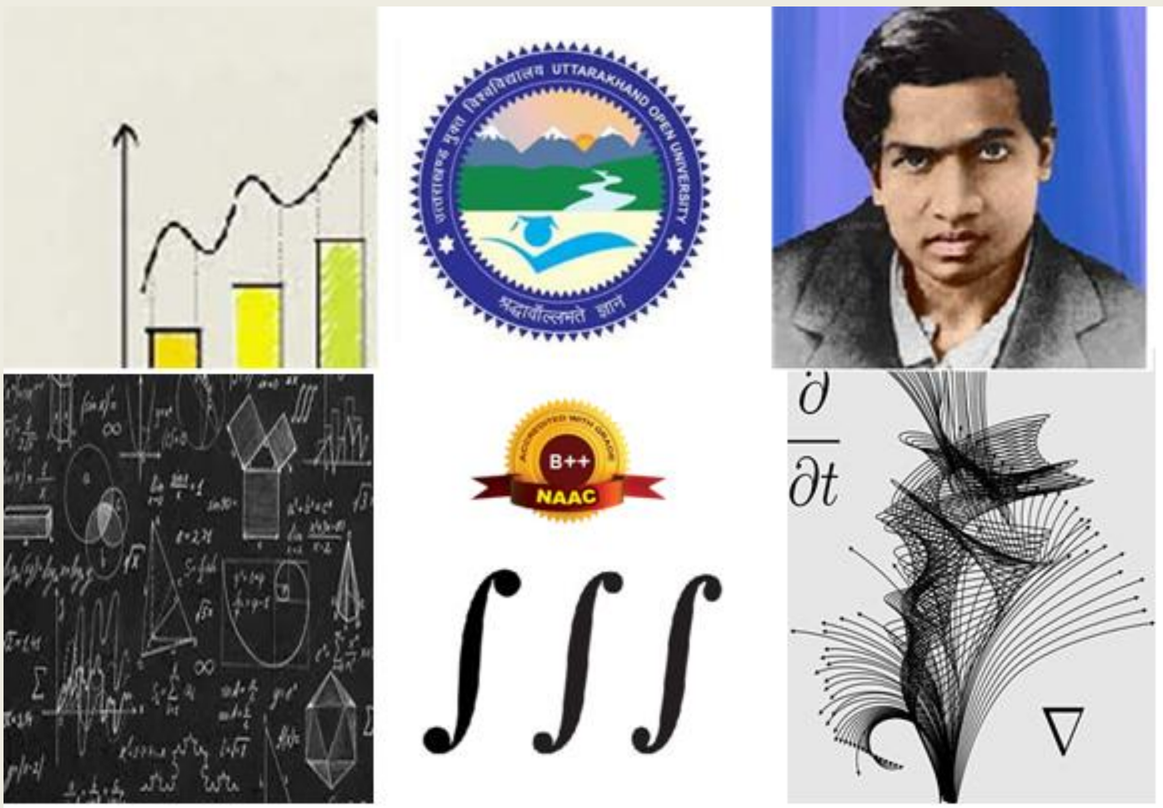
Master of Science
(THIRD SEMESTER)

MAT 601
ADVANCED COMPLEX ANALYSIS



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: ADVANCED COMPLEX ANALYSIS

COURSE CODE: MAT 601

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India, 263139

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MAT 601

COURSE INFORMATION

The self-learning material titled **"Advanced Complex Analysis"** has been specifically designed for M.Sc. (Third Semester) learners at Uttarakhand Open University, Haldwani, to provide them with easy access to high-quality educational resources. The course is organized into fourteen units, each focusing on different key concepts in complex analysis. **Units 1 and 2** introduce the foundational topics of complex numbers, including the concept of functions, limits, and continuity, which are essential for understanding more advanced topics. **Units 3 and 4** delve into the concept of analytic functions, particularly focusing on the Cauchy-Riemann equations and power series, which are crucial for analyzing complex functions. **Units 5 and 6** shift the focus to conformal mapping and Möbius transformations, as well as other types of mappings that preserve angles and are fundamental in complex analysis. **Units 7 and 8** explore complex line integrals and Cauchy's inequalities, along with their various consequences, which are key tools in evaluating integrals and understanding the behaviour of complex functions. Units 9 and 10 cover Cauchy's Theorem and the Cauchy Integral Formula, along with the Maximum and Minimum Modulus Principle and Schwarz Lemma, which provide powerful results about the properties of analytic functions. **Unit 11** is dedicated to singularities, where the behaviour of functions near points of discontinuity is studied. **Units 12, 13, and 14** explain the residue theorem, argument principle, Rouche’s theorem, and the uniqueness of analytic continuation, which are advanced topics that extend the applications of complex analysis.

The material is not only structured for academic learning but is also useful for competitive examinations. It explains fundamental principles and theories in a clear and straightforward manner, with numerous examples and exercises included to help learners easily grasp the subject matter.

Course Name: Advanced Complex Analysis Course code: MAT601

Credit: 4

Syllabus

Algebra and Topology of the complex plane. Geometry of the complex plane, Complex differentiation. : Power series and its convergence, Cauchy-Riemann equations, Harmonic functions, Conformal Mapping: Circle and line revisited, Conformal Mapping, Möbius transformations, Other Mapping, Integration along a contour, The fundamental theorem of calculus, Homotopy, Cauchy's theorem, Cauchy integral formula, Cauchy's inequalities and other consequences, Winding number, Open mapping theorem, Schwarz reflection Principle, Singularities of a holomorphic function, Laurent series, The residue theorem, Argument principle, Rouche's theorem, Uniqueness of analytic continuation.

Refrences

- Ruel V.Churchill, (1960), Complex Variables and Applications, McGraw-Hill, New York.
- S. Ponnusamy, (2011), *Foundations of Complex Analysis* (2nd edition), Narosa Publishing House.
- Murray R. Spiegel, (2009), *Schaum's Outline of Complex Variables*(2nd edition).

Suggested Readings

- L. V. Ahlfors,(1966), *Complex Analysis*, Second edition, McGraw-Hill, New York.
- J.B. Conway, (2000), *Functions of One Complex Variable*, Narosa Publishing House,
- E.T. Copson, (1970), *Introduction to Theory of Functions of Complex Variable*, Oxford University Press.

BLOCK I
ANALYTIC FUNCTIONS

UNIT 1:-Introduction to Complex Numbers

CONTENTS:

1.1 *INTRODUCTION:-*

Complex numbers extend the real number system to include solutions to equations that have no real solutions, such as $x^2+1=0$. A complex number is of the form $z=a+ib$ where a the **real part**, b is the **imaginary part**, and i is the **imaginary unit** with $i^2=-1$. They can be represented on the **complex plane**, with the real part on the horizontal axis and the imaginary part on the vertical axis. The term **"Complex Number"** was coined by C.F. Gauss, and later mathematicians like A.L. Cauchy, B. Riemann, and K. Weierstrass made significant contributions, enriching the subject with their original work. Basic operations with complex numbers, such as addition, subtraction, multiplication, and division, follow specific rules. The modulus and argument provide a polar form, offering an alternative way to express complex numbers, which is particularly useful in advanced mathematics and engineering.

1.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To find the solutions to equations that lack real solutions.
- To represent complex numbers as points or vectors on the complex plane.
- To solved the form of complex numbers.
- To solved the equation of straight line and circle.

1.3 *COMPLEX NUMBERS:-*

Complex numbers were introduced to provide solutions to equations like $x^2 + 1 = 0$, where there are no real solutions. These numbers include both real and imaginary parts and are denoted as $a + ib$, where a, b are the real numbers, is called *Complex Number*. and i represents the imaginary unit, which is defined as the square root of -1, also called i as *imaginary unit*.

If we represent a number in the form $z = x + iy$, then z is called a complex Variable . Here, x and y are called the real and imaginary parts of z respectively. Sometimes we write z as

$$z = (x, y)$$

we also write

$$R(z) = x, I(z) = y$$

If $x = 0, i.e.,\ z = iy$, then z is known as pure imaginary number.

The complex conjugate of a complex number $z = x + iy$ is denoted as $x + iy$ and is equal to $x - iy$. In other words, it involves changing the sign of the imaginary part while leaving the real part unchanged.

$$z = x + iy \quad or \quad \bar{z} = x + iy$$

Example: the conjugate of $-3 - 5i$ is $3 + 5i$.

It is easy to verify that

$$R(z) = x = \frac{z + \bar{z}}{2}, \quad I(z) = y = \frac{z - \bar{z}}{2i}$$

1.4 *EQUALITY OF COMPLEX NUMBERS:-*

The equality of complex numbers follows the same principles as equality of real numbers. Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are considered equal if and only if both their real parts and imaginary parts are equal, i.e., $x_1 = x_2$ and $y_1 = y_2$.

Formally

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$z_1 = z_2 \text{ if and only if } x_1 = x_2, \quad y_1 = y_2.$$

Remark: The phrases "greater than" or "less than" have no meaning in the set of complex numbers.

Fundamental operations with complex numbers:

- **Addition:** To add two complex numbers, add their corresponding real and imaginary parts.

  If

  $$z_1 = a + ib, \quad z_2 = c + id \text{ then}$$
  $$z_1 + z_2 = (a + c) + i(b + d)$$

- **Subtraction:** To subtract one complex number from another, subtract their corresponding real and imaginary parts.

  $$z_1 = a + ib, \quad z_2 = c + id \text{ then}$$
  $$z_1 - z_2 = (a - c) + i(b - d)$$

- **Multiplication:** To divide two complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

  If

  $$z_1 = a + ib, \quad z_2 = c + id \text{ then}$$
  $$z_1.z_2 = (a + ib)(c + id) = ac + iad + bc + i^2bd$$
  $$= (ac - bd) + i(ad + bc)$$

- **Division:** To divide two complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

  If

  $z_1 = a + ib, \quad z_2 = c + id,$ the conjugate of z_2 is $\overline{z_2} = c - id$ then

  $$\frac{z_1}{z_2} = \frac{a + ib}{c + id} . \frac{c - id}{c - id} = \frac{(a + ib)(c - id)}{c^2 + d^2}$$
  $$= \frac{(ac - bd)(bc - ad)}{c^2 + d^2}$$
  $$= \left(\frac{ac - bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right) \text{ if } c^2 + d^2 \neq 0$$

Absolute Value: For a complex number $z = a + ib$, where a the real part is and b is the imaginary part, the absolute value is defined as:

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

$$\therefore \qquad |z|^2 = a^2 + b^2 = (a + ib)(a - ib) = z\bar{z}$$

$$|z|^2 = z\bar{z}$$

Also $$\overline{z_1 . z_2} = \bar{z_1}\ \bar{z_2}$$

Properties of the Absolute Value:

- **Non-negativity:** $| z | \geq 0$
  The absolute value is always non-negative.
- **Zero:** $| z | = 0$

  if and only if z=0 (i.e., both the real and imaginary parts are zero).

- **Multiplicatively:** $|z_1 . z_2 | = |z_1||z_2|$
  The absolute value of the product of two complex numbers is the product of their absolute values.

- **Triangle Inequality:** $|z_1 + z_2 | \leq |z_1| + |z_2|$
  The absolute value of the sum of two complex numbers is less than or equal to the sum of their absolute values.
- **Conjugate:** $| z | = | \bar{z} |$
  The absolute value of a complex number is equal to the absolute value of its conjugate.

1.5 *GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS*:-

A complex number $z = x + iy$ is defined as an ordered pair of real numbers (x, y), where x the real part is and y is the imaginary part.

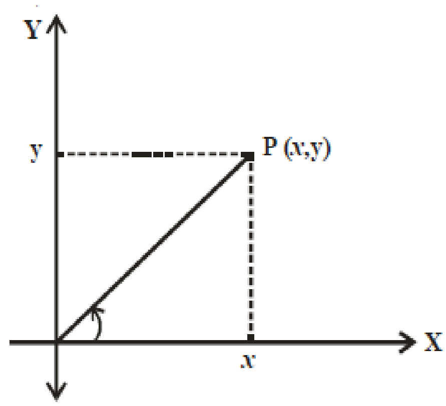

Fig.1

A complex number $z = x + iy$ can be represented by a point P with Cartesian coordinates (x, y) on a rectangular coordinate system, where the X −axis is the real axis and the $Y - a$xis is the imaginary axis.

Each complex number corresponds to a unique point in the plane, and conversely, each point in the plane corresponds to one and only one complex number.

1.6 *COMPLEX PLANE OR ARGAND PLANE*:-

The complex plane, also known as the **Argand plane** or **the z-plane**, is a two-dimensional coordinate system used to represent complex numbers geometrically. **Gauss** was the first to produce in 1799 that complex numbers are represented by points in a plane, then this concept that was developed by **Argand** in 1806. In this plane, each complex number $z = x + iy$ can identify with a point $P = (x, y)$, where x the real part is and y is the imaginary part. The horizontal axis, known as the real axis, contains all points of the form $(x, 0)$, representing real numbers, while the vertical axis, called the imaginary axis, includes points of the form $(0, y)$, representing purely imaginary numbers. Points not on the real axis represent general complex numbers with both real and imaginary parts. The origin $(0,0)$, represents the complex number $0 + i0$. This graphical representation helps in visualizing complex number operations and understanding their properties.

The nonnegative number $|z|$, called the modulus or absolute value of a complex number $z = x + iy$, represents the distance of the complex number z from the origin in the complex plane. It is calculated using the formula:

$$|z| = \sqrt{x^2 + y^2}$$

This is derived from the Pythagorean theorem, considering $z = (x, y)$ as a point in the xy −plane.(see Fig.2.)

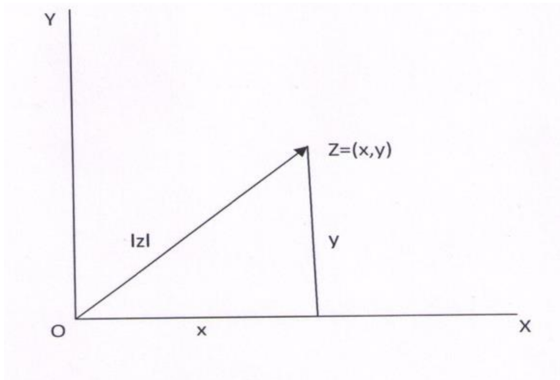

Fig.2.

The distance between two points $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the complex plane is given by the distance formula:

$$|z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula measures the straight-line distance between the points (x_1, y_1) and (x_2, y_2) in the complex plane.

1.7 *POLAR FORM OF COMPLEX NUMBERS*:-

In the complex plane, any complex number $z = x + iy$ can be represented as a point P with coordinates (x, y). The polar coordinates (r, θ) of this point are derived from its Cartesian coordinates (x, y). The modulus r, also known as **the absolute value of z**, is found by taking the square root of the sum of the squares of the real and imaginary parts, giving us $r = \sqrt{x^2 + y^2}$.

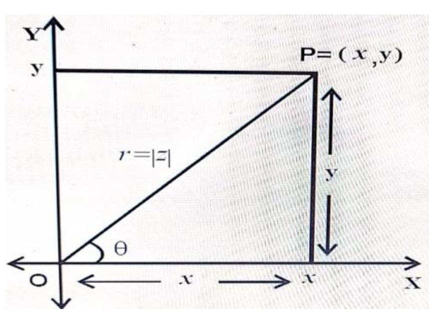

Fig.3.

This represents the distance of the point P from the origin O. The argument θ, often referred to as the amplitude of z, is determined by the angle θ between the line segment OP and the positive real axis. It is calculated as the arctangent of $\theta = \tan^{-1}\frac{y}{x}$, representing the direction of the point P from the origin. Together, the **modulus and argument** provide a comprehensive description of the complex number's location and direction within the complex plane and it is also written as $\theta = amp(z)$ or $\theta = \arg(z)$.

From the figure 3. $x = rcos\theta, y = rsin\theta$

Then $r = \sqrt{x^2 + y^2} = |x + iy| = |z|$

$$\theta = \tan^{-1}\frac{y}{x}$$

It follows

$$z = x + iy = r(cos\theta + isin\theta) = re^{i\theta}$$

It is called polar form of the complex number z and that r, θ are called polar coordinates of z.

- The argument of a complex number z is not unique because it can differ by any integer multiple of 2π.
- The principal value of the argument of a complex number z, denoted as $Arg(z)$, is the value of θ that lies within the interval $-\pi < \theta \leq \pi \; or \; 0 < \theta \leq 2\pi$.
- If $z = 0$, then $arg(z) = arg(0)$ is not defined and $arg(z)$ is defined only if only $z \neq 0$.
- If $Arg(z)$ denoted general value and argument $arg(z)$ denoted principal value, then
$$Arg(z) = arg(z) + 2n\pi \; \forall \; n \in I$$
where $I =$ set of integers.
- If $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}\frac{y}{x}, if \; x > 0, y > 0 \; or \; y \leq 0 \\ \pi + \tan^{-1}\frac{y}{x} if \; x < 0 \; and \; y \geq 0 \\ -\pi + \tan^{-1}\frac{y}{x} if \; x < 0 \; and \; y \geq 0 \\ \frac{\pi}{2} \; if \; x = 0, y > 0 \\ -\frac{\pi}{2} \; if \; x = 0, y < 0 \end{cases}$$

1.8 *PROPERTIES OF ARGUMENTS:-*

Theorem1. The argument of the product of two complex numbers is equal to the sum of their arguments.

$$\arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

Proof: Let z_1 and z_2 be two complex numbers with arguments $\theta_1$and θ_2 respectively. In polar form, these complex numbers can be written as:

$$z_1 = r_1(cos\theta_1 + isin\theta_1)$$
$$z_2 = r_2(cos\theta_2 + isin\theta_2)$$

Now, consider the product $z_1 z_2$:

$$z_1 z_2 = [r_1(cos\theta_1 + isin\theta_1)][r_2(cos\theta_2 + isin\theta_2)]$$

$$= r_1 r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$$

Using the angle addition formulas for sine and cosine:

$$cos(\theta_1 + \theta_2) = cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2$$

$$sin(\theta_1 + \theta_2) = cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2$$

We can rewrite the product as:

$$z_1 z_2 = r_1 r_2[cos(\theta_1 + \theta_2) + sin(\theta_1 + \theta_2)]$$

This shows that the modulus of the product is $r_1 r_2$and the argument of the product is $\theta_1 + \theta_2$.Therefore

$$\arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

Theorem2. The argument of the quotient of complex numbers is equal to the difference of their arguments.

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

Proof: Let z_1 and z_2 be two complex numbers with arguments $\theta_1$and θ_2 and moduli r_1 and r_2. In polar form, these complex numbers can be expressed as:

$$z_1 = r_1(cos\theta_1 + isin\theta_1)$$

$$z_2 = r_2(cos\theta_2 + isin\theta_2)$$

Now, consider the quotient z_1/z_2:

$$\frac{z_1}{z_2} = \frac{r_1(cos\theta_1 + isin\theta_1)}{r_2(cos\theta_2 + isin\theta_2)}$$

Using the properties of complex numbers in polar form, we can write this as:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} . \frac{(cos\theta_1 + isin\theta_1)}{(cos\theta_2 + isin\theta_2)}$$

To simplify the fraction, we multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} . \frac{(cos\theta_1 + isin\theta_1)}{(cos\theta_2 + isin\theta_2)} . \frac{(cos\theta_1 - isin\theta_1)}{(cos\theta_2 - isin\theta_2)}$$

The denominator simplifies as:

$$(cos\theta_2 + isin\theta_2).(cos\theta_2 - isin\theta_2) = cos^2\theta_2 + isin^2\theta_2 = 1$$

This shows that the modulus of the quotient is $\frac{r_1}{r_2}$, and the argument of the quotient is $\theta_1 - \theta_2$. Therefore

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

This completes the proof that the argument of the quotient of two complex numbers is equal to the difference of their arguments.

1.9 *PROPERTIES OF MODULI*:-

Theorem3: The modulus of the product of two complex numbers is the product of their modulus.

$$|z_1.z_2| = |z_1|.|z_2|$$

Proof: $|z_1.z_2|^2 = (z_1.z_2)\overline{(z_1.z_2)} = z_1.z_2.\overline{z_1}.\overline{z_2}$

$$= (z_1.\overline{z_1})(z_2.\overline{z_2}) = |z_1|^2.|z_2|^2$$

$$|z_1.z_2|^2 = |z_1|^2.|z_2|^2$$

⇒

$$|z_1.z_2| = |z_1|.|z_2|$$

Remark: $|z_1| = \left|\frac{z_1}{z_2}.z_2\right| = \left|\frac{z_1}{z_2}\right|.|z_2|,$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|$$

Theorem4: The modulus of the sum of two complex numbers is less than or equal to sum of their moduli.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof: Suppose, $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1 cos\theta_1 + r_2 cos\theta_2) + i(r_1 sin\theta_1 + r_2 sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{(r_1 cos\theta_1 + r_2 cos\theta_2)^2 + (r_1 sin\theta_1 + r_2 sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 + 2r_1 r_2} \qquad for\ \ cos(\theta_1 - \theta_2) \leq 1$$

$$= r_1 + r_2 = |z_1| + |z_2|$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Remark: By induction, it follows that

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|$$

Theorem5: The modulus of the sum of two complex numbers is less than or equal to sum of their moduli.

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Proof: Let, $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 - z_2 = r_1(cos\theta_1 + isin\theta_1) - r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1 cos\theta_1 - r_2 cos\theta_2) + i(r_1 sin\theta_1 - r_2 sin\theta_2)$$

$$|z_1 - z_2| = \sqrt{(r_1 cos\theta_1 - r_2 cos\theta_2)^2 + (r_1 sin\theta_1 - r_2 sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 - 2r_1r_2} \qquad for\ \ cos(\theta_1 - \theta_2) \geq -1$$

$$= r_1 - r_2 = |z_1| - |z_2|$$

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Remark: To prove

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

$$|z_1 - z_2| = |z_1 + (-z_2)|$$

$$\leq |z_1| + |-z_2| \text{ by theorem2}$$

$$= |z_1| + |z_2|$$

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

Hence $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

Theorem6: To prove $|z_1 + z_2| \geq |z_1| - |z_2|$.

Proof: : Let, $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1cos\theta_1 + r_2cos\theta_2) + i(r_1sin\theta_1 + r_2sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{(r_1cos\theta_1 + r_2cos\theta_2)^2 + (r_1sin\theta_1 + r_2sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2cos(\theta_1 - \theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1r_2} \qquad for\ \ cos(\theta_1 - \theta_2) \geq -1$$

$$= r_1 - r_2 = |z_1| - |z_2| \text{ if } r_1 > r_2$$

$$r_1 - r_2 = |z_1| - |z_2|$$

$$|z_1 + z_2| \geq |z_1| - |z_2| \text{ if } |z_1| > |z_2|$$

Theorem7: Parallelogram Law: The sum of squares of the length of diagonals of a parallelogram is equal to the sum of squares of length of its sides, i.e., prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$$

OR

To prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

Proof: Let $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z_1 + z_2 = r_1(cos\theta_1 + isin\theta_1) + r_2(cos\theta_2 + isin\theta_2)$$

$$= (r_1cos\theta_1 + r_2cos\theta_2) + i(r_1sin\theta_1 + r_2sin\theta_2)$$

$$z_1 - z_2 = (r_1cos\theta_1 - r_2cos\theta_2) + i(r_1sin\theta_1 - r_2sin\theta_2)$$

Now $|z_1 + z_2|^2 + |z_1 - z_2|^2 = [(r_1cos\theta_1 + r_2cos\theta_2)^{\mathbf{2}} + i(r_1sin\theta_1 + r_2sin\theta_2)^2] + [(r_1cos\theta_1 - r_2cos\theta_2)^2 + (r_1sin\theta_1 - r_2sin\theta_2)^2]$

$$= [r_1^2 + r_2^2 + 2r_1r_2cos(\theta_1 - \theta_2)] + [r_1^2 + r_2^2 - 2r_1r_2cos(\theta_1 - \theta_2)]$$

$$= [r_1^2 + r_2^2]$$

$$= 2[|z_1|^2 + |z_2|^2]$$

Theorem8: **(Equation of Straight line)**To find the equation of straight line joining two pointsz_1 and z_2 in the complex plane.

Proof: Let the equation of line AB joining the points A (z_1) and B(z_2), suppose point P(z) on it. So

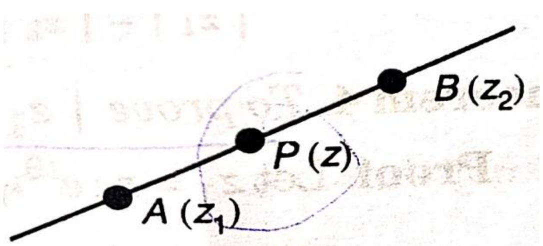

Fig.4.

$$arg\left(\frac{z-z_1}{z_1-z_2}\right) = 0 \text{ or } \pi$$

Consequently $\left(\frac{z-z_1}{z_1-z_2}\right)$ is purely real so that

$$\left(\frac{z-z_1}{z_1-z_2}\right)=\left(\overline{\frac{z-z_1}{z_1-z_2}}\right)=\left(\frac{\bar{z}-\bar{z}_1}{\bar{z}_1-\bar{z}_2}\right)$$

$$(z-z_1)(\bar{z}_1-\bar{z}_2)=(z_1-z_2)(\bar{z}-\bar{z}_1)$$

$$z(\bar{z}_1-\bar{z}_2)-\bar{z}(\bar{z}_1-\bar{z}_2)-z_1\bar{z}_1+z_1\bar{z}_2+z_1\bar{z}_1-z_2\bar{z}_1=0$$

$z(\bar{z}_1-\bar{z}_2)-\bar{z}(\bar{z}_1-\bar{z}_2)+(z_1\bar{z}_2-z_2\bar{z}_1)=0$ is required equation of line.

Theorem9: (**Equation of a Circle)** To show that the equation of circle in the Argand plane can be put in the form

$$z\bar{z}+\bar{z}b+\bar{b}z+c=0$$

where c is real and b is complex constant.

Proof: Suppose a be a complex coordinate of the centre C and r be the radius of circle. Consider any point $P(z)$ on the circle.
Then the length of line $CP=$ radius of circle or

$$|z-a|=r$$

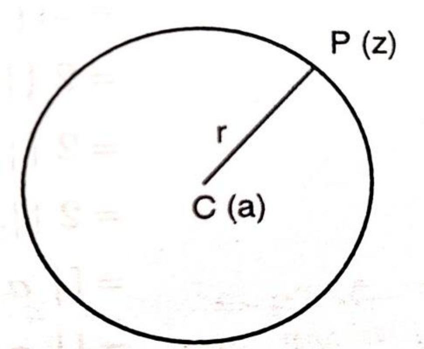

Fig.5.

Squaring both sides, we have

$$|z-a|^2=r^2$$
$$(z-a)(\bar{z}-\bar{a})=r^2$$
$$(z\bar{z}-\bar{a}z+a\bar{a}-a\bar{z})=r^2$$
$$z\bar{z}-\bar{a}z-a\bar{z}+(|a|^2-r^2)=0$$

Taking $-a=b$ and $(|a|^2-r^2)=c=real\ number$

$$z\bar{z}+\bar{z}b+\bar{b}z+c=0$$

where c is real and b is complex constant.

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Prove that $\left|a+\sqrt{a^2-b^2}\right|+\left|a-\sqrt{a^2-b^2}\right|=[|a-b|+|a-b|]^2$.

SOLUTION: Suppose $|z_1+z_2|^2+|z_1-z_2|^2=2[|z_1|^2+|z_2|^2]$

$$|z_1z_2|=|z_1|.|z_2|$$

Now we shall prove the given problem

$$\left[\left|a+\sqrt{a^2-b^2}\right|+\left|a-\sqrt{a^2-b^2}\right|\right]^2$$

$$=\left|a+\sqrt{a^2-b^2}\right|^2+\left|a-\sqrt{a^2-b^2}\right|^2$$
$$+2\left|a+\sqrt{a^2-b^2}\right|\left|a-\sqrt{a^2-b^2}\right|$$

$$=2\left[|a|^2+\left|\sqrt{a^2-b^2}\right|^2\right]+2\left|\left[a+\sqrt{a^2-b^2}\right]\left[a-\sqrt{a^2-b^2}\right]\right|$$

$$=2[|a|^2+|a^2-b^2|]+2|a^2-(a^2-b^2)|$$

$$=2[|a|^2+|a^2-b^2|]+2|b^2|$$

$$=2[|a|^2+|b^2|]+2|a^2-b^2|$$

$$=[|a+b|^2+|a-b|^2]+2|a+b|.|a-b|$$

$$=[|a+b|^2+|a-b|^2]^2$$

Hence

$\left|a+\sqrt{a^2-b^2}\right|+\left|a-\sqrt{a^2-b^2}\right|=[|a-b|+|a-b|]^2$ is required the solution.

EXAMPLE2: Determine the regions of Argand diagram given by

$$|z^2-z|<1.$$

SOLUTION: Let $z=re^{i\theta}$

Then $z^2-z=r^2e^{i2\theta}-re^{i\theta}$

$$=(r^2cos2\theta-rcos\theta)+i(r^2sin2\theta-rsin\theta)$$

$$|z^2-z|^2=(r^2cos2\theta-rcos\theta)^2+(r^2sin2\theta-rsin\theta)^2$$

$$=r^4+r^2-2r^3\cos(2\theta-\theta)$$

But $$|z^2 - z|^2 < 1$$

Hence

$$r^4 + r^2 - 2r^3 \cos\theta < 1$$

or $$r^4 + r^2 - 2r^3 \cos\theta - 1 < 0$$

Hence

$$r^4 + r^2 - 2r^3 \cos\theta - 1 = 0$$

EXAMPLE3: Determine the region of z −plane for which

$$|z - 1| + |z + 1| \le 3.$$

SOLUTION: Let $z = x + iy$

$$|z - 1| + |z + 1| = |x + iy - 1| + |x + iy + 1|$$

$$= \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}$$

But $|z - 1| + |z + 1| \le 3$

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} \le 3$$

$$\sqrt{(x-1)^2 + y^2} \le 3 - \sqrt{(x+1)^2 + y^2}$$

$$(x-1)^2 + y^2 \le 9 + (x+1)^2 + y^2 - 6\sqrt{(x+1)^2 + y^2}$$

$$0 < 4x + 9 - 6\sqrt{(x+1)^2 + y^2}$$

$$6\sqrt{(x+1)^2 + y^2} \le (4x + 9)$$

$$36[(x+1)^2 + y^2] \le 16x^2 + 81 + 72x$$

$$36x^2 + 36 + 36y^2 + 72x \le 16x^2 + 81 + 72x$$

$$36x^2 + 36 + 36y^2 \le 16x^2 + 81$$

$$20x^2 + 36y^2 \le 45$$

$$\frac{x^2}{(9/4)} + \frac{y^2}{(5/4)} = 1$$

EXAMPLE4: Show that the locus of z such that

$$|z-a|.|z+a| = a^2, a > 0$$

is a lemniscate.

SOLUTION: Let $|z^2 - a^2| = a^2$ or $z^2 - a^2 = a^2 e^{i\lambda}$

Put $z = re^{i\theta}$. Then $r^2 e^{i2\theta} - a^2 = a^2 e^{i\lambda}$

This $\Rightarrow$ $r^2 cos2\theta - a^2 = a^2 cos\lambda$

$$r^2 sin2\theta = a^2 sin\lambda$$

Both above equations are squaring and adding

$$(r^2 cos2\theta - a^2)^2 + (r^2 sin2\theta)^2 = a^4$$

$$r^2(r^2 - 2a^2 cos2\theta) = 0$$

But $r \neq 0$ as $z \neq 0$

$r^2 - 2a^2 cos2\theta = 0$ or $r^2 = 2a^2 cos2\theta$

which is lemniscates.

SELF CHECK QUESTIONS

1. Given two complex numbers$z_1 = 3 + 4i, z_2 = 1 - 2i$ perform the following operations:
   a. Find the sum $z_1 + z_2$.
   b. Find the difference $z_1 - z_2$.
   c. Find the product $z_1 \times z_2$.
   d. Find the quoteint of z_1/z_2
   e. Find the magnitude of z_1
   f. Find the conjugate of z_2.
2. Determine whether the following pairs of complex numbers are equal:
   a. $z_1 = 4 + 2i$ and $z_2 = 4 + 2i$ equal
   b. $z_3 = -1 + 4i$ and $z_3 = -1 - 4i$ not equal
   c. $z_5 = 5$ and $z_6 = 7 + 0i$ equal
   d. $z_7 = 0$ and $z_8 = 0 + 0i$ equal
   e. $z_5 = 2i$ and $z_6 = -2i$ not equal
3. Determine whether the following statements about the geometrical representation of complex numbers are true (Answer with "True" or "False"):

a. The complex number $3 + 4i$ is represented by the point $(3, 4)$ on the complex plane. true
b. The magnitude of the complex number $-3 - 4i$ is 5. true
c. The complex number $0 + 0i$ is located at the origin. true
d. The argument of the complex number $1 + i$ is $\frac{\pi}{4}$. true
e. The complex number 4 lies on the imaginary axis. False

4. For the circle $|z| = 1$, the inverse of the point z is :
   a. z
   b. $\bar{z}$
   c. $1/\bar{z}$
   d. None
5. If the amplitude of the complex number z be θ, then amplitude of iz is :
   a. $-\theta$
   b. $\theta + \pi/2$
   c. $\theta + \pi$
   d. None
6. Polar form of complex number $-5 + 5i$ is:
   a. $5\sqrt{2}e^{i\pi/4}$
   b. $5\sqrt{2}e^{-3i\pi/4}$
   c. $5\sqrt{2}e^{3i\pi/4}$
   d. None
7. If a, b, c and u, v, w complex numbers representing vertices of two triangles s.t., $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles.
   a. have the same area
   b. are similar
   c. are congruent
   d. None
8. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram in order iff
   a. $z_1 + z_4 = z_2 + z_3$
   b. $z_1 + z_3 = z_2 + z_4$
   c. $z_1 + z_2 = z_3 + z_4$
   d. None

1.10 *SUMMARY*:-

In this unit we have studied, a complex number is a mathematical entity that extends the concept of one-dimensional real numbers to a two-dimensional number system. It is expressed in the form $a + bi$, where a and b are real numbers, and i is the imaginary unit satisfying $i^2 = -1$.

Here, a is referred to as the real part and b as the imaginary part of the complex number. Complex numbers enable the solution of equations that have no real solutions, such as $x^2 + 1 = 0$. They are fundamental in various fields of mathematics, engineering, and physics, providing a way to describe oscillations, waveforms, and other phenomena.

1.11 *GLOSSARY*:-

- **Complex Number:** A number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit with $i^2 = -1$.
- **Real Part:** The component a in a complex number $a + bi$, representing a real number.
- **Imaginary Part:** The component b in a complex number $a + bi$, representing a real number multiplied by the imaginary unit i.
- **Imaginary Unit (i):** A mathematical constant satisfying $i^2 = -1$.
- Equality of Complex Numbers: Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.
- **Complex Plane (Argand Plane):** A two-dimensional plane for representing complex numbers, where the horizontal axis is the real part and the vertical axis is the imaginary part.
- **Magnitude (Modulus):** The distance from the origin to the point (a, b) in the complex plane, calculated as $|z| = \sqrt{a^2 + b^2}$ for a complex number $z = a + bi$.
- **Argument (Phase):** The angle θ formed with the positive real axis, calculated using $\theta = \tan^{-1}\frac{b}{a}$, (b/a) for a complex number $z = a + bi$.
- **Polar Form:** A representation of a complex number in terms of its magnitude and argument: $z = r(cos\theta + isin\theta)$ or $z = re^{i\theta}$.
- **Conjugate:** The conjugate of a complex number $a + bi$ is $a - bi$.
- **Addition of Complex Numbers:** Combining two complex numbers by adding their real parts and their imaginary parts separately: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- **Multiplication of Complex Numbers:** Multiplying two complex numbers using distributive property and the fact that $i^2 = -1$: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- **Division of Complex Numbers:** Dividing by multiplying the numerator and denominator by the conjugate of the denominator and simplifying: $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$.

1.12 *REFERENCES*:-

- James Ward Brown and Ruel V. Churchill 2009 (Eighth Edition) ,Complex Variables and Applications.
- Elias M. Stein and Rami Shakarchi (2003), Complex Analysis.
- I.John B. Conway1978 (Second Edition), Functions of One Complex Variable.
- Theodore W. Gamelin(2001),Complex Analysis.

1.13 SUGGESTED *READING*:-

- file:///C:/Users/user/Downloads/Paper-III-Complex-Analysis.pdf
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- A.I. Markushevich 2005 (Dover Reprint of 1977 Edition),Theory of Functions of a Complex Variable.
- file:///C:/Users/user/Desktop/1456304480EtextofChapter1Module1%20(1).pdf

1.14 *TERMINAL QUESTIONS:-*

(TQ-1) Let A and B be two complex numbers s.t

$$\frac{A}{B} + \frac{B}{A} = 1$$

Prove that origin and two points represented by A and B form vertices of an equilateral triangle.

(TQ-2) If the complex numbers $sinx + icos2x$ and $cosx - isin2x$ are complex conjugate to each other, then the value of x.

(TQ-3)A relation R on the set of complex numbers is defined by $z_1Rz_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ real. Show that R is an equivalence relation.

(TQ-4) Show that the origin and the point representing the roots of the equation $z^2 + pz + q = 0$ form an equilateral if $p^2 = 3q$.

(TQ-5) Find $argi(x + iy)$ if $arg(x + iy) = \alpha$.

(TQ-6) Show that the triangles whose vertices are z_1, z_2, z_3 and $z'_1 z'_2 z'_3$are directly if

$$\begin{vmatrix} z_1 & z'_1 & 1 \\ z_2 & z'_2 & 1 \\ z_3 & z'_3 & 1 \end{vmatrix} = 0$$

(TQ-7) Show that $|z_1 - z_2| \geq (|z_1| - |z_2|)$

(TQ-8) Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and deduce that $\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$ all the numbers concerned being complex.

(TQ-9) Find the principal value of $arg\ 'i'$.

(TQ-10) Find the principal value of $arg\ (1 + i)$.

1.15 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

1.

a. $4 + 2i$
b. $2 + 6i$
c. $11 - 2i$
d. $-1 + 2i$
e. 5
f. $1 + 2i$

2.

g. equal
h. not equal
i. equal
j. equal
k. not equal

3.

a. true
b. true
c. true
d. true
e. False

4. $1/\bar{z}$

5. $\theta + \frac{\pi}{2}$

6. $5\sqrt{2}e^{3i\pi/4}$

7. are similar

8. $z_1 + z_3 = z_2 + z_4$

TERMINAL ANSWERS (TQ'S)

(TQ-2) $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \ldots$
(TQ-5) $\frac{\pi}{2} + \alpha$
(TQ-9) $\frac{\pi}{2}$
(TQ-10) $\frac{\pi}{4}$

UNIT 2:-Concept of Functions, Limit and Continuity

CONTENTS:

2.1 *INTRODUCTION:-*

In this unit, learners are well-versed in the definitions of limits and continuity for functions of a real variable; in this chapter, we will extend these concepts to functions of a complex variable. This includes exploring the nuances of limits and continuity within the complex plane, where functions involve both real and imaginary components. By understanding these definitions in the context of complex variables, students will gain deeper insights into the behavior and properties of complex functions, building on their existing knowledge from real analysis to navigate the more intricate landscape of complex analysis.

2.2 *OBJECTIVES:-*

After studying this unit, the learner's will be able to

- To Understand and describe the relationship between variables through functions, enabling the modeling of real-world phenomena.
- To verify the continuity of a function of two variables at a point.

Overall apply these fundamental concepts to solve practical problems in various fields such as physics, engineering, economics, and beyond, where understanding change and stability is essential.

2.3 *POINT SET:-*

A point set in the complex plane refers to a gathering of points, each representing a distinct element within the set. These points, commonly referred to as numbers or elements of the set, collectively constitute the spatial arrangement of the set within the two-dimensional complex plane.

2.4 *NEIGHBORHOOD:-*

In the Argand plane (also known as the complex plane), the neighborhood of a point z_0 is defined as the set of points z such that the distance between z_0 and z (denoted as $|z - z_0| < \varepsilon$ is less than some positive real number ε. Mathematically, it can be expressed as $\{z \in \mathbb{C}: |z - z_0| < \varepsilon \}$, where $\mathbb{C}$ represents the complex numbers. This neighborhood represents an open set around the point z, where all points within a certain distance ε from z are included.
The **neighborhood of the point at infinity** in the complex plane is the set of points z s.t. $|z| < k$ where k is any positive real number.

2.5 *LIMIT POINTS:-*

A point z_0 in the complex plane $\mathbb{C}$ is termed a limit point of a set $S \subseteq \mathbb{C}$ if every punctured neighborhood of z_0 contains at least one point of S. A limit point may or may not be an actual member of the set. For instance, all points on the circle $|z| = r$ are limit points of the set $|z| < r$, yet they do not belong to the set itself. Conversely, all points within the circle $|z| = r$ are limit points of the set and they do belong to the set $|z| < r$. This illustrates the nuanced relationship between limit points and sets in the complex plane.

OR

A point $z_0 \in \mathbb{C}$ is called a limit point (or accumulation point) of a subset $S \subseteq \mathbb{C}$ if every open neighborhood of z_0 contains at least one point of S different from z_0 itself. Formally, z_0 is a limit point of S if for every $\epsilon > 0$, there exists a point $z \in S$ such that $0 <| z - z_0 |< \epsilon$. This means that z_0 can be approached arbitrarily closely by points of S.

Theorem1: Let f be a complex valued function defined on D and let $z_0 \epsilon Cl(D)$. If $\lim_{z-z_0} f(z)$ exist, then this limit is unique.

Solution: Suppose for contradiction that the limit is not unique. This means there exist two distinct complex numbers $l_1$and l_2 such that:

$$\lim_{z-z_0} f(z) = l_1, \qquad \lim_{z-z_0} f(z) = l_2$$

Now assume $l_1 \neq l_2$

By the definition of the limit, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that:

$| z - l_1 |< \epsilon$ whenever $0 <| z - z_0 |< \delta_1$

Similarly, there exists a $\delta_2 > 0$ such that:

$| z - l_2 |< \epsilon$ whenever $0 <| z - z_0 |< \delta_2$

Let $\delta = \delta_1 \delta_2$ Then for any z satisfying $0 <| z - z_0 |< \delta$ both inequalities hold:

$$| z - l_1 |< \epsilon, | z - l_2 |< \epsilon$$

Using the triangle inequality, we get:

$$| l_1 - l_2 |= |l_1 - f(z) + f(z) - l_2| \leq |l_1 - f(z)| + |f(z) - l_2| <\in +\in = 2 \in$$

However, by our assumption $l_1 \neq l_2$, so $| l_1 - l_2 |$is a positive constant. By choosing $\frac{|l_1 - l_2|}{3}$ we have

$$| l_1 - l_2 |< 2.\frac{| l_1 - l_2 |}{3} = \frac{2}{3} | l_1 - l_2 |$$

This is a contradiction because $|l_1 - l_2|$ cannot be less than itself. Therefore, our assumption that the limit is not unique must be false.

$$\lim_{z-z_0} f(z) = l_1$$

Hence the limit $\lim_{z-z_0} f(z)$ is unique if it exist.

Theorem2: Let f be the complex valued function defined on D. Suppose, $f(z) = u(x,y) + iv(x,y),\ z_0 = x_0 + iy_0,\ w_0 = u_0 + iv_0$ and $z_0 \in Cl(D)$.

Then $\lim_{z-z_0} f(z) = w_0$ iff $\lim_{z-z_0} u(x,y) = u_0$ and $\lim_{z-z_0} v(x,y) = v_0$

Solution: Given

$$f(z) = u(x,y) + iv(x,y)$$

$$z_0 = x_0 + iy_0$$

$$w_0 = u_0 + iv_0$$

and

$$z_0 \in Cl(D)$$

Forward direction $\left(If\ \lim_{z\to z_0} f(z) = w_0 then\ if\ \lim_{z\to z_0} u(x,y) = u_0\ and\ \lim_{z\to z_0} u(x,y) = v_0\right)$:

Assume

$$\lim_{z\to z_0} f(z) = w_0$$

This means that $\forall \in > 0, \exists\ \delta > 0$ such that whenever $0 < |z - z_0| < \delta$, we have

$$|f(z) - w_0| < \in$$

We can express this in terms of the real and imaginary parts:

$$|u(x,y) + iv(x,y) - (u_0 + iv_0)| < \in$$

Using the properties of absolute values for complex numbers; this can be written as

$$\sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2} < \in$$

Since the square root function is positive and increasing, we have:

$$(u(x,y)-u_0)^2+(v(x,y)-v_0)^2<\in^2$$

This implies:

$|u(x,y)-u_0|<\in$ and $|v(x,y)-v_0|<\in$

Therefore, $\lim_{z\to z_0} u(x,y)=u_0$ and $\lim_{z\to z_0} v(x,y)=v_0$.

Reverse direction $\left(if\ \lim_{z\to z_0} u(x,y)=u_0\ and\ \lim_{z\to z_0} u(x,y)=v_0\ then\ \lim_{z\to z_0} f(z)=w_0\right)$:

Assume

$$\lim_{z\to z_0} u(x,y)=u_0\ and\ \lim_{z\to z_0} u(x,y)=v_0$$

This means that $\forall \in>0, \exists\ \delta_1>0$ such that whenever $0<|z-z_0|<\delta_1$, we have

$$|u(x,y)-u_0|<\frac{\in}{\sqrt{2}}$$

Similarly $\exists\ \delta_2>0$ such that whenever $0<|z-z_0|<\delta_2$, we have

$$|v(x,y)-v_0|<\frac{\in}{\sqrt{2}}$$

Using these inequalities, we get:

$$(u(x,y)-u_0)^2<\left(\frac{\in}{\sqrt{2}}\right)^2=\frac{\in^2}{2}$$

$$(v(x,y)-v_0)^2<\left(\frac{\in}{\sqrt{2}}\right)^2=\frac{\in^2}{2}$$

This implies:

$$|f(z)-w_0|=\left|\big(u(x,y)+iv(x,y)\big)-(u_0+iv_0)\right|<\in$$

Hence, $\lim_{z\to z_0} f(z)=w_0$.

Therefore $\lim_{z\to z_0} f(z)=w_0$ if and only if $\lim_{z\to z_0} u(x,y)=u_0$ and $\lim_{z\to z_0} v(x,y)=v_0$.

SOLVED EXAMPLE

EXAMPLE1: Prove that $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

SOLUTION: Let $f(z) \to l$ (a unique limit) as $z \to z_0$ in any manner in the $\mathbb{C} -$ plane.

Let $f(z) = \frac{\bar{z}}{z}$ and $z \to 0$, along the real axis.

$\therefore\ y = 0, z = x \qquad (\because z = x + iy)$

$\therefore\ \lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{z\to 0}\frac{x}{x} = 1$

Suppose $z \to 0$, along the imaginary axis.

$\therefore x = 0, z = iy \qquad (\because z = x + iy)$

$\therefore\ \lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{z\to 0}\frac{-iy}{iy} = -1$

Hence the limit is not unique along real and imaginary axis.

$\therefore \lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

EXAMPLE2: If $f(z) = z^2$, prove that $\lim_{z\to 0} f(z) = z^2$.

SOLUTION: Let $\epsilon > 0$ given, to find $\delta > 0$ s.t. $|z^2 - {z_0}^2| < \in$ whenever $0 < |z - z_0| < \delta$

Consider $\qquad |z^2 - {z_0}^2| = |(z - z_0)(z + z_0)|$

$$|z + z_0||z - z_0| < \delta|z + z_0|$$

$$= |z - z_0 + 2z_0| \le \delta|z - z_0| + 2\delta|z_0| < \delta\delta + 2\delta|z_0| = \in$$

$\because$ now $\delta > 0$s.t. $\min\left\{\frac{\in}{1+2|z_0|}, 1\right\}$

$\Rightarrow\ |z^2 - {z_0}^2| < \in$

$\Rightarrow \lim_{z\to 0} f(z) = {z_0}^2$

2.6 *INTERIOR, EXTERIOR AND BOUNDARY POINTS:-*

An **interior point** of a set $S \subseteq C$ is a point $z_0 \in S$ where there exists a neighborhood entirely contained within S. An **exterior point** of S is a point z_0 where there exists a neighborhood containing no point of S. A **boundary point** of S is a point z_0 where every neighborhood contains at least one point of S and at least one point of the complement of S. The collection of all such boundary points is referred to as the boundary of S.

2.7 *OPEN SET AND CLOSED SET:-*

A set $S \subseteq \mathbb{C}$ is called as an **open set** if it exclusively comprises interior points, ensuring that for every point in S, there exists a neighborhood contained entirely within S. Conversely, a **closed set** in $\mathbb{C}$ includes all its limit points or possesses no limit points; if a set is closed, its complement is open. Some sets neither qualify as open nor closed, while others exhibit characteristics of both.

Notably, the empty set $\emptyset$ and the entire complex plane C are examples of sets that are simultaneously open and closed. The set $A = \{z: 0 <| z - z_0 | \leq r\}$ exemplifies a set that fails to meet the criteria for either openness or closedness. Moreover, the open disc $| z |< 1$ represents an open set, whereas the closed disc $| z |\leq 1$ constitutes a closed set. Additionally, it's crucial to acknowledge that the intersection of a finite number of open sets remains open, while the arbitrary union of open sets also remains open.

2.8 *CONVEX SET:-*

A set $S \subseteq \mathbb{C}$ is called a convex set if, for any two point $z_1, z_2 \in S,$ the line segment connecting1z1and z2is entirely contained with$S$. Mathematically, this means that $z_1, z_2 \in S$ and any t in the interval $0 \leq t \leq 1$, the point $tz_1 + (1 - t)z_2 \in S$.

2.9 *BOUNDED, UNBOUNDED AND COMPACT SET:-*

Bounded Set: A set $S \subseteq \mathbb{C}$ is bounded if there exists a positive real number M such that for every point $z \in S$, the distance $| z |$ from the origin is less than M. That is, S is contained within some finite region of the complex plane.

Unbounded Set: A set $S \subseteq \mathbb{C}$ is unbounded if for every positive real number$M$, there exists at least one point $z \in S$ such that $| z |\geq M$. This means that the set is not contained within any finite region of the complex plane and can extend infinitely in one or more directions.

Compact Set: A set $S \subseteq \mathbb{C}$ is compact if it is closed (contains all its limit points) and bounded (contained within a finite region). Equivalently, a set is compact if every open cover of the set has a finite subcover.

2.10 *DERIVED SET, CLOSURE OF A SET AND CONNECTED SET:-*

Derived Set: The derived set (or the set of limit points) of a set $S \subseteq \mathbb{C}$ is the set of all points in the complex plane that are limit points of S. A point z_0 is a limit point of S if every neighborhood of z_0 contains at least one point of S other than z_0 itself. The derived set can be denoted as S'.

Closure of a Set: The closure of a set $S \subseteq \mathbb{C}$ is the smallest closed set that contains S. It is denoted by $\bar{S}$ and includes all points of S along with all its limit points. Mathematically, $\bar{S} = S \cup S'$, where S' is the derived set of S. The closure represents the **"completion"** of S by including its boundary points.

Connected Set: A set $S \subseteq \mathbb{C}$ is connected if it cannot be partitioned into two non-empty disjoint open subsets. Intuitively, this means that S is all in **"one piece,"** and there is a continuous path within S between any two points in S. A connected set does not have any isolated parts, making it an essential concept in understanding the topological structure of sets.

2.11 *DOMAIN:-*

A domain is defined as a nonempty, open, and connected subset of the complex plane C. If the set includes its boundary points, it is termed a closed domain. Notably, every neighborhood of a point in the complex plane qualifies as a domain. When a domain is combined with some, none, or all of its boundary points, it is referred to as a region. Consequently, every domain is a type of region, but not all regions qualify as domains, highlighting that domains are a specific subset of regions with stricter criteria.

2.12 *JORDAN ARC(CURVE):-*

The equation $z = z(t) = x(t) + iy(t)$

where $x(t)$ and $y(t)$ are real-valued continuous functions of the real variable t, with t in the interval $[a, b]$, defines a set of points in the complex plane known as a continuous curve. This curve is called a simple curve if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$meaning the curve does not intersect itself. If the curve is such that $t_1 < t_2$ and $z(t_1) = z(t_2)$ implies $t_1 = a$ and $t_2 = b$, then it is a simple closed curve, which means the curve starts and ends at the same point, forming a loop without self-intersections except at the endpoints. Simple curves are often referred to as Jordan curves. A common example of a Jordan curve is a polygon formed by joining a finite number of line segments end to end.

An important property of a bounded infinite set in the complex plane is that it must have at least one limit point within the complex plane. This property is derived from the Bolzano-Weierstrass theorem, which states that every bounded sequence in C has a convergent subsequence. This implies that any bounded infinite set in the complex plane cannot be composed entirely of isolated points; instead, it must contain points arbitrarily close to each other, leading to the presence of limit points. This property is fundamental in understanding the structure and behavior of sets in the complex plane.

Theorem 3. (Bolzano-Weierstrass Theorem)

If a set $S \subseteq \mathbb{C}$ is bounded and contains an infinite number of points, then it must have at least one limit point.

Theorem 4. (Jordan Curve Theorem)

It states that a simple closed Jordan curve divides the Argand plane into two open domains which have the curve as the common boundary. One of these domains is bounded and is known as interior domain, while the other is bounded and is called exterior domain.

2.13 *FUNCTION OF A COMPLEX VARIABLE:-*

A complex variable, symbolized by z, denotes any element within a set S contained in the complex plane $\mathbb{C}$. A function $f: S \to \mathbb{C}$ represents a rule assigning a unique complex value $f(z)$ to each $z \in S$, denoted as $w = f(z)$, where z is considered the independent variable and w is the dependent variable. This function f maps elements from the domain S to the complex plane$\mathbb{C}$, often visualized in another complex plane known as the $w-$plane. If S constitutes a subset of the real line, f is termed a complex function of a real variable. The set S is identified as the domain of f, while the collection of all $f(z)$ for z in S is recognized as the range of f.

OR

A function $f: A \to B$, where A and B are non-empty subsets of the complex numbers, is a rule that assigns to each complex number $z_0 = x_0 + iy_0 \in A$ a unique complex number $w_0 = u_0 + iv_0 \in B$.

The number w_0 is the value of f at z_0, denoted $f(z_0) = w_0$. As z varies in $f(z) = w$ varies in B. This function is a complex-valued function of a complex variable, where w is the dependent variable and z is the independent variable. If S is a subset of A, then $f(S) = \{f(z) \mid, z \in S\}$ is called the image of S under f, and the set $R = \{f(z) \mid z \in A\}$ is called the range of f.

<u>Single and Multiple Valued Function:</u> For any non-zero complex number $z \in \mathbb{C} - \{0\}$, the polar form of z is given by $z = re^{i\theta}$, where $r = \mid z \mid$ is the modulus of z and $\theta \in [-\pi, \pi]$ is the argument of z. This can be expressed as $z = z(r, \theta) = re^{i\theta}$. If we increase the argument θ by 2π, we have:

$$z(r, \theta + 2\pi) = re^{i(\theta+2\pi)} = re^{i\theta}.e^{2\pi i} = re^{i\theta} = z(r, \theta)$$

Thus, $z(r, \theta + 2\pi)$ returns to its original value, demonstrating the periodicity of the complex exponential function with period 2π.

Definition. A function f is said to be single-valued if it satisfies $f(z) = f\big(z(r, \theta)\big) = z(r, \theta + 2\pi)$, meaning the function's value remains unchanged when the argument θ is increased by 2π.

Otherwise, f is said to be a multiple valued function.

Example: $f(z) = z^n, n \in Z$ is said to a single valued function.

Solution: $f(z) = f\big(z(r, \theta)\big) = \left(re^{i\theta}\right)^n$

$$f\big(z(r, \theta + 2\pi)\big) = \left[re^{i(\theta+2\pi)}\right]^n = r^n e^{in\theta} e^{2\pi n i}$$

$$= r^n e^{in\theta} e^{2\pi n i}$$

$$\left\{\because \quad e^{2\pi n i} = 1,\ n \in \mathbb{Z}\right\}$$

$$= \left(re^{i\theta}\right)^n = f\big(z(r, \theta)\big)$$

Note: If $n \notin \mathbb{Z}$ then $f(z) = z^n$ is multiplied valued function.

$\because$ $e^{2\pi ni} \notin 1$, when $n \notin \mathbb{Z}$

Let $f: A \to B$ be a function then

i) If the elements of A are complex numbers and those of B are real numbers, then f is a real-valued function of a complex variable.

ii) If the elements of A are real numbers and those of B are complex numbers, then f is a complex-valued function of a real variable.

Let $f: \mathbb{R} \to \mathbb{R}$ be a function. The graph of f is a subset of $\mathbb{R} \times \mathbb{R}$ and is a two-dimensional object that can be represented well on a two-dimensional page. However, for a function $f: \mathbb{C} \to \mathbb{C}$, the graph is a subset of $\mathbb{C} \times \mathbb{C}$, which is equivalent to the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. This four-dimensional object cannot be represented directly on a two-dimensional plane. Instead, we use two separate planes: one for the z-plane (domain) and another for the w-plane (range).

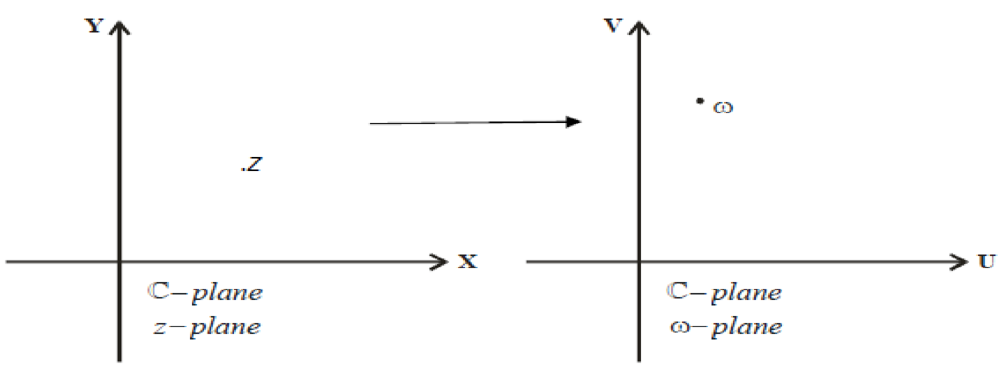

Fig.1.

2.14 *CONTINUITY:-*

A function $f: \mathbb{C} \to \mathbb{C}$ is said to be **continuous** at a point $z_0 \in \mathbb{C}$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $| z - z_0 | < \delta$, it follows that $| f(z) - f(z_0) | < \epsilon$. In other words, small changes in the input z near z_0 result in small changes in the output $f(z)$. The function f is continuous on a set $S \subseteq \mathbb{C}$ if it is continuous at every point in S.

OR

A function $f: D \to \mathbb{C}$ is said to be **continuous** at a point $z_0 \in D$ iff $\forall \in > 0, \exists\, \delta > 0$ s.t $|\, f(z) - f(z_0)\, | < \epsilon$ whenever $z \in D$ and $|\, z - z_0\, | < \in$.

<u>Uniform Continuous:</u> A function $f: D \to \mathbb{C}$ is said to be uniformly continuous D if for every $\in > 0\ \exists\, \delta > 0$ s.t. $\forall z_1$ and z_2 in D.

$$|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \in$$

<u>SOLVED EXAMPLE</u>

EXAMPLE3: If $f(z) = z^2$ then prove that f is continuous at a point $z = i \in \mathbb{C}$.

SOLUTION: Let $f(z) = z^2, z_0 = i$

$\therefore \quad f(i) = i^2 = -1$

$\therefore \quad \lim_{z \to i} z^2 = l^2 = -1$

$\therefore \quad \lim_{z \to i} z^2 = -1 = f(i)$

So f is continuous at a point $z = i$.

EXAMPLE4: Let $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$ prove that f is not continuous at a point $z = i$.

SOLUTION: $f(i) = 0$

$\therefore \lim_{z \to i} f(z) = \lim_{z \to i} l^2 = -1$

$\therefore \lim_{z \to i} f(z) = -1 \neq f(i)$

$\therefore$ f is not continuous at $z = z_0$.

EXAMPLE5: explain the continuity of $f(z) = \frac{z^2}{z^4 + 3z^2 + 1}$ at $z = e^{i\frac{\pi}{4}}$.

SOLUTION: $z = e^{i\frac{\pi}{4}}$

$\Rightarrow \quad z^2 = e^{i\frac{\pi}{2}} = i \Rightarrow z^4 = -1$

$\therefore\ f(z) = -\frac{i}{-1 + 3i + 1} = \frac{1}{3}$

$\therefore$ *the limit exist* $z = e^{i\frac{\pi}{4}}$

$\therefore f(z)$ is continuous at $z = e^{i\frac{\pi}{4}}$.

EXAMPLE6: Determine whether the function $f(x)$ is continuous at $x = 1$ where:

$$f(x) = \begin{cases} x^2 & if\ x \neq 1 \\ 3 & if\ x = 1 \end{cases}$$

SOLUTION: To determine whether $f(x)$ is continuous at $x = 1$, we need to check the three conditions of continuity.

1. **Check if the function is defined at $x = 1$**:
   The function is defined at $x = 1$, and we have:
   $$f(1) = 3$$
2. **Find the limit of the function as x approaches 1**:We need to find $\lim_{x\to 1} f(x)$ Since the function has two different cases, we consider the limit as x approaches 1 from the left ($x \to 1^-$) and from the right ($x \to 1^+$).
   For $x \to 1^-$ (approaching from the left), $f(x) = x^2$, so:
   $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$$
   For $x \to 1^+$ (approaching from the left), $f(x) = x^2$, so:
   $$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1^2 = 1$$
   Since both one-sided limits are equal, the limit lim $\lim_{x\to 1} f(x)$ exists and is equal to 1.
3. **Compare the limit with the function value at $x = 1$**: We have
   $$\lim_{x\to 1} f(x) = 1 \ \ or \ \ \lim_{x\to 1} f(1) = 3$$
   Since $\lim_{x\to 1} f(x) \neq f(1)$, the function is not continuous at $x = 1$.

Hence the function $f(x)$ is not continuous at $x = 1$ because the limit as x approaches 1 does not equal the function value at 1.

EXAMPLE7: Determine whether the function $g(x)$ is continuous at $x = 0$, where:

$$g(x) = f(x) = \begin{cases} \dfrac{\sin(x)}{x} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$$

We need to check the three conditions of continuity at $x = 0$.

1. Check if the function is defined at $x = 0$:
   The function is defined at $x = 0$, and:
   $$g(0) = 1$$

2. Find the limit of the function as x approaches 0:
   We need to find $\lim_{x\to 0} g(x)$. When $x \neq 0$, the function is given by $g(x) = \frac{\sin(x)}{x}$. So, we need to find:
   $$\lim_{x\to 0} \frac{\sin(x)}{x}$$
   This is a standard limit in calculus, and it is known that:
   $$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$
   Compare the limit with the function value at $x = 0$: We have
   $$\lim_{x\to 0} g(x) = 1 \ \ and \ \ g(0) = 1$$
   Since $\lim_{x\to 0} g(x) = g(0)$, the function is continuous at $x = 0$.

The function $g(x)$ is continuous at $x = 0$ because the limit as x approaches 0 equals the function value at 0. There is no discontinuity at this point, and the function behaves smoothly.

SELF CHECK QUESTIONS

1. What is a function?
2. What is a limit?
3. What does it mean for a function to be continuous?
4. What is a complex-valued function?
5. What is the range of a function?

2.15 *SUMMARY:-*

Functions: A function is a relation between two sets, typically denoted as $f: A \to B$, where each element $z \in A$ (the domain) is associated with a unique element $w \in B$ (the codomain). If z is a complex number $z = x + iy$ and $w = u + iv$, then $f(z) = u(x,y) + iv(x,y)$. The number $w = f(z)$ is called the value of the function at z. Functions can be real-valued or complex-valued, depending on whether their output values are real or complex numbers.

Limit: The limit of a function describes the behavior of the function as its input approaches a particular value. For a function $f(z)$ defined on a domain D and a point $z_0 \in \overline{D}$ (the closure of D), we say $\lim_{z\to z_0} f(z) = w_0$ $\forall \in > 0$ and there exists a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, we have $|f(z) - w_0| < \in$ For complex functions, this can be broken down into limits of the real and imaginary parts: $\lim_{z\to z_0} f(z) = w_0$ if and only if $\lim_{z\to z_0} u(x,y) = u_0$ *and* $\lim_{z\to z_0} u(x,y) = v_0$.

Continuity: A function f is continuous at a point z_0 if the limit of $f(z)$ as z approaches z_0 equals the function value at z_0 : $\lim_{z \to z_0} f(z) = f(z_0)$.Continuity can also be described in terms of ∈ and δ:f is continuous at z_0 if for every $\in > 0$, there exists a $\delta > 0$ such that if $|z - z_0| < \delta$, then$|f(z) - f(z_0)| < \in$.

Uniform Continuity: Uniform continuity strengthens the concept of continuity by requiring that the δ in the definition of continuity works uniformly over the entire domain D. A function f is uniformly continuous if for every $\in > 0$, there exists a $\delta > 0$ such that such that for all $z_{1,} z_2 \in D$ if $|z_1 - z_2| < \delta$, then$|f(z_1) - f(z_2)| < \in$.

2.16 *GLOSSARY:-*

- **Function:** A relation between a set of inputs (domain) and a set of possible outputs (co domain), where each input is related to exactly one output. It is typically written as $f(x)$, where x is the input, and $f(x)$ is the output.
- **Domain:** The set of all possible input values (or arguments) for which a function is defined. For example, the domain of $f(x) = 1/x$ is all real numbers except $x = 0$.
- **Piecewise Function:** A function that is defined by different expressions or rules over different parts of its domain. For example:

$$f(x) = \begin{cases} x + 1 & if\ x < 0 \\ x^2 & if\ x \geq 0 \end{cases}$$

- **Limit:** The value that a function approaches as the input approaches a certain point. Limits help describe the behavior of functions at points where they may not be explicitly defined. It is written as $\lim_{x \to a} f(x)$.
- **Left-Hand Limit:** The value that a function approaches as the input approaches a certain point from the left (i.e., from smaller values). It is written as $\lim_{x \to a^-} f(x)$.
- **Right-Hand Limit:** The value that a function approaches as the input approaches a certain point from the right (i.e., from larger values). It is written as $\lim_{x \to a^+} f(x)$.

- **Limit at Infinity:** Describes the behavior of a function as the input grows infinitely large (positive infinity) or infinitely small (negative infinity). It is written as as $\lim_{x\to\infty} f(x)$.or lim $\lim_{x\to-\infty} f(x)$.
- **Continuity:** A function is continuous at a point $x = a$ if the function is defined at a, the limit as x approaches a exists, and the limit equals the function's value at a (i.e., $\lim_{x\to a} f(x) = f(a)$.).
- **Continuous Function:** A function that is continuous at every point in its domain. This means the graph of the function has no breaks, jumps, or holes.
- **Discontinuity:** A point at which a function is not continuous. There are several types of discontinuities:
  - **Removable Discontinuity:** Occurs when the limit $\lim_{x\to a} f(x)$ exists, but is not equal to the function's value $f(a)$. The discontinuity can be "removed" by redefining the function value at $x = a$.
  - **Jump Discontinuity:** Occurs when the left-hand limit and right-hand limit at a point $x = a$ are not equal, leading to a "jump" in the graph.
  - **Infinite Discontinuity:** Occurs when the function approaches infinity or negative infinity as the input approaches a certain point.

2.17 *REFERENCES:-*

- James Ward Brown and Ruel V. Churchill (2009),Complex Variables and Applications.
- Ravi P. Agarwal, Kanishka Perera, and Sandra Pinelas (2011), An Introduction to Complex Analysis.
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.

2.18 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- file:///C:/Users/user/Desktop/1456304615ETextofChapter2Module1%20(1).pdf
- file:///C:/Users/user/Desktop/1456304677ETextofChapter2Module2%20(1).pdf

- https://old.mu.ac.in/wp-content/uploads/2020/12/Paper-III-Complex-Analysis.pdf

2.19 *TERMINAL QUESTIONS:-*

(TQ-1) Find the limit of the sequence $z_n = z^n$ for $|z| < 1$.

(TQ-2) Explain the definition of a function and provide an example.

(TQ-3) What is a limit in the context of a function, and how is it formally defined? Provide an example.

(TQ-4) Define continuity for a function and explain the difference between continuity and uniform continuity.

(TQ-5) Prove that if a function f is uniformly continuous on a set D, then it is also continuous on D.

(TQ-6) Determine whether the function $h(x)$ is continuous at $x = 2$, where:

$$h(x) = \begin{cases} x+2 & if\ x < 1 \\ x^2 & if\ x \geq 1 \end{cases}$$

2.20 *ANSWERS:-*

SELF CHECK ANSWERS

1. A function is a relation that assigns each input exactly one output
2. A limit is the value that a function approaches as the input approaches a certain point.
3. A function is continuous if it does not have any abrupt changes in value and the limit as the input approaches any point equals the function's value at that point.
4. A real-valued function is one where the outputs are real numbers, even if the inputs are complex.
5. A complex-valued function is one where the outputs are complex numbers, even if the inputs are real.

TERMINAL ANSWERS

(TQ-1) 0

(TQ-6) The function is continuous at $x = 2$.

UNIT 3:- Analytic Function (Cauchy Riemann Equation)

CONTENTS:

3.1 *INTRODUCTION:-*

An analytic function is a complex-valued function that is differentiable at every point in its domain and can be locally represented by a convergent power series. A key condition for a function to be analytic is that it satisfies the Cauchy-Riemann equations, which are a set of two partial differential equations that link the partial derivatives of the function's real and imaginary parts. Harmonic functions, which are closely related to analytic functions, are real-valued functions that satisfy Laplace's equation, meaning their second partial derivatives sum to zero. These functions often represent the real or imaginary parts of an analytic function.

3.2 *OBJECTIVES:-*

In this unit, we will explore the differentiability of complex-valued functions through their power series expansions, where a function is termed analytic around a point $z_0 \in \mathbb{C}$. An analytic function $f(z)$ must satisfy certain properties, notably the Cauchy-Riemann equations. Additionally, we will examine the term-by-term differentiation of power series, assuming it is feasible. The unit will also cover the inverse function theorem and introduce harmonic functions. Furthermore, we will discuss the differentiability of well-known complex functions such as e^z, $sin(z), cos(z)$.

3.3 *DEFINITION:-*

A single-valued function $f(z)$ is said to be **differential** at a point z_0 if there exists a neighborhood U around z_0 such that $f(z)$ is differentiable at every point $z \in U$. Mathematically, this can be expressed as: For all

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \text{ exist.}$$

This implies that the limit

$$\lim_{h\to 0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists and is finite, and this condition holds not just at z_0 but at every point in the neighborhood U of z_0.

Definition: The function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and at every point within some neighborhood of z_0.

In a broader context, $f(z)$ is analytic in a region R of the complex plane if it is analytic at every point within R.

The terms "regular" and "holomorphic" are often used interchangeably with "analytic" to describe functions that satisfy these conditions of differentiability in the complex plane.

Singular point: A point $z = z_0$ is said to be a singular point of a function $f(z)$ if $f'(z)$ does not exist.

3.4 *CAUCHY RIEMANN EQUATIONS:-*

The Cauchy-Riemann equations are a set of two partial differential equations which, together with certain continuity conditions, are necessary and sufficient for a complex function to be holomorphic. The Cauchy-Riemann equations are given by:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy$.

3.5 *CONJUGATE FUNCTION:-*

If $f(z) = u + iv$ is analytic and if u and v satisfy Laplace's equation $\nabla^2 V = 0$, then u and v are called conjugate harmonic functions or conjugate function simply.

3.6 *HARMONIC FUNCTION:-*

A function $u(x,y)$ is said to be harmonic function if first and second order partial derivatives of u are continuous and u satisfied Laplace's equation$\nabla^2 V = 0$.

Theorem1: Necessary condition for $f(z)$ to be analytic. If $f(z) = u + iv$ is analytic in a domain D, then u, v satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Provided the four partial derivatives u_x, u_y, v_x, v_y exist.

Proof: A function $w = f(z) = u + iv$ is analytic at a point z if it is differentiable $\frac{dw}{dz}$ exists so that $\frac{dw}{dz}$ has the same value along every path.

i. Along x −axis, $\delta z = \delta x$.

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta x \to 0} \frac{\delta w}{\delta x} = \frac{\partial w}{\partial x} \quad \text{... (1)}$$

ii. Along y −axis, $\delta z = i\delta y$.

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta y \to 0} \frac{\delta w}{i\delta y} = -i\frac{\partial w}{\partial y} \quad \text{... (2)}$$

From (1) and (2), $\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}$$

This $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Theses equations are called Cauchy Riemann equations.

Theorem2: Sufficient condition for $f(z)$ to be analytic. The function $w = f(z) = u + iv$ is analytic in a domain D if

i. u, v are differentiable in D and $u_x = v_y, u_y = -v_x$
ii. The partial derivatives u_x, v_y, u_y, v_x all are continuous in D.

Proof: Let $w = f(z) = u + iv = u(x,y) + iv(x,y) = f(x,y)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad i.e., u_x = v_y, u_y = -v_x \quad \text{... (1)}$$

For $f(z) = u(x,y) + iv(x,y)$ to be analytic in a domain D, u and v must be differentiable with continuous partial derivatives satisfying the Cauchy-Riemann equations, and increments $\delta z, \delta u, \delta v, \delta w$ correspond to the increments $\delta x, \delta y$ of x and y.

So

$$u_x \Rightarrow \delta u = u_x \delta x + u_y . \delta y + \alpha \delta x + \beta \delta y$$

Similarly

$$\Rightarrow \delta v = v_x \delta x + v_y . \delta y + \alpha_1 \delta x + \beta_1 \delta y$$

where $\alpha, \beta, \alpha_1, \beta_1$ all tends to zero as $\delta x \to 0, \delta y \to 0$

Now

$$\frac{\delta w}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y} \qquad \dots (2)$$

$$\delta u + i\delta v = \delta x(u_x + iv_x) + \delta y\left(u_y + iv_y\right) + (\alpha + i\alpha_1)\delta x + (\beta + i\beta_1)\delta y$$

$$= \delta x(u_x + iv_x) + i\delta y\left(-iu_y + v_y\right) + \alpha'\delta x + \beta'\delta y$$

Where $\alpha' = \alpha + i\alpha_1, \beta' = \beta + i\beta_1$

From (1)

$$\delta u + i\delta v = (u_x + iv_x)(\delta x + i\delta y) + \alpha'\delta x + \beta'\delta y$$

Dividing by $\delta x + i\delta y$ and then from(2)

$$\frac{\delta w}{\delta z} = u_x + iv_x + \frac{\alpha'\delta x}{\delta x + i\delta y} + \frac{\beta'\delta y}{\delta x + i\delta y}$$

$$\left|\frac{\delta w}{\delta z} - (u_x + iv_x)\right| = \left|\frac{\alpha'\delta x}{\delta z} + \frac{\beta'\delta y}{\delta z}\right| \le |\alpha'|.\left|\frac{\delta x}{\delta z}\right| + |\beta'|\left|\frac{\delta y}{\delta z}\right|$$

$$\le |\alpha'| + |\beta'| \text{ as } |\delta x| \le |\delta x + i\delta y|$$

$\left|\frac{\delta w}{\delta z} - \frac{\delta w}{\delta x}\right| \le |\alpha| + |\alpha_1| + |\beta| + |\beta_1|$ as $\alpha' = \alpha + i\alpha_1$

But when $\delta z \to 0$, the R.H.S$\to$ 0. Hence

$\lim\limits_{\delta z \to 0} \frac{\delta w}{\delta z} - \frac{\delta w}{\delta x} = 0$ or $\frac{dw}{dz} = \frac{\partial w}{\partial x} = u_x + iv_x$

But u_x, v_x exist. So $\frac{dw}{dz}$ hence w is analytic in D.

Note1. The equation $\frac{dw}{dz} = \frac{\partial w}{\partial x}$ is of vital importance for further study.

Note2. $f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$|f'(z)|^2 = (u_x)^2 + (v_x)^2 = u_x v_y - u_y v_x$$

Note3. $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$

$$d^2u = d[du] = \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}dx\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}dy\right)dy\right]$$

$$= \left[\frac{\partial^2 u}{\partial x^2}(dx)^2 + \frac{\partial^2 u}{\partial x\partial y}dxdy\right] + \left[\frac{\partial^2 u}{\partial y^2}(dy)^2 + \frac{\partial^2 u}{\partial y\partial x}dxdy\right]$$

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x\partial y}dxdy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

Similarly

$$d^2v = \frac{\partial^2 v}{\partial x^2}(dx)^2 + 2\frac{\partial^2 v}{\partial x\partial y}dxdy + \frac{\partial^2 v}{\partial y^2}(dy)^2$$

3.7 *POLAR FORM OF CAUCHY-RIEMANN EQUATIONS:-*

Theorem3: If $f(z) = u + iv$ is an analytic function and $z = re^{i\theta}$ where u, v, r, θ are all real, show that the Cauchy-Riemann Equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Or

To prove that the necessary and sufficient condition for $f(z)$ to be analytic in polar coordinates.

Proof: Let $f(z) = u + iv$ is an analytic function so that Cauchy-Riemann Equation

$$u_x = v_y \quad \ldots (1)$$
$$u_y = -v_x \quad \ldots (2)$$

are satisfied
to prove that , in view of above equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Let $x = r sin\theta, y = r cos\theta$.
Then $r^2 = x^2 + y^2, tan\theta = \frac{y}{x},\ \theta = \tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = cos\theta, \frac{\partial r}{\partial y} = \frac{y}{r} = sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} . \left(\frac{-y}{x^2}\right) = -\frac{sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} . \left(\frac{1}{x}\right) = \frac{cos\theta}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Or

$$\frac{\partial u}{\partial x} = cos\theta . \frac{\partial u}{\partial r} - \frac{sin\theta}{r} . \frac{\partial u}{\partial \theta} \quad \ldots (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial y} = sin\theta . \frac{\partial v}{\partial r} + \frac{cos\theta}{r} . \frac{\partial v}{\partial \theta} \quad \ldots (4)$$

Now, by (1), (3) and (4), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$cos\theta . \frac{\partial u}{\partial r} - \frac{sin\theta}{r} . \frac{\partial u}{\partial \theta} = sin\theta . \frac{\partial v}{\partial r} + \frac{cos\theta}{r} . \frac{\partial v}{\partial \theta} \quad \ldots (5)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = sin\theta\frac{\partial u}{\partial r} + \frac{cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = cos\theta\frac{\partial v}{\partial r} - \frac{sin\theta}{r}\frac{\partial v}{\partial \theta}$$

Since last two equations obtains

$$sin\theta\frac{\partial u}{\partial r} + \frac{cos\theta}{r}\frac{\partial u}{\partial \theta} = cos\theta\frac{\partial v}{\partial r} - \frac{sin\theta}{r}\frac{\partial v}{\partial \theta} \quad \ldots (6)$$

Now the equation (5) multiply by $\cos\theta$ and (6) by $\sin\theta$ obtains

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \dots(7)$$

From (6), we have

$$\frac{sin\theta}{r}.\frac{\partial v}{\partial \theta} + \frac{cos\theta}{r}.\frac{\partial u}{\partial \theta} = -cos\theta.\frac{\partial v}{\partial r} + \frac{sin\theta}{r}.\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \dots(8)$$

The above equations (7) and (8), which is the required results.

Theorem4: Derivative of w in polar form. To prove that

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$$

Proof: the Cauchy-Riemann Equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Let $x = rsin\theta, y = rcos\theta.$

Then $r^2 = x^2 + y^2, tan\theta = \frac{y}{x},\ \theta = \tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = cos\theta, \frac{\partial r}{\partial y} = \frac{y}{r} = sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y^2}{x^2}\right)}.\left(\frac{-y}{x^2}\right) = -\frac{sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y^2}{x^2}\right)}.\left(\frac{1}{x}\right) = \frac{cos\theta}{r}$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}.\frac{\partial \theta}{\partial x} = cos\theta\frac{\partial w}{\partial r} - \frac{sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$\frac{dw}{dz} = cos\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) - \frac{sin\theta}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \quad \dots(1)$$

$$= cos\theta\frac{\partial w}{\partial r} - \frac{sin\theta}{r}\left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)$$

$$= cos\theta\frac{\partial w}{\partial r} - isin\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= (cos\theta - isin\theta)\frac{\partial w}{\partial r} = e^{-i\theta}\frac{\partial w}{\partial r}$$

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} \quad \dots(2)$$

Now from (1), we obtain

$$\frac{dw}{dz} = cos\theta\left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right) - \frac{sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= -i\frac{cos\theta}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) - \frac{sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$-\frac{i}{r}(cos\theta - isin\theta)\frac{\partial w}{\partial\theta} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial\theta}$$
$$\frac{dw}{dz} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial\theta}$$
Which is required the results.

3.8 *ORTHOGONAL SYSTEM:-*

Two families of curves $u(x,y) = c_1, v(x,y) = c_2$ are said to form an **Orthogonal System** if they interest at right angles at each of their points of interaction.

Theorem5: If $f(z) = u + iv$ is an analytic function, in domain D, prove that the curves $u = const., v = const.$ from two orthogonal families.

Proof: Let $f(z) = u + iv$ is an analytic function so that Cauchy-Riemann Equation

$$u_x = v_y \quad \ldots(1)$$
$$u_y = -v_x \quad \ldots(2)$$

are satisfied

To prove that the curves $u(x,y) = const. = c_1$, $v(x,y) = const. = c_2$

Suppose

$m_1 = c_1 = u$(slope of tangent to the curve)

$m_2 = c_2 = v$(slope of tangent to the curve)

Now if we show that $m_1 m_2 = -1$, the result will be proved.

Taking

$$u = c_1$$
$$v = c_2$$
$$du = 0, dv = 0$$
$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0, \qquad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$
$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y}, m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}$$
$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = \frac{u_x v_x}{u_y v_y} = \frac{u_x v_x}{(-v_x)(u_x)} = -1$$

Theorem6: Real and imaginary parts of an analytic function satisfied Laplace's equation. That is to say, if $f(z) = u + iv$ is an analytic function of $z = x + iy$, then u, v satisfy Laplace equation.

Or. If $f(z) = u + iv$ is an analytic function of $z = x + iy$, then u and v both are harmonic functions.

Proof: Let $f(z) = u + iv$ is an analytic function so that Cauchy-Riemann Equation

$$u_x = v_y \quad \ldots(1)$$
$$u_y = -v_x \quad \ldots(2)$$

are satisfied

To prove that $\nabla^2 u = 0, \nabla^2 v = 0$

Differentiating (1) and (2) w.r.t. x and y and adding, we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=0$$

Or

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad or\ \nabla^2 u=0$$

Differentiating (1) and (2) w.r.t. x and y and subtracting, we get

$$\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial y^2}-\left(\frac{\partial^2 v}{\partial x^2}\right)=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad or\ \nabla^2 v=0$$

3.9 *MILNE'S THOMSON METHOD:-*

The Milne-Thomson method is a technique used to derive the Cauchy-Riemann equations in polar coordinates from their Cartesian form. It involves expressing a complex function $f(z)=u(x,y)+iv(x,y)$ in terms of polar coordinates, where $z=re^{i\theta}$, and then applying the chain rule to relate the partial derivatives in Cartesian coordinates to those in polar coordinates. This method ensures that the conditions for analyticity are preserved in the transformation.

We have $z=x+iy$ so that $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2i}$

$$w=f(z)=u+iv=\ u(x,y)+iv(x,y)$$

Or

$$f(z)=u\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)+iv\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)$$

Now let $x=z, y=0$ so that $z=\bar{z}$, we have

$$f(z)=u(z,0)+iv(z,0)$$

$$f'(z)=\frac{dw}{dz}=\frac{\partial w}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$=\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x} \qquad \text{by Cauchy Riemann equation}$$

Let we take

$$\frac{\partial u}{\partial x}=\emptyset_1(x,y)=\emptyset_1(z,0)$$

$$\frac{\partial u}{\partial y}=\emptyset_2(x,y)=\emptyset_2(z,0)$$

We obtain $\quad f'(z)=\emptyset_1(z,0)-\emptyset_2(z,0)$

Now integrating,

$$f(z)=\int[\emptyset_1(z,0)-i\emptyset_2(z,0)]dz+C$$

Where C is constant.

Similarly

$$f(z) = \int [\psi_1(z,0) - i\psi_2(z,0)]dz + C'$$

Where $\psi_1 = \frac{\partial v}{\partial y}, \quad \psi_2 = \frac{\partial v}{\partial x}$.

SOLVED EXAMPLE

EXAMPLE1: Find the analytic function $f(z) = u + iv$ of which the real part $u = e^x(xcosy - ysiny)$.

Solution: Now let

$$\frac{\partial u}{\partial x} = e^x(xcosy - ysiny) + e^x cosy$$
$$\frac{\partial u}{\partial y} = e^x(-xsiny - siny - ycosy)$$

$$\left(\frac{\partial u}{\partial x}\right)_{y=0} = e^x x + e^x = e^x(x+1)$$
$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = e^x 0 = 0$$

$$\phi_1(x,0) = \left(\frac{\partial u}{\partial x}\right)_{y=0} = e^x(x+1)$$
$$\phi_2(x,0) = \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

By Milne's method,

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] + c$$
$$= \int [e^z(z+1) - i.0]dz + c = \int (ze^z + e^z)dz + c$$
$$= (z-1)e^z + e^z + c = ze^z + c$$
$$f(z) = ze^z + c$$

EXAMPLE2: Find the analytic function $f(z) = u + iv$, where $u = e^{-x}[(x^2 - y^2)cosy + 2xy\, siny]$.

SOLUTON: Let $u = e^{-x}[(x^2 - y^2)cosy + 2xy\, siny]$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = -e^{-x}[(x^2 - y^2)cosy + 2xy\, siny] + e^{-x}[2xcosy + 2ysiny]$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^{-x}[-2ycosy - (x^2 - y^2)siny + 2x\, siny + 2xycosy]$$

Substituting $x = z, y = 0$ and so $cosy = 1, siny = 0$, we obtain

$$\phi(z,0) = -e^{-z}(z^2 + 2z), \phi_2(z,0) = 0$$

$$f(z) = \int (\phi_1 - i\phi_2)\, dz = \int (\phi_1 - i.0) = \int \phi_1\, dz$$
$$= -\int e^{-z}(z^2 + 2z)\, dz = e^{-z}(z^2 + 4z + 4) + c$$

EXAMPLE3: If $f(z) = u + iv$ is analytic function and $u - v = e^x(cosy - siny)$, find $f(z)$ in term of z.

SOLUTION: Given $f(z) = u + iv \quad \ldots(1)$

$$u - v = e^x(cosy - siny) \quad \ldots(2)$$

Again from (1)

$$if(z) = iu - v \quad \ldots(3)$$

Adding (1) + (3), we have

$$(1 + i)f = (u - v) + i(u + v)$$

Let we take $u - v = U, u + v = V, (1 + i)f = F(z)$

We get $F(z) = U + iV$

By (2), we obtain

$$U = e^x(cosy - siny)$$

Take $\phi_1(x, y) = \frac{\partial U}{\partial x}, \phi_2(x, y) = \frac{\partial U}{\partial y}$

Then

$$\phi_1(x, y) = e^x(cosy - siny), \phi_2(x, y) = e^x(-siny - cosy)$$

This $\Rightarrow \phi_1(z, 0) = e^z(cos0 - sin0) = e^z, \phi_1(z, 0) = e^x(-sin0 - cos0) = -e^z$

By Milne's method,

$$F(z) = c + \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz = c + \int e^z\,(1 + i)dz$$
$$(1 + i)f = c + (1 + i)e^z$$
$$f(z) = c_1 + e^z$$

EXAMPLE4: To prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}}$.

SOLUTION: $z = x + iy, \bar{z} = x - iy$

$$x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i} = \frac{-i}{2}(z - \bar{z})$$

This $\Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}},\ \frac{\partial y}{\partial \bar{z}} = \frac{i}{2} = -\frac{\partial y}{\partial z}$

Let $f = f(x, y)$. Then $f = f(z, \bar{z})$ also

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$
$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

EXAMPLE5: Show that the harmonic function satisfies the differential equation:

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

SOLUTION: Let we know that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} \qquad \ldots (1)$$

Let u is harmonic function$\Rightarrow \nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Now from (1), we get

$$4\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

EXAMPLE6: If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2|f'(z)|^2$$

SOLUTION: $f(z) = u + iv,\ Rf(z) = u$

$$\frac{\partial}{\partial x}u^2 = 2u\frac{\partial u}{\partial x}$$

Differentiation Again

$$\frac{\partial^2 u^2}{\partial x^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$

Similarly

$$\frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$$

Now adding

$$\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]$$

But u satisfies Laplace's equation, we get

$$\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u.(0)$$
$$= [u_x^2 + v_x^2] \qquad \text{For } u_y = -v_x$$

But

$$f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now at last gives

$$\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2=2|f'(z)|^2$$

EXAMPLE7: Prove that the function $f(z)=xy+iy$ is everywhere continuous but not analytic.

SOLUTION: Let given that $f(z)=xy+iy$

$$\frac{\partial u}{\partial x}=y,\frac{\partial v}{\partial y}=1,\frac{\partial u}{\partial y}=x,\frac{\partial v}{\partial x}=0.$$

From which $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$

This implies that the Cauchy-Riemann equations are not satisfied.

$\Rightarrow$ $f(z)$ is not analytic.

EXAMPLE8: If ϕ and ψ are functions of x and y satisfying Laplace's equation show that $s+it$ is analytic, where

$$s=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},t=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}$$

SOLUTION: Let us suppose $\phi(x,y)$ and $\psi(x,y)$ satisfy Laplace equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Let $s=\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},t=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}$

Now to prove that $s+it$ is analytic function, we obtain to show that

$$\frac{\partial s}{\partial x}=\frac{\partial t}{\partial y},\frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x}$$

$i.e.$,

$$\frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=0,\frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=0$$

$and\ s_x,s_y,t_x,t_y$ all are continuous.

$$\frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)$$

$$=\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\phi}{\partial y\partial x}-\frac{\partial^2\psi}{\partial y^2}=0$$

$$-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0$$

$$\therefore\quad \frac{\partial s}{\partial x}-\frac{\partial t}{\partial y}=0$$

$$\frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)$$

$$\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\psi}{\partial y\partial x}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\partial y}=0$$

$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0$$

Also

$$\frac{\partial s}{\partial y}+\frac{\partial t}{\partial x}=0$$

EXAMPLE9: Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine corresponding analytic function.

SOLUTION: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=3x^2-3y^2+6x$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=-6xy-6y$$

$$\frac{\partial^2 u}{\partial x^2}=6x+6$$

$$\frac{\partial^2 u}{\partial y^2}=-6x-6$$

From the above equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$\therefore$ u is satisfies Laplace's equation. Substituting $x = z, y = 0$ in above equations

$\phi_1(z,0) = 3z^2 + 6z, \phi_2(z,0) = 0.$

By Milne's Thomson method

$$f(z)=\int[\phi_1(z,0)-i\phi_2(z,0)]dz$$

$$\int[3z^2+6z-i.0]\,dz=z^3+3z^2+c$$

EXAMPLE10: If $u = x^3 - 3xy^2$, show that there exists function $v(x,y)$ such that $w = u + iv$ is analytic in a finite region.

SOLUTION: Let given that $u = x^3 - 3xy^2$

$$\frac{\partial u}{\partial y}=-6xy,\frac{\partial u}{\partial x}=3x^2-3y^2$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\left(\frac{\partial u}{\partial y}\right)dx+\left(\frac{\partial u}{\partial x}\right)dy$$

$$6xydx+(3x^2-3y^2)dy=Mdx+Ndy$$

$$\frac{\partial M}{\partial y}=6x=\frac{\partial N}{\partial x}$$

$\therefore$ $Mdx + Ndy$ is exact. So that

$$\int dv=\int 6xydx+\int -3y^2dy=3x^2y-y^3+c$$

$$f(z)=u+iv=(x^3-3xy^2)+i(3x^2y-y^3+c)$$

$$=(x+iy)^3+ic=z^3+ic$$

So that $f(z)$ is analytic in a finite domain.

SELF CHECK QUESTIONS

1. What is an analytic function?
2. State the Cauchy-Riemann equations.
3. What is the significance of the Cauchy-Riemann equations?
4. What is the relationship between analyticity and harmonic functions?
5. Explain the concept of analytic continuation.
6. What is the Milne-Thomson method used for?
7. How does the Milne-Thomson method work?
8. What is the advantage of using the Milne-Thomson method in complex analysis?
9. The function $f(z) = tanz$ is
   a. continuous everywhere
   b. analytic in finite complex plane
   c. analytic everywhere except the points where $cosz = 0$
   d. none
10. A function of x and y possessing continuous partial derivatives of first and second order is called harmonic function if it is satisfied
    a. Euler equation
    b. Laplace equation
    c. Homogenous equation
    d. Lagrange equation
11. An analytic function with constant modulus is:
    a. Variable
    b. May be Variable and constant
    c. Constant
    d. None
12. Which of the following functions is not analytic
    a. $sinz$
    b. $cosz$
    c. $ax^2 + bz + c = 0$
    d. $\frac{1}{z-1}$
13. True/False statements

    (i). Cauchy Riemann equation is sufficient for a function to be analytic.

    (ii). The function $w = |z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

    (iii). An analytic function with constant modulus is constant.

    (iv). If $f(z) = u + iv$ is an analytic function, then being given one of u and v, the other can b determined.

    (v). An analytic function cannot have a constant absolute value without reducing to a constant.

3.10 *SUMMARY:-*

An analytic function, or holomorphic function, is a complex function that is differentiable at every point in its domain, meaning it can be represented by a convergent power series around any point within its domain. These functions satisfy the Cauchy-Riemann equations, ensuring their real and imaginary parts are harmonic. Key properties include infinite differentiability and the ability to be expressed in a power series. Important theorems associated with analytic functions are Liouville's theorem, the maximum modulus principle, and Cauchy's integral theorems. Applications of analytic functions span across conformal mappings, complex integration, and potential theory in physics and engineering.

3.11 *GLOSSARY:-*

- **Analytic Function:** A complex function that is differentiable at every point in its domain. Also known as a holomorphic function.
- **Cauchy-Riemann Equations:** A set of partial differential equations that must be satisfied for a function to be analytic. For $f(z) = u(x, y) + iv(x, y)$, the equations are:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
- **Harmonic Function:** A function $u(x, y)$ or $v(x, y)$ that satisfies Laplace's equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$
- **Analytic Continuation:** The process of extending the domain of an analytic function beyond its original domain while maintaining its analyticity.
- **Complex Plane:** A two-dimensional plane representing complex numbers, with the horizontal axis as the real part and the vertical axis as the imaginary part.
- **Holomorphic:** Another term for an analytic function, emphasizing its differentiability properties.
- **Potential Theory:** A field of study using harmonic and analytic functions to solve problems in physics and engineering related to potentials, such as gravitational or electrostatic potentials.
- **Complex Differentiability:** A condition for a function $f(z)$ to be differentiable with respect to z in the complex plane. This requires the existence of the limit:
$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

- **Holomorphic:** Another term for an analytic function, emphasizing its differentiability and the satisfaction of the Cauchy-Riemann equations.
- **Necessary Condition:** A condition that must be true for a function to be analytic. The Cauchy-Riemann equations are necessary conditions for complex differentiability.
- **Sufficient Condition:** A condition that, if true, guarantees a function is analytic. If a function satisfies the Cauchy-Riemann equations and its partial derivatives are continuous, it is analytic.
- **Complex Plane:** A two-dimensional plane used to represent complex numbers, with the horizontal axis as the real part and the vertical axis as the imaginary part.
- **Partial Derivative:** A derivative of a function with respect to one variable, treating other variables as constants. In the context of the Cauchy-Riemann equations, partial derivatives are taken with respect to x and y.
- **Continuity:** A property of a function where small changes in the input result in small changes in the output. For the Cauchy-Riemann equations to imply analyticity, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ must be continuous.
- **Complex Conjugate:** For a complex number $z = x + iy$, its complex conjugate is $\bar{z} = x - iy$. The Cauchy-Riemann equations involve differentiating with respect to x and y, not $\bar{z}$.
- **Complex Potential:** A complex function representing potential flow in fluid dynamics, where the real part is the velocity potential and the imaginary part is the stream function. The Cauchy-Riemann equations ensure that the flow is irrotational and incompressible.

3.12 *REFERENCES:-*

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- Hilary A. Priestley (2nd Edition, 2019, Oxford University Press), Introduction to Complex Analysis.

3.13 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.

- https://old.mu.ac.in/wp-content/uploads/2020/12/Paper-III-Complex-Analysis.pdf
- Robert E. Greene and Steven G. Krantz (4th Edition, 2020), Function Theory of One Complex Variable.

3.14 *TERMINAL QUESTIONS:-*

(TQ-1) Show that $u = (1/2)\log(x^2 + y^2)$ s harmonic and find its harmonic conjugate.

(TQ-2) Prove that an analytic function with constant real part is constant.

(TQ-3) If $f(z)$ is analytic function with constant modulus, then it is constant.

(TQ-4) If $f(z) = u + iv$ is an analytic function of $z = x + iy$, and ψ is a function of x and y possessing partial differential coefficients of the first two orders, show that

i. $\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \left[\left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2\right]|f'(z)|^2$

ii. $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\right)|f'(z)|^2$

(TQ-5) An incompressible fluid flowing xy-plane has velocity potential

$$\phi = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

Find stream function ψ so that $w = \phi + i\psi$ is analytic.

(TQ-6) Prove that the function $e^x(cosy + isiny)$ is holomorphic and find its derivative.

(TQ-7) Find an analytic function whose real part is $e^x cosy$.

(TQ-8) Show that the function $u = cosxcoshy$ is harmonic and find the harmonic conjugate.

(TQ-9) show that the function $f(z) = |xy|^{1/2}$is not regular at the origin, although the Cauchy Riemann equations are satisfied at the point.

(TQ-10) the Explain the nature of the function $w = f(z) = z^{1/3}$

i. Show that $f(z) = \bar{z}$ is continuous at $z = z_0$ but not analytic at $z = z_0$

ii. Prove that the function $f(z) = \cos(z)$ is continuous at everywhere analytic.

(TQ-11) Prove that continuity is necessary but not sufficient condition for the existence of a finite derivative.

(TQ-12) Find the analytic function $w = u + iv$ if

i. $u = x^3 - 3xy^2$

ii. $u = e^x cosy$

3.15 *ANSWERS:-*

SELF CHECK ANSWERS

1. An analytic function, also known as a holomorphic function, is a complex function that is differentiable at every point in its domain. Furthermore, its derivative must also be continuous. This implies that the function can be represented as a convergent power series in some neighborhood around any point in its domain.

2. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ $\quad i.e., u_x = v_y, u_y = -v_x$

3. The Cauchy-Riemann equations provide necessary and sufficient conditions for a function to be analytic. They ensure that the function is differentiable in the complex sense. If a function satisfies these equations and has continuous first partial derivatives, it is guaranteed to be analytic.

4. Harmonic functions are twice continuously differentiable functions that satisfy Laplace's equation, implying that they are solutions to the heat equation in steady state.

5. Analytic continuation is a technique to extend the domain of a given analytic function beyond its initial domain. This is done by defining the function on a larger domain such that the new function agrees with the original function on their common domain. The extended function remains analytic in the new domain.

6. The Milne-Thomson method is used to find the real and imaginary parts of a complex function, given its complex form. It is particularly useful in fluid dynamics and conformal mapping problems, where it helps to convert complex potential functions into their real and imaginary components.

7. The Milne-Thomson method works by expressing a complex function $f(z) = u(x, y) + iv(x, y)$ in terms of z and its conjugate $\bar{z}$. The method involves writing the complex function $f(z)$ in terms of z and $\bar{z}$, and then using the relationships between the real and imaginary parts to separate them.

8. The advantage of using the Milne-Thomson method is that it provides a systematic way to separate the real and imaginary parts of a complex function, which is essential in many applications, such as solving physical problems in fluid dynamics and electromagnetic. It simplifies the process of finding these parts without directly differentiating the function.

9. c **10.** b **11.** c **12.** d

13. **(i).** F **(ii).**T **(iii).**T **(iv).** T **(v).** T

UNIT 4:- Power Series

CONTENTS:

4.1 *INTRODUCTION:-*

A power series is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-c)^n$ are complex coefficients, z is a complex variable, and c is the center of the series. Power series are fundamental in complex analysis because they represent functions as sums of infinitely many terms that depend on the distance from the center c. The series converges within a certain radius, called the radius of convergence, which can be determined using various convergence tests. Within this radius, a power series can be used to represent analytic functions, providing a powerful tool for understanding their properties and behavior.

4.2 *OBJECTIVES:-*

After studying this unit, learners will be able to

- Understand and identify the structure and components of power series, including their terms, coefficients, and center. They will be able to determine the radius and interval of convergence for power series and apply convergence tests to assess where a series converges.

- Learners will also be able to represent analytic functions as power series, use these series to approximate functions, and compute derivatives and integrals of functions expressed in this form.

4.3 *POWER SERIES:-*

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

are complex coefficients, z is a complex variable, and c is a constant known as the center of the series. It represents a function as a sum of terms involving powers of $(z-c)$. The power series converges within a certain radius of convergence, R, and diverges outside of this radius. Within this radius, the power series can be used to express analytic functions and analyze their properties.

OR

A series of the form

$\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-a)^n$

is called a power series, where a_n, a are a complex constant and z is a complex variable. The second form $\sum a_n (z-a)^n$ ccan be simplified to the first form by substituting $z = \zeta + a$ yielding

$$\sum a_n (z-a)^n = \sum a_n \zeta^n$$

The first form is simpler than the second form. Hence

$\sum_{n=0}^{\infty} a_n z^n$ or simply $\sum a_n z^n$

in our discussion.

4.4 *ABSOLUTE CONVERGENCE OF* $\sum a_n z^n$*:-*

The concept of absolute convergence is crucial when dealing with power series. A series $\sum_{n=0}^{\infty} a_n z^n$ is said to be absolutely convergent if the series formed by taking the absolute value of each term,

$$\sum |a_n z^n| = \sum |a_n||z^n|$$

converges.

The power series $\sum a_n z^n$ is said to be conditionally convergent if $\sum a_n z^n$ is convergent but $\sum |a_n||z|^n$ is not convergent.

4.5 *SOME SPECIAL TEST FOR CONVERGENCE OF SERIES:-*

Convergence tests are essential tools in mathematical analysis to determine whether a series converges (i.e., approaches a finite limit) or diverges. Here are some special tests commonly used to check for the convergence of series:

1. If $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.
2. If $\lim_{n\to\infty} \frac{u_n}{v_n} =$ finite non-zero quantity, then the two series $\sum u_n$ and $\sum v_n$ have identical nature.
3. **Comparison Test:** $\sum u_n$ is absolutely convergent if
$$|u_n| \leq |v_n|$$
4. **Root Test:** For a series u_n:
   Compute $L = \lim_{n\to\infty} |u_n|^{1/n}$
   - If $L < 1$, the series converges absolutely.
   - If $L > 1$, the series diverges.
   - If L=1, the test is inconclusive.
5. **Ratio Test:** For a series u_n:
   Compute $L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ or > 1
   - If $L < 1$, the series converges absolutely.
   - If $L > 1$or $L = \infty$, the series diverges.
   - If L=1, the test is inconclusive.
6. **p-Series Test:** For a series of the form $\frac{1}{n^p}$:
   - The series converges if $p > 1$
   - The series diverges if $p \leq 1$
7. **Dirichlet's Test:** The series $\sum a_n u_n$ is convergent if
   i. $|S_n| = |\sum_{i=1}^{n} a_i| \leq k \forall n$, k being a finite number.
   ii. $\lim_{n\to\infty} u_n = 0$
   iii. $\sum(u_n - u_{n+1})$ is convergent.
8. **Integral Test:** If $f(x)$ is a continuous, positive, decreasing function for $x \geq 1$ and $a_n = f(n)$, then the series $\sum a_n$ converges if and only if the improper integral $\int_1^\infty f(x)dx$ converges.
9. **Absolute Convergence Test:** If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges absolutely.
10. **p-series test:** For a series of the form $\sum \frac{1}{n^p}$

- The series converges if $p > 1$.
- The series diverges if $p \leq 1$.

These tests are commonly used to determine the convergence or divergence of series. Each test has its specific conditions and is useful in different scenarios.

4.6 *RADIUS OF CONVERGENCE OF POWER SERIES:-*

Let the power series $\sum a_n z^n = \sum u_n(z)$,say, $\sum u_n$ is convergent if

$$\lim_{n\to\infty} |u_n|^{1/n} < 1$$

This$\Rightarrow \lim_{n\to\infty} |a_n z^n|^{\frac{1}{n}} < 1 \Rightarrow \lim_{n\to\infty} |a_n|^{\frac{1}{n}}. |z| < 1$

Taking $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, we obtain

$\frac{|z|}{R} < 1$ or $|z| < R$.

Thus $\sum a_n z^n$ is convergent or divergent according as

$|z| < R$ or $|z| > R$.

∴ For a given power series n, there exists a non-negative real number R, known as the radius of convergence. This radius R determines the region in the complex plane where the series converges. Specifically, the series converges absolutely for all complex numbers z such that $| z | < R$ and diverges for $| z | > R$.

Now if we draw a circle of radius R with centre at the origin, then define in one paragraph

i. The power series $\sum a_n z^n$ is convergent for every z within the circle. $(|z| < R)$
ii. The power series $\sum a_n z^n$ is divergent for every z outside the circle. $(|z| > R)$

The circle of radius R, centered at the origin, is called the circle of convergence for the power series $\sum a_n z^n$, and the radius R is referred to as the radius of convergence. There are three possibilities for R:

i. $R = 0$
In case series in convergent only when $z = 0$.

ii. R is finite
In this case series is convergent at every point within this circle and divergent at every point outside it.

iii. R is infinite

iv. In this case series is convergent$\forall\ z$.

4.7 *SUM FUNCTION OF A POWER SERIES:-*

The **sum function of a power series** is a function that represents the value to which the series converges for each point within its radius of convergence. Given a power series of the form:

$$\sum a_n z^n$$

the sum function $f(z)$ is defined as:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where z is a complex variable and a_n are the coefficients of the series.

4.8 *THEOREMS:-*

Theorem1: The power series $\sum a_n z^n$ either

i. Converges for every z

ii. Converges only for $z = 0$

iii. Converges for some values of z

Proof:

i. Let the power series $\sum \frac{z^n}{n!}$.

Comparing this with $\sum u_n(z)$, we find that

$$\frac{u_{n+1}}{u_n} = \frac{z^{n+1}}{(n+1)!}.\frac{n!}{z^n} = \frac{z}{(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}.|z| = 0 < 1$$

Hence the power series $\sum \frac{z^n}{n!}$ is convergent for every z.

ii. Let the power series $\sum z^n\, n! = \sum u_n$, then $\lim_{n\to\infty}|u_n| = \lim_{n\to\infty} n!.|z|^n$

$$= \begin{cases} 0, if\ z = 0 \\ \infty\ \ if\ \ z \neq 0 \end{cases}$$

$\therefore \sum u_n, i.e., \sum n\,!\, z^n$ is convergent if $z = 0$ and divergent if $z \neq 0$.

iii. The power series $\sum z^n$ is convergent if $|z| < 1$ and is not convergent if $|z| \geq 1$.

Theorem2: If the power series $\sum a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for every z for which $|z| < |z_0|$.

Proof: Let the power series $\sum a_n z^n$ is convergent for $z = z_0$ so that $\sum a_n z^n$ is convergent. So

$$\lim_{n\to\infty} \sum a_n z_0^n = 0 \qquad \text{...(1)}$$

To prove that $\sum a_n z^n$ is convergent $\forall\, z$ for which $|z| < |z_0|$.

From (1), there exist a real positive constant $M > 0$, we obtain

$$|a_n z_0^n| \leq M \forall\, n$$

Now

$$|a_n z_0^n| \leq M \left|\frac{z}{z_0}\right|^n$$

But $\sum \frac{|z|^n}{|z_0|^n}$ is convergent $\forall\, z$ s.t.,

$$\frac{|z|}{|z_0|} < 1, \;\; i.e., \quad |z| < |z_0|$$

$\therefore$ By comparison test, $\sum |a_n z_0^n|$ is convergent $\forall\, z\; s.t.\;\; |z| < |z_0|$.

Consequently $\sum a_n z_0^n$ is absolutely convergent $\forall\, z\; s.t. |z| < |z_0|$.

Theorem3: For every power series $\sum a_n z^n$, there exists a number R such that $0 \leq R \leq \infty$ with the following properties:

i. The series converges absolutely for every z such that $|z| < R$.

ii. The series diverges if $|z| > R$.

Proof: Given the power series $\sum_{n=0}^{\infty} a_n z^n$, we need to find the radius of convergence R such that the series converges for $|\, z \,| < R$ and diverges for $|\, z \,| > R$.

i. Convergence within the radius:

Let $r < R$ and z be such that $|\, z \,| = r$. By the definition of the limit superior, for any $\epsilon > 0$, there exists an integer N such that for all $n > N$,

$$|a_n|^{1/n} < \frac{1}{\mathrm{R}} + \in$$

Thus, for $|\, z \,| = r < R$, we have:

$$|a_n z^n| = |a_n||z^n| = |a_n| r^n < \frac{1}{R} + \in^n r^n$$

Since $r < R$, there exists $\epsilon > 0$ such that $r\left(\frac{1}{R} + \epsilon\right) < 1$. Therefore, the terms$|a_n z_0^n|$ are bounded above by a geometric series with a ratio less than 1, ensuring the series converges absolutely.

ii. Divergence outside the radius:

Now, consider $|z| > R$. We want to show that the series $\sum_{n=0}^{\infty} a_n z^n$ diverges in this case. Assume$|z| = r > R$. By the definition of R, for any $\epsilon > 0$, there exists infinitely many n such that:

$$|a_n|^{1/n} < \frac{1}{\mathrm{R}} - \in$$

Choosing $\in = \frac{1}{2R}$, we find

$$|a_n| > \left(\frac{1}{R} - \frac{1}{2R}\right)^n = \left(\frac{1}{2R}\right)^n$$

Thus, for sufficiently large n,

$$|a_n z^n| = |a_n||z^n| = |a_n| r^n > \left(\frac{1}{2R}\right)^n r^n$$

Since $r > R$, the ratio $\frac{r}{2R} > 1$ and the terms $|a_n z^n|$ grow without bound, leading to the divergence of the series.

Theorem 4: To show that the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$, obtained by differentiating power series$\sum a_n z^n$, has the same radius of convergence as the original series $\sum a_n z^n$.

Proof: Let the original power series be $\sum_{n=0}^{\infty} a_n z^n$, and let R be its radius of convergence. According to the Cauchy-Hadamard theorem, the radius of convergence R is given by:

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

Consider the differentiated series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Differentiating term by term, we get:

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

We aim to prove that the series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R as the original series.

The radius of convergence of the differentiated series is given by:

$$\frac{1}{R'} = \lim_{n\to\infty} |n a_n|^{1/(n-1)}$$

To find R', consider: $\lim_{n\to\infty} |n a_n|^{1/(n-1)}$. Using above properties, we can simplify the expression:

$$\lim_{n\to\infty} |n a_n|^{1/(n-1)} = \lim_{n\to\infty} |n|^{1/(n-1)} |a_n|^{1/(n-1)}$$

Since, $\lim_{n\to\infty} n^{1/n} = 1$, we get

$$\lim_{n\to\infty} n^{1/(n-1)} = 1$$

Thus, the radius of convergence R' is:

$$\frac{1}{R'} = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$$

Therefore, $R' = R$, and the differentiated series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence as the original series $\sum_{n=0}^{\infty} a_n z^n$.

SOLVED EXAMPLE

EXAMPLE1: Prove that the series $1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c.(c+1)} z^2 + \cdots$ has unit radius of convergent.

SOLUTION: The given series is

$$1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c.(c+1)} z^2 + \cdots$$

We can write the general term a_n as:

$$a_n = \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{1.2\ldots n.c\,(c+1)\ldots\ldots(c+n-1)}$$

$$a_{n+1} = \frac{a(a+1)\ldots(a+n-1)(a+n)b(b+1)\ldots(b+n-1)(b+n)}{1.2\ldots n\,(n+1)c\,(c+1)\ldots\ldots(c+n-1)(c+n)}$$

Simplifying, we get:

$$\frac{a_{n+1}}{a_n} = \frac{(n+a)(n+b)}{(n+1)(c+n)} = \frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}$$

Now, taking the limit as $n \to \infty$:

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(1+0)(1+0)}{(1+0)(1+0)}\right| = 1$$

$$R = 1$$

EXAMPLE2: Find the radius of convergence of the series $\frac{z}{2} + \frac{1.3}{2.5} z^2 + \frac{1.3.5}{2.5.8} z^3 + \cdots$

SOLUTION: The coefficient of z^n of the given power series is given by

$$a_n = \frac{1.3.5\ldots(2n-1)}{2.5.8\ldots(3n-1)}$$

$$a_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.5.8\ldots(3n-1)(3n+2)}$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{3n-1} = \frac{2}{3}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}\frac{(1+0)}{(1+0)} = \frac{2}{3}$$

$$R = \frac{3}{2}$$

EXAMPLE3: Find the convergence of the series$\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n$.

SOLUTION: Given the series:

$$\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n$$

To apply the ratio test, we find:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \left(\frac{z^2+1}{1+i}\right)^{n+1}}{(n)^2 \left(\frac{z^2+1}{1+i}\right)^n}$$

$$= \frac{(n+1)^2}{(n)^2} \left|\frac{z^2+1}{1+i}\right|$$

$$= \left(\frac{n+1}{n}\right)^2 \left|\frac{z^2+1}{1+i}\right|$$

Now, we take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \left|\frac{z^2+1}{1+i}\right|$$

To express this condition in terms of$z$, we calculate the modulus of the denominator:

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

So the inequality becomes:

$$\left|\frac{z^2+1}{\sqrt{2}}\right| < 1$$

$$|z^2+1| < \sqrt{2}$$

This condition determines the region in the complex plane where the series converges. Specifically, the series converges for all z such that the absolute value of z^2+1 is less than$\sqrt{2}$.

EXAMPLE4: Examine the behavior of power series $\sum_{n=2}^{\infty} \frac{z^n}{n(logn)^2}$ on the circle of convergence.

SOLUTION: The general term of the series is:

$$a_n = \frac{1}{n(logn)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n(logn)^2}{(n+1)[\log(n+1)]^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)\left[\frac{\log n\left(1+\frac{1}{n}\right)}{logn}\right]^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)\left[\frac{\left\{1+log\left(1+\frac{1}{n}\right)\right\}}{logn}\right]^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)\left[1+\frac{1}{nlogn}-\frac{1}{2n^2logn}+\cdots\right]^2}$$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{1}.1 = 1$$

$$R = 1$$

The series converges inside the circle $|z| < 1$ and diverges outside $|z| > 1$. Now, we must examine the series' behavior on the circle of convergence, $|z| = 1$.

Behavior on the Circle $|z| = 1$:

To analyze convergence on the circle $|z| = 1$, also $\sum_{n=2}^{\infty}\frac{z^n}{n(logn)^2}$ is convergent, by Cauchy's condensation test. Hence $\sum_{n=2}^{\infty}\frac{z^n}{n(logn)^2}$ is absolutely convergent $\forall\, z$ on the circle of convergence.

EXAMPLE5: For what value of z, does the series $\sum\frac{1}{(z^2+1)^2}$ convergence, and find its sum.

SOLUTION: The given series is

$$u_n = \frac{1}{(z^2+1)^2}, u_{n+1} = \frac{1}{(z^2+1)^{n+1}}$$

Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{1}{z^2+1}\right|$$

Since the series is convergent if

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$$

or if $\left|\frac{1}{z^2+1}\right| < 1$ if $1 < |z^2+1|$.

Therefore, the series convergent for $|z^2+1| > 1$.

Let $S_n(z)$ be the sum of n terms of given series.

$$S_n = \sum_{1}^{n}\frac{1}{(z^2+1)^n} = \frac{1}{z^2+1} + \frac{1}{(z^2+1)^2} + \cdots ..to\; n\; terms$$

$$= \frac{1}{z^2+1}\frac{\left[1-\frac{1}{(z^2+1)^n}\right]}{1-\frac{1}{z^2+1}} \quad \text{as} \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{1}{z^2}\left[1 - \frac{1}{(z^2+1)^n}\right]$$

$$lim S_n = \frac{1}{z^2} \quad as \quad \frac{1}{|z^2+1|} < 1.$$

EXAMPLE6: Find the domain of convergence of the power series $\sum\left(\frac{2i}{z+i+1}\right)^n$.

SOLUTION: The given series can be rewritten as:

$$\sum\left(\frac{2i}{z+i+1}\right)^n$$

We have $u_n = \left(\frac{2i}{z+i+1}\right)^n, u_{n+1} = \left(\frac{2i}{z+i+1}\right)^{n+1}$

Now

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2i}{z+i+1}\right| = \frac{2}{|z+i+1|}$$

The given series is convergent if $\frac{2}{|z+i+1|} < 1$.

$$|z+i+1| > 2$$

Thus the inequality $|z+i+1| > 2$ describes the set of all complex numbers z such that the distance between z and $-i-1$ is greater than 2. This represents the exterior of a circle in the complex plane with center $-i-1$ and radius 2.

EXAMPLE7: For what values of z does the series $\sum(-1)^n\,(z^n + z^{n+1})$ converge and find its sum.

SOLUTION: First, rewrite the series as follows:

$$u_n = (-1)^n(z^n + z^{n+1})$$

$$u_{n+1} = (-1)^{n+1}(z^{n+1} + z^{n+2})$$

$$= -(-1)^n z(z^n + z^{n+1}) = -zu_n$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}|-z| = |z|$$

The series is convergent if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ or if $|z| < 1$.

Hence the series is convergent inside the circle of radius one and centre at $z = 0$.

Suppose $S_n(z) =$ sum of n terms of the series.

Then

$$S_n = (1 - z + z^2 - z^3 + \cdots to\ n\ terms) + (z - z^2 + z^3 + \cdots to\ n\ terms)$$

$$= (1 - z + z^2 - z^3 + \cdots to\ n\ terms) + z(1 - z + z^2 - z^3 + \cdots to\ n\ terms)$$

$$= (1+z)(1 - z + z^2 - z^3 + \cdots to\ n\ terms)$$

$$= (1+z)\frac{[1-(-z)^n]}{1+z} = 1 - (-z)^n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}[1-(-z)^n] = 1, \ for\ |z| < 1$$

∴ Sum of series =1

SELF CHECK QUESTIONS

1. What is the radius of convergence for the power series$\sum_{n=0}^{\infty}\frac{n!z^n}{n^n}$?
2. For which values of z does the series $\sum_{n=0}^{\infty}\frac{z^n}{3^n}$?
3. What kind of singularity does the function represented by $\sum_{n=0}^{\infty} n!\, z^n$ have at infinity?
4. What is the interval of convergence of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} z^n$?

4.9 *SUMMARY:-*

A power series is an infinite sum of the form $\sum_{n=0}^{\infty} a_n z^n$, where are coefficients and z is a variable. The radius of convergence R, defines the interval where the series converges absolutely: it converges for $|\, z\, |< R$ and diverges for $|\, z\, |> R$, with boundary convergence needing separate verification. The radius is determined by the formula $\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n}$. Power series can be differentiated and integrated term by term, retaining the same radius of convergence, and they are used to represent analytic functions that are infinitely differentiable within their interval of convergence.

Power series are fundamental in many areas of mathematics and engineering, including calculus, differential equations, and complex analysis. They provide a way to represent functions and can be used to approximate functions over certain intervals.

4.10 *GLOSSARY:-*

- **Power Series:** An infinite series of the form $\sum_{n=0}^{\infty} a_n z^n$, where an are coefficients, z is a variable, and n is a non-negative integer.
- **Coefficient:** The a_n in a power series, representing the weight of the n −th term z^n.
- **Variable:** The z in a power series, often representing a complex or real number.
- **Term:** An individual component of the power series,$a_n z^n$.
- **Radius of Convergence (R):** A non-negative number that determines the interval in which the power series converges absolutely. Calculated as
- $\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n}$.
- **Convergence:** The condition where the partial sums of a series approach a finite limit as the number of terms increases.
- **Absolute Convergence:** When the series $\sum|a_n z^n|$ converges. This implies the convergence of the original series.

- **Divergence:** The condition where a series does not converge, meaning the partial sums do not approach a finite limit.
- **Analytic Function:** A function that can be represented by a power series within its radius of convergence. Such functions are infinitely differentiable in this interval.
- **Term-by-Term Differentiation:** The process of differentiating a power series term by term, resulting in a new series $\sum_{n=0}^{\infty} na_n z^{n-1}$ with the same radius of convergence as the original.
- **Term-by-Term Integration:** The process of integrating a power series term by term, resulting in a new series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$plus a constant, with the same radius of convergence as the original.
- **Limit Superior (lim sup):** The limit of the supremum (upper bound) of the tail end of a sequence. Used in calculating the radius of convergence.
- **Interval of Convergence:** The range of values of z for which the power series converges. It includes all z for which |z|<R and may or may not include boundary points where |z|=R.
- **Analytic Function:** A function defined by a power series within its radius of convergence is called an analytic function. Such functions are infinitely differentiable within the interval of convergence.

4.11 *REFERENCES:-*

- Dennis G. Zill and Patrick D. Shanahan(2013 Third Edition), Complex Analysis: A First Course with Applications.
- William F. Trench(2013), Introduction to Real Analysis.
- John B. Conway(2020), A First Course in Analysis.
- Hilary A. Priestley (2nd Edition, 2019, Oxford University Press), Introduction to Complex Analysis.

4.12 *SUGGESTED READING:-*

- file:///C:/Users/user/Desktop/1468562409EText(Ch-6,M-3%20(1).pdf
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- Igor Kriz and Ales Pultr(2013),Introduction to Mathematical Analysis.

4.13 *TERMINAL QUESTIONS:-*

(TQ-1) Prove that the sum function $f(z)$ of the power series $\sum a_n z^n$ represents an analytic function inside the circle of convergence.

(TQ-2) If R_1 and R_2 are the radii of convergence of the power series $\sum a_n z^n$ and $\sum b_n z^n$ respectively, then show that the radius of convergence of the power series $\sum a_n b_n z^n$ is $R_1 R_2$.

(TQ-3) Show that the domain of convergence of the series $\sum\left(\frac{iz-1}{2+i}\right)^n$ is given by $|z+i| < \sqrt{5}$.

(TQ-4) Find the domain of convergence of the following series:

$$\sum \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$$

(TQ-5) Find the radius of convergence of power series $\sum_{n=1}^{\infty} \frac{(n+1)z^n}{(n+2)(n+3)}$.

(TQ-6) Discuss the behavior of power series $\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}$ on the circle of convergence.

(TQ-7) Objectives types questions:

i. If the series $\sum a_n z^n$ is convergent but the series $\sum |a_n z^n|$ is not convergent, then $\sum a_n z^n$ is said to be

a. divergent
b. oscillatory
c. conditionally convergent
d. finite

ii. The radius of the convergence of the series $\sum_{n=1}^{\infty} n^n z^n$ is:

a. 0
b. 1
c. 2
d. 3

iii. The radius of the convergence of the series $\sum_{n=0}^{\infty} z^{n!}$ is

a. 1
b. 2
c. 3
d. 4

iv. The radius of the convergence of the series $\sum 2^n z^{n!}$ is

a. 1
b. 2
c. 0
d. N

v. The alternating series test guarantees convergence if:

a. The terms decrease and have a non-zero limit.
b. The terms decrease and have a limit of zero.
c. The terms increase and have a limit of zero.
d. The terms are positive and bounded.

(TQ-8) True/False type Questions

a. A power series can converge conditionally at a point on its circle of convergence. T
b. The radius of convergence of a power series can be zero. T
c. A power series can be used to represent a function that is not differentiable. F
d. The power series $\sum_{n=0}^{\infty} z^n$ converges for all complex numbers z. F
e. If a power series converges at $z = 3$, it must also converge at $z = -3$. F

(TQ-9) If $\sum_{n=0}^{\infty} a_n$ converges, then prove that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ tends to $f(1)$ as $z \to 1$ in such a manner that $\frac{|1-z|}{1-|z|}$ remains bounded.

(TQ-10) Find the radii of convergence of the following power series:

a. $\sum \frac{2+in}{2^n} z^n$ b. $\sum 2^{\sqrt{n}} z^n$

c. $\sum \left(1+\frac{1}{n}\right)^{n^2} z^2$ d. $\sum \frac{z^n}{2^n+1}$

$e.$ $\sum \frac{z^n}{n^n}$ f. $\sum \frac{2^{-n} z^n}{1+in^2}$

g. $\sum \frac{(n!)^2}{(2n)!} z^n$ h. $\sum \frac{n!}{n^n} z^n$

(TQ-11) Investigate the behavior of $\sum \frac{z^n}{n}$ on the circle of convergence.

(TQ-12) Examine the behavior of the power series $\sum_{n=0}^{\infty} \frac{z^{4n}}{1+4n}$ on the circle of convergence.

(TQ-13) For what values of z does the series $\sum (-1)^n (z^n + z^{n+1})$ converge and find its sum.

(TQ-14) Prove that the series $1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c.(c+1)} z^2 + \cdots$ has unit radius of convergent.

(TQ-15) Show that the domain of convergence of series $\sum \left(\frac{iz-1}{2+i}\right)^n$ is given by $|z+i| < \sqrt{5}$.

(TQ-16) If the radius of the convergence of power series $\sum_{n=0}^{\infty} a_n z^n$ is a positive real number R, then prove that the function f(z) defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $|z| < R$.

4.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. 1
2. 3
3. Essential
4. $|z| < 1$

TERMINAL ANSWERS

(TQ-4) The series is absolutely convergent for every finite value of z.

(TQ-5) $R = 1$

(TQ-7) 1.(c) 2.(a) 3.(a) 4.(a) 5.(b)

(TQ-8) a.T b.T c.F d.F e.F

(TQ-10) a.$R = 2$ b.$R = 1$ c.$R = 1/e$ d.$R = 2$
e. $R = \infty$ f. $R = 2$ g. $R = 4$ h. $R = e$

(TQ-13) 1

BLOCK II
CONFORMAL MAPPING

UNIT 5:-Conformal Mapping

CONTENTS:

5.1 *INTRODUCTION:-*

Conformal mapping is a fundamental concept in complex analysis that focuses on functions which preserve angles between intersecting curves. These mappings are characterized by being analytic functions with non-zero derivatives in their domains, ensuring that small shapes are mapped similarly, though their size may change. The key property of conformal mappings is their ability to maintain the local geometry of figures, specifically the angles and the orientation of intersections, making them invaluable for simplifying and analyzing complex shapes and patterns. The applications of conformal mapping extend across various scientific and engineering fields. In fluid dynamics, for example, conformal maps are used to transform complex flow patterns into simpler ones, aiding in the analysis and solution of potential flow problems. In electromagnetism, they help simplify the geometry of problems, making it easier to solve Maxwell's equations in complex domains. Additionally, conformal maps are crucial in cartography, where they preserve angles, making them useful for navigation and map projections like the Mercator projection, which represents the globe on a flat surface.

5.2 *OBJECTIVES:-*

After studying this unit, learners will be able to

- To understand the fundamental concepts and transformation of mapping.
- To identify and apply conformal transformation.
- To prove theoretical results.

These objectives make conformal mapping a versatile and powerful tool across various disciplines, allowing for the transformation and analysis of complex systems and shapes while preserving essential geometric properties.

5.3 *CONFORMAL MAPPING:-*

A transformation or mapping defined by the equations $u = u(x, y)$ and $v = v(x, y)$ establishes a correspondence between points in the xy −plane and the uv −plane. If each point in the xy −plane uniquely maps to a point in the uv −plane and vice versa, this transformation is termed a one-to-one (bijective) transformation. The points corresponding in the two planes are considered images of each other. Through such a transformation, regions or curves in the xy − plane are mapped onto corresponding regions or curves in the uv −plane, effectively representing one plane's features in terms of another.

Jacobian transformation:

Let's consider a transformation from coordinates (x, y) to (u, v) defined by the functions:

$$u = u(x, y) and v = v(x, y).$$

The Jacobian matrix J for this transformation is given by derivatives of the functions:

$$J = \frac{\partial(x, y)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

The determinant of the Jacobian matrix, denoted as $det(J)$ or $\mid J \mid$, is:

$$det(J) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

By Cauchy –Riemann Equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$det(J) = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \left(-\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$\left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = \left|\frac{\partial w}{\partial x}\right|^2 = \left|\frac{dw}{dx}\right|^2 = |f'(z)|^2 \quad for \quad \frac{dw}{dx} = \frac{\partial w}{\partial x}$$

Thus

$$\frac{\partial(x,y)}{\partial(x,y)} = |f'(z)|^2 \text{ , if f(z) is analytic.}$$

5.4 *TRANSFORMATION OF CONFORMAL:-*

In complex analysis, a transformation $u = u(x, y), v = v(x, y)$ that maps two curves $C_1$and C_2, intersecting at a point $P(z_0)$ in the z-plane, onto curves $C_1{}'$and $C_2{}'$, intersecting at $P'(z_0)$ in the w-plane, is classified based on how it preserves angles. If the angle between $C_1$and C_2 at z_0 is the same as the angle between $C_1{}'$and $C_2{}'$atw_0, the transformation is called isogonal. An isogonal transformation maintains the magnitude of the angles but not necessarily their orientation. If the transformation also preserves the sense of rotation (i.e., the orientation), it is called conformal. This means that conformal transformations maintain both the magnitude and the clockwise or counterclockwise nature of the angles.

For example, the transformation $w = \bar{z}$, where $\bar{z}$ is the complex conjugate of z , is isogonal because it preserves the angles but reverses the orientation (flipping clockwise to counterclockwise), distinguishing it from conformal transformations which preserve both the angle magnitude and orientation.

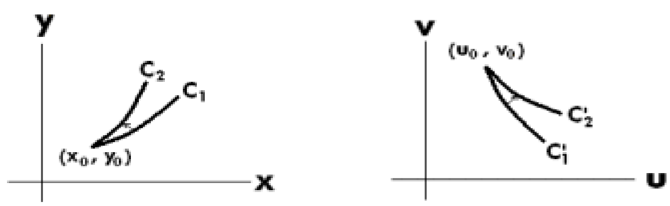

Fig.1

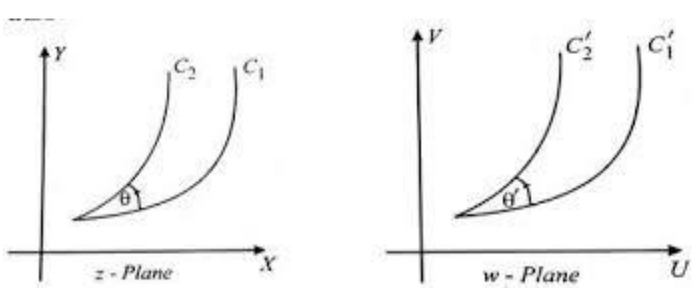

Fig.2

5.5 *SOME GENERAL TRANSFORMATIONS:-*

Transformations involving complex numbers can perform various geometric operations on figures in the complex plane. Here are explanations of some common types of transformations:

1. **Translation:** The transformation of the form $w = z + a$, where a is a complex constant, is known as a translation. This transformation shifts all points in the z −plane uniformly in the direction and magnitude specified by a.

   EXAMPLE: Let a rectangular domain R be bounded by $x = 0, y = 0, x = 2, y = 1$. Determine the region R'of w-plane into which R is mapped under the transformation $w = z + (1 - 2i)$.

   SOLUTION: Let the given transformation

$$w = z + (1 - 2i)$$
$$u + iv = x + iy + (1 - 2i)$$

$$\Rightarrow \quad u = x + 1, v = y - 2$$

By the map $u = x + 1$, the line $x = 0, x = 2$ are mapped respective on the lines $u = 1, u = 3$.

Again the map $v = y - 2$, the lines $y = 0, y = 1$ are mapped on $v = -2,\ v = -1$ respectively.

Since the required image R' bounded by $u = 1, u = 3, v = -2,\ v = -1$ in w-plane as shown in figure.

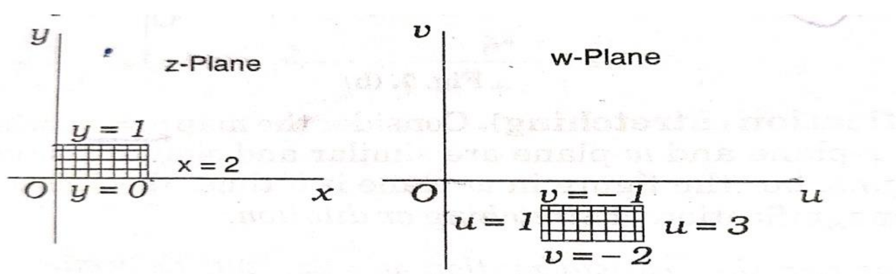

Fig.3

2. **Rotation:** The transformation $w = e^{i\theta}z$, where θ is a real constant, is known as a rotation. Here, iθ represents a complex number on the unit circle (with modulus 1) corresponding to an angle θ in radians. This transformation rotates figures in the z −plane by an angle θ around the origin. If $\theta > 0$, the rotation is anti-clockwise; if $\theta < 0$, the rotation is clockwise.

   EXAMPLE: Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w −plane corresponding to triangular region R bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in z-plane.

SOLUTION: Let $w = ze^{i\pi/4}$ gives $u + iv = (x + iy)\frac{(1+i)}{\sqrt{2}}$

$\Rightarrow \quad u = \frac{1}{\sqrt{2}}(x - y), v = \frac{1}{\sqrt{2}}(x + y) \quad \dots(1)$

Substituting $x = 0$ in above equation

$$u = -\frac{1}{\sqrt{2}}y, v = \frac{1}{\sqrt{2}}y$$

$$v = -u$$

Again substituting $y = 0$ in above equation

$$u = \frac{1}{\sqrt{2}}x, v = \frac{1}{\sqrt{2}}x$$

$$v = u$$

Substituting $x + y = 1$ in (1), we obtain

$$v = \frac{1}{\sqrt{2}}$$

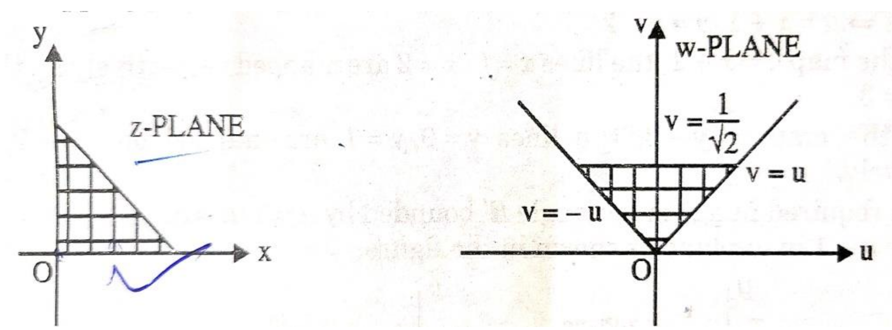

Fig.4

3. **Stretching (Magnification):** The transformation $w = az$, where a is a real constant, is called stretching (or scaling). This transformation stretches or contracts figures in the z-plane depending on the value of a. If $a > 1$, figures are stretched (scaled up) away from the origin, and if $0 < a < 1$, figures are contracted (scaled down) towards the origin. This transformation alters the size but not the shape of the figures.

EXAMPLE: Consider the transformation $w = 2z$ and determine the region R' in w −plane corresponding to triangular region R bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in z-plane is mapped under the map.

SOLUTION: Let $w = 2z$ gives $u + iv = 2x + i2y$

$\Rightarrow \quad u = 2x, v = 2y$

$\Rightarrow \quad x = 0, u = 2x \Rightarrow u = 0$

$\Rightarrow \quad y = 0, v = 2y \Rightarrow v = 0$

$\Rightarrow \quad x + y = 1, u = 2x, v = 2y \Rightarrow u + v = 2(x + y) = 2.1 = 2$

$\Rightarrow \quad u + v = 2$

Hence the required image R' bounded by $u = 0, v = 0, u + v = 2, \; v = -1$ as shown in figure.

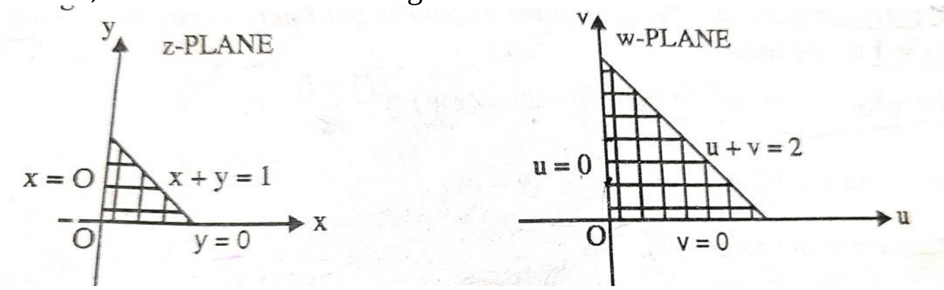

Fig.5

4. Inversion: The mapping $w = \frac{1}{z}$ is known as inversion. This transformation maps each point zzz in the complex plane to a new point www such that the product of their magnitudes is 1 (i.e., $|z||w| = 1$) and the argument (angle) is preserved but reversed in sign. The origin, however, is mapped to infinity, and vice versa. Inversion changes the scale of figures based on their distance from the origin, inverting them relative to the unit circle. For example, points inside the unit circle are mapped outside and vice versa.

EXAMPLE: Consider the map $w = \frac{1}{z}$ and determine the region R' in w-plane of infinite strip R-bounded by $\frac{1}{4} < y < \frac{1}{2}$.

SOLUTION: Let $w = \frac{1}{z}$ gives $u + iv = \frac{x-iy}{x^2+y^2}$

$\Rightarrow \quad u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2} \quad \Rightarrow \frac{u}{v} = -\frac{x}{y} \Rightarrow x = -\frac{uy}{v}$

$\Rightarrow \quad v = -\frac{y}{x^2+y^2} \quad \Rightarrow \quad v = -\frac{y}{\frac{y^2u^2}{v^2}+y^2} = -\frac{v^2}{y(u^2+v^2)}$

$\Rightarrow \quad y = -\frac{v}{u^2+v^2}$

$\Rightarrow \quad y < \frac{1}{2} \quad \Rightarrow \quad -\frac{v}{u^2+v^2} < \frac{1}{2} \quad \Rightarrow -v < \; u^2 + v^2 \quad \Rightarrow \; u^2 + (v + 1)^2 > 1$

$\Rightarrow \quad y < \frac{1}{4} \quad \Rightarrow \quad -\frac{v}{u^2+v^2} > \frac{1}{4} \quad \Rightarrow -4v < \; u^2 + v^2 \quad \Rightarrow \; u^2 + (v + 2)^2 > 2^2$

So

$\frac{1}{2} < y < \frac{1}{4} \quad \Rightarrow \quad u^2 + (v + 2)^2 > 2^2$ and $u^2 + (v + 1)^2 > 1$

$u^2 + (v + 2)^2 = 2^2$ and $u^2 + (v + 1)^2 = 1$

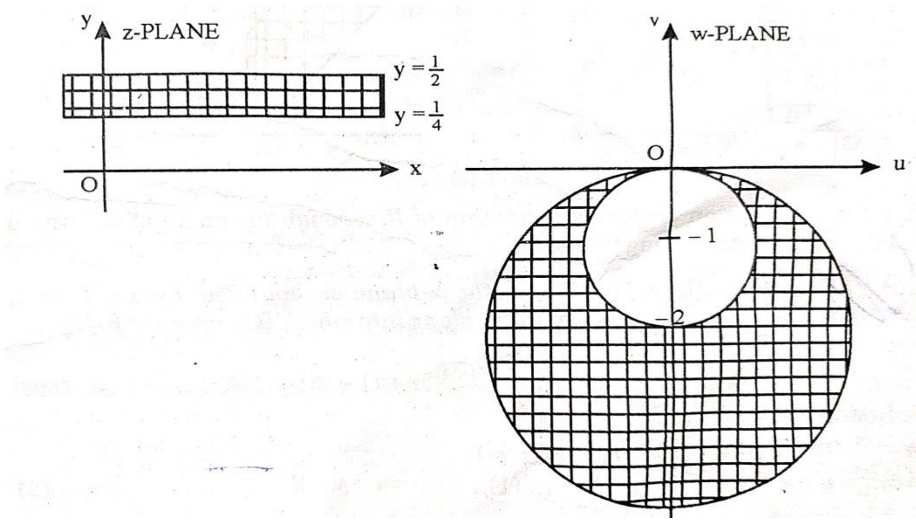

Fig.6

Theorem1: To prove that at each point z of a domain where $f(z)$ is analytic and $f'(z) \neq 0$,the mapping $w = f(z)$ is conformal.

Proof: Let $w = f(z)$ be an analytic function in a domain D on the z −plane, and z_0 be a point in the interior of D. Consider two curves C_1 and C_2 in the z −plane that intersect at z_0) . These curves are mapped to the w- plane under the function f, resulting in the curves $C_1^{'}$ and $C_2^{'}$ intersecting at $w_0 = f(z_0)$. Let z_1 and z_2 be points on C_1 and C_2, respectively, close to z_0. These points are mapped to $w_1 = f(z_1)$ and $w_2 = f(z_2)$on $C_1^{'}$ and $C_2^{'}$, respectively.

distance between z_1 and z_0= distance between z_2 and $z_0 = r$. So we can write

$$z_1 - z_0 = re^{i\theta_1}, z_2 - z_0 = re^{i\theta_2}$$

The tangent at z_0 to the curves C_1 and C_2 make angles α_1, α_2 with real axis to that $\theta_1 \to \alpha_1, \theta_2 \to \alpha_1 \; as \; r \to 0$.

Also the tangent at w_0 to the curves $C_1{}'$ and $C_2{}'$ make angles β_1, β_2 with real axis and let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, z_2 - z_0 = \rho_2 e^{i\phi_2}$$

Where $\phi_1 \to \beta_1 \; as \rho_1 \to 0$, $\phi_2 \to \beta_2 \; as \; \rho_2 \to 0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}}$$

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$$

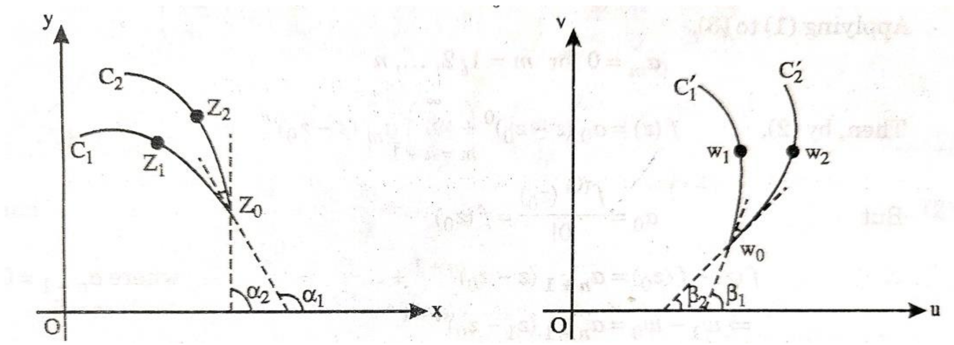

Fig.7

So our assumption $f'(z_0) \neq 0$. we write $f'(z_0) = Re^{i\lambda}$

$$Re^{i\lambda} = \lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$$

Now equating modulus and argument

$$R = \lim_{z_1 \to z_0} \frac{\rho_1}{r}, \lambda = \lim(\phi_1 - \theta_1) = \beta_1 - \alpha_1$$

$$\beta_1 = \lambda + \alpha_1$$

Similarly

$$\beta_2 = \lambda + \alpha_2$$

So

$$\beta_1 - \beta_2 = \alpha_1 - \alpha_2$$

This proof demonstrates that the angle between the images of the curves $C_1{}'$ and $C_2{}'$ at the point w_0 in the www-plane is both equal in magnitude and identical in sign to the angle between the original curves C_1 and C_2 at the point z_0 in the $z-$ plane. Consequently, this shows that the transformation $w = f(z)$ preserves both the size and orientation of angles, thus confirming that the transformation is conformal.

Theorem2: To the study of conformal property when $f'(z) = 0$ if $f(z)$ is a regular function of z.

Solution: Let us consider the transformation $w = f(z)$ is conformal at $z = z_0$ if $f(z)$ is analytic at $z = z_0$ and $f'(z) \neq 0$. Let us examine the case when $f'(z) = 0$. Suppose that $f'(z)$ has a zero of order n at z_0 so that

$$f'(z) = (z - z_0)^n \phi(z)$$

where $\phi(z)$ is analytic and $\phi(z_0)\ ) \neq 0$.

$$\Rightarrow \quad f'(z_0) = f''(z_0) = \cdots f^{(n)}(z_0) = 0, f^{(n+1)}(z_0) \neq 0 \quad \ldots (1)$$

Expending by Taylor theorem

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \quad \ldots (2)$$

where

$$a_m = \frac{f^{(m)}(z_0)}{m!} \quad \ldots (3)$$

Now we apply (1) to (3)

$$a_m = 0 \quad for\ m = 1,2,3 \dots n$$

By (2)

$$f(z) = a_0(z - z_0)^0 + \sum_{m=n+1}^{\infty} a_m\,(z - z_0)^m$$

Since $$a_0 = \frac{f^{(0)}(z_0)}{m!} = f(z_0)$$

∴ $$f(z) - f(z_0) = a_{n+1}(z - z_0)^{n+1} + \cdots$$

Where $a_{n+1} \neq 0$

⇒ $$w_1 - w_0 = a_{n+1}(z - z_0)^{n+1}$$

Now we taking

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, z_1 - z_0 = re^{i\theta_1}, a_{n+1} = \alpha e^{i\lambda}$$

We obtain

$$\rho_1 e^{i\phi_1} = \alpha r_1^{n+1} e^{i[(n+1)\theta_1 + \lambda]}$$

⇒ $lim\phi_1 = lim[(n+1)\theta_1 + \lambda] = (n+1)a_1 + \lambda$

Similarly

$$lim\phi_2 = (n+1)a_2 + \lambda$$

The curves $C_1{}'$ and $C_2{}'$ intersect at w_0 with an angle that is $(n+1)$ times the angle between the curves C_1 and C_2 intersecting at z_0, then the conformal property fails at z_0.

Theorem3: If the mapping $w = f(z)$ is conformal, then show that $f(z)$ is an analytic function of z.

Proof: Given the conformal transformation $w = u(x,y) + iv(x,y) = f(z)$., where u and v are the real and imaginary parts of $f(z)$, the line elements ds in the z −plane and $d\sigma$ in the w-plane respectively so that

$$ds^2 = dx^2 + dy^2, d\sigma^2 = du^2 + dv^2$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Squaring and adding

$$d\sigma^2 = du^2 + dv^2$$

$$= \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)dx^2 + \left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)dy^2 + 2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]dxdy$$

So the transformation is conformal and hence the ratio $\frac{d\sigma^2}{ds^2}$ is of direction-independent and Compare with

$$ds^2 = dx^2 + dy^2$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0}$$

$\Rightarrow$ $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \quad ..(1)$$

and $$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \quad ...(2)$$

From(2),

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial y}} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y}} = \lambda$$

$$\frac{\partial u}{\partial x} = \lambda \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\lambda \quad ...(3)$$

Substituting these values in (1), we obtain

$$\lambda^2\left(\frac{\partial v}{\partial y}\right)^2 + \lambda^2\left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$(\lambda^2 - 1)\left[\left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2\right] = 0$$

$\Rightarrow$ $\lambda^2 - 1 = 0 \ \Rightarrow \lambda = \pm 1$

Using in(3), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \ \ when\ (\lambda = 1) \quad ..(4)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \ \ when\ (\lambda = -1)$$

Hence the equations (4) are Cauchy Riemann equations.

SOLVED EXAMPLE

EXAMPLE1: Show that $w = iz + i$ map half plane $x > 0$ onto half plane$v > 1$.

SOLUTION: Gives $w = iz + i$

$$u + iv = i(x + iy) + i$$

$$\Rightarrow \left.\begin{matrix} u = -y \\ v = x + 1 \end{matrix}\right\} \Rightarrow \ x = v - 1$$

Using $x > 0$

$\Rightarrow$ $v - 1 > 0 \Rightarrow v > 1$

EXAMPLE2: Consider the map $w = ze^{i\pi/4}\sqrt{2}$ and determine the region R' of w-plane into which the rectangular region R bounded by $x = 0, y = 0, x = 2, y = 3$ in z-plane is mapped under this map.

SOLUTION: Let $w = ze^{i\pi/4}\sqrt{2}$ gives

$$u + iv = (x + iy)\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{2}$$

$$\Rightarrow \quad u = x - y, v = x + y \quad \dots (1)$$

Substituting $x = 0$ in (1), $u = -y, v = y$ so that $v = -u$

Substituting $y = 0$ in (1), $u = x, v = x$ so that $v = u$

Substituting $y = 3$ in (1), $u = x - 3, v = x + 3$ so that $v - 3 = u + 3$ or $v = u + 6$

Substituting $x = 2$ in (1), $u = 2 - y, v = 2 + y$ or $2 - u = v - 2$ or $v = 4 - u$

The region R' is a rectangle in the uv-plane bounded by the lines $v = u$, $v = -u, v = u + 6$ and $v = 4 - u$, as depicted in the following figures.

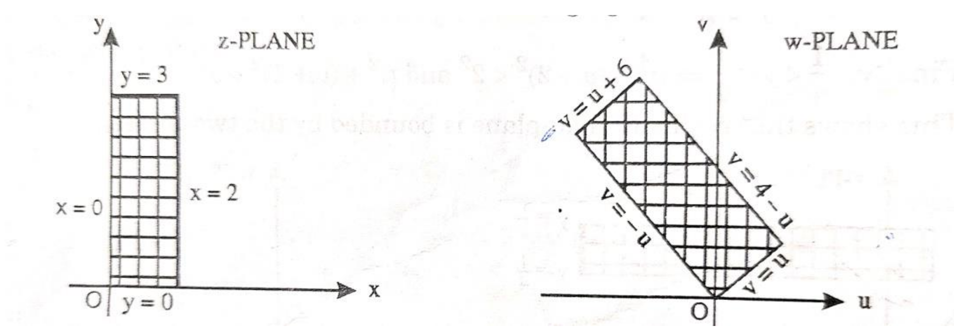

Fig.8

This transformation performs a rotation of R through an angle of 45^0 and magnification of length of magnitude$\sqrt{2}$.

EXAMPLE3: A rectangle region R in z −plane is bounded by $x = 0, y = 0, x = 2, y = 1$, determine the region R' of the w-plane into which R is mapped under the transformation: $w = ze^{i\pi/4}\sqrt{2}.z + (1 - 2i)$

SOLUTION:$w = ze^{i\pi/4}\sqrt{2}.z + (1 - 2i)$

$$u + iv = (1 + i)(x + iy) + (1 - 2i)$$

$$u = x - y + 1 \quad \dots (1)$$

$$v = x + y - 2 \quad \dots (2)$$

Substituting $y = 0$ in (1) & (2), we get

$\Rightarrow \quad u = x + 1, v = x - 2 \Rightarrow \quad u = (v + 2) + 1 \ \Rightarrow u = v + 3$

$\Rightarrow \quad \frac{u}{3} + \frac{v}{(-3)} = 1$

Substituting $y = 2$ in (1) & (2), we obtain

$\Rightarrow \quad u = x, v = x - 1 \Rightarrow \quad u = v + 1$

$\Rightarrow \quad \frac{u}{1} + \frac{v}{(-1)} = 1$

Substituting $x = 0$ in (1) & (2), we have

$\Rightarrow \quad u = -y + 1, v = y - 2 \Rightarrow \quad v = (1 - u) - 2$

$\Rightarrow \quad \frac{u}{(-1)} + \frac{v}{(-1)} = 1$

Substituting $x = 2$ in (1) & (2), we have

$\Rightarrow \quad u = 3 - y, v = y \Rightarrow \quad u = 3 - v$

$\Rightarrow \quad \frac{u}{3} + \frac{v}{3} = 1$

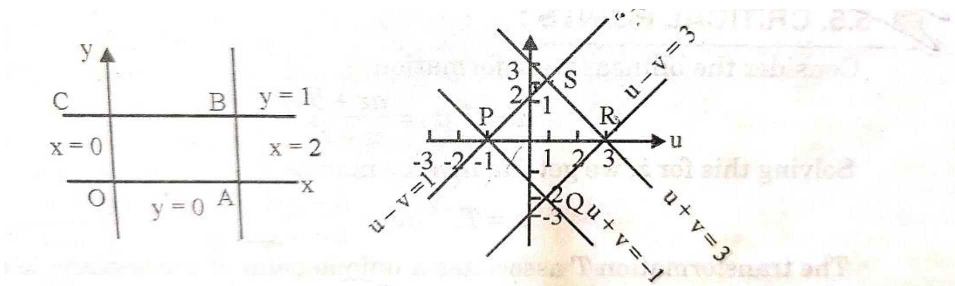

Fig.9

Where R is rectangle OABC.

R' is rectangle PQRS. R' Is image of R.

SELF CHECK QUESTIONS

1. What is a conformal mapping?
2. What is preserved under a conformal mapping?
3. What must be non-zero for a function to be conformal at a point? Does a conformal mapping preserve the magnitude of angles or their orientation? What type of function must f(z) be for it to be conformal?
4. How does a conformal mapping affect the local distance between points?
5. What happens to the angles between curves under a conformal mapping?

5.6 *SUMMARY:-*

Conformal mapping is a mathematical technique in complex analysis where a function preserves local angles and the shapes of infinitesimally small figures, though it may alter their size and position. For a function f(z) to be conformal, it must be holomorphic (complex differentiable) with a non-zero derivative throughout its domain. Common examples include linear transformations, Möbius transformations, and exponential functions, which map complex domains in ways that preserve angle relationships. Conformal mappings are crucial in solving complex boundary value problems and have applications in fields like fluid dynamics and electromagnetic theory.

5.7 GLOSSARY:-

- **Conformal Mapping:** A function that locally preserves angles and the shapes of infinitesimal figures between complex domains, although it may alter their size and position.
- **Holomorphic:** A function that is complex differentiable at every point in its domain.
- **Derivative:** In complex analysis, the derivative of a function $f(z)$ at a point z is given by $f'(z)$. For a mapping to be conformal, $f'(z)$ must be non-zero.
- **Angle Preservation:** The property of a conformal mapping where the angle between two intersecting curves is preserved under the transformation.
- **Möbius Transformation:** A specific type of conformal mapping of the form $f(z) = \frac{cz+d}{az+b}$ where $ad - bc \neq 0$. It maps lines and circles in the complex plane to other lines or circles.
- **Unit Disk:** The set of all complex numbers z such that $|z| < 1$. It is often used as a domain for conformal mappings.
- **Upper Half-Plane:** The set of all complex numbers z such that $Im(z) > 0$. It is commonly mapped to other domains, such as the unit disk.
- **Branch Point:** A point where a function (such as the logarithm) fails to be single-valued and requires a branch cut to define it properly.
- **Branch Cut:** A curve or line in the complex plane where a multivalued function (like the logarithm) is discontinuous. It is used to make a function single-valued.
- **Exponential Function:** The function $f(z) = e^z$, which is conformal everywhere in the complex plane except at infinity.
- **Logarithm Function:** The complex function $f(z) = log(z)$, which maps complex numbers to a strip in the complex plane and introduces a branch cut for proper definition.

- **Linear Transformation:** A function of the form $f(z) = az + b$ where a and b are constants, and $a \neq 0$. This is a simple type of conformal mapping involving translation and scaling.
- **Fractional Linear Transformation:** Also known as a Möbius transformation, it is of the form $f(z) = \frac{cz+d}{az+b}$, which transforms circles and lines in the complex plane.
- **Reflection:** A type of conformal mapping that involves reflecting points through a line or circle in the complex plane.
- **Singularity:** A point where a function ceases to be well-behaved, such as where it is not holomorphic or where its derivative is zero. Singularities affect conformality at those points.
- **Domain:** The set of all points in the complex plane where a function is defined. The nature of the mapping often depends on the domain being considered.

5.8 *REFERENCES:-*

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- James Ward Brown and Ruel V. Churchill (2016), Complex Variables and Applications.

5.9 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- file:///C:/Users/user/Desktop/1468564002EText(Ch-9,M-1%20(1).pdf
- X. Zhang and J. Li (2017), Recent Advances in Conformal Mapping Techniques.
- Y. Wang and S. Kumar (2019), Conformal Mapping and its Applications in Fluid Dynamics.

5.10*TERMINAL QUESTIONS:-*

(TQ-1) Prove that if $w = x + i\frac{b}{a}y, 0 < a < b$ the inside of the circle $x^2 + y^2 = a^2$corresponds to the inside of an ellipse in the w-plane, but that the transformation is not conformal.

(TQ-2) Consider the transformation $w = z + (1 - i)$ and determine the region D' of w-plane corresponding to the rectangle D in z-plane bounded by $x = 0, y = 0, x = 1, y = 2$.

(TQ-3) What is the region of w-plane into which the rectangular region in z-plane bounded by the lines $x = 0, y = 0, x = 1, y = 2$ is mapped under the map $w = z + (2 - i)$?

(TQ-4) Find the image of rectangle $x = 0, y = 0, x = 1, y = 2$ in z-plane under the map $w = (1 + i)z + (2 - i)$

(TQ-5) True/False type Questions

a. A conformal mapping always preserves distances between points in the complex plane.
b. The function f(z)= (z+2)/(z−2) is a conformal mapping.
c. Conformal mappings are linear transformations of the complex plane.
d. The complex logarithm function f(z)=log(z) is conformal everywhere in the complex plane.
e. Conformal mappings can map the entire complex plane onto itself.
f. Conformal mappings preserve the orientation of angles between intersecting curves.
g. The function $f(z) = e^{iz}$ is a conformal mapping.
h. In the case of an isogonal mapping, the magnitude of angles is preserved but not necessarily the sense of angles.
i. The mapping $w = \bar{z}$ is isogonal but not conformal.

(TQ-6) Show that the transformation $w = \frac{1}{z}$ transforms circles in z-plane to circles of w-plane. What type of circles in z-plane will be transformed into straight lines of w-plane?

(TQ-7) Show that the region $|z - a| \leq R$ ia mapped conformally on $w \leq 1$ by the transformation$w = \frac{R(z-c)e^{i\alpha}}{R^2-(z-a)(\bar{c}-\bar{a})}$, where α is real and $z = c$ is the point which is transformed into the origin.

5.11 *ANSWERS:-*

SELF CHECK ANSWERS

1. A conformal mapping is a function that preserves angles locally between curves. Mathematically, if $f(z)$ is a conformal mapping, it means that f is holomorphic (complex differentiable) and its derivative $f'(z)$ is non-zero. The property of preserving angles

means that infinitesimally small shapes are preserved under the transformation, though their sizes and positions might change.

2. Angles
3. Derivative
4. Mapping
5. Analytic
6. Scales
7. They are preserved

TERMINAL ANSWERS

(TQ-2) D' is rectangle bounded by $u = 1, v = -1, u = 2$ and $v = 1$. It leads to translation of D.

(TQ-3) Rectangle in w-plane bounded by lines $u = 2, v = -1, u = 3$ and $v = 1$.

(TQ-4) Re Image in w-plane is rectangle bounded by $u + v = 1, u - v = 3, u + v = 3$ and $u - v = 1$. It performs rotation and magnification.

(TQ-5)

a.F	b.T	c.F	d.F	e.F
f.T	g.T	h.T	i.T	

UNIT 6:- Möbius transformations and other Mapping

CONTENTS:

6.1 *INTRODUCTION:-*

Möbius transformations are a fundamental class of conformal mappings in complex analysis, characterized by the formula $f(z) = \frac{cz+d}{az+b}$ with complex coefficients satisfying $ad - bc \neq 0$. These transformations are notable for their ability to map circles and lines in the complex plane to other circles and lines while preserving angles and the cross ratio, making them essential in geometry and mathematical physics. In addition to Möbius transformations, complex analysis also encompasses a variety of other mappings such as linear transformations, exponential functions, and more general holomorphic functions. These mappings are crucial for modeling and solving problems in various fields, including fluid dynamics, electromagnetism, and conformal geometry, by maintaining essential properties like angle preservation and local structure.

6.2 *OBJECTIVES:-*

After studying this unit, learners will be able to

- To understand how Möbius transformations and other mappings preserve angles and the structure of geometric figures, such as lines and circles, which is crucial in conformal geometry and various applications.
- To analyze how these transformations map different regions of the complex plane to one another, providing insights into the topological and analytic properties of these regions.

6.3 *MÖBIUS TRANSFORMATION (BILINEAR TRANSFORMATION*:-

A Möbius transformation, also known as a bilinear transformation, is a function defined on the extended complex plane (including the point at infinity) of the form:

$$f(z) = \frac{cz+d}{az+b} \qquad \dots (1)$$

where $a, b, c,$ and d are complex numbers such that $ad - bc \neq 0$. The transformation (1) can be written as

$$cwz + wd - az - b = 0, ad - bc \neq 0$$

The transformation $ad - bc \neq 0$ is called the determinant of the transformation. The transformation (1) is said to be the normalized if $ad - bc = 1$.

Let w_1 and w_2 be the values of w corresponding to z_1 and z_2 in (1), then

$$w_2 - w_1 = \frac{az_2 + b}{cz_2 + d} - \frac{az_1 + b}{cz_1 + d}$$

$$= \frac{(ad - bc)(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)} = 0 \; if \; ad - bc = 0$$

$\therefore \quad w_2 - w_1 = 0 \; if \; ad - bc = 0.$

This shows that w is constant.

6.4 *CRITICAL POINTS*:-

Let $f(z)$ be a non-constant analytic function defined on a domain D. A point $z_0 \in D$ is called a critical point of the function $f(z)$ if the derivative of f at z_0, denoted by $f'(z_0)$, is equal to zero, i.e., $f'(z_0) = 0$.

This definition can be illustrated with the example function $w = f(z) = 2z^3 - 1$ .The derivative of $f(z)$ is $f'(z) = 6z^2$ .Setting the derivative equal to zero, $6z^2 = 0$, we find that $z = 0$ is the only critical point in this case. Thus, $z = 0$ is a critical point of the transformation $w = 2z^3 - 1$.

It is also noted that a constant function f(z)=c for some constant c has no critical points. This is because the derivative of a constant function is zero everywhere, and a critical point requires the function to be non-constant at some point in the domain.

6.5 *INVARIANT AND FIXED POINTS*:-

Fixed points, or invariant points, of a transformation $w = f(z)$ are points z in the domain of f that satisfy the condition $z = f(z)$. In other words, a point z is a fixed point if it remains unchanged under the transformation f; that is, the output of the transformation is the point itself. These points are obtained by solving the equation $z = f(z)$ and are crucial for understanding the behavior and properties of the transformation, including identifying equilibrium states and analyzing the stability of the system described by the function f.

The invariant points of the transformation $w = \frac{az+b}{cz+d}$ ... (1) is given by $w = \frac{az+b}{cz+d}$ or $cz^2 - (a-d)z - b = 0$ or $z = \frac{(a-d)\pm\sqrt{M}}{2c}$ ... (2)

where $M = (a-d)^2 + 4bc$. According as $M = 0$ or $M \neq 0$.

Case1: Suppose $c = 0$ and $d \neq 0$. Then from (1), we get

$$w = \frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}$$

The fixed point is obtained by

$$z = \frac{a}{d}z + \frac{b}{d} \text{ or } z = \frac{b}{d-a} \qquad \text{... (3)}$$

If $a - d \neq 0$, the equation (2) has one fixed point at infinity and another fixed point that is finite, while if $a - d = 0$, the transformation has only one fixed point, which is at infinity. Thus we have the following results:

i. If $c \neq 0$ and $M \neq 0$, two finite fixed points.
ii. If $c \neq 0$ and $M = 0$, one finite fixed point.
iii. If $c \neq 0$ and $a - d = 0$, only one fixed point i.e., infinity.
   In this case $w = z + \frac{b}{d}$
iv. If $c \neq 0$ and $a - d \neq 0$, only one finite and other is infinity.

6.6 *CROSS RATIO*:-

The cross ratio of four distinct points z_1, z_2, z_3, z_4 is a complex number defined as:

$$(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

Despite the 24 permutations of four points z_1, z_2, z_3, z_4 leading to different arrangements of the cross ratio formula, there are actually only six distinct cross ratios. This is because the cross ratio is invariant under permutations of the four points and under Möbius transformations, which means many permutations result in the same value. The six distinct cross ratios are:

$$(z_1, z_2, z_3, z_4), (z_1, z_2, z_4, z_3), (z_1, z_3, z_2, z_4)$$

$$(z_1, z_3, z_4, z_2), (z_1, z_4, z_2, z_3), (z_1, z_4, z_3, z_2)$$

6.7 *PARABOLIC*:-

A linear fractional (bilinear) transformation with exactly one fixed point z_0 is known as a parabolic transformation and can be expressed as

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + h \quad \text{if } z_0 \neq \infty$$

$$w = z + h \text{ if } z_0 = \infty$$

A linear fractional (bilinear) transformation with two fixed points z_1 and z_2 can be expressed as

$$\frac{w-z_1}{w-z_2} = \frac{k(z-z_1)}{(z-z_2)} \text{ if } z_1, z_2 \neq \infty$$

If $z_2 = \infty$, then it becomes $w - z_1 = k(z - z_1)$.

A transformation with two fixed points is known as hyperbolic if $k > 0$, and elliptic if $k = e^{i\alpha}$ and loxodromic if $k = \alpha e^{i\alpha}$, where $\alpha \neq 1,\ \alpha \neq 0$; α and α both are real numbers and $\alpha > 0$.

6.8 *THEOREMS*:-

Theorem1: (Geometrical Inversion) Every bilinear transformation is the resultant of bilinear transformations with simple geometric imports.

Proof: Let the bilinear transformation

$$w = \frac{az+b}{cz+d} \qquad \dots (1)$$

Where

$$ad - bc \neq 0, c \neq 0$$

In equation (1), we obtain

$$w = \frac{a}{c} . \frac{z + \left(\frac{b}{a}\right)}{z + \left(\frac{d}{c}\right)} = \frac{a}{c}\left[1 + \frac{\frac{b}{a} - \frac{d}{c}}{z + \frac{d}{c}}\right]$$

$$w = \frac{a}{c} + \frac{bc - ad}{c^2} . \frac{1}{z + \frac{d}{c}}$$

Taking

$$z_1 = z + \frac{d}{c}, z_2 = \frac{1}{z_1}, z_3 = \frac{bc - ad}{c^2} . z_2$$

we get $w = \frac{a}{c} + z_3$ which is similar to

$$z_1 = z + \frac{d}{c}$$

Now z_1, z_2, z_3 are of the form

$$w = z + \alpha, w = \frac{1}{z}, w = \beta z$$

Where

1. $w = z + \alpha$ represents **translation**.
2. $w = \frac{1}{z}$ represents **inversion**.
3. $w = \beta z$ represents **dilation**.

This proves that every bilinear transformation is the resultant of bilinear transformations.

Theorem2: To show that the resultant (or product) of two bilinear transformations is a bilinear transformation.

Solution: Let the bilinear transformation

$$w = \frac{az+b}{cz+d} \qquad \dots(1)$$

and

$$\zeta = \frac{a_1 w + b_1}{c_1 w + d_1} \qquad \dots(2)$$

where

$$a_1 d_1 - b_1 c_1 \neq 0$$

Substituting the value of w in (2), we get

$$\zeta = \frac{a_1\left(\frac{az+b}{cz+d}\right) + b_1}{c_1\left(\frac{az+b}{cz+d}\right) + d_1}$$

$$\zeta = \frac{z(aa_1 + b_1 c) + (b_1 d + a_1 b)}{z(c_1 a + d_1 c) + (d_1 d + c_1 b)}$$

Now we can write

$$A = aa_1 + b_1 c, B = b_1 d + a_1 b$$
$$C = c_1 a + d_1 c, D = d_1 d + c_1 b$$

$$\zeta = \frac{Az + B}{Cz + D}$$

Here

$$AD - BC = (aa_1 + b_1 c)(d_1 d + c_1 b) - (c_1 a + d_1 c)(c_1 a + d_1 c)$$
$$= (a_1 d_1 - b_1 c_1)(ad - bc) \neq 0$$

Thus

$$\zeta = \frac{Az + B}{Cz + D} \quad s.t. AD - BC \neq 0$$

This is bilinear transformation and is called resultant (or product) of transformation.

Theorem3: (Preservance of cross ratio) To prove that the cross ratio remains invariant under a bilinear transformation.

Solution: Suppose w_1, w_2, w_3, w_4 be the images of z_1, z_2, z_3, z_4 respectively

$$w = \frac{az+b}{cz+d} \qquad \dots(1)$$

Where

$$ad - bc \neq 0, c \neq 0$$

If we prove that

$$\frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)} = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

From (1), we get

$$w_r - w_s = \frac{az_r + b}{cz_r + d} - \frac{az_s + bc}{cz_s + d}$$

$$w_r - w_s = \frac{(ad - bc)(z_r - z_s)}{(cz_r + d)(cz_s + d)}$$

$\Rightarrow$

$$(w_4 - w_1)(w_2 - w_3) = \frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)} \cdot \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

$$(w_4 - w_1)(w_2 - w_3) = \frac{(ad - bc)^2(z_4 - z_1)(z_2 - z_3)}{(cz_4 + d)(cz_1 + d)(cz_2 + d)(cz_3 + d)}$$

Similarly,

$$(w_2 - w_1)(w_4 - w_3) = \frac{(ad - bc)^2(z_2 - z_1)(z_4 - z_3)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

Dividing last two equations, we obtain

$$\frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)} = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

which is proves that.

Theorem4: (Fixed points of a bilinear transformation) To prove that in general there are two values of z (invariant points) for which w=z, but there is only one if

$$(a - d)^2 + 4bc = 0$$

Show that if there are distinct invariant points p and q, the transformation may be put in the form **(normal form)**

$$\frac{(w - p)}{(w - q)} = k\frac{(z - p)}{(z - q)}$$

and that, if there is only one invariant point p, the transformation may be put in the form**(normal form)**

$$\frac{1}{w - p} = \frac{1}{z - p} + K$$

Proof: Let the bilinear transformation

$$w = \frac{az + b}{cz + d} \qquad \dots(1)$$

i. The invariant points are

$$w = z \qquad i.e., \qquad z = \frac{az + b}{cz + d}$$

This gives $cz^2 - (a-d)z - b = 0$ ... (2)
Solving

$$z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}$$

Taking

$$p = \frac{(a-d) + \sqrt{(a-d)^2 + 4bc}}{2c}$$
$$q = \frac{(a-d) - \sqrt{(a-d)^2 + 4bc}}{2c}$$

If $(a-d)^2 + 4bc = 0$, then

$$p = q = \frac{a-d}{2c}$$

Hence there is only one invariant point namely
$p = \frac{a-d}{2c}$ if $(a-d)^2 + 4bc = 0$.

ii. Let p and q are distinct invariant points of the transformation(1) So from (2), we get

$$cp^2 - (a-d)p - b = 0$$
$$cq^2 - (a-d)q - b = 0$$

From above equations

$$cp^2 — ap = b - pd$$
$$cq^2 — aq = b - qd$$
$$\frac{(w-p)}{(w-q)} = \frac{\left(\frac{az+b}{cz+d}\right) - p}{\left(\frac{az+b}{cz+d}\right) - q}$$

$$= \frac{(az+b) - pcz - pd}{(cz+d) - cqz - qd} = \frac{(a-pc)z + b - pd}{(c-cq) + b - qd}$$

$$= \frac{(a-pc)z + cp^2 — ap}{(c-cq) + cq^2 — aq}$$
$$= \frac{(a-pc)(z-p)}{(c-cq)(z-q)}$$

Taking $k = \frac{(a-pc)}{(c-cq)}$, we obtain

$$\frac{(w-p)}{(w-q)} = k\frac{(z-p)}{(z-q)}$$

iii. Suppose there be only one variant p so that

$$p = \frac{a-d}{2c}, cp^2 — ap = b - pd$$

Now,

$$\frac{1}{w-p} = \frac{1}{\left(\frac{az+b}{cz+d}\right) - p} = \frac{cz+d}{(a-pc)z + b - pd}$$

Solving this equation

$$\frac{1}{w-p}=\frac{c}{a-cp}+\frac{1}{z-p}$$

Taking $K=\frac{c}{a-cp}$, we obtain

$$\frac{1}{w-p}=\frac{1}{z-p}+K$$

Theorem5: If α and β are two given points and k is a constant, show that the equation

$$\left|\frac{z-\alpha}{z-\beta}\right|=k$$

represent the circle.

Proof: The given equation is

$$\left|\frac{z-\alpha}{z-\beta}\right|=k \qquad \ldots(1)$$

From (1), we obtain

$$\frac{|z-\alpha|^2}{|z-\beta|^2}=k^2 \quad or \quad \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\beta)(\bar{z}-\bar{\beta})}=k^2$$

$$(z\bar{z}-\alpha\bar{z}-\alpha\bar{z}+\alpha\bar{\alpha})=k^2(z\bar{z}-\beta\bar{z}-\beta\bar{z}+\beta\bar{\beta})$$

$$(1-k^2)z\bar{z}-(\alpha-\beta k^2)\bar{z}-(\bar{\alpha}-\bar{\beta}k^2)z=\beta\bar{\beta}k^2-\alpha\bar{\alpha}$$

$$z\bar{z}-\left(\frac{\alpha-\beta k^2}{1-k^2}\right)\bar{z}-\overline{\left(\frac{\alpha-\beta k^2}{1-k^2}\right)}z+\frac{|\alpha|^2-k^2|\beta|^2}{1-k^2}=0 \quad \ldots(2)$$

It is the form

$$z\bar{z}+b\bar{z}+z\bar{b}+c=0, where\ c\ is\ real$$

The equation (2) express a circle if $k\neq 1$.

Consequently (1) express a circle if $k\neq 1$.

SOLVED EXAMPLE

EXAMPLE 1: Find the bilinear transformation which maps the points $z_1=2, z_2=i, z_3=-2$ into the points $w_1=1, w_2=i, w_3=-1$.

SOLUTION: The transformation is given by

$$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}=\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$$

Substituting values in above equation are

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)}=\frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$=\left(\frac{z-2}{z+2}\right)\left(\frac{4-1+4i}{4+1}\right)\left(\frac{1-1-2i}{1+1}\right)$$

$$\frac{w-1}{w+1}=\left(\frac{4-3i}{5}\right)\left(\frac{z-2}{z+2}\right)$$

$$\frac{w-1+w+1}{(w-1)-(w+1)}=\frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

$$\frac{w}{-1} = \frac{3z(3-i)+2(1+3i)}{-iz(3-i)-6(3-i)} = \frac{-(3z+2i)(3-i)}{(iz+6)(3-i)}$$

$$w = \frac{3z+2i}{iz-6}$$

EXAMPLE 2: Find the bilinear transformation which transforms the unit circle $|z| = 1$ into real axis in such a way that the points $1, i, -i$ are mapped into $0, 1, \infty$ respectively. Into what regions the interior and exterior of the circle are mapped.

SOLUTION: The given values are

$$z_1 = 1, z_2 = i, z_3 = -i$$
$$w_1 = 0, w_2 = 1, w_3 = -\infty$$

Now the required transformation is:

$$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)} = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$$

$$\frac{(w-0)(1-\infty)}{(w-\infty)(1-0)} = \frac{(z-1)(i+i)}{(z+i)(i-1)}.\ but\ \frac{(1-\infty)}{(w-\infty)} = \lim_{n\to\infty}\left(\frac{1-n}{w-n}\right) = 1$$

$$w = \frac{(z-1)2i}{(z+i)(i-1)} = \frac{(z-1)2}{(z+i)(1+i)} = \frac{(z-1)(1-i)2}{(z+i)(1-i^2)}$$
$$= \frac{(z-1)(1-i)}{(z+i)}$$

$$w = \frac{(z-1)(1-i)}{(z+i)} \qquad \ldots(1)$$

Now from (1) for z, we obtain

$$wz + wi = z(1-i) - (1-i) \Rightarrow z = -\frac{[w+(-i-1)]i}{w-(1-i)}$$

$$iz = \frac{w-(1+i)}{w-(1-i)} \qquad \ldots(2)$$

So $|z| = 1 \ \Rightarrow\ |iz| = 1$.

From (2), we get

$$\left|\frac{w-(1+i)}{w-(1-i)}\right| = 1$$

$$|w-(1+i)|^2 = |w-(1-i)|^2$$

$\Rightarrow \quad (u-1)^2 + (v-1)^2 = (u-1)^2 + (v+1)^2 \quad \ldots(3)$

$\Rightarrow \quad -4v = 0 \Rightarrow v = 0 \Rightarrow Real\ axis\ of\ w - plane$

$\therefore \quad |z| = 1 \quad corresponds\ real\ axis\ of\ w - plane$

Now for interior of $|z| = 1$, we obtain $|z| < 1$.

From (3), we have

$$\frac{(u-1)^2+(v-1)^2}{(u-1)^2+(v+1)^2} < 1$$

$$(u-1)^2 + (v-1)^2 < (u-1)^2 + (v+1)^2$$

$\Rightarrow \quad -4v < 0 \Rightarrow v > 0 \Rightarrow upper\ half\ plane\ of\ w - plane.$

Similarly $|z| = 1$ and $|z| < 1$ corresponds to lower half of w −plane.

EXAMPLE 3: If $z = 2w + w^2$, then show that $|w| = 1$ corresponds to a cardioids of z −plane.

SOLUTION: The given equation

$$z = 2w + w^2 \quad \ldots (1)$$

$$|w| = 1 \quad or \;\; w = e^{i\phi} \quad \ldots (2)$$

$$\Rightarrow \quad z + 1 = (w+1)^2 = \left(1 + e^{i\phi}\right)^2 = \left[e^{i\phi/2}\left(e^{-i\phi/2} + e^{i\phi/2}\right)\right]^2$$

$$\Rightarrow \quad = e^{i\phi}\left(2cos\left(\frac{\phi}{2}\right)\right)^2 \Rightarrow \; z + 1 = \left(4cos^2\left(\frac{\phi}{2}\right)\right)e^{i\phi}$$

Taking $z + 1 = re^{i\theta}, i.e.,\; pole\ at\;\; z = -1$, we obtain

$$re^{i\theta} = 4cos^2\left(\frac{\phi}{2}\right)e^{i\phi} \Rightarrow r = 4cos^2\left(\frac{\phi}{2}\right), \theta = \phi$$

$$r = 4cos^2\left(\frac{\phi}{2}\right) = 2(1 + cos\theta)$$

$$r = 2(1 + cos\theta)$$

EXAMPLE 4: Find the image of the circle $|z - 2| = 2$ under the Mobius transformation $w = \frac{z}{z+1}$.

SOLUTION: $w = \frac{z}{z+1}$ gives $wz + w = z$

$$z(w - 1) = -w$$

$$z = \frac{w}{1 - w} \quad \ldots (1)$$

Now $|z - 2| = 2$ gives $(z - 2)(\bar{z} - 2) = 2^2 \quad \ldots (2)$

From (1), we have

$$z - 2 = \frac{w}{1 - w} - 2 = \frac{3w - 2}{1 - w}$$

$$\bar{z} - 2 = \frac{\overline{w}}{1 - \overline{w}} - 2 = \frac{3\overline{w} - 2}{1 - \overline{w}}$$

Substituting these values in (2), we get

$$\frac{(3w - 2)(3\overline{w} - 2)}{(1 - w)(1 - \overline{w})} = 4$$

$$9w\overline{w} + 4 - 6(w + \overline{w}) = 4[1 + w\overline{w} - (w + \overline{w})]$$

$$5w\overline{w} - 2(w + \overline{w}) = 0$$

$$5(u^2 + v^2) - 2(2u) = 0$$

$$u^2 + v^2 - \frac{4}{5}u = 0$$

So the circle of centre $(-g, -f) = \left(\frac{2}{5}, 0\right)$

Radius $= (g^2 + f^2 - c)^{1/2} = \frac{2}{5}$

EXAMPLE 5: Show that the transformation $w = \frac{5-4z}{4z-2}$ transform the circle $|z| = 1$ into a circle of radius unity in w −plane and find the centre of the circle.

SOLUTION: the given transformation is

$$w = \frac{5 - 4z}{4z - 2}$$

$$4zw - 2w = 5 - 4z \quad or \quad z = \frac{2w+5}{4w+4}$$

$|z| = z\bar{z} = 1$ Corresponds to $\left[\frac{2w+5}{4(w+1)}\right]\left[\frac{2\bar{w}+5}{4(\bar{w}+1)}\right] = 1$

$$4w\bar{w} + 25 + 10(w + \bar{w}) = 16(w\bar{w} + 1 + w + \bar{w}) \qquad ...(1)$$

But $w = u + iv, \bar{w} = u - iv, w\bar{w} = u^2 + v^2, w + \bar{w} = 2u$ ... (2)

Hence from (1), we get

$$u^2 + v^2 - \frac{3}{4} = 0$$

Now comparing with $u^2 + v^2 + 2gu + 2fv + c = 0$, we obtain centre $= (-g, -f) = \left(-\frac{1}{2}, 0\right)$ and radius $[g^2 + f^2 - c]^{1/2}$i.e.,

$$= \left(\frac{1}{4} + 0 + \frac{3}{4}\right)^{1/2} = 1$$

EXAMPLE 6: Find the condition that the transformation $w = \frac{az+b}{cz+d}$ transforms a straight line $z - plane$ into the unit circle $w - plane.$

SOLUTION: The transforms a straight line of $z - plane$ is

$$\left|\frac{z-p}{z-q}\right| = 1 \qquad ...(1)$$

The unit circle in w-plane is $|w| = 1$ ... (2)

So

$$w = \frac{a\left(z + \frac{b}{a}\right)}{c\left(z + \frac{d}{c}\right)}$$

Taking $p = -\frac{b}{a}, q - \frac{d}{c}$, we have

$$w = \frac{a(z-p)}{c(z-q)}$$

Hence $|w| = \left|\frac{a}{c}\right| \left|\frac{z-p}{z-c}\right| = \left|\frac{a}{c}\right| = 1$, if $|a| = |c|$

SELF CHECK QUESTIONS

1. Define the cross ratio for four complex numbers.
2. Why is the cross ratio invariant under Möbius transformations?
3. How many distinct cross ratios can be formed from four points?
4. What is a fixed point in the context of a transformation?
5. Differentiate between elliptic, hyperbolic, and loxodromic transformations in terms of their fixed points and behavior.
6. In what contexts or fields are Möbius transformations particularly useful?
7. Give an example of how Möbius transformations can be applied in geometry or physics.

6.9 *SUMMARY:-*

A Möbius transformation, also known as a linear fractional transformation, is a complex function of the form $w = (cz + d)/(az + b)$, where $a, b, c,$ and d are complex numbers with $ad - bc \neq 0$. These transformations map the extended complex plane, including the point at infinity, onto itself and are conformal, meaning they preserve angles. Möbius transformations can transform circles and lines into other circles or lines and are characterized by their fixed points, which can be one, two, or sometimes none on the finite plane, depending on the nature of the transformation. They are classified into parabolic, elliptic, hyperbolic, and loxodromic types based on the configuration of their fixed points and the trace of the transformation matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. The cross ratio, a key invariant under Möbius transformations, plays a crucial role in understanding the relative geometry of four points in the complex plane. These transformations have significant applications in complex analysis, geometry, and theoretical physics.

6.10 *GLOSSARY:-*

- **Möbius transformation:** A Möbius transformation, also known as a bilinear transformation, is a function defined on the extended complex plane (including the point at infinity) of the form:
  $$f(z) = \frac{cz + d}{az + b} \qquad \text{... (1)}$$
  where a, b, c, and d are complex numbers such that $ad - bc \neq 0$
- **Conformal:** A property of transformations that preserve angles locally, meaning the shape of infinitesimally small figures is preserved, though their size may not be.
- **Fixed Point:** A point z that remains unchanged under a transformation, i.e., $f(z) = z$. In Möbius transformations, fixed points are solutions to the equation $(z) = \frac{az+b}{cz+d}$.
- **Extended Complex Plane:** Also known as the Riemann sphere, it is the complex plane plus the point at infinity. Möbius transformations naturally act on this extended plane.
- **Cross Ratio:** An invariant under Mobius transformation, defined for four distinct points z_1, z_2, z_3, z_4 as $(z_1, z_2, z_3, z_4) = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$. It uniquely determines the relative positions of the four points.

- **Parabolic Transformation:** A type of Möbius transformation with exactly one fixed point, characterized by the condition $(a+b)^2 = 4(ad - bc)$ and typically represented as $T(z) = \frac{z-z_0}{1-cz}$ .where c is a constant.
- **Elliptic Transformation:** A Möbius transformation with two fixed points and associated with rotations. It occurs when $|\ a + d\ | < 2$.
- **Hyperbolic Transformation:** A Möbius transformation with two fixed points, characterized by real eigenvalues and associated with stretching along one direction. It occurs when $|\ a + d\ | > 2$.
- **Loxodromic Transformation:** A transformation with two fixed points, involving a combination of rotation and dilation. It occurs when $|\ a + d\ | \neq 2$ and the transformation is neither purely elliptic nor hyperbolic.
- **Invariance:** A property where certain aspects of a geometric configuration, such as the cross ratio in Möbius transformations, remain unchanged under specific transformations.
- **Bilinear Transformation:** Another name for Möbius transformation, emphasizing its representation as a ratio of two linear functions.
- **Riemann Sphere:** A model of the extended complex plane where every point on the complex plane corresponds to a point on the sphere, and the point at infinity is represented by the north pole of the sphere.
- **Angle Preservation:** A key property of Möbius transformations, which maintain the angles between intersecting curves after transformation.
- **Transformation Matrix:** The matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ associated with a Möbius transformation, encapsulating the coefficients that define the transformation.
- **Identity Transformation:** The Möbius transformation $T(z) = z$, which leaves every point unchanged. It serves as the identity element in the group of Möbius transformations**.**

These terms are fundamental to understanding Möbius transformations and their applications in complex analysis and geometry.

6.11 *REFERENCES:-*

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6.12 *SUGGESTED READING:-*

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- Y. Wang and S. Kumar (2019), Conformal Mapping and its Applications in Fluid Dynamics.

6.13 *TERMINAL QUESTIONS:-*

(TQ-1) To show that the set of all bilinear transformations forms a non-abelian group under the product of transformations.

(TQ-2) To prove that the cross ratio (z_1, z_2, z_3, z_4) is real iff the four points z_1, z_2, z_3, z_4 lie on a circle or on a straight line.

(TQ-3) To prove that the every bilinear transformation with two finite fixed points α, β can be put in the form:

$$\frac{w-\alpha}{w-\beta} = \lambda\left(\frac{z-\alpha}{z-\beta}\right)$$

(TQ-4) If α, β are the inverse points of a circle, then prove that the equation of a circle can be written as $\left|\frac{z-\alpha}{z-\beta}\right| = k, k \neq 1$ and k is real constant.

(TQ-5) To prove that the every bilinear transformation maps circles or straight lines into circles or straight lines.

Or

Find the condition that the transformation $w = \frac{cz+d}{az+b}$ transforms the unit circle in $w - plane$ into straight line of $z - plane$.

(TQ-6) To prove that the general linear transformation of a circle $|z| \leq \rho$ into a circle $|w| \leq \rho'$ is

$$w = \rho\rho' e^{i\lambda} \frac{(z-\alpha)}{(\bar{\alpha}z-\rho^{2'})}\ s.t. |\alpha| < \rho.$$

(TQ-7) To prove that the region $|z-a| \leq R$ is mapped conformally on $|w| \leq 1$ by the transformation

$$w = \frac{R(z-c)e^{i\alpha}}{R^2 - (z-a)(\bar{c}-\bar{a})}$$

where α is real and $z = c$ is the point which is transformed into the origin.

(TQ-8) If the transformation $z = \frac{i-w}{i+w}$, show that half of w −plane given by $v \geq 0$ corresponds to the circle $|z| \leq 1$ in $z -$ plane.

(TQ-9) Discuss the application of the transformation $w = \frac{iz+1}{z+i}$ to the areas in the z-plane which are respectively inside and outside the unit circle with its centre at the origin.

(TQ-10) Show that the transformation $w = \frac{5-4z}{4z-2}$ transforms the circle $|z| = 1$ into a circle of radius unity in $w - plane$ and find the centre of circle.

(TQ-11) Find the linear maps for $z = 0, -i, -1$ and corresponding values of w are $w = i, 1, 0$.

(TQ-12) Find bilinear transformation which maps $0, i, -i$ of z-plane to $1, -1, 0$ of w-plane.

(TQ-13) True/False Type questions.

- **a.** A Möbius transformation maps circles and lines in the complex plane to circles and lines.
- **b.** A Möbius transformation can always be written in the form of a linear fractional transformation.
- **c.** Two Möbius transformations are considered equivalent if they differ by a scaling factor.
- **d.** The composition of two Möbius transformations is itself a Möbius transformation.
- **e.** The coefficients a,b,c, and d in a Möbius transformation are allowed to be zero, as long as $ad - bc \neq 0$.
- **f.** The Möbius transformation $w = \frac{z-1}{z+1}$ maps the real axis to itself.

(TQ-14) Objectives Type Questions.

1. A transformation of type $w = az + \beta$, where α and β are complex constants is the resultant of

a. Magnification and translation
b. Magnification, rotation and translation
c. Rotation and translation
d. None

2. Critical points of $w = \frac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta - \beta\gamma \neq 0$ are

a. $-\frac{\delta}{\gamma}$
b. $-\frac{\delta}{\gamma}$ and ∞
c. $-\frac{\delta}{\gamma}$ and 0
d. None

3. If $w = f(z)$ represents a conformal mapping of a domain D, then f(z) is
   a. continuous in D
   b. analytic in D
   c. not analytic in D
   d. none

4. The transformation $w = iz + i$ maps the half plane $x > 0$ into
   a. half plane $u > 0$
   b. half plane $v < 1$
   c. the half plane $v > 1$
   d. none

5. The set of all bilinear transformations under the product of transformations form a:
   a. Semi-group
   b. Non-abelian group
   c. Abelian group
   d. None

(TQ-15) If $(a-d)^2 + 4bc \neq 0$, then for transformation $w = \frac{az+b}{cz+d}$ there exist unequal numbers α, β such that

$$\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$$

where k is constant. Show also the radius of the circle in the w-plane corresponding to the circle in the z-plane whose diameter is the line joining the points $z = \alpha, z = \beta$ is $\left|\frac{\alpha-\beta}{2\cos\theta}\right|$, where θ is the argument of k.

6.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. The cross ratio of four points $(z_1, z_2; z_3, z_4) = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$
2. The cross ratio is invariant under Möbius transformations because such transformations preserve the projective properties of points, meaning they maintain the relative geometry of the configuration.
3. Despite 24 permutations, only six distinct cross ratios can be formed from four points, due to the symmetries and invariances in the definition of the cross ratio.
4. A fixed point of a transformation is a point that remains unchanged under the transformation, i.e., $f(z) = z$.
5. **Elliptic:** Has two fixed points with a rotation around them.
   Hyperbolic: Has two fixed points with one attracting and the other repelling, characterized by dilation or contraction along a line.
   Loxodromic: Has two fixed points, and the transformation involves a combination of rotation and dilation/contraction.

6. Möbius transformations are used in complex analysis, geometric function theory, hyperbolic geometry, computer graphics, and in the study of conformal mappings.
7. In geometry, Möbius transformations can map the upper half-plane onto the unit disk, facilitating the study of hyperbolic geometry. In physics, they are used in relativity theory to model transformations in space-time.

TERMINAL ANSWERS

(TQ-11) $w = -\frac{i(z+1)}{z-1}$ **(TQ-12)** $w = -\frac{i(z+i)}{(3z-i)}$

(TQ-13) a.T b.T c.F d.T e.F f.T

(TQ-14) 1.a 2.b 3.b 4.c 5.c

BLOCK III
COMPLEX INTEGRATION

UNIT 7:- Complex Line Integral

CONTENTS:

7.1 *INTRODUCTION:-*

Complex integration involves integrating functions of a complex variable along a path or contour in the complex plane. The integral of a complex function $f(z)$ along a contour γ is computed using a parameterization of γ, transforming the problem into a standard real integral. Key results in complex integration include Cauchy's Theorem, which states that the integral of a holomorphic function over a closed contour is zero, and Cauchy's Integral Formula, which relates function values to contour integrals. The Residue Theorem further simplifies the evaluation of integrals by relating them to residues at singularities within the contour. These principles not only facilitate the evaluation of integrals but also provide insights into the analytic properties of complex functions.

7.2 *OBJECTIVES:-*

In this unit we will study about the Compute line integrals of complex-valued functions along curves in the complex plane, integration along piecewise smooth path, including using parameterizations and contour integration techniques.

7.3 *CONNECTED SET, OPEN AND CLOSED DOMAIN (REGION):-*

Connected Set: A set S in the Argand plane (complex plane) is said to be connected if for any two points within the set, there exists a continuous curve (or path) that lies entirely within S and connects these two points.

Open Domain: An open domain is a specific type of connected set that is also open. A set D is an open domain if it is connected and, additionally, every point in D has a neighborhood completely contained within D. In other words, D is open and does not include its boundary points.

Closed Domain: If a set D is an open domain and you include all its boundary points, the resulting set is called a closed domain. In other words, a closed domain includes both the open domain and its boundary, making it closed and still connected.

7.4 *JORDAN ARC:-*

The equation $z = z(t) = x(t) + iy(t)$

where $x(t)$ and $y(t)$ are real-valued continuous functions of the real variable t, with t in the interval $[a, b]$, defines a set of points in the complex plane known as a continuous curve. This curve is called a simple curve if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$meaning the curve does not intersect itself. If the curve is such that $t_1 < t_2$ and $z(t_1) = z(t_2)$ implies $t_1 = a$ and $t_2 = b$, then it is a simple closed curve, which means the curve starts and ends at the same point, forming a loop without self-intersections except at the endpoints. Simple curves are often referred to as Jordan curves. A common example of a Jordan curve is a polygon formed by joining a finite number of line segments end to end.

An important property of a bounded infinite set in the complex plane is that it must have at least one limit point within the complex plane. This property is derived from the Bolzano-Weierstrass theorem, which states that every bounded sequence in C has a convergent subsequence. This implies that any bounded infinite set in the complex plane cannot be composed entirely of isolated points; instead, it must contain points arbitrarily close to each other, leading to the presence of limit points. This property is fundamental in understanding the structure and behavior of sets in the complex plane.

Theorem 1. (Bolzano-Weierstrass Theorem)

If a set $S \subseteq \mathbb{C}$ is bounded and contains an infinite number of points, then it must have at least one limit point.

Theorem 2. (Jordan Curve Theorem)

It states that a simple closed Jordan curve divides the Argand plane into two open domains which have the curve as the common boundary. One of these domains is bounded and is known as interior domain, while the other is bounded and is called exterior domain.

7.5 *RECTIFIABLE AND REGULAR CURVES:-*

A rectifiable curve is a curve whose length can be measured and is finite. In mathematical terms, a curve is rectifiable if its total length is finite. Here's a detailed breakdown of the concept:

Rectifiable Curve: A curve γ in the complex plane (or in Euclidean space) is called rectifiable if its length is finite. This means that if γ is parameterized by a continuous function $\gamma:[a,b]\to\mathbb{C}$ (or more generally in $\mathbb{R}^n$), the length of the curve can be computed and is finite.

The length L of a rectifiable curve parameterized by γ(t) from t=a to t=b is given by:

$$L = \int_a^b |\gamma'(t)| dt$$

Here, $\gamma'(t)$ denotes the derivative of $\gamma(t)$ with respect to t, and $|\cdot|$ denotes the modulus (or absolute value) in the complex plane or Euclidean space.

Regular Curve: A regular curve is a smooth curve where the tangent vector is never zero. Formally, if a curve is parameterized by $\gamma(t)$ with t in some interval $[a, b]$, then $\gamma(t)$ is a regular curve if its derivative $\gamma'(t)$ is never zero for t in $[a, b]$. This means that:

$$\gamma'(t) \neq 0 \forall\ t\ in\ [a, b]$$

This condition ensures that the curve does not have any sharp corners or cusps and is smooth throughout.

7.6 *CONTOUR:-*

Contour: A simple curve in the complex plane is called a contour if it is piecewise smooth, meaning it can be broken down into a finite number of smooth (regular) segments, where each segment is smooth and has a finite length. A contour is a curve that can be traversed in a specific direction, and it is always rectifiable, meaning its length is finite.

Closed Contour: A simple closed curve, often referred to as a closed contour, is a contour that forms a closed loop. It returns to its starting point, and like any contour, it is piecewise smooth with a finite number of smooth segments. Since it is a closed loop, it also has a finite length. An example of a closed contour is the curve is

$$z(t) = cost + isint = e^{it}, \qquad 0 \le t \le 2\pi$$

7.7 *SIMPLY AND MULTI CONNECTED DOMAINS:-*

A domain is called **simply connected** if every closed curve within it can be continuously contracted to a single point without leaving the domain, implying that the domain has no holes or obstructions. Conversely, a domain is **multiply connected** if it is not simply connected, meaning it contains one or more holes, which prevents certain closed curves from being shrunk to a point without crossing these holes.

7.8 *WEIERSTRASS M-TEST:-*

If $|u_n(z)| \le M_n$ for each z in a domain D, where M_n is a sequence of constants independent of z, and the series $\sum M_n$ is uniformly convergent, then the series $\sum u_n(z)$ converges uniformly on D. This result follows from the Weierstrass M-test, which states that if a series of functions is bounded by a uniformly convergent series of constants, then the original series converges uniformly.

7.9 *COMPLEX LINE INTEGRAL:-*

Let $f(z)$ be a complex-valued function defined on a smooth curve γ in the complex plane, parameterized by $z(t)$ where t varies over an interval $[a, b]$, the complex line integral of $f(z)$ along γ is defined as:

$$\int_\gamma f(z)\,dz = \int_a^b f\big(z(t)\big).z'(t)dt = \sum_{j=1}^{n-1} \int_i^{j+1} f\big(z(t)\big).z'(t)dt$$

Where $z(t) = x(t) + iy(t)$ and $z'(t) = \frac{dz}{dt} = x'(t) + iy'(t)$.
The limit $a = t_0 < t_1 < t \ldots < t_{n-1} = b$ is called the complex line integral of f over x.

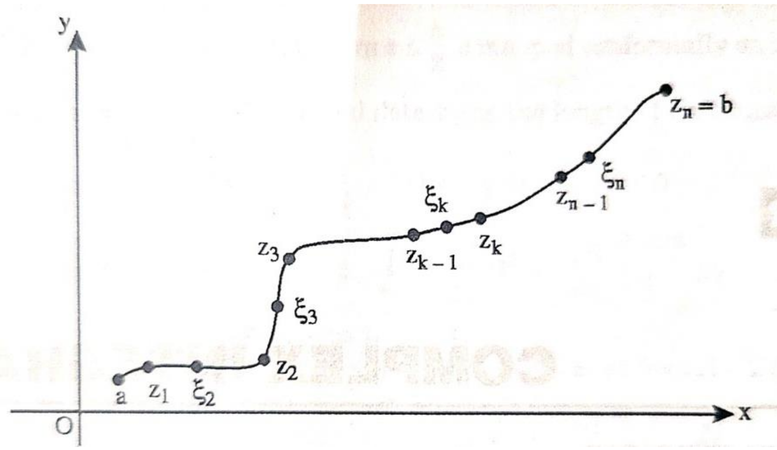

Fig.1

Suppose $f(z)$ is continuous at every point of a closed curve C having a finite length, i.e., C is rectifiable curve.
Divide C into n parts by means of points

$$z_0, z_1, z_2, \ldots, z_n$$

Suppose $a = z_0, b = z_n$
We choose a point ξ_k on each arc joining z_{k-1} to z_k.
Now from the sum

$$S_n = \sum_{r=1}^{n} f(\xi)\,(z_r - z_{r-1})$$

When the sum S_n of the integrals of $f(z)$ over subdivisions of the rectifiable curve C converges to a fixed limit that is independent of the mode of subdivision, this limit is denoted by

$$\int_a^b f(z)\,dz = \int_C f(z)dz$$

which is called the complex line integral, or line integral of $f(z)$ along the curve C.

<u>Connection between Real and Complex line integral:</u> If $f(z) = u(z) + iv(z)$ then the complex line integral $\int_x f(z)\,dz$ can be expressed as

$$\int_x f(z)\,dz = \int_x udx - vdy + i\int_x u\,dy + vdx$$

The connection between real and complex line integrals can be expressed through the decomposition of the complex line integral into real and imaginary components. If $f(z) = u(z) + iv(z)$ where u and v are real-valued functions representing the real and imaginary parts of

f respectively, then the complex line integral $\int_\gamma f(z)\,dz$ along a curve γ can be expressed as:

$$\int_\gamma f(z)\,dz = \int_\gamma (udx - vdy) + i\int_\gamma (u\,dy + vdx)$$

Here:

- $\int_\gamma (udx - vdy)$ Represents the real part of the complex line integral.
- $\int_\gamma (u\,dy + vdx)$ Represents the imaginary part of the complex line integral.

7.10 *FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS:-*

If $f(z)$ be a single valued function in a simple connected domain D. If $a, b\epsilon D$, then $\int_a^b f(z)dz = F(b) - F(a)$, where $F(z)$ is an infinite integral of $f(z)$.

Proof: By the definition of infinite integral, we get

$$F(z) = \int_a^b f(t)dt$$

$$F(b) - F(a) = \int_{z_0}^b f(t)\,dt - \int_{z_0}^a f(t)\,dt = \int_{z_0}^b f(t)dt + \int_a^{z_0} f(t)dt$$

$$= \int_a^b f(t)dt$$

Or

$$F(b) - F(a) = \int_a^b f(z)\,dz$$

SOLVED EXAMPLE

EXAMPLE1: Evaluate $\int_x \frac{1}{z}dz$, where $x(t) = e^{it}, t \in [0,2\pi]$.

SOLUTION: By complex line integral

$$\int_x \frac{1}{z}dz = \int_a^b f\big(x(t)\big).x'(t)dt$$

Here, $f(z) = \frac{1}{z}$, $x(t) = e^{it}, a = 0, b = 2\pi$

$\therefore$ $f[(x(t))] = \frac{1}{e^{it}}$, $x'(t) = ie^{it}$

$$\int_x \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{it}}.ie^{it}\,dt = \int_0^{2\pi} i\;dt = i[t]_0^{2\pi} = 2\pi i$$

EXAMPLE2: Find the length of the curve $x(t) = 4e^{it}, t \in [0,2\pi]$.
SOLUTION: The length of

$$x = L(x) = \int_0^{2\pi} |4i\, e^{it}|\, dt = \int_0^{2\pi} |4||i||e^{it}| dt$$

$$= \int_0^{2\pi} 4\, dt = 4 \int_0 \frac{2\pi}{dt} = 8\pi$$

EXAMPLE3: Find the length of the curve $x(t) = i(1+i)t, t \in [0,4]$.
SOLUTION: The given curve is

$$x(t) = i(1+i)t$$
$$x'(t) = i(1+i)$$

∴ Length

$$x = L(x) = \int_0^4 |(1+i)|\, dt$$

$$= \int_0^4 \sqrt{1^2 + 1^2}\, dt$$

$$= \int_0^4 \sqrt{2}\, dt = \sqrt{2}\, [t]_0^4 = 4\sqrt{2}$$

EXAMPLE4: Find the value of integral

$$\int_0^{1+i} (x - y + ix^2)\, dz$$

a. along the straight line from $z = 0$ to $z = 1 + i$.
b. Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 0$ to $z = 1 + i$.

SOLUTION: Suppose $z = x + iy$

$$dz = dx + idy$$

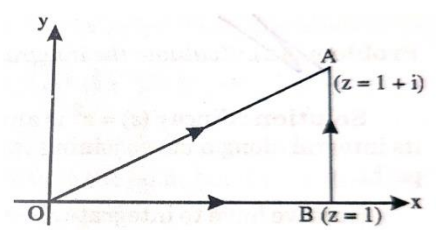

Fig.2

a. Let OA is straight line joining $z = 0$ to $z = 1 + i$.
So $y = x \text{ on } OA$

$$dy = dx$$

$$\int_{OA} (x - y + ix^2)\, dz = \int_0^1 (x - y + ix^2)(dx + idx)$$

$$= i(1+i) \int_0^1 x^2\, dx = \frac{i(1+i)}{3} = \frac{i-1}{3}.$$

b. The real axis from $z = 0$ to $z = 1$ is the line OB, $y = 0$ on OB and so $z = x, dz = dx$.

$$\int_{OB} (x - y + ix^2)\, dz = \int_0^1 (x - 0 + ix^2) dx$$
$$= \int_0^1 (x + ix^2) dx = \frac{1}{2} + \frac{i}{3}$$

Now BA is the line parallel to imaginary axis from $z = 0$ to $z = 1 + i$.
$x = 1$ on BA so that $dx = 0, dz = idy$ on BA.

$$\int_{BA} (x - y + ix^2)\, dz = \int_0^1 (x - y + i) dy$$
$$= \left[(1 + i) - \frac{1}{2}\right] i = -1 + \frac{i}{2}$$

$$\int_{OBA} (x - y + ix^2)\, dz$$
$$= \int_{OB} (x - y + ix^2)\, dz + \int_{BA} (x - y + ix^2)\, dz$$
$$= \left(\frac{1}{2} + \frac{i}{3}\right) + \left(\frac{i}{2} - 1\right) = -\frac{1}{2} + \frac{5i}{6}$$

EXAMPLE5: Evaluate $\int_C \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $z = t^2 + it$.

SOLUTION:

$$I = \int_0^{4+2i} \bar{z}\, dz \qquad \text{... (1)}$$

along the curve C given curve by $z = t^2 + it$

$$dz = (2t + i)dt, \quad \bar{z} = (t^2 - it)$$
$$\bar{z}dz = (t^2 - it)(2t + i)dt = (2t^3 - it^2 + t)dt \qquad \text{... (2)}$$
$$z = 0 \Rightarrow t = 0 \text{ and } z = 4 + 2i \Rightarrow t^2 = 4, t = 2 \Rightarrow t = 2$$

From (1) and (2) ,we get

$$I = \left(\frac{2}{4} t^4 - \frac{i}{3} t^3 + \frac{1}{2} t^2\right)_0^2 = 8 - \frac{8i}{3} + 2 = 10 - \frac{8}{3} i$$

SELF CHECK QUESTIONS

1. What is a complex line integral, and how is it defined for a complex function $f(z)$ along a curve C?
2. How can the complex line integral $\int_C f(z) dz$ be expressed in terms of real and imaginary components?
3. Under what conditions is the complex line integral independent of the path taken between two points?
4. Explain the physical or geometric interpretation of the real and imaginary parts of a complex line integral.

7.11 *SUMMARY:-*

A complex line integral is an integral of a complex-valued function $f(z)$ along a path or contour C in the complex plane, defined as $\int_C f(z)dz$. It is computed by parameterizing the contour C with $z(t)$ and integrating $f(z(t)) \cdot z'(t)$ with respect to t over the interval of the parameter. This integral can be expressed in terms of real and imaginary parts, and it plays a crucial role in complex analysis by connecting the evaluation of integrals to the properties of analytic functions and their singularities. Key results such as the Fundamental Theorem of Line Integrals and the Residue Theorem are often used to simplify and compute these integrals.

7.12 *GLOSSARY:-*

- **Complex Line Integral:** An integral of a complex-valued function along a contour in the complex plane, defined as $\int_C f(z)dz$, where C is the path or curve and $f(z)$ is a complex function.
- **Contour (or Path):** A continuous and differentiable curve in the complex plane along which the complex line integral is computed.
- **Parameterization:** A representation of the contour C by a continuous function $z(t)$ where t varies over an interval$[a, b]$. The integral is then evaluated using this parameterization.
- **Complex Function:** A function $f(z)$ where z is a complex variable and $f(z)$ maps z to another complex number. It can be expressed as $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions.
- **Differentialdz:** The differential element of the complex variable z, representing an infinitesimal change along the contour. In parameterized form, $dz = z'(t)dt$.
- **Fundamental Theorem of Line Integrals:** A theorem stating that if $f(z)$ is analytic and $F(z)$ is an antiderivative of $f(z)$, then $\int_C f(z)dz = F(z_1) - F(z_2)$, where C is a path from z_0 to z_1.
- **Analytic (or Holomorphic) Function:** A function f(z) that is complex differentiable at every point in its domain, meaning it has a derivative at every point in an open set.
- **Simply Connected Domain:** A domain in which any closed contour can be continuously shrunk to a point without leaving the domain. It ensures that the integral of an analytic function around any closed curve is zero.
- **Real and Imaginary Parts:** For a complex function $f(z) = u(x, y) + iv(x, y)$, $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of $f(z)$, respectively. The complex line integral can be split into integrals involving these parts.

7.13 *REFERENCES:-*

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- Steven G. Krantz (2018), Complex Variables: A Physical Approach with Applications and MATLAB Tutorials.
- Barry Simon (2015), Complex Analysis: A Comprehensive Course in Analysis, Part 2B.

7.14 *SUGGESTED READING:-*

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- file:///C:/Users/user/Downloads/Paper-III-Complex-Analysis.pdf
- file:///C:/Users/user/Desktop/Unit-3_Complex-Integration.pdf
- file:///C:/Users/user/Desktop/1468564049EText(Ch-9,M-2%20(2).pdf
- A.I. Markushevich 2005 (Dover Reprint of 1977 Edition),Theory of Functions of a Complex Variable.

7.15 *TERMINAL QUESTIONS:-*

(TQ-1) Evaluate $\int_C z^2 dz$, where C is the straight line joining the origin O to the point $P(2,1)$ in the complex plane.

(TQ-2) Evaluate $\oint_C In\, z dz$, where C is unit circle $|z| = 1$ taken is countor clockwise sense.

(TQ-3) Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0), (1,0), (1,1), (0,1)$

(TQ-4) Evaluate $\oint_C (z-a)^n dz$, where a is a given complex number, n is any integer and C is a circle of radius R centered at a and oriented anticlockwise.

(TQ-5) Find the integral of the function $f(z) = \frac{1}{z}$, taken over a circle of Radius R.

(TQ-6) Evaluate $\int_C zdz\ (ab\ initio)$

(TQ-7) Evaluate $\int_C dz\ (ab\ initio)$

(TQ-8) Evaluate $\int_0^{1+i} zdz$ along the line $z = 0$ to $z = 1 + i$.

(TQ-9) Evaluate the integral $\int_C \bar{z} dz$, where C is the straight line from (1,0) to (1,1).

(TQ-10) Evaluate $\int_0^{1+i} z^2\, dz$.

(TQ-11)True/False Questions

- **a.** The circle $z = cost + isint,\quad 0 \le t \le 2\pi$ is a simple closed Jordan curve.
- **b.** If $f(z)$ is conformally continuous in a domain D, then it is not necessarily continuous in D,.
- **c.** If a function $f(z)$ is differentiable at $z = z_0$, then it is necessarily continuous there.
- **d.** A contour is continuous chain of a finite number of regular arcs. T
- **e.** If $f(z)$ be continuous in a simply connected domain D and $\int_\Gamma f(z)dz = 0$, where Γ is any rectifiable closed Jordan curve in D, then $f(z)$ is analytic in D.T
- **f.** For the indefinite integral of a function $f(z)$ to exist in a simply connected domain D, it is not necessary that $f(z)$ be analytic in D.
- **g.** Let G be the simply connected domain, and let $f(z)$ be a single valued analytic function such that $\int_C f(z)dz = 0$, where L is any closed rectifiable curve continuous in G.
- **h.** A function $f(z)$ possesses a indefinite integral in a simply closed connected domain G iff the function $f(z)$ is analytic in G.
- **i.** A function $f(z)$ is called an integral function or entire function if it is analytic in finite complex integral.
- **j.** Zeros of an analytic function are isolated.
- **k.** The integral $\int_C \frac{1}{z-a} dz$over a closed contour C that does not enclose a is zero.

(TQ-12) Prove that the function $\left[c\left(1+\frac{1}{z}\right)\right]$ can be expended in series of the type $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficients of both of z^n and z^{-n}, are $\frac{1}{2\pi}\int_0^{2\pi} sin(2c\cos\theta)\, sinn\theta\, d\theta$.

(TQ-13) Objectives type Questions:

1. A simple closed Jordan curve divides the argand plane into….. open domain which have the curve as common boundary.
   a. two
   b. three
   c. four
   d. eight
2. The path of the definite integral $\int_a^b f(z)\, dz$ is:
   a. the line segment joining the points $z = a$ and $z = b$.
   b. any curve joining the points $z = a$ and $z = b$.
   c. any circle such that the points $z = a$ and $z = b$.
   d. any rectangle whose two vertices are the points $z = a$ and $z = b$.

3. $\int_C dz$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to :
   a. z
   b. $z - a - b$
   c. $a - b - z$
   d. $b - a$
4. $\int_C |dz|$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to :
   a. $|b - a|$
   b. b-a
   c. arc length of L
   d. 0
5. If $f(z)$ is analytic in simply connected domain D and C is any closed continuous rectifiable curve in D, then $\int_C f(z)dz$ is equal to:
   a. 0
   b. 1
   c. C
   d. D
6. Let $f(z)$ be a continuous on a contour L of length l and let $|f(z)| \le M$ on L, then $\left|\int_C f(z)\ dz\right| = A$ where
   a. $A \le Ml$
   b. $A > Ml$
   c. $A \ge Ml$
   d. None
7. If $f(z)$ is analytic in simply connected domain Genclosed by a rectifiable Jordan CurveL and Let $f(z)$ be continuous on L. Then $\int_L \frac{f(z)}{z - z_0} dz$ is:
   a. $2\pi i\, f(z_0)$
   b. $2\pi i\, f'(z_0)$
   c. $2\pi f(z_0)$
   d. None

7.16 *ANSWERS:-*

<u>SELF CHECK ANSWERS</u>

1. A complex line integral is an integral of a complex-valued function $f(z)$ along a curve C in the complex plane. It is defined as $\int_C f(z)dz = \int_a^b f\big(z(t)\big).z'(t)dt$, where $z(t)$ is a parameterization of C and t varies over an interval $[a, b]$.

2. If $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions, and the curve C is parameterized by $z(t) = x(t) +$

$iy(t)$, then the complex line integral can be expressed as $\int_C f(z)dz = \int_a^b f(u(x(t), y(t)) + iv\big(x(t), y(t)\big)(x'(t) + y'(t))dt$.This separates into the real part $\int_C (udx - vdy)$ and the imaginary part $i\int_C (udy + vdx)$.

3. The complex line integral is independent of the path if f(z) is analytic (holomorphic) in a simply connected domain that includes the paths and the endpoints. This is a consequence of the Cauchy-Goursat theorem, which states that the integral around any closed contour in such a domain is zero.
4. The real part of a complex line integral can be interpreted as the work done by a vector field with components u and −v along the path C, while the imaginary part corresponds to the work done by a vector field with components v and u along the same path. This duality arises from treating the complex function f(z) as comprising two interrelated real-valued functions.

TERMINAL ANSWERS

(TQ-1) $\frac{1}{3}(2 + 11i)$ **(TQ-2)** $2\pi i$

(TQ-3) $-1 + i$ **(TQ-4)** $2\pi i$

(TQ-5) $2\pi i$ **(TQ-6)** $\frac{b^2 - a^2}{2}$

(TQ-7) 0 **(TQ-8)** i

(TQ-9) $i + \frac{1}{2}$ **(TQ-10)** $\frac{1}{2}(1 + i)^3$

(TQ-11)

a.T	b.F	c.T	d.T	e.T	f.F
g.T	h.T	i.T	j.T	k.T	

(TQ-13)

1.a	2.b	3.d	4.a	5.a	6.a
7.a					

UNIT 8:- Cauchy's Theorem and Cauchy's Integral Formula

CONTENTS:

8.1 *INTRODUCTION:-*

Cauchy's Theorem is a fundamental result in complex analysis, stating that if a function $f(z)$ is analytic (holomorphic) within and on a simple closed curve C in a simply connected domain, then the contour integral of $f(z)$ over C is zero. This theorem highlights the profound property of analytic functions, ensuring that the value of an integral around a closed path depends solely on the function's behavior within the path, leading to significant results like Cauchy's Integral Formula and the development of residue theory.

Overall, Cauchy's Theorem is a powerful result that not only facilitates the evaluation of complex integrals but also provides deep insights into the nature of analytic functions.

8.2 *OBJECTIVES:-*

The objectives of Cauchy's Theorem and Cauchy's Integral Formula in complex analysis are to establish the fundamental property that the contour integral of an analytic function over a closed curve in a simply connected domain is zero, and to provide a method for determining the function's value and its derivatives inside the contour using the values on the contour. These results serve as foundational tools for further study, enabling the development of series representations, the calculation of residues, and the exploration of the relationship between the analytic and geometric properties of functions and their domains.

8.3 *GREEN'S THEOREM:-*

Let C be a positively oriented, simple closed curve in the plane, and let D be the region enclosed by C. Suppose $P(x, y)$ and $Q(x, y)$ are functions with continuous partial derivatives on an open region containing D. Then Green's Theorem states:

$$\int_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here: $P(x, y)$ and $Q(x, y)$ are the components of a vector field $F = (P, Q)$.

Theorem: If $f(z)$ is analytic with $f'(z)$continuous within and on a simple closed contour C, then

$$\int_C f(z)dz = 0 \qquad ...(1)$$

Proof: Let $C: [a, b] \to G$ is a smooth curve.

Now we assume $f(z) = u(x, y) + iv(x, y)$ and $C = x(t) + iy(t), a \leq t \leq b$. Then

$$\int_C f(z)dz = \int_C \big(u(x, y) + iv(x, y)\big)(dx + idy)dz$$
$$= \int_a^b (udx - vdy) + i\int_a^b (udy + vdx) \qquad ...(2)$$

Given that $f(z) = u(x, y) + iv(x, y)$ is an analytic function with $f'(z)$ continuous, the first order partial derivative and v are continuous.

Now from (2), we get

$$\int_C f(z)dz = \iint_D (-v_x - u_y)\, dxdy + i \iint_D (u_x - v_y)\, dxdy$$

Where D is the region enclosed by C. By Cauchy Riemann Equations i.e., $u_x = v_y, u_y = -v_x$, we get

$$\int_C f(z)dz = \iint_D (u_y - u_y)\, dxdy + i \iint_D (u_x - u_x)\, dxdy$$

$$\int_C f(z)dz = \iint_D (0)\, dxdy + i \iint_D (0)\, dxdy$$

$$\int_C f(z)dz = 0$$

8.4 *CAUCHY'S THEOREM:-*

If the function $f(z)$ is analytic and single valued inside and a simple closed contour C, then

$$\int_C f(z)dz = 0$$

Proof:

Lemma: Given $\epsilon > 0$, it is possible to divide the region inside a simple closed contour C into a finite number of smaller regions, either complete squares $C_n$or partial squares D_n, such that within each mesh there exists a point z_0, where the following conditions are satisfied:

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon\ \forall z\ \ in\ mesh \quad \ldots(1)$$

Proof the Lemma:

Gourset's Lemma: Suppose the lemma is not true. It means that the lemma fails at least one mesh. Subdivide this mesh by means of line joining the middle points of the opposite sides. If there is still at least one part which does not satisfy the condition (1). Again subdivide that part in the same way. This process comes to an end after a finite number of steps, when the condition (1) is satisfied for every subdivision, or the process may go on indefinitely. In the second case, we obtain the sequence of squares (each contained the proceeding ones) which has z_0 as it limit point at which the condition (1) is not satisfied. Of course, z_0 is an interior point of C. Since the condition (1) is not satisfied at z_0 and so

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right| \nless \varepsilon where\ |z-z_0|<\delta$$

δ being a small number of depending upon ε.

Given that $f(z)$ is not differentiable at z_0, and z_0 is an interior point of C, you can conclude that $f(z)$ is not analytic at z_0. This contradicts the initial assumption that $f(z)$ is analytic at every interior point of C. Therefore, the lemma must be true. From the lemma, we get

$$\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)=\eta<\varepsilon$$

And $\eta \to \infty$ as $z \to z_0$

So

$$f(z)=(z-z_0)\eta(z)+f(z_0)+(z-z_0)f'^{(z_0)} \qquad \dots(2)$$

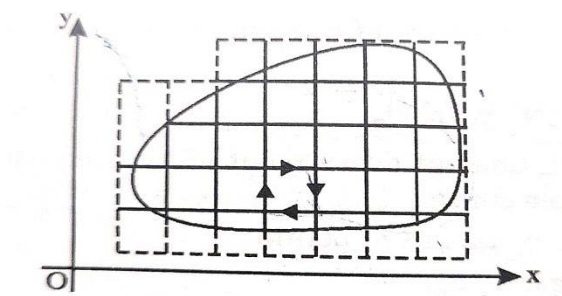

Fig.1

Proof of this theorem. Divide the interior of C into complete squares $C_1, C_2, \dots \dots \dots \dots, C_n$ and partial squares $D_1, D_2, \dots \dots \dots \dots, D_n$, where D_i part of its boundary along the boundary of has C.

Consider the integral

$$\sum_{r=1}^{n}\int_{C_r} f(z)dz+\sum_{r=1}^{n}\int_{D_r} f(z)dz$$

where the path of every integral being in anti-clockwise direction.

In the complete sum, integration along each straight side of each square (whether complete or partial) is taken twice in opposite directions, causing all such integrals to cancel out; thus, only the integrals along the curved boundaries of the partial squares remain, as these are described only once. The integrals which are left behind sum equal to

$$\int_C f(z)dz$$

$$\int_C f(z)dz = \sum_{r=1}^{n} \int_{C_r} f(z)dz + \sum_{r=1}^{n} \int_{D_r} f(z)dz \qquad \ldots(3)$$

From(2), we get

$$\int_C f(z)dz = \int_{C_r} [f(z_0) + (z - z_0)\eta + (z - z_0)\, f'(z_0)]\, dz$$

$$= [f(z_0) - z_0\, f'(z_0)] \int_{C_r} dz + f'(z_0) \int_{C_r} zdz + \int_{C_r} (z - z_0)\eta dz$$

Now using

$$\int_{C_r} dz = 0 = \int_{C_r} zdz$$

We express as

$$\int_{C_r} f(z)dz = 0 = \int_{C_r} (z - z_0)\eta dz$$

Now from (3), we have

$$\int_C f(z)dz = \sum_{r=1}^{n} \int_{C_r} (z - z_0)\eta dz + \sum_{r=1}^{m} \int_{D_r} (z - z_0)\eta dz$$

$$\left|\int_C f(z)dz\right| \le \sum_{r=1}^{n} \left|\int_{C_r} (z - z_0)\eta dz\right| + \sum_{r=1}^{m} \left|\int_{D_r} (z - z_0)\eta dz\right|$$

$$\le \sum_{r=1}^{n} \int_{C_r} |z - z_0|\, |\eta|\, |dz| + \sum_{r=1}^{m} \int_{D_r} |z - z_0|\, |\eta|\, |dz|$$

$$< \sum_{r=1}^{n} \varepsilon \int_{C_r} |z - z_0|\ \ |dz| + \varepsilon \sum_{r=1}^{m} \int_{D_r} |z - z_0|\, |dz| \quad \ldots(4)$$

as $|\eta| < \varepsilon$.

Let l_n, A_n denote the length of the side and the area of the complete square C_n, respectively. Similarly, let l'_n and A'_ndenote the length of the side and the area of the partial square D_n, respectively.Then from the equation (4),

$$\left|\int_C f(z)dz\right| < \sum_{r=1}^{n} \varepsilon l_r\sqrt{2}\int_{C_r} |dz| + \sum_{r=1}^{m} \varepsilon l'_r\sqrt{2}\int_{D_r} |dz|$$

$$\left[Since\ |z-z_0| \le l_r\sqrt{2} = diagonal\ of\ square\ C_r\ \right]$$

$$= \sum_{r=1}^{n} \varepsilon l_r\sqrt{2}\,.4\,l_r + \sum_{r=1}^{m} \varepsilon l'_r\sqrt{2}\,.(4\,l_r + s_r)$$

$$\left[For\ \int_{C_r} |dz| = perimeter\ of\ square\ C_r\right]$$

$$= 4\varepsilon\sqrt{2}\left[\sum_{r=1}^{n} A_r^2 + \sum_{r=1}^{m} A'_r\right] + \varepsilon\sqrt{2}\sum_{r=1}^{m} l'_r\, s_r$$

$$= 4\varepsilon\sqrt{2}.A + \varepsilon\sqrt{2}\sum_{r=1}^{m} l'_r\, s_r$$

where $A =$ total area of square of side l with which the region was originally covered. Also let l be total length of boundary of C. then

$$\left|\int_C f(z)dz\right| < 4\varepsilon\sqrt{2}.A + \varepsilon\sqrt{2}\sum_{r=1}^{m} l\, s_r$$

$$= 4\varepsilon\sqrt{2}.A + 4\varepsilon lL\sqrt{2}$$

$$\left|\int_C f(z)dz\right| < \varepsilon\left[4\varepsilon\sqrt{2}.A + 4\varepsilon lL\sqrt{2}\right]$$

Since $\varepsilon \to 0$, we obtain

$$\int_C f(z)dz = 0.$$

8.5 *EXTENSION OF CAUCHY'S THEOREM:-*

Corollary1: If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ along any rectifiable curve joining two points z_1 and $z_2$within D is independent of the path taken between z_1 and z_2.

Proof: If $A(z_1)$ and $B(z_2)$ are two points in a simply connected domain D, and they are joined by two curves C_1 and C_2 the integral of an analytic function $f(z)$ along these curves is independent of the specific path taken,then by Cauchy's theorem

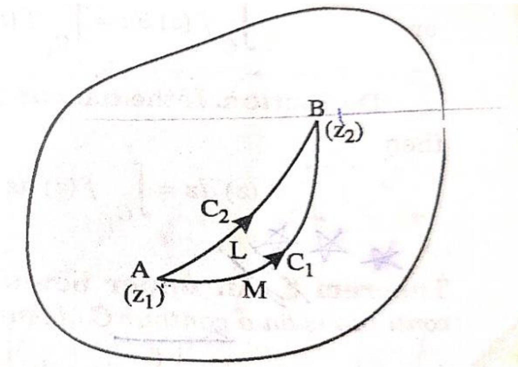

Fig.2

$$\int_{ALBMA} f(z)dz = 0$$

$$\int_{ALB} f(z)dz + \int_{BMA} f(z)dz = 0$$

$$\int_{ALB} f(z)dz - \int_{AMB} f(z)dz = 0$$

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Corollary2: If $f(z)$ is analytic in the annular region D bounded by C and C_1, then the integral of $f(z)$ around C and C_1 is related by:

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz$$

Proof: If we join a point A on the contour C to a point E on the contour C_1, then by Cauchy's theorem, we get

$$\int_{ABCDAEFGEA} f(z)dz = 0$$

Or

$$\int_{ABCDA} f(z)dz + \int_{AE} f(z)dz + \int_{EFGE} f(z)dz + \int_{EA} f(z)dz = 0$$

But $\int_{AE} f(z)dz = -\int_{EA} f(z)dz$ and $\int_{AE} f(z)dz + \int_{EA} f(z)dz = 0$

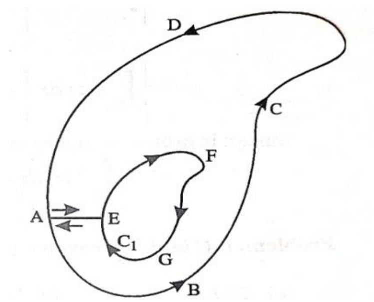

Fig.3

So

$$\int_{ABCDA} f(z)dz + \int_{EFGE} f(z)dz = 0$$

$$\int_{C} f(z)dz + \int_{EFGE} f(z)dz = 0$$

$$\int_{C} f(z)dz - \int_{EGFE} f(z)dz = 0$$

Hence

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz$$

If the contour C contains non-intersecting $C_1, C_2 \dots\dots . C_n$, then

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots \int_{C_n} f(z)dz$$

8.6 *AN UPPER BOUND FOR COMPLEX INTEGRAL:-*

If a function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, then $\left|\int_C f(z)dz\right| \le Ml$.

Proof: Given that divide the contour C into n parts by means of points $z_0, z_1, z_2, \dots \dots z_n$. we choose a point ξ_r on each arc joining z_{r-1} to z_r. From the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1})$$

Also $(z_r - z_{r-1}) \to 0 as\, n \to \infty$.

Now we define $\int_C f(z)dz = \lim_{n\to\infty} S_n$

$$|S_n| = \left|\sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1})\right| \le \sum_{r=1}^{n} |f(\xi_r)|\,|(z_r - z_{r-1})|$$

$$\le \sum_{r=1}^{n} M\,|(z_r - z_{r-1})|$$

Let $n \to \infty$ and noting (1), we obtain

$$\left|\int_C f(z)dz\right| \le \lim_{n\to\infty} M \sum_{r=1}^{n} |(z_r - z_{r-1})| \quad \dots (2)$$

But

$$\lim_{n\to\infty} M \sum_{r=1}^{n} |(z_r - z_{r-1})|$$

$$= \lim_{n\to\infty} [|z_1 - z_0| + |z_2 - z_1| + \cdots + |z_n - z_{n+1}|]$$

$$= \lim_{n\to\infty} [chord\ z_1 z_0 + chord\ z_2 z_1 + \cdots + chord\ z_n z_{n-1}]$$

$$= \lim_{n\to\infty} [arc\ z_1 z_0 + arc\ z_2 z_1 + \cdots + arc\ z_n z_{n-1}]$$

$$= \text{arc length of contour } C = l$$

Using (2), we get

$$\left|\int_C f(z)dz\right| \le Ml$$

8.7 *CAUCHY INTEGRAL FORMULA:-*

If $f(z)$ is analytic within and on a closed contour C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Proof: Let $f(z)$ is analytic within and on a closed contour C and a is interior point of C.

To prove that

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

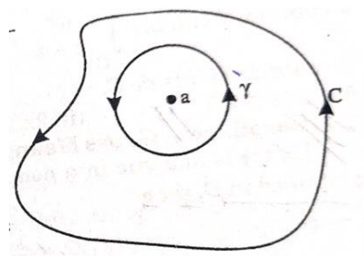

Fig.4

Let the circle γ about the centre $z = a$ of small radius $r\ s.t.$, $|z-a| = r$ does not intersect the curve C. Hence by the corollary to Cauchy's theorem, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz \qquad \ldots(1)$$

$$\int_C \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)-f(a)}{z-a}dz + \int_\gamma \frac{f(z)}{z-a}dz \quad \ldots(2)$$

Since $f(z)$ is analytic within the contour C, it is also continuous at any point $z = a$. This means that for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(a)| < \varepsilon \quad \ldots(3)$ whenever $||z-a| < \delta \ldots..(4)$. We can choose a radius r less than δ so that for all points z on a circle γ centered at a with radius r, the condition $|z - a| < \delta$ is satisfied, ensuring $|f(z) - f(a)| < \epsilon$ for all z on γ. For any point z on γ, z can be expressed as $z - a = re^{i\theta}$, where θ ranges from 0 to 2π.

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a)re^{i\theta}id\theta}{re^{i\theta}} = 2\pi i f(a)$$

In view of (2), we have

$$\left|\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a)\right| = \left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right|$$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}|dz| < \frac{\varepsilon}{r}\int_\gamma |dz| = \frac{\varepsilon}{r}.2\pi r$$

$$\left|\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a)\right| < 2\pi\varepsilon$$

Since $\varepsilon \to 0$, we obtain

$$\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a) = 0$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Remarks:

1. $|a-b| < \varepsilon \Rightarrow a-b=0$
2. $\int_\gamma |dz| = 2\pi.radius$
3. From equation(1) and (2), we get
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \qquad \ldots(5)$$

Corollary1: (Gauss Mean Value Theorem) If $f(z)$ is analytic in a Domain D and if the circular domain $|z-z_0| \le \rho$ is contained in D, then the value of $f(z)$ at z_0 is given by the average of its values of the boundary of the circle $|z-z_0| = \rho$.Mathematically, this is expressed as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right)d\theta$$

Proof: From the equation (1) and (5), we get

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz$$

$$|z-z_0| = \rho \Rightarrow z - z_0 = \rho e^{i\theta} \Rightarrow dz = \rho i e^{i\theta}d\theta$$

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})\rho i e^{i\theta}d\theta}{\rho e^{i\theta}}$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})d\theta$$

Theorem1: (Extension of Cauchy's Integral formula to multiply connected regions): If $f(z)$ is analytic in a ring shaped region bounded by two closed curves C_1 and C_2 and a is a point in the region between C_1 and C_2.

$$f(a) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-a}dz$$

where C_2 is outer curve.

Proof: The circle γ centered at $z = a$ with radius r is chosen such that it lies entirely within the annular region bounded by the closed curves C_1 and C_2, ensuring that the function $\frac{f(z)}{z-a}$ is analytic in the region enclosed by C_1, C_2 and γ.

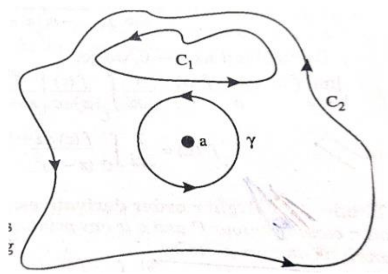

Fig.5

By corollary Cauchy's theorem

$$\int_{C_2} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz + \int_{\gamma} \frac{f(z)}{z-a}dz$$

Where the integral along each curve is taken in anti-clockwise direction. Using Cauchy's integral formula,

$$\int_{C_2} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz + 2\pi i\, f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-a}dz$$

Theorem2: (Cauchy's Integral formula for the derivative of an analytic function): If $f(z)$ is analytic within and on a closed contour C and a is any lying in it, then

$$f'(a) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^2}dz$$

Or

Using Cauchy's integral formula to find first derivative of an analytic function $f(z)$ at $z = z_0$.

Proof: Let $a + h$ be a point in the neighborhood of a point a, then by Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+h)}dz$$

Now

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{h}\left[\frac{1}{z-a-h} - \frac{1}{z-a}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\left(1-\frac{h}{z-a}\right)^{-1} - 1\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\frac{h}{z-a} + \left(\frac{h}{z-a}\right)^2 + \left(\frac{h}{z-a}\right)^3 + \cdots\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\left[\frac{1}{z-a} + \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^3} + \cdots\right]dz$$

Taking $h \to 0$, we obtain

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\left[\frac{1}{z-a} + 0 + 0 \ldots\right]dz$$

$$f'(a) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^2}dz$$

8.8 *HIGHER ORDER DERIVATIVES:-*

If $f(z)$ is a function of analysis within and on a closed contour C and a is any point within C then derivatives of all orders are analytic and are given by

$$f^n(a) = \frac{n!}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{n+1}}dz$$

Proof: From the previous theorem 2, we obtain

$$f^{(1)}(a) = f'(a) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^2}dz$$

This proves that the result is true for $n = 1$. Let us consider the result is true for $n = m$ so that

$$f^{(m)}(a) = f'(a) = \frac{m!}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{m+1}}dz$$

Let $a + h$ be the point in neighborhood of a, then

$$\frac{f^{(m)}(a+h) - f^{(m)}(a)}{h}$$
$$= \frac{m!}{2\pi i}\int_C f(z)\,\left[\frac{1}{(z-a-h)^{m+1}} - \frac{1}{(z-a)^{m+1}}\right]dz$$

$$= \frac{m!}{2\pi ih}\int_C \frac{f(z)}{(z-a)^{m+1}}\,\left[\left(1-\frac{1}{z-a}\right)^{-(m+1)} - 1\right]dz$$

$$= \frac{m!}{2\pi ih}\int_C \frac{f(z)}{(z-a)^{m+1}}\,\left[\frac{h(m+1)}{z-a} + \frac{(m+1)(m+2)}{2!}\left(\frac{h}{z-a}\right)^2 + \cdots\right]dz$$

Taking $h \to 0$

$$\lim_{h\to 0}\frac{f^{(m)}(a+h) - f^{(m)}(a)}{h} == \frac{m!}{2\pi ih}\int_C \frac{f(z)}{(z-a)^{m+1}}\,\left[\frac{(m+1)}{z-a} + 0 + \cdots\right]dz$$

$$f^{(m+1)}(a) = \frac{(m+1)m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+2}}\,dz$$

$$f^{(m+1)}(a) = \frac{(m+1)!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+2}}\,dz$$

This proves that we have already seen that the result is true for $n = 1$ and shown that it being true for $n = m$ implies it is true for $n = m + 1$, it follows that the result is true for all positive integers n.

Since $f^{(1)}(a), f^{(2)}(a), f^{(3)}(a) \ldots\ldots\ldots\ldots.. \; all\ exist$
Consequently $f^{(1)}(a), f^{(1)}(a) \ldots\ldots\ldots all\ are\ analytic\ within\ C.$

8.9 *POISSON'S INTEGRAL FORMULA:-*

If $f(z)$ is analysis within and on a circle C defined by $|z| = R$ and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2 - a\bar{a})f(z)}{(z-a)(R^2 - z\bar{a})}dz$$

Hence deduce the Poisson's formula

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_C \frac{(R^2 - r^2)f(Re^{i\phi})d\phi}{(R^2 - 2Rrcos(\theta - \phi) + r^2)}dz$$

where $a = re^{i\theta}$ is any point inside the circle $|z| = R$.

Proof: Let us suppose $f(z)$ is analysis within and on a circle C defined by $|z| = R$ and $a = re^{i\theta}$ is any point A inside C so that $0 < r < R$

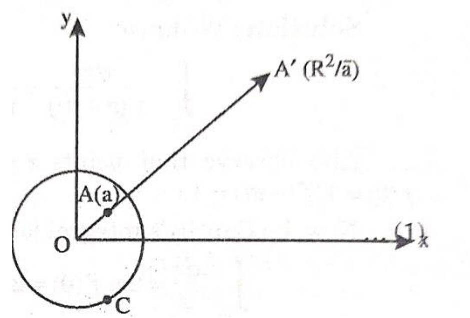

Fig.6

Let $A'(a')$ of $A(a)$ w.r.t. the circle C is given by $a' = \frac{R^2}{\bar{a}}$ which outside the circle C, by Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - a}dz \quad \text{... (1)}$$

Since $f(z)$ is analysis within and upon the circle C and so $\frac{f(z)}{z-a'}$ is within and on a circle C, by Cauchy's theorem

$$\int_C \frac{f(z)}{z - a'}dz = 0 \quad \text{... (2)}$$

From (1) and (2), we obtain

$$f(a) - 0 = \frac{1}{2\pi i}\int_C \left[\frac{f(z)}{z - a} - \frac{f(z)}{z - a'}\right]dz$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(a - a')f(z)}{(z - a)(z - a')}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(a - \frac{R^2}{\bar{a}}\right) f(z)}{(z-a)\left(z - \frac{R^2}{\bar{a}}\right)} dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(a\bar{a} - R^2) f(z)}{(z-a)(\bar{a}z - R^2)} dz$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(a\bar{a} - R^2) f(z)}{(z-a)(\bar{a}z - R^2)} dz \qquad \ldots(3)$$

This proves first result.

Let $|z| = R$ is expressible as $z = Re^{i\phi}$.

Also $a = re^{i\theta}$ so that $\bar{a} = re^{-i\theta}$

Now $R^2 - a\bar{a} = R^2 - re^{i\theta}.re^{-i\theta} = R^2 - r^2 \qquad \ldots(4)$

$$(z-a)(R^2 - z\bar{a}) = \left(Re^{i\phi} - re^{i\theta}\right)\left(R^2 - Re^{i\phi}re^{-i\theta}\right)$$

$$= Re^{i\phi}\left(R - re^{i(\theta-\phi)}\right)\left(R - re^{-i(\theta-\phi)}\right)$$

$$= Re^{i\phi}\left[R^2 + r^2 - rR\left(e^{i(\theta-\phi)} - e^{-i(\theta-\phi)}\right)\right]$$

$$= Re^{i\phi}[R^2 + r^2 - 2rRcos(\theta - \phi)] \qquad \ldots(5)$$

$$dz = d\left(Re^{i\phi}\right) = Rie^{i\phi}d\phi \qquad \ldots(6)$$

Putting the value of (4), (5) and (6) in (3), we get

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2 - r^2) f(z)\left(Re^{i\phi}\right) i d\phi}{(R^2 - 2Rrcos(\theta - \phi) + r^2)Re^{i\phi}} dz$$

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_C \frac{(R^2 - r^2) f\left(Re^{i\phi}\right) d\phi}{(R^2 - 2Rrcos(\theta - \phi) + r^2)} dz$$

SOLVED EXAMPLE

EXAMPLE1: Evaluate $\int_C \frac{e^{2z}dz}{(z+1)^4}$, where C is $|z| = 3$.

SOLUTION: Suppose

$$f^n(a) = \frac{n!}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{n+1}} dz$$

Put $a = -1, n = 3$

$$f^3(-1) = \frac{3!}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^4} dz$$

Taking $f(z) = e^{2z}$, then $f^n(z) = 2^n e^{2z}$

$$f^3(-1) = 2^3 e^{-2} = \frac{8}{e^2}$$

$$\frac{8}{e^2} = \frac{3!}{2\pi i}\int_{C_2} \frac{e^{2z}}{(z-a)^4} dz$$

$$\frac{8\pi i}{3e^2} = \int_C \frac{e^{2z}}{(z-a)^4} dz$$

EXAMPLE2: Using Cauchy's integral Formula, Calculate the following integrals.

- **i.** $\int_C \frac{z}{(9-z^2)(z+i)} dz$, where C is the circle $|z| = 2$ described in positive sense.
- **ii.** $\int_C \frac{1}{z(z+\pi i)} dz$, where C is the circle $|z + 3i| = 1$
- **iii.** $\int_C \frac{cosh(\pi z)dz}{z(z^2+1)} dz$, where C is the circle $|z| = 2$
- **iv.** $\int_C \frac{e^{az}}{z(z-\pi i)} dz$, where C is the ellipse $|z-2| + |z+2| = 6$
- **v.** Evaluate $\int_C \frac{dz}{z-2}$, where C is $|z| = 3$.

SOLUTION: By Cauchy's integral Formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} dz \Rightarrow \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \quad ...(1)$$

where $z = a$ is a point inside contour C and $f(z)$ is analytic within and upon C.

i.

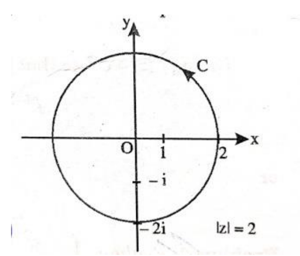

Fig.7

Suppose

$$I = \int_C \frac{z}{(9-z^2)(z+i)} dz$$

Now we

$$I = \int_C \frac{z}{(9-z^2)}dz$$

Then

$$I = \int_C \frac{f(z)}{(z-(-i))}dz = 2\pi i f(-i) = \frac{2\pi}{9+1} = \frac{\pi}{5}$$

Hence $f(z)$ is analytic within and upon C s.t. $|z| = 2$ and $z = -i$ lies inside C.

ii. Let

$$I = \int_C \frac{1}{z(z+\pi i)}dz$$

Take $f(z) = \frac{1}{z}$

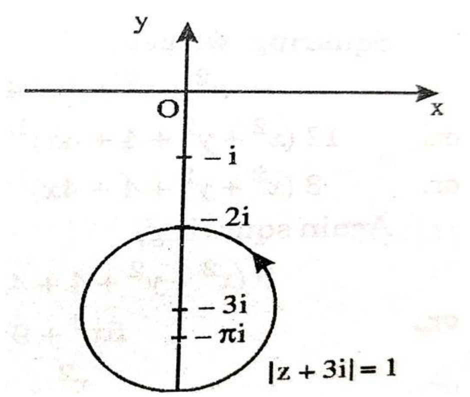

Fig.8

$$I = \int_C \frac{1}{z(z-(-\pi i))}dz$$

$$= 2\pi i f(-\pi i)$$

$$= 2\pi i \left(\frac{1}{-\pi i}\right) = -2$$

Hence $z = -i\pi$ lies inside C and $f(z)$ is analytic within C.

iii. Let

$$\int_C \frac{cosh(\pi z)dz}{z(z^2+1)}dz$$

Take $f(z) = cosh(\pi z) = cosh(i\pi z)$ and C is $|z| = 2$.

Then

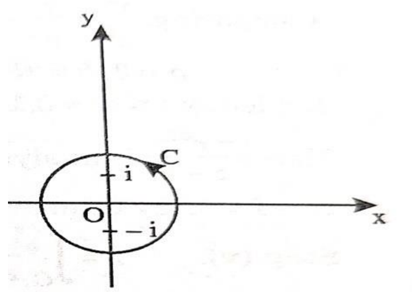

Fig.9

$$I = \int_C \frac{f(z)}{z(z^2+1)} dz$$

$$I = \int_C \left[\frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}\right] f(z) dz \quad ...(2)$$

$$\frac{1}{z(z-i)(z+i)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$

$$A = \frac{1}{(z-i)(z+i)} = 1 \; at \; z = 0$$

$$B = \frac{1}{z(z+i)} = -\frac{1}{2} \; at \; z = i$$

$$C = \frac{1}{z(z-i)} = -\frac{1}{2} \; at \; z = -i$$

when $z = 0, i, -i$ are the points inside C.
From the equation $(1) + (2)$, we obtain

$$I = 2\pi i[Af(0) + Bf(i) + Cf(-i)]$$

$$= 2\pi i \left[f(0) - \frac{1}{2} f(i) - \frac{1}{2} f(-i)\right]$$

$$= 2\pi i \left[cos(0) - \frac{1}{2} cos(i^2\pi) - \frac{1}{2} f(-i^2\pi)\right]$$

$$= 2\pi i \left[1 + \frac{1}{2} - \frac{1}{2}\right] = 2\pi i$$

iv. Let

$$\int_C \frac{e^{az}}{z(z-\pi i)} dz,$$

C is the ellipse $|z-2| + |z+2| = 6$

$$[(x-2)^2 + y^2]^{1/2} = 6 - [(x+2)^2 + y^2]^{1/2}$$

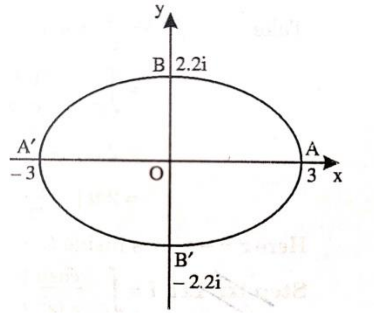

Fig.10

Squaring both sides,

$$x^2 + y^2 + 4 - 4x = 36 + (x^2 + y^2 + 4 + 4x) - 12[(x+2)^2 + y^2]^{1/2}$$

$$12[(x+2)^2 + y^2]^{1/2} = 36 + 8x$$

$$3(x^2 + y^2 + 4 + 4x)^{1/2} = 9 + 2x$$

Again squaring

$$9(x^2 + y^2 + 4 + 4x)^{1/2} = 81 + 36x + 4x^2$$

$$5x^2 + 9y^2 = 45$$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $a^2 = 9, b^2 = 5 \Rightarrow a = 3, b = 2.2 appox$

So $\frac{e^{az}}{z(z-\pi i)}$ is analytic within and upon C.

$\therefore$ $I = 0$, by Cuchy's theorem

v. Suppose

$$\int_C \frac{dz}{z-2} = \int_C \frac{f(z)dz}{z-2}$$

Then $a = 2, f(z) = 1$. C is circle $|z| = 3$ whose centre is at $z = 0$ and radius $R = 3$.

Since $a = 2$ lies inside C.

Now from (1) and (2) ,we get

$$I = 2\pi i f(a) = 2\pi i f(2) = 2\pi i(1) = 2\pi i$$

For $f(z) = 1 \Rightarrow f(2) = 1$.

EXAMPLE3: Evaluate by Cauchy's integral formula

$$\int_C \frac{dz}{z(z-i\pi)}$$

where C is $|z+3i|=1$.

SOLUTION: Let

$$\int_C \frac{dz}{z(z+i\pi)} = \frac{1}{i\pi}\left[\int_C \frac{dz}{z} - \int_C \frac{dz}{(z+i\pi)}\right]$$

Also

The distance from $z=0$ to the center of the circle $-3i$ is $|\ 0+3i\ |=3$. Since $3>1, z=0$ is not inside the circle.

The distance from $z=i\pi$ to the center of the circle $-3i$ is $|\ i\pi+3i\ |=|\ \pi+3\ |\ i\ |=\pi+3$. Since $\pi \approx 3.14 > 1\pi \approx 3.14 > 1, z=i\pi$ is also not inside the circle.

Since neither of the poles is inside the contour C, the function $\frac{1}{z(z-i\pi)}$ is analytic inside and on C.
Therefore, the integral of this function over the closed contour C is zero by Cauchy's Theorem:
$\int_C \frac{dz}{z} = 2\pi i f(0) = 2\pi i$ and $\int_C \frac{dz}{(z+i\pi)} = 2\pi i f(-\pi i) = 2\pi i$
$\therefore$

$$\int_C \frac{dz}{z(z+i\pi)} = \frac{2\pi i - 2\pi i}{\pi i} = 0$$

EXAMPLE4: Find the value of $\int \frac{1}{z} dz$ where C is circle $= e^{i\theta}, 0 \le \theta \le 2\pi$.
SOLUTION: The circle $z=e^{i\theta}$ represents a unit circle in the complex plane, parameterized by θ ranging from 0 to 2π. The parameterization is:

- $z(\theta) = e^{i\theta}$
- $dz = \frac{dz}{d\theta} d\theta = ie^{i\theta} d\theta$

The integral becomes:

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} . ie^{i\theta} d\theta$$

Simplifying the expression inside the integral:

$$= \int_0^{2\pi} i\, d\theta$$

Since i is a constant, the integral simplifies to:

$$= \int_0^{2\pi} i\, d\theta = i[\theta]_0^{2\pi} = i(2\pi - 0) = 2\pi i$$

Therefore, the value of the integral $\int_C \frac{1}{z} dz$, where C is the circle $z=e^{i\theta}$ for $0 \le \theta \le 2\pi$.
EXAMPLE5: Evaluate $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz$

where C is the boundary of square whose sides lie along the lines $x = \pm 2, y = \pm 2$ and it is described in positive sense, where $|x_0| < 2$.

SOLUTION: By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

and

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

This ⇒

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

and

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a)$$

where $z = a$ lies inside C and $f(z)$ is analytic within and upon C.

SELF CHECK QUESTIONS

1. How does Cauchy's Theorem relate to the concept of an analytic function?
2. What is the significance of the domain being "simply connected" in Cauchy's Theorem?
3. Can Cauchy's Theorem be applied if the function is not analytic on the contour?
4. What are the conditions for applying Cauchy's Theorem?
5. How does Cauchy's Theorem lead to Cauchy's Integral Formula?

8.10 *SUMMARY:-*

In this unit we have studied the Cauchy's Theorem and Cauchy's Integral Formula are foundational results in complex analysis. Cauchy's Theorem states that if a function $f(z)$ is analytic in a simply connected domain D and on a simple closed contour C within D, then the integral of $f(z)$ around C is zero, emphasizing the independence of path in analytic regions. Building on this, Cauchy's Integral Formula provides a specific value for the function at any point z_0 inside C by expressing it as $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$, showing that the value of an analytic function inside a contour can be determined entirely by its values on the contour. Together, these results highlight the deep connections between the values of an analytic function within a region and on its boundary, enabling further theorems and applications in complex analysis.

8.11 *GLOSSARY:-*

- **Analytic Function (Holomorphic Function):** A function $f(z)$ that is complex differentiable at every point in its domain.
- **Simply Connected Domain:** A region in the complex plane without holes, where any closed curve can be continuously contracted to a point within the region.
- **Contour (Path):** A directed curve in the complex plane often used in the context of integrating functions along its length.
- **Simple Closed Contour:** A contour that does not intersect itself and encloses a well-defined region.
- **Cauchy's Theorem:** A fundamental theorem stating that if a function is analytic within and on a simple closed contour in a simply connected domain, then the integral of the function over that contour is zero.
- **Cauchy's Integral Formula:** A powerful result in complex analysis that expresses the value of an analytic function inside a contour in terms of an integral over the contour. It is given by$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0} dz$, where z_0 is a point inside the contour C.
- **Residue:** The coefficient of $(z - z_0)^{-1}$ in the Laurent series expansion of a function around a point z_0, significant in evaluating integrals around singularities.
- **Laurent Series:** A representation of a complex function that generalizes the Taylor series, allowing terms with negative powers of$(z - z_0)$.
- **Pole:** A type of singularity where a function behaves like $\frac{1}{(z-z_0)^n}$ near $z = z_0$, causing the function to go to infinity.
- **Singularity:** A point at which a function is not analytic, which can be a pole, essential singularity, or removable singularity.
- **Integral Path Independence:** A property in simply connected domains where the integral of an analytic function depends only on the endpoints, not the path taken.
- **Integral along a Contour:** The sum (integral) of the values of a function along a contour in the complex plane, taking into account the direction of traversal.
- **Winding Number:** The number of times a contour wraps around a point, relevant in calculating residues and applying the argument principle.
- **Argument Principle:** A theorem relating the number of zeros and poles of a meromorphic function inside a contour to the integral of the function's logarithmic derivative around the contour.

8.12 *REFERENCES:-*

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8.13 *SUGGESTED READING:-*

- R. A. Adams and J. J. Fournier (2014),Introduction to Complex Analysis.
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- file:///C:/Users/user/Desktop/1468561978EText(Ch-5,M-3.pdf

8.14 *TERMINAL QUESTIONS:-*

(TQ-1) Prove Cauchy's Theorem for a simple closed contour in a simply connected domain.

(TQ-2) Derive and prove Cauchy's Integral Formula for an analytic function.

(TQ-3) Derive Cauchy's Integral Formula for higher-order derivatives.

(TQ-4) Prove the generalized version of Cauchy's Theorem for multiple contours within a simply connected domain.

(TQ-5) Prove the generalized Cauchy Integral Formula for a multi-connected domain with multiple singularities.

(TQ-6) State Cauchy’s theorem.

(TQ-7) If $f(z)$ is analytic with $f'(z)$continuous within and on a simple closed contour C, then prove that

$$\int_C f(z)dz = 0$$

(TQ-8) If $f(z)$ is analytic in a simply connected domain D, then prove that the integral of $f(z)$ along any rectifiable curve joining two points z_1 and $z_2$within D is independent of the path taken between z_1 and z_2.

(TQ-9) If a function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, then prove that

$$\left|\int_C f(z)dz\right| \leq Ml.$$

(TQ-10) If $f(z)$ is analytic within and on a closed contour C, and if a is any point within $C,$ then prove that

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

(TQ-11) Using Cauchy's integral formula to find first derivative of an analytic function $f(z)$ at $z = z_0$.

(TQ-12) Prove that $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}$, where C is contour containing $z = a$.

8.15 *ANSWERS:-*

SELF CHECK ANSWERS

1. Cauchy's Theorem highlights that if a function is analytic within a simply connected domain, the line integral of the function along any closed contour within that domain is zero. This property shows the independence of the integral from the specific path taken, illustrating that the integral depends only on the endpoints, which are the same for a closed contour, thus resulting in zero.
2. The domain being simply connected ensures that there are no holes or isolated singularities within the domain. This is important because if the domain were not simply connected, a function might have singularities that could affect the value of the integral. In a simply connected domain, any two paths between the same points can be continuously deformed into each other without leaving the domain, ensuring that the integral around a closed loop is zero if the function is analytic.
3. No, Cauchy's Theorem cannot be directly applied if the function has singularities on the contour C. The theorem requires the function to be analytic on and inside the contour C. If there are singularities on the contour, the integral might not even be defined in 1. the usual sense, and different techniques or theorems like the residue.
4. Answer: The key conditions for applying Cauchy's Theorem are:
    a. The function $f(z)$ must be analytic (holomorphic) in a domain D.
    b. The domain D must be simply connected.

c. The contour C must be a simple closed contour (a loop without self-intersections) within D.

5. Cauchy's Integral Formula is a consequence of Cauchy's Theorem. It provides a way to evaluate the value of an analytic function inside a contour using an integral over the contour itself. Specifically, for $f(z)$ analytic in a domain D containing a closed contour C and z_0 inside C, Cauchy's Integral Formula states:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

This formula is derived using Cauchy's Theorem, where the function $\frac{f(z)}{z-z_0}$ is analytic inside C except at $z = z_0$, where it has a simple pole.

UNIT 9:- Cauchy's Inequalities and other Theorem

CONTENTS:

9.1 *INTRODUCTION:-*

In this previous unit, we have studied about Cauchy's Theorems and Inequalities, are foundational concepts in complex analysis, providing key insights into the behavior of analytic functions. Cauchy's Theorem states that the integral of an analytic function over a closed contour is zero, highlighting the independence of the path in such integrals. Cauchy's Integral Formula further extends this by expressing the value of a function at a point inside the contour in terms of an integral around the contour, also allowing the calculation of derivatives.

In this Unit we will study about Cauchy's Inequalities give upper bounds for the derivatives of an analytic function based on the maximum value of the function on a surrounding contour. These results collectively help in understanding the properties of analytic functions, such as growth, smoothness, and the nature of singularities, and have broad applications in mathematics and physics.Morera's Theorem, Liouville's Theorem, Taylor's Theorem, and Laurent's Series are pivotal results in complex analysis that extend our understanding of analytic and meromorphic functions. Morera's Theorem provides a converse to Cauchy's Theorem, stating that if a function is continuous on a domain and its integral over every closed contour within that domain is zero, then the function is analytic. Liouville's Theorem asserts that any bounded entire function (analytic on the entire complex plane) must be constant, a fundamental result with implications in complex function theory and number theory. Taylor's Theorem in complex analysis expresses an analytic function as an infinite power series centered at a point, detailing the function's behavior in a neighborhood of that point. Laurent's Series generalizes this concept, representing functions with singularities in terms of both positive and negative powers, providing a powerful tool for analyzing functions with isolated singularities. Together, these theorems and series offer deep insights into the structure and properties of complex functions.

9.2 *OBJECTIVES:-*

The objectives of Cauchy's Inequalities, Morera's Theorem, Liouville's Theorem, Taylor's Theorem, and Laurent's Series are to provide foundational tools and insights in complex analysis: Cauchy's Inequalities aim to bound the derivatives of analytic functions within a region, highlighting their growth constraints; Morera's Theorem serves to confirm the analyticity of functions based on contour integrals, acting as a converse to Cauchy's Theorem; Liouville's Theorem establishes that any bounded entire function must be constant, offering critical insights into the behavior of complex functions; Taylor's Theorem provides a method to represent analytic functions as power series, facilitating local analysis and approximation; and Laurent's Series extends this representation to functions with isolated singularities, allowing detailed study of their behavior near such points.

9.3 *MORERA'S THEOREM:-*

If $f(z)$ is a continuous function in a domain D and if for every closed Contour C in the domain D,

$$\int_C f(z)\,dz = 0$$

Then $f(z)$ is analytic within D.

Proof: Let z_0 be the fixed point in D and z a variable point inside the domain D. the value of the integral $\int_{z_0}^{z} f(t)dt$ is independent of the curve joining z_0 to z and depends only z.

Now we write

$$F(z) = \int_{z_0}^{z} f(t)dt$$

Let $z + h$ be a point in the neighborhood of z. Consider the difference quotient:

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(t)dt - \int_{z_0}^{z} f(t)dt$$

$$= \int_{z_0}^{z+h} f(t)dt + \int_{z}^{z_0} f(t)dt = \int_{z}^{z+h} f(t)dt$$

By the continuity of f, for $|h|$ small enough, $f(t)$ is close to $f(z)$ for t between z and $z + h$. Specifically, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|h| < \delta$, then $|f(t) - f(z)| < \epsilon$ for all t between z and $z + h$. Therefore

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{z}^{z+h} f(t)dt - f(z)\right|$$

$$= \frac{1}{|h|}\left|\int_{z}^{z+h} [f(t) - f(z)]dt\right|$$

This can be bounded by:

$$\leq \frac{1}{|h|}\int_{z}^{z+h} |f(t) - f(z)||dt| < \frac{\varepsilon}{|h|}|h|$$

Since

$$[|f(t) - f(z)| < \varepsilon \; for \; |t - z| < \delta \; because \; of \; continuity \; of \; f(z)]$$

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| < \varepsilon \; which \; \varepsilon \to 0$$

Thus $\lim\limits_{h\to 0} \frac{F(z+h)-F(z)}{h} - f(z) = 0$ or $F'(z) = F(z)$.

Since $F(z)$ is the antiderivative of $f(z)$ and $F(z)$ is analytic (as it is defined by an integral of a continuous function), $f(z)$, being the derivative of an analytic function, is also analytic.

There $F'(z)$ i.e., $f(z)$ is analytic in D.

Thus, we have shown that if $f(z)$ is continuous in D and the integral of $f(z)$ over every closed contour in D is zero, then $f(z)$ is analytic in D. This completes the proof of Morera's Theorem.

Remark: If $f(z)$ is analytic in a simply connected region D of the complex plane, show that there exists a function $F'(x)$ analytic in D, and such that $F'(z) = f(z)$ for z in D.

9.4 *CAUCHY'S INEQUALITY:-*

If $f(z)$ is analytic within and on a circle C, given by $|z - a| = R$ and if $|f(z)| \leq M$ for every z on C, then the magnitude of the n −th derivative of f at a is bounded by

$$|f^n(a)| \le \frac{Mn!}{R^n}$$

Proof: Let $|z-a| = R \Rightarrow z - a = Re^{i\theta} \Rightarrow dz = iRe^{i\theta}d\theta \Rightarrow |dz| = Rd\theta$
According to Cauchy's Integral Formula, the $n-$th derivative of f at a can be expressed as:

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}$$

$$|f^n(a)| \le \frac{n!}{2\pi i}\int_C \frac{|f(z)||dz|}{|z-a|^{n+1}} \le \frac{Mn!}{2\pi R^{n+1}}\int_0^R Rd\theta$$

$$= \frac{Mn!}{2\pi R^{n+1}} 2\pi R$$

$$|f^n(a)| \le \frac{Mn!}{R^n}$$

Remark: If we take $a_n = \frac{f^n(a)}{n!}$, then $|a_n| \le \frac{M}{R^n}$

Integral Function: A function $f(z)$ is called an integral function or entire function if it is analytic in every finite region.

9.5 *LIOVILLE'S THEOREM:-*

State and prove Liouvlille's theorem.

or

If $f(z)$ is an entire function is bounded for all values of z, then it is constant.

or

If a function $f(z)$ is analytic for a finite value of z, and is bounded, then $f(z)$ is constant.

or

If f is regular in hole $z-$plane and if $|f(z)| < k\forall z$, then $f(z)$ must be constant.

Proof: Let a and b be two arbitrary distinct points in the complex plane. Consider a large circle C centered at the origin with radius R such that C encloses both a and b. The equation of C is $|z| = R$ so that $z = Re^{i\theta}, |dz| = Rd\theta$.
$F(z)$ is bounded $\forall z \Rightarrow |f(z)| < M\forall z$ where $M > 0$.
By Cauchy's integral formula for $f(a)$ and $f(b)$,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz, \quad f(b) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}dz$$

Simplify the integrand:

$$f(a) - f(b) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) f(z)\, dz$$

$$= \left(\frac{a-b}{2\pi i}\right)\int_C \frac{f(z)}{(z-a)(z-b)}dz$$

The modulus of the integral can be bounded as follows:

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi} \int_C \frac{|f(z)|}{(|z|-|a|)(|z|-|b|)} |dz|$$

The length of the contour C is 2πR:

$$\leq \frac{M|a-b|.\, 2\pi R}{2\pi(R-|a|)(R-|b|)}$$

Simplifying, we get:

$$|f(a) - f(b)| \leq \frac{M|a-b|.\, R}{(R-|a|)(R-|b|)} \to 0 \; as \; R \to \infty.$$

This implies that $f(a) - f(b) = 0 \; or \; f(a) = f(b)$, Since a and b were arbitrary points in the complex plane, this means that $f(z)$ is constant everywhere in C.

$$\left[\int_C |dz| = circumfrence \; of \; the \; circle \; C = 2\pi R\right]$$

Thus, we have proved that if an entire function $f(z)$ is bounded for all z, then $f(z)$ must be constant. This completes the proof of Liouville's Theorem.

Example: If $|f(\zeta)|$ has maximum $(r) on \, |\xi - a| = r < \rho$, then if $a_n = \frac{f^n(a)}{n!}$, show that $|a_n| \leq \frac{M(r)}{r^n}$ and from it deduce Liouville's theorem.

Solution: To prove $|a_n| \leq \frac{M}{r^n}$ (Refer the remark of Cauchy inequality)

Let $f(z)$ is analytic and bounded $\forall z$.

To prove that $f(z) = const.$

$$|f^n(a)| \leq \frac{Mn!}{R^n}$$

Where $|z - a| = R$, by (i) part

Taking $n = 1, |f'(a)| < \frac{M}{R}$. R is given to be large. Hence making $R \to \infty, |f'(a)| \leq 0$. But $|f'(a)| \geq 0$.

This $\Rightarrow$ $f'(a) = 0$. Also a is arbitrary.

Hence $f'(z) = 0 \forall z \; or \; \frac{df}{dz} = 0$, integrating, $f(z) = const.$

9.6 *TAYLOR'S THEOREM:-*

If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then at every point z inside C,

$$f(z) = \sum_{n=0}^{\infty} f^n\,(a) \frac{(z-a)^n}{n!} \quad or \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

where $a^n = \frac{f^n(a)}{n!}$

.

Proof: Let $f(t)$ be analytic within a circle C whose equation is $|t - a| = R$. Let z be any point within C s.t. $|z - a| = r < R$.

By Cauchy integral formula,

$$f(t) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dz$$

Notice that $t - z = (t-a) - (z-a)$. So,

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)-(z-a)}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1} dt$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\cdots+\left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1}\left(1-\frac{z-a}{t-a}\right)^{-1}\right]dt$$

Using the formula, $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}$,we have

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{1!} + \cdots + (z-a)^n\frac{f^n(a)}{n!} + U_{n+1} \quad \ldots(1)$$

Where $U_{n+1} = \frac{(z-a)^{1+n}}{2\pi i}\int_C \frac{f(t)dt}{(t-z)(t-a)^{n+1}}$

$\therefore$

$$|U_{n+1}| = \frac{|z-a|^{1+n}}{2\pi}\int_C \frac{|f(t)||dt|}{(|t-a|-|z-a|)|t-a|^{n+1}}$$

$$\leq \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}.\frac{1}{(R-r)}.2\pi R$$

where $M = max.|f(t)|$ on C.

Or

$$|U_{n+1}| \leq M.\left(\frac{r}{R}\right)^{n+1}.\frac{1}{1-\left(\frac{r}{R}\right)} \to 0 \; as \; n \to \infty.$$

For

$$\lim_{n\to\infty}\left(\frac{r}{R}\right)^{n+1} = 0 \; as \frac{r}{R} < 1.$$

$\therefore$

$$\lim_{n\to\infty} U_{n+1} = 0$$

$$f(z) = \lim_{n\to\infty}\left[f(a) + (z-a) + f'(a) + (z-a)^2\frac{f''(a)}{1!} + \cdots + (z-a)^n\frac{f^n(a)}{n!})\right]$$

Or

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n\frac{f^n(a)}{n!} = \sum_{n=0}^{\infty}a_n(z-a)^n \qquad \ldots(2)$$

Where $a_n = \frac{f^n(a)}{n!}$.
This is the Taylor series expansion of $f(z)$ around the point $z = a$.

Note:
The above theorem can also be related as:
Let $f(z)$ be analytic at all points within a circle C_0 with its centre at z_0 and radius R. Let z any point inside C_0. Then prove that

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^n(z_0)$$

Deduction:

i. Since z is a point within the circle $|t-a| = R$ such that $|z-a| = r < R$ so that we can take $z = a + h, h = z - a$. Substituting in (2), we obtain

$$f(a+h) = \sum_{n=1}^{\infty} \frac{h^n}{n!} f^n(z_0)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

This is alternative form to Taylor's Series.

ii. If we write $a = 0$ in (2), then we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z)^n \qquad\qquad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

This is known as *Maclaurin's Series*.

iii. The series converges for $|z-a| < R$, where RRR is the radius of convergence, defined as the distance from a to the nearest singularity of the function $f(z)$. The convergence of the series on the circle$|z-a| = R$, may or may not occur.

9.7 *LAURENT'S THEOREM:-*

Suppose a function $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R' $(R > R')$. If z is any point of annulus, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{(t-a)^{-n+1}}$$

Proof: Let $f(t)$ be analytic in the closed ring bounded by two concentric circles C and C' of the centre a and radii R and R' $(R > R')$. Then if z is any point within the ring space, then

$$R' < |z-a| = r < R.$$

Here we shall make the following facts:

i. $\frac{1}{1-b} = (1-b)^{-1} = 1 + b + b^2 + \cdots + b^n + \frac{b^{n+1}}{1-b}$

ii. $\left[1 - \frac{t-a}{z-a}\right]^{-1} = \frac{1}{1-\left[\frac{t-a}{z-a}\right]} = \frac{z-a}{z-t}$

iii. $\lim_{n\to\infty}\left(\frac{r}{R}\right)^n = 0 = \lim_{n\to\infty}\left(\frac{R'}{r}\right)^n \; as \frac{r}{R} < 1, \frac{R'}{r} < 1$

iv. $\int_C |dt| = 2\pi$. Radius of a circleC=circumference.

By extension to Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z} - \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{t-z}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)-(z-a)} - \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{(z-a)-(t-a)}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1 - \frac{z-a}{t-a}\right]^{-1} dt - \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left[1 - \frac{t-a}{z-a}\right]^{-1} dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \cdots + \left(\frac{z-a}{t-a}\right)^n + \left(\frac{z-a}{t-a}\right)^{n+1}\left[1 - \frac{z-a}{t-a}\right]^{-1}\right] dt$$
$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{t-a}\left[1 + \left(\frac{t-a}{z-a}\right) + \left(\frac{t-a}{z-a}\right)^2 + \cdots + \left(\frac{t-a}{z-a}\right)^n + \left(\frac{t-a}{z-a}\right)^{n+1}\left[1 - \frac{t-a}{z-a}\right]^{-1}\right]$$

Taking

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{(t-a)^{-n+1}} = -a_n$$

$$f(z) = [a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots + (z-a)^n a_n + U_{n+1}] + \left[\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^n} + V_{n+1}\right] \quad \ldots (1)$$

where

$$U_{n+1} = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z)}\left(\frac{z-a}{t-a}\right)^{n+1} dt$$

$$V_{n+1} = \frac{1}{2\pi i}\int_C \frac{f(t)}{(z-t)}\left(\frac{t-a}{z-a}\right)^{n+1} dt$$

Let $M = max.|f(t)|$ on C, $M' = max.|f(t)|$ on C'

$$|U_{n+1}| \le \frac{1}{2\pi}\int_C |f(t)| \left|\frac{z-a}{t-a}\right|^{n+1} \frac{|dt|}{(|t-a|-|z-a|)}$$
$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1} \frac{2\pi R}{(R-r)}$$

or

$$|U_{n+1}| \le M\left(\frac{r}{R}\right)^{n+1} . \frac{1}{1-\left(\frac{r}{R}\right)} \to 0 \quad as\, n \to \infty$$

Hence $\lim_{n\to\infty} U_{n+1} = 0$

$$|V_{n+1}| \le \frac{1}{2\pi}\int_{C'} |f(t)| \left|\frac{t-a}{z-a}\right|^{n+1} \frac{|dt|}{(|z-a|-|t-a|)}$$
$$\le \frac{M'}{2\pi}\left(\frac{R'}{r}\right)^{n+1} \frac{2\pi R'}{(r-R')}$$
$$\le M'\left(\frac{R'}{r}\right)^{n+1} . \frac{1}{\left(\frac{r}{R'}\right)-1} \to 0 \quad as\, n \to \infty$$

Hence $\lim_{n\to\infty} V_{n+1} = 0.$

Making $n \to \infty$ in (1) and noting the above facts,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-a)^n} \qquad \text{... (2)}$$

Deduction: Take C_0 a circle whose equation is

$$R' < |t-a| = R_0 < R.$$

Then

$$a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{-n+1}} = -a_n$$

From (2), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} (z-a)^{-n} a_{-n}$$
$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=-1}^{-\infty} (z-a)^n a_n = \sum_{n=-\infty}^{-\infty} (z-a)^n a_n$$

Or

$$f(z) = \textstyle\sum_{n=-\infty}^{-\infty} (z-a)^n a_n \text{ with } a_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(t)dt}{(t-a)^{n+1}}$$

9.8 *UNIQUENESS OF LAURENT EXPENSION:-*

Suppose the we have obtained in any manner or as the definition of $f(z)$, the formula

$$f(z) = \sum_{n=-\infty}^{-\infty} (z-a)^n A_n \, , R' < |z-a| < R$$

Is the series necessarily identical with the Laurent's series?

Proof: The given series

$$f(z) = \sum_{n=-\infty}^{-\infty} (z-a)^n A_n \, , R' < |z-a| < R \quad \dots (1)$$

To prove that is identical with Laurent's expansion. Laurent series is an expansion of a complex function that includes terms with both positive and negative powers of $(z-a)$. It has the general form:

$f(z) = \sum_{n=-\infty}^{-\infty} (z-a)^n a_n \;\dots (2)$ with $a_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(t)dt}{(t-a)^{n+1}}$

Now $A_n = a_n$, the equation to C_0 is $|t-a| = r, i.e., t-a = re^{i\theta}, \; R' < r < R.$

$$a_n = \frac{1}{2\pi i} \int_{C_0} \sum_{m=-\infty}^{-\infty} (z-a)^n A_m (t-a)^m \frac{f(t)dt}{(t-a)^{n+1}}$$

$$a_n = \frac{1}{2\pi i} \sum_{m=-\infty}^{-\infty} A_m \int_{C_0} (t-a)^{m-n-1} dt$$

$$= \frac{1}{2\pi i} \sum_{m=-\infty}^{-\infty} A_m \int_0^{2\pi} r^{m-n-1} e^{i(m-n-1)\theta} i r e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{-\infty} A_m r^{m-n} \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

If $\quad m \neq n, \int_0^{2\pi} e^{i(m-n)\theta} d\theta$

$$= \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = 0 \; as \; e^{2p\pi i=1}$$

If $\quad m = n, \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \int_0^{2\pi} e^0 d\theta = 2\pi$

$$a_n = \frac{1}{2\pi i} A_n . r^{n-n} . 2\pi = A_n$$

SOLVED EXAMPLE

EXAMPLE1: Obtain the Taylor's and Laurent's Series which represents the function $\frac{z^2-1}{(z+2)(z+3)}$ in the series.

(i) $|z| < 2$

(ii) $2 < |z| < 3$

(iii) $|z| > 3$

SOLUTION: Let the given series is

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) When $|z| < 2$, then $\frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$1 + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$-\frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_0^{\infty}(-1)^n \frac{z^n}{2^n} - \frac{8}{3}\sum_0^{\infty}(-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_0^{\infty}(-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n$$

(iv) When $2 < |z| < 3$, then $\frac{2}{|z|} < 1, \frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$-\frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{0}^{\infty}(-1)^n \frac{2^n}{z^n} - \frac{8}{3}\sum_{0}^{\infty}(-1)^n \frac{z^n}{3^n}$$
$$= 1 + \sum_{0}^{\infty}(-1)^n \left[\frac{32^n}{z^{n+1}} - \frac{8z^n}{3^{n+1}}\right]$$

This is Laurent's series in the annuals$2 < |z| < 3$.

(v) When $|z| > 3$, then $\frac{3}{|z|} < 1, \frac{2}{|z|} < \frac{2}{3} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$
$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\sum_{0}^{\infty}(-1)^n \frac{2^n}{z^n} - \frac{8}{3}\sum_{0}^{\infty}(-1)^n \frac{3^n}{z^n}$$
$$= 1 + \sum_{0}^{\infty}\frac{(-1)^n}{z^{n+1}}[3.2^n - 3^n.8]$$

EXAMPLE2: Find the Taylor's series expansion of the function $f(z) = \frac{z}{z^4+9}$ around $z = 0$. Find also radius of convergence.

SOLUTION: The given series is

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\left(1 + \frac{z^4}{9}\right)^{-1}$$
$$= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z^4}{9}\right)^n$$
$$\left[For\ (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots = \sum_{n=0}^{\infty}(-1)^n (x)^n\right]$$
$$f(z) = \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n \frac{z^{4n}}{3^{2n}} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{4n+1}}{3^{2n+2}}$$
$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{4n+1}}{3^{2n+2}} \quad ...(1)$$

This is Taylor expansion of $f(z)$.

PartII: Let

$$f(z) = \sum_{n=0}^{\infty} u_n(z) \qquad ...(2)$$

From (1) and (2), we get

$$u_n(z) = (-1)^n \frac{z^{4n+1}}{3^{2n+2}}$$

$$\left|\frac{u_{n+1}(z)}{u_n(z)}\right| = \left|\frac{z^{4n+5}}{3^{2n+4}} \cdot \frac{3^{2n+2}}{3^{4n+1}}\right| = \left|\frac{z^4}{3^2}\right| = \frac{|z^2|^2}{3^2}$$

Series is convergent if $\left|\frac{u_{n+1}}{u_n}\right| < 1$

If

$$\frac{|z^2|^2}{3^2} < 1 \; or \; |z| < 3^{2/4} \; or \quad |z| < \sqrt{3}$$

Radius of convergence$= \sqrt{3}$.

EXAMPLE3: Find the Laurent series of the function $f(z) = \frac{1}{(z^2-4)(z+1)}$ valid in the region. $1 < |Z| < 2$.

SOLUTION: The given series is

$$f(z) = \frac{1}{(z^2-4)(z+1)}$$

$$f(z) = \frac{1}{(z-2)(z+2)(z+1)}$$

$$f(z) = \frac{1}{12(z-2)} + \frac{1}{4(z+2)} - \frac{1}{3(z-2)}$$

$$f(z) = \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1} - \frac{1}{3}\left(1+\frac{1}{z}\right)^{-1}$$

$$f(z) = \frac{1}{24}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{1}{3z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}.$$

EXAMPLE4: Expand $sinz$ in a Taylor's series about $z = \frac{\pi}{4}$.

SOLUTION: Let the series

$$f(z) = \sum_{n=0}^{\infty} a_n\left(z-\frac{\pi}{4}\right)^n \quad ...(1)$$

where
$$a_n = \frac{f^n\left(\frac{\pi}{4}\right)}{n!} \quad ...(2)$$

$$f(z) = sinz$$

$$f^n(z) = sin\left(z+\frac{n\pi}{2}\right)$$

$$f^n\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)$$

Now substituting value in (1), we get

$$f(z) = \sum_{n=0}^{\infty} sin\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)\frac{\left(z-\frac{\pi}{4}\right)^n}{n!}$$

EXAMPLE5: (a) Expand $f(z) = \frac{1}{z}$ as a Taylor's series about $z = 1$.

(b) Determine Laurent's expansion of the function $f(z) = \frac{sinz}{\left(z-\frac{\pi}{4}\right)^3}$ in the annulus $0 < \left|z - \frac{\pi}{4}\right| < 1$.

SOLUTION: (a) $f(z) = \frac{1}{z}$

$$f^n(z) = \frac{(-1)^n n!}{z^{n+1}}$$
$$f^n(1) = \frac{(-1)^n n!}{1^{n+1}}$$

If $a_n = \frac{(-1)^n n!}{n!}$, then $a_n = (-1)^n$

For Taylor's expansion about $z = 1$,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-1)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

(b) Let the given series is

$$f(z) = \frac{sinz}{\left(z-\frac{\pi}{4}\right)^3}$$

For Taylor's expansion about $z = \frac{\pi}{4}$, we write

$$f(z) = \sum_{n=0}^{\infty} a_n\left(z-\frac{\pi}{4}\right)^n + \sum_{n=1}^{\infty} \frac{b_n}{\left(z-\frac{\pi}{4}\right)^n}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{\left(z-\frac{\pi}{4}\right)^{n+1}}$$

And

$$b_n = a_{(-n)}.C\ is\ \left|z-\frac{\pi}{4}\right| = 1$$
$$z - \frac{\pi}{4} = e^{i\theta}, dz = ie^{i\theta}d\theta$$
$$f(z) = \frac{sin\left(\frac{\pi}{4}+e^{i\theta}\right)}{e^{i3\theta}}\ by\ (1)$$

Substituting this value in (3), we have

$$a_n = \frac{1}{2\pi i}\int_C \frac{sin\left(\frac{\pi}{4}+cos\theta+isin\theta\right)ie^{i\theta}d\theta}{e^{i3\theta}.(e^{i\theta})^{n+1}}$$
$$= \frac{1}{2\pi}\int_C sin\left(\frac{\pi}{4}+cos\theta+isin\theta\right)ie^{-i\theta(n+3)}d\theta$$
$$= \frac{1}{2\pi}\int_C \{sin\phi.\cosh(sin\theta)+icos\phi.sinh(sin\theta)\}\{cosm\theta - isinm\theta\}d\theta$$

Where

$$m = n + 3, \phi = \frac{\pi}{4} + cos\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} [sin\phi . cosh(sin\theta)\cos(m\theta) + icos\phi . \sinh(sin\theta)\sin(m\theta)]d\theta$$

where $\phi = \frac{\pi}{4} + cos\theta, m = n + 3, b_n = a_{(-n)}$

EXAMPLE6: If $0 < |z - 1| < 2$, then express

$$f(z) = \frac{z}{(z-1)(z-3)}$$

in a series of positive and negative powers of $(z - 1)$.

SOLUTION: Suppose$u = z - 1$, then

$0 < |u| < 2$, so that $\frac{|u|}{2} < 1$

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$A = \left[\frac{z}{z-3}\right]_{z=1} = \frac{1}{1-3} = -\frac{1}{2}, B = \left[\frac{z}{z-1}\right]_{z=3} = \frac{3}{3-1} = \frac{3}{2}$$

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)} = \frac{3}{2(u-2)} - \frac{1}{2u}$$

$$= \frac{3}{4}\left(1 - \frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

$$= -\frac{3}{4}\sum_{n=0}^{\infty}(u/2)^n - \frac{1}{2u} = -\frac{3}{4}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} - \frac{1}{2(z-1)}$$

EXAMPLE7: Prove that $log\, z = (z - 1) - \frac{(z-1)^2}{2!} + \cdots, |z - 1| < 1.$

SOLUTION: Let $f(z) = logz$. By Taylor's Theorem

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{f^n(a)}{n!}$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

Taking $a = 1$, we get $f(1) = log1 = 0$

$$f'(z) = \frac{1}{z}, f''(z) = -\frac{1}{z^2}$$

$$f(1) = 0, f'(1) = 1, f''(1) = -1$$

$$f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \cdots$$

$$= 0 + (z-1) + \frac{(z-1)^2}{2!} + \cdots$$

EXAMPLE8: Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for:

a. $|z| > 3$
b. $0 < |z + 1| < 2.$

SOLUTION: $f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z+3}\right)$

a. when $|z|>3 \Rightarrow \frac{|z|}{3}>1 \Rightarrow \frac{3}{|z|}<1 \Rightarrow \frac{1}{|z|}<\frac{1}{3}<1$

$$f(z)=\frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1}-\left(1+\frac{3}{z}\right)^{-1}\right]=\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{z^n}-\frac{3^n}{z^n}\right]$$

$=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{z^{n+1}}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{z^n}.$

b. When $0<|z+1|<2$, substitute $z+1=t$, then

$$f(z)=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{t+2}\right],\quad |t|<1 \Rightarrow \left|\frac{t}{2}\right|<\frac{1}{2}<1$$

$$=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{2}\left(\frac{1}{t+2}\right)^{-1}\right]$$

$$=\frac{1}{2}\left[\frac{1}{t}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{t}\right)^2\right]$$

$$=\frac{1}{2}\left[\frac{1}{z+1}-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z+1)^n}{2^n}\right]$$

SELF CHECK QUESTIONS

1. How can Cauchy's inequalities be used to bound the n-th derivative of an analytic function within a given radius?
2. State Morera's theorem.
3. Explain how Morera's theorem is used to show that a function is analytic.
4. Question: State Liouville's theorem.
5. Question: How does Liouville's theorem relate to the fundamental theorem of algebra?
6. How is the radius of convergence of a Taylor series determined?

9.9 *SUMMARY:-*

In this unit we have studied the Cauchy's inequalities provide upper bounds for the derivatives of an analytic function within a given radius. Specifically, if a function $f(z)$ is analytic within a disk $|z|\leq R$ and the absolute value of the function is bounded by M on the boundary $|z|=R$, then the absolute value of the $n-$th derivative of f at the center is bounded by $\frac{n!M}{R^n}$. These inequalities are crucial for estimating the size of derivatives and understanding the growth behavior of analytic functions. Morera's theorem states that if a continuous function f on a domain D has an integral equal to zero around every closed contour within D, then f is analytic in D. This theorem is significant because it provides a criterion for analyticity based on the behavior of line integrals, allowing one to prove that a function is analytic by verifying that the integral of the function around all closed paths in the domain is zero. Morera's theorem is often used to demonstrate the analyticity of functions when direct verification of the Cauchy-Riemann equations is difficult. Liouville's theorem asserts that any entire function (analytic over the entire complex plane) that is bounded must be constant, which has implications like proving the fundamental theorem of algebra. Taylor's theorem states that if a function f is analytic at a point a, it can be expanded in a Taylor series $f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} A_n(z-a)^n$, with the radius of convergence determined by the distance to the nearest singularity. Laurent's theorem generalizes this to functions analytic in an annulus, expressing them as a Laurent series $f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$, including both positive and negative powers, useful for representing functions with singularities. These theorems are fundamental in understanding the behavior and properties of analytic functions in complex analysis.

9.10 *GLOSSARY:-*

- **Cauchy's Inequality (General Form):** For a function f analytic inside and on a simple closed contour C in the complex plane, if M is the maximum value of |f(z)| on C, then for any z inside C, we have
  - $|f^n(a)| \leq \frac{Mn!}{R^n}$
- where R is the radius of the largest circle centered at z that is contained within C.
- **Cauchy's Integral Formula:** If f is analytic inside and on a simple closed contour C, then for any point a inside C,
- $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$
- **Cauchy's Theorem:** If f is analytic on and inside a simple closed contour C, then
- $\int_C f(z) dz = 0.$
- **Cauchy's Integral Theorem:** An extension of Cauchy's Theorem stating that if f is analytic on a simply connected domain, then the integral of f around any closed contour within this domain is zero.
- **Cauchy's Formula for Derivatives:** For a function f analytic inside and on a simple closed contour C, the n-th derivative of f at a point z0 inside C is given by
- $f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}$
- **Morera's Theorem:** A theorem in complex analysis that provides a criterion for a function to be analytic based on its integral over contours. It states:

- If f is continuous on a region D and if the integral of f over every closed contour within D is zero, then f is analytic on D.

- **Closed Contour:** A path in the complex plane that starts and ends at the same point. It does not intersect itself and can be simple (non-intersecting) or more complex.
- **Contour Integral:** The integral of a complex-valued function along a contour in the complex plane. If f is integrated along a path γ, it is written as $\int_\gamma f(z)dz$.
- **Region D:** A subset of the complex plane where a function f is defined. For Morera's Theorem, D must be an open region (meaning every point in D has a neighborhood contained in D).
- **Zero Integral Condition:** In the context of Morera's Theorem, this condition means that the integral of f around every closed contour within the region D must be zero for f to be analytic in D.
- **Zero Integral Condition:** In the context of Morera's Theorem, this condition means that the integral of f around every closed contour within the region D must be zero for f to be analytic in D.
- **Holomorphic Function:** Another term for an analytic function, emphasizing that it is complex differentiable. In many contexts, the terms "analytic" and "holomorphic" are used interchangeably.
- **Taylor's Theorem:** If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then at every point z inside C,

$$f(z) = \sum_{n=0}^{\infty} f^n(a)\frac{(z-a)^n}{n!} \quad or \quad f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n,$$

where $a^n = \frac{f^n(a)}{n!}$
- **Convergence of the Taylor Series:** A Taylor series converges to the function f if, as n approaches infinity, the remainder term $R_n(x)$ approaches zero. If the Taylor series converges to f for all x in a neighborhood of a, f is said to be analytic at a.
- **Analytic Function:** A function is analytic at a point if its Taylor series converges to the function in some neighborhood around that point. This means the function can be represented exactly by its Taylor series in that region.
- **Radius of Convergence:** The radius R of the largest disk centered at a within which the Taylor series converges to f. Outside this radius, the series may diverge.
- **Higher-Order Derivatives:** Derivatives of f of order greater than one, which are used in constructing higher-degree Taylor polynomials. For instance, $f^2(a)$ is the second derivative evaluated at a.

- **Polynomial Approximation:** The process of approximating a function by a polynomial of degree n derived from the Taylor series expansion, where the polynomial matches the function and its derivatives up to order n at a specific point.

9.11 *REFERENCES:-*

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- Ravi P. Agarwal, Kanishka Perera, and Sandra Pinelas (2011), An Introduction to Complex Analysis.
- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.

9.12 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- https://old.mu.ac.in/wp-content/uploads/2020/12/Paper-III-Complex-Analysis.pdf
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9.13 *TERMINAL QUESTIONS:-*

(TQ-1) State and prove Liouvlille's theorem.

(TQ-2) If $f(z)$ is a continuous function in a domain D and if for every closed Contour C in the domain D, then prove that

$$\int_C f(z)\,dz = 0$$

where $f(z)$ is analytic within D.

(TQ-3) If $f(z)$ is analytic within and on a circle C, given by $|z-a| = R$ and if $|f(z)| \leq M$ for every z on C, then prove that the magnitude of the n −th derivative of f at a is bounded by

$$|f^n(a)| \leq \frac{Mn!}{R^n}$$

(TQ-4) If $f(z)$ is an entire function is bounded for all values of z, then it is constant.

or

If a function $f(z)$ is analytic for a finite value of z, and is bounded, then $f(z)$ is constant.

or

If f is regular in hole $z-$plane and if $|f(z)| < k \forall z$, then $f(z)$ must be constant.

(TQ-5) If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then prove that at every point z inside C,

$$f(z) = \sum_{n=0}^{\infty} f^n(a) \frac{(z-a)^n}{n!} \quad or \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

where $a^n = \frac{f^n(a)}{n!}$

(TQ-6) If a function $f(z)$ is analytic for all values of z and as $|z| \to \infty$, $|f(z)| = A|z|^k$, then prove that f(z) is polynomial of degree $\leq k$.

(TQ-7) Find the Laurent series of the function $f(z) = \frac{1}{z^2(1-z)}$ about $z = 0$.

(TQ-8) If $f(z) = \sum_0^\infty a_n z^n \ (|z| < R)$ and $M(r)$ is the upper bound of $|f(z)|$ on the circle $|z| = r, (r < R)$, then prove that $|a_n| r^n \leq M(r) \forall n$.

(TQ-9) If the function f(z) is analytic and single valued in $|z-a| < R$ prove that when $0 < r < R$.

$f'(a) = \frac{1}{\pi r} \int_0^{2\pi} P(\theta)\, e^{-i\theta} d\theta$ where $P(\theta)$ is real part of $f\left(a + re^{i\theta}\right)$. Also prove that $\frac{f^n(a)}{n!} = \frac{1}{\pi r^n} \int_0^{2\pi} P(\theta)\, e^{-i\theta} d\theta$.

(TQ-10) Prove that $log\, z = (z-1) - \frac{(z-1)^2}{2!} + \cdots, |z-1| < 1$.

9.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. Cauchy's inequalities provide an upper bound for the $n-$th derivative of an analytic function at the origin in terms of the maximum value M of the function on the circle of radius R.
2. If $f(z)$ is a continuous function in a domain D and if for every closed Contour C in the domain D,

$$\int_C f(z)\, dz = 0$$

3. Morera's theorem can be used to prove that a function is analytic by showing that the integral of the function around any closed

contour in the domain is zero, indicating the existence of an ant derivative and hence analyticity.

4. If f is entire (analytic on the entire complex plane) and bounded, then f is constant.
5. Liouville's theorem can be used to prove the fundamental theorem of algebra by showing that a non-constant polynomial cannot be bounded, implying it must have a root.
6. The radius of convergence R is determined by the distance to the nearest singularity of the function from the expansion point a.

UNIT 10:- Maximum and Minimum Modulus Principle and Schwarz Lemma

CONTENTS:

10.1 *INTRODUCTION:-*

The Maximum and Minimum Modulus Principles and the Schwarz Lemma are fundamental results in complex analysis. The Maximum Modulus Principle states that if a function is holomorphic on a domain, the maximum of the modulus of the function on a closed bounded subset of that domain occurs on the boundary. Conversely, the Minimum Modulus Principle asserts that if a function is non-constant and holomorphic, the minimum modulus cannot occur in the interior unless the function is constant. The Schwarz Lemma provides a powerful result for holomorphic functions that map the unit disk to itself: if a function fixes the origin, then the magnitude of the function is at most as large as the magnitude of the input, and the magnitude of the derivative at the origin is at most one, with equality only if the function is a rotation. These principles are essential tools in the study of holomorphic functions and their behavior.

10.2 *OBJECTIVES:-*

The objectives of the Maximum and Minimum Modulus Principles and the Schwarz Lemma in complex analysis are to understand and characterize the behavior of holomorphic functions. The Maximum Modulus Principle aims to identify where the maximum modulus of a holomorphic function occurs, emphasizing that it cannot happen in the interior of the domain unless the function is constant. The Minimum Modulus Principle similarly helps in determining where the minimum modulus occurs, ensuring that a non-constant holomorphic function cannot achieve its minimum modulus in the interior. The Schwarz Lemma, on the other hand, provides constraints on holomorphic functions mapping the unit disk to itself, particularly those fixing the origin. It sets bounds on the function's magnitude and the magnitude of its derivative at the origin, offering insight into the function's growth and behavior. Collectively, these results help in analyzing the constraints and characteristics of holomorphic functions in various domains.

10.3 *MAXIMUM MODULUS PRINCIPLE:-*

Suppose $f(z)$ is analytic within and a simple closed Contour C and $f(z)$ is not constnt. Then $|f(z)|$ reaches its maximum value on C (and not inside C), that is to say, if M is maximum value of $|f(z)|$ on and within $|f(z)| < M$ for every z inside C.

Proof: We prove this theorem by the method of contradiction. Analyticity of f(z) declares that f(z) is continuous within and on C. Consequently we assume f(z) is analytic within and on C and is not constant. Suppose that |f(z)| attains its maximum value M at some point a inside C. Therefore,

$$max. |f(z)| = |f(a)| = M \qquad \dots (1)$$

and

$$|f(z)| \leq M \forall z\; within C \qquad \dots (2)$$

Since $f(z)$ is not constant, by the continuity of $f(z)$, there must exist a point b inside C such that $|\ f(b)\ | < M$.

Let $\varepsilon > 0$ be s.t. $|f(b)| = M - \varepsilon$

By the continuity of $|f(z)|$ is continuous at $z = b$ and so $|[|f(z)| - |f(b)|]| < \varepsilon/2$ whererever $|z - b| < \delta$.

Since $|[|f(z)| - |f(b)|]| \geq |f(z)| - |f(b)|$

$$|f(z)| - |f(b)| \leq |[|f(z)| - |f(b)|]| < \varepsilon/2$$

$$|f(z)| - |f(b)| < \frac{\varepsilon}{2}$$

$$|f(z)| < |f(b)| + \frac{\varepsilon}{2}$$

This implies that within the circle γ centered at b with radius δ,

$$= M - \varepsilon + \frac{\varepsilon}{2} = M - \frac{\varepsilon}{2}$$

$$|f(z)| < M - \frac{\varepsilon}{2} \forall s.t. |z - b| < \delta$$

we draw a circle γ withcentre at b and radius δ. Then (3) shows that

$$|f(z)| < M - \frac{\varepsilon}{2} \forall z \ inside \ \gamma$$

Consider a circle Γ'centred at a with radius $r =| b - a |$. By Cauchy's Integral Formula, for f analytic on and inside Γ'.

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma'} \frac{f(z)}{z - a} dz$$

On $\Gamma', z - a = re^{i\theta}$.

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f\left(a + re^{i\theta}\right)\frac{rie^{i\theta}}{re^{i\theta}} d\theta$$

If we measure θ in anti-clock-wise direction and if

$$< QPR = \alpha, then$$

$$f(a) = \frac{1}{2\pi}[\int_0^{\alpha} + \int_{\alpha}^{2\pi}] f\left(a + re^{i\theta}\right) d\theta$$

$$\therefore \qquad |f(z)| \le \tfrac{1}{2\pi}\int_0^{\alpha}\left|f\left(a + re^{i\theta}\right)\right| d\theta + \tfrac{1}{2\pi}\int_{\alpha}^{2\pi}\left|f\left(a + re^{i\theta}\right)\right| d\theta$$

$$< \frac{1}{2\pi}\int_0^{\alpha}\left(M - \frac{\varepsilon}{2}\right) d\theta + \frac{1}{2\pi}\int_0^{\alpha}(M) d\theta \ \ from \ above \ equation$$

$$= \left(M - \frac{\varepsilon}{2}\right)\frac{\alpha}{2\pi} + \frac{M(2\pi - \alpha)}{2\pi} = M - \frac{\alpha\varepsilon}{2\pi}$$

Then $\qquad M = |f(a)| < M - \frac{\alpha\varepsilon}{2\pi}$

$M < M - \frac{\alpha\varepsilon}{2\pi}$. A contraction.

which is a contradiction because $\frac{\alpha\varepsilon}{2\pi} > 0$. Thus, M cannot be the maximum value inside C.

Therefore, the maximum value of $|f(z)|$ must occur on the boundary C. This completes the proof of the Maximum Modulus Principle.

10.4 *MINIMUM MODULUS PRINCIPLE:-*

Suppose $f(z)$ is analytic within and on a closed contour C and let $f(z) \neq 0$ inside C.

Suppose further that f(z) is not constant. Then $|f(z)|$ attains its minimum value at a point on the boundary of C, that is to say, if m is minimum value of $|f(z)|$ inside and on C, then

$$|f(z)| > m \ \forall z \ inside \ C$$

Proof: Let $f(z)$ be analytic within and on the closed contour C, and let $f(z) \neq 0$ inside$C$. Since $f(z)$ is analytic and non-zero, the function $\frac{1}{f(z)}$ is also analytic within$C$.By the Maximum Modulus Principle, since 1/f(z) is analytic within C and continuous on C, the function $\left|\frac{1}{f(z)}\right|$ attains its maximum value on the boundary of C. Let this maximum value be M.

Consequently, since $\left|\frac{1}{f(z)}\right|$ reaches its maximum value C, it follows that $|f(z)|$ attains its minimum value $\frac{1}{M}$.Thus, if m is the minimum value of |f(z)| inside and on C, then $|f(z)| \geq m$ on the boundary of C.

Since $\frac{1}{f(z)}$ attains its maximum value on the boundary of C, $|f(z)|$ must attain its minimum value on the boundary of C. Therefore, $|f(z)|$ is greater than m for every z inside C, which implies $|f(z)| > m$ for all z inside C.

This completes the proof of the Minimum Modulus Principle.

10.5 *SCHWARZ LEMMA:-*

Suppose

i. F(z) analytic in a domain defined by $|z| < 1$
ii. $|f(z)| < 1$

iii. $|f(0)| = 0$

Then $|f(z)| < |z|$, $|f'(z)| < 1$ equality holds if f(z) is a linear transformation

$$w = f(z) = ze^{i\alpha}$$

where α is real constant.

Hence prove that $|f'(0)| < 1$

Proof: Let c be the circle defined by $|z| = r$ such that $r < 1$. By assumption (i), $f(z)$ is analytic within and upon the circle c, then by Taylor theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

For any point z within c

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \qquad \ldots(1)$$

Using the assumption (iii), $i.e., |f(0)| = 0$, we get $a_0 = 0$. Now (1) reduces to

Given $f(0) = 0$, we have $a_0 = 0$. Thus,

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

Taking $\frac{f(z)}{z} = F(z)$, we obtain

$F(z) = a_1$ for $z = 0$, $\quad i.e., \ F(0) = a_1$

Let $z = a$ be any point on c so that $a = re^{i\alpha}$, where α is real.

Now $F(z)$ is analytic within and upon c and so by Maximum modulus theorem, $|F(z)|$ attains its maximum value on c, say at $z = a$ and not within c. Then

$$|F(a)| = max|F(z)|$$
$$|F(z)| < max|F(z)|$$

$$= \max value\ of \ \left|\frac{f(z)}{z}\right| < \frac{1}{|z|} = \frac{1}{r}, by\ (ii)$$

$$|F(z)| < \frac{1}{r}\ as\ r \to 1$$

$$|F(z)| < 1$$

$$\left|\frac{f(z)}{z}\right| < 1$$

$$|f(z)| < |z|$$

Substituting $z = 0$ in above equation

$|F(0)| < 1$. But $F(0) = \alpha_1$

Hence

$$|\alpha_1| < 1$$

$\Rightarrow f'(z) = a_1 z + a_2 z^2 + \cdots \Rightarrow f'(0) = a_1$

$\Rightarrow |f'(0)| = |a_1| < 1 \Rightarrow |f'(0)| < 1.$

This completes the proof.

In general, since a is arbitrary, we have

$$|f(z)| \leq |z| \text{ for all } z \text{ for which } |z| \leq 1.$$

If $|f(z)| = |z|$ at some point z with $|z| < 1$, then $|F(z)|$ attains its maximum value on the interior of the domain. By the Maximum Modulus Principle, $F(z)$ must be constant. Therefore, there exists a constant $\alpha \in R$ such that

$$F(z) = \frac{f(z)}{z} = e^{i\alpha}$$

Thus, $f(z) \leq |z|$ and $|f'(0)| \leq 1$, with equality holding if and only if $f(z) = ze^{i\alpha}$ for some real constant α. This completes the proof of the Schwarz Lemma.

10.6 *OPEN MAPPING THEOREM:-*

The Open Mapping Theorem is a fundamental result in complex analysis that asserts that a non-constant analytic (holomorphic) function maps open sets to open sets.

Theorem: If f is a non-constant analytic function on an open set $U \subset C$, then $f(U)$ is an open set in C.

Proof: Let f be a non-constant analytic function on an open set U⊂C. Assume f does not map U to an open set. We will reach contradiction. Suppose $f(U)$ is not open. This means there exists a point $w0 \in f(U)$ such that there is no neighborhood around w_0 entirely contained within $f(U)$.

Since $w_0 \in f(U)$, there exists $z_0 \in U$ such that $f(z_0) = w_0$.Because f is analytic and non-constant, f can be expressed locally around z_0 by its Taylor series expansion:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots$$

The Maximum Modulus Principle states that if f is non-constant and analytic in an open set, then the maximum of $| f(z) |$ cannot occur in the interior of the set unless f is constant.

If f were to fail to map U to an open set, f would be unable to achieve local maximum or minimum values on the boundary of small neighborhoods around z_0, contradicting the principle.

Because f is non-constant, the image of any sufficiently small open neighborhood around z_0 will contain an open disk centered at w_0. This implies that $w_0$cannot be an isolated point in $f(U)$ and that every point in $f(U)$ has a neighborhood entirely contained within $f(U)$.

Thus, $f(U)$ must be open, proving the theorem.

SOLVED EXAMPLE

EXAMPLE1: Let $f(z) = z^2 + 3z + 2$. Show that the maximum value of $| f(z) |$ in the disk $| z | \leq 1$ is attained on the boundary $| z | = 1$.

SOLUTION: The given equation is

$$f(z) = z^2 + 3z + 2$$

and the domain $| z | \leq 1$.

The function $f(z)$ is a polynomial and hence analytic everywhere, including within and on the disk $| z | \leq 1$.

$$| f(z) | \; on \;\; | z | = 1$$

Let $z = e^{i\theta}$, where θ range $0 \; to \; 2\pi$.

$$f(e^{i\theta}) = e^{2i\theta} + 3e^{i\theta} + 2$$

The modulus of this expression is:

$$\left| f(e^{i\theta}) \right| = \left| e^{2i\theta} + 3e^{i\theta} + 2 \right|$$

Since $f(z)$ is analytic within and on $|z| \leq 1$, the Maximum Modulus Principle states that the maximum value of $|f(z)|$ is attained on the boundary $|z| = 1$.

To compare, we evaluate specific points on the boundary:

$$|f(1)| = |(1)^2 + 3.1 + 2| = |6| = 6$$

$$|f(-1)| = |(-1)^2 + 3.(-1) + 2| = |1 - 3 + 2| = |0| = 0$$

We can see that the modulus of $f(z)$ is highest at $z = 1$ on the boundary.

By the Maximum Modulus Principle, the maximum value of $|f(z)|$ in the disk $|z| \leq 1$ is indeed attained on the boundary $|z| = 1$, and the maximum value is 6.

EXAMPLE2: Let $f(z)$ be an analytic function on the unit disk $|z| < 1$, with $f(0) = 0$ and $|f(z)| \leq 1/2$ for all $|z| < 1$. Prove that $|f(z)| \leq 1/2\,|z|$ and $|f'(0)| \leq 1/2$.

SOLUTION: Let the $f(z)$ be an analytic function on the unit disk $|z| < 1, f(0) = 0, |f(z)| \leq \frac{1}{2} \forall |z| < 1$.

Now we apply the Schwarz Lemma, which states that if $g(z)$ is analytic on $|z| < 1, g(0) = 0$, and $|g(z)| < 1$ for all $|z| < 1$, then $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$.

Define $g(z) = f(z)/1/2 = 2f(z)$. Then $g(z)$ satisfies the conditions of the Schwarz Lemma: g is analytic on $|z| < 1, g(0) = 2f(0) = 0, |g(z)| \leq 1$ for all $|z| < 1$.

By the Schwarz Lemma: $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$

Since $g(z) = 2f(z)$

$$|2f(z)| \leq |z| \quad \Rightarrow \quad |f(z)| \leq \frac{1}{2}|z|$$

and for the derivative:

$$|g'(0)| = |2f'(0)| \leq 1 \quad \Rightarrow |f'(0)| \leq \frac{1}{2}$$

EXAMPLE3: Let f be analytic and bounded by 1 in the unit disc and $f\left(\frac{1}{2}\right) = 0$ Estimate $\left|f\left(\frac{3}{4}\right)\right|$.

SOLUTION: Given: $f\left(\frac{1}{2}\right) = 0$, then $|f(z)| \leq 1$ for $|z| < 1$.
We want to estimate $\left|f\left(\frac{3}{4}\right)\right|$.
Using the Schwarz-Pick Lemma:

$$\frac{|f(z_1) - f(z_2)|}{\left|1 - \overline{f(z_2)}f(z_1)\right|} \leq \frac{|z_1 - z_2|}{|1 - \bar{z}_2 z_1|}$$

For any $z_1, z_2 \epsilon D$

$$\frac{|f(3/4) - f(1/2)|}{\left|1 - \overline{f(1/2)}f(3/4)\right|} \leq \frac{|3/4 - 1/4|}{\left|1 - \overline{f(1/2)}f(3/4)\right|}$$

Since $f(1/2) = 0$, this simplifies to:

$$|f(3/4)| \leq \frac{\left|\frac{3}{4} - \frac{1}{2}\right|}{\left|1 - \overline{f\left(\frac{1}{2}\right)}\, f\left(\frac{3}{4}\right)\right|}$$

Calculate the numerator:

$$\left|\frac{3}{4} - \frac{1}{2}\right| = \left|\frac{1}{4}\right| = \frac{1}{4}$$

Calculate the denominator:

$$\left|1 - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\right| = \left|1 - \frac{3}{8}\right| = \left|\frac{5}{8}\right| = \frac{5}{8}$$

Thus:

$$f\left(\frac{3}{4}\right) \leq \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{1}{4}.\frac{8}{5} = \frac{2}{5} = 0.4$$

Therefore, the estimate for $|\ f(3/4)\ |$ is:

$$|f(3/4)| \leq 0.4$$

This shows that under the given conditions, $|\ f(3/4)\ |$ can be estimated to be less than or equal to 0.4.

EXAMPLE4: Let $f(z) = z^2 + 1$ be a holomorphic (analytic) function on the closed disk $|\, z \,| \leq 1$. Determine the maximum value of $|\, f(z) \,|$ on the disk $|\, z \,| \leq 1$.

SOLUTION: The Maximum Modulus Principle states that if $f(z)$ is a non-constant holomorphic function on a domain D, then the maximum of $|\, f(z) \,|$ occurs on the boundary of D.

Given $f(z) = z^2 + 1$, we need to evaluate $|\, f(z)|$ on the boundary $|\, z \,| = 1$:

On the boundary, $|\, z \,| = 1$, so $z = e^{i\theta}$ for $\theta \in [0, 2\pi)$.

$$f(z) = z^2 + 1 = e^{2i\theta} + 1$$

The modulus is

$$|f(z)| = \left|e^{2i\theta} + 1\right|$$

Using the identity $|\, a + b \,| \leq |\, a \,| + |\, b \,|$, we get:

$$\left|e^{2i\theta} + 1\right| \leq \left|e^{2i\theta}\right| + |1| = 1 + 1 = 2$$

The maximum occurs when $e^{2i\theta} = -1$, which gives

$$|f(z)| = |-1 + 1| = 2$$

The maximum value of $|\, f(z) \,|$ on the disk $|\, z \,| \leq 1$ | is 2, and it occurs on the boundary $|\, z \,| = 1$ when $z = -1$.

EXAMPLE5: Let $f(z) = z^2 - 2z + 2$ be a holomorphic function on the closed disk $|\, z \,| \leq 2$. Determine the minimum value of $|\, f(z) \,|$ on the disk $|\, z \,| \leq 2$.

SOLUTION: The Minimum Modulus Principle states that if $f(z)$ is a non-constant holomorphic function on a domain D, the minimum value of $|\, f(z) \,|$ in the interior of D occurs at a point where $f(z) = 0$ or on the boundary.

Now to solve

$$f(z) = z^2 - 2z + 2 = 0$$

$$\Rightarrow \qquad z = \frac{2 \pm \sqrt{4-8}}{2} = 2 \pm i$$

The roots $1 + i$ and $1 - i$ are inside the disk $| z |\leq 2|z|$.

On the boundary,$| z |= 2$:

$$|f(z)| = |z^2 - 2z + 2|$$

We need to evaluate $| f(z) |$ for $| z |= 2$. Consider $z = 2e^{i\theta}$, and find:

$$|f(z)| = \left|4e^{2i\theta} - 4e^{i\theta} + 2\right|$$

The exact calculation is complex, but we only need to know that $| f(z) |$ on the boundary is larger than the minimum at the root points.

At $z = 1 + i, f(1 + i) = 0$, so $| f(1 + i) |= 0$.

Similarly, at $z = 1 - i, f(1 - i) = 0$, so $| f(1 - i) |= 0$.

The minimum value of $| f(z) |$ is 0, which occurs at the points $z = 1 + i$ and $z = 1 - i$.

EXAMPLE6: Let $f(z)$ be an analytic function on the unit disk $| z |< 1$ such that $f(0) = 0$ and $| f(z) |\leq| z |$ for all $| z |< 1$. Show that $f(z) = cz$ for some constant $| c |\leq 1$.

SOLUTION: Schwarz's Lemma states that if f is analytic on the unit disk $| z |< 1, f(0) = 0$, and $| f(z) |\leq 1$ for all $| z |< 1$, then:

1. $| f(z) |\leq| z |$ for all $| z |< 1$.

2. If equality holds for some $z \neq 0$, then $f(z) = cz$ for some constant c with $| c |\leq 1$.

Given $| f(z) |\leq| z |$, we already satisfy the condition.

Now consider $f(z)$ under these conditions. If $| f(z) |=| z |$ for some nonzero z, Schwarz's Lemma implies $f(z) = cz$.

Since $f(0) = 0$ and $| f(z) |\leq| z |$, it follows by Schwarz's Lemma that $f(z) = cz$ for some c with $| c |\leq 1$.

Therefore, $f(z) = cz$ where $| c |\leq 1$, satisfying all conditions of Schwarz's Lemma.

<u>SELF CHECK QUESTIONS</u>

1. What is the Maximum Modulus Principle, and how does it relate to the boundary of a domain?
2. Can the Maximum Modulus Principle be applied to non-analytic functions? Why or why not?
3. How would you prove the Maximum Modulus Principle using contradiction?
4. Give an example of a function where the Maximum Modulus Principle is applied. What are the implications for the maximum value of the modulus?
5. Why does the Maximum Modulus Principle imply that a non-constant analytic function cannot achieve its maximum modulus in the interior of a domain?
6. How does the Minimum Modulus Principle follow from the Maximum Modulus Principle?
7. State the Schwarz Lemma and explain its significance in complex analysis.
8. Under what conditions does the Schwarz Lemma hold?
9. How can the Schwarz Lemma be generalized, and what are the implications of such generalizations?

10.7 *SUMMARY:-*

The Maximum Modulus Principle and Minimum Modulus Principle are fundamental results in complex analysis that state if $f(z)$ is a non-constant analytic function on a domain D and continuous on the closure of D, then the maximum (or minimum) modulus of $f(z)$ occurs on the boundary of D rather than in its interior. The Schwarz Lemma further refines this idea, stating that if a function $f(z)$ is analytic in the unit disk $|z| < 1$, satisfies $f(0) = 0$, and $|f(z)| < 1$ for all z in the disk, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, with equality holding if and only if $f(z)$ is of the form $f(z) = ze^{i\alpha}$ for some real constant α. These principles are crucial in understanding the behavior and constraints of analytic functions within specified domains.

10.8 *GLOSSARY:-*

- **Maximum Modulus Principle:** States that if $f(z)$ is a non-constant analytic function on a domain D and continuous on the closure of D, then the maximum value of $|f(z)|$ occurs on the boundary of D, not in its interior.

- **Analytic Function:** A function that is complex differentiable at every point in its domain.
- **Domain:** An open connected set in the complex plane.
- **Modulus:** The absolute value of a complex number $z = x + iy$, denoted as $|z|$ and defined as $\sqrt{x^2 + y^2}$.
- **Minimum Modulus Principle:** States that if $f(z)$ is a non-constant analytic function on a domain D and continuous on the closure of D, and $f(z) \neq 0$ inside D, then the minimum value of $|f(z)|$ occurs on the boundary of D.
- **Continuous Function:** A function f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.
- **Schwarz Lemma:** States that if $f(z)$ is analytic in the unit disk $|z| < 1$, $f(0) = 0$, and $|f(z)| < 1$ for all z in the disk, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, with equality if and only if $f(z) = ze^{i\alpha}$ for some real constant α.
- **Unit Disk:** The set of all points in the complex plane whose distance from the origin is less than 1, denoted as $\{z \in C : |z| < 1\}$.
- **Equality Condition:** In the context of the Schwarz Lemma, equality $|f(z)| = |z| \; or \; |f'(0)| = 1$ holds if and only if $f(z) = ze^{i\alpha}$, where α is a real constant.
- **Linear Transformation:** A function of the form $f(z) = az + b$, where a and b are constants.
- **Argument Principle:** A theorem in complex analysis that relates the number of zeros and poles of a meromorphic function inside a closed contour to the change in the argument of the function along the contour.
- **Meromorphic Function:** A function that is analytic except at a set of isolated points, which are poles of the function.
- **Holomorphic Function:** Another term for an analytic function, a function that is complex differentiable at every point in its domain.
- **Contour:** A piecewise smooth curve in the complex plane along which an integral is evaluated.
- **Simple Closed Contour:** A contour that does not intersect itself and forms a closed loop.

Understanding these terms and concepts provides a deeper insight into the behavior and properties of analytic functions as described by the Maximum and Minimum Modulus Principles and the Schwarz Lemma.

10.9 *REFERENCES:-*

- Ravi P. Agarwal, Kanishka Perera, Sandra Pinelas (2011)- Maximum Modulus Principle in An Introduction to Complex Analysis.
- Graziano Gentili, Caterina Stoppato, Daniele C. Struppa -(2022) Maximum Modulus Theorem and Applications in Regular Functions of a Quaternionic Variable Springer, 2022.

10.10 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- https://old.mu.ac.in/wp-content/uploads/2020/12/Paper-III-Complex-Analysis.pdf
- file:///C:/Users/user/Desktop/1468563787EText(Ch-8,M-3.pdf
- file:///C:/Users/user/Desktop/1468562099EText(Ch-5,M-4.pdf

10.11 *TERMINAL QUESTIONS:-*

(TQ-1) State and prove the principle of maximum modulus.

(TQ-2) Find the maximum modulus of $z^2 - z$ in the disc $|z| \leq 1$.

(TQ-3) Show that the maximum modulus of e^z is always assumed on the boundary of the compact domain.

(TQ-4) Suppose f, g both are analytic in a compact domain D. Show that $|f(z)| + |g(z)|$ take it's maximum on the boundary.

(TQ-5) Let f be a non-constant analytic function on the closed unit disc $D = \{z \in C: | z | \leq 1\}$. Prove that $| f(z) |$ attains its maximum value on the boundary $\partial D = \{z \in C: | z | = 1\}$ and not in the interior $D = \{z \in C: | z | < 1\}$. Use this to show that if f is analytic in D, bounded by 1, and $f(0) = 0$, then $| f(z) | \leq | z | \; for\ all\ z \in D$.

(TQ-6) Show that among all functions, which are analytic and bounded by 1, in the unit disc, Max $\left|f'\left(\frac{1}{3}\right)\right|$ is assumed, when $f\left(\frac{1}{3}\right) = 0$.

(TQ-7) If f is a non-constant analytic function on an open set $U \subset C$, then prove that $f(U)$ is an open set in C.

(TQ-8) Suppose further that $f(z)$ is not constant. Then $|f(z)|$ attains its minimum value at a point on the boundary of C, that is to say, if m is minimum value of $|f(z)|$ inside and on C, then

$$|f(z)| > m \ \forall z \ inside \ C$$

10.12 *ANSWERS:-*

<u>SELF CHECK ANSWERS</u>

1. The Maximum Modulus Principle states that if a function f is analytic and non-constant in a domain D, then the maximum value of $| f(z) |$ on the closure of D occurs on the boundary of D. This means that the modulus of f(z) cannot attain its maximum value inside the domain unless the function is constant.
2. No, the Maximum Modulus Principle cannot be applied to non-analytic functions because the principle relies on the properties of analytic (holomorphic) functions, specifically their continuity and differentiability. Non-analytic functions do not necessarily have these properties, and thus the principle does not hold for them.
3. To prove the Maximum Modulus Principle by contradiction, assume that $| f(z) |$ attains its maximum value M at some point a inside the domain D. Consider a small circle centered at a. By analyticity, $f(z)$ is continuous and there exists a point b within the circle such that $| f(b) | < M$. This contradicts the assumption that M is the maximum value. Therefore, $| f(z) |$ must attain its maximum value on the boundary of D.
4. Consider the function $f(z) = z$ on the unit disk $| z | < 1$. The modulus $| f(z) | = | z |$. The maximum value of $| z |$ in the unit disk is 1, which occurs on the boundary $| z | = 1$. The implication is that the modulus $| z |$ does not reach 1 within the disk but only on the boundary.
5. If a non-constant analytic function $f(z)$ could achieve its maximum modulus in the interior of the domain, this would contradict the principle that $f(z)$ attains its maximum on the boundary. This is because an analytic function is open and cannot have a local maximum in the interior unless it is constant.
6. Answer: The Minimum Modulus Principle can be derived from the Maximum Modulus Principle by considering the function $g(z) =$

$1/f(z)$, where $f(z)$ is analytic and non-zero. Applying the Maximum Modulus Principle to $g(z)$, we find that $|g(z)|$ attains its maximum on the boundary, implying that $|f(z)|$ attains its minimum on the boundary.

7. The Schwarz Lemma states that if f is an analytic function on the unit disk $|z|< 1$ with $f(0) = 0$ and $|f(z)|< 1$, then $|f(z)|\leq|z|$ for all z in the disk, and $|f'(0)|\leq 1$. If $|f(z)|=|z|$ for some $z \neq 0$, then $f(z) = e^{i\theta}, z$ for some real θ. This lemma is significant because it provides bounds for the function and its derivative at the origin, showing that an analytic function cannot grow too fast within the unit disk.
8. The Schwarz Lemma holds for an analytic function f on the unit disk $|z|< 1$ that satisfies $f(0) = 0$ and $|f(z)|< 1$ for all z in the disk.
9. The Schwarz Lemma can be generalized to the Schwarz-Pick theorem, which deals with functions mapping the unit disk to itself and not necessarily fixing the origin. The generalization provides bounds on the function and its derivative in terms of the hyperbolic metric. The implications are broader control over the behavior of analytic functions mapping the unit disk into itself.

BLOCK IV
SINGULARITIES AND RESIDUE

UNIT 11:- Singularities

CONTENTS:

11.1 *INTRODUCTION:-*

In complex analysis, singularities are points at which a complex function ceases to be analytic. There are different types of singularities, classified primarily into three categories: removable singularities, poles, and essential singularities. A removable singularity is a point where the function can be redefined to make it analytic. Poles are points where the function goes to infinity in a specific manner, characterized by the function behaving like $\frac{1}{z-z_0}$ near the singularity z_0. Essential singularities are points where the function exhibits chaotic behavior, as described by the Weierstrass–Casorati theorem and Picard's theorem. Understanding these singularities is crucial for analyzing the behavior of complex functions, especially in the context of Laurent series and residue calculus, which are fundamental tools in complex analysis for evaluating integrals and solving differential equations.

11.2 *OBJECTIVES:-*

Singularities in complex analysis are critical for understanding the behavior and structure of complex functions. They represent points where a function ceases to be well-behaved, such as where it becomes infinite or undefined. Analyzing singularities helps in classifying functions into categories like poles, essential singularities, or branch points, and understanding their local behavior and global implications. This analysis is vital for solving complex integrals, developing conformal mappings, and exploring the analytic continuation of functions, thereby enhancing the comprehension of complex functions' intricate properties and their applications in various scientific fields.

11.3 *ZERO OF AN ANALYTIC FUNCTION:-*

A zero of an analytic function is a point z_0 in the complex plane where the function $f(z)$ evaluates to zero, i.e.,$f(z_0) = 0$. If $f(z)$ is analytic at z_0, then z_0 is called a zero of the function.

Zeros are important in the study of complex functions because they provide critical information about the function's behavior and its structure. For instance, the order or multiplicity of a zero, defined as the highest power of $(z - z_0)$ that divides $f(z)$, indicates how the function behaves near that zero. Analyzing zeros helps in understanding the function's growth, mapping properties, and in solving complex differential equations. They also play a significant role in the residue theorem and in the factorization of analytic functions.

An analytic function $f(z)$ in a domain D can be expanded as a Taylor series around a point $a \in D$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \quad ...(1), \quad a_n = \frac{f^n(a)}{n!} \quad ...(2)$$

If the coefficients $a_0, a_1, a_2 = \cdots a_{m-1}$ are all zero. But $a_m \neq 0$, then $f(z)$ has a zero of order m at $z = a$. so that

$$f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0, f^m(a) \neq 0$$

In this case we say that $f(z)$ has a zero of order m at $z = a$.

From (1) takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n = \sum_{n=0}^{\infty} a_{n+m} (z - a)^{n+m}$$

$$= (z - a)^n \sum_{n=0}^{\infty} a_{n+m} (z - a)^n$$

Taking

$$\sum_{n=0}^{\infty} a_{n+m}(z-a)^n = \phi(z) \qquad \dots (3)$$

$$f(z) = (z-a)^m \phi(z)$$

From (3),

$$\phi(a) = \left[\sum_{n=0}^{\infty} a_{n+m}(z-a)^n\right]_{z=a} = \left[a_m + \sum_{n=1}^{\infty} a_{n+m}(z-a)^n\right]_{z=a} = a_m$$

But $a_m \neq 0$ so that $\phi(a) \neq 0$.
Thus
An analytic function $f(z)$ is said to have a zero of order m if $f(z)$ is expressible as

$$f(z) = (z-a)^m \phi(z)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$. $f(z)$ is said to have a simple zero at $z = a$ if $z = a$ is a zero of order one.

11.4 *SINGULAR POINTS:-*

A singularity of a complex function $f(z)$ is a point z_0 where $f(z)$ ceases to be analytic. For example if $f(z) = \frac{1}{(z-2)}$, then $z = 2$ is a singularity of $f(z)$. Singular points are categorized into different types based on their characteristics:

1. **Isolated Singularity:** A point $z = a$ is an isolated singularity of a complex function $f(z)$ if:
   - **i.** $f(z)$ is not analytic at $z = a$.
   - **ii.** There exist a neighborhood around $z = a$ (excluding a itself) where $f(z)$ is analytic.

In other words, $z = a$ is an **isolated singularity** if $f(z)$ fails to be analytic only at $z = a$, but $f(z)$ is analytic in some punctured neighborhood around $z = a$.
Conversely, if every deleted neighborhood of $z = a$ contains other singularities of $f(z)$, then $z = a$ is called a **non-isolated singularity**. This means $z = a$ is part of a set of singularities that accumulate around it.

2. **Definition:** For a function $f(z)$ with an isolated singularity at $z = a$, there exists a deleted neighborhood $0 < | z - a | < r$ where $f(z)$ is analytic. In this neighborhood, $f(z)$ can be expanded using a Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

The series $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ is called the principal part of the Laurent series. Three cases arise based on the nature of the Singularity.

- **I. Removable Singularity:** If the principal part of a function $f(z)$ at an isolated singularity $z = a$ contains no terms (i.e., $b_n = 0$ for all

n), then $z = a$ is called a removable singularity. In this case, the function $f(z)$ can be expressed as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

and $f(z)$ can be analytically extended to $z = a$.

Or

A singularity $z = a$ of a function $f(z)$ is called a **removable singularity** if $f(z)$ can be analytically extended to $z = a$. Formally:

A singularity $z = a$ is a **removable singularity** of $f(z)$ if: $\lim_{z \to a} f(z)$ exists finitely.

This means that even though $f(z)$ is not analytic at $z = a$, it can be defined or redefined at $z = a$ such that $f(z)$ becomes analytic at $z = a$.

EXAMPLE1: Consider $f(z) = \frac{sin(z)}{z}$. The point $z = 0$ is a removable singularity. Although $\frac{sin(z)}{z}$ is not defined at $z = 0$, it can be extended to an analytic function by defining $f(0) = 1$.The extended function $f(z) = \frac{sin(z)}{z}$ for $z \ = 0$ and $f(0) = 1$ is analytic everywhere.

II. **Pole:** A singularity $z = a$ of a function $f(z)$ is called a pole of order m if the principal part of the Laurent series expansion around $z = a$ contains a finite number of terms, specifically up to $(z - a)^{-m}$. More precisely:

If $b_n = 0$ for all n s.t. $n > m$, where m is positive integer, then $z = a$ pole of order m of $f(z)$.

Thus if $z = a$ is a pole of order m of the function $f(z)$, then $f(z)$ have the expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Or

A function $f(z)$ is said to have pole of order n if it is expressible as

$$f(z) = \frac{\phi(z)}{(z-a)^n}$$

where $\phi(z)$ is analytic and $\phi(z) \neq 0$.

Simple Pole: A pole of order 1 is called a **simple pole**.

Residue of a function f(z) at a simple pole z=a is defined as

$$\lim_{z \to a} (z-a)\, f(z) = Res(z = a)$$

Or

$$Res\ (z = a) = \lim_{z \to a} \frac{\phi(z)}{\psi'(z)}$$

where $f(z) = \frac{\phi(z)}{\psi'(z)}$.

EXAMPLE2: For $f(z) = \frac{1}{(z-1)^2}$, $z = 1$ is a pole of order 2. Near $z = 1$, $f(z)$ grows without bound as z approaches 1.

III. **Essential singularity**: A singularity $z = a$ of a function $f(z)$ is called an essential singularity if the Laurent series expansion around $z = a$ has an infinite number of terms in the principal part. Specifically:If $b_n \neq 0$ for independently many values of n so that the series

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

contains an infinite number of terms, then $z = a$ is an essential singularity of $f(z)$.

In other words, near an essential singularity, the function $f(z)$ exhibits highly irregular and chaotic behavior.

EXAMPLE3: For $f(z) = e\ 1/(z-1)$, the point $z = 1$ is an essential singularity. As z approaches 1, $f(z)$ exhibits highly irregular behavior, such as taking on virtually every complex value near $z = 1$.

Theorem1: If $f(z)$ has a pole at $z = a$, then $|f(z)| \to \infty,\ z \to a$.

Proof: Suppose $f(z)$ has a pole of order m at $z = a$. By definition, the Laurent series expansion of $f(z)$ around $z = a$ is given by:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

This can be rewritten as:

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_n}{(z-a)^m}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{1}{(z-a)^m}[b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m]$$

As $z \to a$, the expression within the square brackets on the right-hand side approaches b_m because the terms $b_1(z-a)^{m-1}, b_2(z-a)^{m-2}, \ldots$ all vanish and the only non-vanishing term is b_m.Therefore

$$f(z) \approx \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_n}{(z-a)^m}$$

As $z \to a$, the term $\frac{b_n}{(z-a)^m}$ dominates the expression since the other terms involving $a_n(z-a)^n$ and the lower-order principal part terms become negligible in comparison. Thus, the magnitude of $f(z)$ is primarily determined by $\frac{b_n}{(z-a)^m}$

Hence, as $z \to a$,

$$|f(z)| \approx \left|\frac{b_n}{(z-a)^m}\right| \to \infty$$

Therefore, $|\ f(z)\ | \to \infty$ as $z \to a$, proving that $f(z)$ tends to infinity near the pole $z = a$.

Theorem2: If an analytic function $f(z)$ has a pole of order m at $z \to a$, then $\frac{1}{f(z)}$ has a zero of order m at $z \to a$ and conversely.

Proof: Suppose an analytic function $f(z)$ has a zero of order m at $z \to a$ this that

$$f(z) = \frac{\phi(z)}{(z-a)^m} \qquad \dots(1)$$

where $\phi(a) \neq 0$ and $\phi(z)$ is analytic.

We need to prove that $\frac{1}{f(z)}$ has a zero of order m at $z = a$.

$$\frac{1}{f(z)} = \frac{(z-a)^m}{\phi(z)} = (z-a)^m \psi(z)$$

Where $\frac{1}{\phi} = \psi$ and $\psi(a) \neq 0$.

This implies that $\frac{1}{f}$ has a zero of order m at $z = a$ so that $\frac{1}{f(z)} = (z-a)^m g(z)$, where $g(z)$ is analytic and $g(a) \neq 0$.

Therefore

$$f(z) = \frac{1}{(z-a)^m g(z)} = \frac{h(z)}{(z-a)^m}$$

where $\frac{1}{f(z)} = h(z)$.

This shows that $f(z)$ can be written as $\frac{h(z)}{(z-a)^m}$ where $h(z)$ is an analytic $h(a) \neq 0$ $f(z)$ has a pole of order m at $z = a$.

Thus, if $f(z)$ has a pole of order m at $z = a$, then $1/f(z)$ has a zero of order m at $z = a$, and conversely.

11.5 *ZEROS ARE ISOLATED:-*

Theorem3: (***Zeros Are Isolated***) **Zeros of an analytic function is isolated.**

Proof: Suppose $f(z)$ is an analytic function in a domain D, and suppose $f(z)$ has a zero at $z = a$. We aim to show that this zero is isolated, meaning there exists a neighborhood around $z = a$ where $f(z)$ has no other zeros except $z = a$.

Since $f(z)$ is analytic at $z = a$, it can be expressed as a Taylor series around $z = a$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where $a_n = \frac{f^n(a)}{n!}$.

If $f(z)$ has a zero of order m at $z = a$, then $a_0 = a_1 = a_2 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$.
Therefore, the Taylor series can be written as:

$$f(z) = (z-a)^n \sum_{n=0}^{\infty} b_n (z-a)^n$$

where $b_0 = a_m \neq 0$.
Define

$$g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

Notice that $g(z)$ is analytic and $g(a) = b_0 \neq 0$.

By the continuity of $g(z)$, there exists a neighborhood around $z = a$ such that $g(z) \neq 0$ in this neighborhood. Therefore, in this neighborhood, the function $f(z)$ can be written as:

$$f(z) = (z-a)^m g(z)$$

Since $g(z) \neq 0$ in this neighborhood, $f(z) = 0$ if and only if $(z-a)^m = 0$ which implies $z = a$.
Thus, there is a neighborhood around $z = a$ where $f(z)$ has no zeros other than $z = a$. This proves that the zero at $z = a$ is isolated.

11.6 *POLES ARE ISOLATED:-*

Theorem4: Poles of an analytic function are isolated.
Proof: Suppose $\boldsymbol{f(z)}$ is an analytic function in a domain $\boldsymbol{D}$ and has a pole at $\boldsymbol{z = a}$. We need to show that this pole is isolated, meaning there exists a neighborhood around $\boldsymbol{z = a}$ where $\boldsymbol{f(z)}$ has no other poles except $\boldsymbol{z = a}$.

By definition, if $z = a$ is a pole of $f(z)$ of order m, then near $z = a$, the function $f(z)$ can be expressed as:

$$f(z) = \frac{h(z)}{(z-a)^m}$$

where $h(z)$ is analytic and $h(a) \neq 0$.
Let's consider the function $g(z) = (z-a)^m f(z)$. This function $g(z)$ is constructed to remove the pole at $z = a$:

$$g(z) = (z-a)^m = \frac{h(z)}{(z-a)^m}$$

Since $h(z)$ is analytic and non-zero at $z = a$, $g(z)$ is analytic at $z = a$.
By the continuity of $h(z)$, there exists a neighborhood U around $z = a$ such that $h(z) \neq 0$ in U. Because $g(z) = h(z), g(z)$ is analytic and non-zero in this neighborhood U.
Given that g(z) is analytic and non-zero in U, we have:

$$f(z) = \frac{g(z)}{(z-a)^m}$$

Therefore, the pole at $z = a$ is isolated because there is a neighborhood around $z = a$ where $f(z)$ has no other poles except $z = a$.
Hence, we have shown that poles of an analytic function are isolated.

11.7 *THE POINT AT INFINITY:-*

The point at infinity is treated differently from the finite points in the complex plane. The extended complex plane (or Riemann sphere) includes the point at infinity, denoted as ∞. This extended plane allows for a more comprehensive understanding of the behavior of complex functions, especially in terms of their singularities.

Key Concepts:

- **Extended Complex Plane:** The extended complex plane is the complex plane C together with the point at infinity, ∞. It is often visualized as the Riemann sphere, where the complex plane is projected onto a sphere such that the point at infinity is the north pole of the sphere.
- **Neighborhood of Infinity:** A neighborhood of ∞ consists of all points z in C such that $| z |> R$ for some large $R > 0$. Essentially, as $| z |$ grows without bound, z approaches ∞.
- **Behavior at Infinity:** To study the behavior of a function $f(z)$ at ∞, we often consider the transformed function $g(w) = f(w_1)$ and examine its behavior as $w \to 0$. The point $z = \infty$ in the original function corresponds to $w = 0$ in the transformed function.
- **Poles and Essential Singularities at Infinity:**
- **Pole at Infinity:** A function $f(z)$ has a pole of order m at ∞ if $f(w_1)$ has a zero of order m at $w = 0$. Mathematically, $f(z)$ behaves like z^m as $z \to \infty$.
- **Essential Singularity at Infinity:** A function $f(z)$ has an essential singularity at ∞ if $f(w_1)$ has an essential singularity at $w = 0$.
- **Removable Singularity at Infinity:** A function $f(z)$ has a removable singularity at ∞ if $f(w_1)$ is analytic at $w = 0$. This means $f(z)$ approaches a finite limit as $z \to \infty$.

Examples:

- **Pole at Infinity:** The function $f(z) = z_1$ has a simple pole at $z = \infty$ because $f(w_1) = w$, which has a zero of order 1 at $w = 0$.

- **Essential Singularity at Infinity:** The function $f(z) = e^z$ has an essential singularity at $z = \infty$ because $f\left(\frac{1}{w}\right) = e^{1/w}$ has an essential singularity at $w = 0$.
- **Removable Singularity at Infinity:** The function $f(z) = \frac{1}{1+z}$ has a removable singularity at $z = \infty$ because $f\left(\frac{1}{w}\right) = \frac{1}{1+\frac{1}{w}} = \frac{w}{1+w}$, which is analytic at $w = 0$.

11.8 *LIMITING POINT OF ZEROS:-*

A point $z_0 \in \mathbb{C} \cup \{\infty\}$ is called a limiting point (or accumulation point) of zeros of an analytic function $f(z)$ if, for any $\epsilon > 0$, there are infinitely many zeros of $f(z)$ within the distance ϵ from z_0.

Theorem5: Let $\boldsymbol{f(z)}$ be an analytic function in a simply connected region $\boldsymbol{D}$. Let $\boldsymbol{a_1}, \boldsymbol{a_2} \dots \boldsymbol{a_n} \dots$ be a sequence of zeros having a as its limit point, $\boldsymbol{a}$ being the interior point of $\boldsymbol{D}$. Then either $\boldsymbol{f(z)}$ vanishes identically or else has an isolated essential singularity at $\boldsymbol{z = a}$.

Proof: Suppose $f(z)$ is analytic in a simply connected region D and $\{\boldsymbol{a_n}\}$ is a sequence of zeros of $f(z)$ converging to a, where a is an interior point of D.

The function $f(z) = 0$ in D, the sequence $\{\boldsymbol{a_n}\}$ being zeros converging to a, poses no contradiction since every point in D is a zero.

Assume $f(z)$ is not identically zero. By the Identity Theorem, an analytic function that is zero on a sequence of points with an accumulation point in D must be identically zero unless it has a singularity at the accumulation point.

Here, a is the limit point of zeros of $f(z)$, meaning $f(z)$ cannot be analytic at a. Therefore, a must be a singularity of $f(z)$.

11.9 *LIMIT POINT OF POLES:-*

A point $\boldsymbol{z_0}$ is called a **limiting point of poles** of a function $f(z)$ if every neighborhood of $\boldsymbol{z_0}$ contains infinitely many poles of $f(z)$.

Theorem6: If $z = a$ is a limit point of the sequence of poles of an analytic function $f(z)$, then $z = a$ is an essential singularity (non-isolated essential singularity) of $f(z)$.

Proof: Suppose $f(z)$ is analytic in a domain D except for some poles. Let $\{a_n\}$ be a sequence of poles of $f(z)$ such that $a_n \to a$ as $n \to \infty$.

Since a is a limit point of poles, every neighborhood of a contains infinitely many poles of $f(z)$.

By definition, $z = a$ is a singularity of f(z) because it is a point where $f(z)$ is not analytic.If $z = a$ were a pole, then $f(z)$ would have a Laurent series expansion around a with only finitely many negative powers:

$$f(z) = \frac{b_m}{(z-a)^m} + \frac{b_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{b_1}{(z-a)^1} + analytic\ part$$

However, if a were a pole, then $f(z)$ would not have an infinite number of poles accumulating at a. Poles are isolated by definition.

Since $z = a$ is not isolated (it has an infinite number of poles arbitrarily close to it), a cannot be a pole.

According to complex analysis, the only type of singularity where poles can accumulate and still fit the behavior of having infinitely many poles in every neighborhood is an essential singularity.

Thus, $z = a$ must be an essential singularity of $f(z)$, specifically a non-isolated essential singularity. This means that $f(z)$ exhibits highly irregular behavior near a, consistent with the presence of infinitely many poles accumulating at a.

11.10 *CHARACTERIZATION OF POLYNOMIALS:-*

Theorem7: The order of zero of the polynomial equals the order of its first non-vanishing derivative.

Or

The order (or multiplicity) of a zero of a polynomial $p(z)$ at $z = a$ is equal to the order of the first non-zero derivative of $p(z)$ at $z = a$.

Proof: A polynomial $p(z)$ has a zero of order m at $z = a$ if:

$$p(a) = p'(a) = p''(a) = \cdots = p^{m-1}(a) = 0 \text{ and } p^{(m)} \neq 0.$$

In other words, $p(z)$ can be written locally around $z = a$ as:

$$p(z) = (z-a)^m q(z)$$

where $q(z)$ is a polynomial and $q(a) \neq 0$.

The derivatives of $p(z)$ are:

$$p'(z) = m(z-a)^{m-1}q(z) + (z-a)^m q'(z)$$

$$p'^{\prime(z)} = m(m-1)(z-a)^{m-2}q(z) + 2m(z-a)^{m-1}q'(z) + (z-a)^m q''(z)$$

At $z = a, p(a) = p'(a) = \cdots = p^{m-1}(a) = 0$, but $p^m(a)$ will be non-zero because

$$p^m(z) = m!\,q(z) \text{ and } p^m(a) = m!\,q(a) \neq 0$$

Therefore, the order of the zero at $z = a$ is exactly m, which corresponds to the smallest k for which $p^k(a) \neq 0$.

EXAMPLE4: For the polynomial $p(z) = (z-2)^3(z-5)$.

At $z = 2$, $p(z)$ has a zero of order 3.

The first non-zero derivative at $z = 2$ is the third derivative, which will not be zero. This confirms that the order of the zero at $z = 2$ is indeed 3.

Theorem8: (Due to Riemann) Removable singularity. If $z = a$ is an isolated singularity of $f(z)$ and if $f(z)$ is bounded on some deleted neighborhood of a, then a is a removable singularity.

Proof: Given that $f(z)$ is bounded on some deleted neighborhood $N(a)$ of a, Let M be the maximum value of $|f(z)|$ on a circle C defined by $|z - a| = r$, where the radius r is chosen so small that C lies entirely within $N(a)$, the Laurent series expansion for a function $f(z)$ with an isolated singularity at $z = a$, the Laurent series expansion around $z = a$ is:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^n \qquad \dots (1)$$

Where

$$b_n = \frac{1}{2\pi i}\int_C (z-a)^{n-1} f(z) dz$$

$$|b_n| \le \frac{M}{2\pi}\int_C |z-a|^{n-1}|dz| = \frac{Mr^{n-1}}{2\pi}.2\pi r = Mr^n$$

$$|b_n| \le \; Mr^n \text{ which} \rightarrow \text{as } r \rightarrow 0$$

$$b_n = 0 \forall n$$

Thus, the boundedness of $f(z)$ on the deleted neighborhood implies that $z = a$ is a removable singularity of $f(z)$,by definition, this proves that $z = a$ is removable singularity.

Theorem9: (Weierstrass Theorem) Essential Singularity. If $z = a$ is an essential singularity of $f(z)$, prove that any positive number, r, ε and any number c, there is a point in the circle $|z - a| < r$ at which $|f(z) - c| < \varepsilon$.

In other words, in any arbitrary neighborhood of an essential singularity, there exists a point (and therefore an infinite number of points) at which the function differs as little as we please form any –assigned number.

Proof: Suppose the theorem is false. Then there exist positive numbers

r and ε and a complex number c, such that$|f(z) - c| > \varepsilon$ for all z with $|z - z_0| < r$ so that $\frac{1}{|f(z)-c|} < \varepsilon$ whenever $|z - z_0| < r$.

Use of theorem (2), we see that the function $\frac{1}{|f(z)-c|}$ has a removable singularity so that principal part of Laurent's expansion for $\frac{1}{f(z)-c}$ about the point z_0 does not contain any term so that

$$\frac{1}{f(z)-c} = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \dots (1)$$

If $a_n \neq 0$, we define $\frac{1}{f(z_0)-c} = a_0$ so that $f(z_0) = c + (1/a_0)$. It means that $\frac{1}{f(z)-c}$ is analytic and non-zero at z_0.

∴ As a result of which $f(z)$ itself is analytic at z_0 . Contrary to be initial assumption that z_0 is an essential singularity of $f(z)$.

Again if $a_n = 0$ for $n = 0,1,2, \dots m-1$ then (1) becomes

$$\frac{1}{f(z)-c} = \sum_{n=m}^{\infty} a_n (z-z_0)^n$$

$$= a_m (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots$$

$$= (z-z_0)^m [a_m + a_{m+1}(z-z_0) + \cdots]$$

$$= (z-z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z-z_0)^n$$

This proves the theorem.

SOLVED EXAMPLE

EXAMPLE5: Find the singularities of the function $\frac{e^{c/(z-a)}}{e^{z/a}-1}$, indicating the character of each singularity.

SOLUTION: Given

$$f(z) = \frac{e^{\frac{c}{(z-a)}}}{e^{\frac{z}{a}} - 1}$$

i. Let we write $\exp(z/a)$ in place of $e^{\frac{z}{a}}$.

Now then

$$f(z) = \frac{e^{\frac{c}{(z-a)}}}{e^{\frac{z}{a}} - 1} = \frac{exp\left(\frac{e}{z-a}\right)}{exp\left(1 + \frac{z-a}{a}\right) - 1} = \frac{e^{\frac{c}{(z-a)}}}{e.e^{\frac{(z-a)}{a}} - 1}$$

$$= e^{\frac{c}{(z-a)}} \left[1 - e.e^{\frac{(z-a)}{a}}\right]^{-1}$$

$$= e^{\frac{c}{(z-a)}\left[1-e.\left\{1+\frac{(z-a)}{a}+\left(\frac{(z-a)}{a}\right)^2.\frac{1}{2!}+\cdots\right\}\right]^{-1}}$$

$$= \left[1+\frac{c}{z-a}+\left(\frac{c}{z-a}\right)^2.\frac{1}{2!}+\cdots\right]$$
$$\times\left[1+e\left\{1+\frac{(z-a)}{a}+\left(\frac{(z-a)}{a}\right)^2.\frac{1}{2!}+\cdots\right\} + e^{2\left\{1+\frac{(z-a)}{a}+\left(\frac{(z-a)}{a}\right)^2+\cdots\right\}}\right]$$

Hence by the definition $z = a$ is an essential singularity.

ii. $$f(z) = \frac{exp\left(\frac{e}{z-a}\right)}{exp\left(\frac{z}{a}\right)-1}$$

Evidently denominator has zero of order 1at

$$e^{z/a} = 1 = e^{2n\pi i}, i.e., z = 2n\pi ia.$$

So the function $f(z)$ has a pole of order one at each point $z = 2n\pi ia$ (where $n = 0, \pm1, \pm2, \dots\dots\dots$).

EXAMPLE6: Specify the nature of singularity at $z = -2$ of $f(z) = (z-3)sin\left(\frac{1}{z+2}\right)$.

SOLUTION: The given singularity is

$$f(z) = 0 \quad or \quad (z-3)\,sin\left(\frac{1}{z+2}\right) = 0$$

This implies that

$\Rightarrow \quad z = 3$ and $sin\left(\frac{1}{z+2}\right) = 0 = sin0$

$\Rightarrow \quad \frac{1}{z+2} = n\pi + (-1)^n(0) = n\pi$ and $sin\left(\frac{1}{z+2}\right) = 0 = sin0$

$\Rightarrow \quad z + 2 = \frac{1}{n\pi} \Rightarrow \quad z = -2 + \frac{1}{n\pi}$

$\Rightarrow \quad z = -2 + \frac{1}{n\pi}$ for $n = 0,1,2,3 \dots\dots$

Limit points of zeros is $z = -2$.

Hence the isolated singularity is $z = -2$.

EXAMPLE7: Find zeros and pole of $\left(\frac{z+1}{z^2+1}\right)^2$.

SOLUTION: The given singularity is

$$f(z) = \left(\frac{z+1}{z^2+1}\right)^2$$

I. The zeros of $f(z)$ is given by

$$(z+1)^2 = 0$$
$$z = -1, -1$$

So $z = -1$ is a zero of order 2.

II. The poles of $f(z)$ is given by

$$(z^2+1)^2=0$$
$$(z+i)^2(z-i)^2=0$$
$$z=-i,-i,i,i$$

Hence $z=-i$ and $z=i$ both are poles of order2.

EXAMPLE8: Find Laurent series of $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$ about singularity $z=-2$ and indicate nature of singularity.

SOLUTION: Let

$$sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots$$

So that

$$f(z)=(z-3)\left[\frac{1}{(z+2)}-\frac{1}{3!}.\frac{1}{(z+2)^3}+\frac{1}{5!\,(z+2)^5}+\cdots\right]$$

which represents Laurent's series.

Now the zeros of $f(z)$ is

$$(z-3)sin\frac{1}{(z+2)}=0\Rightarrow z-3=0, sin\frac{1}{(z+2)}=0$$

$\Rightarrow$

$$sin\frac{1}{(z+2)}=0=sin(n\pi)\ \Rightarrow\frac{1}{(z+2)}=n\pi$$

$\Rightarrow\ z+2=\frac{1}{n\pi}\Rightarrow z=\frac{1}{n\pi}-2$ for $n=1,2,3,4,\dots$ ....

Now

$$z=\frac{1}{\infty}-2=-2$$

EXAMPLE9: Determine the nature of the singularity at z=0 for the function:

$$f(z)=\frac{sin(z)}{z}$$

SOLUTION: The function

$$f(z)=\frac{sin(z)}{z}$$

is not defined at $z=0$ because of the division by zero. This suggests a possible singularity at $z=0$.

So the limit as z approaches 0:

$$\lim_{z\to 0}\frac{sin(z)}{z}$$

Using the standard limit $\lim_{z\to 0}\frac{sin(z)}{z}=1$, we have

$$\lim_{z\to 0}\frac{sin(z)}{z}=1$$

Since the limit exists and is finite, the singularity at $z=0$ is a removable singularity. The function can be redefined at $z=0$ as $f(0)=1$ to remove the singularity and make the function analytic at that point.

EXAMPLE10: Determine the nature and order of the singularity at $z = 0$ for the function:

$$g(z) = \frac{1}{z^3}$$

SOLUTION: The function $g(z) = \frac{1}{z^3}$ is undefined at $z = 0$, indicating a potential singularity at $z = 0$.
The function can be written as,

$$g(z) = \frac{1}{z^3}$$

Here, the function clearly approaches infinity as z approaches 0.
Since the function can be expressed as
$g(z) = \frac{1}{(z-0)^3}$,
It is evident that this is a pole of order 3.The singularity at $z = 0$ is a pole of order 3.

SELF CHECK QUESTIONS

1. What is a singularity in the context of complex analysis?
2. What are the three main types of isolated singularities?
3. Define a removable singularity.
4. What does it mean for zeros to be isolated?
5. Explain the relationship between the limit points of zeros and essential singularities.
6. How does the Weierstrass theorem describe a removable singularity?

11.11 *SUMMARY:-*

In this unit we have studied the singularities in complex analysis are points where a function is not analytic. They can be classified into three main types: removable singularities, poles, and essential singularities. A removable singularity is one where the function can be redefined to make it analytic. Poles are points where the function approaches infinity, with the order of the pole indicating how fast it diverges. Essential singularities are characterized by chaotic behavior, where the function takes on almost every possible value infinitely often in any neighborhood around the singularity. According to Picard's theorem, near an essential singularity, a function assumes every complex value, with at most one exception, infinitely often. Singularities play a crucial role in understanding the behavior and properties of complex functions.

11.12 *GLOSSARY:-*

- **Analytic Function:** A function that is locally given by a convergent power series.

- **Singularity:** A point at which a function is not analytic.
- **Isolated Singularity**: A singularity where there exists a neighborhood around the point such that there are no other singularities within that neighborhood.
- **Removable Singularity:** A point $z = a$ where a function is not analytic but can be redefined to be analytic. If the Laurent series of $f(z)$ around $z = a$ contains no negative power terms, it is a removable singularity.
- **Pole:** A singularity where a function approaches infinity as the variable approaches the point. The order of the pole indicates the highest power of the negative term in the Laurent series expansion.
- **Simple Pole:** A pole of order one.
- **Essential Singularity:** A point where a function exhibits chaotic behavior such that, in any neighborhood around the singularity, the function takes on nearly every possible value infinitely often. The Laurent series has an infinite number of negative power terms.
- **Laurent Series:** A representation of a complex function as a series that includes both positive and negative powers of $(z - a)$.
- **Principal Part:** The part of the Laurent series that contains the negative powers of $(z - a)$.
- **Limit Point of Zeros:** A point where a sequence of zeros accumulates. If such a point is within the domain of an analytic function, the function must either be identically zero or have an essential singularity at that point.
- **Picard's Theorem:** A theorem stating that near an essential singularity, a function takes on every possible complex value, with at most one exception, infinitely often.
- **Weierstrass Theorem:** In the context of removable singularities, this theorem states that if a function is bounded near an isolated singularity, then the singularity is removable.
- **Deleted Neighborhood:** A neighborhood around a point, excluding the point itself.
- **Non-Isolated Singularity:** A singularity where every neighborhood of the point contains other singularities.
- **Simply Connected Region:** A region without holes, where any loop can be continuously contracted to a point within the region.

11.13 *REFERENCES:-*

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11.14 *SUGGESTED READING:-*

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11.15 *TERMINAL QUESTIONS:-*

(TQ-1) If $f(z) = \sum_{n=1}^{\infty} \frac{z^2}{4+n^2z^2}$, show that $f(z)$ is finite and continuous for all real values of z but f(z) cannot be expanded in a Maclaurin's series. Show that $f(z)$ possesses Laurent's expansion valid in succession of the ring spaces.

(TQ-2) Show the function $cosecz$ has a simple pole at $z = \infty$.

(TQ-3) Show that the function e^{-1/z^2} has no singularities.

(TQ-4) Show that the function e^z has isolated essential singularity at $z = \infty$.

(TQ-5) Show that the function $e^{1/z}$ actually takes every value except zero an infinite number of times in the neighborhood of $z = 0$.

(TQ-6) Prove that a function which have no singularity in the finite part of the planes or at infinity is constant.

(TQ-7) To show that a function which has no singularity in the finite part of the plane and has a pole of order n at infinity is a polynomial of degree n.

(TQ-8) If $z = a$ is an essential singularity of $f(z)$, prove that any positive number, r, ε and any number c, there is a point in the circle $|z - a| < r$ at which $|f(z) - c| < \varepsilon$.

(TQ-9) If $z = a$ is an isolated singularity of $f(z)$ and if $f(z)$ is bounded on some deleted neighborhood of a, then a is a removable singularity.

(TQ-10) If $f(z)$ has a pole at $z = a$, thenprove that $|f(z)| \to \infty,\ z \to a$.

(TQ-11) Objectives Types Questions:

1. If a function is analytic at all points of a bounded domain except at infinity many points, then these points are called:
   a. Zeros
   b. Singularities
   c. Poles
   d. Simple points
2. A function which has poles as its only singularities in the finite part of the plane is
   a. An analytic function
   b. A meromorphic function
   c. An entire function
   d. A finite function
3. The number of zeros of the function $f(z) = \sin\left(\frac{1}{z}\right)$ is:
   a. 3
   b. 4
   c. Infinite
   d. None
4. The number of poles of the function $f(z) = \tan\frac{1}{z}$ is:
   a. 2
   b. 4
   c. Infinite
   d. None
5. What type of singularity is characterized by the fact that a function's Laurent series contains only a finite number of negative power terms?
   a. Removable Singularity
   b. Pole
   c. Essential Singularity
   d. Isolated Singularity

(TQ-12) True/ False Types Questions:

1. A removable singularity is characterized by a Laurent series with no negative power terms.
2. At a pole, the Laurent series of the function has an infinite number of negative power terms.
3. An essential singularity is a point where the Laurent series of the function has only a finite number of negative power terms.
4. A function with a pole at z_0 can be expressed as $\frac{f(z)}{(z-z_0)^n}$ where f(z) is analytic and non-zero at z_0.
5. The behavior of a function near an isolated singularity can be classified as removable, pole, or essential.
6. The residue of a function at a removable singularity is always zero.
7. Every isolated singularity of a function is either a removable singularity, a pole, or an essential singularity.

8. The function $f(z) = e^{1/(z-1)}$ has a removable singularity at $z = 1$.

(TQ-13) Find kind of the singularities of the following

a. $\frac{cot\pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.
b. $\tan(1/z)$ at $z = 0$.
c. $cosec\left(\frac{1}{z}\right)$ at $z = 0$.
d. $sin\left[\frac{1}{1-z}\right]$ at $z = 1$.
e. $\frac{1}{sinz-cosz}$ at $z = \frac{\pi}{4}$.
f. $sinz - cosz$ at $z = \infty$.
g. $\frac{e^z}{z^2+4}$
h. $f(z) = \frac{1-e^z}{1+e^z}$ at $z = \infty$.
i. $f(z) = z\ cosecz$ at $z = \infty$.

11.16 *ANSWERS:-*

SELF CHECK ANSWERS

1. A singularity is a point at which a complex function is not analytic. Singularities are points where the function fails to be differentiable or undefined.
2. The three main types of isolated singularities are:
   - Removable singularities
   - Poles
   - Essential singularities
3. A removable singularity at $z = a$ is a point where a function $f(z)$ is not defined or not analytic, but can be made analytic by defining or redefining $f(a)$ appropriately. If $f(z)$ is bounded in some deleted neighborhood of a, then a is a removable singularity.
4. Zeros of a function $f(z)$ are said to be isolated if each zero has a neighborhood in which it is the only zero of the function. This implies that there are no other zeros of $f(z)$ arbitrarily close to it.
5. If a function $f(z)$ is analytic in a simply connected region D and has a sequence of zeros $a_1, a_2, a_3, \dots \dots$ that accumulate at a point a within D, then either $f(z)$ is identically zero in D, or $z = a$ is an isolated essential singularity of $f(z)$.
6. The Weierstrass theorem states that if $z = a$ is an isolated singularity of $f(z)$ and if $f(z)$ is bounded on some deleted neighborhood of a, then a is a removable singularity.

TERMINAL ANSWERS

(TQ-11) 1.b 2.b 3.c 4.c 5.d

(TQ-12) 1.T 2.F 3.F 4.T 5.T
6.T 7.T 8.T

(TQ-13)

a. $z = a$
b. $z = 0$
c. $z = 0$
d. $z = 1$
e. $z = \frac{\pi}{4}$
f. $z = \infty$.
g. $z = 2i, -2i$
h. $z = \infty$
i. $z = \infty$.

UNIT 12:- The residue theorem

CONTENTS:

12.1 *INTRODUCTION:-*

The residue theorem is a powerful tool in complex analysis, which simplifies the evaluation of contour integrals. It states that if a function $f(z)$ is analytic inside and on a simple closed contour $C,$ except for a finite number of isolated singularities within $C,$ then the integral of $f(z)$ around C is 2πi times the sum of the residues of f at those singularities. Formally, $\int_C f(z)dz = 2\pi i \sum Res(f, z_k)$ where $Res(f, z_k)$ denotes the residue of f at the singularity z_k. This theorem greatly facilitates the computation of complex integrals, especially in cases where directly evaluating the integral is difficult.

12.2*OBJECTIVES:-*

After studying this unit, learners will be able to:

- Understand and verify the Cauchy residue theorem, comprehending its significance and applications in evaluating contour integrals with isolated singularities.

- Analyze Jordan's lemma, gaining insights into its role in simplifying the evaluation of integrals involving exponential functions and semicircular contours.
- Solve integration problems around the unit circle, applying the concepts learned to compute integrals of complex functions effectively.

12.3 *CAUCHY'S RESIDUE THEOREM:-*

Theorem1: Let $f(z)$ be a function that is analytic inside and on a simple closed contour C, except for a finite number of isolated singularities $z_1, z_2 \dots \dots \dots \dots z_n$ inside C. Then prove that the integral of $f(z)$ around C is given by:

$$\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res(z = z_r)$$

where $Res(z = z_r)$ denotes the residue of f at the singularity z_k.

Proof: Let $\gamma_1, \gamma_2, \dots \dots, \gamma_n$ are the circles with centres at $z_1, z_2, \dots \dots, z_n$ respectively and radii within C and do not overlap. $f(z)$ is analytic within the annulus bounded by these circles and curve C, by the corollary to Cauchy's theorem,

$$\int_C f(z)dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \dots + \int_{\gamma_n} f(z)\,dz$$

But $\frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz = residue\ of\ f(z) at\ z = z_1$

$$= Res(z = z_1)$$

$$\int_{\gamma_1} f(z)\,dz = 2\pi i Res(z = z_1)$$

Using(1), we obtain

$$\int_C f(z)dz = 2\pi i Res(z = z_1) + \dots + 2\pi i Res(z = z_n)$$

$$= 2\pi i \sum_{r=1}^{n} Res(z = z_1)$$

This completes the proof that the integral of $f(z)$ around a simple closed contour C is $2\pi i$ times the sum of residues of f at the isolated singularities inside C.

Theorem2: If a function $f(z)$ is analytic except at finite number of singularities (including that at infinity), then the sum of residues of these singularities is zero.

Proof: Let C be a closed contour that encloses all the singularities of $f(z)$ in the finite part of the complex plane, except that at infinity. By sum $\sum R$ residues at all the singularities of $f(z)$, the integral of $f(z)$ around C is given by:

$$\int_C f(z)dz = 2\pi i \sum R$$

Also

$$-\frac{1}{2\pi i}\int_{\gamma_1} f(z)\, dz = Res(z = \infty)$$

Adding these equations, we obtain

$$Res(z = \infty) + \sum R = 0$$

This completes the proof that the sum of the residues of f(z) at all singularities in the finite plane, including the singularity at infinity, is zero.

SOLVED EXAMPLE

EXAMPLE1: Evaluate the residue of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at 1,2,3 and infinity and show that their sum is zero.

SOLUTION: Suppose the given residue is

$$f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

Residue at $z = 1$: To find the residue of $f(z)$ at $z = 1$, we first rewrite $f(z)$ as:

$$\lim_{z\to 1}(z-1)\, f(z) = \lim_{z\to 1}\frac{z^2}{(z-2)(z-3)}$$

$$= \frac{z^2}{(1-2)(1-3)} = \frac{1}{2}$$

Residue at $z = 2$: To find the residue of $f(z)$ at $z = 2$, we first rewrite $f(z)$ as:

$$\lim_{z\to 1}(z-2)\, f(z) = \lim_{z\to 1}\frac{z^2}{(z-1)(z-3)} = -4$$

Residue at $z = 3$:

$$\lim_{z\to 1}(z-3)\, f(z) = \lim_{z\to 1}\frac{z^2}{(z-1)(z-2)} = \frac{9}{2}$$

Residue at $z = +\infty$:

$$\lim_{z\to 1}(-z)\, f(z) = \lim_{z\to 1}\frac{z^z}{z(z-1)(z-2)(z-3)} = -1$$

Sum of residues$= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0.$

Therefore, the sum of residues of $f(z)$ at $z = 1,2,3,$ and infinity is indeed zero.

EXAMPLE2: Evaluate the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z \to \infty$.
SOLUTION: Suppose the given residue is

$$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$
$$= \left(1 - \frac{1}{z}\right)^{-1}\left(1 - \frac{2}{z}\right)^{-1}\left(1 - \frac{3}{z}\right)^{-1}$$
$$= \left(1 + \frac{1}{z} + \cdots\right)\left(1 + \frac{2}{z} + \cdots\right)\left(1 + \frac{3}{z} + \cdots\right)$$
$$= 1 + \frac{6}{z} + higher\ power\ of\ \frac{1}{z}$$

Thus,

$$Res(f, \infty) = -6$$

Therefore, the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z \to \infty$ is -6.

EXAMPLE3: Evaluate the residue of $f(z)$ where $f(z) = \frac{e^z}{z^2(z^2+9)}$ at $z = 0, -3i, +3i$.

SOLUTION: To find the residues of the function $f(z) = \frac{e^z}{z^2(z^2+9)}$at the given points $z = 0, z = -3i$, and $z = 3i$, follow these steps:

1. **Residue at $z = 0$:** The point $z = 0$ is a pole of order 2. To find the residue at a pole of order 2, use the formula:

$$Res(f, z_0) = \lim_{z\to z_0}\frac{d}{dz}[(z - z_0)f(z)]$$

Here, $z_0 = 0$ so we need

$$Res(f, z_0) = \lim_{z\to z_0}\frac{d}{dz}z^2\frac{e^z}{z^2(z^2+9)}$$

Simplify inside the limit:

$$z^2\frac{e^z}{z^2(z^2+9)} = \frac{e^z}{(z^2+9)}$$

Now, differentiate:

$$\frac{d}{dz}\frac{e^z}{(z^2+9)} = \frac{e^z(z^2+9) - e^z 2z}{(z^2+9)^2} = \frac{e^z(z^2+9-2z)}{(z^2+9)^2} = \frac{9}{81} = \frac{1}{9}$$

Evaluate at $z = 0$:

$$Res(f,0) = \frac{e^0(0^2+9-2.0)}{(0^2+9)^2} = \frac{9}{81} = \frac{1}{9}$$

2. **Residue at $\boldsymbol{z = 3i}$**: The point $z = 3i$ is a simple pole. To find the residue at a simple pole, use the formula:

$$Res(f,z_0) = \lim_{z\to z_0}\frac{d}{dz}[(z-z_0)f(z)]$$

Here, $z_0 = 3i$ so we need

$$Res(f,3i) = (z-3i)\frac{e^z}{z^2(z^2+9)}$$
$$(z^2+9) = (z-3i)(z+3i)$$
$$Res(f,3i) = \frac{e^z}{z^2(z+3i)}$$

Evaluate at $z = 3i$:

$$Res(f,3i) = \frac{e^{3i}}{(3i)^2(3i+3i)} = \frac{e^{3i}}{-54i}$$

3. **Residue at $\boldsymbol{z = -3i}$**:
Similarly, $z = -3i$ is a simple pole. To find the residue:

$$Res(f,z_0) = \lim_{z\to z_0}\frac{d}{dz}[(z-z_0)f(z)]$$

Here, $z_0 = -3i$ so we need

$$Res(f,-3i) = (z+3i)\frac{e^z}{z^2(z^2+9)}$$
$$(z^2+9) = (z-3i)(z+3i)$$
$$Res(f,3i) = \frac{e^z}{z^2(z-3i)}$$

Evaluate at $z = -3i$:

$$Res(f,-3i) = \frac{e^{-3i}}{(-3i)^2(-3i-3i)} = \frac{e^{3i}}{54i}$$

Hence

- Residue at $z = 0$: $\frac{1}{9}$
- Residue at $z = 3i$: $\frac{e^{3i}}{-54i}$
- Residue at $z = -3i$: $\frac{e^{3i}}{54i}$

EXAMPLE4: Using residue theorem, evaluate $\int_C \frac{e^z}{z(z-1)^2} dz$ where C is circle $|z| = 2$.

SOLUTION: To evaluate the integral $\int_C \frac{e^z}{z(z-1)^2} dz$ where C is circle $|z| = 2$. Using the residue theorem, we need to find the residues of the integrand inside the contour C.

The integrate $\frac{e^z}{z(z-1)^2}$ has singularities at $z = 0$ and $z = 1$.

- $z = 0$ is a simple pole.
- $z = 1$ is a pole of order 2.

Both poles are inside the contour $| z |= 2$.

Residue at $z = 0$:

For the simple pole at $z = 0$:

$$Res \frac{e^z}{z(z-1)^2}\ 0 = \lim_{z\to 0} \frac{e^z}{z(z-1)^2}\ z = \lim_{z\to 0} \frac{e^z}{(z-1)^2} = \frac{e^0}{(0-1)^0} = 1$$

Residue at $z = 1$:

For the pole of order 2 at $z = 1$:

$$Res \frac{e^z}{z(z-1)^2}, 1 = \lim_{z\to 1} \frac{d}{dz}(z-1)^2 \frac{e^z}{z(z-1)^2} = \lim_{z\to 0} \frac{d}{dz} \frac{e^z}{z}$$

Now, differentiate:

$$\frac{d}{dz}\frac{e^z}{z} = \frac{ze^z - e^z}{z^2}$$

Evaluate at $z = 1$:

$$Res \frac{e^z}{z(z-1)^2}, 1 = \frac{1e^1 - e^1}{1^2} = 0$$

The residue theorem states:

$$\int_C f(z)dz = 2\pi i\, Res(f, z_i)$$

Sum of residues inside C:

$$Res \frac{e^z}{z(z-1)^2}, 0 + Res \frac{e^z}{z(z-1)^2}, 1 = 0 + 1 = 1$$

Therefore, the integral is:

$$\int_C \frac{e^z}{z(z-1)^2}dz = 2\pi i \cdot 1 = 2\pi i$$

So, the value of the integral is $2\pi i$.

12.4 *COMPUTATION OF RESIDUE AT FINITE POLE:-*

1. **Residue of $f(z)$ at a simple pole $z = a$.**
   i. $Res\ (z = a) = \lim_{z\to a}(z-a)\, f(z)$
   ii. Let $f(z) = \frac{\phi(z)}{\psi(z)}$ have a simple pole at $z = a$, where $\psi(z) = (z-a)F(z)$ and $F(a) \neq 0$.
   Then, residue of $f(z)$ at $z = a$

$$= \lim_{z\to a}(z-a)f(z)$$

$$= \lim_{z\to a}(z-a)\frac{\phi(z)}{\psi(z)} \qquad \left[from\frac{0}{0}\right]$$

$$= \lim_{z\to a}\frac{(z-a)\left[\phi(a) + (z-a)\phi'^{(a)} + \frac{(z-a)^2}{2!}\phi''^{(a)} + \cdots\right]}{\psi(a) + (z-a)\psi'^{(a)} + \frac{(z-a)^2}{2!}\psi''^{(a)} + \cdots}$$

[by Taylor's theorem]

$$= \lim_{z\to a}\frac{\left[\phi(a) + (z-a)\phi'^{(a)} + \frac{(z-a)^2}{2!}\phi''^{(a)} + \cdots\right]}{\psi'(a) + \frac{(z-a)^2}{2!}\psi''^{(a)} + \cdots} = \frac{\phi(a)}{\psi'(a)}$$

[for $(a) = 0$]

Hence

$$Res\ (z = a) = \frac{\phi(a)}{\psi'(a)}$$

2. **Residue at a pole of order m:**

Theorem3: To prove that the residue of $\frac{\phi(z)}{(z-a)^m}$ at $z = a$ is $\frac{\phi^{(m-1)}(a)}{(m-1)!}$.

Proof: Let $\phi(z)$ be analytic at $z = a$, and let

$$f(z) = \frac{\phi(z)}{(z-a)^m} \qquad \ldots(1)$$

where $m \geq 1$. Then the residue of $f(z)$ at $z = a$ is given by:
Residue of $f(z)$ at $z = a$ is b_1, where b_1 is given by

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^m}dz$$

$$= \frac{1}{(m-1)!}.\frac{(m-1)!}{2\pi i}\int_C \frac{\phi(z)}{(z-a)^{m-1+1}}dz$$

$$= \frac{1}{(m-1)!}.\phi^{(m-1)}(a)$$

[by Cauchy's integral formula]

From (1), we obtain
$Res\ (z = a) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$ as $z \to a$.

Theorem4: Liouville's theorem. If a function is analytic at every point and finite at infinity, then it must be constant.

Proof: Let $f(z)$ be the given function. Let a and b be any two distinct points, then the only singularities of the function

$$F(z) = \frac{f(z)}{(z-a)(z-b)} \text{ are } z = a \text{ and } z = b,$$

and possibly at infinity. But

$$Res(z = \infty) = \lim_{z\to\infty} -zF(z)$$

$$Res(z = \infty) = \left[\lim_{z\to\infty}\frac{-z}{(z-a)(z-b)}\right] \times \left[\lim_{z\to\infty} f(z)\right] = 0$$

$$Res(z = \infty) = 0$$

Since the sum of all the residues is zero and so

$$Res(z = a) + Res(z = b) + Res(z = \infty) = 0$$

$$\lim_{z\to\infty}(z-a)F(z) + \lim_{z\to\infty}(z-b)F(z) + 0 = 0$$

$$\frac{f(a)}{(a-b)} + \frac{f(b)}{(b-a)} = 0 \ \ or \ \ f(a) = f(b)$$

where $f(z)$ is constant.

12.5 *WORKING RULE (FOR COMPUTING THE RESIDUE):-*

To compute the residue of a function at a given point, follow these steps:

1. **Identify the Type of Pole:**
   Determine if the point $z = a$ is a simple pole, a pole of higher order, or an essential singularity.
2. **Simple Pole (Pole of Order 1):**
   If $z = a$ is a simple pole, the residue can be found using the limit:

$$Res(z = a) = \lim_{z-a}(z - a)\, f(z)$$

3. **Pole of Order m:**
   If $z = a$ is a pole of order m, the residue can be computed using the formula:
   $$Res(f, a) = \frac{1}{(m-1)!}\lim_{z-a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)f(z)]$$
4. **Removable Singularity:**
   If $z = a$ is a removable singularity, the residue is zero because the function is analytic at that point after removing the singularity.
5. **Residue for Functions Expressed as Laurent Series:**
   If the function can be expressed as a Laurent series around $z = a$:
   $$\int_C f(z)dz = 2\pi i \sum_{r=1}^{n} Res(r = r_r)$$

SOLVED EXAMPLE

EXAMPLE5: Find the residue

$$\frac{z^3}{(z-1)^4(z-2)(z-3)} \text{ at } z = 1,2,3.$$

SOLUTION: Suppose

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$$

Now we take

$\phi(z) = \frac{z^3}{(z-2)(z-3)}$, then

$$f(z) = \frac{\phi(z)}{(z-1)^4}$$

$$Res(z = 1) = \frac{\phi^{(3)}(1)}{3!} \qquad \dots(1)$$

Breaking $\phi(z)$ into partial fractions

$$\phi(z) = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

$$\phi'(z) = z + 5 - \frac{8}{(z-2)^2} + \frac{27}{(z-3)^2}$$

$$\phi''(z) = -\frac{16}{(z-2)^3} + \frac{54}{(z-3)^3}$$

$$\phi^3(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\phi^3(1) = 48 - \frac{162}{16} = \frac{303}{8}$$

Now using in (1), we have

$$Res(z=1)=\frac{303}{8}=\frac{101}{16}$$

$$Res(z=2)=\lim_{z-2}(z-2)\,f(z)$$

$$Res(z=2)=\lim_{z-2}\frac{z^3}{(z-1)^4(z-3)}=\frac{8}{1\times(-1)}=-8$$

$$Res(z=3)=\lim_{z-2}(z-3)\,f(z)$$

$$Res(z=3)=\lim_{z-3}\frac{z^3}{(z-1)^4(z-2)}=\frac{27}{16\times(1)}=\frac{27}{16}$$

EXAMPLE6: Find the residue of $\frac{1}{(z^2+1)^3}$ at $z=i$.

SOLUTION: Let

$$f(z)=\frac{1}{(z^2+1)^3}=\frac{\phi(z)}{(z-i)^3}$$

Where $\phi(z)=\frac{1}{(z+i)^3}, \phi'(i)=\frac{-3}{(z+i)^4}, \phi''^{(z)}=\frac{12}{(z+i)^5}$

$$\phi''(i)=\frac{12}{(i+i)^5}=\frac{12}{(2i)^5}=\frac{3}{8i}$$

$$Res(z=i)=\frac{\phi''(i)}{2!}=\frac{3}{16!}$$

Hence $z=i$ is the pole of order 3.

EXAMPLE7: Find the residue of $\frac{z^3}{(z^2-1)}$ at $z=\infty$.

SOLUTION: Let

$$f(z)=\frac{z^3}{(z^2-1)}$$

Then

$$f(z)=\frac{z^3}{z^2}\left(1-\frac{1}{z^2}\right)^{-1}$$

$$=z\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\cdots\right]$$

$$f(z)=z+\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^5}+\cdots$$

$$Res(z=\infty)=-\left(coefficient\ of\,\frac{1}{z}\right)=-1=-1$$

EXAMPLE8: Find the residue of $f(z) = \frac{2z+1}{(z^2+a^2)^2}$ at $z = ia$.

SOLUTION: Let

$$f(z) = \frac{2z+1}{(z+ia)^2(z-ia)^2} = \frac{\phi(z)}{(z-ia)^2},$$

$$\phi(z) = \frac{1}{(z+ia)^2}$$

$f(z)$ has a pole of order 2 at $z = ia$

$$Res(z = ia) = \lim_{z\to ia} \frac{\phi'(z)}{1} = \lim_{z\to ia} \frac{-2}{(z+ia)^3} = -\frac{2}{(2ia)^3} = \frac{i}{4a^3}$$

Theorem5: If AB is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a| = r$ and if $\lim_{z\to a}(z-a)\,f(z) = k$ (constant), then $\lim_{r\to 0}\int_{AB} f(z)\,dz = i(\theta_1 - \theta_2)k$.

Proof: Let AB be the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|\,z-a\,| = r$, and suppose $\lim_{z\to a}(z-a)f(z) = k$, where k is a constant. We want to prove that:

$$\lim_{r\to 0}\int_{AB} f(z)\,dz = i(\theta_1 - \theta_2)k.$$

The arc AB on the circle $|\,z-a\,| = r$ can be parameterized as:

$$z(\theta) = a + re^{i\theta}, \qquad \theta_1 \le \theta \le \theta_2$$

The differential dz is:

$$dz = ire^{i\theta}d\theta$$

The integral over the arc AB is:

$$\int_{AB} f(z)\,dz = \int_{\theta_1}^{\theta_2} f(z(\theta))\ ire^{i\theta}d\theta$$

Substitute $z(\theta) = a + re^{i\theta}$ into $f(z)$:

$$= \int_{\theta_1}^{\theta_2} f(a + re^{i\theta}\,)\ ire^{i\theta}d\theta$$

$\lim_{z\to a}(z-a)\,f(z) = k$, substitute $z(\theta) = a + re^{i\theta}$and take the limit as $r \to 0$:

$$\lim_{r\to 0}\left(a + re^{i\theta} - a\right) f\left(a + re^{i\theta}\right) = \lim_{r\to 0} re^{i\theta} f\left(a + re^{i\theta}\right) = k$$

So,

$$\lim_{r\to 0} r f\left(a + re^{i\theta}\right) = \frac{k}{e^{i\theta}}$$

Substitute this into the integral:

$$= \lim_{r\to 0} \int_{\theta_1}^{\theta_2} \frac{k}{e^{i\theta}}\; ie^{i\theta} d\theta = \lim_{r\to 0} \int_{\theta_1}^{\theta_2} kid\theta$$

The r terms cancel out, and the integral simplifies to:

$$= \lim_{r\to 0} \int_{\theta_1}^{\theta_2} kid\theta$$

The integral is straightforward:

$$= \lim_{r\to 0} \int_{\theta_1}^{\theta_2} kid\theta = ki(\theta_2 - \theta_1)$$

Hence,

$$\lim_{r\to 0} \int_{AB} f(z)\, dz = i(\theta_1 - \theta_2)k$$

This completes the proof.

Theorem6: If AB is an arc $\alpha \le \theta \le \beta$ of the circle $|z| = R$ and if $\lim_{R\to\infty} zf(z) = k$(constant), then

$$\lim_{R\to 0} \int_{AB} f(z)\, dz = i(\beta - \alpha)k.$$

Proof: Let AB be the arc $\alpha \le \theta \le \beta$ of the circle $| z - a | = r$, and suppose $\lim_{z\to a}(z-a)f(z) = k$, where k is a constant. We want to prove that:

$$\lim_{R\to 0} \int_{AB} f(z)\, dz = i(\beta - \alpha)k.$$

The arc AB on the circle $| z - a | = r$ can be parameterized as:

$$z(\theta) = a + Re^{i\theta}, \qquad \alpha \le \theta \le \beta$$

The differential dz is:

$$dz = iRe^{i\theta} d\theta$$

The integral over the arc AB is:

$$\int_{AB} f(z)\, dz = \int_{\alpha}^{\beta} f(z(\theta))\; iRe^{i\theta} d\theta$$

Substitute $z(\theta) = a + Re^{i\theta}$ into $f(z)$:

$$= \int_{\theta_1}^{\theta_2} f(a + Re^{i\theta}\,)\; iRe^{i\theta} d\theta$$

$\lim_{z \to a}(z-a)\,f(z) = k$, substitute $z(\theta) = a + Re^{i\theta}$ and take the limit as $R \to 0$:

$$\lim_{R\to 0}\left(a + Re^{i\theta} - a\right) f\left(a + Re^{i\theta}\right) = \lim_{R\to 0} Re^{i\theta} f\left(a + Re^{i\theta}\right) = k$$

So,

$$\lim_{R\to 0} rf\left(a + Re^{i\theta}\right) = \frac{k}{e^{i\theta}}$$

Substitute this into the integral:

$$= \lim_{R\to 0}\int_{\alpha}^{\beta} \frac{k}{e^{i\theta}}\; ie^{i\theta} d\theta = \lim_{r\to 0}\int_{\alpha}^{\beta} kid\theta$$

The R terms cancel out, and the integral simplifies to:

$$= \lim_{r\to 0}\int_{\alpha}^{\beta} kid\theta$$

The integral is straightforward:

$$= \lim_{r\to 0}\int_{\alpha}^{\beta} kid\theta = ki(\beta - \alpha)$$

Hence,

$$\lim_{R\to 0}\int_{AB} f(z)\, dz = i(\beta - \alpha)k.$$

This completes the proof.

12.6 *JORDAN'S INEQUALITY:-*

Jordan's Inequality is an inequality that provides bounds on the sine function in terms of the angle θ. It is defined as follows:
For any angle θ in the interval

$$\frac{2}{\pi}\theta \leq sin\theta \leq \theta$$

Theorem7: If $f(z)$ is analytic except at finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to 0}\int_{\Gamma} e^{imz} f(z)\, dz = 0, m > 0$$

where Γ denotes the semi-circle $|z| = R, I(z) > 0$.

Proof: Given

- $f(z)$ is analytic except at a finite number of singularities.
- $f(z) \to 0$ uniformly as $|\, z \,| \to \infty$.

The semicircle Γ in the upper half-plane can be parametrized as:

$$z(\theta) = Re^{i\theta}, \qquad 0 \leq \theta \leq \pi$$

The differential dz is:

$$dz = iRe^{i\theta}d\theta$$

The integral over Γ becomes:

$$\int_{\Gamma} e^{imz} f(z)\, dz = \int_{0}^{\pi} e^{imRe^{i\theta}}\, f(Re^{i\theta}) iRe^{i\theta} d\theta$$

The exponential term e^{imz} for $z = Re^{i\theta}$ is:

$$e^{imz} = e^{imRe^{i\theta}} = e^{imR(cos\theta + isin\theta)} = e^{-imRsin\theta} e^{mRcos\theta}$$

Thus, the integral becomes:

$$\int_{0}^{\pi} e^{-imRsin\theta} e^{mRcos\theta}\, f\left(Re^{i\theta}\right).iRe^{i\theta} d\theta$$

Consider the modulus of the integral:

$$\left|\int_{\Gamma} e^{imz} f(z)\, dz\right| \leq \left|\int_{0}^{\pi} e^{-imRsin\theta} e^{mRcos\theta} f\left(Re^{i\theta}\right).iRe^{i\theta}\right| d\theta$$

Since $\left|e^{mRcos\theta}\right| = 1$ and $\left|e^{-mRsin\theta}\right| = e^{-mRsin\theta}$, this simplifies to:

$$\left|\int_{\Gamma} e^{imz} f(z)\, dz\right| \leq \int_{0}^{\pi} e^{-imRsin\theta}\, \left|f\left(Re^{i\theta}\right)\right| d\theta$$

Given that $f(z) \to 0$ uniformly as $\mid z \mid \to \infty$, for large R, $\left|f\left(Re^{i\theta}\right)\right|$becomes small uniformly for all θ.

Therefore, for large R, we have:

$$\left|\int_{\Gamma} e^{imz} f(z)\, dz\right| \leq R.\in.\int_{0}^{\pi} e^{-imRsin\theta}\, d\theta$$

where ϵ is a small constant because $f(z)$ is small for large R.

The integral $\int_{0}^{\pi} e^{-imRsin\theta}\, d\theta$ tends to zero as R increases because $e^{-imRsin\theta}$ decays rapidly except near $\theta = 0$ and $\theta = \pi$, where $sin\theta$ is small.

Therefore, the whole expression tends to zero:

$$\lim_{R\to 0} \int_{\Gamma} e^{imz} f(z)\, dz = 0$$

Thus, we have shown that:

$$\lim_{R\to 0} \int_{\Gamma} e^{imz} f(z)\, dz = 0, m > 0$$

12.7 *INTEGRATION ROUND THE UNIT CIRCLE:-*

We proceed to evaluate the integrals of the type

$$\int_{0}^{2\pi} f(sin\theta, cos\theta) d\theta$$

If we take $z = e^{i\theta}$, then the above takes the form

$$\int_{C} \phi(z)\, dz. \text{ For } \frac{z + z^{-1}}{2} = cos\theta, \frac{z - z^{-1}}{2} = sin\theta$$

Where C is the unit circle $|z| = 1$.

SOLVED EXAMPLE

EXAMPLE9: Evaluate $\int_0^{2\pi} \frac{d\theta}{1+acos\theta}, a^2 < 1$.

SOLUTION: Let $I = \int_0^{2\pi} \frac{d\theta}{1+acos\theta}, a^2 < 1$

Let C be a circle $|z| = 1$.

$$z = e^{i\theta}, dz = e^{i\theta} id\theta = izd\theta$$

$$d\theta = \frac{dz}{iz}, cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{1 + z^2}{2z}$$

$$I = \int_C \frac{1}{\left[1 + \frac{a(1+z^2)}{2z}\right]} \left(\frac{dz}{iz}\right) = \frac{2}{i} \int \frac{dz}{2z + a + az^2}$$

$$I = \frac{2}{ai} \int_C f(z) dz$$

where $$f(z) = \frac{1}{\left[z^2 + \frac{2z}{a} + 1\right]}$$

Now

$$z^2 + \frac{2z}{a} + 1 = 0 \quad or \quad az^2 + 2z + a = 0$$

or

$$z = \frac{-2 \pm \sqrt{4 - 4a^2}}{2a} = \frac{-1 \pm \sqrt{1 - a^2}}{a}$$

Take

$$\alpha = \frac{-1 + \sqrt{1 - a^2}}{a}, \beta = \frac{-1 - \sqrt{1 - a^2}}{a}$$

Than

$\alpha\beta = 1$. Evidently $|\alpha| < 1$. And $|\beta| > 1$.

$z = \alpha$ is a simple pole lying inside C.

$$Res(z = \alpha) = \lim_{z \to \alpha} (z - \alpha) f(z) = \lim_{z \to \alpha} (z - \alpha) \frac{1}{(z - \alpha)(z - \beta)}$$

$$= \lim_{z \to \alpha} \frac{1}{(z - \beta)} = \frac{1}{(\alpha - \beta)} = \frac{a}{2\sqrt{1 - a^2}}$$

$$\int_C f(z) dz = 2\pi i . Res(z = \alpha) = \frac{2\pi i a}{2\sqrt{1 - a^2}} = \frac{\pi i a}{\sqrt{1 - a^2}}$$

$$I = \frac{2}{ai} \int_C f(z) dz = \left(\frac{2}{ai}\right) \frac{\pi i a}{\sqrt{1 - a^2}} = \frac{2\pi}{\sqrt{1 - a^2}}$$

EXAMPLE10: Evaluate $\int_0^\pi \frac{ad\theta}{a^2+sin^2\theta}$, where $a > 0$.

SOLUTION: Given

$$I = \int_0^\pi \frac{ad\theta}{a^2 + sin^2\theta}$$

Then

$$I = \int_0^\pi \frac{2ad\theta}{2a^2 + 2sin^2\theta} = \int_0^\pi \frac{2ad\theta}{2a^2 + 1 - cos2\theta}$$

$$= \int_0^\pi \frac{adt}{2a^2 + 1 - cost}, \qquad substituting\ 2\theta = t$$

$$= \int_0^\pi \frac{adt}{2a^2 + 1 - \frac{1}{2}(e^{it} + e^{-it})}$$

Substituting $z = e^{it}$ so $dz = e^{it}dt$, we have

$$I = \int_C \frac{2a}{2(2a^2 + 1) - (z + z^{-1})} \frac{dz}{iz}$$

where c is unit circle $|z| = 1$

$$I = \frac{2a}{i} \int_C \frac{dz}{2(2a^2 + 1)z - z^2 - 1} = 2ai \int_C \frac{dz}{z^2 - 2(2a^2 + 1)z + 1}$$

$$I = \int_C f(z)dz \qquad \dots (1)$$

$$f(z) = \frac{1}{z^2 - 2(2a^2 + 1)z + 1}$$

The poles $f(z)$ are given as

$$z^2 - 2(2a^2 + 1)z + 1 = 0$$

$$z = \frac{2(2a^2 + 1) \pm \sqrt{4(2a^2 + 1)^2 - 4}}{2}$$

$$= (2a^2 + 1) \pm \sqrt{(2a^2 + 1)^2 - 1}$$

$$= (2a^2 + 1) \pm 2a\sqrt{a^2 + 1}$$

Now taking

$$\alpha = (2a^2 + 1) + 2a\sqrt{a^2 + 1}$$

$$\beta = (2a^2 + 1) - 2a\sqrt{a^2 + 1}$$

we obtain $z = \alpha, \beta$. Evidently $|\alpha| > 1$ and $|\beta| > 1$.

So $f(z)$ has only simple pole $z = \beta$ lying within C.

$$Res(z=\beta)=\lim_{z\to\alpha}(z-\beta)\,f(z)=\lim_{z\to\alpha}(z-\beta)\;\frac{1}{(z-\alpha)(z-\beta)}$$

$$=\frac{1}{(\beta-\alpha)}=\frac{1}{-4a\sqrt{a^2+1}}$$

By Cauchy's residue theorem

$$\int_C f(z)dz=2\pi i(sum\ of\ residue\ within\ C)=\frac{2\pi i}{-4a\sqrt{a^2+1}}$$

From (1), we obtain

$$I=\frac{2ia.\,2\pi i}{-4a\sqrt{a^2+1}}=\frac{\pi}{(a^2+1)^{1/2}}$$

EXAMPLE11: Evaluate the contour integration: $\int_0^{\pi}\left(\frac{1+2cos\theta}{5+4cos\theta}\right)d\theta.$

SOLUTION: Given

$$I=\int_0^{\pi}\left(\frac{1+2cos\theta}{5+4cos\theta}\right)d\theta$$

Then

$$I=\frac{1}{2}\int_0^{2\pi}\left(\frac{1+2cos\theta}{5+4cos\theta}\right)d\theta$$

Now we take the circle c as $|z=1|, z=e^{i\theta}, dz=e^{i\theta}, z=izd\theta$

$$I=\frac{1}{2}\int_C\frac{\left(1+2e^{i\theta}\right)}{5+2\left(z+\frac{1}{z}\right)}\left(\frac{dz}{iz}\right)=\frac{1}{2i}\int_C\frac{(1+2z)dz}{5z+2z^2+2}$$

$$=\frac{1}{4i}\int_C\frac{(1+2z)dz}{\frac{5z}{2}+z^2+1}=\frac{1}{4i}\int_C f(z)dz$$

where $f(z)=\frac{(1+2z)}{\frac{5z}{2}+z^2+1}$. So the poles are $\frac{5z}{2}+z^2+1=0.$

$$z^2+5z+1=0\ \Rightarrow z=-\frac{5\pm3}{4}=-2,-\frac{1}{2}=\alpha,\beta$$

$z=\alpha$ lies outside $|z|=2>1. z=\beta$ lies inside C.

$$Res(z=\beta)=\lim_{z\to\alpha}(z-\beta)\,f(z)=\lim_{z\to\alpha}(z-\beta)\;\frac{(1+2z)}{(z-\alpha)(z-\beta)}$$

$$=\lim_{z\to\alpha}\frac{(1+2z)}{(z-\alpha)}=\frac{1+(-1)}{(\beta-\alpha)}=\frac{0}{\beta-\alpha}=0.$$

$$\int_C f(z)dz=2\pi i.\ Res(z=\beta)=2\pi i.0=0$$

Hence

$$I = 0$$

EXAMPLE12: By the method of contour integration, prove that

$$\int_0^{2\pi} e^{cos\theta} . cos(sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}$$

where n is a positive integer.
OR
Prove that

$$\int_0^{2\pi} e^{cos\theta} . cos(n\theta - sin\theta)\, d\theta = \frac{2\pi}{n!}$$

SOLUTION: Now let $|z| = 1$ denote the circle C and

$$I = \int_0^{2\pi} e^{cos\theta} . cos(n\theta - sin\theta)\, d\theta$$

$$I = \int_0^{2\pi} e^{cos\theta}\, e^{i(sin\theta - n\theta)} d\theta$$

$$I = \int_0^{2\pi} \exp[cos\theta + i(sin\theta - n\theta)]\, d\theta = \int_0^{2\pi} \exp\left[\left(e^{i\theta} - in\theta\right)\right] d\theta$$

$$= \int_C exp(z)\, e^{-in\theta} \frac{dz}{iz}, \quad z = e^{i\theta}$$

$$I = \frac{1}{i} \int_C \frac{e^z}{z^{n+1}} dz = \frac{1}{i} \int_C f(z)\, dz$$

where $f(z) = \frac{e^z}{z^{n+1}}$ at $z = 0$ of order $n + 1$.

$$Res(z = 0) = \frac{1}{n!} \frac{d^n}{dz^n} [e^z]_{z=0} = \frac{1}{n!} . e^0 = \frac{1}{n!}$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i (sum\ of\ residues\ within\ C) = \frac{2\pi i}{n!}$$

From (1), we obtain

$$I = \frac{1}{i} \frac{2\pi i}{n!}$$

12.8 *EVALUATION OF INTEGRALS OF THE TYPE:-*

$\int_{-\infty}^{\infty} f(z) dz$ where the function $f(z)$ is s.t. no pole of f(z) lies on the real line, but pole lies in the upper half of z-plane. We evaluate the above integrals by considering them along a closed contour C consisting of

i. semi circle γ s.t. $|z| = R$ in the upper half plane.
ii. real axis from $-R$ to R.

The integral over the closed contour C is:

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(z)dz$$

Taking limits as $R \to \infty, \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(z)dz$

According to the Cauchy's Residue Theorem, the integral over the closed contour C is equal to $2\pi i$ times the sum of the residues of $f(z)$ inside C:

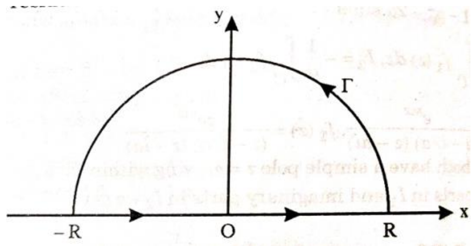

Fig.1.

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i(Sum\ of\ residues\ within\ C)$$

SOLVED EXAMPLE

EXAMPLE13: Prove that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$.

SOLUTION: Recall that the integral

$$\int_C f(z)dz, \qquad where\ f(z) = \frac{1}{1+z^2}$$

$$\int_C f(z)dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)dz$$

Where C is closed contour consisting of Γ. Let $f(z)$ has only one simple pole at $z = i$ inside C. So

$$Res(z = i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(sum\ of\ residues\ within\ C) = \frac{2\pi i}{2i} = \pi$$

$\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{1}{1+z^2} = 0$ by theorem6

$$\lim_{R\to\infty}\int_\Gamma f(z)dz = i(\pi - 0)(0) = 0$$

From $R \to \infty$ in (1), we obtain

$$\pi = \int_{-\infty}^{\infty} f(x)\,dx + 0 \; or \; \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_{0}^{\infty} \frac{dx}{1+x^2} = \pi \; or \; \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

EXAMPLE14: Prove that the contour integration $\int_0^\infty \frac{cosmx}{x^2+a^2}dx = \frac{\pi e^{-ma}}{2a}$ where $m \geq 0, a \geq 0$.

SOLUTION: First, express the cosine function using Euler's formula:

$$cosmx = \frac{e^{imx} + e^{-imx}}{2}$$

Thus, the integral becomes:

$$\int_0^\infty \frac{cosmx}{x^2+a^2}dx = \frac{1}{2}\int_0^\infty \frac{e^{imx}}{x^2+a^2}dx + \int_0^\infty \frac{e^{-imx}}{x^2+a^2}dx$$

Consider the complex function:

$$f(z) = \frac{e^{imx}}{z^2+a^2}$$

The poles of $f(z) = \frac{e^{imx}}{x^2+a^2}$ are at $z = \pm ia$. $f(z)$ has only simple one pole $z = ia$ inside C.

$$Res(z = ia) = \lim_{z \to ia}(z - ia)f(z) = \lim_{z \to ia}(z - ia)\frac{e^{imx}}{(z+ia)(z-ia)}$$

$$Res(z = ia) = \frac{1}{2\pi i}e^{-ma}$$

$\lim_{|z| \to \infty} \frac{1}{x^2+a^2} = 0$. Hence by Jordan's Lemma.

$$\lim_{R \to \infty} \int_\Gamma \frac{e^{imx}}{z^2+a^2}dz = \lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(sum\ of\ residues\ with\ C) = \frac{2\pi i e^{-ma}}{2ia}$$

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{a}e^{-ma}$$

Now making $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{a}e^{-ma} \quad or \quad \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

Now equating real parts, we obtain

$$\int_{-\infty}^{\infty} \frac{cosmx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma} \quad ...(1)$$

$$\int_{0}^{\infty} \frac{cosx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma} \quad ...(2)$$

Deductions:

1. Taking $m = a = 1$, we get
$$\int_{0}^{\infty} \frac{cosx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-1} = \frac{\pi}{2e}$$
2. Taking $m = 1$ in (1), we obtain
$$\int_{0}^{\infty} \frac{cosx}{x^2 + 4} dx = \frac{\pi}{a} e^{-a}$$
3. Taking $a = 2, m = 1$ in (2) , we have
$$\int_{0}^{\infty} \frac{cosx}{x^2 + 4} dx = \frac{\pi}{4} e^{-2} = \frac{\pi}{4e^2}$$
4. Taking $m = 1$ in 92) ,we get
$$\int_{0}^{\infty} \frac{cosx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}$$

EXAMPLE15: Apply the method of calculus of residues to prove that

$$\int_{0}^{\infty} \frac{\log(1 + x^2)}{1 + x^2} dx = \pi log2$$

SOLUTION: Consider the complex function

$$f(z) = \frac{\log(z + i)}{1 + z^2}$$

We use a keyhole contour that consists of:

- A line segment along the real axis from $0 < \epsilon < 1$ and $R > 1$.
- A large semicircle $C_R$centered at the origin in the upper half-plane of radius R.
- A line segment along the real axis from $-R$ to $-\epsilon$.
- A small semicircle $C\epsilon$ around the origin in the upper half-plane.

The function $f(z)$ has branch points at $z = \pm i$. The branch cut is usually taken along the imaginary axis from $z = -i$ to $z = i$.

$f(z)$ has poles at $z = \pm i$ where $1 + z^2 = 0$

Let's calculate the residue at $z = i$:

$$Res(f(z), z = i) = \lim_{z \to i} (z - i) \frac{\log(z + i)}{1 + z^2}$$

Since $1 + z^2 = (i - z)(i + z)$, we simplify this to:

$$Res(f(z), z = i) = \frac{\log(2i)}{2i} = \frac{\log(2e^{i\pi/2})}{2i} = \frac{\log 2 + (i\pi/2)}{2i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i(Res(z = i)) = \frac{2\pi i}{2ia}[\log 2 + (i\pi/2)]$$

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = [\log 2 + (i\pi/2)]$$

Now let $R \to 0$ so

$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{1+x^2} dx = \pi[\log 2 + (i\pi/2)]$$

Equating real parts from both sides, we get

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x+i)}{1+x^2} dx = \pi[\log 2]$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x+i)}{1+x^2} dx = \int_0^{\infty} \frac{\log(x+i)}{1+x^2} dx = \pi[\log 2]$$

EXAMPLE16: Show that $\int_0^\infty \frac{xsinax}{x^2+k^2} dx = \frac{\pi}{2} e^{-ak}, (where\ a > 0, k > 0)$.

SOLUTION: We begin by considering the complex function:

$$f(z) = \frac{ze^{iaz}}{z^2 + k^2}$$

where z is a complex variable. This function has poles at $z = \pm ik$.

To evaluate the real integral, consider integrating $f(z)$ over a contour that consists of:

A line segment along the real axis from $-R$ to R.

A semicircle Γ_R in the upper half-plane $(radius\ R)$.

We close the contour in the upper half-plane because e^{iaz} decays rapidly as $Im(z)$ increases when $a > 0$.

The only pole of $f(z)$ in the upper half-plane is at $z = ik$. The residue of $f(z)$ at $z = ik$ is:

$$Res(f(z), z = ik) = \lim_{z \to ik} (z - ik) \frac{ze^{iaz}}{z^2 + k^2}$$

Since $z^2 + k^2 = (z - ik)(z + ik)$, the residue simplifies to:

$$Res(f(z), z = ik) = \frac{ike^{-ak}}{2ik} = \frac{e^{-ak}}{2}$$

By the residue theorem, the contour integral around the closed contour C is:

$$\int_C f(z)dz = 2\pi i \times \frac{e^{-ak}}{2} = \pi i e^{-ak}$$

$$\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \pi i e^{-ak}$$

Now → ∞ , we obtain

$$\int_{-\infty}^{R} f(x)dx = \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2 + k^2} dx = \pi ie^{-ak}$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{xsinax}{x^2 + k^2} dx = 2\int_{0}^{\infty} \frac{xsinax}{x^2 + k^2} dx = \pi e^{-ak}$$

$$\int_{0}^{\infty} \frac{xsinax}{x^2 + k^2} dx = \frac{1}{2} \pi e^{-ak} \qquad \dots (1)$$

Deduction:

1. Substituting $k = 1$, in (1), we get
$$\int_{0}^{\infty} \frac{xsinax}{x^2 + k^2} dx = \frac{1}{2} \pi e^{-a}$$
2. Substituting $k = 1$, in (1), we get
$$\int_{0}^{\infty} \frac{xsinx}{x^2 + k^2} dx = \frac{1}{2} \pi e^{-k}$$
3. Substituting $a = k = 1$, in (1), we get
$$\int_{0}^{\infty} \frac{xsinx}{x^2 + 1} dx = \frac{1}{2} \pi e^{-1} = \frac{\pi}{2e}$$

SELF CHECK QUESTIONS

1. What is a residue?
2. How do you find the residue at a simple pole?
3. What are the conditions for applying the Residue Theorem?
4. If a function $f(z)$ has no singularities inside the contour C, what is the value of the contour integral?
5. How does the Residue Theorem simplify the evaluation of integrals?
6. What happens if the contour C encloses multiple singularities?

12.9 *SUMMARY:-*

In this unit we have studied the Residue Theorem in complex analysis states that if a function is analytic inside and on a closed contour, except for a finite number of isolated singularities, the contour integral of the function around the contour is equal to 2πi2πi times the sum of the residues of the function at those singularities. This theorem provides a powerful tool for evaluating complex integrals by reducing the problem to calculating residues at the singular points within the contour.

12.10 *GLOSSARY:-*

- **Residue Theorem**: A fundamental theorem in complex analysis stating that the contour integral of a function around a closed curve can be calculated as 2πi times the sum of the residues at the singularities enclosed by the contour.

- **Residue**: The coefficient of the $1/z - z_k$ term in the Laurent series expansion of a function around a singularity z_k, representing the contribution of the singularity to the contour integral.
- **Contour**: A closed, oriented path in the complex plane typically used to describe the boundary around which the integral is taken.
- **Singularity**: A point where a complex function is not analytic, which may be a pole, essential singularity, or other types of non-removable discontinuity.
- **Laurent Series**: A representation of a complex function as a power series that includes terms of negative powers, used to describe functions near singularities.
- **Pole**: A type of singularity where the function approaches infinity as the variable approaches the pole; classified by the order of the pole, indicating how quickly the function diverges.
- **Analytic Function**: A function that is differentiable at every point in a given domain, implying it is smooth and has a well-defined derivative.

12.11 *REFERENCES:-*

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12.12 *SUGGESTED READING:-*

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- Fischer, W. (2012). Complex Analysis: A Brief Introduction (1st ed.). World Scientific.
- Mathews, J. H., & Howell, R. W. (2012). Complex Analysis for Mathematics and Engineering (6th ed.). Jones & Bartlett Learning.
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12.13 *TERMINAL QUESTIONS:-*

(TQ-1) Using residue theorem, evaluate $\int_C \frac{e^z dz}{z(z-1)^2}$ with C is circle $|z| = 2$.

(TQ-2) Prove that

$$\lim_{z\to\infty} -zf(z) = Res(z = \infty)$$

Provided f(z) is analytic at $z = \infty$.

(TQ-3) Determine the order of the pole and values of residues of the function

i. $cosecz$

ii. $\frac{z+3}{z^2-2z}$

(TQ-4) If $\phi(z)$ and $\psi(z)$ are two regular functions and $z = a$ is onced repeated root of $\psi(z) = 0$ and $\phi(a) \neq 0$, then prove that the residue $\frac{\phi(z)}{\psi(z)}$ at $z = a$ is

$$\frac{6\phi'(a)\psi''^{(a)} - 2\phi(a)\,\psi'''(a)}{3[\psi'''(a)]^2}$$

(TQ-5) Prove that $\int_0^{2\pi} \frac{cos^2 3\theta d\theta}{1-2pcos2\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p}, 0 < p < 1$

(TQ-6) Evaluate$\int_0^{\pi} \frac{d\theta}{2+sin^2\theta}$.

(TQ-7) Prove that $\int_0^{2\pi} \frac{(1+2cos\theta)^2 cosn\theta d\theta}{3+2cos\theta}$ n being positive integer.

(TQ-8) Prove that $\int_0^{2\pi} \frac{sin^2\theta d\theta}{a+bcos\theta} = \frac{2\pi}{b^2}\left[a - \sqrt{(a^2 - b^2)}\right]$, where $a > b > 0$

(TQ-9) Prove that $\int_0^{2\pi} \frac{cos2\theta d\theta}{5+4cos\theta} = \frac{\pi}{6}$

(TQ-10) Evaluate $\int_0^{\pi} \frac{1+2cos\theta}{4+5cos\theta} d\theta$

(TQ-11) Prove that $\int_0^{2\pi} e^{-cos\theta} cos(n\theta + sin\theta) d\theta = \frac{2\pi(-1)^n}{n!}, n$ being positive integer.

(TQ-12) Prove that $\int_{-\pi}^{\pi} \frac{acos\theta}{a+cos\theta} d\theta \ = 2\pi a\left[1 - \frac{a}{\sqrt{a^2-1}}\right], a > 1$

(TQ-13) show that $\int_0^{\pi} tan(\theta + ia)\, d\theta = i\pi$, where $R(a) > 0$.

(TQ-14) Prove that if $a > 0$, $\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$.

(TQ-15) Prove by the contour integration $\int_0^{-\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$

(TQ-16) Show that $\int_0^{\infty} \frac{xsinaxdx}{x^2+a^4} = \frac{\pi}{2} e^{-ak}$, (where $a > 0, k > 0$).

(TQ-17) Prove that $\int_0^{\infty} \frac{logxdx}{(1+x^2)^2} = -\frac{\pi}{4}$.

(TQ-18) Prove that $\int_0^{\infty} \frac{sinmxdx}{x} = \frac{\pi}{2}$

(TQ-19) Show that $\int_0^{\infty} \frac{sin^2 mxdx}{x^2(x^2+a^2)} = \frac{\pi}{4a^3}(e^{-2ma} - 1 + 2ma)$where $m > 0, a > 0$

(TQ-20) Prove that if $0 < a < 1$, then

i. $\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \frac{\pi}{sin\pi a}$

ii. $\int_0^\infty \frac{x^{a-1}}{1-x} dx = \pi cot\pi a.$

12.14 *ANSWERS:-*

SELF CHECK ANSWERS

1. A residue is the coefficient of $1/z - z_k$ in the Laurent series expansion of a function around a singularity z_k.
2. For a simple pole at z_k, the residue can be found using:
$$Res(f, z_k) = \lim_{z \to z_k} z_k\,(z - z_k)f(z)$$
3. The function $f(z)$ must be analytic inside and on a simple, closed contour C, except at a finite number of isolated singularities inside C.
4. If there are no singularities inside C, the contour integral is zero.
5. Answer: Instead of directly evaluating the contour integral, which may be complicated, the Residue Theorem allows the integral to be computed by summing the residues of the function at the singularities inside the contour and multiplying by $2\pi i$.
6. The integral is calculated by summing the residues at all the enclosed singularities and multiplying the result by $2\pi i$.

TERMINAL ANSWERS

(TQ-1) $2\pi i$

(TQ-3) i. $Res(z = 0) = 1$ ii. $Res(z = 0) = -\frac{3}{2},\ Res(z = 2) = \frac{5}{2}$

(TQ-6) $\frac{\pi}{\sqrt{6}}$ **(TQ-10)** $\frac{4\pi}{5}$

UNIT 13:- Argument Principle and Rouche's Theorem

CONTENTS:

13.1 *INTRODUCTION:-*

The **Argument Principle** in complex analysis relates the change in the argument of a **meromorphic function** around a closed contour to the difference between the number of its zeros and poles inside the contour, quantified as $\frac{1}{2\pi}$ imes the total change in argument. **Rouché's Theorem** provides a method for determining the number of zeros of a function inside a contour by comparing it to another function, stating that if one function is dominated by another on the contour, they have the same number of zeros inside. Both theorems are essential for analyzing the distribution of zeros and poles of analytic functions in complex domains. The **Argument Principle** aims to determine the net number of zeros and poles of a meromorphic function inside a closed contour by analyzing the change in the argument of the function along the contour, providing a relationship between this change and the count of singularities within. **Rouché's Theorem** seeks to facilitate the counting of zeros of a function within a contour by comparing it to a simpler function, asserting that if the simpler function dominates the difference between the two functions on the contour, both functions will have the same number of zeros inside. Both principles are used to analyze and understand the behavior of complex functions and their zeros and poles in a given region.

13.2 *OBJECTIVES:-*

The objectives of the Argument Principle and Rouche's Theorem in complex analysis are to provide students with the tools to analyze and determine the number of zeros and poles of analytic functions within a contour. The Argument Principle aims to connect the change in the argument of a function along a contour to the number of zeros and poles inside, while Rouche's Theorem helps in comparing functions to establish when they have the same number of zeros within a given region. Mastery of these principles enables students to solve complex problems involving zero distributions, simplify functions for easier analysis, and understand the impact of function perturbations on zero locations.

13.3 *ARGUMENT PRINCIPLE:-*

The Argument Principle is indeed a powerful tool for determining the number of zeros and poles of a meromorphic function inside a given contour. Here's a summary of the relevant observations and theorem:

Meromorphic Functions: A function $f(z)$ is meromorphic if it is analytic except for isolated poles. Specifically:

- Analytic functions are a subset of meromorphic functions, where the function is analytic everywhere in its domain (i.e., no poles).
- Rational functions, defined as $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials, are meromorphic across the entire complex plane, with poles corresponding to the roots of $q(z)$.
- Meromorphic functions do not have essential singularities, only poles.

Argument Principle: If $f(z)$ is analytic on and inside a simple closed contour C and meromorphic in the region enclosed by C, then:

- $f(z)$ can have at most a finite number of zeros and poles inside C.
- The function $g(z) = \frac{1}{f(z)}$ is also meromorphic in this region, and thus $f(z)$ has a finite number of poles inside Cbecause $g(z)$ will have a finite number of zeros inside C.

The Argument Principle states that the difference between the number of zeros and poles of $f(z)$ inside C is given by:

$$\boldsymbol{Number\ of\ zeros - Number\ of\ poles = \frac{1}{2\pi}\Delta arg\ f(z)}$$

Where $\boldsymbol{\Delta arg\ f(z)}$ is the total change in the argument of $f(z)$ as z traverses C once. This principle provides a way to calculate the number of zeros and poles of $f(z)$ based on the behavior of $f(z)$ around the contour.

Theorem1:(Number of poles and zeros of a meromorphic function) Let $f(z)$ by analytic inside and on a simple closed curve C except for a finite number of poles inside C, and let $f(z) \neq 0$ on C. Prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$$

where N and P are respectively the number of zeros and the number of poles of $f(z)$ inside C. A pole of zero of order n is counted n times.

Proof: Let $f(z)$ be analytic inside and on a simple closed contour C except for a finite number of poles inside C. Assume $f(z) \neq 0$ on C. To use the Argument Principle, we decompose C into smaller contours around each pole and zero of $f(z)$. Also suppose that $f(z)$ has a zero of order n at $z = b$ inside C. Then we prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$$

Specially, Let γ_1 and Γ_1 be non-overlapping circles inside C with their centres at $z = a$ and $z = b$ respectively. The by the corollary of Cauchy's Theorem,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f'(z)}{f(z)}dz \qquad ...(1)$$

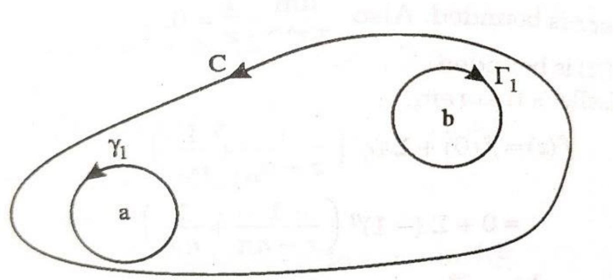

Fig.1.

Suppose $f(z)$ has a zero of order p at $z = a$, $f(z)$ can be written as:

$$f(z) = \frac{g(z)}{(z-a)^p} \quad \ldots(2)$$

where $g(z)$ is analytic and non zero within and γ_1.

Taking both side logarithmic of $f(z)$, we obtain

$$log f(z) = log\ g(z) + p\ log(z-a)$$

Now differentiating, w.r.t.to z,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - p\ log(z-a)$$

$$\int_{\gamma_1} \frac{f'(z)}{f(z)} dz = \int_{\gamma_1} \frac{g'(z)}{g(z)} dz - p \int_{\gamma_1} \frac{1}{(z-a)} dz$$

$$= \int_{\gamma_1} \frac{g'(z)}{g(z)} dz - 2\pi i p \quad \ldots(3)$$

Since $g(z)$ is analytic and $g'(z)$ is also analytic, so $g'(z)/g(z)$ is analytic and within on γ_1. Hence by Cauchy's theorem , we get

$$\int_{\gamma_1} \frac{g'(z)}{g(z)} dz = 0$$

So

$$\int_{\gamma_1} \frac{f'(z)}{f(z)} dz = 0 - 2\pi i p \quad \ldots(4)$$

Suppose $f(z)$ has a zero of order p at $z = b$, $f(z)$ can be written as:

$$f(z) = (z-b)^n \phi(z) \quad \ldots(5)$$

Where $\phi(z)$ is analytic and $\phi'(z)/\phi(z)$ within on Γ_1. Hence by Cauchy's theorem

$$\int_{\Gamma_1} \frac{\phi'(z)}{\phi(z)} dz = 0 \quad \ldots(6)$$

Taking both side logarithmic of $f(z)$, we obtain

$$\log f(z) = n \log(z-b) + log\phi(z)$$

Differentiating this, we obtain

$$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

Integrating along Γ_1 and the integral $\frac{\phi\prime(z)}{\phi(z)}$ around Γ_1 is zeros by Cauchy's theorem because $\frac{\phi\prime(z)}{\phi(z)}$ is analytic on Γ_1 , from (6), we get

$$\int_{\Gamma_1} \frac{f'(z)}{f(z)} dz = n \int_{\Gamma_1} \frac{dz}{(z-b)} dz = 2\pi i n$$

$$\int_{\Gamma_1} \frac{f'(z)}{f(z)} dz = 2\pi i n \qquad \ldots (7)$$

Writing (1) with the help of (4) and (7), we obtain

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -p + n \qquad \ldots .(8)$$

Now we suppose that $f(z)$ has the pole of order p_m at $z = a_m$ for $m = 1,2 \ldots \ldots r$ and n_m at $z = a_m$ for $m = 1,2 \ldots \ldots s$ within C. Specifically, let $\gamma_1, \gamma_2, \ldots \ldots . \gamma_r$ and $\Gamma_1, \Gamma_2, \ldots \ldots . \Gamma_s$ be small circles around the poles of $f(z)$. By summing up the contributions from all the poles and zeros, from (8), we get

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -\sum_{m=1}^{r} p_m + \sum_{m=1}^{s} n_m$$

Taking

$$\sum_{m=1}^{r} p_m = P, \quad \sum_{m=1}^{s} n_m = N$$

Hence

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

Theorem2: (Principle of Argument): If $f(z)$ is analytic and on C, then

$$N = \frac{1}{2\pi} . \Delta_C \arg f(z)$$

Proof: By the Argument Principle, we have:

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P \qquad \ldots (1)$$

where $P = Number\ of\ pole\ inside\ C, N = number\ of\ zeros\ inside\ C.$
Suppose $f(z)$ has no poles inside C , then $P = 0$ and from (1) , we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N \qquad \text{... (2)}$$

We can relate this to the change in the logarithm of $f(z)$, the function $log f(z)$ (where log denotes the principal branch of the complex logarithm) has a change in its value around C given by:

$$\int_C \frac{f'(z)}{f(z)} dz = [\log f(z)]_C = \Delta_C \, log\, f(z) \qquad \text{... (3)}$$

From the equation (2) and (3), we obtain

$$2\pi i N = [\log f(z)]_C = \Delta_C \, log\, f(z) \qquad \text{... (4)}$$

where Δ_C is the variation of $\log f(z)$ and $f(z)$ as z moves once round C.

$$log\, f(z) = log|f(z)| + iarg\, f(z)$$

For

$$\log(x + iy) = \frac{1}{2}(x^2 + y^2) + itan^{-}(y/x)$$

$$\Delta_C \, log\, f(z) = log|f(z)| + iarg\, f(z)$$

But

$$\Delta_C \, log|f(z)| = 0 \; as \; log|f(z)| \; is\; single\; valued.$$

Hence

$$\Delta_C \, log\, f(z) = i.\Delta_C \, arg\, f(z)$$

From(2), we get

$$2\pi i N = i.\Delta_C \, arg\, f(z)$$

$$N = \frac{1}{2\pi}.\Delta_C \, arg\, f(z)$$

This completes the proof of the Principle of Argument.

SOLVED EXAMPLE

EXAMPLE1: If $f(z) = z^5 - 3iz^2 + 2zi - 1$, then $\int_C \frac{f'(z)}{f(z)} dz$, where C encloses zero of f(z)?

SOLUTION: Let given $f(z)$ has 5 zeros, So

$$N = 5$$

Put $P = 0$, by theorem (1), we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P = 5 - 0 = 5$$

$$\int_C \frac{f'(z)}{f(z)} dz = 10\pi i$$

EXAMPLE2: Evaluate the integral $\int_C \frac{f'(z)}{f(z)} dz$ when $f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3}$ and C is the circle $|z| = 3$, taken in the positive sense.

SOLUTION: The zeros of $f(z)$ occur where the numerator is zero, $i.e.$,

$$z^2 + 1 = 0 \;\Rightarrow z^2 = -1 \Rightarrow z = \pm i$$

So, $f(z)$ has zeros at $z = i$ and $z = -i$. Since the numerator is squared, both zeros are of order 2.

The poles of $f(z)$ occur where the denominator is zero, $i.e.$,

$$z^2 + 3z + 2 = 0$$

Solving the quadratic equation:

$$z^2 + 3z + 2 = (z+1)(z+2) = 0$$

gives poles at $z = -1$ and $z = -2$. Since the denominator is cubed, both poles are of order 3.

- The zeros $z = i$ and $z = -i$ have magnitudes $|\, i \,| = |\, -i \,| = 1$, so they are both inside the contour.
- The poles $z = -1$ and $z = -2$ have magnitudes $|\, -1 \,| = 1$ and $|\, -2 \,| = 2$, so they are also inside the contour.

Using the integral formula

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$$

where N is the number of zeros (counted with multiplicity) inside C and P is the number of poles (counted with multiplicity) inside C.

Zeros inside C:

$$z = i\ (order\ 2),$$
$$z = -i\ (order\ 2)$$

Thus,

$$N = 2 + 2 = 4$$

Poles inside C:

$$z = -1 (order\ 3)$$
$$z = -2 (order\ 3)$$

Thus,

$$P = 3 + 3 = 6$$

Now substitute the values into the formula:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P = 4 - 6 = -2$$

Thus, the integral evaluates to:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = -2$$

$$\int_C \frac{f'(z)}{f(z)}\,dz = -2 \times 2\pi i = -4\pi i$$

13.4 *ROUCHE'S THEOREM:-*

If $f(z)$ and $g(z)$ are analytic and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ both have the same number of zeros inside C.

Proof: Given $f(z)$ and $g(z)$ are analytic on and inside a simple closed curve $|g(z)| < |f(z)|$ on C.

To prove that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

(i). **Prove that neither f(z)nor f(z)+g(z) has zeros on C.**

Assume $f(z)$ has a zero at $z = a$ on C. Then $f(a) = 0$. But according to the given condition, $\mid g(a) \mid < \mid f(a) \mid = 0$, which implies $\mid g(a) \mid = 0$. Therefore, $f(a) + g(a) = 0 + 0 = 0$, leading to $\mid f(a) \mid = \mid g(a) \mid$. This contradicts the assumption that $\mid g(z) \mid < \mid f(z) \mid$ on
Now, assume $f(z) + g(z)$ has a zero at $z = a$ on C. Then $f(a) + g(a) = 0$, C. which implies $f(a) = -g(a)$ and $\mid f(a) \mid = \mid g(a) \mid$. Again, this contradicts the assumption $\mid g(z) \mid < \mid f(z) \mid$ on C.
Thus, neither $f(z)$ nor $f(z) + g(z)$ has zeros on C.
(ii). **Show that $N_1 = N_2$ where N_1 and N_2 are the number of zeros of $f(z)$ and $f(z) + g(z)$ inside C, respectively.**
Since f and $f + g$ are analytic inside and on C, they have no poles inside C.By the Argument Principle, the number of zeros N_1 of $f(z)$ inside C is given by:

$$\frac{1}{2\pi i}\int_C \frac{f'}{f} dz = N_1$$

Similarly, the number of zeros N_2 of $f + g$ inside C is given by:

$$\frac{1}{2\pi i}\int_C \frac{f' + g'}{f + g} dz = N_2$$

Subtracting these two integrals, we have

$$\frac{1}{2\pi i}\int_C \left(\frac{f'+g'}{f+g} - \frac{f'}{f}\right) dz = N_2 - N_1 \qquad \dots (1)$$

express $g/f = \phi$ so that $g = \phi f$, where $|g/f| < 1 = \phi < 1$

$$\frac{f' + g'}{f + g} - \frac{f'}{f} = \frac{\phi'}{1 + \phi}$$

Now using (1), we have

$$N_2 - N_1 = \frac{1}{2\pi i}\int_C \frac{\phi'}{1 + \phi}\, dz$$

$$N_2 - N_1 = \frac{1}{2\pi i}\int_C \phi'\,(1 + \phi)^{-1} dz \qquad \dots (2)$$

Since $|\phi| < 1$, we can expand $(1 + \phi)^{-1}$as a binomial series is possible and the binomial expansion thus obtained is uniformly convergent and hence term by term integration is permissible. Hence

$$\int_C \phi'\,(1 + \phi)^{-1} dz = \int_C \phi'\,(1 - \phi + \phi^2 - \phi^3 + \cdots) dz$$

$$= \int_C \phi'\, dz - \int_C \phi'\phi\, dz + \int_C \phi'\phi^2\, dz - \int_C \phi'\,\phi^3 dz + \cdots \qquad \dots (3)$$

The functions f and g both are analytic within and on C and $g(z) \neq 0$ for any point on C. Hence $\frac{g}{f} = \phi$ is analytic and non-zero for any point of C. Therefore ϕ and its all derivatives are analytic. By Cauchy's integral theorem, each integral on R.H.S. of (3). Consequently

$$\int_C \phi'\,(1 + \phi)^{-1} dz = 0$$

From (2), we get

$$N_2 - N_1 = 0 \; or \; N_2 = N_1$$

Hence the number of zeros of $f(z)$ and $f(z) + g(z)$ inside C is the same, i.e., $N_2 = N_1$.

13.5 *FUNDAMENTAL THEOREM OF ALGEBRA:-*

Every polynomial degree of n has exactly n zeros.

Or

Prove that the polynomial equation.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, a_n \neq 0, n \geq 1$$

has exactly n roots.

Proof: Given polynomial is

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, a_n \neq 0$$

we want to show that it has exactly n roots.

Now we take

$$f(z) = a_n z^n, g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$$

Let C be the circle $|z| = r$ where $r > 1$.

$$|g(z)| \leq |a_0| + |a_1| r + |a_2| r^2 + \cdots + |a_{n-1}| r^{n-1}$$
$$\leq |a_0| r^{n-1} + |a_1| r^{n-1} + |a_2| r^{n-1} + \cdots + |a_{n-1}| r^{n-1}$$
$$= (|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|) r^{n-1}$$

But $\quad |f(z)| = |a_n r^n| = |a_n| r^n$

$$\therefore \quad \left|\frac{g(z)}{f(z)}\right| \leq \frac{(|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|) r^{n-1}}{|a_n| r^n}$$
$$= \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| r}$$

Now if $|g(z)| \leq |f(z)|$ so that $\left|\frac{g(z)}{f(z)}\right| < 1$, then

$$\frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| r} < 1$$

$\Rightarrow$

$$r > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| r}$$

As r becomes large, the terms involving r in the denominator tend to zero. Hence, for sufficiently large r, we have:

$$|g(z)| \leq |f(z)| r^n = |f(z)| \text{ on } C$$

By applying Rouche's Theorem, we find that the polynomial $P(z) = f(z) + g(z)$ has the same number of zeros inside the circle $|z| < r$ as the polynomial $f(z) = a_n z^n$. Since $f(z)$ is a polynomial of degree n, it has exactly n zeros (counting multiplicities) inside any sufficiently large circle. Consequently, the given polynomial $P(z)$ also has exactly n zeros in the complex plane.

SOLVED EXAMPLE

EXAMPLE3: Using Rouche's theorem determine the number of zeros of the polynomial $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ in $|z| < 1$.

SOLUTION: Let $P(z) = z^{10} - 6z^7 + 3z^3 + 1$

$$f(z) = -6z^7, g(z) = z^{10} + 3z^3 + 1$$

Then $$P(z) = f(z) + g(z)$$

Let the circle C defined $|z| = 1$.

Then $f(z)$ and $g(z)$ both are analytic within and upon C.

$$\left|\frac{g}{f}\right| = \left|\frac{z^{10} + 3z^3 + 1}{6z^7}\right| \leq \frac{|z^{10}| + 3|z^3| + 1}{6|\,z^7|}$$

$$= \frac{1^{10} + 3(1)^3 + 1}{6(1)^7} < 1$$

$$= \frac{5}{6} < 1$$

$$\left|\frac{g}{f}\right| < 1 \quad or \quad |g| < |f|$$

By Rouche's Theorem, $f(z) = -6z^7$ and $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ have the same number of zeros inside $|\,z\,| < 1$.

The function $f(z) = -6z^7$clearly has 7 zeros inside the unit circle (all at $z = 0$).

Hence the polynomial $P(z) = z^{10} - 6z^7 + 3z^3 + 1$has exactly 7 zeros inside the unit circle $|\,z\,| < 1$.

EXAMPLE4: Use Rouche's theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

SOLUTION: We are given the equation $z^5 + 15z + 1 = 0$ and need to demonstrate that it has one root in the disc $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$. using Rouche's theorem.

Let $|z| = 2$ represent the circle C_1. Then we have

$$z^5 + 15z + 1 = 0$$

Take $f(z) = z^5$ and $g(z) = 15z + 1$

Then

$$\left|\frac{g}{f}\right| = \left|\frac{15z + 1}{z^5}\right| = \frac{15|z| + 1}{|z|^5} = \frac{15.2 + 1}{2^5} = \frac{31}{32} < 1$$

Since $|\,g(z)\,| < |\,f(z)\,|$ on C_1 by Rouche's theorem, $P(z) = f(z) + g(z) = z^5 + 15z + 1$ and $f(z) = z^5$ have the same number of zeros inside $|\,z\,| < 2$. The polynomial $f(z) = z^5$has five zeros inside $|\,z\,| < 2$ (all at $z = 0$). Therefore, $z^5 + 15z + 1 = 0$ has five zeros inside $|\,z\,| = 2$.

Consider the circle C_2 defined by $|z| = 3/2$. We need to determine the number of zeros of $z^5 + 15z + 1$ inside $|z| < 3/2$

Take $f(z) = 15z, g(z) = z^5 + 1$.
Thus

$$\left|\frac{g}{f}\right| = \left|\frac{z^5 + 1}{15z}\right| \leq \frac{|z|^5 + 1}{15|z|} = \frac{\left(\frac{3}{2}\right)^5 + 1}{15\left(\frac{3}{2}\right)} = \frac{275}{720} < 1$$

Since $\mid g(z) \mid < \mid f(z) \mid$ on C_2, by Rouche's theorem, $PP(z) = f(z) + g(z) = z^5 + 15z + 1$ and $f(z) = 15z$ have the same number of zeros inside $\mid z \mid < 3/2$. The polynomial $f(z) = 15z$ has one zero at z=0. Therefore, $z^5 + 15z + 1 = 0$ has exactly one zero inside $\mid z \mid = 3/2$. Therefore, we conclude that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $\mid z \mid < 3/2$ and four roots in the annulus $3/2 < \mid z \mid < 2$.

SELF CHECK QUESTIONS

1. What does the Argument Principle state about the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$ for a function $f(z)$ analytic inside and on a contour C.
2. If $f(z) = z^2 - 1$ and C is the unit circle, what is the number of zeros of $f(z)$ inside C using the Argument Principle?
3. State Rouche's Theorem and how it helps in counting zeros of $f(z)$ and $g(z)$ inside a contour C.
4. If $f(z)$ is analytic on and inside C except for a simple pole at $z = a$, how does the Argument Principle account for this pole when counting zeros?

13.6 *SUMMARY:-*

In this unit we have studied the Argument Principle and Rouche's Theorem are fundamental results in complex analysis used to count the zeros of analytic functions within certain regions. The Argument Principle states that for an analytic function $f(z)$ inside and on a simple closed contour C, the change in the argument of $f(z)$ around C is 2π times the difference between the number of zeros and the number of poles inside C. Rouche's Theorem provides a method for comparing two analytic functions $f(z)$ and $g(z)$ on a contour C. If $\mid g(z) \mid < \mid f(z) \mid$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. Both theorems are powerful tools for determining the number of zeros of complex functions without explicitly solving for them.

13.7 *GLOSSARY:-*

- **Argument Principle:** A result in complex analysis that relates the number of zeros and poles of an analytic function inside a simple closed contour C to the integral of $f'(z)/f(z)$ around C. Specifically, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ equals the number of zeros minus the number of poles of $f(z)$ within C.

- **Rouche's Theorem:** A theorem in complex analysis that helps determine the number of zeros of a function inside a contour. If $f(z)$ and $g(z)$ are analytic on and inside a simple closed contour C, and if $| f(z) - g(z) | < | f(z) |$ on C, then $f(z)$ and $g(z)$ have the same number of zeros inside C.
- **Zeros:** Points where a function $f(z)$ is equal to zero. In the context of these theorems, zeros are counted with their multiplicities.
- **Poles:** Points where a function f(z) goes to infinity. Poles are also counted with their multiplicities when applying the Argument Principle.
- **Contour C:** A simple closed curve in the complex plane that is used to define the boundary of a region where properties of the function are analyzed.
- **Analytic Function:** A function f(z) that is differentiable at every point in its domain, meaning it can be expressed locally by a convergent power series.
- **Integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$:** The integral used in the Argument Principle to calculate the difference between the number of zeros and poles of a function $f(z)$ inside the contour C.
- **Comparison Function:** In Rouche's Theorem, a function $g(z)$ is compared with $f(z)$ to determine if they have the same number of zeros inside a contour C.

13.8 *REFERENCES:-*

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- Ahlfors, L. V. (2017). Complex Analysis. Harvard University Press (Updated Edition).
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13.9 *SUGGESTED READING:-*

- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
- file:///C:/Users/user/Desktop/1468563753EText(Ch-8,M-2%20(1).pdf

- L. L. Stewart (2015). Advanced Complex Analysis. Cambridge University Press.
- M. R. Douglas, K. A. Johnson, & M. D. McCarty (2011). Complex Analysis: An Introduction to the Theory. Wiley.

13.10 *TERMINAL QUESTIONS:-*

(TQ-1) Let $f(z)$ by analytic inside and on a simple closed curve C except for a finite number of poles inside $C,$ and let $f(z) \neq 0$ on $C.$ Prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P$$

where N and P are respectively the number of zeros and the number of poles of $f(z)$ inside $C.$ A pole of zero of order n is counted n times.

(TQ-2) If $f(z)$ is analytic and on C, then prove that

$$N = \frac{1}{2\pi}.\Delta_C \arg f(z)$$

(TQ-3) State and prove the principle of Argument.

(TQ-4) State and prove Rouche's Theorem.

Or

If $f(z)$ and $g(z)$ are analytic and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ both have the same number of zeros inside $C.$

(TQ-5) Prove that polynomial equation.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, a_n \neq 0, n \geq 1$$

has exactly n roots.

(TQ-6) Show that one root of the equation $z^4 + z + 1 = 0$ lies in the first quadrant.

(TQ-7) Show that roots of the equation $z^6 - 9z^2 + 11 = 0$ all lies in the circles $|z| = 1$ and $|z| = 3.$

(TQ-8) Prove that one root of the equation $z^4 + z^3 + 1 = 0$ lies in the positive quadrant.

(TQ-9) Show that the equation $z^4 + 2z^3 + 3z^2 + 4z + 5 = 0$ has no real purely imaginary roots and that it has one complex root in each quadrant.

(TQ-10) Show that the equation $e^{-z} = z - (1 + i)$ has one root in the first quadrant.

(TQ-11) State Rouche's theorem and use it to find the number of zeros of the polynomial $2z^4 - 2z^3 + 2z^2 + 2z + 11 = 0$ inside the circle $|z| = 1.$

(TQ-12) State Rouche's theorem and apply it to determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$, that lies inside the circle$|z| = 1$.

13.11 *ANSWERS:-*

SELF CHECK ANSWERS

1. The Argument Principle states that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$ equals the number of zeros minus the number of poles of $f(z)$ inside C.
2. The number of zeros inside C is $1\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = 2$, since $f(z) = z^2 - 1$ has 2 zeros inside the unit circle.
3. Rouche's Theorem states that if $| f(z) - g(z) | < | f(z) |$ on C, then $f(z)$ and $g(z)$ have the same number of zeros inside C.
4. The Argument Principle considers the residue at the pole, which affects the count of zeros. Specifically, the number of zeros is adjusted by subtracting the number of poles, including their orders, from the total integral result.

UNIT 14:- Uniqueness of analytic continuation

CONTENTS:

14.1 *INTRODUCTION:-*

Analytic continuation is a fundamental concept in complex analysis, where the idea is to extend the domain of an analytic function beyond its original region of definition. The uniqueness of analytic continuation asserts that if two analytic functions agree on a common domain, then they must agree everywhere on their domain of convergence. This property highlights the rigidity of analytic functions: once an analytic function is defined on a small region, its behavior on a much larger region is completely determined. The principle stems from the fact that analytic functions are highly constrained by their local behavior. Specifically, if two analytic functions are equal on a set that has an accumulation point within their domain of definition, they must be equal on the entire connected domain where both functions are analytic. This concept is crucial in various areas of mathematics and physics, as it ensures that the analytic continuation of a function is unique, making the extended function well-defined.

14.2 *OBJECTIVES:-*

The objectives of studying the uniqueness of analytic continuation are to understand the fundamental concept of extending an analytic function beyond its initial domain while ensuring that the extended function remains uniquely determined. This involves exploring the uniqueness theorem, which asserts that if two analytic functions agree on a common domain with an accumulation point, they must coincide throughout their entire domain of definition. Additionally, the objectives include applying this principle to solve complex analysis problems, recognizing the role of accumulation points, and connecting this concept to broader topics such as the identity theorem and holomorphic functions, all while working through illustrative examples to solidify understanding.

14.3 *ANALYTIC CONTINUATION:-*

Let $f_1(z)$ be a function that is analytic in a domain D_1. If there exists another function $f_2(z)$ that is analytic in a different domain D_2, and D_2 overlaps with D_1 in a region D_{12}, where $f_1(z) = f_2(z)$ for all z in $D12$, then $f_2(z)$ is called the analytic continuation of $f_1(z)$ from D_1 into D_2 via D_{12}.

Equivalently, $f_1(z)$ can be viewed as the analytic continuation of $f_2(z)$ from D_1 into D_2 via D_{12}.

Fig.2

For analytic continuation, it is sufficient that the domains $D_1$and D_2 have even a small arc in common, as long as the function values agree on that common part.

For example, consider an arc ABC that is shared by both D_1 and D_2. If a function $f_1(z)$ is analytic in D_1 and another function $f_2(z)$ is analytic in D_2, and if $f_1(z) = f_2(z)$ for all z on the $arc\ ABC,$ then $f_2(z)$ is the analytic continuation of $f_1(z)$ from D_1into D_2 via the $arc\ ABC$.

An Alternate definition: If $f(z)$ is analytic in a domain S_1 and if $f(z)$ is also analytic in a domain S_2 continuation of $f(z)$ in the domain S_2.

SOLVED EXAMPLE

EXAMPLE1: Let $f(z) = \sum_{n=0}^{\infty} z^n\ , \phi(z) = \frac{1}{1-z}$.

SOLUTION: The function $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic within the unit circle $|\ z\ | < 1$, where it equals the function $\phi(z) = \frac{1}{1-z}$. While $\varphi(z)$ is analytic at all points in the complex plane except at $z = 1$, where it has a simple pole, it matches $f(z)$ within $|\ z\ | < 1$. Therefore, $\varphi(z)$ provides the analytic continuation of $f(z)$ over the entire complex plane, except at the singularity $z = 1$.

EXAMPLE2: Let $f_1(z) = \sum_{n=0}^{\infty} z^n$, $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$.

SOLUTION: The first power series $f_1(z) = \sum_{n=0}^{\infty} z^n$ is convergent inside the circle R_1defined by $|\ z\ | = 1$ and has the sum $\frac{1}{1-z}$. The second power series $f_2(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$ is Geometric series with first term $\frac{1}{2}$ and a common ratio $\frac{1+z}{2}$ converging when $\left|\frac{1+z}{2}\right| < 1$ or equivalent $|z+1| < 2$. The sum of the second series is given by

$$f_2(z) = \frac{1}{2} . \frac{1}{1 - \frac{(1+z)}{2}} = \frac{1}{1-z}$$

Hence, $f_1(z)$ simplifies to the same function as $f_2(z), i.e., \frac{1}{1-z}$.

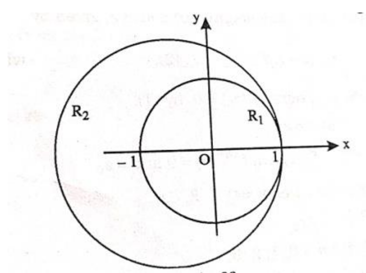

Fig.1

The function $f_1(z)$ is analytic inside the circle R_1 where $|\ z\ | = 1$, and $f_2(z)$ is analytic inside the circle R_2 where $|\ z + 1\ | = 2$. Both functions are equal in the region where R_1 and R_2 overlap, meaning $f_1(z) = f_2(z)$ in this common region. Therefore, $f_2(z)$ extends the domain of the analytic function $f_1(z)$ from R_1 to the larger domain R_2, making $f_2(z)$ the analytic continuation of $f_1(z)$ from R_1 in to R_2.

Definition: An analytic function f defined on a domain *D* is known as a function element, which is denoted by (f, D).

14.4 *COMPLETE ANALYTIC FUNCTION:-*

Suppose $f(z)$ is an analytic function within a domain D. By forming all possible analytic continuations of the pair (f, D), and subsequently all possible continuations of the resulting function elements $(f_1, D_1) \dots \dots .. (f_n, D_n)$, we eventually construct a function $F(z)$. This function $F(z)$ is defined such that for any point $v, F(v)$ represents the value obtained by the analytic continuation to v, with:

$$F(z) = \begin{cases} f_1(z) if\ z \in D_1 \\ f_1(z)\ if\ z \in D_2 \\ \dots\dots\dots\dots\dots\dots. \\ f_n(z)\ if\ z \in D_n \end{cases}$$

The function $F(z)$ constructed in this way is known as a **complete analytic function**. During the process of analytic continuation, it is possible to encounter a closed curve beyond which further continuation is not possible. This closed curve is referred to as the **natural boundary** of the complete analytic function. Points lying outside this natural boundary are considered **singularities** of the complete analytic function, where the function ceases to be analytic.

Theorem1: If $f(z)$ is analytic in a domain R and $f(z) = 0$ at all points on $arc\ PQ$ inside R, then $f(z) = 0$ throughout R.

Solution: Suppose $f(z)$ is analytic within a domain R. Let PQ be an arc inside RR such that f(z)=0 for all points z on the arc PQ. This is the given condition, s.t.

$$f(z) = 0 \forall z\ on\ PQ \qquad \dots (1)$$

We need to prove that $f(z) = 0$ throughout the entire domain R. Consider an arbitrary point z_0 on the arc PQ. Since $f(z)$ is analytic at z_0,we can describe a circle C centered at z_0 within which $f(z)$ can be expanded as Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \dots (2)$$

where $a_n = \frac{f^n(z_0)}{n!}$.

Since $z_0$lies on PQ, by the given condition $f(z_0) = 0$, by (1)

$\Rightarrow \qquad f(z) = 0$ at $z = z_0$

$\Rightarrow \qquad f(z), f'(z), f''(z), \dots, f^n(z) = 0$ at $z = z_0$

$\Rightarrow \qquad f^n(z_0) = 0$ when $n = 0,1,2,3, \dots$

Here $f^0(z_0) = f(z)$

$\Rightarrow \qquad a_n = 0$, for $n = 0,1,2,3, \dots$

Form(2), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} 0(z - z_0)^n = 0$$

Hence, $f(z) = 0$ for all points z inside the circle C.
Since $f(z) = 0$ inside the circle C centered at z_0, and z_0 was an arbitrary point on PQ, this argument can be extended across the entire domain R. By the identity theorem, since $f(z) = 0$ in an open set (the arc PQ and its interior), $f(z)$ must be identically zero throughout R.
Thus, $f(z) = 0$ throughout the domain R. This completes the proof of Theorem.

Theorem2: If a function $f(z)$ and all its derivatives vanish at a point a, then $f(z)$ and all its derivatives will vanish at all points in the domain of a.

Proof: By Taylor theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{f^n(z_0)}{n!}$. for $n = 0,1,2,3, \ldots$
By assumption, $a_0 = a_1 = a_2 = a_3 = \cdots = 0$.
Hence $f(z), f'(z), f''(z)$ all vanish at all points of the domain.
Thus, if a function $f(z)$ and all its derivatives vanish at a point a, then $f(z)$ and all its derivatives will vanish at all points in the domain containing a.

14.5 *UNIQUENESS OF ANALYTIC CONTINUATION:-*

The uniqueness of analytic continuation is a fundamental concept in complex analysis. It states that if two analytic functions agree on a non-discrete subset of a domain, they must be identical on the entire domain to which they can both be analytically continued.

Theorem3: There cannot be one more than one continuation of analytic continuation $f_2(z)$ in to the same domain.

Proof: Let $f_1(z)$ be analytic in a domain $D_1$and let $f_2(z)$ and $g_2(z)$ be two analytic continuations of $f_1(z)$ from D_1 into a domain D_2 .Assume via D_{12} which is common to both D_1 and D_2.
If we show that $f_2(z) = g_2(z)$ throughout D_2, the result will follow
By the definition of analytic continuation

i. $f_1(z) = f_2(z) \forall z \in D_{12}$ and $f_1(z)$ is analytic in D_2.

ii. $f_1(z) = g_2(z) \forall z \in D_{12}$ and $g_2(z)$ is analytic in D_2.

Now from (i) and (ii), we get

$$f_2(z) = f_1(z) = g_2(z) \forall z \in D_{12}$$
$$f_2(z) = g_2(z) \forall z \in D_{12}$$
$$(f_2 - g_2)(z) = 0 \forall z \in D_{12}$$

f_2 and g_2 are analytic in D_2.

Thus we see that $(f_2 - g_2)(z)$ vanishes in D_{12} which is a part of D_2. Also the function of analytic in D_2. Hence we must have

$$(f_2 - g_2)(z) = 0 \forall z \in D_2$$
$$f_2(z) = g_2(z) \forall z \in D_2$$

14.6 *SCHWARTZ'S REFLECTION PRINCIPLE:-*

Theorem5: Suppose that $f_1(z)$ is analytic in the region R_1 and that $f_1(z)$takes only real values on the parts LMN of the real axis. Then the Swartz's reflection principle states that the analytic continuation $f_2(z)$of $f_1(z)$into the domain R_2(considered as the mirror images of R_1 with LMN as mirror) is given by $f_2(z) = \overline{f_1(z)}$.

Proof: Suppose that $f_1(z)$is analytic in the region R_1 and that $f_1(z)$takes only real values on the parts LMN of the real axis so that

$$f_2(z) = \overline{f_1(z)} \text{ on the line LMN} \quad \text{….(1)}$$

Let R_2be the mirror image R_1 with LMN as mirror and let

$$f_2(z) = \overline{f_1(z)} \quad \text{…(2)}$$

To prove that $f_2(z)$ is analytic continuation of $f_1(z)$from R_1 into R_2 via the line LMN, it is enough to show that

i. $f_2(z) = g_2(z)$ on the line LMN
ii. $f_2(z)$ is analytic in R_2.

(i) . on the line LMN, $z = x, \bar{z} = x$ so that $z = \bar{z}$

Now from (1) , we get $f_2(z) = \overline{f_1(z)}$ on the line LMN

Using (2), we have $f_1(z) = f_2(z)$ on the line LMN

Hence prove result (i)

(ii). Let $f_1(z) = f_1(x + iy) = u_1(x, y) + iv_1(x, y)$ ... (3)

Analyticity of $f_1(z)$ in the domain R_1 implies that Cauchy-Riemann equations are satisfied

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y}, \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x} \quad \text{... (4)}$$

and Compute the partial derivatives are continuous so

From(3), we obtain

$$f_1(\bar{z}) = f_1(x - iy) = u_1(x, -y) + iv_1(x, -y)$$
$$\overline{f_1(z)} = f_2(z) = u_1(x, -y) + iv_1(x, -y)$$

Now $f_2(z)$ is analytic in R_2, we obtain

If

$$\frac{\partial u_1}{\partial x} = \frac{\partial(-v_1)}{\partial(-y)}, \frac{\partial u_1}{\partial(-y)} = -\frac{\partial(-v_1)}{\partial x}$$

Or equivalently if

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y}, \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}$$

which is true by (4),

For

$$\frac{\partial(-v_1)}{\partial(-y)} = \frac{\partial v_1}{\partial y}, \frac{\partial(-v_1)}{\partial x} = -\frac{\partial v_1}{\partial x}$$

These satisfy the Cauchy-Riemann equations, showing$f_2(z)$ is analytic in R_2.

SOLVED EXAMPLE

EXAMPLE3: If $\overline{f(z)} = f(z)$, then prove that $f(x)$ is real.

SOLUTION: Suppose $f_1(z)$ is an analytic function and satisfies:

$$f(\bar{z}) = f(z)$$

Let

$$f(z) = f_1(x + iy) = u(x, y) + iv(x, y)$$
$$\overline{f(z)} = f_1(x - iy) = u(x, -y) + iv(x, -y)$$

Taking conjugate of it and Let $f(z)$ be written as:

$$\overline{f(z)} = u_1(x, -y) + iv_1(x, -y)$$
$$\overline{f(z)} = f(z) = u(x, y) + iv(x, y)$$

So that

$$u(x, y) + iv(x, y) = u(x, -y) + iv(x, -y)$$

Substituting $y = 0$, we have

$$u(x, 0) + iv(x, 0) = u(x, 0) + iv(x, 0)$$
$$2iv(x, 0) = 0 \; or \; v(x, 0) = 0$$

Substituting $y = 0$ in $f(x + iy) = u(x, y) + iv(x, y)$

$$f(x) = u(x, 0) + iv(x, 0)$$
$$= u(x, 0) \text{ as } v(x, 0) = 0$$
$$f(x) = u(x, 0)$$

Hence $f(x)$ is real.

EXAMPLE4: Show that the power series $\sum_{n=0}^{\infty} z^{3n}$ cannot be continued analytically beyond the circle $|z| = 1$.

SOLUTION: Let

$$|u_n(z)|^{1/n} = |z^{3n}|^{1/n} = |z^3| = |z|^3$$

Hence the series is convergence is $|z| < 1$.

Circle of convergence is $|z| = 1$.

$$f(z) = \sum_{n=0}^{q} z^{3n} + \sum_{n=q+1}^{\infty} z^{3n} = f_1(z) + f_2(z)$$

Take a point p of affix $z = re^{\frac{2\pi pi}{3q}} (r > 1)$.

Now proceed as in Problem 2, $f(z) \to \infty as\ r \to 1$.

For $f_1(z) =$ finite quantity and $f_2(z) \to \infty$ as $r \to 1$.

EXAMPLE5: Show that the series

$$\frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \cdots += \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

and

$$\frac{1}{2-i} + \frac{z-i}{(2-i)^2} + \frac{(z-i)^2}{(2-i)^3} + \cdots = \sum_{n=0}^{\infty} \frac{(z-1)^n}{(2-i)^{n+1}}$$

are analytic continuation of each other.

SOLUTION: Consider the first series:

$$f_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

This is a geometric series where the first term $a = 1/2$ and the common ratio $r = \frac{z}{2}$, the sum of an infinite geometric series $\sum_{n=0}^{\infty} ar^n$is given by:

$$f_1(z) = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{z}{2}} = \frac{\frac{1}{2}}{\frac{2-z}{2}} = \frac{1}{2-z}$$

Consider the second series:

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{(2-i)^{n+1}}$$

This is a geometric series where the first term $a = \frac{1}{2-i}$ and the common ratio $r = \frac{z-i}{2-i}$. The sum of an infinite geometric series $\sum_{n=0}^{\infty} ar^n$ is given by:

$$f_2(z) = \frac{a}{1-r} = \frac{\frac{1}{2-i}}{1-\frac{z-i}{2-i}} = \frac{\frac{1}{2-i}}{\frac{2-z}{2-i}} = \frac{1}{2-z}$$

We found that:

$$f_1(z) = \frac{1}{2-z}$$

And

$$f_2(z) = \frac{1}{2-z}$$

Since both functions $f_1(z)$ and $f_2(z)$ are equal, the series $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{(z-1)^n}{(2-i)^{n+1}}$ are indeed analytic continuations of each other. This shows that they represent the same analytic function over their domains of convergence.

EXAMPLE6: Show that the function defined by

$$f_1(z) = \int_0^{\infty} t^3 e^{-zt}\, dt$$

is analytic at all points z for which $R(z) > 0$. Find also a function which is analytic continuation of $f_1(z)$ into the left hand place $R(z) < 0$, where $R(z)$ means the real part of z.

SOLUTION: Consider the integral:

$$f_1(z) = \int_0^{\infty} t^3 e^{-zt}\, dt$$

Integration by parts

$$f_1(z) = \left[t^3\left(\frac{e^{-zt}}{-z}\right) - 3t^2\left(\frac{e^{-zt}}{z^2}\right) + 6t\left(\frac{e^{-zt}}{-z^3}\right) - 6\left(\frac{e^{-zt}}{z^4}\right)\right]_{t=0}^{\infty}$$

$$f_1(z) = \frac{6}{z^4} \text{ if } R(z) > 0$$

Let $$f_2(z) = \frac{6}{z^4}$$

$$f_2(z) = f_1(z) \text{ for } R(z) > 0$$

Hence $f_1(z)$ is required analytic continuation of $f_2(z)$.

EXAMPLE7: Show that the circle of convergence of the power series $\sum_{n=0}^{\infty} z^n$ is a natural boundary for its sum function.

Or

Show that the function $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ can be obtained outside the circle of convergence of the power series.

SOLUTION: The given power series is:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

This is a geometric series with the first term $a = 1$ and the common ratio $r = z$. The sum S of an infinite geometric series is given by:

$$S = \frac{a}{1-r} = \frac{1}{1-z}$$

provided |z|<1. So, within the circle of radius 1 centered at the origin, the series converges to:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

The radius of convergence RR for the series $\sum_{n=0}^{\infty} z^n$ can be found using the formula:

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

where $a_n = z^n$. We have:

$$\frac{a_{n+1}}{a_n} = z$$

So

$$\frac{1}{R} = |z| \Rightarrow R = 1$$

Hence, the power series $\sum_{n=0}^{\infty} z^n$ converges for $|\, z \,| < 1$, which means the circle $|\, z \,| = 1$ is the boundary of convergence.

SELF CHECK QUESTIONS

1. What is the statement of the Uniqueness Theorem for analytic continuation?
2. Why does the existence of an accumulation point in the shared domain matter in the uniqueness theorem?
3. Explain the role of connectedness in the uniqueness theorem for analytic continuation.
4. What is the implication of the uniqueness theorem for the extension of an analytic function?
5. What are the practical implications of the uniqueness of analytic continuation in complex analysis?
6. How does the identity theorem relate to the uniqueness of analytic continuation?

14.7 *SUMMARY:-*

The uniqueness of analytic continuation is a fundamental principle in complex analysis, stating that if two analytic functions coincide on any non-discrete subset of their domain, then they must be identical on their entire domain. This means that an analytic function is uniquely determined by its values on any small region, as long as this region has an accumulation point within the domain. The principle underscores the rigidity of analytic functions, where local behavior completely dictates global behavior, allowing a function defined on a small part of its domain to be uniquely extended to a larger domain.

14.8 *GLOSSARY:-*

- **Analytic Function:** A complex function that is differentiable at every point in its domain. This implies that the function can be locally represented by a convergent power series.
- **Analytic Continuation:** The process of extending the domain of an analytic function beyond its original domain, while preserving the function's analyticity.
- **Domain:** The set of all points in the complex plane where a function is defined and analytic.
- **Non-Discrete Subset:** A subset of a domain that has an accumulation point, meaning that within any neighborhood of a point in this subset, there are infinitely many other points from the subset.
- **Accumulation Point:** A point in the complex plane where any neighborhood around it contains infinitely many points from a given set.
- **Uniqueness Theorem:** A principle in complex analysis stating that if two analytic functions coincide on a non-discrete subset of their domain, they must be identical throughout the entire domain.
- **Rigidity:** The concept that the behavior of analytic functions is strictly determined by their values in a small region, allowing no flexibility in how they can be extended.
- **Power Series:** A series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where anan are coefficients and z_0 is the center of the series. An analytic function can be expressed as a power series around any point in its domain.
- **Local Behavior:** The behavior or properties of a function in a small neighborhood around a specific point.
- **Global Behavior:** The overall behavior or properties of a function across its entire domain.
- **Identity Theorem:** A theorem in complex analysis that states if two analytic functions agree on an infinite set of points that has an accumulation point within their domain, then they must be identical on the entire domain.

- **Extension:** The process of enlarging the domain of a function while maintaining its original properties, particularly analyticity.
- **Holomorphic Function:** Another term for an analytic function, often used interchangeably in the context of complex analysis.
- **Isolated Singularity:** A point where a function is not analytic but is analytic in some punctured neighborhood around that point. Analytic continuation can sometimes resolve isolated singularities.

- **Analytic Continuation Along a Path:** A method of extending an analytic function by moving along a path in the complex plane, ensuring that the function remains analytic at each point.
- **Convergence Radius:** The radius within which a power series converges to the corresponding analytic function. The concept is crucial in understanding the limits and possibilities for analytic continuation.
- **Removable Singularity:** A point where a function is not initially analytic, but where it can be redefined so that the function becomes analytic. Analytic continuation can often "fill in" these points, ensuring the uniqueness of the function.
- **Schwarz Reflection Principle:** A method used in analytic continuation, especially when the original function is defined on a domain with a boundary. It reflects the function across the boundary, maintaining analyticity and ensuring uniqueness.

14.9 *REFERENCES:-*

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14.10 *SUGGESTED READING:-*

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- Goyal and Gupta (Twenty first edition 2010), Function of complex Variable.
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14.11 *TERMINAL QUESTIONS:-*

(TQ-1) Define analytic continuation.

(TQ-2) Define natural boundary.

(TQ-3) show that two function $f_1(z)$ and $f_2(z)$ are equal at all points of a line L in a region D in which they are holomorphic; the functions are equal at all points of D.

(TQ-4) Prove that the unit circle $|z| = 1$ is a natural boundary of the function $f(z) = \sum_{n=1}^{\infty} z^{n!}$

(TQ-5) If $f(z)$ is analytic in a domain R and $f(z) = 0$ at all points on $arc\ PQ$ inside R, thenprove that $f(z) = 0$ throughout R.

(TQ-6) Show that the function

$$f(z) = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \cdots$$

Can be continued analytically outside the circle of convergence.

(TQ-7): If a function $f(z)$ and all its derivatives vanish at a point a, then prove that $f(z)$ and all its derivatives will vanish at all points in the domain of a.

14.12 *ANSWERS:-*

SELF CHECK ANSWERS

1. The Uniqueness Theorem for analytic continuation states that if two analytic functions $f(z)$ and $g(z)$ are defined on a connected domain D and if there exists a subset $S \subset D$ containing an accumulation point where $f(z) = g(z)$ for all $z \in S$, then $f(z) = g(z)$ for all $z \in D$. This means that an analytic function is uniquely determined by its values on any subset that has an accumulation point within the domain.

2. The existence of an accumulation point ensures that the agreement of the two functions on the shared subset is not just coincidental but instead implies that the functions must be identical on the entire connected open set U.
3. Connectedness of the domain U is crucial because if the domain were not connected, the agreement of the functions on a subset might not necessarily extend to the entire domain. The theorem relies on the ability to "continue" the agreement throughout a single connected component.
4. The uniqueness theorem implies that an analytic function, if it can be analytically continued from a given domain to a larger domain, must be unique on that larger domain as long as the continuation is well-defined and the domain is connected.
5. The practical implications include the fact that once an analytic function is known in a small region, it is uniquely determined in the entire domain of definition, allowing for consistent extension and prediction of the function's behavior beyond the initially known region.
6. The identity theorem is a specific case of the uniqueness theorem. It states that if two analytic functions agree on a set that has a limit point within their domain of definition, then they must be identical on the entire connected component of the domain. This theorem underpins the uniqueness of analytic continuation.

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