^2 + \Sigma\, my^2$

But $\Sigma\, mx^2$ is the moment of inertia I_y of the lamina about OY and $\Sigma\, my^2$ is the moment of inertia I_x of the lamina about OX.

Therefore, $I_z = I_y + I_x$

or $I_z = I_x + I_y$

Example 3: Show that the moment of inertia I of a thin square plate PQRS (Figure) of uniform thickness about an axis passing through the centre O and perpendicular to the plane of the plate is $(I_1 + I_2)$ or $(I_3 + I_4)$ or $(I_1 + I_3)$.



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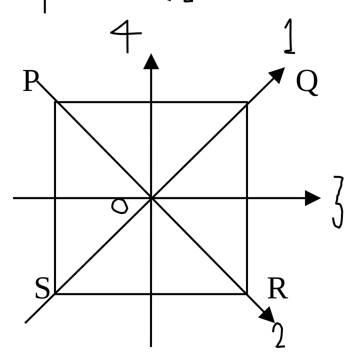
Solution:



Let I_1, I_2, I_3 and I_4 are the moments of inertia about axes 1, 2, 3 and 4 respectively which are in the plane of the plate.

By the theorem of perpendicular axes, we have-

$I = I_1 + I_2 = I_3 + I_4$

By symmetry of square plate, $I_1 = I_2$ and $I_3 = I_4$

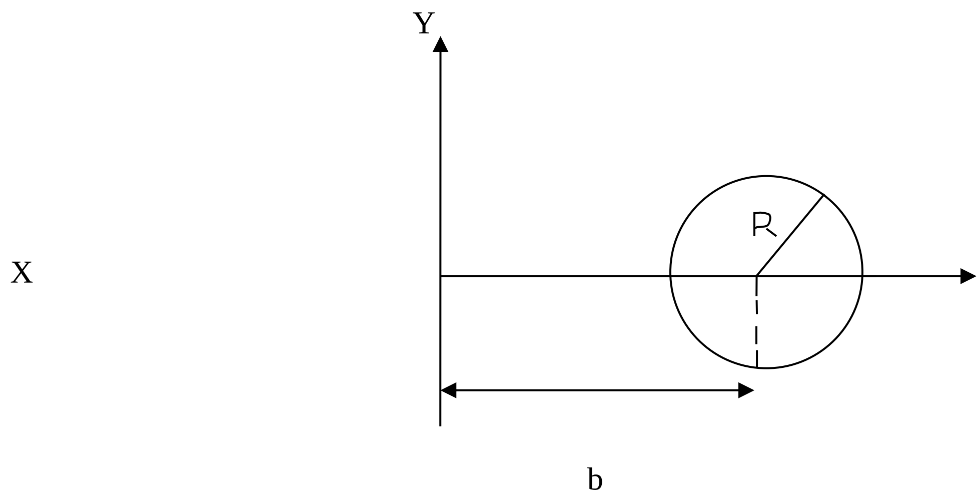
Therefore, $I = 2I_1 = 2\, I_3$

or $I_1 = I_3$

Thus $I_1 = I_2 = I_3 = I_4$

and $I = I_1 + I_2 = I_1 + I_3$

Self Assessment Question (SAQ) 4: The figure represents a disc of mass M and radius R, lying in XY- plane with its centre on X-axis at a distance 'b' from the origin. Determine the moment of inertia of the disc about Y-axis if its moment of inertia about a diameter is $\frac{MR^2}{4}$.



3.6 SUMMARY: -

In the present unit, we have studied about equations of motion and derived all the three equations of motion. In the unit, we have also studied Newton's laws of rotational motion. According to the first law of rotational motion "unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered" while the second law states "the rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque". According to Newton's third law of rotational motion "when a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation". Sometimes it is difficult to calculate the moments of inertia of some specific bodies. In this unit, we have also studied and derived the general theorems on moment of inertia. These theorems are known as theorem of parallel axes and theorem of perpendicular axes. If 'I_{cm}' be the moment of inertia of a body about a parallel axis through its centre of mass, 'M' be the mass of the body and 'r' be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$. This is the theorem of parallel axes. If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then $I_z = I_x + I_y$. This is the theorem of perpendicular axes. These theorems make easy to find out the moments of inertia of those specific bodies. We have included examples and self assessment questions (SAQs) to check your progress.

3.7 GLOSSARY: -

Velocity- a vector physical quantity whose magnitude gives speed

Acceleration- increase of velocity

Unless- except, if not

External- exterior, outer

Rotational- the action of moving in a circle

Unaltered- unchanged, unaffected

Facilitate- make easy, smooth the progress of

3.8 : -TERMINAL QUESTIONS :-

1. Explain the equations of motion.

2. What are Newton's laws of rotational motion? Explain.

3. Discuss and derive general theorems on moment of inertia.

4. Calculate the moment of inertia of mass M and length L about an axis perpendicular to the length of the rod and passing through a point equidistant from its midpoint and one end.

5. Give the statement of theorem of parallel axes. Also derive this theorem.

6. State and establish the theorem of perpendicular axes.

3.9 ANSWERS: -

Self Assessment Questions (SAQs):

1. Since the car is moving with constant speed, therefore acceleration of car $a = 0$.

Here, $u = 30$ Km/hr $= 25/3$ m/sec, $t = 1$ hr $= 3600$ sec

Using second equation of motion, $s = ut + \frac{1}{2} a t^2$

$s = (25/3) \times 3600 + \frac{1}{2}(0) \times (3600)^2 = 3 \times 10^4$ m $= 30$ Km

2. Given $u = 6 \times 10^6$ m/sec, $a = -1.26 \times 10^{14}$ m/sec^2, $v = 0$

Using third equation of motion, $v^2 = u^2 + 2as$

$(0)^2 = (6 \times 10^6)^2 + 2(-1.26 \times 10^{14})\, s$

or $s = 0.143$ m

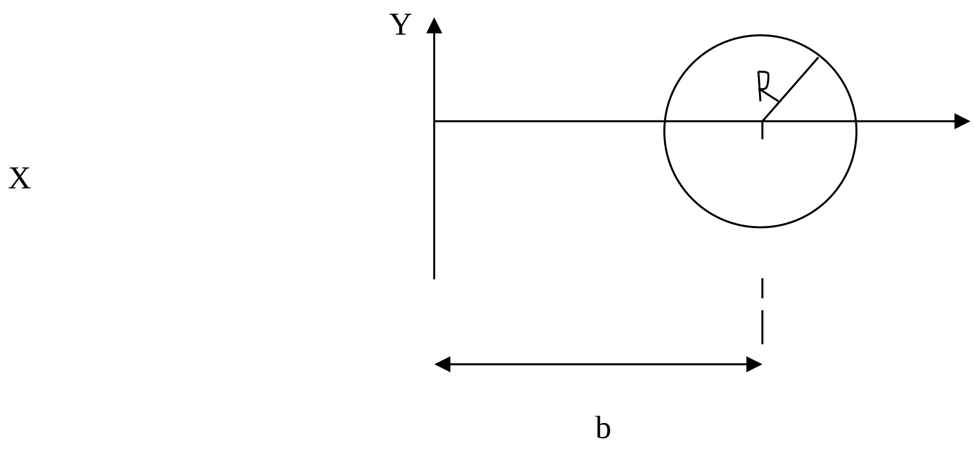
3. Here acceleration $= at^2$

Using $v = u + at$

$v = u + (at^2)\, t = u + at^3$

Hence relation (c) is valid.

4.



By the theorem of parallel axes, the moment of inertia of disc about Y-axis is-

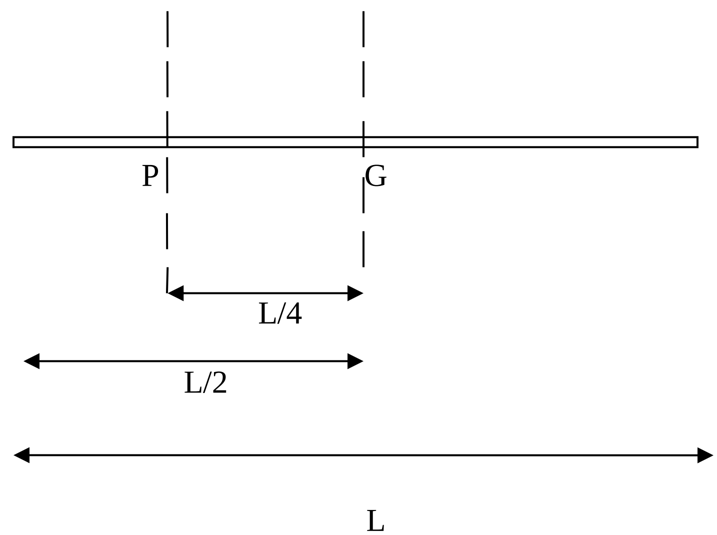
$I = I_{cm} + Mb^2$

$= \frac{MR^2}{4} + Mb^2$

$= M\,(\frac{R^2}{4} + b^2)$

Terminal Questions:

4. The moment of inertia of rod about an axis passing through centre of mass and perpendicular to length $I_{cm} = \frac{ML^2}{12}$



The moment of inertia of rod about an axis passing through P and perpendicular to length by theorem of parallel axes is-

$I = I_{cm} + Mr^2$

Here r = PG = L/4

Therefore, $I = (ML^2/12) + M\,(L/4)^2 = (7/48)\,ML^2$

2.10 REFERENCES: -

1. Mechanics- DS Mathur, S Chand and Company Ltd., New Delhi

2. Mechanics- JK Ghose, Shiva Lal Agarwal and Company, Delhi

3. Elements of Mechanics- JP Agrawal and Satya Prakash, Pragati Prakashan, Meerut

3.11 SUGGESTED READING: -

1. Concepts of Physics, Part I, HC Verma, Bharati Bhawan, Patna

2. Mechanics and Wave Motion – DN Tripathi and RB Singh, Kedar Nath Ram Nath, Meerut

3. Modern Physics, Beiser, Tata McGraw Hill

4. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons

UNIT 4: FORMULATION OF MOMENT OF INERTIA

CONTENTS:

4.1 *INTRODUCTION: -*

In the previous unit, we have studied theorem of parallel axes and theorem of perpendicular axes. These theorems make easy the calculations of moment of inertia in some typical cases. In general, the moment of inertia of a body is calculated as the sum $\Sigma\, mr^2$ for all particles making up the body by integration or other means. In this unit, we shall formulate and derive the moment of inertia for some simple symmetric systems like rod, rectangular lamina, circular lamina, solid sphere and cylinder.

4.2 OBJECTIVES: -

After studying this unit, you should be able to-

- Understand the formulation and derivation of moment of inertia
- Solve problems based on moment of inertia
- Apply the formulae of moment of inertia

4.3 FORMULATION AND DERIVATION OF MOMENT OF INERTIA: -

The moment of inertia of a continuous homogeneous body of a definite geometrical shape can be calculated by (i) first obtaining an expression for the moment of inertia of an infinitesimal element of the same shape of mass dm about the given axis i.e. $dm.r^2$, where r is the distance of the infinitesimal element from the axis and then (ii) integrating this expression over appropriate limits so as to cover the whole body. In fact sometimes the theorem of parallel and perpendicular axes are also used to calculate the moment of inertia about an axis when the moment of inertia of the body about some other axis has first been calculated. Thus,

$$I = \int dm.r^2$$

where the integral is taken over the whole body.

4.3.1 MOMENT OF INERTIA OF A THIN UNIFORM ROD: -

(i) About an axis passing through its centre of mass and perpendicular to its length

Let PQ be a thin uniform rod of mass per unit length m. Let RS be the axis passing through the centre of mass C of the rod and perpendicular to its length PQ.

Let us consider an element of length dr at a distance r from centre of mass C.

The mass of the element, dm = m.dr

The moment of inertia of element about the axis through C $= dm.r^2$

$= m\,dr\,r^2$

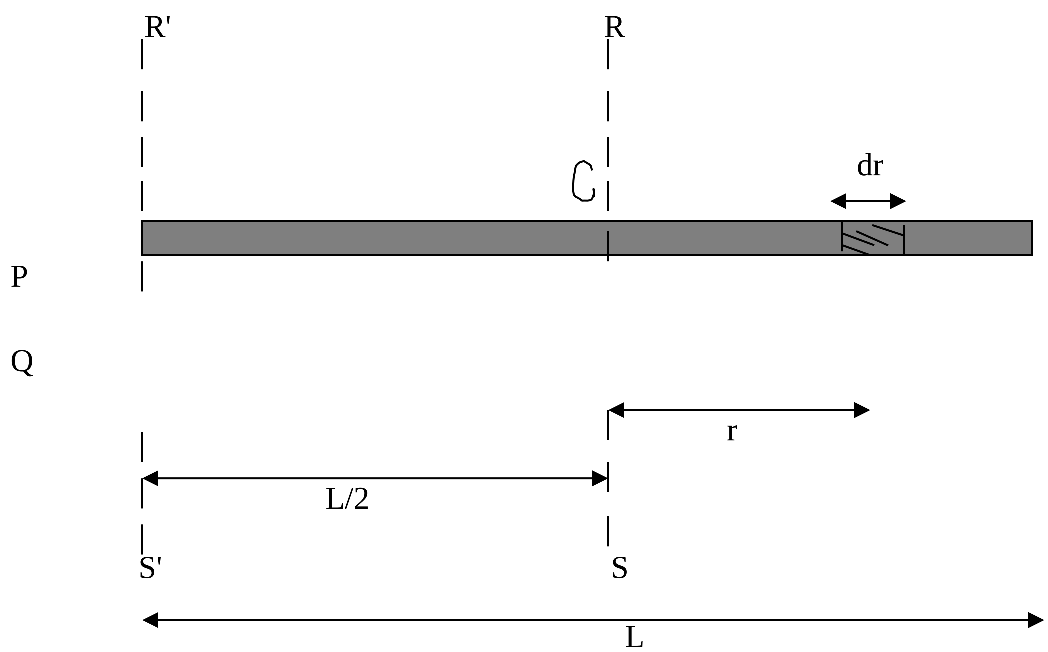


Figure 1

The moment of inertia of the whole rod about axis RS is the sum of the moments of inertia of all such elements lying between r = - L/2 at P and r = L/2 at Q. Hence the moment of inertia

$I_{cm} = \int_{-L/2}^{L/2} mr^2 dr$

$= 2m \int_0^{L/2} r^2 dr = \frac{mL^3}{12} = \frac{(mL)L^2}{12}$

$= \frac{ML^2}{12}$

where $mL = M$, the total mass of the rod

Thus, $I_{cm} = \frac{ML^2}{12}$
…..(1)

(ii) About an axis passing through its one end and perpendicular to its length

Let R' S' be the axis passing through the end P of the rod (Figure 1). The moment of inertia I about a parallel R' S' axis passing through one end (using theorem of parallel axes)-

$I = I_{cm} + M\,(CP)^2$

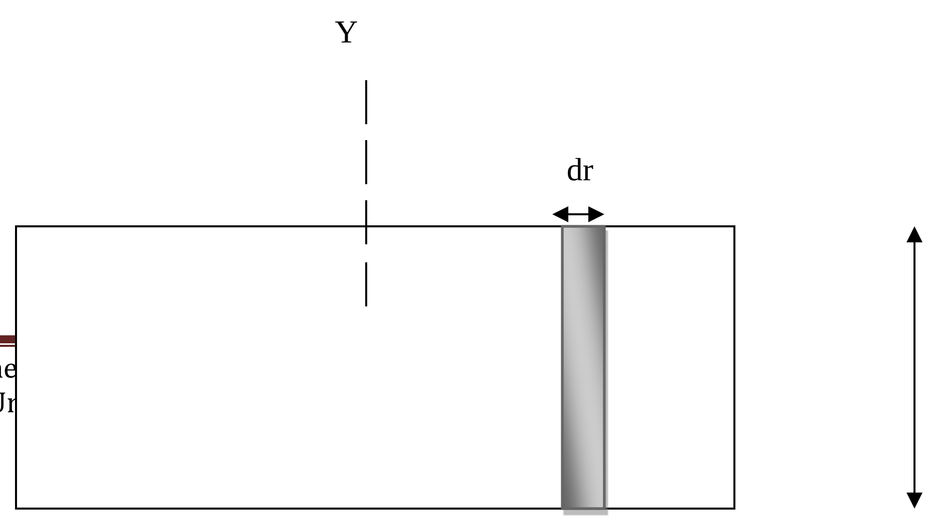
$= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$

$= \frac{ML^2}{3}$

4.3.2 MOMENT OF INERTIA OF A RECTANGULAR LEMINA: -

(i) About an axis in its own plane, parallel to one of the sides and passing through the centre of mass

Let PQRS be a rectangular lamina of mass M, length l and breadth b with O as its centre of mass. Let the mass per unit area of the lamina is σ. Let us consider a strip of width dr parallel to the given axis Y' at a distance r from it.



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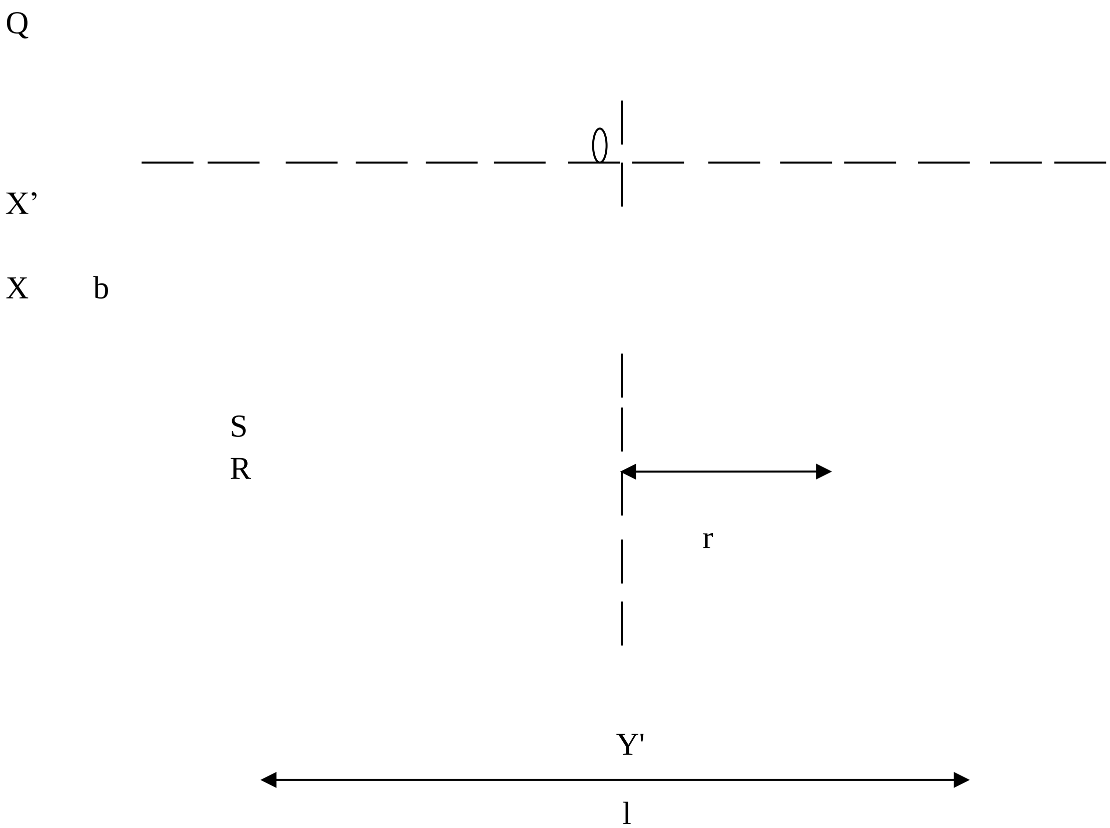


Figure 2

Area of the strip = b dr

Mass of the strip m = (bdr) σ

Moment of inertia of strip about the axis YY' = mr^2

$= (bdr)\, \sigma\, r^2$

$= \sigma\, b\, r^2\, dr$

The moment of inertia of the whole lamina about axis YY′

$I_y = \int_{-l/2}^{l/2} \sigma b r^2 dr = 2\int_0^{l/2} \sigma b r^2 dr$

$= 2\sigma b \int_0^{l/2} r^2 dr = \frac{\sigma b l^3}{12} = \frac{(\sigma b l) l^2}{12}$

i.e. $I_y = \frac{Ml^2}{12}$ …..(1)

where σbl = M, the total mass of the lamina

Similarly, the moment of inertia of the lamina about an axis XX' parallel to the side of length l and passing through the centre will be-

$I_x = \frac{Mb^2}{12}$ …..(2)

(ii) About an axis perpendicular to its plane and passing through the centre of mass

The moment of inertia of the lamina about an axis passing through the centre C and perpendicular to the plane of the lamina

$I = I_x + I_y$ (using theorem of perpendicular axes)

$= \frac{Mb^2}{12} + \frac{Ml^2}{12}$

$= M\left(\frac{l^2+b^2}{12}\right)$ …..(3)

4.3.3 MOMENT OF INERTIA OF A CIRCULAR LEMINA: -

(i) About an axis passing through its centre and perpendicular to its plane

Let O be the centre of the circular lamina and σ the mass per unit area. Let the lamina be supposed to be composed of a number of thin circular strips. Let us consider one such strip of radius r and thickness dr.

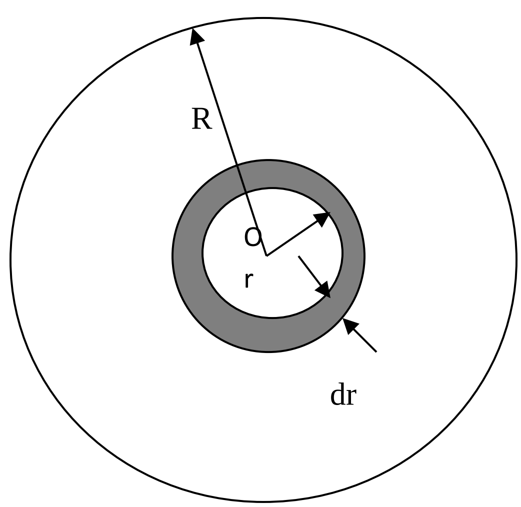


Figure 3

The circumference of the strip = 2π r
Area of the strip = 2π r dr
Mass of the strip m = σ (2π r dr)
The moment of inertia of the strip about an axis passing through O and perpendicular to the plane of the lamina dI = mass × (distance)^2 = m × r^2

$$= \sigma(2\pi r\, dr)\, r^2$$
$$= \sigma(2\pi r^3 dr)$$

The moment of inertia of whole lamina $I = \int dI$

$$= \int \sigma(2\pi r^3 dr)$$

or $I = 2\pi\sigma \int r^3 dr$ …..(1)

If circular lamina is solid, then its moment of inertia $I = 2\pi\sigma\int_0^R r^3 dr$

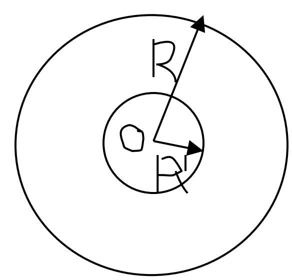
$$= 2\pi\sigma\frac{R^4}{4} = \frac{1}{2}R^2(\pi R^2\sigma)$$
$$= \frac{1}{2}R^2 M$$

where $\pi R^2\sigma = M$, mass of the disc

Thus, $I = \frac{1}{2} M R^2$ …..(2)

If lamina is having concentric hole and R' and R be the its internal and external radii then the moment of inertia $I = 2\pi\sigma\int_{R'}^{R} r^3 dr$

$$= \frac{2\pi\sigma}{4}[R^4 - R'^4]$$
$$= \frac{2\pi\sigma}{4}(R^2 + R'^2)(R^2 - R'^2)$$
$$= \pi(R^2 - R'^2)\,\sigma\left\{\frac{1}{2}(R^2 + R'^2)\right\}$$
$$= \frac{1}{2} M (R^2 + R'^2)$$



where $\pi(R^2 - R'^2)\,\sigma = M$, mass of the lamina

Figure 4

Thus, $I = \frac{1}{2} M (R^2 + R'^2)$ …..(3)

(ii) About any diameter

Let us consider two mutually perpendicular diameters AB and CD of the lamina. The lamina is symmetrical about both diameters AB and CD.
In accordance with the theorem of perpendicular axis, the moment of inertia about diameter

$I_D = \frac{I}{2}$, where I is the moment of inertia of the disc, about an axis. Through it's center and perpendicular to it's plane.

For solid lamina, $I_D = \frac{1}{2}\left(\frac{1}{2}MR^2\right)$

$= \frac{1}{4}MR^2$

For circular lamina with concentric hole,

$I_D = \frac{1}{4}M(R^2 + R'^2)$

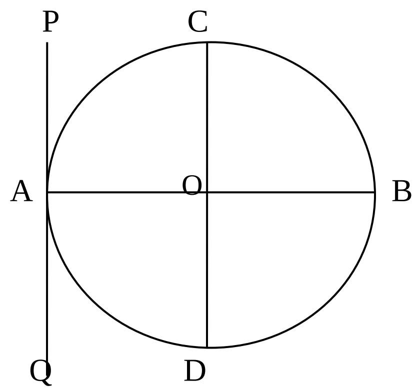


Figure 5

(iii) About a tangent in its plane

Let PQ be a tangent to the lamina in its plane and let it be parallel to the diameter CD.

Using theorem of parallel axes, the moment of inertia of lamina about PQ = Moment of inertia of lamina about CD + $(OA)^2$

or $I_T = I_D + MR^2$. M.

For a solid lamina, $I_T = \frac{1}{4}MR^2 + MR^2$

$= \frac{5}{4}MR^2$

For a lamina having a concentric hole,

$I_T = \frac{1}{4}M(R^2 + R'^2) + MR^2$

$= \frac{1}{4} M\,(5R^2 + R'^2)$

(iv) About a tangent perpendicular to its plane

Using theorem of parallel axes, moment of inertia of lamina about a tangent perpendicular to its plane $I'_T = I + MR^2$

For a solid lamina, $I'_T = \frac{1}{2} M R^2 + MR^2 = \frac{3}{2} MR^2$

For a circular lamina having concentric hole, $I'_T = \frac{1}{2}M(R^2 + R'^2) + MR^2$

$$= \; = \frac{1}{2}M(3R^2 + R'^2)$$

4.3.4 MOMENT OF INERTIA OF A SOILD SPHERE: -

(i) About a diameter

Let us consider a sphere of radius R and mass M with centre at O. Let ρ be the density of the material of the sphere.

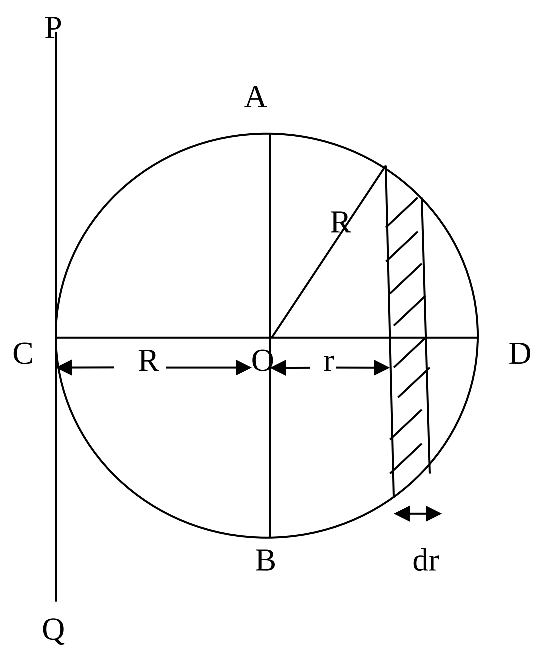


Figure 7

Let us divide the sphere into a number of thin discs by planes perpendicular to the diameter CD and let us consider one such elementary disc of thickness dr at a distance r from O.

Obviously, the radius of the elementary disc = $\sqrt{R^2 - r^2}$

Volume of the elementary disc = its area × its thickness

= $\pi(R^2-r^2)$ dr

Mass of the elementary disc m = volume × density

= $\pi(R^2-r^2)$ dr ρ

The moment of inertia of the disc about diameter CD-

dI = $\frac{1}{2}$ [mass of the disc × (radius of the disc)^2]

= $\frac{1}{2}\pi(R^2-r^2)$ dr $\rho\times(R^2-r^2)$

= $\frac{1}{2}\pi(R^2-r^2)^2$ dr ρ …..(1)

The moment of inertia of the whole sphere about CD, I = $\int_{-R}^{+R} dI$

= $\int_{-R}^{+R}\frac{1}{2}\pi(R^2 - r^2)^2\rho\, dr$

= $\int_0^R \pi(R^2 - r^2)^2\rho\, dr$

= $\pi\rho\int_0^R (R^4 + r^4 - 2R^2r^2)dr$

= $\frac{8}{15}\pi\rho R^5$

= $\frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$

= $\frac{2}{5} M R^2$, where $\frac{4}{3}\pi R^3\rho$ = M, the mass of the sphere

Therefore, I = $\frac{2}{5} M R^2$

(ii) About a tangent

Let PQ be a tangent to the sphere parallel to the diameter AB and at a distance R (the radius of the sphere) from it.

Using theorem of parallel axes, the moment of inertia of the solid sphere about a tangent PQ-

I_T = moment of inertia of sphere about CD + $M\times(OC)^2$

$= \frac{2}{5} M R^2 + MR^2 = \frac{7}{5} MR^2$

4.3.5 MOMENT OF INERTIA OF A SOILD CYLINDER : -

Let us consider a solid cylinder of mass M, radius R and length L.

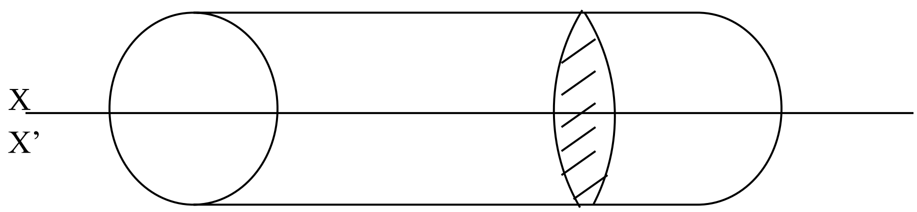


Figure 8

Volume of the cylinder = $\pi R^2 L$

Density of the cylinder $\rho = \frac{M}{\pi R^2 L}$

or $M = (\pi R^2 L)\rho$

(i) Moment of inertia of solid cylinder about geometrical axis

Let us suppose that the cylinder be formed of a large number of co-axial discs of equal radius R, the axis being the geometrical axis (XX') of the cylinder. Let us consider one such disc of mass m and radius R.

Moment of inertia of the disc about geometrical axis XX' = $\frac{1}{2}$ m R^2

Therefore, the moment of inertia of the whole cylinder about its geometrical axis XX',

$$I_1 = \Sigma\left(\frac{1}{2} mR^2\right)$$

$$= \frac{1}{2} R^2 \Sigma m$$

$$= \frac{1}{2} R^2 M$$

where $\Sigma m = M$, the mass of the cylinder

Therefore, $I_1 = \frac{1}{2} MR^2$ …..(1)

(ii) Moment of inertia about an axis passing through centre and perpendicular to geometrical axis

Let us consider an axis YY' perpendicular to geometrical axis XX' and passing through centre O of the cylinder. Let us suppose that the cylinder to be formed of coaxial discs of equal radius R.

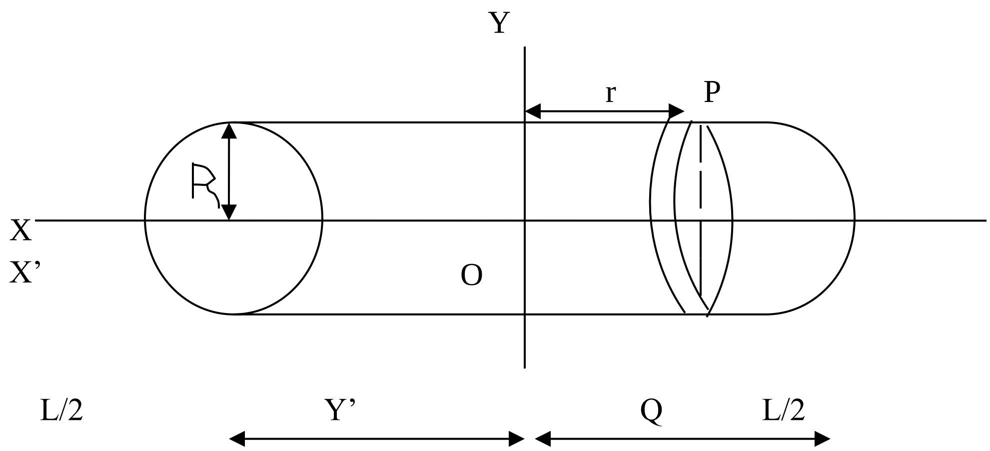


Figure 9

Let us consider one such disc of radius R and thickness dr at a distance r from the axis.

Area of disc = πR^2

Volume of the disc = (πR^2) dr

Mass of the disc $m = (\pi R^2)\, dr\, \rho$

Moment of inertia of disc about its diameter PQ = $\frac{1}{4}$ [mass of the disc × (radius of the disc)^2]

$$= \frac{1}{4} (\pi R^2 dr \rho) R^2$$

$$= \frac{1}{4} (\pi R^4 \rho dr)$$

Using theorem of parallel axes, the moment of inertia of the disc about axis YY' –

dI_2 = (Moment of inertia of disc about its diameter PQ) + m r^2

$$= \frac{1}{4}\pi R^4 \rho dr + (\pi R^2 dr) \rho r^2$$

$$= \pi R^2 \rho \left(\frac{R^4}{4} + r^2\right) dr$$

The moment of inertia of whole cylinder about axis YY'-

$$I_2 = \int_{-L/2}^{+L/2} dI_2$$

$$= \int_{-L/2}^{+L/2} \pi R^2 \rho \left(\frac{R^4}{4} + r^2\right) dr$$

$$= 2 \pi R^2 \rho \int_0^{L/2} \left(\frac{R^4}{4} + r^2\right) dr$$

$$= (\pi R^2 L \rho)\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

$$= M \left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

where $\pi R^2 L \rho = M$, the mass of cylinder

Therefore, $I_2 = M \left[\frac{R^2}{4} + \frac{L^2}{12}\right]$

Example 1: The mass and radius of a solid circular disc are 500 Kg and 1 metre respectively. Calculate its moment of inertia about its axis.

Solution: The mass of circular disc M = 500 Kg

The radius of circular disc R = 1 m

Moment of inertia of the disc about its axis $I = \frac{1}{2} MR^2$

$= \frac{1}{2}$ (500) $(1)^2$ = 250 Kg-m^2

Example 2: Determine the moment of inertia of a rectangular lamina of 2 Kg about an axis perpendicular to its plane and passing through the centre of mass. The length and breadth of lamina are 100 cm and 50 cm respectively.

Solution: Mass of lamina M = 2 Kg, l = 100 cm = 1 m, b = 50 cm = 0.5 m

Moment of inertia of lamina about an axis perpendicular to its plane and passing through the centre of mass I = $M\left(\frac{l^2+b^2}{12}\right)$

$$= 2\left[\frac{1^2+0.5^2}{12}\right] = 0.21 \text{ Kg-m}^2$$

Self Assessment Question (SAQ) 1: The moment of inertia of a disc about a tangent perpendicular to the plane of disc is-

(a) $\frac{3}{2}$ M R^2 (b) $\frac{5}{4}$ M R^2 (c) M R^2 (d) $\frac{1}{2}$ M R^2

Self Assessment Question (SAQ) 2: The moment of inertia of a thin rod of mass M and length L about an axis passing through one end and perpendicular to length is-

(a) ML^2 (b) $\frac{ML^2}{12}$ (c) $\frac{ML^2}{4}$ (d) $\frac{ML^2}{3}$

4.4 SUMMARY:-

In the present unit, we have formulated and derived the expressions for moment of inertia of thin uniform rod, rectangular lamina, circular lamina, solid sphere and solid cylinder. Moment of inertia of a thin uniform rod about an axis passing through its centre of mass and perpendicular to its length is expressed as $\frac{ML^2}{12}$, where M is the mass of the rod and L its length while the moment of inertia of the rod about an axis passing through its one end perpendicular to its length is derived as $\frac{ML^2}{3}$. The moment of inertia of a rectangular lamina of mass M, length l and breadth b about an axis perpendicular to its plane and passing through the centre of mass is given as $M\left(\frac{l^2+b^2}{12}\right)$.We have established the expression for moment of inertia of a circular lamina of mass M and radius R about an axis passing through its centre and perpendicular to its plane as $\frac{1}{2}$ M R^2. We have formulated the moment of inertia of a solid sphere about a diameter as $\frac{2}{5}$ M R^2 and about a tangent as $\frac{7}{5}$ M R^2. We have also derived the formulae for moment of inertia of a solid cylinder. After studying the unit, we can solve problems based on moment of inertia. We have also included examples and self assessment questions (SAQs) in the unit to check your progress.

4.5 GLOSSARY

Continuous- unbroken

Homogeneous- uniform

Infinitesimal- microscopic, extremely small

Appropriate- suitable, proper

Uniform- consistent, homogeneous

4.6 TERMINAL QUESTIONS

1. Derive the ratio of moment of inertia of a rectangular bar about an axis passing through one of its ends and through its centre.

2. Establish the expression for moment of inertia of a rectangular lamina about an axis in its own plane parallel to one of the sides and passing through the centre of mass.

3. Calculate the moment of inertia of a solid sphere about (i) a diameter and (ii) a tangent

4. Calculate the moment of inertia of a circular lamina about an axis passing through its centre and perpendicular to its plane. The mass of the lamina is 300 gm and radius 50 cm.

5. Calculate the moment of inertia of a solid cylinder about its own axis.

4.7 ANSWERS

Self Assessment Questions (SAQs):

1. Using theorem of parallel axis, $I = \frac{1}{2} M R^2 + M R^2$

$$= \frac{3}{2} M R^2$$

Hence option (a) is correct

2. Option (d) is correct

Terminal Questions:

4. Mass of lamina M = 300 gm = 0.3 Kg, Radius R = 50 cm = 0.5 m

Required moment of inertia $I = \frac{1}{2} M R^2$

$$= \frac{1}{2} \times 0.3 \times 0.5$$

$$= 0.075 \text{ Kg-m}^2$$

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BLOCK II
MOTION OF RIGID BODY

UNIT 5: - MOTION OF RIGID BODY

CONTENTS:

5.1 INTRODUCTION: -

A rigid body is an idealized mechanical system consisting of a continuous distribution of mass in which the distance between any two material points remains invariant under the action of external forces. Consequently, the configuration of a rigid body in three-dimensional space is completely determined by six independent parameters: three specifying the position of its center of mass and three specifying its orientation. When external forces act on a rigid body, its motion can be decomposed into translational motion of the center of mass and rotational motion about the center of mass. These forces give rise to changes in the linear momentum and angular momentum of the system, governed by Newton's laws and the principles of rotational dynamics. Since most physical bodies can be approximated as rigid bodies, the mathematical analysis of rigid body motion is fundamental for determining trajectories and stability. Such analyses have significant applications in areas such as robotic kinematics, spacecraft attitude dynamics, and computational models used in simulations and video game physics. This unit will discuss about the concepts of Motion of rigid body.

5.2 OBJECTIVES: -

After studying this unit, the learner's will be able to

1. To study the stability of rigid body rotation by analysing equilibrium solutions of Euler's equations and applying linear stability theory.
2. To define and compute the inertia tensor and determine its principal axes and principal moments through eigenvalue analysis.
3. To analyze the decomposition of motion into translational motion of the center of mass and rotational motion about the center of mass using vector and tensor methods.
4. To derive expressions for linear and angular momentum of a rigid body in both space-fixed and body-fixed reference frames.

5.3 RIGID BODY DYNAMICS: -

A rigid body is defined as the one which the distance between any two particles is fixed. Such a body in general has a motion which is a superposition of both translation and rotation.

The theory of rigid body dynamics was developed to address the limitations of classical mechanics in describing the motion and kinematics of extended bodies, which cannot be treated as point masses and are capable of rotational motion under the action of external forces. When forces act on a rigid body, its motion may occur in two distinct forms:

(i) Translatory Motion

(ii) Rotational Motion

5.3.1 TRANSLATORY MOTION: -

A body is said to have translation motion if the position of any line inscribed on the body remains parallel to its original position during its motion. It may be noted that during translation motion, all particles of a rigid body will have the same velocity with reference to some reference frame, at any instant of time. In fact, velocity vector may charge from time to time.

5.3.2 ROTATIONAL MOTION: -

A body is said to have rotational motion, if the body rotates such that along some straight line, all the particle of the body have zero velocity with respect to some reference frame. Such a line of stationary particle is called the axes of rotation. During rotation, the distance of any point of the body from the axes of rotation remain unaltered .

5.4 ANGULAR VELOCITY: -

Consider the motion of rigid body during an infinitesimal interval such that as shown in figure. During this interval, suppose we have a translation displacement ds of all points in the body together with a rotational displacement – so about an axis through the base point A' fixed in the body. The order of performing the translation and rotation is immaterial.

In figure we have shown that translation occurs first, followed by rotation. Thus a typical point -P moves top' and the base point A moves to A', each undergoes the same displacement ds, then follow rotation by an amount and θ moving P' to P''. However, A' does not move as it is on the six of rotation. Now we shall define the angular velocity of the body denoted by $\vec{w}$, as

$\vec{w} = \log_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

It may be noted that the angular velocity in general changes continuously with time both in magnitude and direction. Also, the angular velocity of the body is different in different frame of references. Therefore, while specifying the angular velocity of a body, one has to specify the reference frame too, otherwise the inertial frame is assumed.

5.5 RIGID BODY ROTATION ABOUT A FIXED AXIS: -

Consider a rigid body which is constrained to rotate about a fixed axis AA', that is fixed in an inertial frame OXYZ (as in fig.) for convenience, use shall assume that for convenience, we shall assume that the origin is on AA', and z-axis coincides with AA'.

Let $\vec{v}$ be the absolute velocity of a typical point P of the body, whose position vector is $\vec{r}$ w.r.t inertial frame, and $\vec{w}$, the angular velocity of the body. Then $\vec{w}$ has a fixed direction and the path described by the point P is a circle of radius r sin $\emptyset$, with center at B.

Here, $\emptyset$ is the angle between AA' and $\vec{r}$. If P experiences a displacement s of the are described by P when the body rotates through an angle θ, during the interval Δt , then

$\Delta s \ = (BP)\ \Delta\theta \ = \Delta t$ = r sin $\emptyset$. $\Delta\theta$

And the magnitude of the velocity of P is

$|\vec{v}\,| \ = \log_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \ \log_{\Delta t \to 0}$ r sin $\emptyset$ $\frac{\Delta\theta}{\Delta t}$ = r sin $\emptyset$.w

Since we write $\vec{v} = \vec{w} \times \vec{r}$ …(1)

Here, the vector $\vec{w}$ = w. $\vec{R}$ is the angular velocity of the body.

The acceleration $\vec{a}$ of p can be obtained by differential equation w.r.t time t.

Thus, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\overrightarrow{w} \times \overrightarrow{r})}{dt} = \frac{d\vec{w}}{dt} \times \vec{r} + \vec{w} \times \frac{d\vec{r}}{dt}$

If we define the vector $\frac{d\vec{w}}{dt}$by $\vec{\alpha}$, called angular acceleration of that body

So above equation become

$\vec{a}$=$\vec{\alpha} \times \vec{r} + \vec{w} \times \vec{v}$ (since $\vec{v} = \frac{d\vec{r}}{dt}$)

$$\vec{a}=\vec{\alpha} \times \vec{r} + \vec{w} \times (\vec{w} \times \vec{r}) \qquad \ldots (2)$$

We also have,

$\vec{\alpha} = \alpha\hat{k} = \frac{d\vec{w}}{dt}\,\hat{k} = \vec{\dot{w}}\,\hat{k} = \ddot{\theta}$ k

In equation (2), the term $\vec{w} \times (\vec{w} \times \vec{r})$, a vector that is directed radially increased from point P towards the instantaneous axis of rotation and perpendicular to it, it is called Centripetal acceleration, while the term $\vec{\alpha} \times \vec{r} = \vec{\dot{w}} \times \vec{r}$ is called the Tangential acceleration.

5.6 SUMMARY: -

Rigid body dynamics deals with the motion of a body in which the distance between any two particles remains constant (i.e., no deformation occurs). A rigid body can undergo two types of motion: Translational motion , Rotational motion ,Or a combination of both . The analysis involves applying Newton's laws to extended bodies, considering both linear and angular quantities like force, torque, momentum, and energy.

In translatory motion, every point of the rigid body moves with the same velocity and acceleration.

In rotational motion, the rigid body rotates about an axis, and different points have different velocities depending on their distance from the axis.

Angular velocity measures how fast a body rotates. When a rigid body rotates about a fixed axis, all particles move in circles cantered on that axis.

5.7 GLOSSARY: -

Rigid Body: An ideal body in which the distance between any two particles remains constant, regardless of motion.

Translatory Motion: Motion in which all points of a body move with the same velocity and acceleration.

Rotational Motion: Motion in which a body rotates about an axis, and particles move in circular paths around that axis.

Axis of Rotation: An imaginary line about which a body rotates.

Angular Displacement (θ): The angle through which a body rotates about a fixed axis.

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5.10 TERMINAL QUESTIONS: -

1. Define a rigid body. Explain the different types of motion of a rigid body with suitable examples.

2. Explain translatory motion of a rigid body. Derive the equation of motion using Newton's second law.

3. Discuss the motion of a rigid body about a fixed axis. Derive the expression for rotational kinetic energy.
4. Define moment of inertia. Derive its expression for a system of particles and discuss its physical significance.
5. Why do different points of a rotating rigid body have different linear velocities but same angular velocity?
6. A wheel rotates with angular velocity ω. Find the linear velocity of a point at distance r from the center.

UNIT 6: - D' ALEMBERT'S PRINCIPLE

CONTENTS:

6.1 :-*INTRODUCTION: -*

The study of the motion of rigid bodies is a fundamental part of classical mechanics. Unlike a particle, a rigid body consists of a collection of particles whose relative positions remain fixed throughout the motion. The analysis of such bodies requires the application of Newton's laws together with the principles governing the translational and rotational motion of systems of particles.

This unit begins with the equation of motion of a particle**,** which forms the foundation for understanding the dynamics of more complex systems. It then introduces the concept of a rigid body and explains the role of impressed (external) forces acting on it. To simplify the analysis of the motion of rigid bodies, D'Alembert's Principle is presented, converting a dynamic problem into an equivalent problem of static equilibrium.

The unit further discusses the angular momentum of a system of particles and develops the general equations of motion for a rigid body. The motion of the centre of inertia (centre of mass) is then examined to distinguish between translational and rotational motion. Finally, the scalar equations of motion and the motion of a rigid body relative to its centre of inertia are derived, providing a systematic framework for analysing the dynamics of rigid bodies under the action of external forces.

By the end of this unit, learners will be able to formulate and apply the equations governing the motion of rigid bodies, understand the significance of D'Alembert's principle, analyse the motion of the centre of inertia, and solve problems involving translational and rotation3al dynamics using both vector and scalar approaches.

6.2:-*OBJECTIVES: -*

After studying this unit, students will be able to:

1. Define impressed (external) forces and identify their role in the motion of rigid bodies.
2. Understand the concept of the equation of motion of a particle and apply Newton's laws to describe its motion.
3. Explain the characteristics of a rigid body and distinguish it from a particle.
4. Analyse the motion of the centre of inertia (centre of mass) under the action of external forces.

5. Apply the principles of rigid body dynamics to solve numerical and theoretical problems in mechanics.

6.3:- EQUATION OF MOTION OF A PARTICLE

If (x, y, z) be the co-ordinates of a moving particle of mass m, at any time t and x, y, z, be the components of forces parallel to three axes, then by Newton's second law $(P = mf)$ the equations of motion of the particle are $m\frac{d^2x}{dt^2} = X, m\frac{d^2y}{dt^2} = Y, m\frac{d^2z}{dt^2} = Z.$

6.4:- RIGID BODY

A rigid body is an idealized object made up of material particles whose relative positions remain unchanged during motion. In other words, the distance between any two particles of the body remains constant at all times.

For a rigid body, the following assumptions are made:

1. The force of interaction between any two particles acts along the straight line joining them.
2. The forces of action and reaction between any pair of particles are equal in magnitude and opposite in direction, in accordance with Newton's third law.

In principle, the motion of a rigid body can be analyzed by writing the equations of motion for each individual particle. However, these equations involve several unknown internal forces, such as the mutual interactions between the particles, making the analysis complicated.

To overcome this difficulty, D'Alembert introduced a powerful method that allows the required equations of motion to be obtained without writing separate equations for each particle and without explicitly considering the unknown internal forces. This approach is based on the fundamental principle that the internal forces and reactions within a system of rigid bodies in motion are in equilibrium with one another and therefore do not influence the overall motion of the system.

6.5:- IMPRESSED FORCES

The forces acting on a body from sources outside the body are known as **external** or **impressed forces**. These forces are responsible for producing or influencing the motion of the body. A common example of an impressed force is the **weight of a body**, which is the gravitational force exerted by the Earth. Similarly, when a body is attached to a string, the **tension** exerted by the string on the body is also considered an impressed force because it originates from an external agent.Effective forces. D'Alembert, defined the product of the mass m of a particle and its acceleration f to be the effective force on the particle. Thus, $m\frac{d^2x}{dt^2}, m\frac{d^2y}{dt^2}, m\frac{d^2z}{dt^2}$ are the effective forces on the particle parallel to the axes, and $m\frac{d^2s}{dt^2}, m\frac{v^2}{\rho}$ are the effective forces on the particle along the tangent and normal.

Also $\left(-m\frac{d^2x}{dt^2}\right)$ etc, are called reversed effective forces.

6.6: D' ALEMBERT'S PRINCIPLE

Statement of D' Alembert's Principle

The reversed effective forces at each point of the system and the impressed (external) forces on the system are in equilibrium.

Proof Let a rigid body be in motion. At time t let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{P}$ and $\mathbf{Q}$ the external and internal forces respectively acting on it, then by Newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{P} + \mathbf{Q} \text{ or } \mathbf{P} + \mathbf{Q} - m\frac{d^2\mathbf{r}}{dt^2} = 0$$

i.e. the three forces $\mathbf{P}, \mathbf{Q}$ and $\left(-m\frac{d^2r}{dt^2}\right)$ are in equilibrium. Applying the same reasoning to every particle of the system (right body), the forces

$$\Sigma\mathbf{P}, \Sigma\mathbf{Q} \text{ and } \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right)$$

are in equilibrium, the summation extending to all the particles. But internal forces acting on the system (rigid body) form pairs of equal and opposite forces, therefore their vector sum must vanish, that is,

$$\Sigma\mathbf{Q} = 0.$$

Hence, the forces $\Sigma\mathbf{P}$ and $\Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium.

This is,

$$\Sigma\left(-m\frac{d^2r}{dt^2}\right) + \Sigma\mathbf{P} = 0$$

Hence, the reversed effective forces acting at each point of the system and the impressed (external) forces on the system are in equilibrium.

Thus, D' Alembert's principle is proved.

6.7: - ANGULAR MOMENTUM OF A SYSTEM OF PARTICLE

Let $\mathbf{r}$ be the position vector, of a particle of mass m relative to a point O, then the vector sum

$$\mathbf{H} = \Sigma\mathbf{r} \times m\mathbf{v} = \Sigma m\mathbf{r} \times \mathbf{v}$$

is called angular momentum (or moment of momentum) of the system about O.

Centroid of a system Let $\mathbf{r}$ be the position vector, of any particle m of the system relative to a point O, then the point with position vector

$$\bar{\mathbf{r}} = \frac{\Sigma m\mathbf{r}}{\Sigma m}$$

is defined as the centroid of the system.

Also if $\mathbf{v}$ be the velocity of the particle m and $\mathbf{V}$ the velocity vector of the centroid, then

$$\mathbf{V}\frac{d\mathbf{r}}{dt} = \frac{\Sigma m\frac{d\mathbf{r}}{dt}}{\Sigma m} = \frac{\Sigma m\mathbf{v}}{\Sigma m}$$

6.8: - GENERAL EQUATIONS OF A MOTION

From D' Alembert's principle to deduce the general equations of motion of a rigid body.

General Vector Equations of a rigid body : A rigid body moves under the action of given forces. At time t let $\mathbf{r}$ be the position vector of a particle mass m and $\mathbf{P}$ the external force acting on it.

Then by D′ Alembert's principle

$$\Sigma\left[-m\left(\frac{d^2\mathbf{r}}{dt^2}\right)\right] + \Sigma\mathbf{P} = 0 \ \text{ or } \ \Sigma m\left(\frac{d^2\mathbf{r}}{dt^2}\right) = \Sigma\mathbf{P}$$

Multiplying vectorially by $\mathbf{r}$, we get

Thus, we have

$$\Sigma \mathbf{r} \times m \frac{d^2\mathbf{r}}{dt^2} \qquad = \Sigma \mathbf{r} \times \mathbf{P}$$

$$\Sigma m \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{P} \qquad (1)$$

$$\Sigma m \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{r} \times \mathbf{P} (2)$$

and

These two are is general vector equations of motion of a rigid body. General scalar equations of a rigid body: We deduce below the six scalar equations of motion of a rigid body from the above two vector equations of motion.

Let the components of **r** be $[x, y, z]$ and the components of **P** be $[X, Y, Z]$.

Then and so that

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\mathbf{P} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$$
$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

Substituting for $\mathbf{P}, \mathbf{r}$ and $\frac{d^2\mathbf{r}}{dt^2}$ in equations (1) and (2), we have respectively

$$\Sigma m \left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \right) = \Sigma (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$$

and $\Sigma m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \right) = \Sigma (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$

Equating coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we have

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma X \qquad (1)$$

$$(\dots . .) \Sigma m \frac{d^2y}{dt^2} = \Sigma Y, \qquad (1)$$

$$\Sigma m \frac{d^2 z}{dt^2} = \Sigma m, \tag{3}$$

$$\Sigma m \left(y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right) = \Sigma(yZ - zY), \tag{4}$$

$$\Sigma m \left(z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right) = \Sigma(zX - xZ) \tag{5}$$

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = \Sigma(xY - yX), \tag{6}$$

Equations (1) to (6) are in general the scalar equations of motion of any rigid body. Equations (1), (2), (3) show that the sums of the components of the effective forces parallel to the axes of co-ordinates are respectively equal to the sums of the components of the external (impressed) forces parallel to the same axes.

Equations (4), (5) and (6) show that the sums of the moments of the effective forces about co-ordinate axes are respectively equal to the sums of the moments of the impressed forces about the same axes.

6.9: -MOTION OF THE CENTRE OF INERTIA

To show that the centre of inertia of a body moves if all the me mere collected at it, and as if all the external forct directions parallel

Proof: Let $\bar{\mathbf{r}}$ be the position vector of the centre of inertia and $\mathbf{r}$ the position vector of a particle of mass m of a rigid body of mass M.

Then by definition of centroid

$$\mathbf{r} = \frac{\Sigma m \mathbf{r}}{\Sigma m'}$$

$$\therefore \frac{d^2 \bar{\mathbf{r}}}{dt^2}$$

$$\therefore \frac{\Sigma m \frac{d^2 \mathbf{r}}{dt^2}}{\Sigma m} = \frac{\Sigma m \frac{d^2 \mathbf{r}}{dt^2}}{M}$$

Vector equation for the motion of a rigid body is

$$\Sigma m \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{P}$$

Substituting in this for $\frac{d^2\mathbf{r}}{dt^2}$ from above we have

$$M \frac{d^2\overline{\mathbf{r}}}{dt^2} = \Sigma \mathbf{P}. \qquad (1)$$

This is vector form of the equation of motion of a particle of mas M placed at the centre of inertia of the body and acted upon by the external forces $\Sigma\mathbf{P}$.

6.10: - SCALAR EQUATION

To obtain the corresponding scalar form.
Let the components of $\overline{\mathbf{r}}$ be $[\bar{x}, \bar{y}, \bar{z}]$, then

$$\overline{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$$

so that

$$\frac{d^2\overline{\mathbf{r}}}{dt^2} = \frac{d^2\bar{x}}{dt^2}\mathbf{i} + \frac{d^2\bar{y}}{dt^2}\mathbf{j} + \frac{d^2\bar{z}}{dt^2}\mathbf{k}$$

and the equation (1) reduces to

$$M\left(\frac{d^2\bar{x}}{dt^2}\mathbf{i} + \frac{d^2\bar{y}}{dt^2}\mathbf{j} + \frac{d^2\bar{z}}{dt^2}\mathbf{k}\right) = \Sigma(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$$

Equating coefficient of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we have

$$m\frac{d^2\bar{x}}{dt^2} = \Sigma X \qquad (i)$$

$$m\frac{d^2\bar{y}}{dt^2} = \Sigma Y \qquad (ii)$$

$$m\frac{d^2\bar{z}}{dt^2} = \Sigma Z \qquad (iii)$$

But equations (i), (ii) and (iii) are the equations of motion of a particle, of mass M, placed at the centre of inertia of the body, and acted on by forces $\Sigma X, \Sigma Y, \Sigma Z$ parallel and equal to the external forces acting on the different points of the body.

This proves the proposition.

6.11: - MOTION RELATIVE TO THE CENTER OF INERTIA

To show that the motion of a body about its centre of inertia is the same as it would be if the center of inertia were fixed and the same forces acted on the body.

At time t let $\overline{\mathbf{r}}$ be the position vector of the center of inertia of the rigid body of mass M. Let m be the mass of a particle whose position vector referred to a fixed origin is r and whose position vector with regard to the centre of

inertia is $\mathbf{r}'$: then, so that

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\overline{\mathbf{r}}}{dt^2} + \frac{d^2\mathbf{r}'}{dt^2}$$

Moment vector equation of motion for a rigid body is

or

$$\Sigma m(\overline{\mathbf{r}} + \mathbf{r}') \times \left(\frac{d^2\overline{\mathbf{r}}}{dt^2} + \frac{d^2\mathbf{r}'}{dt^2}\right) = \Sigma\left(\overline{\mathbf{r}}' + \mathbf{r}'\right) \times \mathbf{P}$$

Or But $\quad \Sigma m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \Sigma\mathbf{r} \times \mathbf{P}$

Or $\quad \mathbf{r} \times m\frac{d^2\overline{\mathbf{r}}}{dt^2} + \overline{\mathbf{r}} \times \Sigma m\frac{d^2\mathbf{r}'}{dt^2} + \frac{d^2\overline{\mathbf{r}}}{dt^2} \times \Sigma(m\mathbf{r}') + \Sigma m\mathbf{r}' \times \frac{d^2\mathbf{r}'}{dt^2}$

$$\frac{\Sigma m\mathbf{r}'}{\Sigma m} = 0$$

being the position of the center of the inertia G referred G as origin.

Thus,

$$\Sigma m\mathbf{r}' = 0,$$

so that

$$\Sigma m \frac{d^2\mathbf{r}'}{dt^2} = 0$$

Also, we know that

$$m \frac{d^2\bar{\mathbf{r}}}{dt^2} = \Sigma\mathbf{P}$$

Hence, the last equation reduces to

$$\Sigma m\mathbf{r}' \times \frac{d^2\mathbf{r}'}{dt^2} = \Sigma\mathbf{r}' \times \mathbf{P}$$

This is the vector equation of motion of a rigid body of we have regarded centre of gravity as a fixed point.

6.12: - SCALAR FORM

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of motion of a rigid body if we have (x', y', z') coordinates of the particle m with regard to G, then

$$\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$$

or

$$\frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2x'}{dt^2}\mathbf{i} + \frac{d^2y'}{dt^2}\mathbf{j} + \frac{d^2z'}{dt^2}k$$

Hence, equation (1) becomes

$$\Sigma m(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})\times\left(\frac{d^2x'}{dt^2}\mathbf{i}+\frac{d^2y'}{dt^2}\mathbf{j}+\frac{d^2z'}{dt^2}\mathbf{k}\right)$$
$$=\Sigma(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})\times(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})$$
$$\text{or }\ \Sigma\left[m\left(y'\frac{d^2z'}{dt^2}-z'\frac{d^2y'}{dt^2}\right)\mathbf{j}+m\left(z'\frac{d^2x'}{dt^2}-x'\frac{d^2z'}{dt^2}\right)\mathbf{j}\right.$$
$$\left.+m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)\mathbf{k}\right]$$
$$=\Sigma[(y'Z-z'Y)\mathbf{i}+(z'X-x'Z)\mathbf{j}+(x'Y-y'X)\mathbf{k}]$$

Equating coefficients if **i**, we have

$$\Sigma m\left(yt'\frac{d^2z'}{dt^2}-\mathbf{z}'\frac{d^2y'}{dt^2}\right)=\Sigma(y'Z-z'Y)$$

and two more similar equations can be had by equating coefficients of j and **k**. This is the equation which would have been obtained if we had regarded centre of gravity as a fixed point. Hence, the proposition.

Note. In the above two articles, we have shown that motion of translation and the motion of rotation can be considered independently. Hence, these two articles prove the independence of translation and rotation. Thus, in the case of the motion of a rigid body above two articles establish the principle of the independence of translation and rotation.

EXAMPLE: A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle to the horizon, and a man of mass M', starting form the upper end walks down the plank so that it does not move: show that he gets to the other end in time,

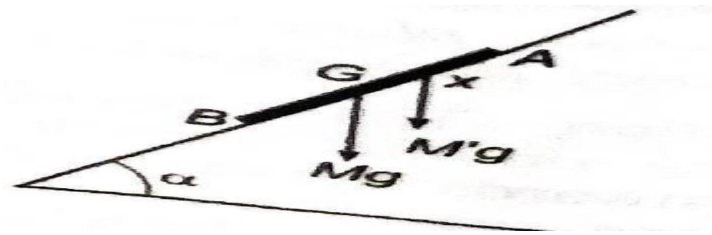
$$\sqrt{\left[\frac{\mathbf{2}\boldsymbol{M'a}}{(\boldsymbol{m}+\boldsymbol{M'})\boldsymbol{g}\sin\boldsymbol{\alpha}}\right]}$$

where a is the length of the plank.

Solution: AB is the plank of mass M resting along the line of greatest slope of the inclined plane. The weight of the plank acts through G such that

$$AG=\frac{1}{2}a.$$

The man starts from A; let him move down through a distance x in time t. If $\bar{x}$ be the distance of C.G. of the plank and the man in this position from A, then



$$\bar{x} = \frac{M \cdot \frac{1}{2} a + M' x}{M + M'}.$$

Differentiating this relation twice w.r.t., ' t ' we get

$$\frac{d^2\bar{x}}{dt^2} = \frac{M'}{M + M'} \cdot \frac{d^2 x}{dt^2} \qquad (i)$$

Now for the motion of translation of the body, we should consider simply the motion of its C.G. supposing as if all the external forces act on it.

Thus, equation of motion of the C.G. along the plane gives

$$(M + M')\frac{d^2\bar{x}}{dt^2} = Mg\sin\alpha + M'g\sin\alpha$$

i.e.,

$$M'\frac{d^2x}{dt^2} = (M + M')g\sin\alpha$$
$$\frac{d^2x}{dt^2} = \frac{M + M'}{M'} g\sin\alpha$$
$$\frac{dx}{dt} = \frac{M + M'}{M'} g\sin\alpha . t$$

from Eq. (1) or

the constant of integration vanishes because when $t = 0, \frac{dx}{dt} = 0$.

Integrating agin,

$$x = \frac{M + M'}{M'} g\sin\alpha \cdot \frac{t^2}{2},$$

the constant again vanishes because when $t = 0, x = 0$.
Thus,

$$t = \sqrt{\left\{\frac{2xM'}{(M + M')g\sin\alpha}\right\}}.$$

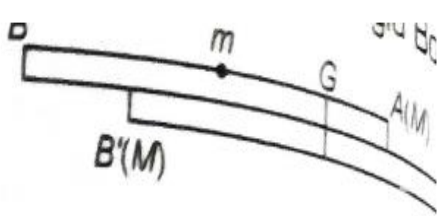
Putting $x = a$, the time of getting to the other end of the plank

$$= \sqrt{\left\{\frac{2aM'}{(M + M')g\sin\alpha}\right\}}$$

EXAMPLE 1 : A rough uniform board, of mass m and length $2a$, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time. Solution The board and the man constitute a material system. The external forces acting on that system are the weights of the board and man acting vertically downwards, the reaction of the horizontal plane acting vertically upwards (the mutual actions, reactions and fractional forces between the man and the board being internal forces are not to be taken into account).

We thus see that there are no external forces on the system in the horizontal direction, therefore, by D'Alembert's principle, the C.G. of the system would remain at rest.

As a matter of fact, as the man moves forward, the board slips backwards, thus keeping the position of the C.G. unchanged. AB is the position of board when the man is at A.



In this position distance of the system from

$$A = \frac{M \times 0 + m \times a}{M + m} = \frac{ma}{M + m}.$$

Let $A'B'$ be the position of the board when the man reaches, she other end of the board. If the board slips through a distance x in this time (i.e., $AA' = x$), then in this position distance of C.G. of the system from

$$A = \frac{M(2a - x) + m(a - x)}{M + m}$$

But the distance of C.G. is the same from A or or

$$\therefore \frac{ma}{M + m} = \frac{M(2a - x) + m(a - x)}{M + m}$$

$ma = M(2a - x) + m(a - x)$
or

$$(M + m)x = 2Ma$$
$$x = \frac{2Ma}{M + m}$$

This gives the required distance moved by the board.

6.13: - SUMMARY

This unit presented the fundamental principles governing the dynamics of particles and rigid bodies. It began with the equation of motion of a particle**,** which is based on Newton's second law and forms the foundation for the study of dynamics. The concept of **a** rigid body was then introduced as a system of particles whose mutual distances remain constant during motion.

The unit explained impressed (external) forces, which are responsible for producing or changing the motion of a body, while emphasizing that the internal forces between the particles of a rigid body do not affect its overall motion. To simplify the analysis of rigid body dynamics, D'Alembert's

Principle was discussed. This principle transforms a problem of dynamics into an equivalent problem of static equilibrium by introducing inertia forces.

The concept of angular momentum for a system of particles was developed, and its relationship with the external moments acting on the system was established. Using these ideas, the for a rigid body were derived, providing a complete description of its general equations of motion translational and rotational motion.

The motion of the centre of inertia (centre of mass) was studied, showing that it depends only on the resultant external force acting on the body. The scalar equations of motion were then obtained from the corresponding vector equations, making them suitable for practical problem-solving. Finally, the motion of a rigid body relative to its centre of inertia was examined to separate the translational motion of the body from its rotational motion.

Overall, this unit established the theoretical framework required to analyse the dynamics of rigid bodies, combining Newton's laws, D'Alembert's principle, angular momentum, and the motion of the centre of inertia to solve a wide range of mechanical problems.

6.14: - GLOSSARY

Angular Momentum: The rotational equivalent of linear momentum, representing the tendency of a particle or rigid body to continue rotating about a point or axis.

Centre of Inertia (Centre of Mass): The point at which the entire mass of a body may be considered to be concentrated for the purpose of analyzing its motion.

D'Alembert's Principle: A principle that converts a problem of dynamics into an equivalent problem of static equilibrium by introducing inertia forces.

Dynamics: The branch of mechanics that studies the motion of bodies and the forces producing that motion.

Equation of Motion: A mathematical equation that relates the forces acting on a body to its motion, usually based on Newton's second law.

External (Impressed) Forces: Forces acting on a body from outside the system, such as gravity, tension, friction, or applied loads.

Force: A push or pull that changes or tends to change the state of rest or motion of a body.

Inertia: The property of a body by which it resists any change in its state of rest or uniform motion.

Inertia Force: A fictitious force equal to the product of mass and acceleration, acting opposite to the direction of acceleration in D'Alembert's principle.

Internal Forces: Forces of interaction between the particles of the same rigid body, which do not affect the overall motion of the body.

Moment (Torque): The turning effect of a force about a point or an axis, equal to the product of the force and the perpendicular distance from the point or axis to the line of action of the force.

Newton's Second Law: The law stating that the rate of change of momentum of a body is equal to the applied external force and acts in the direction of that force.

Particle: An idealized body having mass but negligible dimensions, whose motion can be described by the movement of a single point.

Rigid Body: An idealized body whose shape and size remain unchanged during motion because the distances between its constituent particles remain constant.

Rotational Motion: Motion in which every particle of a body moves in a circular path about a fixed axis.

Scalar Equation of Motion: An equation expressing the motion of a body in terms of scalar components along specified coordinate axes.

System of Particles: A collection of two or more particles considered together for the analysis of their collective motion.

Translation: The motion of a body in which all its particles move the same distance in the same direction at the same time.

Vector Equation of Motion: An equation representing the motion of a body in vector form, accounting for both magnitude and direction of forces and acceleration.

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6.16: - TERMINAL QUESTIONS: -

1. Derive the general equations of motion of a rigid body using Newton's laws.
2. State and prove D'Alembert's principle. Discuss its applications in mechanics.
3. Derive the expression for the angular momentum of a system of particles and establish its relation with the external moment of forces.
4. Obtain the equations governing the motion of the centre of inertia and explain their physical significance.
5. Explain the assumptions made in the study of a rigid body.
6. Describe the concept of impressed forces with suitable examples.
7. State and explain D'Alembert's principle.
8. Define angular momentum of a system of particles and discuss its significance.

UNIT 7- VERTUAL WORK

CONTENTS:

7.1 INTRODUCTION

In the study of Mathematics, the concept of *virtual work* provides a powerful and elegant method for analysing the equilibrium of mechanical systems. Instead of resolving all forces into components and applying multiple equilibrium equations, the method of virtual work allows us to determine equilibrium by considering the work done by forces during an imagined (virtual) displacement of the system.

This unit introduces the definition of virtual work, explains the principle of virtual work, and identifies the forces that can be omitted while forming the equation of virtual work. These ideas greatly simplify the analysis of systems involving constraints, reactions, and complex force distributions.

7.2 OBJECTIVES

After studying this unit learner will be able

1. To Understand the concept of virtual displacement and distinguish it from actual displacement.

2. To Define virtual work and explain its physical significance in mechanical systems.

3. To State and apply the principle of virtual work to problems involving equilibrium.

4. To Identify the conditions of equilibrium using the method of virtual work.

7.3 DEFINITION OF VIRTUAL WORK

When several forces act on a body and cause it to move, they perform actual work. But if the forces are in equilibrium, the body does not move, and therefore no real work is done. However, if we *assume* that the forces in equilibrium undergo a very small imaginary displacement and then calculate the work done during this imagined movement, the displacement is called a virtual displacement, and the work done during it is known as virtual work**.**

7.4 THE PRINCIPLE OF VIRTUAL WORK

If a system of forces acting on a body is in equilibrium, then the total virtual work done by all the forces during any small imaginary (virtual) displacement of the body is zero.

In other words, when a body is in equilibrium:

Total Virtual Work=0

A virtual displacement is not a real motion; it is only an imagined, infinitesimally small and kinematically possible displacement that is consistent with the constraints of the body.

The principle helps in determining unknown forces or reactions without considering the actual motion of the body.

Theorem : The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation. Proof. Let a system of forces $\mathbf{F}_1, \ldots \ldots, \mathbf{F}_n$ act at the points of a rigid body whose

Proof. Let a system of forces $\mathbf{F}_1, \ldots \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \ldots \ldots, \mathbf{r}_n$. Suppose this system of forces is equivalent to a single force $\mathbf{R} = \Sigma \mathbf{F}_1$ acting at O, together with a couple of moment $\mathbf{G} = \Sigma \mathbf{r}_1 \times \mathbf{F}_1$. Then during any small displacement of the body consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O,the sum of the works done by these forces

$$= \Sigma \mathbf{F}_1 \cdot d\mathbf{r}_1 = \Sigma \mathbf{F}_1 \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r}_1)$$
$$= \mathbf{u} \cdot \Sigma \mathbf{F}_1 + \mathbf{e} \cdot \Sigma \mathbf{r}_1 \times \mathbf{F}_1$$
$$= \mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G}. (1)$$

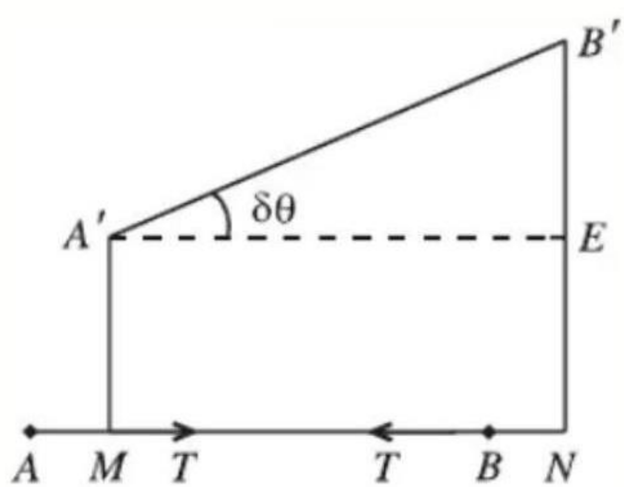
The condition is necessary. Suppose the given system of forces is in equilibrium. Then $\mathbf{R} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$. Therefore, from (1), the sum of the works done by the forces is zero. Hence the condition is necessary. The condition is sufficient. Suppose the sum of the works done by the forces during any small displacement is zero. Then to prove that the forces are in equilibrium. We have, from (1)

$$\mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G} = 0 \tag{2}$$

7.5 FORCE WHICH CAN BE OMITTED THE EQUATION OF VIRTUAL WORK

The principle of virtual work provides a highly effective technique for solving problems related to the equilibrium of forces. Its major advantage over other methods is that certain forces do not need to be included when writing the virtual work equation, which simplifies the calculations considerably. Below, we discuss and prove the types of forces that can be omitted when formulating the equation of virtual work (i) The work done by the tension of an inextensible string is zero during a small displacement.

Let AB be an inextensible string of length l joining two points A and B of a rigid body. Let T be the tension in the string AB. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is inextensible, therefore $A'B' = AB = l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.



On account of the tension in the string AB, there are two forces each equal to T acting on A and B in opposite directions as shown in the figure. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during this displacement

$= T \cdot AM - T \cdot BN$ [Note that the dis
$= T \cdot (AB - MB) - T \cdot (MN - MB)$
$= T \cdot (AB - MN)$
$= T \cdot (AB - A'E) = T \cdot (AB - A'B' \cos \delta\theta)$

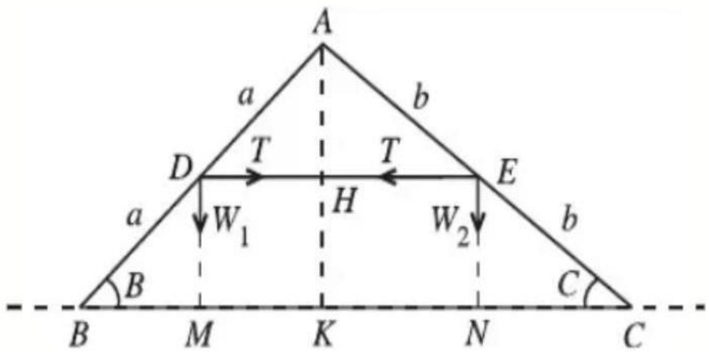
$= T \cdot l(1 - \cos \delta\theta)$
$= T \cdot l \left[1 - \left\{1 - \frac{(\delta\theta)^2}{2!} + \cdots\right\}\right]$, expanding $\cos \delta\theta$ in powers of $\delta\theta$
$= T \cdot l \cdot 0$, to the first order of small quantities
$= 0.$

Example 5: Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is

$$\frac{W}{\tan B + \tan C}$$

where W is the total weight of the rods AB and AC.

Solution: AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC



respectively. Let

$$AB = 2a \text{ and } AC = 2b$$

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore, the total weight $W = W_1 + W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B + \delta B$ and C changes to $C + \delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE = DH + HE = a\cos B + b\cos C$$

the height of any point of the line DE above BC

$$= DM = a\sin B$$

The equation of virtual work is

$$-T\delta(a\cos B + b\cos C) - W\delta(a\sin B) = 0$$

or $aT\sin B\delta B + bT\sin C\delta C - aW\cos B\delta B = 0$
or $a(W\cos B - T\sin B)\delta B = bT\sin C\delta C$.

From the figure,

$$DM = a\sin B \text{ and } EN = b\sin C$$

Since $DM = EN$, therefore $a\sin B = b\sin C$.

$$\begin{aligned} \therefore \quad & \delta(a\sin B) = \delta(b\sin C) \\ \text{or} \quad & a\cos B\delta B = b\cos C\delta C. \end{aligned} \tag{2}$$

Dividing (1) by (2), we have

$$\frac{W\cos B - T\sin B}{\cos B} = \frac{T\sin C}{\cos C}$$

or $W - T\tan B = T\tan C$
or $T(\tan B + \tan C) = W$
or $T = \frac{W}{\tan B + \tan C}$.

CHECK YOUR PROGRESS

MULTIPLE CHOICE QUESTIONS

1. Virtual work is defined as the work done by:

A. Actual displacement of the body
B. Imaginary small displacement of the body
C. Gravity only
D. Frictional forces

2. The principle of virtual work states that a body is in equilibrium if:

A. The sum of forces is zero
B. The sum of moments is zero

C. Total virtual work done by all forces is zero
D. The body moves with uniform velocity

3. Which forces can be omitted when applying the virtual work method?

A. Active forces
B. Internal forces and reactions along constraints
C. Gravitational forces
D. All applied forces

4. Virtual displacement is:

A. Always along the direction of motion
B. Infinitesimally small and consistent with constraints
C. A large actual movement
D. The displacement of centre of gravity only

5. In a solid of revolution, the centre of gravity lies:

A. Always at the edge
B. On the axis of symmetry
C. Outside the solid
D. At the centroid of generating curve only

7.6 SUMMARY

1. Definition of Virtual work: Virtual work is the work done by a system of forces during an imagined, infinitesimally small, and kinematically possible displacement of the body, called virtual displacement. This displacement is not real but is assumed only for the purpose of analyzing equilibrium.

2. Principle of Virtual Work: The principle of virtual work states that if a body is in equilibrium under a system of forces, then the total virtual work done by all the forces during any virtual displacement is zero.

Total Virtual Work=0

This principle provides a powerful method to test equilibrium and determine unknown forces.

3. **Forces Which Can Be Omitted in Forming the Equation of Virtual Work**

Certain forces do not contribute to virtual work and can be excluded from the virtual work equation. These include:

1. **Internal forces** in a rigid body, because their virtual work cancels out in pairs.
2. **Reaction forces perpendicular to the virtual displacement**, such as forces at smooth surfaces or hinges.
3. **Constraint forces** that do not cause motion in the direction of the virtual displacement.

Omitting these forces simplifies the analysis and makes the method of virtual work highly efficient for solving equilibrium problems.

7.7 GLOSSARY

Work : The product of a force and the displacement in the direction of that force.
Work = Force × Displacement.

Virtual displacement: A small, imagined, reversible displacement of a system that is consistent with the constraints.

Virtual work: The work done by a force during a virtual displacement.
It is not real work but an imagined work used for analysis.

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7.9 SUGGESTED READING

1. **Engineering Mechanics: Statics** – J. L. Meriam & L. G. Kraige *Excellent for understanding theoretical principles and solving analytical problems.*

2. **Engineering Mechanics** – R. K. Bansal *Useful for Indian students; provides numerous solved examples and practice problems.*

3. **Engineering Mechanics: Statics and Dynamics** – R. C. Hibbeler *Provides strong conceptual explanation with real-life engineering applications.*

4. **A Textbook of Engineering Mechanics** – S. S. Bhavikatti *Covers equilibrium, friction, centroid, and structural applications in a simple, student-friendly style.6*

5. **Engineering Mechanics**– A. K. Tayal *Straightforward explanations suitable for undergraduate mathematics and engineering courses.*

7.10 TERMINAL QUESTIONS

1. Explain the concept of virtual work and derive the condition for equilibrium.

2. Derive the formula for the centre of gravity of a semi-circular lamina**.**

3. Discuss the centre of gravity of a solid of revolution using integration.

4. State and prove the principle of virtual work for a rigid body.

5. Explain the forces that can be omitted in the virtual work method, with proof.

5.11 ANSWERS

MCQ1. B MCQ 2. C MCQ 3. B MCQ 4. B MCQ 5 . B

UNIT 8: - Hamilton's Principal

CONTENTS:

8.1 *INTRODUCTION: -*

Hamilton's Principle, also known as the **Principle of Stationary Action**, is one of the most fundamental principles of classical mechanics. It was introduced by the Irish mathematician and physicist **Sir William Rowan Hamilton** and provides a variational approach to describing the motion of mechanical systems. According to this principle, the actual path followed by a system between two fixed configurations during a given time interval is the one for which the **action integral** is stationary (usually a minimum) compared with all other possible paths. The action is defined as the time integral of the Lagrangian, where the Lagrangian is the difference between the kinetic and potential energies of the system. Hamilton's Principle forms the theoretical foundation of Lagrangian and Hamiltonian mechanics and provides a unified method for deriving the equations of motion of particles and rigid bodies. Owing to its elegance and generality, it plays a central role in classical mechanics and serves as the basis for advanced fields such as quantum mechanics, field theory, relativity, and analytical dynamics.

8.2 *OBJECTIVES: -*

- Understand the concept and significance of Hamilton's Principle in classical mechanics.
- Define the action integral and explain its physical meaning.
- Describe the relationship between the Lagrangian and the action integral.
- State Hamilton's Principle and explain the principle of stationary action.
- Derive the equations of motion using Hamilton's Principle.
- Explain the connection between Hamilton's Principle and the Euler–Lagrange equations.
- Apply Hamilton's Principle to simple mechanical systems.
- Distinguish Hamilton's Principle from Newtonian and Lagrangian formulations of mechanics.
- Recognize the role of Hamilton's Principle in analytical mechanics and variational methods.

8.3 *ACTUAL PATH AND VARIED PATH: -*

Assume a particle of mass m moves between two fixed points A and B from time t_0 at A to time t_1 at B, following the curve APB. Such a curve along which the particle actually travel is called actual path of the particle.

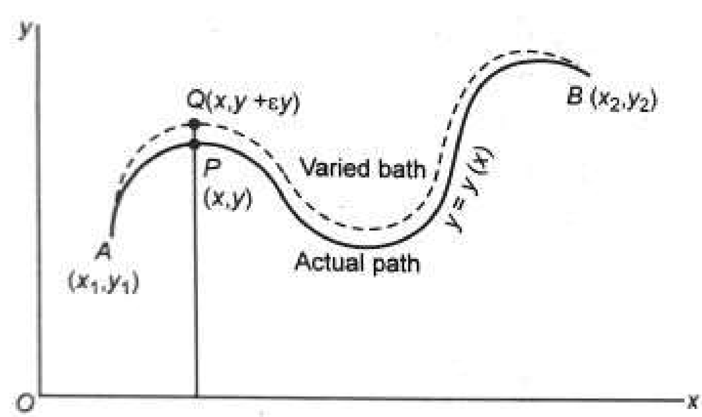

Fig.1

Actual and Varied Paths in Hamilton's Principle. The solid curve represents the actual path of a particle from the initial point $A(x_1, y_1)$ to the final point $A(x_2, y_2)$. The dashed curve represents a nearby varied path obtained by introducing a small variation $\delta y = \epsilon\eta$. At any point $P(x, y)$ on the actual path, the corresponding point on the varied path is $Q(x, y + \epsilon\eta)$. The variation is zero at the end points, ensuring that both paths have the same initial and final positions. This concept forms the basis of Hamilton's Principle, according to which the actual path makes the action integral stationary.
Similarly, a rigid body moving from configuration A at time t_0 to configuration B at time t_1 has two paths: real and variable.In fact, each particle in the body has both an actual journey and a variable path.

8.4 *VARIATION METHOD (CALCULUS OF VARIATION): -*

Let the function

$$f((y, y', x)$$

where y is dependent variable, x an independent variable

$$y' = \frac{dy}{dx}$$

Now $y = y(x)$ in order that the definite integral

$$I = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

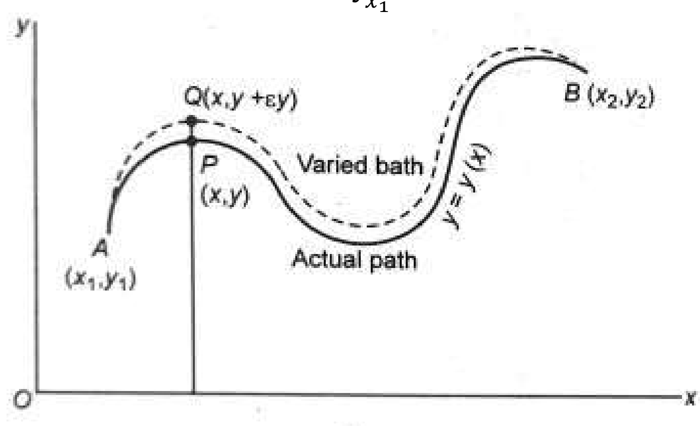

Fig.2

From the figure
Let

$$I^{\oplus} = \int_{x_1}^{x_2} [f(y, y', x) + \left(\epsilon y \frac{\partial}{\partial y} + \epsilon y' \frac{\partial}{\partial y'}\right) f$$
$$+ \frac{1}{2!}\left(\epsilon y \frac{\partial}{\partial y} + \epsilon y'^{\frac{\partial}{\partial y'}}\right)^2 f]dx$$

Take first Variation

$$= I + \int_{x_1}^{x_2} \left(\epsilon y \frac{\partial}{\partial y} + \epsilon y' \frac{\partial}{\partial y'}\right) f\, dx \ldots..$$

$$I^{\oplus} - I = \int_{x_1}^{x_2} \left(\epsilon y \frac{\partial}{\partial y} + \epsilon y' \frac{\partial}{\partial y'}\right) f\, dx$$

Since I is stationary, so that

$$\epsilon \int_{x_1}^{x_2} \left(y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'}\right) f\, dx = 0$$
$$\int_{x_1}^{x_2} \left(y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'}\right) dx = 0$$

$$\int_{x_1}^{x_2} y \frac{\partial f}{\partial y} dx + \int_{x_1}^{x_2} y' \frac{\partial f}{\partial y'} dx = 0$$

Now the second Integral, we obtain

$$\int_{x_1}^{x_2} y \frac{\partial f}{\partial y} dx + \left[\left(y \frac{\partial f}{\partial y'}\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx\right] = 0$$

Hence,

$$\int_{x_1}^{x_2} y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx = 0$$
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

Since $y \neq 0$ expect at x_1 and x_2.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

is known as Euler's Lagrange's Equation.
If y is absent from f then

$$\frac{\partial f}{\partial y'} = constant.$$

8.5 *ANOTHER FORM OF EULER-LAGRANGE EQUATION: -*

The Euler's Equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$y'\frac{\partial f}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$y'\frac{\partial f}{\partial y} - \left[\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - \frac{dy'}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] = 0$$

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}\frac{dy'}{dx}\right) - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

$$\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$$

This is the another form of Euler's Lagrange's Equations.
If y is absent from f then

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = constant.$$

8.6 *GENERAL CASE: -*

By considering the function

$$f(y_1, y_2, \dots \dots . y_n\,; y'_2 \dots y'_n : x)$$

where $y_1, y_2, \dots \dots . y_n$ are n dependent variables; x an independent variable and

$$y_r' = \frac{dy_r}{dx} \qquad (r = 1,2 \dots . n)$$

Now we are find the relation between y_r and $x i.e.,$ relation $y_r = r_r$ such that the definite integral

$$I = \int_{t_0}^{t_1} f(y_r; y_r'; x)dx$$

is stationary
Suppose

$$y_r = y_r(x)$$

Which is called Actual path.
Let $(y_r + \epsilon y_r; x)$ be a position on the varied path corresponding to a point $(y_r; x)$ on the actual path, where only y varies and x does not vary. The integral that corresponds to I on the varied path is

$$I^{\oplus} = \int_{t_0}^{t_1} f(y_r + \epsilon y_r; y'_r + \epsilon y'_r; x)\, dx$$

$$= \int_{t_0}^{t_1} [f(y_r + \epsilon y_r) + \sum_{r=1}^{n} \left(\epsilon_r y_r \frac{\partial}{\partial y_r} + \epsilon_r y'_r \frac{\partial}{\partial y'_r}\right) f$$

$$+ \sum_{r=1}^{n} \left(\epsilon_r y_r \frac{\partial}{\partial y_r} + \epsilon_r y'_r \frac{\partial}{\partial y'_r}\right)^2 f$$

$$I^{\oplus} - I = \sum_{r=1}^{n} \epsilon_r \int_{t_0}^{t_1} \left(y_r \frac{\partial f}{\partial y_r} + y'_r \frac{\partial f}{\partial y'_r}\right)$$

If I is stationary, the first variation must be zero

$$\int_{t_0}^{t_1} \left(y_r \frac{\partial f}{\partial y_r} + y'_r \frac{\partial f}{\partial y'_r}\right) = 0$$

$$\int_{t_0}^{t_1} \left(y_r \frac{\partial f}{\partial y_r}\right) dx + \int_{t_0}^{t_1} \left(y'_r \frac{\partial f}{\partial y'_r}\right) dx = 0$$

$$\int_{t_0}^{t_1} y_r \frac{\partial f}{\partial y_r} dx + \left[\left\{y_r \frac{\partial f}{\partial y'_r}\right\}_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(y'_r \frac{\partial f}{\partial y'_r}\right) dx\right] = 0$$

Then the middle term vanishes

$$\int_{t_0}^{t_1} y_r \left[\frac{\partial f}{\partial y_r} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_r}\right)\right] = 0$$

$$y_r \left[\frac{\partial f}{\partial y_r} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_r}\right)\right] = 0$$

But $y_r \neq 0$ except at t_0 and t_1.

$$\frac{\partial f}{\partial y_r} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_r}\right) = 0 \qquad (r = 1,2, \ldots., n)$$

These are n different equations whose solutions.

$$y_1 = y_1(x), y_2 = y_2(x), \ldots.. y_n = y_n(x)$$

8.7 *HAMILTION'S PRINCIPLE (PROOF BASED ON CALCULUS OF VARIATION): -*

Hamilton's principle states that in case of conservative, holonomic dynamical system specified by n generalized coordinates

$$\int_{t_0}^{t_1} L dt = 0$$

has necessarily and extreme value minimum or maximum.

Solution: We obtain

$$L = T - V$$

where T and V are the kinetic and potential energies.

Since, T is a function of qs' and $\dot{qs}'$ and V is a function of qs', therefore L is the function of qs' and $\dot{qs}'$.

Now let $L(q_1, q_2, , \dots \dots q_n; \dot{q_1}, \dot{q_2} \dots \dots \dot{q_n}; t)$, where $q_1, q_2, , \dots \dots q_n$ are dependent variables, t an independent variable, and $\dot{q_r} = \frac{dq_r}{dt} (r = 1,2, \dots . n)$

Hence

$$\int_{t_0}^{t_1} L dt$$

Will have an extreme value minimum or maximum provided that.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_r}}\right) - \frac{\partial L}{\partial q_r} = 0$$

Hence

$$\int_{t_0}^{t_1} L dt$$

has necessarily an extreme value when the dynamical system is conservative and holonomic.

The integral $\int_{t_0}^{t_1} L dt$ expressed as Hamilton's Principle function, and is usually denoted by S.

The above principal is defined as

$$\int_{t_0}^{t_1} \delta L dt \quad or \quad \delta \int_{t_0}^{t_1} L dt = 0 \quad or \quad \delta S = 0$$

On actual path as compared with the varied path.

8.8 *HAMILTION'S PRINCIPLE (PROOF BASED ON D'ALEMBERT'S PRINCIPLE): -*

According to Alembert's Principle, the external and reverse effective forces acting on the body are in equilibrium; hence the sum of their virtual work is zero, which is

$$\sum (F - m\ddot{r}) . \delta r = 0$$
$$\sum F . \delta r = \sum m\ddot{r}\, \delta r$$

Now,$\sum F . \delta r$ is the total virtual work done by external forces F due to the imagined displacement; call this work Delta δw as we do in derivation of leg ranges equation.

$\sum F.\delta r$ is the entire virtual work done by external forces F due to the imagined displacement; designate this work Delta δw as we do in deriving the Lagrange's equations.
Also

$$\sum m\ddot{r}\,\delta r = \sum\left[\frac{d}{dt}(m\dot{r}.\delta r) - m\dot{r}.\delta\dot{r}\right]$$

$$= \sum\frac{d}{dt}(m\dot{r}.\delta r) - \sum\left(\frac{1}{2}m\delta\dot{r}^2\right)$$

Hence, the above equation becomes

$$\sum\frac{d}{dt}(m\dot{r}.\delta r) - \sum\left(\frac{1}{2}m\delta\dot{r}^2\right)\delta r = \delta W$$

$$\sum\delta\left(\frac{1}{2}m\dot{r}^2\right) + \delta W = \sum\frac{d}{dt}(m\dot{r}.\delta r)$$

$$\sum\delta t + \delta W = \sum\frac{d}{dt}(m\dot{r}.\delta r)$$

Integrating w.r.t. 't' between t_0 to t_1, we obtain

$$\int_{t_0}^{t_1}(\delta T + \delta W)dt = \left[\sum(m\dot{r}.\delta r)\right]_B^A$$

Because the varied path intersects at both A and B, the variation $\delta r = 0$ at both sites, resulting in the right-hand side of the equation being zero.
Hence

$$\int_{t_0}^{t_1}(\delta T + \delta W)dt = 0 \ or \ \ \delta\int_{t_0}^{t_1}(T + W)dt = 0$$

This is the general form of the Hamilton's principle for any type of external force.
Particularly,

$$\int_{t_0}^{t_1}\delta(T - V)dt = 0$$

$$\int_{t_0}^{t_1}\delta L dt = 0$$

$$\delta\int_{t_0}^{t_1}L dt = 0$$

This is called as Hamilton's principle.

This means that

$$\int_{t_0}^{t_1}L dt$$

has a stationary value and an actual path when compared to the varied path.

8.9 *PRINCIPLE OF LEAST ACTION: -*

Principle of least action states that if T is kinetic energy at a time t, of a conservative hollonomic dynamical system specified by n generalised coordinates, then the integral

$$\int_{t_0}^{t_1} 2T dt$$

has necessarily an extreme value, minimum or maximum, on actual path as compared with the valid path as the system passes from one configuration at a time t_0 to another configuration at a time t_1.

Let L be the Lagrangian and T represent the potential energy at time t, then we get

$$L = T - V$$
$$T + V = E$$

By Hamilton's principle we obtain

$$\int_{t_0}^{t_1} \delta L dt = 0$$
$$\int_{t_0}^{t_1} \delta (T - V) dt = 0$$
$$\int_{t_0}^{t_1} \delta (2T - E) dt = 0$$

Since $\delta E = 0$ and E is Constant.

$$\int_{t_0}^{t_1} \delta (2T) dt = 0$$
$$\delta \int_{t_0}^{t_1} (2T) dt = 0$$

This result is known as **Principle of least action.**

The integral $\int_{t_0}^{t_1} 2T$ is defined as action and is commonly denoted by A. Therefore, the principle of least action can be expressed as

$$\delta A = 0.$$

when a result, the principle of wave textion states that the action in the actual path is minimal when compared to the varied path, when the system transitions from one configuration to another.

8.10 *DISTINCTION BETWEEN HAMILTON'S PRINCIPLE AND PRINCIPLE OF LEAST ACTION: -*

Hamilton's Principle	Principle of Least Action
• Hamilton's Principle states that the actual motion of a mechanical system between two fixed times makes the **action integral stationary** (minimum, maximum, or saddle value).	• The Principle of Least Action states that the actual path followed by a system is the one for which the **action integral is minimum** compared with neighboring paths.
• It is a **more general variational principle**.	• It is a **special case** of Hamilton's Principle.
• It allows the action to be minimum, maximum, or stationary.	• It considers only the minimum action.
• It applies to a wide range of conservative mechanical systems and forms the basis of **Lagrangian and Hamiltonian mechanics**.	• It provides a physical interpretation of nature selecting the path of minimum action and is commonly used as an introductory concept.
• It is widely used in advanced classical mechanics, quantum mechanics, and field theory.	• It is mainly used to explain the concept of variational methods and the evolution of mechanical systems.
• It leads directly to the **Euler–Lagrange equations** and Hamilton's equations of motion.	• It also leads to the Euler–Lagrange equations when the action is minimized, but is viewed as a restricted interpretation.

8.11 *LAGRANGE'S EQUATIONS DERIVED FROM HAMILTON'S PRINCIPLE: -*

The Hamilton's Principle is expressed as

$$\int_{t_0}^{t_1} \delta L dt = 0$$

Since, L is function of qs' and $\dot{q}s'$, therefore

$$\delta L = \sum \frac{\partial L}{\partial q_r} \delta q_r + \sum \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r$$

Hence

$$\int_{t_0}^{t_1} \delta L dt = 0$$

may be written as

$$\int_{t_0}^{t_1} \sum_{r=1}^{n} \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) dt = 0$$

$$\int_{t_0}^{t_1} \sum_{r=1}^{n} \left[\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \frac{d}{dt} \delta q_r \right] dt = 0$$

$$\int_{t_0}^{t_1} \sum_{r=1}^{n} \left[\frac{\partial L}{\partial q_r} \delta q_r dt + \int_{t_0}^{t_1} \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r} \right] \delta q_r dt = 0$$

$$\int_{t_0}^{t_1} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r} \partial q_r dt + \left[\frac{\partial L}{\partial q_r} \partial q_r \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\sum_{r=1}^{n} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) \delta q_r \right] dt = 0$$

Using the method of integration

$$\int_{t_0}^{t_1} \left[\sum_{r=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} \right\} \delta q_r \right] dt = 0$$

$\forall\ vanishes \delta q_r = 0\ at\ t_0\ and\ \ t_1$

$$\sum_{r=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} \right\} \delta q_r = 0$$

But $\delta q_1, \delta q_2, \dots\dots, \delta q_n$ are all arbitrary and independent of each other; hence

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0, \qquad r = 1,2, \dots . n$$

Which is familiar Lagrange's Equations.

SOLVED EXAMPLE

EXAMPLE1: Use the variational method to show that the shortest curve joining to fixed point is a straight line.

SOLUTION: Let the shortest curve joining to fixed point $A(x_1, y_1)$ and $A(x_2, y_2)$

Since

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

$$= \int_{x_1}^{x_2} f(y')dx$$

where

$$f(y') = \sqrt{1 + y'^2}$$

Since y is absent from f, for at s stationary (here minimum), we should have from Euler-Lagrange's equation

$$\frac{\partial f}{\partial y'} = constant$$

$$\frac{\partial}{\partial y'}\sqrt{1 + y'^2} = constant$$

$$\frac{y'}{\sqrt{1 + y'^2}} = constant$$

$$y' = b$$

$$\frac{dy}{dx} = b \quad or \quad y = a + bx$$

Thus

$$y = a + bx$$

EXAMPLE2: On the curves can the functional

$$V[y(x) = \int_0^1 (y'^2 + 12xy)\, dx$$

with $y(0) = 0, y(1) = 1$ be extremized.

SOLUTION: Let

$$I = \int_0^1 (y'^2 + 12xy)dx = \int_0^1 f(x, y, y')$$

where

$$f(x, y, y') = y'^2 + 12xy$$

For the functional, i.e., integral I to be extremized, the Euler's Lagrange's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

must be satisfied.

Here,

$$f = y'^2 + 12xy,$$

thus

$$12x - \frac{d}{dx}(2y') = 0 \quad \Rightarrow \quad \frac{d}{dx}(y') = 6x.$$

On integrating, we get

$$y' = 3x^2 + a$$

or

$$\frac{dy}{dx} = 3x^2 + a,$$

where a is a constant.

Again integrating, we get

$$y = x^3 + ax + b,$$

where b is also a constant. .... (1)

Now, using the conditions $y(0) = 0$ and $y(1) = 1$ quation (1), we get

$$y(0) = 0 = b$$

and

$$y(1) = 1 = 1 + a + b.$$

then

$$a = 0$$

and

$$b = 0.$$

Hence, the equation of the required curve is

$$\boxed{y = x^3.}$$

EXAMPLE3: A particle moves on a smooth curve, joining the two fixed points A and B, under gravity, starting from rest from A. Find the form of the path in order that the time from A to B is minimum.

SOLUTION: Take the fixed point A as origin and horizontal and vertical through A as co-ordinate axes, and let (x_1, y_2) be the co-ordinates of the other extremity B.

At a time t, let v be the velocity in a position P whose co-ordinates are (x, y) and actual distance s measured from the origin A.

By energy equation, we have

$$\frac{1}{2}mv^2 = mgy$$

or

$$v = \sqrt{2gy}$$

or

$$\frac{ds}{dt} = \sqrt{2gy}.$$

Integrating,

$$t = \int \frac{ds}{\sqrt{2gy}} = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

since

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Hence,

$$t = \frac{1}{\sqrt{2g}} \int_0^{x_1} f(y, y')\,dx$$

where

$$f(y, y') = \sqrt{\frac{1+y'^2}{y}}.$$

Since x is absent in f, therefore for t to be stationary (here minimum) we should have

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

That is,

$$\sqrt{\frac{1+y'^2}{y}} - y'\left(\frac{y'}{\sqrt{y}\sqrt{1+y'^2}}\right) = \text{constant}.$$

$$\frac{1}{\sqrt{y(1+y'^2)}} = \text{constant}.$$

Let the constant be

$$\frac{1}{\sqrt{2b}},$$

where b is a constant.

Therefore,

$$y(1+y'^2) = 2b.$$

Since

$$y' = \tan\psi,$$

we have

$$y = 2b\cos^2\psi$$

or

$$y = b(1+\cos 2\psi) \qquad \cdots (1)$$

And as

$$\frac{dy}{dx} = \tan\psi,$$

we have

$$dx = \cot\psi\, dy$$
$$= \cot\psi(-4b\cos\psi\sin\psi\, d\psi)$$
$$= -4b\cos^2\psi\, d\psi$$

or

$$dx = -2b(1+\cos 2\psi)\, d\psi.$$

Integrating

$$x = -2b\left(\psi + \frac{sin2\psi}{2}\right) + a$$

where a is constant.

$$x = a - b(2\psi + sin2\psi)$$
$$x = b(1 + cos2\psi)$$

EXAMPLE4: A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin. Deduce from the principle of energy and least action that the differential equation of the path is

$$c^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\,[\,h - \phi(r)\,].$$

SOLUTION:

Let T and V be the kinetic and potential energies.

$$T + V = h$$

or

$$T = h - V$$

$$= h - \phi(r),$$

since $V = \phi(r)$ is given.

Also,

$$\frac{1}{2}v^2 = h - \phi(r),$$

where v is the velocity.

Hence,

$$v = \sqrt{2\{h - \phi(r)\}}.$$

The action

$$A = \int_{t_0}^{t_1} 2T\,dt$$

by definition,

$$= \int_{t_0}^{t_1} 2 \cdot \frac{1}{2}v^2 dt = \int_{t_0}^{t_1} v\,ds,$$

since

$$v = \frac{ds}{dt}.$$

Therefore,

$$A = \sqrt{2}\int_{t_0}^{t_1} \{h - \phi(r)\}^{1/2}\,ds.$$

Also,

$$ds = \left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2} dr,$$

since

$$\frac{ds}{dr} = \left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}.$$

Hence,

$$A = \sqrt{2}\int_{t_0}^{t_1} \{h - \phi(r)\}^{1/2}\left\{1 + r^2\theta'^2\right\}^{1/2} dr,$$

where

$$\theta' = \frac{d\theta}{dr}.$$

Thus,

$$A = \sqrt{2}\int_{t_0}^{t_1} f(\theta', r)\, dr,$$

where

$$f(\theta', r) = \{h - \phi(r)\}^{1/2}\left\{1 + r^2\theta'^2\right\}^{1/2}.$$

Since θ is absent from f,

$$\frac{\partial f}{\partial \theta'} = c \quad \text{(constant)}.$$

Hence,

$$\frac{\partial}{\partial \theta'}\left[\{h - \phi(r)\}^{1/2}\{1 + r^2\theta'^2\}^{1/2}\right] = c.$$

or

$$\{h - \phi(r)\}^{1/2} \cdot \frac{1}{2}\{1 + r^2\theta'^2\}^{-1/2}(2r^2\theta') = c.$$

or

$$\{h-\phi(r)\}^{1/2}\cdot\frac{r^2}{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}}=c.$$

Therefore,

$$c^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]=r^4\left[h-\phi(r)\right].$$

This is the desired equation.

EXAMPLE4: A particle of unit mass moves along OX under a constant force f, starting from rest at the origin at time $t = 0$. If T and V are the kinetic and potential energies of the particle, calculate

$$\int_0^{t_1}(T-V)dt$$

Evaluate this for the varied motion in which the position of the particle is given by

$$x=\frac{1}{2}ft^2+\varepsilon ft(t-t_0)$$

where ε is a constant; and show that the result is in agreement with Hamilton's principle. What are the essential features of the varied motion that ensure this agreement?

SOLUTION: The particle actually moves along OX with acceleration f, starting from rest; therefore its actual path is

$$x=\frac{1}{2}ft^2,$$

so that

$$\dot{x}=ft.$$

Hence,

$$T=\frac{1}{2}\dot{x}^2=\frac{1}{2}f^2t^2.$$

Since the particle moves from $t = 0$ to $t_1 = 0$, the end points of the actual motion are

$$0 \quad \text{and} \quad \frac{1}{2}ft_0^2.$$

The potential at a distance x, at time t, is

$$V = -fx = -f\left(\frac{1}{2}ft^2\right) = -\frac{1}{2}f^2t^2.$$

Therefore,

$$T - V = \frac{1}{2}f^2t^2 - \left(-\frac{1}{2}f^2t^2\right) = f^2t^2.$$

Thus, in the actual path,

$$\int_0^{t_0}(T-V)\,dt = \int_0^{t_0} f^2t^2dt = \frac{1}{3}f^2t_0^3.$$

The varied path is

$$x = \frac{1}{2}ft^2 + \varepsilon ft(t-t_0) = \frac{1}{2}\left[(1+2\varepsilon)ft^2 - 2\varepsilon ft_0t\right].$$

Putting $t = 0$ and $t = t_0$, we see that the end points of the varied motion are also

$$0 \quad \text{and} \quad \frac{1}{2}ft_0^2.$$

Hence, the end points of the actual path and the varied path coincide.

Velocity in the varied path is

$$(1+2\varepsilon)ft - \varepsilon ft_0.$$

Hence,

$$T = \frac{1}{2}\left[(1+2\varepsilon)ft - \varepsilon ft_0\right]^2$$

$$= \frac{1}{2}\left[(1+4\varepsilon+4\varepsilon^2)f^2t^2 - (2\varepsilon+4\varepsilon^2)f^2t_0t + \varepsilon^2f^2t_0^2\right].$$

Also,

$$V = -f\left[\frac{1}{2}\{(1+2\varepsilon)ft^2 - 2\varepsilon ft_0t\}\right]$$

Therefore,

$$T - V = \frac{1}{2}\left[(2+6\varepsilon+4\varepsilon^2)f^2t^2 - (4\varepsilon+4\varepsilon^2)f^2t_0t + \varepsilon^2f^2t_0^2\right].$$

Hence,

$$\int_0^{t_0} (T - V)\, dt = \frac{1}{3}\left(1 + \frac{1}{2}\varepsilon^2\right) f^2 t_0^3.$$

This integral is minimum when

$$\boxed{\varepsilon = 0.}$$

The minimum value is

$$\boxed{\frac{1}{3} f^2 t_0^3,}$$

which is the same as that obtained for the actual pat ius, the result is in agreement with **Hamilton's**

SELF CHECK QUESTIONS

1. What is Hamilton's Principle? State its mathematical expression.
2. Define the action integral in Hamilton's Principle.
3. Explain the significance of the Lagrangian function in Hamilton's Principle.
4. Derive the Euler–Lagrange equation from Hamilton's Principle.
5. What conditions must be satisfied by the end points in Hamilton's Principle?
6. Distinguish between the actual path and the varied path in Hamilton's Principle.
7. Explain the concept of stationary action. Why is the action said to be stationary rather than always minimum?

8.12 *SUMMARY:-*

This unit introduces the **calculus of variations** and its application in analytical mechanics. It begins by explaining the concepts of the **actual path** and the **varied path**, which form the basis of variational methods. The **Variation Method** is then developed to determine the stationary value of a functional, leading to the **Euler–Lagrange equation** and its alternative form for solving variational problems. The **general case** of the Euler–Lagrange equation is discussed for systems involving several dependent variables and generalized coordinates. The chapter then presents **Hamilton's Principle** with two independent proofs—one based on the **Calculus of Variations** and the other derived from **D'Alembert's Principle**. It further explains the **Principle of Least Action**, its relationship with Hamilton's Principle, and highlights the **distinction between Hamilton's Principle and the Principle of Least Action**. Finally, the unit demonstrates how **Lagrange's equations of motion** are derived directly from Hamilton's Principle, establishing a powerful and elegant formulation of classical mechanics that serves as the foundation for advanced studies in analytical mechanics, quantum mechanics, and modern theoretical physics.

8.13 *GLOSSARY: -*

- **Actual Path:** The **actual path** is the real trajectory followed by a mechanical system or particle under the action of applied forces. It satisfies the equations of motion and makes the action integral stationary according to Hamilton's Principle.
- **Varied Path:** A **varied path** is any hypothetical path that differs infinitesimally from the actual path while having the same fixed end points. It is used in the calculus of variations to compare the action with that of the actual path.
- **Variation Method:** The **variation method** is a mathematical technique used to determine the function that makes a functional stationary (minimum, maximum, or stationary). It forms the basis of the calculus of variations and analytical mechanics.
- **Functional:** A **functional** is a quantity whose value depends on an entire function rather than on individual variables. In mechanics, the action integral is an example of a functional.
- **Variation:** A **variation** is an infinitesimal change in the dependent variable or function while keeping the independent variable fixed. It is represented by the symbol δ.
- **Euler–Lagrange Equation**: The **Euler–Lagrange equation** is the fundamental differential equation obtained from the calculus of variations that determines the function for which a functional is stationary.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = C$$

- **Another Form of Euler–Lagrange Equation (Beltrami Identity):** When the functional does not explicitly depend on the independent variable, the Euler–Lagrange equation can be written as

$$F - y'\frac{\partial F}{\partial y'} = C = Constant$$

- **General Case:** The general case extends the Euler–Lagrange equation to systems involving several dependent variables or generalized coordinates, producing one Euler–Lagrange equation for each coordinate.

$$\delta \int_{0}^{t_1} (T - V) dt = 0$$

- **Calculus of Variations: The** calculus of variations **is a branch of D'Alembert's Principle:** D'Alembert's Principle states that the total virtual work done by the impressed forces and the inertial forces is zero for a system in dynamic equilibrium. It provides an alternative derivation of Hamilton's Principle.
- **Action:** The action is the time integral of the Lagrangian between two fixed instants.

$$\boldsymbol{A = \int_{t_2}^{t_1} (T) dt}$$

- **Lagrangian:** The Lagrangian is defined as the difference between kinetic energy and potential energy.

$$\boldsymbol{L = T - V}$$

- **Principle of Least Action:** The Principle of Least Action states that, among all possible paths satisfying the boundary conditions, the actual path makes the action least (or more generally stationary) under suitable conditions.
- **Hamilton's Principle vs. Principle of Least Action:** Although closely related, Hamilton's Principle requires the action to be stationary, whereas the Principle of Least Action emphasizes that the action is minimum in many physical situations.
- **Stationary Action:** An action is said to be stationary when its first variation is zero, i.e.,

$$\boldsymbol{\delta S = 0}.$$

A stationary value may be a minimum, maximum, or saddle point.

- **Generalized Coordinates:** Generalized coordinates are independent variables used to describe the configuration of a mechanical system while satisfying its constraints.

- **Virtual Displacement:** A virtual displacement is an infinitesimal imagined displacement consistent with the constraints of the system at a given instant.
- **Fixed End Points:** Fixed end points are boundary conditions in which the initial and final positions of the system remain unchanged during the variation.

  - **Conservative System:** A conservative system is one in which the total mechanical energy remains constant because the forces are derivable from a potential function.
  - **Boundary Conditions:** Boundary conditions specify the values of the dependent variables at the initial and final points of the interval over which the variation is performed.
  - **Variational Principle:** A variational principle is a physical law expressed in terms of making a functional stationary rather than directly using differential equations.
  - **Analytical Mechanics:** Analytical mechanics is a branch of classical mechanics that describes the motion of systems using energy principles and generalized coordinates instead of force-based Newtonian methods.

8.14 *REFERENCES: -*

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8.15 *SUGGESTED READING:-*

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8.16 *TERMINAL QUESTIONS:-*

(TQ-1) Derive the alternative form (Beltrami Identity) of the Euler–Lagrange equation when the integrand is independent of the independent variable.

(TQ-2) Derive the Euler–Lagrange equations for a functional involving several dependent variables.

(TQ-3) Explain the general case of the calculus of variations for multiple generalized coordinates.

(TQ-4) State and prove Hamilton's Principle using the calculus of variations.

(TQ-5) State D'Alembert's Principle and derive Hamilton's Principle from it.

(TQ-6) Show that Hamilton's Principle can be obtained directly from D'Alembert's Principle.

(TQ-7) Compare the derivation of Hamilton's Principle using the calculus of variations and D'Alembert's Principle.

(TQ-8) Explain the Principle of Least Action and derive its mathematical expression.

(TQ-9) Discuss the relationship between Hamilton's Principle and the Principle of Least Action.

(TQ-10) Explain the physical interpretation of stationary action and least action with suitable examples.

(TQ-11) Derive the system of Euler–Lagrange equations for

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx.$$

(TQ-12) Starting from the action integral

$$S = \int_{t_1}^{t_2} (T - V)\, dt,$$

Derive Hamilton's Principle and obtain the Euler–Lagrange equations.

(TQ-13) Using the Euler–Lagrange equation, determine the curve that minimizes the functional

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx.$$

(TQ-14) Determine the curve that extremizes the functional

$$I = \int_0^1 (y'^2 + 12xy)\, dx,$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 1.$$

(TQ-15) Prove that if

$$F = F(y, y'),$$

Then

$$F - y'\frac{\partial F}{\partial y'} = C,$$

Where C is constant.

(TQ-16) Show that the integral $\int_0^{t_1} L dt$ is Greater for valid path the actual path but the result is in agreement with Hamilton principle.

8.17 *ANSWERS:-*

(TQ-11) $\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_i}\right) = 0 \qquad r = 1,2, \dots . n$

(TQ-12) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \qquad r = 1,2, \dots . n$

(TQ-13)

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

(TQ-14) $y = x^3$

UNIT 9: - Hamilton's Equations of Motion

CONTENTS:

9.1 *INTRODUCTION: -*

Classical mechanics can be expressed in various ways. Newton's rules describe motion in terms of forces, whereas Lagrangian mechanics employs generalized coordinates and energies. Hamiltonian mechanics is another powerful formulation that reduces the equations of motion to a set of first-order differential equations known as Hamilton's equations. This method is widely applied in classical mechanics, quantum mechanics, statistical mechanics, and current physics.

9.2 *OBJECTIVES: -*

After studying this unit, learners will be able to:

- Understand generalized velocities and generalized momentum.
- Define the Hamiltonian function.
- Derive Hamilton's equations of motion.
- Explain the relationship between Lagrangian and Hamiltonian formulations.
- Deduce Euler-Lagrange equations from Hamilton's equations.
- Apply Hamiltonian methods to mechanical systems.

9.3 *GENERALISIED VELOCITES, LAGRANGIAN AND GENERALSIED MOMENTUM: -*

Assume a holonomic dynamical system defined by n generalized co-ordinates $q_1, q_2, \ldots q_n$ travels under conservative forces, with T and V representing the kinetic and potential energies.

Then, T is a function of both $qs'$and qs' while V is a function of only $qs,$ where $\dot{q}_r \left(=\frac{dq_r}{dt}\right)$ is the generalised velocity $(r = 1, 2, \ldots, g)$.
The system's Lagrangian, usually denoted by Land, is defined as $L = T - V$. T is a function of $qs'$and $\dot{q}s'$, while V is a function of qs' only. This implies that L is a function of $qs'$and $\dot{q}s'$.
The generalized momentum of the system is defined as - and is generally denoted as $\partial L/dr$ by p_r.
Now

$$p_r = \frac{\partial L}{d\dot{q}_r}$$

9.4 *HAMILTONIAN: -*

The Hamiltonian of a dynamical system is denoted by H and is defined by

$$H = -L + \sum_{r=1}^{n} p_r \, \dot{q}_r$$

where $p_r = \frac{\partial L}{\partial \dot{q}_r}$ $(r = 1,2,3, \ldots \ldots n)$
Since $\frac{\partial L}{\partial \dot{q}_r}$ in general contains $qs'$and qs', the n equations $p_r = \frac{\partial L}{\partial \dot{q}_r}(r = 1,2,3, \ldots \ldots n)$

9.5 *HAMILTON'S EQUATIONS OF MOTION: -*

Suppose a holonomic dynamical system, defined by n generalized coordinates $q_1, q_2 \ldots \ldots \ldots \ldots . q_n$ move under the influence of conservative forces, then the Hamiltonian

$$H = -L + \sum_{r=1}^{n} p_r \, \dot{q}_r$$

where $p_r = \frac{\partial L}{\partial \dot{q}_r} \, (r = 1,2,3, \dots \dots n)$
Since H as the function of $qs'$and qs' we get

$$\delta H = -\sum_{r=1}^{n} \left(\frac{\partial L}{\partial q_r} \partial q_r + \frac{\partial L}{\partial \dot{q}_r} \partial \dot{q}_r \right) + \sum_{r=1}^{n} \dot{q}_r \delta p_r + p_r \delta \dot{p}_r$$

$$= -\sum_{r=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial q_r} \right) \partial q_r + \ p_r \ \partial \dot{q}_r \right] + \sum_{r=1}^{n} (\dot{q}_r \delta p_r + p_r \delta \dot{p}_r)$$

Using Lagrange Equation$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0$

$$= -\sum_{r=1}^{n} \left[\frac{d}{dt} (p_r) \partial q_r + \ p_r \ \partial \dot{q}_r \right] + \sum_{r=1}^{n} (\dot{q}_r \delta p_r + p_r \delta \dot{q}_r)$$

$$= \sum_{r=1}^{n} (-\dot{p}_r \delta q_r + \dot{q}_r \delta p_r) \qquad \dots (1)$$

Also

$$\delta H = \sum_{r=1}^{n} \left(\frac{\partial H}{\partial q_r} \partial q_r + \frac{\partial H}{\partial p_r} \partial \dot{q}_r \right) \qquad \dots (2)$$

From (1) and (2), we get

$$\sum_{r=1}^{n} (-\dot{p}_r \delta q_r + \dot{q}_r \delta p_r) = \sum_{r=1}^{n} \left(\frac{\partial H}{\partial q_r} \partial q_r + \frac{\partial H}{\partial p_r} \partial \dot{q}_r \right)$$

But δp_r and δq_r are arbitrary and independent of each other : Therefore , δp_r and δq_r respectively we have

$$\dot{p}_r = -\frac{\partial H}{\partial q_r}, \dot{q}_r = \frac{\partial H}{\partial p_r}$$

where $r = 1,2,3, \dots \dots n$
which is known as Hamilton's equations of motion, or Hamilton's Canonical equations.

IF THE LAGRANGE L DOES NOT INVOLVE T EXPLICITY, THE HAMILTONIAN H IS CONSTANT AND EQUAL TO THE TOTAL ENERGY H.

Since T is a quadratic homogeneous function of qs', by Euler's theorem on partial differential of homogeneous functions, we should have

$$\sum_{r=1}^{n} \dot{q}_r \frac{\partial T}{\partial \dot{q}_r} = 2T$$

Now

$$H = -L + \sum_{r=1}^{n} p_r \, \dot{q}_r$$

where $p_r = \frac{\partial L}{\partial \dot{q}_r}$ $(r = 1,2,3, \dots \dots n)$ by definition

$$= -L + \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r$$

$$= -L + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \dot{q}_r$$

Since

$$\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T}{\partial \dot{q}_r}$$

$$= -L + 2T$$

$$= (T - V) + 2T$$

Since $L = T - V$ by definition

$$= T + V = h$$

9.6 *TO DEDUCE EULER'S EQUATIONS FROM HAMILTON'S EQUATIONS: -*

Eular's equations are established for the motion about a fixed, then we get

$$2T = (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

we have

$$L = T - V = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) - V$$

where

$$\omega_1 = \dot{\theta} sin\phi - \dot{\Psi} sin\theta cos\phi$$
$$\omega_2 = \dot{\theta} cos\phi - \dot{\Psi} sin\theta sin\phi$$
$$\omega_3 = \dot{\phi} + \dot{\Psi} cos\theta$$

where $\omega_1, \omega_2, \omega_3$ do not involve explicitly and L does not involve t explicitly hence the total energy

$$H = T + V$$

$$H = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) - V$$

By the definition

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \omega_1}\frac{\partial \omega_1}{\partial \dot{\theta}} + \frac{\partial L}{\partial \omega_2}\frac{\partial \omega_2}{\partial \dot{\theta}} + \frac{\partial L}{\partial \omega_3}\frac{\partial \omega_3}{\partial \dot{\theta}}$$

$$= A\omega_1 sin\phi + B\omega_2 cos\phi + C\omega_3 . 0$$

Similarly

$$p_\phi = \frac{\partial L}{\partial \phi} = C\omega_3$$

and

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = -A\omega_1 sin\theta cos\phi + B\omega_2 sin\theta sin\phi + C\omega_3 cos\theta$$

Solving three equations $\omega_1, \omega_2, \omega_3$ are

$$\omega_1 = \frac{1}{A}\left[p_\theta sin\phi + \left(p_\phi cos\theta - p_\psi\right)\frac{cos\phi}{sin\theta}\right]$$
$$\omega_2 = \frac{1}{B}\left[p_\theta cos\phi + \left(p_\phi cos\theta - p_\psi\right)\frac{sin\phi}{sin\theta}\right]$$
$$\omega_3 = \frac{1}{C}p_\phi$$

Now the Hamilton's two equations are

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} \text{ and } \dot{\phi}\frac{\partial H}{\partial p_\phi}$$

Gives

$$C\dot{\omega} = -\left[\frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial \phi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial \phi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial \phi}\right] - \frac{\partial V}{\partial \phi}$$
$$= \left[A\omega_1.\frac{1}{A}B\omega_2 + B\omega_2\left(-\frac{A}{B}\omega_1\right) + C\omega_3.0\right] - \frac{\partial V}{\partial \phi}$$
$$= (A - B)\omega_1\omega_2 - \frac{\partial V}{\partial \phi}$$
$$C\frac{\partial \omega_1}{\partial t} - (A - B)\omega_1, \omega_2 = N$$
$$-\frac{\partial V}{\partial \phi} = N$$

Which is known as Euler's third familiar dynamical equations.

SOLVED EXAMPLES

EXAMPLE1: Prove that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

where H is the Hamilton's function.

SOLUTION: Let $q_1, q_2, \dots, q_n$ be generalized co-coordinates1; then the Hamilton's equations are

$$\dot{p}_r = -\frac{\partial H}{\partial q_r}, \dot{q}_r = \frac{\partial H}{\partial p_r}$$

Since H as the function of $qs'$and qs' we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{r=1}^{n}\left(\frac{\partial H}{\partial q_r}\partial \dot{q}_r\right) + \sum_{r=1}^{n}\frac{\partial H}{\partial p_r}\delta \dot{p}_r$$

$$= \frac{\partial H}{\partial t} + \sum_{r=1}^{n} (-\dot{p}_r)\dot{q}_r + \sum_{r=1}^{n} \dot{q}_r \, \dot{p}_r = \frac{\partial H}{\partial t}$$

EXAMPLE2: Write the Hamilton equation for a particle of mass m moving in a plane under a force which is some function of the distance from the origin.

SOLUTION: Since the force is a function of r, the potential V is also a function of r and thus dependent on θ, resulting in V=V(r). The velocity components are radial (r ̇) and transversal (r θ).

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r))$$

from which

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } p_e = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Since L does not explicitly involve t, hence the Hamiltonion H is

$$H = T + V$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= \frac{1}{2} m\left[\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2r^2}\right] + V(r)$$

$$= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + V(r)$$

Hence, the Hamilton's equations are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p_\theta} = \frac{\partial H}{\partial \theta} = 0$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

Which is four Hamilton Equations.

EXAMPLE3: Use Hamilton equation to find the equation of motion of the simple pendulum

SOLUTION: Suppose l be the length of string, and m the mass of the bob, forming a simple pendulum. At time t, the string be inclined at an angle θ to the downward drawn vertical.

velocity of bob= $l\dot{\theta}$

$$T = \frac{1}{2} ml^2\dot{\theta}^2 \text{ and } V = -W = mglcos\theta$$

$$L = T - V = \frac{1}{2} ml^2\dot{\theta} + mglcos\theta$$

From which

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

Hence the Hamilton's equations is given by

$$H = T + V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\ cos\theta$$

$$= \frac{1}{2}m\frac{p_\theta^2}{m^2l^2} - mglcos\theta = \frac{l}{2ml^2}p_\theta^2 - mglcos\theta$$

Now

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -mglsin\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

Differentiating

$$\ddot{\theta} = \frac{1}{ml^2}\dot{p_\theta} = -\frac{1}{ml^2}mgl\ sin\theta$$

$$= -\frac{g}{l}sin\theta$$

Thus

$$\ddot{\theta} = -\frac{g}{l}sin\theta$$

Hence which is the equation of simple pendulum.

EXAMPLE4: If $2T = \dot{\theta}^2 + \theta^2\phi^2$ and $V = \frac{1}{2}n^2\theta^2$, prove that Hemilton's equations give

$$\theta^2 = a^2cos^2(nt+\alpha) + b^2sin^2(nt+\alpha)$$

and $\quad tan(\phi+\beta) = \frac{b}{a}tan(nt+\alpha)$

where a, b, α, β are constants.

SOLUTION: Let

$$L = T - V = \frac{1}{2}(\theta^2 + \theta^2\phi^2) - \frac{1}{2}n^2\theta^2$$

So that

$$P_\theta = \frac{\partial L}{\partial \theta} = \dot{\theta} \text{ and } P_\phi = \frac{\partial L}{\partial \phi} = \dot{\phi}\theta^2$$

Therefore, the Hamiltonian H is

$$H = T + V = \frac{1}{2}(\theta^2 + \theta^2\phi^2 + n^2\theta^2)$$

$$= \frac{1}{2}\left[p_\theta^2 + \left(\frac{p_\theta^2}{\theta^2}\right) + n^2\theta^2\right]$$

Hence Hamilton's equations are

$$\dot{p_\theta} = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{\theta^3} - n^2\theta \qquad \ldots (1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta \qquad \dots(2)$$

$$\dot{p_\phi} = \frac{\partial H}{\partial \phi} = 0 \qquad \dots(3)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\theta^2} \qquad \dots(4)$$

Fromm (3), we get

$$p_\phi = constant = cn$$

Differentiating (2), we obtain

$$\ddot{\theta} = \dot{p}_\theta = \left(\frac{p_\phi^2}{\theta^3}\right) - n^2\theta$$

Now integrating

$$\dot{\theta^2} = -\frac{c^2n^2}{\theta^2} - n^2\theta^2 + 2en^2$$

where e is constant.

$$= \frac{n^2}{\theta^2}[-c^2 - \theta^4 + 2e\theta^2]$$

$$= \frac{n^2}{\theta^2}[e^2 - c^2 + (\theta^2 - e)^2]$$

$$\dot{\theta} = -\frac{n}{\theta}[(e^2 - c^2) + (\theta^2 - e)^2]^{1/2}$$

Taking negative sign

Integrating

$$2nt + 2\alpha = -\int \frac{2\theta}{(e^2 - c^2) + (\theta^2 - e)^2}$$

Where α is constant.

$$= cos^{-1}\left[\frac{\theta^2 + e}{\sqrt{e^2 - c^2}}\right] \quad or \quad \theta^2 - e = \sqrt{e^2 - c^2}\cos(2nt + 2\alpha)$$

$$\theta^2 = \sqrt{e^2 - c^2}\cos(2nt + 2\alpha) + e$$

$$= \frac{1}{2}(a^2 + b^2) + \frac{1}{2}(a^2 - b^2)\cos(2nt + 2\alpha)$$

Where a, b are constant, taking $e = \frac{1}{2}(a^2 + b^2)$ and $c = ab$

$$= \frac{1}{2}a^2\{1 + 2cos2(nt + \alpha)\} + \frac{1}{2}b^2\{1 - 2cos2(nt + \alpha)\}$$

$$= a^2cos^2(nt + \alpha) + b^2sin^2(nt + \alpha)$$

This is one result.

Again from (4), we obtain

$$\dot{\phi} = \frac{p_\phi}{\theta^2} = \frac{cn}{\theta^2}$$

$$= \frac{cn}{a^2cos^2(nt + \alpha) + b^2sin^2(nt + \alpha)}$$

$$= cn\frac{sec^2(nt + \alpha)}{a^2 + b^2tan^2(nt + \alpha)}$$

Again integrating

$$\phi + \beta = \frac{c1}{ab}\left\{tan^{-1}\frac{b}{a}tan(nt+\alpha)\right\}$$

$$= \left\{tan^{-1}\frac{b}{a}tan(nt+\alpha)\right\}$$

$$\tan(\phi+\beta) = \frac{b}{a}$$

This is second result.

SELF CHECK QUESTIONS

1. What is the Hamiltonian?
2. What are generalized coordinates?
3. What is generalized momentum?
4. State Hamilton's equations of motion.
5. Define the Hamiltonian function.

9.7 *SUMMARY:-*

Hamilton's equations of motion are a reformulation of classical mechanics that describe the motion of a mechanical system in terms of generalized coordinates and generalized momenta. They are obtained from the Hamiltonian function, which is derived from the Lagrangian using the Legendre transformation. Unlike Lagrange's equations, Hamilton's formulation expresses the dynamics of a system as two coupled first-order differential equations, making it simpler to analyze complex mechanical systems. This formulation provides a complete description of the system's motion in phase space and forms the theoretical foundation for many advanced branches of physics, including quantum mechanics, statistical mechanics, and dynamical systems.

9.8 *GLOSARY:-*

- **Hamiltonian (H):**A function that describes the total energy of a system in terms of generalized coordinates and generalized momenta.
- **Hamilton's Equations:** A pair of first-order differential equations that describe the time evolution of a mechanical system.
- **Generalized Coordinates q_i:**Independent variables used to specify the configuration of a mechanical system.

- **Generalized Momenta p_i:** Quantities conjugate to the generalized coordinates, defined by $\frac{\partial L}{\partial \dot{q}_r}$.
- **Lagrangian (L):** A function defined as the difference between kinetic energy and potential energy, $(L = T - V)$.
- **Legendre Transformation:** A mathematical transformation used to convert the Lagrangian into the Hamiltonian.
- **Canonical Variables:** The generalized coordinates and their corresponding generalized momenta, represented as $\boldsymbol{p_i}, \boldsymbol{q_i}$.
- **Canonical Equations:** Another name for Hamilton's equations of motion expressed in terms of canonical variables.
- **Phase Space:** A multidimensional space in which each point represents the complete state of a system using coordinates and momenta.
- **Degrees of Freedom:** The number of independent coordinates required to describe the configuration of a system.
- **Conservative System:** A system in which the total mechanical energy remains constant because only conservative forces act.
- **Kinetic Energy (T):** The energy possessed by a body due to its motion.
- **Potential Energy (V):** The energy possessed by a body due to its position or configuration.
- **Action Integral:** The time integral of the Lagrangian, whose stationary value determines the actual path of motion.
- **Hamilton's Principle:** The principle stating that the actual motion of a system makes the action integral stationary.
- **Euler–Lagrange Equation:** The fundamental equation of Lagrangian mechanics from which Hamilton's equations are derived.
- **Time-Independent Hamiltonian:** A Hamiltonian that does not explicitly depend on time and is conserved in conservative systems.
- **Trajectory:** The path followed by a system in phase space as it evolves with time.
- **Classical Mechanics:** The branch of physics that studies the motion of macroscopic bodies under the action of forces.
- **Dynamical System:** A system whose state changes with time according to specified physical laws.

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9.11 *TERMINAL QUESTIONS:-*

(TQ-1) Distinguish between the Lagrangian and Hamiltonian formulations of mechanics with suitable examples.

(TQ-2) Explain the applications and advantages of Hamiltonian mechanics in classical mechanics, quantum mechanics, statistical mechanics, and celestial mechanics.

(TQ-3) Use Hamilton equation to find the equation of motion of a top.

(TQ-4) Use Hamilton equation to find the equation of motion of a particle in space.

(TQ-5) Use Hamilton equation to find the equation of motion of a top.

(TQ-6) equation of a motion of a particle moving in three dimensions in a force field of a potential V.

(TQ-6) Determine the motion, of a spherical pendulum, by Sing Hamilton's equations.

UNIT 10: - Central Orbit

CONTENTS:

10.1 *INTRODUCTION: -*

In classical mechanics, a central orbit is the route taken by a particle traveling under the influence of a central force a force that is always directed toward or away from a fixed point and whose magnitude is determined solely by its distance from that point. This fixed position is referred to as the center of force. Because the force applies along the line connecting the particle and the center, the motion is limited to a plane, and the particle's angular momentum is conserved. Celestial mechanics, atomic physics, and orbital theory all heavily rely on central forces. Common examples include gravitational pull between heavenly entities and electrostatic attraction between charged particles. Depending on the nature of the central force (e.g., inverse-square law), the resulting orbits can take the form of circles, ellipses, parabolas, or hyperbolas. The study of central orbits contributes to a better understanding of planetary motion, scattering issues, and the stability of systems controlled by central interactions.

This unit will discuss about the concepts of central force and central orbit, the differential equation governing a central orbit, the rate of description of sectorial area, the nature of elliptic orbits, and the ideas of apse and apsidal distance.

10.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

- To understand the concept of a central Force.
- To study the nature of a central orbit,
- To derive and analyze the differential equation of a central orbit.
- To understand the conservation of angular momentum.
- To examine the rate of description of sectorial area for a particle in a central force field.
- To explore different types of orbits, especially elliptic orbits, and understand the conditions under which they occur.
- To study the concepts of apse and apsidal distance.

10.3 *CENTRE FORCE: -*

A force which is always directed toward a fixed point is called Central force and fix point is known as centre force.

Or

A central force is a force that always acts along the line joining a particle to a fixed point, called the center of force, and whose magnitude depends only on the distance of the particle from that point. This means the force is directed either toward the center (attractive) or away from it (repulsive), and it does not depend on the direction of motion.

Mathematically, a central force F can be written as:

$$\vec{F} = F(r)\hat{r}$$

where F(r) depends only on the radial distance r and $\hat{r}$ is the unit vector toward the center.

Because the force is always central, the torque on the particle is zero, which implies that angular momentum is conserved. As a result, the motion of the particle lies in a plane, and the particle obeys the law of equal areas in equal times, which is fundamental in orbital mechanics.

10.4 *CENTRE ORBIT: -*

A central orbit is the path described by a particle moving under the action of central force. The motion of a planet about the sun is an important example of a central orbit.

Theorem: A central orbit is always a plane curve.

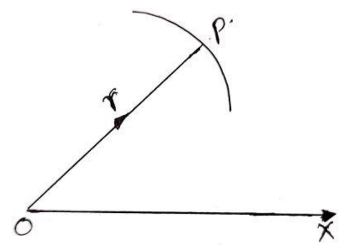


Fig.1

Assume that the center of force O is the vectors' origin. Let P be the particle's position in a central orbit at any time t, and let $\overrightarrow{OP} = r$. The acceleration vector of the particle at point P is $\frac{d^2r}{dt^2}$. This is because the particle moves under the action of the central force, with the center at O. The only force acting on the particle at P is along the line OP or PO, so the acceleration vector of P is parallel to the vector OP.

$$\frac{d^2r}{dt^2} \text{ is parallel to } r \Rightarrow \frac{d^2r}{dt^2} \times r = 0$$

$$\frac{d^2r}{dt^2} \times r + \frac{dr}{dt} \times \frac{dr}{dt} = 0 \qquad \left[\frac{dr}{dt} \times \frac{dr}{dt} = 0\right]$$

$$\frac{d}{dt}\left(\frac{dr}{dt} \times r\right) = 0$$

$$\frac{dr}{dt} \times r = a\ constant\ vector = h,$$

Now taking the product

$$r \cdot \left(\frac{dr}{dt} \times r\right) = r \cdot h$$

But

$$r \cdot h = 0$$

which shows that r is always perpendicular to a constant vector h.

10.5 *DIFFERENTIAL EQUATION OF CENTRE ORBIT: -*

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane two obtained the differential equation of the path.

Let a particle move in a plane with an acceleration P which is always directed to a fix point O in the plane take the centre of force O as the whole lead OX the initial line and R theta the polar coordinate of the position P of the moving particle at any instant t.

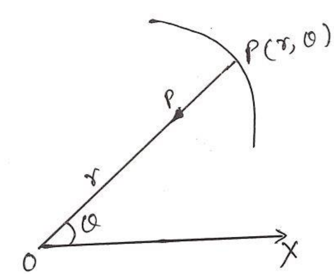


Fig.2

Since the acceleration of the particle is always directed towards the pole O there for the particle has only the radial acceleration and the transfer component of the acceleration of the particle is always zero so the radial acceleration is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \qquad \ldots (1)$$

(- Sign has been taken because the radial acceleration P is in the direction of r decreasing) and the transverse acceleration $i.e.,$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \qquad \dots(2)$$

From (2), we have

Integrating, we obtain

$$r^2\frac{d\theta}{dt} = constant = h$$

Let $\qquad r = \frac{1}{u}$

Now

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$$

Also

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}.u^2h = -h\frac{du}{d\theta}$$

and

$$\frac{d^2r}{dt^2} = -h\ \frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h\ \frac{d^2u}{d\theta^2}(u^2h) = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting in (1), we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u^2}(u^2h)^2 = -P$$

$$h^2u^2\frac{d^2u}{d\theta^2} + \frac{1}{u^2}h^2u^3 = P$$

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

Hence, which is the differential equation of the central Orbit in polar form.

Pedal Form: If P is the length of the perpendicular drawn from the origin to the tangent at the point P, we obtain

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

But

$$u = \frac{1}{r}, therefore\ \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}\ , \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

So

$$\frac{1}{p^2} = u^2 + \frac{1}{r^4}\left(\frac{du}{d\theta}\right)^2$$

Differentiating w.r.t. x, we get

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$-\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}.\frac{P}{h^2u^2}$$

$$-\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{d\theta} = \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2u^2}\right)$$

$$\frac{1}{p^3}\frac{dp}{dr} = \left(\frac{1}{r^2}\right)\left(\frac{P}{h^2u^2}\right) = u^2.\frac{P}{h^2u^2} = \frac{P}{h^2}$$

$$P = \frac{h^2}{p^3}\frac{dp}{dr}$$

which is the differential equation of a Central Orbit in pedal form.

10.6 *RATE OF DESCRIPTION OF THE SECTORIAL AREA: -*

If every Central Orbit the factorial area traced out by the radius vector to the centre of the force increases uniformly by unit of the time and the linear velocity where is inversely as the perpendicular from the centre upon the tangent to the path.

Let O be the board, and OX be the initial line. Let $P(r,\theta)$ and $Q(r+\delta r, \theta + \delta\theta)$ represent the particle's position in a central orbit at time t and $t + \delta t$ respectively.

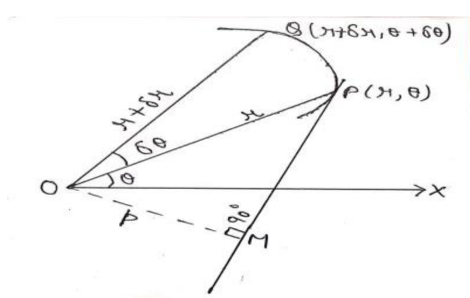


Fig.3

The sectorial area OPQ defined by the particle in time

$\delta t =$ area of the ΔOPQ

$[\therefore \textit{the point } Q \textit{ is very close to } P \textit{and take this limit } Q \to P]$

$$= \frac{1}{2} OP.OQ\, sin < POQ = \frac{1}{2} r(r + \delta r) sin\delta\theta$$

$\therefore$ the rate of description of the sectorial area

$$= \lim_{\delta t \to 0} \frac{sectorial\ area\ OPQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r(r + \delta r) sin\delta\theta}{\delta t}$$

$$= \frac{1}{2} r(r + \delta r). \frac{sind\theta}{\delta\theta}. \frac{\delta\theta}{\delta t} = \frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right) = \frac{1}{2} h$$

$$\left[\because r^2 \left(\frac{d\theta}{dt}\right) = h\right]$$

Thus the rate of sectorial area is constant and is equal to $h/2$.

The rate of the description of the factorial area is also called the aerial velocity of the particle about the fixed point O

Again for a central Orbit we obtain

$$r^2 \left(\frac{d\theta}{dt}\right) = h$$

$$r^2 \left(\frac{d\theta}{ds} \frac{ds}{dt}\right) = h$$

$$r^2 \left(\frac{d\theta}{ds} v\right) = h$$

But

$$r \frac{d\theta}{ds} = sin\phi$$

where ϕ is the angle between the radius vector and tangent.

$$r^2 \left(\frac{d\theta}{ds}\right) = \phi sin\phi = p$$

where p is the length of the perpendicular drawn from the pole O on the tangent at P.

Putting $r^2\left(\frac{d\theta}{ds}\right)=p$

$$\Rightarrow \qquad r^2\left(\frac{d\theta}{ds}v\right)=h \Rightarrow vp=h$$

$$v=\frac{h}{p}$$

$$v\propto 1/p$$

The linear velocity at point P changes inversely with the perpendicular distance from the fixed point to the tangent of the path.

$$v^2=\frac{h^2}{p^2}$$

But

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2$$

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]$$

This is the linear velocity at any point of the path of a central orbit.

10.7 *ELLIPTIC ORBIT: -*

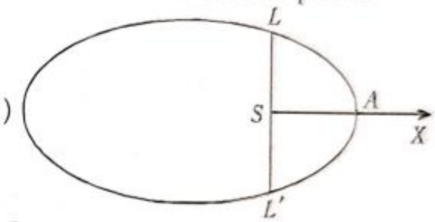
A particle moves in ellipse under a for which is always directed towards its focus to find

- the law of force
- the velocity at any point of its path
- the periodic time

We know that the polar equation of an ellipse referred to its focus S as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

or $$u = \frac{1}{l} + \frac{e}{l}\cos\theta, \qquad \ldots(1)$$



where $u = 1/r$. Differentiating, we have

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

(i) Law of Force: We know that the differential equation of a central orbit referred to the centre of force as pole is

$$\frac{P}{h^2 u^2} = u + \frac{d^2u}{d\theta^2}$$

where P is the central acceleration assumed to be attractive.

Now here $$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= h^2u^2\left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right], \text{ substituting for } u \text{ and } d^2u/d\theta^2$$

$$= \frac{h^2u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2},$$

where $\mu = h^2/l$ or $h^2 = \mu l$.

$\therefore$ $$P \propto \frac{1}{r^2}.$$

Hence the acceleration varies inversely as the square of the distance of the particle from the focus.

(ii) Velocity: We know that the velocity in a central orbit is given by

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

$\therefore$ here, $$v^2 = h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(\frac{-e}{l}\sin\theta\right)^2\right]$$

$$= h^2\left[\frac{1}{l^2} + \frac{2e}{l^2}\cos\theta + \frac{e^2}{l^2}\right] = \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\,\frac{e\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right]$$

$$= \mu\left[2u - \frac{1-e^2}{l}\right] = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right].$$

If $2a$ and $2b$ are the lengths of the major and the minor axes of the ellipse, we have

$$l = \text{the semi-latus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

$$\therefore \qquad \frac{1-e^2}{l} = \frac{1}{a}.$$

$$\therefore \qquad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

which gives the velocity of the particle at any point of its path.

Thus, the aforementioned equation indicates that the velocity's magnitude at any point along the path is only contingent upon the distance from the focus and is unaffected by the direction of motion.

(iii) Periodic time: We know that in a center orbit, the rate of description of the factorial area is constant and equals $h/2$. Let T be the time period for one complete revolution, which is the time taken by the particle to describe the entire ellipse. The factorial area trace used to describe the entire ellipse is equal to the ellipse's total area.

$$T\left(\frac{h}{2}\right) = \textit{the whole area of the ellipse} = \pi ab$$

$$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}}$$

$$T = \frac{2\pi ab}{\sqrt{\{\mu(b^2/a)\}}}$$

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}},$$

Hence the time period for one complete revolution to $a^{3/2}$, being a semi-major axis.

10.8 *HYPERBOLIC AND PARABOLIC ORBITS: -*

(i) **Hyperbolic Orbit:** In the case of hyperbola, we have $e > 1$.

Also $$l = \frac{b^2}{a} = \frac{a^2(e^2-1)}{a} = a(e^2-1).$$

Proceeding as in article 4, we have $P = \mu/r^2$, where $h^2 = \mu l$.

[Note that this result does not depend upon the value of e].

Also proceeding as in establishing the result (4) of article 4, we have here

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right]$$

or $$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right].$$ Note that here $v^2 > 2\mu/r$.

(ii) **Parabolic Orbit:** In this case $e = 1$.

Proceeding as in article 4, we have here $P = \mu/r^2$ and $v^2 = 2\mu/r$.

10.9 *VELOCITY FROM INFINITY: -*

In relation with the central orbit, the phrase velocity from infinity at any place refers to the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the influence of the attractive force in accordance with the orbit's law.
Assume a particle moves from rest to infinity in a straight path under the influence of the central attractive acceleration P, which directs it to the centre of force O.
Assume Q represents the particle's position at any time t, with $OQ = r$, and v is its velocity at Q. The expression for acceleration at point Q is $v(dv/dr)$.

$$v\frac{dv}{dr} = -P$$
$$vdv = -Pdr$$

Now the integrating, we get

$$\int_0^V vdv = -\int_\infty^a P\,dr$$
$$\frac{1}{2}V^2 = -\int_\infty^a P\,dr$$
$$V^2 = -2\int_\infty^a P\,dr$$

This gives the velocity from infinity at distance a from the center of the force well moving under the central acceleration P associated with the orbit.

10.10 *VELOCITY IN A CIRCLE: -*

The term velocity in a circle at any point in the central orbit refers to the velocity required to describe a circle passing through that point and falling under the action of the prescribed force connected with the orbit.

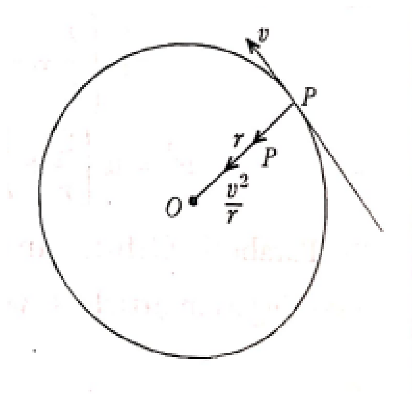


Fig.4

Assume that the center of force O is the pole. let PV the central acceleration directed towards O at any point P of the orbit where op is equal to hour suppose we is the velocity in the circle at P then velocity at the point P of a particle which move under the same Center Acceleration P in a circle with center at o but for a circle with center at the pole the radius vector op is also normal to the circle.

The central radial acceleration P= the inward normal acceleration v^2/ρ.

$$P = v^2/\rho$$

Thus, when moving under a central attractive acceleration P, the velocity V in a circle at a distance a from the center of force is provided by

$$V^2 = a.[P]_{r=a}$$

10.11 *GIVEN CENTRAL ORBIT, TO FIND THE LAW OF FORCE: -*

CaseI: *The equation of the orbit being given in polar form (r, θ).*

We know that the center of the force is referred to as Pole, the differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \qquad \text{... (1)}$$

where P is the central acceleration as you to be attractive.

From the given orbit equation, we can simply calculate u and $\frac{d^2u}{d\theta^2}$ and substitute the value (1). Thus, if P is positive, the force is attracting; if P is negative, the force is repulsive.

CaseI: *The equation of the orbit being given in padel form (p,r).*

The differential equation of (p,r) form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P \qquad \dots (2)$$

Form the given equation we find the dp/dr and substitute in (2), we find P.

SOLVED EXAMPLES

EXAMPLE1: A particle describe the curve $r^n = a^n cosn\theta$ under a force to the pole. Find the law of the force and obtain the law of force under which a cardioid can described.

SOLUTION: First part: The given equation is

$$r^n = a^n cosn\theta$$

Substituting $r = 1/u$, we get

$$\frac{1}{u^n} = a^n \cos n\theta \quad \text{or} \quad a^n u^n = \sec n\theta.$$

Taking both side logarithm in above equation

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n \sec n\theta \tan n\theta \quad \text{or} \quad \frac{du}{d\theta} = u \tan n\theta.$$

Differentiating again w.r.t. θ, we get

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta} \tan n\theta + u(\sec^2 n\theta) \,.\, n$$

$$= u \tan n\theta \,.\, \tan n\theta + un \sec^2 n\theta \qquad [\because du/d\theta = u \tan n\theta]$$

$$= u\tan^2 n\theta + un\sec^2 n\theta.$$

The differential equation of Central orbit is given by

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\ (u + u\tan^2 n\theta + un\sec^2 n\theta)$$

$$= h^2u^3\ (\sec^2 n\theta + n\sec^2 n\theta) = h^2u^3\ (1+n)\sec^2 n\theta$$

$$= h^2\ (1+n)\,u^3\ .(a^n u^n)^2$$

$$= h^2a^{2n}\,(1+n)\,u^{2n+3} = \frac{h^2a^{2n}\,(1+n)}{r^{2n+3}}.$$

Hence $P \propto \frac{1}{r^{2n+3}i}.e.,$ the force varies inversely as the $(2n+3)th$ the power of the distance from the pole.

Second part: Substituting $n = I/2$, we obtain

$$r^{1/2} = a^{1/2}\cos\frac{1}{2}\theta$$

Squaring both side, we obtain

$$r = a\cos^2\frac{1}{2}\theta$$

$$r = \frac{1}{2}a\,.2\cos^2\frac{1}{2}\theta = \frac{1}{2}a\,(1+\cos\theta),$$

which is the equation of Cardioid.

Now putting $n = \frac{1}{2}$ in the value of P, we get

$$P \propto \frac{1}{r^{1+3}}\quad i.e.,\ P \propto \frac{1}{r^4}.$$

EXAMPLE2: A particle describes the curve $r^n = A\,cosn\theta + Bsinn\theta$ under a force to the pole. Find the law of force.

SOLUTION: We know that the given equation is

$$r^n = A\ cosn\theta + Bsinn\theta$$

Let $A = k\cos\alpha$ and $B = k\sin\alpha$, where k and α are constants.

Then $r^n = k(\cos\alpha\cos n\theta + \sin\alpha\sin n\theta) = k\cos(n\theta - \alpha)$.

Replacing r by $1/u$, we have

$$r^n = u^{-n} = k\cos(n\theta - \alpha).$$

$$\therefore \quad -n\log u = \log k + \log\cos(n\theta - \alpha).$$

Differentiating both sides w.r.t. 'θ', we have

$$\frac{-n}{u}\frac{du}{d\theta} = -n\tan(n\theta - \alpha) \quad \text{or} \quad \frac{du}{d\theta} = u\tan(n\theta - \alpha).$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\cdot\tan(n\theta - \alpha) + un\sec^2(n\theta - \alpha)$$

$$= u\tan^2(n\theta - \alpha) + un\sec^2(n\theta - \alpha).$$

The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$P = h^2u^2[u + u\tan^2(n\theta - \alpha) + un\sec^2(n\theta - \alpha)]$$

$$= h^2u^3[\sec^2(n\theta - \alpha) + n\sec^2(n\theta - \alpha)]$$

$$= (1+n)h^2u^3\sec^2(n\theta - \alpha)$$

$$= (1+n)h^2u^3(ku^n)^2$$

$$= \frac{(1+n)h^2k^2}{r^{2n+3}}.$$

Hence $P \propto \frac{1}{r^{2n+3}}.e.,$ the force is inversely as the $(2n+3)th$ the power of the distance from the pole.

10.12 *APSE AND APSIDAL DISTANCE: -*

Apse: *An Apse is a point on the central Orbit at which the radius vector from the centre of the force to the point has a maximum and minimum value.*

Apsidal Distance: *The length of the radius vector at an Apps is called an apsidal distance*

Apsidal Angle: *The angle between two consecutive apsidal distances is called an apsidal angle.*

THEOREM: At the radius vector is perpendicular to the tangent i.e., at an apse the article moves at right angles to the radius vector.

SOLUTION: From the definition of apse, r is max. or min. at an apse $i.e.,$ $u = 1/r$ is max. or min at an apse.

$$\frac{du}{d\theta} = 0$$

But

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 = \frac{1}{r^2}$$

$$p = r \text{ or } r sin\phi = r$$

$$sin\phi = 1 \text{ or } \phi = 90°$$

Hence at the radius vector is perpendicular to the tangent i.e., at an apse the article moves at right angles to the radius vector.

10.13 *PROPERTY OF APSE LINE:-*

Theorem: *If the central acceleration P is a single valued function of the distance, every apse-lime divides the orbit into equal and symmetrical portions, and thus there can only be two apsidal distances.*

Proof: Because the central acceleration P is a single-valued function of r, the particle's acceleration remains constant at any given distance r.
The differential equation for a center orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

$$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\frac{P}{u^2}$$

Multiplying both sides by $2du/d\theta$ and integrating w.r.t.$\theta,$ we get

$$v^2=h^2\left[\frac{d^2u}{d\theta^2}+u\right]=2\int\frac{P}{u^2}du+C$$

$$v^2=C-2\int Pdr$$

The equation (1) demonstrates that if P is a single valued function of distance r, the particle's velocity remains constant at the same distance r and is independent of direction of motion.

As a result, we can see that both velocity and acceleration are constant at the same distance from the center. As a result, if the particle's velocity is reversed at an apse, it will form a symmetrical orbit on both sides of the apse line.

Now, when the particle reaches a second apse, the route is symmetrical about this second apsidal distance as well. However, this is only conceivable if the following (third) apsidal distance is equal to the previous one (first) and the angle between the first and second apsidal distances is equal to the angle between the second and third apsidal distances. As a result, if the distance's central acceleration is a single valued function, the apsidal distances are limited to two options. The apsidal angle, which is the angle formed by any two consecutive apsidal distances, is also constant.

10.14 *AREAL VELOCITY: -*

Areal velocity is defined as the rate at which the radius vector of a particle traveling under a central force sweeps away area in relation to the center of force.

$$Areal\ Velocity=\frac{dA}{dt}$$

For motion under a **central force**, the areal velocity is **constant**, i.e.,

$$\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}=constant$$

This result is known as **Kepler's second law** and follows from the **conservation of angular momentum**.

10.15 *CHARACTERISTICS OF CENTRAL ORBITS: -*

A central orbit is the path of a particle moving under a force that is always oriented toward (or away from) a fixed point known as the center of force.

Main Characteristics

- **Force acts along the radius vector:** The force is constantly directed either toward or away from the center.
- **Motion is limited to the plane:** The particle's orbit is in a fixed plane.
- **Conservation of Angular Momentum:** Angular momentum about the center is constant.
- **Constant areal velocity:** The radius vector sweeps out equal areas at regular periods of time.
- **The equation of orbit is second-order:** The second-order differential equation describes the orbit.
- **Possible orbital forms:** Depending on the type of force, the orbit can be circular, elliptical, parabolic, or hyperbolic.

SELF CHECK QUESTIONS

1. Explain why the torque is zero in central force motion.
2. State Kepler's second law and relate it to central force motion.
3. Write the differential equation of a central orbit.
4. What is meant by bounded and unbounded orbits?
5. Distinguish between circular and elliptic orbits.
6. What condition must be satisfied for a circular orbit?
7. Explain the significance of apsidal points.
8. What is the relation between force law and nature of orbit?
9. What is meant by inverse square law of force?
10. Write any two applications of central orbit theory.
11. Show that the areal velocity of a particle moving under a central force is constant.
12. Obtain the expression for angular momentum of a particle in a central orbit.
13. Show that for an inverse square law force, the orbit is a conic section.
14. Find the condition for circular orbit under a central force.
15. Derive the expression for apsidal distance.

10.16 *SUMMARY: -*

In this unit, we have studied the concept of **central force**, which always acts along the radius vector toward or away from a fixed point. We discussed the idea of a **central orbit**, describing the path traced by a particle under such a force, and showed that the motion is confined to a plane with conserved angular momentum. The **differential equation of a central orbit** was derived to determine the shape of the orbit under a given force law. We then examined the **rate of description of the sectorial area**, establishing that the areal velocity remains constant, in accordance with Kepler's second law. Further, we studied the **elliptic orbit** as an important solution for inverse square law forces, such as gravitation. Finally, the concepts of **apse**, **apsidal distance**, and their properties were explained, which describe the nearest and farthest points of the orbit from the center of force.

10.17 *GLOSSARY: -*

- **Central Force:** A force that always acts along the radius vector toward or away from a fixed point.
- **Center of Force:** The fixed point with respect to which the force on the particle is directed.
- **Central Orbit:** The path traced by a particle moving under the action of a central force.
- **Radius Vector:** The line joining the moving particle to the center of force.
- **Angular Momentum:** The moment of momentum of a particle about the center of force, which remains conserved in central force motion.
- **Areal Velocity:** The rate at which the radius vector sweeps out area with respect to time; it is constant for central orbits.
- **Differential Equation of Orbit:** A second-order differential equation that determines the shape of the orbit under a central force.
- **Sectorial Area:** The area swept by the radius vector in a given interval of time.
- **Elliptic Orbit:** A closed orbit in the shape of an ellipse formed under an inverse square central force.
- **Apse:** A point on the orbit where the particle is at its nearest or farthest distance from the center of force.
- **Apsidal Distance:** The distance of an apse from the center of force.
- **Line of Apsides:** The straight line joining the two apses of an orbit.
- **Periapsis:** The point of minimum distance of the particle from the center of force.
- **Apoapsis:** The point of maximum distance of the particle from the center of force.
- **Inverse Square Law:** A law in which the force varies inversely as the square of the distance from the center.

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10.20 *TERMINAL QUESTIONS: -*

(TQ-1): Find the law of the force towards the pole under which the curve $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is described.

(TQ-2): The velocity at any point of the central orbit is one by $(1/n)$ of words it would be for a circular orbit at the same distance show that the central force varies as $\frac{1}{r^{(2n^2+1)}}$and that the equation of the orbit is

$$r^{n^2-1} = a^{n^2-1}.cos(n^2 - 1)\theta$$

(TQ-3): A particle describes the curve $r^2 = a^2 cos2\theta$ under the force to the pole. Find the law of force.

(TQ-4): A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.

Or

A particle describes the curve $r = 2acos\theta$ under the force to the pole. Find the law of force.

(TQ-5): Find the law of the force towards the pole under which the curve $r^n = a^n cosn\theta$ can be described, if $n = 2$ then show that curve is lemniscate of Bernoulli.

(TQ-6): A particle moves with the central acceleration which varies inversely as the cube of the distance it be projected from an apse at a distance of from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a show that the equation to its path is $r\ cos\left(\frac{\theta}{\sqrt{2}}\right) = a$.

(TQ-7): A particle is projected from an apse at a distance a with the velocity from the Infinity under the action of the central acceleration $\frac{\mu}{r^{2n+3}}$. prove that the equation of the path is $r^n = a^n cosn\theta$.

(TQ-8): Explain the rate of description of sectorial area in a central orbit.

(TQ-9): Obtain the equation of an elliptic orbit under an inverse square law of force.

(TQ-10): Derive the differential equation of a central orbit.

(TQ-11): Write a detailed note on the importance of central force motion in classical mechanics.

(TQ-12): Derive the expression for time period of motion in a central orbit.

10.21 *ANSWERS: -*

(TQ-1):$P \propto 1/r^2$

(TQ-3): $P \propto 1/r^7$

(TQ-4): $P \propto 1/r^5$

(TQ-5): $P \propto 1/r^{2n+3}, P \propto 1/r^7$

UNIT 11: - THEORY OF SMALL OSCILLATION

CONTENTS:

11.1 INTRODUCTION: -

The theory of small oscillations is an important topic in classical mechanics that deals with the motion of systems slightly disturbed from their positions of stable equilibrium. Many physical systems such as pendulums, vibrating molecules, coupled springs, bridges, electric circuits, and mechanical structures exhibit oscillatory motion when displaced by small amounts.

When the displacement from equilibrium is small, the equations of motion can be simplified by neglecting higher-order nonlinear terms. This leads to linear differential equations whose solutions describe harmonic motion. The study of such motions helps in understanding stability, natural frequencies, normal modes, and resonance phenomena.

The theory of small oscillations provides a strong link between analytical mechanics, applied mathematics, and modern physics.

Many physical systems, such as pendulums, springs, or molecules, oscillate around an equilibrium position. When the displacement is very small, the motion can be explained by a linear approximation. This situation is called a small oscillation.

In the theory of small oscillations, we assume that the system is slightly disturbed from its equilibrium position and then undergoes oscillatory motion.

Equilibrium Position: Let u consider the generalized coordinates are $q_1, q_2, q_3 \ldots .. q_n$. the condition in which the potential energy is minimum is called equilibrium position.

$\frac{\partial V}{\partial q_i}$ =0 where V is potential Energy

11.2 OBJECTIVES: -

After completing this unit, the learner's will be able

1. To understand the concept of stable equilibrium.
2. To derive equations of motion for small oscillations.
3. To apply generalized coordinates in oscillatory systems.
4. To use Lagrange's equations for coupled systems.
5. To determine natural frequencies of vibration.
6. To study normal coordinates and normal modes.

11.3 EQUILIBRIUM OF A MECHANICAL SYSTEM

A system is said to be in equilibrium when all generalized forces vanish.

If $q_1, q_2, q_3 \ldots\ldots q_n$ are generalized coordinates, equilibrium occurs when: $\frac{\partial V}{\partial q_i}$ =0 where V is potential Energy the body.

11.4: - STABLE EQUILIBRIUM

The equilibrium is stable if the potential energy is minimum at the equilibrium position.

Mathematically, $\partial^2 V > O$ for small displacements.

11.4: - SMALL OSCILLATIONS ABOUT EQUILIBRIUM

Suppose the equilibrium position is: $q_i = q_i{}^0$

Let the system be slightly displaced: : $q_i = q_i{}^0 + \eta_i$

where η_i are small quantities.

11.5: -LAGRANGE'S EQUATIONS FOR SMALL OSCILLATIONS

Let the position of the system be defined by n independent co- ordinates $q_1, q_2, q_3 \ldots\ldots q_n$ and let these be so chosen that they vanish in the position of equilibrium. Then if the system makes small oscillation about this position, the coordinate will remain small through the motion and so will the velocity $\dot{q}_\iota, \dot{q}_2, \dot{q}_3 \ldots\ldots\ldots \dot{q}_n$, The kinetic energy T is given by

$$2T= a_{11} , \dot{a}_\iota{}^2 + 2a_{12}\dot{q}_\iota\, \dot{q}_2, + a_{22}\, \dot{q}_2{}^2 + \ldots\ldots\ldots a_{nn}\, \dot{q}_n{}^2 \qquad \ldots.(1)$$

Where the a's are the function of the coordinates.

We may suppose that these functions are expand in powers of the co-ordinates, and as in small oscillations, we only required the dynamical equations to be corrected to the first order of the small quantities which they contain, we may neglect the variable parts of these coefficients and take them to be constants. On the hypothesis T does not contain the coordinate explicitly and

$\frac{\partial T}{\partial q_r} = 0$ (r= 1,2,3,,,,,n)

Let v be the potential energy in a position which is only a slight departure from the equilibrium position.

Then V is a function of the co-ordinates, which can be expanded in powers of the co-ordinates thus:

$$2V = 2v_0 + 2\,c_1\,q_1 + 2\,c_2\,q_2 + \ldots\ldots\ldots\ldots\, 2\,c_n\,q_n + c_{11}{q_1}^2 + 2c_{12}\,q_1\,q_2 + \ldots\ldots + c_{nn}\,{q_n}^2 \qquad \ldots(3)$$

But since the potential energy is stationary in the position of equilibrium, we must have:

$$\frac{\partial v}{\partial q_1} = \frac{\partial v}{\partial q_2} \ldots\ldots\ldots \frac{\partial v}{\partial q_n} = 0 \qquad \ldots(4)$$

When the coordinate vanishes

So that $c_1 = c_2 = c_n = 0$

And $2V = 2v_0 + c_{11}{q_1}^2 + 2c_{12}\,q_1\,q_2 + \ldots\ldots + c_{nn}\,{q_n}^2$ …(5)

it being sufficient for our purpose to retain the quadratic term and neglect all higher powers. Further, if the position were one of unstable equilibrium, the system would not oscillate about it.

we may therefore assume that the position is one of stable equilibrium, so that $v_0$represents a minimum value of V and the quadratic terms in (5) must be positive for all values of the co-ordinates. The quadratic expression (1) for the kinetic energy is also essentially positive for all values of the velocities, and the conditions for this are that the discriminant.

$$\begin{vmatrix} a_{11} & a_{12} & . & . & a_{n1} \\ a_{21} & a_{22} & . & . & a_{n2} \\ a_{31} & a_{32} & . & . & a_{nn} \end{vmatrix}$$

And the like --determinant obtained by erasing the first row and column and repeating this process shall all be positive. We have a like set of condition for the coefficient in (5).

The typical Lagrange equation formed from (1) and (5). Taking account of (2) is

$$\frac{d}{dT} = \left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial v}{\partial q_r} = 0 \qquad (r= q,2,3, \ldots\ldots, n) \qquad \ldots(6)$$

And this gives a set of n equations.

$$a_{11}\, \ddot{q}_1 + a_{21}\, \ddot{q}_2 + \ldots\ldots + a_{n1}\, \ddot{q}_n + c_{11}\, q_{11} + c_{21}\, q_2 + \ldots\ldots + c_{n1}\, q_n = 0$$

$$a_{12}\, \ddot{q}_1 + a_{22}\, \ddot{q}_2 + \ldots\ldots + a_{n2}\, \ddot{q}_n + c_{12}\, q_{11} + c_{22}\, q_2 + \ldots\ldots + c_{n2}\, q_n = 0$$

$$a_{1n}\, \ddot{q}_1 + a_{2n}\, \ddot{q}_2 + \ldots\ldots + a_{nn}\, \ddot{q}_n + c_{1n}\, q_{11} + c_{2n}\, q_2 + \ldots\ldots + c_{nn}\, q_n = 0 \qquad \ldots(7)$$

Where these are Lagrange's equations for the small oscillations and since there are n equation of the second order their complete solution will contain 2n arbitrary constant.

Take as a trial solution

$$q_{1=}\; M_1 \sin(pt + \in), \qquad q_{2=}\; M_2 \sin(pt + \in), ,\ldots.. \; q_{n=}\; M_n \sin(pt + \in), \ldots(8)$$

By substituting (7) we get

$(a_{11}\, p^2$-$c_{11})$ M_1 + $(a_{21}\, p^2$-$c_{21})$ M_2 +…..+$(a_{n1}\, p^2$-$c_{n1})$ M_n =0

$(a_{12}\, p^2$-$c_{12})$ M_1 + $(a_{22}\, p^2$-$c_{22})$ M_2 +…..+$(a_{n2}\, p^2$-$c_{n2})$ M_n =0

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$(a_{1n}\, p^2$-$c_{1n})$ M_1 + $(a_{2n}\, p^2$-$c_{2n})$ M_2 +….+$(a_{nn}\, p^2$-$c_{nn}\,)$ M_n =0 …(9)

In order that (9) may have a solution in which not all of M_1, M_2,…, M_n are zero. We must have

$$\begin{vmatrix} a_{11}\, p^2 - c_{11} & a_{21}\, p^2 - c_{21} & \ldots\ldots\ldots\ldots & a_{n1}\, p^2 - c_{n1} \\ a_{12}\, p^2 - c_{12} & a_{22}\, p^2 - c_{22} & \ldots\ldots\ldots\ldots & a_{n2}\, p^2 - c_{n2} \\ a_{1n}\, p^2 - c_{1n} & a_{2n}\, p^2 - c_{2n} & \ldots\ldots\ldots\ldots & a_{nn}\, p^2 - c_{nn} \end{vmatrix}$$

This is an equation of the nth degree in p^2 its all roots will be real and positive provided v is essential positive. We shall confirm our attention to case in which these roots are distinct. Thus, there appear to be 2n values for p of the form $\pm\, p_1, \pm\, p_2,\ \ldots.\ \ p_n$, but we need not concern ourselves with the negative value, because a positive and negative pair exponential term in the solution of the form

Ae^{ipt} + $Be^{-\,iBt}$ which are rearrange in the form M sin(pt+∈). Hence, we confirm our consideration the positive root. Taking any one root pr and substitute in (9). We have a set of n equation for the corresponding values of the ratio of M_1, M_2,……, M_n, and they give

$$\frac{M_1}{I_1\,(pr)} = \frac{M_2}{I_2\,(pr)} = \ldots\ldots\ldots\ldots = \frac{M_n}{I_n\,(pr)} = \alpha_r \quad (\text{r=1,2,3,..n}) \qquad \ldots(11)$$

Where I_1 , I_2, ……. I_n , denote the minor of any row and column in (10). Equation (10) may be called period or frequency equations as it determines the periods.

11.6 SUMMARY: -

This unit introduces the fundamental concepts of equilibrium in classical mechanics and explains how mechanical systems behave when subjected to small disturbances. The important points covered in this chapter are summarized below:

Equilibrium of a Mechanical System: A mechanical system is said to be in equilibrium when the resultant force and the resultant moment acting on it are zero. In equilibrium, the system remains at rest or moves with constant velocity unless disturbed by an external force.

Conditions of Equilibrium: For a conservative mechanical system, equilibrium positions are obtained by setting the first derivative of the potential energy with respect to the generalized coordinates equal to zero:

Stable Equilibrium: A system is in stable equilibrium if, after a small displacement, it tends to return to its original equilibrium position. Stable equilibrium corresponds to a minimum value of potential energy, and the second derivative of the potential energy is positive:

Unstable and Neutral Equilibrium: If the potential energy is maximum, the equilibrium is unstable, whereas if the potential energy remains constant for small displacements, the equilibrium is neutral.

Small Oscillations: When a system is slightly displaced from its stable equilibrium position, it performs oscillatory motion known as **small oscillations**. The restoring force is approximately proportional to the displacement, leading to simple harmonic motion.

Linearization of Equations of Motion: For small displacements, nonlinear equations of motion can be approximated by linear differential equations using Taylor series expansion. This simplifies the analysis considerably.

The study of equilibrium provides the foundation for understanding the static behavior of mechanical systems, while the analysis of stable equilibrium and small oscillations explains how systems respond to slight disturbances. These concepts are essential for the design and analysis of stable mechanical structures, vibration control, and dynamic systems in physics and engineering.

11.7 GLOSSARY: -

Mechanical System

A collection of particles or rigid bodies whose motion is governed by the laws of mechanics.

Equilibrium

The state in which the resultant force and moment acting on a system are zero, so the system remains at rest or in uniform motion.

Static Equilibrium

Equilibrium in which the system remains at rest because all forces and moments balance each other.

Dynamic Equilibrium

Equilibrium in which a system moves with constant velocity while the net force acting on it is zero.

Conservative Force

A force for which the work done depends only on the initial and final positions, not on the path taken.

Small Oscillation

Oscillatory motion produced by a small displacement from a stable equilibrium position.

Amplitude

The maximum displacement of an oscillating particle from its equilibrium position.

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11.10 TERMINAL QUESTIONS: -

1. Discuss the equilibrium of a mechanical system. Derive the conditions for equilibrium using the principle of minimum potential energy.

2. Explain stable, unstable, and neutral equilibrium with appropriate diagrams and practical examples.

3. Derive the equation of motion for small oscillations about a stable equilibrium position and obtain the expressions for the natural frequency and time period.

3. Explain the theory of small oscillations for a conservative mechanical system and discuss its assumptions and limitations.

4.m Using Taylor's series expansion, derive the linearized equation of motion for small oscillations about an equilibrium position.

BLOCK-IV

CANONICAL FORM

UNIT 12: - SPECIAL TRANSFORMATION

CONTENTS:

12.1 *INTRODUCTION: -*

Classical mechanics, one of the foundational pillars of mathematical physics, is concerned with the motion of physical systems under the influence of forces. Its formulation, from Isaac Newton's laws to the analytical frameworks developed by Joseph-Louis Lagrange and William Rowan Hamilton, relies heavily on the concept of transformations. These transformations provide powerful tools for simplifying physical problems, revealing symmetries, and identifying conserved quantities.

In mathematical terms, a transformation in classical mechanics is a mapping between coordinate systems or phase spaces that preserves certain structural or physical properties of a system. If a system is described by generalized coordinates qi, and momenta pi, a transformation may take the form:

$$(q_i, p_i) \longrightarrow (Q_i, P_i)$$

where the new variables often provide a more convenient description of the system.

The study of special transformations is essential because many physical laws are invariant under specific classes of transformations. This invariance reflects fundamental symmetries of nature and leads directly to conservation laws, as formalized in Emmy Noether's theorem.

12.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able to

1. Understand the concept of transformations in phase space and their role in simplifying mathematical and physical problems.
2. Explain the importance of special transformations in advanced mechanics and mathematical analysis.
3. Distinguish between point transformations, contact transformations, and canonical transformations**.**
4. Derive the general form of canonical transformations using Hamiltonian mechanics**.**
5. Express transformation using Poisson brackets and verify canonical condition.

12.3 *ROLE OF TRANSFORMATIONS IN CLASSICAL MECHANICS*

There are some types of transformation which serve several key purposes

(i) **Simplification of Equations of Motion:** By choosing appropriate coordinates, complex dynamical systems can often be reduced to simpler forms.

(ii) **Identification of Symmetries:** Transformations reveal invariance properties of physical systems, such as rotational or translational symmetry.

(iii) **Connection Between Reference Frames:** Physical laws must retain their form under transformations between inertial frames (Galilean transformations).

(iv) **Transition to Advanced Formulations:** Canonical transformations enable the transition from Lagrangian to Hamiltonian mechanics.

12.4 *SPECIAL TRANSFORMATIONS IN CLASSICAL MECHANICS*

(i) Galilean Transformations: Galilean transformations describe the relationship between two inertial frames moving with constant relative velocity.

If one frame moves with velocity v along the x-axis, the transformation is:

$x' = x - v\,t,\ y' = y,\ z' = z,\ t' = t.$

(ii) Rotational Transformations: Rotations change the orientation of the coordinate system while preserving distances and angles.

In matrix form:

$$r' = Rr,$$

where R is an orthogonal matrix.

(iii) Canonical Transformations: For solving many complex problems in mechanics, it is often found to be advantageous to transform from one set of generalized coordinates $q_1, q_2, \ldots,$ qn and generalized momenta p_1, p_2 pn on another set of variables $Q_1, Q_2, \ldots, Q_n$ and $p_1, p_2, \ldots, p_n$

A transformation of the type

$\vec{Q} = \vec{Q}(\vec{q}, \vec{p}), \qquad \vec{P} = \vec{P}(\vec{q}, \vec{p}) \qquad (1)$

is called a canonical transformation, if and only if there exists a function of the type F ($\vec{q}$, $\vec{p}$) satisfying the property.

dF (q, p) = $\sum$ P p d q p - $\sum$n P p (q, p) (2)

However, a time-dependent transformation

Q=Q (q, p, t), P=P (q,p,t)

is said to be a canonical transformation if and only if any one of the following three equivalent properties is found to be true:

(i) The transformation is canonical in the sense of equation (2)

if dF ($\vec{q}$, $\vec{p}$) = $\sum_{p=1}^{n} p_p \, d \, q_p$ - $\sum_{p=1}^{n} p_p \, d \, Q_p$ ($\vec{q}$, $\vec{p}$) for every value of time.

(ii) There exists a function F($\vec{q}$,$\vec{p}$,t) such that for every arbitrary fixed time, say t=t_0 , satisfying

d F (q, p, t) = = $\sum_{p=1}^{n} p_p \, d \, q_p$ - $\sum_{p=1}^{n} p_p$ (q, p,t_0) d Q p (q, p, t_0) (4)

12.5 POINT TRANSFORMATION

A transformation is called a point transformation when:

(i) New coordinates depend only on old coordinates and time.

(ii) They do **not** depend on velocities or momenta.

Thus Q_i = Q_i (q, t)

In another word a point transformation is a transformation in which the new generalized coordinates are expressed only as functions of the old generalized coordinates and time.

The importance of point transformation are Simplifies equations of motion, helps choose convenient coordinates, used in constrained systems, important in Lagrangian and Hamiltonian mechanics and converts one coordinate system into another.

Theorem: Suppose the configuration of a dynamical system is defined by the generalized coordinates $q_1, q_2, \ldots, q_n$, whose behavior under the action of a given force is described by the Hamiltonian H ($\vec{q}, \vec{p}$, t) with the generalized momenta $\vec{p}$. If a new set of generalized coordinates $Q_1, Q_2, \ldots, Q_n$ are introduced which are related to $\vec{q}$ by the point-transformation

$Q_j = Q_j(\vec{q}, t)$ **...(1)**

then the corresponding momenta $\vec{P}$ and the Hamiltonian K are given by

$P_j = \sum_{p=1}^{n} p_p \ (\partial q_p(\vec{Q}, t) / \partial Q_j)$ **...(2)**

and $K = H - \sum_{p=1}^{n} p_p \ \partial q_p(\vec{Q}, t) / \partial t$ **...(3)**

Solution: For a dynamical system with n degree of freedom, the Hamiltonian is defined as

$H = \sum_{p=1}^{n} \dot{q}_p \, p_p - L$ or $L = \sum \dot{q}_p p_p - H$

We have already observed the lagrangian is invariant under point transformation.

So we can write $\sum p_p \dot{q}_p - H = \sum p_p \dot{Q}_p - k$

⇨ $\sum_{p=1}^{n} p_p \frac{d\,q_p}{dt} - H = \sum p_p \frac{d\,q_p}{dt} - K$

⇨ $\sum_{p=1}^{n} p_p \ d\,q_p - H\,dt = \sum P\,p\,d\,q\,p - k\,dt$ …(4)

But form calculas we have

$$d\,q_p = \sum_{j=1}^{n} \partial q_p \frac{(\vec{Q,t})}{\partial Q_i} \, dQ_j + \frac{\partial q_p(\vec{Q,t})}{dt} \, dt \qquad \ldots (5)$$

Substituting equation (5) in equation (4)

$$\sum_{p=1}^{n} p_p \left[\sum_{j=1}^{n} \partial q_p \frac{(\vec{Q,t})}{\partial Q_j} \, dQ_i + \frac{\partial q_p(\vec{Q,t})}{dt} \, dt\right] - H\,dt$$

$= \sum_{j=1}^{n} p_j \ d\,q_{Q_j} - K\,dt$

⇨ $\sum_{p=1}^{n} \sum_{j=1}^{n} p_p \ d\,q_p \frac{(\vec{Q,t})}{\partial Q_j} \, dQ_j + = \sum_{p=1}^{n} p_p \frac{\partial q_p(\vec{Q,t})}{dt} \, dt - H\,dt$

$= \sum_{j=1}^{n} p_j \ d\,q_{Q_j} - K\,dt$ …(6)

Equating the coefficient of dQ_i and dt on both side of equation (6) we get

$$p_j = \sum_{j=1}^{n} p_p \text{ d } q_p \frac{(\overrightarrow{Q,t})}{\partial Q_j}$$

$$And \ \ \text{K} = \text{H} - \sum_{p=1}^{n} \ p_p \frac{\partial q_p \, (\overrightarrow{Q,t})}{dt} \ \ \text{dt}$$

Note : It may be noted that if the point transformation from $\vec{q} \ to \ \vec{Q}$ is independent of time then K= H, while if the transformation depend on time the K ≠ H.

12.6 :- SUMMARY: -

In classical mechanics, transformations are used to recast the description of motion into new variables (coordinates and momenta) while keeping the physical content unchanged. They are central to both Lagrangian and Hamiltonian mechanics and simplify the equations of motion, reveal symmetries, and help find conserved quantities. transformations allow one to change from one set of generalized coordinates (and momenta) to another, often making the equations easier to solve. In addition to general canonical transformations, special classes such as point transformations, coordinate transformations, and symmetry transformations (e.g., translations, rotations) are particularly important. A point transformation is a special kind of coordinate change in configuration space that re-expresses the generalized coordinates q_i .as new coordinates Qi(q,t) without introducing extra degrees of freedom.

12.7:- GLOSSARY: -

Canonical Transformation: A change of canonical coordinates (q,p)→(Q,P) that preserves the form of Hamilton’s equations.

Point Transformation: A special canonical transformation in which only the generalized coordinates in configuration space change, qi→Qi(q,t) while the momenta are updated consistently via the Legendre transform so that the equations of motion stay covariant.

Configuration Space: The space of all possible generalized coordinates q_i of a system point transformations act on configuration space.

Phase Space: The space spanned by all generalized coordinates qi and their conjugate momenta pi canonical transformations are transformations of phase space.

Restricted Canonical Transformation: A canonical transformation that does not depend explicitly on time; many textbook treatments focus only on this sub-class.

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12.10 :- TERMINAL QUESTIONS: -

1. Distinguish between point transformation and general canonical transformation in classical mechanics, with examples.,

2. Define a Point Transformation and explain why it is a special case of a canonical transformation.
3. Show that a point transformation preserves the form of the Euler–Lagrange equations.
4. Give one physical example where point transformation simplify the equation of motion

UNIT 13: - LAGRANGE AND POISSON BRACKETS

CONTENTS:

13.1 *INTRODUCTION: -*

In advanced classical mechanics, particularly within the domain of Analytical Mechanics, the notions of Lagrange brackets and Poisson brackets are essential for the study of canonical transformations and the geometric structure of phase space. These mathematical constructs facilitate the transition from the Lagrangian formulation to the Hamiltonian framework and serve as an important link to modern developments in theoretical physics.

In a system with n degrees of freedom, the state of the system is described by: Generalized coordinates: p_i , Generalized momenta: q_i

The pair (p_i, q_i) defines a point in the phase space**,** which is the natural setting for Hamiltonian Mechanics.

13.2 *OBJECTIVES: -*

After studying this unit, the learner's will be able

1. To understand the concept of generalized coordinates and momenta in analytical mechanics.
2. To explain the origin and significance of Lagrange brackets and Poisson brackets.
3. To define Lagrange brackets and derive their mathematical expressions.
4. To define Poisson brackets and understand their algebraic structure.

13.3 *LAGRANGE BRACKETS*

Lagrange brackets arise when we consider transformations between two sets of generalized coordinates:

$(q_i, p_i) \rightarrow (Q_i, P_i)$

Let u and v be any two members of the set of variables $p_1, p_2, p_3 \ldots\ldots\ldots p_n$ and $q_1, q_2, q_3 \ldots\ldots\ldots\ldots q_{n,}$ then the Lagrange bracket associated with the transformation

$\vec{Q}=\vec{Q}(\vec{q}, \vec{p},t)$, $\vec{P}= \vec{P}(\vec{q}, \vec{p},t)$ ------ (1)

denoted by $[u, v]$ is defined as

$[u, v]= \sum_{i=1}^{n}\left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$ ……(2)

Theorem 13.3.1: The transformation $\vec{Q}=\vec{Q}(\vec{q}, \vec{p},t)$, $\vec{P}= \vec{P}(\vec{q}, \vec{p}, t)$ …(1) is a canonical transformation if and only if the Lagrange brackets $[q_i, q_j]$, $[q_i, p_j]$, $[p_i, p_j]$ satisfy $[q_i, q_j] = 0$, $[q_i, p_j]= \partial_{ij}$, $[p_i, p_j]=0$…(2)when ∂_{ij} is a kroncker delta with the property ,

$$\partial_{ij}=\begin{cases}1, i = j \\ 0, i \neq j\end{cases} \quad \ldots (3)$$

Solution: Let us consider that t is a constant, then the transformation (1) canonical if and only if, there exist a function f such that

df= $\sum_j p_j dq_j$- $\sum_p p_p dQ_p$

df= $\sum_j p_j dq_j$ - $\sum_p \left(\sum_j \frac{\partial Q_p}{\partial_{q_j}} d_{q_j} + \sum_j \frac{\partial Q_p}{\partial_{p_j}} d_{p_j} \right)$

$$\Rightarrow df = \sum_j \left(p_j - \sum_k P_k \frac{\partial Q_k}{\partial q_j} \right) dq_j + \sum_j \left(-\sum_k P_k \frac{\partial Q_k}{\partial p_j} \right) dp_j \quad (4)$$

Now, we define $A_j = p_j - \sum_k p_k \frac{\partial Q_k}{\partial q_j}$
(5)

And $B_j = -\sum_k p_k \frac{\partial Q_k}{\partial p_j}$ (6)

So that eqn (4) becomes,

$$df = \sum_j A_j dq_j + \sum_j B_j d\beta_j \quad (7)$$

But the necessary and sufficient conditions that there exist a function f such that eqn (7) is a perfect differential are given by

$$\frac{\partial A_j}{\partial q_i} = \frac{\partial A_i}{\partial q_j} \quad (8)$$

$$\frac{\partial A_j}{\partial p_i} = \frac{\partial B_i}{\partial q_j} \quad (9)$$

$$\frac{\partial B_j}{\partial p_i} = \frac{\partial B_i}{\partial p_j}$$

Substituting $eq^n(5)$ into $eq^n(8)$,

we get $\frac{\partial}{\partial q_i}\left(p_j - \frac{\delta_R}{R} p_R \frac{\partial Q_k}{\partial q_j} \right) = \frac{\partial}{\partial q_j}\left(p_i - \sum_R P_R \frac{\partial Q_k}{\partial q_i} \right)$

$$\Rightarrow \frac{\partial p_j}{\partial q_i} - \frac{\sum_k}{} \frac{\partial p_k}{\partial q_j} - \frac{\sum_k}{} p_k \frac{\partial^2 Q_k}{\partial q_i \partial q_j} = \frac{\partial p_i}{\partial q_j} - \frac{\sum_k}{} \frac{\partial Q_k}{\partial q_i} - \frac{\sum_k}{} \frac{\partial^2 Q_k}{\partial q_j \partial q_i}$$

$$\Rightarrow -\frac{\sum_k}{} \frac{\partial p_k}{\partial q_j} - \sum_k p_k \frac{\partial^2 Q_k}{\partial q_i \partial q_j} = -\frac{\sum_k}{} \frac{\partial Q_k}{\partial q_i} - \frac{\sum_k}{} \frac{\partial^2 Q_k}{\partial q_j \partial q_i}$$

$$\Rightarrow -\sum_k \frac{\partial p_k}{\partial q_i} \frac{\partial Q_k}{\partial q_j} = -\frac{\sum_k}{} \frac{\partial p_k}{\partial q_i}$$

$\Rightarrow \frac{s}{k} \frac{\partial P_k}{\partial q_j} \frac{\partial Q_k}{\partial q_i} - \frac{s}{k} \frac{\partial P_k}{\partial q_i} \frac{\partial Q_k}{\partial q_j} = 0$ (Franforming)

$$\Rightarrow \frac{j}{k} \frac{\partial Q_k}{\partial q_i} \frac{\partial P_k}{\partial q_j} - \frac{S}{K} \frac{\partial Q_k}{\partial q_j} \frac{\partial P_R}{\partial q_i} = 0 \qquad (11)$$

Which is just the Lagrange's bracket $[q_i, q_j]$ and thus not obtained
$[q_i, q_j] = 0$ (12)

Similarly, substituting (5) into (9) and (6) into (10) we get. $[q_i, p_j] = \delta_{ij}$ and $[p_i, p_j] = 0$

Note : Lagrange bracket does not hold commutative law.

$$[u, v] = \sum_i \left(\frac{\partial Q_i}{\partial u} \frac{\partial P_i}{\partial v} - \frac{\partial Q_i}{\partial v} \frac{\partial p_i}{\partial u} \right)$$
$$= -\sum_i \left(\frac{\partial Q_i}{\partial v} \frac{\partial P_i}{\partial u} - \frac{\partial Q_i}{\partial u} \frac{\partial P_i}{\partial v} \right)$$
$$= -[v, u]$$

13.4 POISSON BRACKETS

Poisson brackets are a fundamental concept in Hamiltonian mechanics, used to describe the relationships between dynamical variables in phase space. For two functions f(q,p,t) and g(q,p,t) the Poisson bracket measures how one quantity changes with respect to another within the system.

They satisfy important properties such as linearity, antisymmetric, the Jacobi identity, and the Leibniz rule, which give them a strong algebraic structure similar to a Lie algebra. The fundamental Poisson brackets between coordinates and momenta form the basis of canonical mechanics.

Let $f = f(\mathrm{q}, \mathrm{p}, t)$ be some general function on phase space. Then along the trajectories of motion,

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t}$$

We define the quantity

$$[H, f] = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}\right),$$

as the Poisson bracket of H and f (NB: the sign (which is entirely a matter of convention) has been reversed as to the presentation in the lectures). Generally, we define the Poisson bracket of f and g as

$$[f, g] = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Note $[f, g] = -[g, f]$. If f is not an explicit function of t, we then have

$$\frac{df}{dt} = [f, H]$$

and therefore f is a conserved quantity if the Poisson bracket of H and f vanishes and therefore f is a conserved quantity if the Poisson bracket of H and f vanishes. This should ring bells with your knowledge of quantum mechanics, where conserved quantities in quantum mechanics are those which commute with the Hamiltonian, and also with Ehrenfest's theorem:

$$\frac{d}{dt}\langle O\rangle = \langle [O, H]\rangle + \left\langle\frac{\partial O}{\partial t}\right\rangle,$$

where these are now the commutators of quantum mechanics.
Let us enumerate some properties of Poisson brackets:

$$[f, g] = -[g, f],$$
$$[f_1 + f_2, g] = [f_1, g] + [f_2, g],$$
$$[f_1 f_2, g] = f_1[f_2, g] + f_2[f_1, g],$$
$$\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right].$$

Also note that

$$[q_k, f] = \frac{\partial f}{\partial p_k}$$
$$[p_k, f] = \frac{\partial f}{\partial q_k}$$

We therefore have

$$[q_i, q_j] = [p_i, p_j] = 0$$
$$[q_i, p_j] = \delta_{ij}$$

relationships which should look very familiar from quantum mechanics. It can also be shown that (the proof is not difficult, just a little fiddly)

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

This relationship is called Jacobi's identity.
If it is not already blindingly obvious, then let us state explicitly that in Poisson brackets you see the classical precursors of the commutators of quantum mechanics.

Poisson brackets play an important role in constructing conserved quantities in Hamiltonian mechanics. Suppose f and g are constants of the motion, so

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} = 0$$
$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} = 0$$

Then $[f, g]$ is also a conserved quantity. To prove this, note that

$$\frac{d}{dt}[f, g] = [[f, g], H] + \frac{\partial}{\partial t}[f, g]$$

From the Jacobi identity,

$$[[f,g],H] = [f,[g,H]] + [g,[H,f]]$$

and from the general properties of Poisson brackets $\frac{\partial}{\partial t}[f,g] = \left[\frac{\partial f}{\partial t},g\right] + \left[f,\frac{\partial g}{\partial t}\right]$. It then follows that

$$\frac{d}{dt}[f,g] = \left[\frac{\partial f}{\partial t},g\right] + \left[f,\frac{\partial g}{\partial t}\right] + [f,[g,H]] + [g,[H,f]]$$

$$= \left[\frac{\partial f}{\partial t} - [H,f],g\right] + \left[f,\frac{\partial g}{\partial t} - [H,g]\right]$$

$$= \left[\frac{df}{dt},g\right] + \left[f,\frac{dg}{dt}\right]$$

$$= 0$$

and so $[f,g]$ is a conserved quantity.
We can therefore use Poisson brackets to generate new conserved quantities from old: the Poisson brackets of two known conserved quantities will generate another, potentially new, conserved quantity. As there are only a finite number of conserved quantities, this of course cannot go on forever, but it may still both generate new conserved quantities and illuminate the relationship between existing ones.

As an example, let us compute the Poisson brackets of the angular momentum operators $L_i = \epsilon_{ijk} r_j p_k$. This is an exercise in being careful with summation convention and Poisson brackets. We have

$$\begin{aligned}
[L_i, L_j] &= [\epsilon_{ilm} r_l p_m, \epsilon_{jqr} r_q p_r] \\
&= \epsilon_{ilm}\epsilon_{jqr}[r_l p_m, r_q p_r] \\
&= \epsilon_{ilm}\epsilon_{jqr}\big(p_m[r_l, r_q p_r] + r_l[p_m, r_q p_r]\big) \\
&= \epsilon_{ilm}\epsilon_{jqr}\big(p_m r_q[r_l, p_r] + p_m p_r[r_l, r_q] + r_l r_q[p_m, p_r] + r_l p_r[p_m, r_q]\big) \\
&= \epsilon_{ilm}\epsilon_{jqr}\big(p_m r_q \delta_{lr} - r_l p_r \delta_{mq}\big) \\
&= \epsilon_{ilm}\epsilon_{jql} p_m r_q - \epsilon_{ilm}\epsilon_{jmr} r_l p_r \\
&= \big(\delta_{mj}\delta_{iq} - \delta_{mq}\delta_{ji}\big) p_m r_q - \big(\delta_{ir}\delta_{lj} - \delta_{ij}\delta_{rl}\big) r_l p_r \\
&= r_i p_j - r_j p_i \\
&= \epsilon_{ijk} L_k \quad (4.41)
\end{aligned}$$

13.5: - SUMMARY

Lagrange brackets and Poisson brackets are important concepts in advanced classical mechanics, especially in the study of canonical transformations and Hamiltonian dynamics. They provide powerful mathematical tools for describing the motion of dynamical systems in generalized coordinates and momenta.

The **Lagrange bracket** is defined in terms of old coordinates and momenta with respect to new variables. It helps determine whether a transformation between two sets of canonical variables preserves the structure of Hamilton's equations. Lagrange brackets are mainly used in the theory of canonical transformations.

Poisson brackets are widely used to express equations of motion, conservation laws, and symmetry principles.

13.6: - GLOSSARY

Analytical Mechanics – A branch of mechanics that uses mathematical methods such as Lagrangian and Hamiltonian formulations to study motion.

Canonical Coordinates – Generalized coordinates qi and corresponding momenta pi used in Hamiltonian mechanics.

Canonical Transformation – A transformation from one set of canonical variables to another that preserves the form of Hamilton's equations.

Generalized Coordinates – Independent variables used to describe the configuration of a mechanical system.

Lagrange Bracket – A quantity used in canonical transformation theory, defined between two variables in terms of old coordinates and momenta.

Symmetry – A property of a system that remains unchanged under certain transformations.

Time Evolution – Change of a physical quantity with time, often expressed using Poisson brackets.

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13.9: - TERMINAL QUESTIONS: -

1. Define Lagrange brackets and explain their mathematical form.

2. What are Poisson brackets? Derive the general expression for Poisson brackets of two functions.

3. State and prove the important properties of Poisson brackets.

4. Explain the relation between Lagrange brackets and Poisson brackets.

5. Distinguish between Lagrange brackets and Poisson brackets.

UNIT 14: - MOTION OF TOP

CONTENTS:

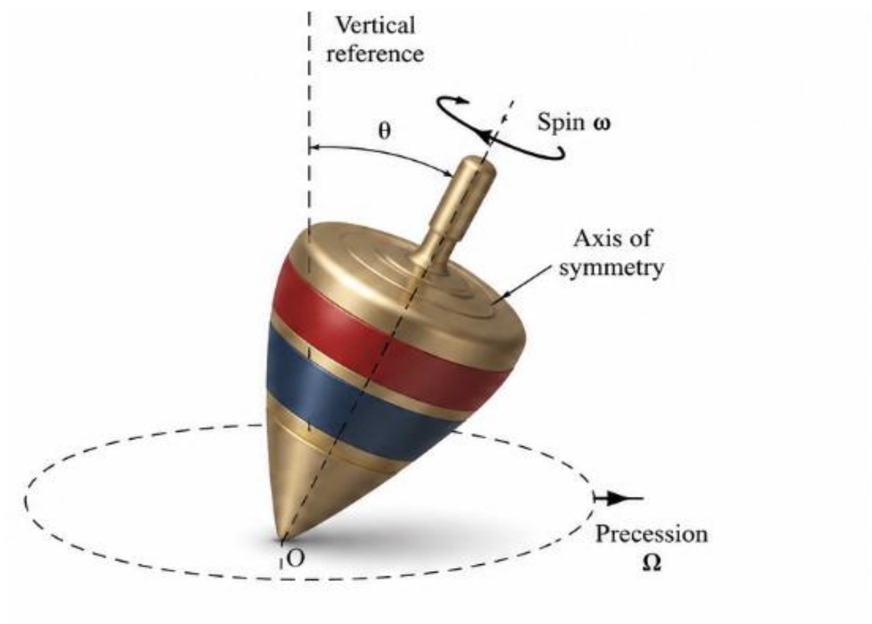
14.1 *INTRODUCTION: -*

The motion of a top is one of the most important applications of the dynamics of rigid bodies. A top is a rigid body that rotates about a fixed point and possesses an axis of symmetry. Unlike the motion of a particle, the motion of a top involves both translation of orientation and rotation, making it an ideal example for understanding three-dimensional rotational dynamics.

When a top is set into motion, it spins rapidly about its own axis. If the axis of rotation is inclined to the vertical and the top is subjected to an external force such as gravity, its axis does not remain fixed. Instead, it undergoes precession, a slow rotation of the axis about the vertical direction. Under certain conditions, the axis also exhibits a small oscillatory motion known as nutation. Thus, the motion of a top generally consists of three simultaneous motions:

1. Spin about its own axis,
2. Precession of the axis around the vertical,
3. Nutation, the oscillation of the inclination angle.

The study of the motion of a top provides an excellent illustration of the principles of rigid body dynamics, including Euler's equations **of** motion, angular momentum, moment of inertia, and torque. It explains how external forces influence the rotational motion of a body and why rapidly spinning bodies exhibit remarkable stability.



The theory of the top forms the foundation for understanding the behaviour of gyroscopes, spinning satellites, spacecraft attitude control systems, navigation instruments, and many engineering and aerospace applications. It also has significant applications in robotics, mechanical engineering, and modern control systems.

In this unit, we shall:

- **(i)** Introduce the concept of a symmetric top and its characteristics.
- **(ii)** Derive the equations governing the motion of a top.
- **(iii)** Study the concepts of spin, precession, and nutation.
- **(iv)** Analyze the motion of a heavy symmetric top under the action of gravity.
- **(v)** Discuss the conditions for steady precession and special cases of motion.
- **(vi)** Examine the practical applications of gyroscopic motion in science and engineering.

Understanding the motion of a top not only strengthens the concepts of rotational dynamics but also provides insight into many natural phenomena and technological systems involving rotating bodies.

14.2 *OBJECTIVES: -*

After studying this unit, students will be able to:

1. Define a top and distinguish its different types.
2. Explain spin, precession, and nutation.
3. Apply Euler's equations to analyse the motion of a rigid body.
4. Determine the conditions for steady precession.
5. Solve numerical problems related to the dynamics of a symmetric top.
6. Recognize the applications of gyroscopic principles in engineering, navigation, and aerospace systems.

14.3 EQUATION OF MOTION OF A TOP (DERIVED FORM EULER'S EQUATIONS

A **top** is a rigid body terminating in a sharp point, called the apex. It is a rotating rigid body with one point fixed in space. A symmetric top possesses an axis of symmetry in its mass distribution; therefore, both the centre of mass and the fixed point lie on the axis of symmetry.

We assume that the top is spinning with its apex in contact with a perfectly rough horizontal plane in a uniform gravitational field such that slipping at the point of contact is prevented. Consequently, the motion takes place about a fixed-point coinciding with the point of contact.

Suppose a top spins with its vertex O fixed in contact with a floor rough enough to prevent slipping. Let OC be the axis of the top and G its centre of gravity where $OG = h$.

Let Ox, Oy, Oz be the fixed axes and OA, OB, OC the principal axes at O. As usual let C denote the moment of inertia about OC and A the moment of inertia about OA, and as the top is symmetrical about OC, therefore, $B = A$.

At zero time, OC is in the plane zOx. At time t let OC be inclined at an angle θ to Oz and the plane zOC making the angle ψ with the fixed plane zOx. Let $\omega_1, \omega_2, \omega_3$ be the angular velocities of the top about OA, OB, OC respectively, and L, M, N the moments of the external forces about these axes. The external forces acting on the top are its weight mg acting vertically downwards through G in the direction parallel to zO; and the reactions acting at O.

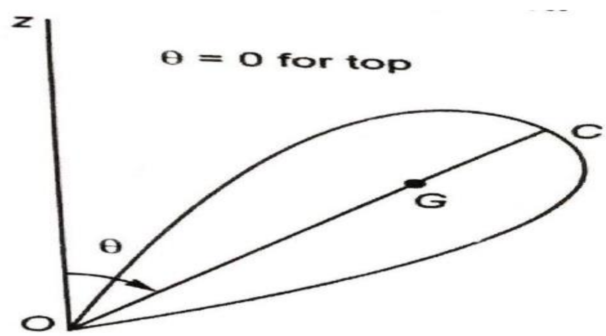


Fig.

The direction cosines of Oz with respect to OA, OB, OC are (§ 9.8 page 388). or

$$-\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$$
$$-\sin\theta, 0, \cos\theta$$

as in case of top$\phi = 0$
Hence, the weight $(-mg)$, acting at $G(0,0,h)$, has its resolved parts $mg\sin\theta, 0, -mg\cos\theta$ parallel to OA, OB, OC respectively.

The reaction act at the origin 0(0,0,0).

Hence using formula L= $\sum(y_1Z_1 - z_1Y_1)$ and similar formulae for M,N, we find that L= 0, M = h mg sin θ , N = 0.

Hence Euler's Equations are

$$A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 = 0$$
$$B\dot{\omega}_2 - (C-A)\,\omega_3\omega_1 = h\,mg\sin\theta$$
$$C\dot{\omega}_3 - (A-B)\,\omega_1\omega_2 = 0.$$

Also we are familiar with Euler's geometrical relations

$$\omega = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi, \quad \omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi, \quad \omega_3 = \dot{\phi} + \dot{\psi}\cos\theta$$

which reduce to $\omega_1 = -\dot{\psi}\sin\theta, \omega_2 = \dot{\theta}; \omega_3 = \dot{\psi}\cos\theta$, because for top $\phi = 0$.

Here, $B = A$, therefore, Euler's third equation gives

$$\dot{\omega}_3 = 0 \quad \text{or} \quad \omega_3 = n\,(\text{constant}), \text{ or } \dot{\psi}\cos\theta = n$$

And substituting these values of $\omega_1, \omega_2, \omega_3$

Euler's first equation $\qquad A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 = 0$

becomes $\qquad A\frac{d}{dt}[-\dot{\psi}\sin\theta] - (A-C)\,\dot{\theta}\dot{\psi}\cos\theta - = 0 \qquad$ as $B = A$

or $\qquad -A\ddot{\psi}\sin\theta - 2A\,\dot{\theta}\dot{\psi}\cos\theta + C\,\dot{\theta}\psi\cos\theta = 0$

$$-A\ddot{\psi}\sin\theta - 2A\,\dot{\theta}\dot{\psi}\cos\theta + C\,n\dot{\theta} = 0 \qquad \ldots(1)$$

And Euler's second equation

$$B\dot{\omega}_2 - (C-A)\,\omega_3\omega_1 = mgh\sin\theta$$

becomes $\qquad A\frac{d}{dt}(\dot{\theta}) + (C-A)\,\dot{\psi}^2\sin\theta\cos\theta = mgh\sin\theta \qquad$ as $B = A$

or $\qquad A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + C\dot{\psi}^2\sin\theta\cos\theta = mgh\sin\theta$

or $\qquad A\ddot{\theta} + A\dot{\psi}^2\sin\theta\cos\theta + C\,n\dot{\psi}\sin\theta = mgh\sin\theta \qquad \ldots(2)$

Equations (1) and (2) are the equations of motion of the top and are of second order. Further, we proceed to deduce from them the first order equations.

Multiply equation (1) by Sin θ and integrating, we get

$$A\dot{\psi}\sin^2\theta + C\,n\cos\theta = D\,(\text{constant}) \qquad \ldots(3)$$

Also, multiply Eq. (1) by $2\dot{\psi}\sin\theta$ and Eq. (2) by $2\dot{\theta}$ and subtract, then we get

$$2A\ddot{\theta}\,\dot{\theta} + 2A\,\dot{\theta}\,\dot{\psi}^2\sin\theta\cos\theta + 2A\,\dot{\psi}\,\ddot{\psi}\sin^2\theta = 2mgh\sin\theta$$

Integrating. $\qquad A\,(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + 2\,mgh\cos\theta = E\,(\text{const}) \qquad \ldots(4)$

Equation (3) and (4) are also the equations of motion of the top but are of the first order. The equations (1), (2), (3), (4) are known as the equations of motion of the top. While equations (1) (2) are of the second order, equation (3) and (4) are of the first order.

The motion due to change in θ called nutation and due to Ψ change in is called precession. The general motion of the top about its fixed vertex o is a combination of these two.

14.4: -EQUATION OF MOTION OF A TOP (DEDUCED FROM LAGRENGE'S EQUATIONS

In case of the motion of a top B= A, Ø =0 and

w_1 = -$\dot{\Psi}$ sin θ, w_2 = $\dot{\theta}$, w_3 = n $\dot{\Psi}$ cos θ.

Hence the Kinetics energy is

$$T = \frac{1}{2} m [A(\omega_1^2 + \omega_2^2) + C\omega_3^2] \qquad \text{since } B = A$$

$$= \frac{1}{2} m [A(\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2) + C\dot{\psi}^2 \cos^2\theta] \qquad \text{by the above.}$$

And the work function w is

W=- mgh cos θ.

Where h is distance of CG from the vertex O and θ is the inclination of OC to the vertical Oz.

Here, the generalized co-ordinates are θ and Ψ.

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$ which gives

$$\frac{d}{dt}(mA\dot{\theta}) - \{mA\dot{\psi}^2 \sin\theta\cos\theta - C\dot{\psi}^2 \cos\theta\sin\theta\} = mgh\sin\theta$$

or
$$A\ddot{\theta} - A^2\dot{\psi}\sin\theta\cos\theta + C\, n\dot{\psi}\sin\theta = mgh\sin\theta \qquad \ldots(1)$$

And Lagrange's ψ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \frac{\partial W}{\partial \psi}$

which gives
$$\frac{d}{dt}(mA\dot{\psi}\sin^2\theta + C\dot{\psi}\cos^2\theta) = 0$$

or
$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \text{ (const.)} \qquad \ldots(2)$$

Equation (1) and (2) are familiar equations of the motion of a top.

14.5: - EQUATION OF MOTION OF A TOP (DUDUCED FROM THE PRINCIPLE OF ENERGY AND MOMENTUM)

In case of motion of a top B = A and Ø = 0 hence,

w_1 = $-\dot{\Psi}$ sin θ , $w_2 = \dot{\theta}$, w_3 = n $\dot{\Psi}$ cos θ.

K.E = $\frac{1}{2}\left[(w_1{}^2 + w_2{}^2) + Cw_3{}^2\right]$ Since B = A

$\frac{1}{2}$ $[A(\dot{\Psi}^2 sin^2$ θ +θ ^{2}) + $Cn^2]$ by the above

Potential energy y mgh cos θ where h is the distance of C.G from the vertex O and (1) inclination of OG to verticle. By the principle of energy, sum of the kinetic and the potential energy remains constant throughout motion. Hence, by the principle of energy we must have

$\frac{1}{2}$ $[A(\dot{\Psi}^2 sin^2$ θ +θ ^{2}) - $Cn^2]$ + m g h cos θ = constant

A ($\dot{\theta}^2$+ $\dot{\Psi}^2 sin^2$ θ) + 2 m g h cos θ = E … (1)

Again, we notice that the external force weight is parallel to OZ and the reaction at O cut Oz ; so that sum of their moments about Oz is zero. Hence, by the principle of momentum about Oz is same throughout the motion .

Aw_1, Aw_2, Cw_3 are the moments of momentum of the top about OA, OB, OC respectively; and the direction cosine of OZ about these axes are

\- Sin θ, 0, cos θ.

Resolving this momentum about Oz, we have moment of momentum about Oz as

- Aw_1Sin θ + Cw_3 cos θ.

Hence, by principle of momentum we must have,

- Aw_1Sin θ + Cw_3 cos θ = constant.

- A Sin θ ($\dot{\Psi}$ sin θ) + Cn cos θ = constant, by the above.

$A\,\dot{\Psi} sin^2 \ + Cn$ cos θ = D constant … (2)

Equation (1) and (2) are familiar equations of motion of a top.

14.6: - STEADY MOTION

The motion of a top to be steady if it spings about its vertex O in such a way that its axis OC makes the same angle with the vertical Oz throughout the motion.

Thus, in case of steady motion θ = α (constant); hence , from equation

$\dot{\Psi}$ cos θ = n . We have $\dot{\Psi}$ = w (constant), showing that the angular velocity about the vertical Oz is constant.

So, when the motion is steady.

Putting θ = α, $\ddot{\theta}$ = 0 and $\dot{\Psi}$ = w in equation

A$\ddot{\theta}$ – A $\dot{\Psi}^2$ Sin θ cos θ + C n $\dot{\Psi}$ sin θ- m g h Sin θ =0

$we\ ahve\ \ \omega^2$A cos α – C n ω + m g h = 0 …(1)

giving $\omega =$ Cn $\pm \frac{\sqrt{C^2n^2 - 4\,A\,m\,g\,h\,cos\,\alpha}}{2Acos\alpha}$

Thus, in general there are two values of the processional velocity ω; the two processional velocities, call them ω_1, ω_2 are real and distinct provided that $C^2n^2 > \ 4\,A\,m\,g\,h\,cos\,\alpha$

This is the necessary condition for the steady motion to exist.

14.7 : - SUMMARY

This unit introduces the dynamics of a rotating rigid body (top) and derives its equations of motion using different analytical approaches. The principal objective is to understand the rotational behavior of a top under the action of external forces, particularly gravity.

Equation of Motion of a Top (Derived from Euler's Equations)

Euler's equations describe the rotational motion of a rigid body about a fixed point in terms of its principal moments of inertia and angular velocity components. By applying Newton's second law for rotational motion, the equations of motion of a top are obtained. These equations explain how the angular velocity changes under the influence of external torque and form the basis for studying gyroscopic motion.

Equation of Motion of a Top (Derived from Lagrange's Equations)

The Lagrangian approach provides an elegant and systematic method for deriving the equations of motion. By expressing the kinetic and potential energies of the top in terms of generalized coordinates (Euler angles) and applying Lagrange's equations, the motion of the top can be described without directly considering constraint forces. This method is particularly useful for complex mechanical systems.

Equation of Motion of a Top (Deduced from the Principle of Energy and Angular Momentum)

The equations of motion can also be obtained using the conservation principles of energy and angular momentum. In the absence of non-conservative forces, the total mechanical energy remains constant, while the angular momentum changes only due to external torque. These conservation laws simplify the analysis of the motion and provide important physical insights into precession and nutation.

Steady Motion of a Top

A top is said to be in steady motion when its axis rotates with constant angular velocity while maintaining a constant inclination with the vertical.

14.8: - GLOSSARY

Angular Momentum

The rotational equivalent of linear momentum, defined as the product of the moment of inertia and angular velocity. It is represented by $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$.

Angular Velocity

The rate at which a body rotates about an axis, usually denoted by $\boldsymbol{\omega}$ and measured in radians per second (rad/s).

Angular Acceleration

The rate of change of angular velocity with respect to time, denoted by $\boldsymbol{\alpha}$.

Axis of Symmetry

The axis about which a top has identical mass distribution on both sides.

Euler Angles

Three angles (ϕ, θ, ψ) used to describe the orientation of a rigid body in three-dimensional space.

Euler's Equations

Fundamental equations governing the rotational motion of a rigid body about a fixed point in body-fixed coordinates.

Moment of Inertia

A measure of a body's resistance to changes in rotational motion about a given axis.

14.9 :- REFERENCES

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14.10: - SUGGESTED READING: -

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14.11: - TERMINAL QUESTIONS: -

1. Derive the equations of motion of a symmetric top using Euler's equation**s** and explain the physical significance of each term.

2. Using Lagrange's equations, derive the equations of motion of a heavy top. Show how generalized coordinates (Euler angles) are employed in the derivation.

3. Deduce the equations of motion of a top from the principle of conservation of energy and angular momentum**.** Compare this approach with the Euler and Lagrangian methods.

4. What is steady motion (steady precession) of a top? Derive the condition for steady precession and discuss the factors affecting its stability.

5. Discuss the **motion of a heavy symmetric top under gravity**. Explain the phenomena of **precession** and **nutation** with the help of the equations of motion and suitable diagrams.

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