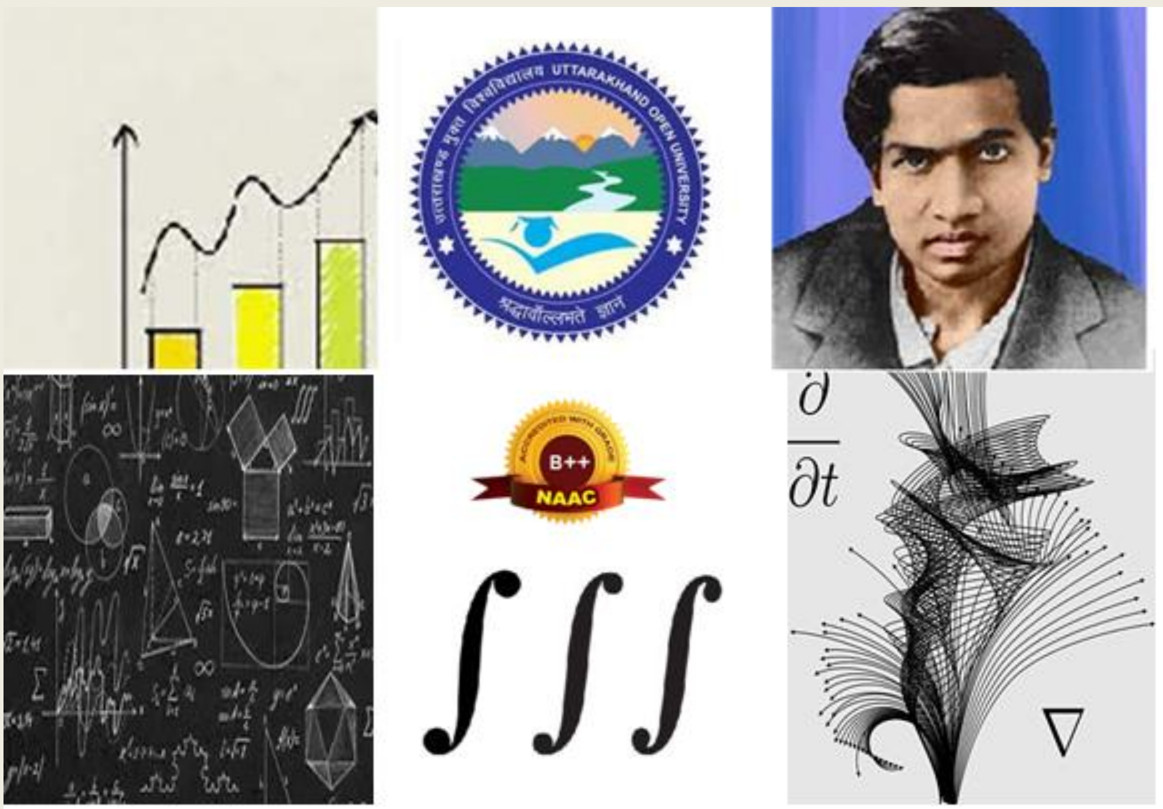
Master of Science
(FIRST SEMESTER)

MAT 504
ADVANCED DIFFERENTIAL EQUATIONS-I



DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY
HALDWANI, UTTARAKHAND
263139

COURSE NAME: ADVANCED DIFFERENTIAL EQUATIONS-I

COURSE CODE: MAT 504

Department of Mathematics
School of Science
Uttarakhand Open University
Haldwani, Uttarakhand, India, 263139

BOARD OF STUDIES-FEBRUARY 2023

Editor

Dr. Kamlesh Bisht
Assistant Professor (AC)
Department of Mathematics,
School of Sciences, Uttarakhand Open University

Unit Writer	Block	Unit
Dr. Jyoti Rani Assistant Professor Department of Mathematics Uttarakhand Open University	**I, III, IV**	**01 ,02,07,08,12, 13,14**
Dr. Richa Tewari Assistant Professor Department of Mathematics M.P.G.College Haldwani	**III**	**09,10,11**
Dr. Narendra Kumar Singh Assistant Professor Department of Mathematics M.P.G.College Haldwani	**II**	**03,05**

Dr. Sumit Pant
Assistant Professor
Department of Mathematics & Statistics CMS Banasthali Vidhyapith, Banasthali Tonk Rajasthan

	II	**04,06**

Course Title and Code	**: ADVANCED DIFFERENTIAL EQUATIONS- I (MAT-504)**
Copyright	**: Uttarakhand Open University**
Edition	**: 2023**

CONTENTS
MAT 504

COURSE INFORMATION

The present self learning material **"Advanced Differential Equations-I"** has been designed for M.Sc. (First Semester) learners of Uttarkhand Open University, Haldwani. This self learning material is writing for increase learner access to high-quality learning materials.This course is divided into 14 units of study. The first two units are devoted to Existence & Nature of Solution and Degree & Exactness of Differential Equation and Principle of Duality. Unit 3 and Unit 4 explained to a concept of Linear Differential Equation and Variation of Parameters. Unit 5 and Unit 6 are focussed on the topic Ordinary, Regular & Singular Points and Second Order Differential Equation. The aim of Unit 7 and 8 are to introduce the concept of Trajectories and Integral Curves and Damped Linear Oscillator. Unit 9 and Unit 10 explain the concept of Fundamental Existence Theorem and Differential Equations with Periodic Solution. Unit 11explain the Method of Bogoliubov & Krylov. Unit 12, 13 and 14 will explain the Chebyshev Polynomials and Legendre Polynomials, Bessel Functions and Hermite Polynomials and Leguerre Polynomials. This material also used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate numbers of illustrative examples and exercises have also been included to enable the leaner's to grasp the subject easily.

BLOCK I
EXISTENCE AND NATURE OF SOLUTIONS

UNIT 1:- EXISTENCE & NATURE OF SOLUTION

CONTENTS:

1.1 *INTRODUCTION:-*

In the previous class you have already studied about basics of differential equations. The concept of differential equations has a long history, with roots dating back to the 17th century. Many mathematicians contributed to the development of differential equations, including Isaac Newton, Gottfried Leibniz, Leonhard Euler, Joseph-Louis Lagrange, and Pierre-Simon Laplace. In particular, Newton and Leibniz are credited with the development of calculus, which provided the mathematical framework for differential equations. Newton also used differential equations to describe the motion of objects under the influence of gravity, which is now known as Newton's law of motion. However, it is difficult to attribute the invention of differential equations to a single person, as the concept has evolved over time with the contributions of many mathematicians.

A differential equation is an equation that involves an unknown function and one or more of its derivatives. It is used to describe the relationship between a function and their rates of change. There are two main types of differential equations: ordinary differential equations (ODEs) and partial differential equations (PDEs). An ODE involves derivatives of a single variable, while a PDE involves derivatives of multiple variables.
Differential equations are used in many areas of science and engineering, including physics, biology, economics, and finance. They are particularly useful for modeling dynamic systems, such as the motion of objects or the behavior of populations over time. Solving a differential equation involves finding a function that satisfies the equation. This can be done analytically or numerically, depending on the complexity of the equation and the desired level of accuracy. Analytical solutions involve finding a closed-form expression for the function, while numerical solutions involve approximating the function using numerical methods.

1.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To analyze and predict the behavior of these systems over time.
- To provide solutions to problems that cannot be solved using other mathematical techniques.
- To understand the definition of differential equation.

1.3 *DIFFERENTIAL EQUATION:-*

An Equation involving derivatives of differentials of one or more dependent variables with respect to one or more independent variables is called ***Differential Equations.***

For Example:

$$\frac{dy}{dx} = (x + sinx) \qquad \dots (1)$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \qquad \dots (2)$$

$$y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}} \qquad \ldots(3)$$

$$k(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2} \qquad \ldots(4)$$

$$\partial^2 v/\partial t^2 = k(\partial^3 v/\partial x^3)^2 \qquad \ldots(5)$$

and

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 + \partial^2 u/\partial x^2 = 0 \qquad \ldots(6)$$

1.4 *ORDINARY DIFFERENTIAL EQUATION:-*

A differential Equation (Art.1.3) given in (1), (2), (3) and (4) involve only one independent variable is called an ***Ordinary Differential Equations***

1.5 *PARTIAL DIFFERENTIAL EQUATION:-*

The equation (Art.1.3) given in (5) and (6) involve partial derivatives with respect to more than one independent variable is called a ***Partial Differential Equation***.

1.6 *ORDER OF A DIFFERENTIAL EQUATION:-*

The ***order of a differential equation*** is order of highest derivative differential equation.

In Art.(1.1) shown that the equation (2) is of 4^{th} order, equation (1) and (3) are of 1^{st} order, equations (4) and (6) are of the second order and equation (5) is of the third order.

1.7 *DEGREE OF A DIFFERENTIAL EQUATION:-*

The ***Degree of a differential equation*** is power of the height order derivative term in the differential equation.

In Art.(1.1) given the equation (1), (2) and (6) are of first degree. Making equation (3) free from fractions, we describe$y\, dy/dx = \sqrt{x}(dy/dx)^2 + k$, which is of 2^{nd} degree.

1.8 *LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION:-*

A differential equation is said to be Linear if

(i) Every dependent variable and every derivative in involved occurs in the first degree only.

(ii) No products of dependent variable and /or derivatives occur.

A differential equation which is not a linear is called the ***non-linear differential equation.***

For Example:

1.$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+9y=0$. is linear.

2. $\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^4+6y=3$. is non-linear because in 2nd term is not of degree one.

1.9 *SOLUTION OF DIFFERENTIAL EQUATION AND FAMILY OF CURVES:-*

Any relation between the dependent and independent variables, which when substituted in differential equation, reduces it to an identity is known as ***Solution of differential equation or integral of differential equation***. A solution of differential equation does not include the derivatives of the dependent variable with respect to the independent variable or variables.

FAMILY OF CURVES: An n-parameter family of curves is a set of relations of the form

$$\{(x,y): f(x,y,c_1,c_2,\dots,c_n)=0\},$$

Where f is real value function of $x,y,c_1,c_2,\dots,c_n$ and $c_i(i=1,2,\dots n)$ range over an interval of real values.

For Example: 1. Let $x^2+y^2=c$ is one parameter family if c takes all non-negative real values.

2. again we take the set of circles, obtained by $(x-c_1)^2+(x-c_2)^2=c_3$ is three parameter family if c_1 take all real values and c_3 takes non-negative real values.

1.10 *COMPLETE PRIMITIVE (GENERAL SOLUTION), PARTICULAR SOLUTION AND SINGULAR SOLUTIONS:-*

Definitions

Suppose $$F(x,y,y_1,y_2,\dots,y_n)=0 \qquad \dots(1)$$
be an nth order differential equation.

(i) A solution of equation (1) containing n independent arbitrary constants is called a general solution.

(ii) A solution of equation (1) obtained by giving particular values to the arbitrary constants in general solution is known as particular solution or integral solution.

(iii) A solution of equation (1) which cannot be described from any general solution of (1) by any n independent arbitrary constants is called singular solution.

For Example:

Suppose $$y = c_1 e^x + c_2 e^x \qquad \ldots (2)$$

is the general solution of $$y'' - 3y' + 2y = 0 \qquad \ldots (3)$$

Where c_1 and c_2 are independent arbitrary constants. Some particular solution of (3) are obtained by $y = e^x + 2e^x, \quad y = e^x - 2e^x$.

SOLVED EXAMPLES

EXAMPLE1: If $y = (A/x) + B$, then show that $(d^2y/dx^2) + (2/x) \times (dy/dx) = 0$.

SOLUTION: Given that

$$(d^2y/dx^2) + (2/x) \times (dy/dx) = 0 \qquad \ldots (1)$$

$$y = (A/x) + B \qquad \ldots (2)$$

Now differentiating equation (2) with respect to x,

$$dy/dx = -A/x^2 \qquad \ldots (3)$$

Again differentiating (3) with respect to x, $d^2y/dx^2 = (2A/x^3)$

Putting the value of dy/dx and d^2y/dx^2 in (1), we get

$(2A/x^3) + (2/x) \times -A/x^2 = 0$ or $0 = 0$

Hence eq. (2) is the solution of (1).

EXAMPLE 2: Find the differential equation of the family of curves$y = e^{mx}$, where m is arbitrary constant.

SOLUTION: Now given that the family of curves

$$y = e^{mx} \qquad \ldots (1)$$

Differentiating (1) w.r.t. x, we have

$$dy/dx = me^{mx} \qquad \ldots (2)$$

From (1) and (2) $dy/dx = my$

$$\Rightarrow \qquad m = (1/y) \times (dy/dx) \qquad \ldots (3)$$

$$my = dy/dx$$

$$my = me^{mx}$$

$$\log y = mx$$

So

$$m = \frac{\log y}{x} \qquad \ldots (4)$$

Eliminating m from (3) and (4)

$$(1/y) \times (dy/dx) = (1/x) \times \log y.$$

EXAMPLE 3: Find the differential equation satisfied by family of circles$x^2 + y^2 = a^2$, a being an arbitrary constant.

SOLUTION: Let us consider the equation of any circle passing through the origin and whose centre is on the x-axis is given by

$x^2 + y^2 + 2gx = 0$, where g being arbitrary constant. ... (1)

Differentiating (1) with respect to x, we have

$$2x + 2y\frac{dy}{dx} + 2g = 0 \qquad \text{... (2)}$$

From(1) $$2gx = -(x^2 + y^2)$$

$$2g = -\frac{(x^2 + y^2)}{x}$$

Now substituting the value of $2g$ in equation (2), we obtain

$$2x + 2y\frac{dy}{dx} - \frac{(x^2 + y^2)}{x} = 0.$$

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0.$$

EXAMPLE 4: Find the differential equation of the family of the curves$y = e^x(Acosx + Bsinx)$, where A and B are arbitrary constant.

SOLUTION: Let

$$y = e^x(Acosx + Bsinx) \qquad \text{... (1)}$$

Differentiating (1) $y' = e^x(-Asinx + Bcosx) + e^x(Acosx + Bsinx)$

$$y' = e^x(-Asinx + Bcosx) + y, \quad \text{from (1)} \qquad \text{... (2)}$$

Again Differentiating (2)

$$y'' = -e^x(Acosx + Bsinx) + e^x(-Asinx + Bcosx) + y' \quad \text{... (3)}$$

Now From (2), we have

$$e^x(-Asinx + Bcosx) = y' - y. \qquad \text{... (4)}$$

Hence eliminating the value of A and B from (1), (3) and (4), we have

$y'' = -y + y' - y + y'$ or $y'' - 2y' + 2y = 0$

1.11*WRONKSKIAN:-*

Definition: The Wronkskian of n functions $y_1(x), y_2(x), \ldots . y_n(x)$, is denoted by $W(x)$ or $W(y_1, y_2, \ldots . y_n)(x)$ and is defined to be the determinant

$$W(y_1, y_2, \ldots. y_n)(x) = W(x) = \begin{vmatrix} y_1 & y_2, & \ldots. y_n \\ y'_1, & y'_2, & \ldots. y'_n \\ \vdots & \vdots & \ldots. \quad \vdots \\ y_1^{(n-1)} & y_2^{(n-1)}, \ldots. & y_n^{(n-1)} \end{vmatrix}$$

1.12 *LINEARLY DEPENDENT AND INDEPENDENT SET OF SOLUTIONS:-*

Definition: the n function $y_1(x), y_2(x), \ldots. y_n(x)$ are linearly dependent if $\exists$ constants $c_1, c_2, \ldots. c_n$(not all zero), such that

$$c_1y_1 + c_2y_2 + \cdots\cdots + c_ny_n = 0 \qquad \ldots (1)$$

If, however, identity (1) implies that $c_1 = c_2 = \cdots = c_n = 0$,
Then $y_1, y_2, \ldots. y_n$ are said to be linearly independent.

1.13*EXISTANCE AND UNIQUENESS THEOREM:-*

Consider the second order differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x) \qquad \ldots (1)$$

Where $a_0(x), a_1(x), a_2(x)$ and $r(x)$ are continuous functions on an interval I and $a_0(x) \neq 0$ for each$x \in I$. Let c_1 and c_2 be arbitrary real numbers and $x_0 \in I$.Then $\exists$ a unique solution of (1) satisfying $y(x_0) = c_1$ and $y'(x_0) = c_2$. This solution $y(x)$ is described over the interval I.

Note1: The above theorem is an existence theorem because it says that the initial value problem does have a solution. It is also a uniqueness theorem, because it says that there is only one solution. Clearly, this theorem also applies to an associated homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

Note2: In this unit, we shall assume without proof, the above basic theorem for initial value problems associated with linear differential equations.

Note3: The conditions of existence and uniqueness theorem cannot be further relaxed. For example, if $a_0(x) = 0 \ \forall \ x \in I$, then the solution of (1) may not be unique or may not be exist at all.

Note4: Existence and uniqueness theorem can be extended to an nth order linear differential equation.

THEOREM I: State the existence and uniqueness theorem for nth order differential equation

$$L(y)(x) = y^{(n)} + p_1(x)y^{(n-1)}(x) \dots\dots . + p_n(x)y(x) = 0, x \in I,$$

which is a linear homogeneous equation.

SOLUTION: Statement of the existence and uniqueness theorem fo$L(y)(x) = y^{(n)} + p_1(x)y^{(n-1)}(x) \dots\dots . + p_n(x)y(x) = 0, x \in I$ ... (1)

Suppose $p_1, p_2 \dots p_n$ be obtained and continuous on an interval I which contains a point x_0.Let $a_0, a_1 \dots a_{n-1}$ be n constants. then $\exists$ a unique solution ϕ on $I$of (1) satisfying the conditions.

$$\phi(x_0) = a_0, \quad \phi'(x_0) = a_1, \dots , \quad \phi^{(n-1)}(x_0) = a_{n-1}$$

Let $\phi_1(x), \dots\dots \phi_n(x)$ are n solution of $L(y)(x) = 0$ given in (1) and suppose that $c_1, c_2 \dots c_n$ are n arbitrary constants. Since $L(\phi_1) = L(\phi_2) = \cdots\cdots = L(\phi_n) = 0$, and L is linear operator, we get

$$L(c_1\phi_1 + c_2\phi_2 + \cdots\cdots + c_n\phi_n) = c_1L(\phi_1) + \cdots\cdots\cdots + c_nL(\phi_n) = 0$$

$\therefore$ n solutions $\phi_1, \cdots\cdots\cdots, \phi_n$ are linearly independent and $c_1, c_2 \dots c_n$ are constants, then

$$c_1\phi_1 + c_2\phi_2 + \cdots\cdots + c_n\phi_n = 0, x \in I \Rightarrow c_1 = c_1 = \cdots c_n = 0.$$

THEOREM II: Show that there are three linearly independent solutions of the third order equation $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0, x \in I$ where p_1, p_2 and p_3 are functions, defined and continuous on an interval I.

SOLUTION: Let $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0, x \in I$ ... (1)

Using theorem I we conclude that $\exists$ solutions $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$ of (1) such that for $x_0 \in I$.

$$\left.\begin{array}{lll} \phi_1(x_0) = 0, & \phi'_1(x_0) = 0 & \phi_1''(x_0) = 0 \\ \phi_2(x_0) = 0, & \phi'_2(x_0) = 1 & \phi_2''(x_0) = 0 \\ \phi_3(x_0) = 0, & \phi'_3(x_0) = 1 & \phi_3''(x_0) = 0 \end{array}\right\} \quad \dots (2)$$

and we proceed to prove that ϕ_1, ϕ_2 and ϕ_3 are linearly independent. Let

$$c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) = 0, x \in I \quad \dots (3)$$

For some constants c_1, c_2 and c_3. At $x = x_0$, from (3), we get

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) + c_3\phi_3(x_0) = 0, x \in I \quad \dots (4)$$

Now differentiating (3) w.r.t. x and replacing x by x_0 yields

$$c_1\phi'_1(x_0) + c_2\phi'_2(x_0) + c_3\phi'_3(x_0) = 0, x \in I \quad \dots (5)$$

Now again differentiating (3)twice w.r.t. x and replacing x by x_0 yields

$$c_1\phi''_1(x_0) + c_2\phi''_2(x_0) + c_3\phi''_3(x_0) = 0, x \in I \quad \dots (6)$$

Using (2) in (4), (5) and (6) , we obtain

$$c_1 = c_2 = c_3 = 0.$$

Hence ϕ_1, ϕ_2 and ϕ_3 are linearly independent.

THEOREM III: Let ϕ be any solution of $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0, x \in I$. Here p_1, p_2 and p_3 are the functions defined and continuous on an interval I. Further, let ϕ_1, ϕ_2 and ϕ_3 be there linearly independent solutions of the given equation. Prove that constants c_1, c_2 and c_3 exist such that

$$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3, x \in I.$$

SOLUTION: Let $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0, x \in I$ ... (1)
Using the existence and uniqueness theorem states in theorem I, at $x = x_0 \in I$, $\exists$ constants a_1, a_2 and a_3 such that

$$\phi(x_0) = a_1,\ \phi'(x_0) = a_2 \text{ and } \phi''(x_0) = a_3$$

The solutions ϕ_1, ϕ_2 and ϕ_3 are given by theorem II. Now we define a function ψ on I such that $\psi(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x), x \in I$. Clearly (1) and

$$\psi(x_0) = a_1,\ \psi'(x_0) = a_2 \text{ and } \psi''(x_0) = a_3$$

Since that two solutions ϕ and ψ of (1) have the same initial conditions. Hence by the existence and uniqueness theorem, it follows that $\phi(x) = \psi(x)$ for $x \in I$.

THEOREM IV: If $y_1(x)$ and $y_2(x)$ are any two solutions of $a_0(x)y''(x) + a_1(x)y'(x) + a_1(x)y(x) = 0$, then the linear combination $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants, is also a solution of the given equation.

SOLUTION: Suppose

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \quad \text{... (1)}$$

$\therefore y_1(x)$ and $y_2(x)$are the solution of (1), we get

$$a_0(x){y_1}''(x) + a_1(x){y_1}'(x) + a_2(x)y_1(x) = 0 \quad \text{... (2)}$$

$$a_0(x){y_2}''(x) + a_1(x){y_2}'(x) + a_2(x)y_2(x) = 0 \quad \text{... (3)}$$

Let

$$u(x) = c_1y_1(x) + c_2y_2(x) \quad \text{... (4)}$$

Hence differentiating (4) twice, w.r.t.x, we obtain

$$u'^{(x)} = c_1{y'}_1(x) + c_2{y'}_2(x) \text{and } u''^{(x)} = c_1{y''}_1(x) + c_2{y''}_2(x) \quad \text{... (5)}$$

From (4) and (5)

$$\Rightarrow a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) = a_0(x)\left[c_1{y''}_1(x) + c_2{y''}_2(x)\right] + a_1(x)[c_1{y'}_1(x) + c_2{y'}_2(x)] + a_2(x)[c_1y_1(x) + c_2y_2(x)]$$

$$= c_1[a_0(x){y_1}''(x) + a_1(x){y_1}'(x) + a_2(x)y_1(x)] + c_2[a_0(x){y_2}''(x) + a_1(x){y_2}'(x)a_2(x)y_2(x)] \quad \text{... (6)}$$

Putting the value of (2) and(3) in (6)

$$= c_1.0 + c_2.0$$

Thus

$$a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) = 0$$

Prove that$u(x)$, i.e., $c_1y_1(x)+c_2y_2(x)$ is also solution of (1).

THEOREM V: Two solutions $y_1(x)$ and $y_2(x)$ of the equation, $a_0(x)y''+a_1(x)y'+a_1(x)y=0, a_0(x)\neq 0, x\in[a,b]$ are linearly dependent if and only if their Wronskian is identically zero.

SOLUTION:

Necessary Condition: Let $y_1(x)$ and $y_2(x)$ be linearly independent, two constants c_1 and c_2, not both zero, such that

$$c_1y_1(x)+c_2y_2(x)=0\ \forall\ x\in[a,b] \qquad \ldots(1)$$

$$c_1y'_1(x)+c_2y'_2(x)=0\ \forall\ x\in[a,b] \qquad \ldots(2)$$

Since c_1 and c_2 cannot be zero simultaneously, the equation (1) and (2) possess non-zero solutions for which the condition is

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x)\end{vmatrix}=0\ \forall\ x\in[a,b]$$

∴ $W(x)\equiv 0$on (a,b)(Wronskian is identically zero)

Sufficient Condition: Let us consider Wronskian is identically zero on $(a,b)$and let

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x)\end{vmatrix}\equiv 0\ on\ [a,b] \qquad \ldots(3)$$

Suppose$x=x_0\in[a,b]$.

Hence from the equation (3), we obtain

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x)\end{vmatrix}=0 \qquad \ldots(4)$$

Since the equation (4) for the existence of two constants k_1 and k_2, both not zero i.e.,

$$k_1y_1(x_0)+k_2y_2(x_0)=0 \qquad \ldots(5)$$

$$k_1y'_1(x_0)+k_2y'_2(x_0)=0 \qquad \ldots(6)$$

And consider $\quad y(x)=k_1y_1(x)+k_2y_2(x) \qquad \ldots(7)$

now $y(x)$ being a linear combination of $y_1(x)$ and $y_2(x)$ is also given equation.(refer theorem IV)

Differentiating equation (7), we get

$$y'(x)=k_1y'_1(x)+k_2y'_2(x) \qquad \ldots(8)$$

Again (7) $\Rightarrow$ $y(x_0)=k_1y_1(x_0)+k_2y_2(x_0)=0$ ... (9) from (5)

$\Rightarrow$ $y(x_0)=k_1y_1(x_0)+k_2y_2(x_0)=0$ ... (10)from(6)

Hence $y(x)\equiv 0$ on $[a,b]$. And by (7)

$$k_1y_1(x)+k_2y_2(x)=0, \forall\ x\in[a,b]$$

Where k_1 and k_2 are constants, both not zero.

Hence, be def., $y_1(x)$ and $y_2(x)$ are linearly independent.

THEOREM VI: ***ABEL'S FORMULA***

Let the function p_1 and p_2 in

$$L(y)(x) = y''^{(x)} + p_1(x)y'^{(x)} + p_2 y(x) = 0, x \in I \quad \ldots (1)$$

be defined and continuous on an interval$I$. Let $\phi_1$any ϕ_2 be two linearly independent solutions of (1) existing on I containing a point x_0. then

$$W(\phi_1, \phi_2)(x) = exp\left(-\int_{x_0}^{x} p_1(x)dx\right) W(\phi_1, \phi_2)(x_0) \quad \ldots (2)$$

Proof: Given

$$y''^{(x)} + p_1(x)y'^{(x)} + p_2 y(x) = 0, x \in I$$

now $$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi'_1 & \phi'_2 \end{vmatrix} = \phi_1\phi'_2 - \phi'_1\phi_2 \ldots (3)$$

From(3)

$$W'(\phi_1, \phi_2) = \phi'_1\phi'_2 + \phi'_1\phi'_2 - (\phi''_1\phi_2 + \phi'_1\phi'_2)$$

$$W'(\phi_1, \phi_2) = \phi_1\phi''_2 - \phi''_1\phi_2 \quad \ldots (4)$$

Hence ϕ_1 and ϕ_2 satisfying (1), we have

$$\phi''_1 + p_1\phi'_1 + p_2\phi_1 = 0 \Rightarrow \phi''_1 = -p_1\phi'_1 - p_2\phi_1$$

and

$$\phi''_1 + p_1\phi'_2 + p_2\phi_2 = 0 \Rightarrow \phi''_2 = -p_1\phi'_2 - p_2\phi_2$$

Putting the value of ϕ''_1 and ϕ''_2 in (4), we get

$$W'(\phi_1, \phi_2) = \phi_1(-p_1\phi'_1 - p_2\phi_1) - \phi_2(-p_1\phi'_2 - p_2\phi_2)$$

or

$$W'(\phi_1, \phi_2) = -p_1(\phi_1\phi'_2 - \phi'_1\phi_2) =$$

$-p_1 W(\phi_1, \phi_2)$ from(3)

Hence, $W(\phi_1, \phi_2)$ satisfied a first order linear homogeneous equation $W' + p_1 W = 0, x \in I$

or $\frac{dW}{dx} = -p_1 W$ or $\frac{dW}{W} = -p_1 dx$ or $\log W - \log c = -\int_{x_0}^{x} p_1 dx$

so $$W(\phi_1, \phi_2)(x) = c\ exp\left(-\int_{x_0}^{x} p_1(x)dx\right) \ldots (5)$$

where c is constant .

Putting $x = x_0$ in (5), we have

$$c = W(\phi_1, \phi_2)(x_0)$$

Hence we get the required result.

SOLVED EXAMPLES

EXAMPLE1: Prove that the function $y = cx^2 + x + 3$ is a solution, though not unique, of the initial value problem $x^2y'' - 2xy' + 2y = 6$ with $y(0) = 3, y'^{(0)} = 1$ on $(-\infty, \infty)$.

SOLUTION: Suppose $x^2y'' - 2xy' + 2y = 6$ ... (1)

and $y(x) = cx^2 + x + 3$ ... (2)

Now differentiating (2), we have

$y' = 2cx + 1$ and $y'' = 2c$ ... (3)

From (1)

L.H.S.⇒ $x^2y'' - 2xy' + 2y = x^2(2c) - 2x(2cx + 1) + 2(cx^2 + x + 3)$, using (2) and (3)

$= 6$=R.H.S. of (1)

From (2) and (3)

$$y(0) = (c \times 0) + 0 + 3 = 3$$

and

$$y'^{(0)} = (2c) \times (0) + 1 = 1.$$

Comparing (1) with

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$$

here

$a_0(x) = x^2,\ a_1(x) = -2x,\ a_2(x) = 2$ and $r(x) = 6$ which are continuous on $(-\infty, \infty)$.

Since $a_0(x) = x^2 = 0$ for $x = 0 \in (-\infty, \infty)$, therefore, the solution $y(x) = cx^2 + x + 3$ is not unique. Hence we see that $y = cx^2 + x + 3$ is a solution for any real value of c.

EXAMPLE2: Find the unique solution of $y'' = 1$ satisfying $y(0) = 1$ and $y'(0) = 2$.

SOLUTION: The given equation $y'' = \frac{d^2y}{dx^2} = 1$ ... (1)

Integrating (1)

$$y' = \frac{dy}{dx} = x + c_1 \quad \text{... (2)}$$

Again integrating (2)

$$y = x^2/2 + c_1x + c_2 \quad \text{... (3)}$$

Putting $x = 0$ in equation(2) and (3) and using $y(0) = 1$ and $y'(0) = 2$, then we have

$c_1 = 2$ and $c_2 = 1$.

Hence from (3) becomes $y = x^2/2 + 2x + 1$ ... (4)

Comparing (1) with

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$$

We get

$a_0(x) = 1,$ $a_1(x) = 0,$ $a_2(x) = 0$ and $r(x) = 1$

Since these all are continuous in $(-\infty, \infty)$ and Since $a_0(x) \neq 0$ for each $x \in (-\infty, \infty)$.Hence by existence and uniqueness theorem, equation (4) is unique.

EXAMPLE3: To show that solutions $\phi_1(x) = e^{2x}, \phi_2(x) = xe^{2x}$ and $\phi_3(x) = x^2e^{2x}$ are linearly independent solutions of $y''' - 6y'' + 12y' - 8y = 0$ on an interval $0 \leq x \leq 1$.

SOLUTION: Suppose

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} e^{2x} & xe^{2x} & x^2\ e^{2x} \\ d(e^{2x})/dx & d(xe^{2x})/dx & d(x^2e^{2x})/dx \\ d^2(e^{2x})/dx & d^2(xe^{2x})/dx & d^2(x^2e^{2x})/dx \end{vmatrix}$$

Or

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (1+2x)e^{2x} & (2x+2x^2)e^{2x} \\ 4e^{2x} & (4+4x)e^{2x} & (2+8x+4x^2)e^{2x} \end{vmatrix} \quad \ldots(1)$$

$\therefore$ it is not very easy to evaluate R.H.S. of (1). We chose $0 \in [0,1]$. Then, from (1),

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{vmatrix} = 2$$

By Abel's formula, we have

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-p_1(x-x_0)} = W(\phi_1, \phi_2, \phi_3)(x_0)$$

Here $p_1 = -6$ and $x_0 = 0$. Hence, (3) reduces to $W(\phi_1, \phi_2, \phi_3)(x) = 2e^{6x}$, using (2)

1.14 *FUNDAMENTAL SET OF SOLUTIONS:-*

Definition: Any set $y_1, y_2 \ldots \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth order differential equation

$$(d^ny/dx^n) + p_1(x)(d^{n-1}y/dx^{n-1}) + p_2(x)(d^{n-2}y/dx^{n-2}) + \cdots + p_n(x)y(x) = 0, \quad x \in I$$

is said to be a fundamental set of solutions on the interval I.

SOLVED EXAMPLES

EXAMPLE1: Prove that the solutions e^x, e^x, e^{2x} of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ are linearly independent and hence or otherwise solve the given equation.

SOLUTION: The given equation $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ Or $y''' - 2y'' - y' + 2y = 0$ ... (1)

Consider

$y_1 = e^x,\ y_2 = e^{-x}$ and $y_3 = e^{2x}$ ... (2)

$y'_1 = e^x,\ y''_1 = e^x$ and $y'''_1 = e^x$ ... (3)

$y_1''' - 2y_1'' - y'_1 + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0$, from (2) and (3)

Hence

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= (e^x \quad e^{-x} \quad e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$$

$$\begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix} = -6e^{2x}$$

Finally y_1, y_2 and y_3 are linearly independent.

EXAMPLE2: Show that $sin2x$ and $cos2x$ form a set of fundamental solutions of $y'' + 4y = 0$ and hence find the general solution of this equation.

SOLUTION: Let $y'' + 4y = 0$ ... (1)

and $y_1(x) = sin2x,\quad y_2(x) = cos2x$ ... (2)

Now $y'_1(x) = 2cos2x,\quad y_2(x) = -4sin2x$ ... (3)

$y''(x) + 4y(x) = -4sin2x + 4sin2x = 0$, from (2) and (3)

Hence we can prove that $y_1(x) = sin2x$ and $y_2(x) = cos2x$ is the solution of (1). So the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is obtained by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} sin2x & cos2x \\ 2cos2x & -2sin2x \end{vmatrix}$$

$$= -2(sin^2 2x + cos^2 2x) = -2 \neq 0.$$

Finally $W(x) \neq 0$, $sin2x$ and $cos2x$ are linearly independent solution of (1).

SELF CHECK QUESTIONS

Choose the Correct Option:

(SCQ-1) The differential equation of family of circles of radius r whose centre lie on the x-axis, is

(a) $y(dy/dx) + y^2 = r^2$

(b) $y\{(dy/dx) + 1\} = r^2$

(c) $y^2\{(dy/dx)+1\}=r^2$
(d) $y^2\{(dy/dx)^2+1\}=r^2$

(SCQ-2) Linear combinations of solutions of an ordinary differential equation are solutions if the differential equation is

(a) Linear non-homogeneous
(b) Linear homogeneous
(c) Non-linear homogeneous
(d) Non-linear non-homogeneous

(SCQ -3)which of the following pair of the functions is not a linear independent solutions of $y''+9y=0$?

(a) $sin3x, sin3x-cos3x$
(b) $sin3x+cos3x, 3sinx-4sin^3x$
(c) $sin3x, sin3x\, cos3x$
(d) $sin3x+cos3x, 4cos^3x-3cosx$

(SCQ-4) Let $y=\phi(x)$ and $y=\psi(x)$ be solutions of $y''-2xy'+(sinx^2)y=0$, such that$\phi(0)=1, \phi'^{(0)}=1$ and $\psi(0)=1, \psi'(0)=2$. The value of Wromhian $W(\phi,\psi)$at $x=1$ is

(a) 0
(b) 1
(c) e
(d) e^2

(SCQ-5) For which of the following functions $y_1(x)$ and $y_2(x)$, continuous functions $p(x)$ and q(x) can be determined on $[-1,1]$ such that $y_1(x)$ and $y_2(x)$ give two linearly independent solutions of $y''+p(x)y'+q(x)y=0, x\in[-1,1]$.

(a) $y_1(x)=xsinx,\ y_2(x)=cosx$
(b) $y_1(x)=xe^x,\ y_2(x)=sinx$
(c) $y_1(x)=e^{x-1}, y_2(x)=\ e^{x-1}$
(d) None of these

(SCQ-6) Let $y_1(x)$ and $y_2(x)$ defined on $[0,1]$ be twice continuously differentiable functions satisfying $y''^{(x)}+y'^{(x)}+y(x)=0$. Let $W(x)$ be Wronskian of y_1 and y_2 and satisfy$W(1/2)=0$. Then

(a) $W(x)=0$ for $x\in[0,1]$
(b) $W(x)>0$ for $x\in[0,1/2]$
(c) $W(x)<0$ for $x\in[1/2,1]$
(d) None of these

(SCQ-7) Order and degree, respectively of the differential equation of the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$ are:

(a) 1,1
(b) 1,2
(c) 1,3
(d) None of these

(SCQ-8) The order of the differential equation $y'''' - 3(y''')^2 + 4y'' - 5y' + 6y = 0$ is

(a) 3
(b) 5
(c) 4
(d) None of these

1.15 *SUMMARY*

In this unit we have studied the differential equation which contains at least one derivative of an unknown function, order of a differential equation is the highest derivative present in the differential equation Order of Differential Equation. Definition, Degree of a Differential Equation, Linear and non- linear differential Equation, Solution of differential Equation and Family of curve, Complete Primitive (General Solution). Particular Solution and Singular Solutions, The Wronskian, Linearly dependent and independent set of functions, Existence of uniqueness theorem, Fundamental set of solutions.

1.16 *GLOSSARY:-*

Differential Equation: A mathematical equation that relates a function or a set of functions to their derivatives. It explains the rate of change of a quantity.

Ordinary Differential Equation (ODE): A differential equation involving a one independent variable and its derivatives. It models various dynamic systems like motion, growth, and decay.

Partial Differential Equation (PDE): A differential equation involving one or more independent variables and their partial derivatives. It is used to explain phenomena in fields like physics, engineering, and fluid dynamics.

1.17 *REFERENCES:-*

- Earl A. Coddington (1961).An Introduction to Ordinary Differential Equations, Dover Publications.
- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- Purna Chandra Biswal (2012) Ordinary Differential Equations.
- Shair Ahmad, Antonio Ambrosetti (2015) A text book on Ordinary Differential Equation.

1.18 *SUGGESTED READING:-*

- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- N.P.Bali (2006). Goldan Differential Equations.
- B. Rai, D. P. Choudhury & H. I. Freedman (2013). A Course in Ordinary Differential Equations (2nd edition). Narosa.
- George F. Simmons (2017). Differential Equations with Applications and Historical Notes (3rd edition). CRC Press. Taylor & Francis.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.

1.19 *TERMINAL QUESTIONS:-*

(TQ-1) Find the differential equation of the family of the curves $y = Ae^{3x} + Be^{5x}$; for different values of A and B.

(TQ-2) Show that $Ax^2 + By^2 = 1$ is the solution of $x[y(d^2y/dx^2) + (dy/dx)^2] = y(dy/dx)$.

(TQ-3) Show that $y = x + xlogx - 1$ is the unique solution of $xy'' - 1 = 0$ satisfying $y(1) = 0$ and $y'(1) = 2$.

(TQ-4) Define linearly dependent and independent set of functions.

(TQ-5) Show that the linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x sinx$ and $e^x cosx$.

(TQ-6) Prove that the functions $1, x, x^2$ are linearly independent. Hence form the differential equation whose solutions are $1, x, x^2$

1.20 *ANSWERS:-*

SELF CHECK ANSWERS (SCQ'S)

(SCQ-1) d	(SCQ-2) b	(SCQ-3) c
(SCQ-4) c	(SCQ-5) c	(SCQ-6) a
(SCQ-7) c	(SCQ-8) c	

TERMINAL ANSWERS (TQ'S)

(TQ-1) $y'' - 8y' + 15y = 0$

UNIT 2: - DEGREE & EXACTNESS OF THE DIFFERENTIAL EQUATION AND PRINCIPLE OF DUALITY

CONTENTS:

2.1 *INTRODUCTION:-*

In this previous unit, you have already studied

- About the differential equations and its type.
- About the general solutions of various differential equations with suitable examples.
- About the existence & uniqueness theorem with examples.

In this unit we will discuss about the degree of a differential equation tells us the highest order of the derivative involved, while exactness is a property specific to first-degree ODEs that allows for straightforward integration.

2.2 *OBJCETIVES:-*

After studying this unit you will be able to

- Student will be able to solve first order first degree differential equations utilizing the standard techniques.
- Determine the first order and first degree depend on the specific context in which they are being used, and they are often used in different types of problems and situations.
- Student will be able to solve standard form of first order.
- Define a Pfaffian differential equation.

2.3 *DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE:-*

The differential equation of first degree and first order can always be defined as, namely

$$\frac{dy}{dx} = f(x, y).$$

or

$$M(x, y)dx + N(x, y)dy = 0.$$

where M and N are the functions of x and y or are constants.
Since this equation being first order, its general solution will contain only one arbitrary constant. We now talk about the various methods to solve such equations.

2.4 *VARIABLES SEPERABLE:-*

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, then the variables are said to be ***Separable***.

Working Rule:

Step1: Suppose $\frac{dy}{dx} = f_1(x)f_2(y)$ ... (1)

where $f_1(x)$ is the function of only x and$f_1(y)$ is the function of only y.

Step2: from (1), we get

$$\frac{dy}{f_2(y)} = f_1(x)dx \qquad \text{... (2)}$$

Step3: Integrating both sides of equation (2) , we obtain

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx \ + c \qquad \text{... (3)}$$

Where c is arbitrary constant.

Note1. Remember to add an arbitrary constant c on one side (only).If arbitrary constant c is not added, then the solution derives will not be general solution.

Note2. The solution of differential equation must be expressed in the form as simple as possible.

Note3. Remember that

i. $\log x + \log y = \log xy$

ii. $\log x - \log y = \log \frac{x}{y}$

iii. $n \log x = \log x^n$

iv. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left[\frac{(x+y)}{(1-xy)}\right]$

v. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left[\frac{(x-y)}{(1+xy)}\right]$

SOLVED EXAMPLES

EXAMPLE1. Solve $(1 + x^2)dy = (1 + y^2)dx$

SOLUTION: The given equation $(1 + x^2)dy = (1 + y^2)dx$

Now separating variables

$$\frac{dy}{(1 + y^2)} = \frac{dx}{(1 + x^2)} \qquad \text{... (1)}$$

Integrating both sides in (1)

$$\Rightarrow \int \frac{dy}{(1 + y^2)} = \int \frac{dx}{(1 + x^2)} \quad = \quad \tan^{-1} y = \tan^{-1} x + \tan^{-1} c$$

$\Rightarrow$ where c is constant.

$$\Rightarrow \qquad \tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

$\Rightarrow$ $\tan^{-1} \frac{(y-x)}{(1+yx)} = \tan^{-1} c$ Using $\left\{\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left[\frac{(x-y)}{(1+xy)}\right]\right\}$

$$\Rightarrow \qquad \frac{y-x}{1+yx} = c$$

EXAMPLE2. Solve $\frac{dy}{dx} = sin(x + y) + cos(x + y)$ ... (1)

SOLUTION. Suppose $x + y = u$.

Then differentiating both side

$$1+\frac{dy}{dx}=\frac{du}{dx}$$
$$\frac{dy}{dx}=\frac{du}{dx}-1$$

Substituting these value in equation (1)

$\Rightarrow$ $$\frac{du}{dx}-1=sinu+cosu$$

$\Rightarrow$ $$\frac{du}{(1+cosu)+sinu}= \ dx, \ \ seperating\ variables$$

$\Rightarrow$ $$\frac{du}{2cos^2\frac{u}{2}+2sin\frac{u}{2}cos\frac{u}{2}}= \ dx$$

$\Rightarrow$ $$\frac{\frac{1}{2}sec^2\frac{u}{2}}{1+tan\frac{u}{2}}du= \ dx.$$

$\therefore$ Integrating both sides, we get

$\Rightarrow$ $$\log\left(1+tan\frac{1}{2}u\right)=x+c \qquad ...(2)$$

Putting the value of u in equation (2)

$\Rightarrow$ $$\log\left(1+tan\frac{1}{2}(x+y)\right)=x+c$$

EXAMPLE3. Solve $\frac{dy}{dx}=e^{x+y}+x^2e^y$

SOLUTION. Given $\frac{dy}{dx}=e^{x+y}+x^2e^y$

$\Rightarrow$ $$\frac{dy}{dx}=e^x.e^y+x^2e^y$$

$\Rightarrow$ $$\frac{dy}{dx}=e^y(e^x+x^2)$$

Separating variables

$\Rightarrow$ $$\frac{dy}{e^y}=(e^x+x^2)dx$$

Integrating both sides

$\Rightarrow$ $$\int\frac{dy}{e^y}=\int(e^x+x^2)dx$$

$\Rightarrow$ $$\int e^{-y}dy=\int(e^x+x^2)dx$$

$\Rightarrow$ $$\frac{e^{-y}}{-1}=e^x+\frac{x^3}{3}+c$$

$\Rightarrow$ $$e^x+\frac{x^3}{3}+e^{-y}+c=0$$ is required solution.

EXAMPLE4. Solve the following differential equations:

a. $sec^2xtany\,dx+sec^2ytanxdy=0$

b. $\frac{dy}{dx}=\frac{sinx+xcosx}{y(2logy+1)}$

c. $y - x\left(\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

d. $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

e. $\frac{dy}{dx} = xy + x + y + 1$

SOLUTION.

a. Given $sec^2xtany\,dx + sec^2ytanxdy = 0$

$\Rightarrow \quad sec^2ytanxdy = -\ sec^2xtany\,dx$

Separating variables

$\Rightarrow \quad \frac{sec^2y}{tany}dy = -\frac{sec^2x}{tanx}dx$

Integrating both sides

$\Rightarrow \quad \int \frac{sec^2y}{tany}dy = -\int \frac{sec^2x}{tanx}dx$

$\Rightarrow \quad \log tany = -\log tanx + c_1 \qquad [c_1 = \log c]$

Finally

$\Rightarrow \quad \log tany + \log tanx = c_1 = \log c$

$\Rightarrow \quad \log tany\ tanx = \log c$

$\Rightarrow \quad tany\ tanx = c$

b. Let $\frac{dy}{dx} = \frac{sinx+xcosx}{(2ylogy+1)}$

$\Rightarrow \quad (sinx + xcosx)dx = (2ylogy + 1)dy$

$\Rightarrow \quad \int(sinx + xcosx)dx = \int(2ylogy + 1)dy \qquad \ldots(1)$

Now

$\Rightarrow \quad \int(xcosx)dx = x\ sinx + cosx$

Also

$\Rightarrow \quad \int(ylogy)dy = (logy) \times (y^2/2) - \int\{(1/y) \times (y^2/2)\}\,dy$

$\Rightarrow \quad = (logy)(y^2/2) - (y^2/4)$

Putting the value of $\int(xcosx)dx$ and $\int(ylogy)dy$ in (1)

$\Rightarrow \quad -cosx + xsinx + cosx = 2\left\{\left(\frac{y^2}{2}\right)logy - y^2/4\right\} + y^2/2 + c$

$\Rightarrow \quad xsinx = y^2\ logy + c$

c. Let $y - x\left(\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

$\Rightarrow \quad xy - x^2\frac{dy}{dx} = y$

$\Rightarrow \quad -x^2\frac{dy}{dx} = y - xy$

$\Rightarrow \quad -x^2\frac{dy}{dx} = y(1 - x)$

$\Rightarrow \quad x^2\frac{dy}{dx} = y(x - 1)$

$\Rightarrow \quad \int \frac{dy}{y} = \int (1-x)dx = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$\Rightarrow \quad logy = logx + \frac{1}{x} + c$

d. Given $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

$\Rightarrow \quad x^2(1-y)dy + y^2(1+x)dx = 0$

$\Rightarrow \quad \frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$

or

$\Rightarrow \left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \left(\frac{1}{x^2} + \frac{1}{x}\right) dy = 0$

Integrating both sides

$\Rightarrow \int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy = \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dy$

$\Rightarrow -\frac{1}{y} - logy - \frac{1}{x} + logx = c$

$\Rightarrow log\frac{x}{y} - \left(\frac{1}{x} + \frac{1}{y}\right) = c$

e. $\frac{dy}{dx} = xy + x + y + 1$

$\Rightarrow \frac{dy}{dx} = (x+1)(y+1)$

$\Rightarrow \int \frac{dy}{(1+y)} = \int (x+1)dx$

$\Rightarrow \quad log(1+y) = \frac{x^2}{2} + x + c$

$\Rightarrow \quad \frac{x^2}{2} + x - log(1+y) + c = 0$

2.5 *HOMOGENEOUS EQUATIONS:-*

A differential equation of first order and first degree is said to be homogeneous if

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Working rule:

Suppose

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \dots(1)$$

Let $\frac{y}{x} = u$ i.e., $y = ux$ ... (2)

Now differentiating (2) w.r.t.x,

$$\frac{dy}{dx} = u + x\frac{du}{dx} \qquad \dots(3)$$

Putting the value of (2) and (3) in (1)

$$u + x\frac{du}{dx} = f(u) \quad \text{or} \quad x\frac{du}{dx} = f(u) - u$$

Separating variable x and u, we get

$$\frac{dx}{x} = \frac{du}{f(u) - u}$$

So

$$\int \frac{dx}{x} = \int \frac{du}{f(u) - u}$$

$$logx + c = \frac{du}{f(u) - u}$$

Where c is an arbitrary constant and after integrating, replace u by y/x.

SOLVED EXAMPLES

EXAMPLE1. Solve $(x^2 - y^2)dx + 2xydy = 0$

SOLUTION: The given equation can be defined as

$\Rightarrow \quad (x^2 - y^2)dx + 2xydy = 0$

$\Rightarrow \quad (x^2 - y^2)dx = -2xydy$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy} \qquad \dots(1)$

$\Rightarrow$ Putting $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$ in (1), we have

$\Rightarrow \quad u + x\frac{du}{dx} = -\frac{(x^2 - (ux)^2)}{2xy} = -\frac{(x^2 - u^2x^2)}{2ux^2} = -\frac{(1 - u^2)}{2u}$

$\Rightarrow \quad x\frac{du}{dx} = -\frac{(1 + u^2)}{2u}$

Separating variables

$\Rightarrow \quad \frac{2u}{(1 + u^2)}du = -\frac{1}{x}dx$

$\Rightarrow$ integrating, we have $\log(1 + u^2) = -\log x + \log c$

$\Rightarrow \quad \log(1 + u^2) = \log\frac{c}{x}$

$\Rightarrow \quad (1 + u^2) = \frac{c}{x}$

$\Rightarrow \quad 1 + \frac{y^2}{x^2} = \frac{c}{x}.$

EXAMPLE2. Solve$x^2ydx - (x^3 + y^3)dy = 0.$

SOLUTION: The given equation

$\Rightarrow \quad x^2ydx - (x^3 + y^3)dy = 0.$

$\Rightarrow \quad x^2ydx = (x^3 + y^3)dy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2y}{(x^3+y^3)} \quad \Rightarrow$ Putting $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow \quad u + x\frac{du}{dx} = \frac{x^2ux}{(x^3+u^3x^3)} = \frac{x^3u}{x^3(1+u^3)} = \frac{u}{(1+u^3)}$

$\Rightarrow \quad x\frac{du}{dx} = \frac{u}{(1+u^3)} - u \quad \Rightarrow \quad x\frac{du}{dx} = -\frac{u^3}{(1+u^3)}$, separating variables

$\Rightarrow \quad \frac{(1+u^3)}{u^3}du = -\frac{1}{x}du \quad \Rightarrow \quad \left(\frac{1}{u^3} + \frac{1}{u}\right)du = -\frac{1}{x}du$

$\Rightarrow$ Integrating, we have $\quad -\frac{u^{-3}}{3} + \log u = -\log x + \log c$

$\Rightarrow \quad -\frac{1}{3u^3} + \log u = -\log x + \log c$

$\Rightarrow \quad \log u + \log c + \log x = \frac{1}{3u^3}$

$\Rightarrow \quad \log uxc = \frac{1}{3u^3} \quad \Rightarrow \quad \log uxc = \frac{1}{3u^3}$

$\Rightarrow$ Putting the value of $u = \frac{y}{x}$

$\Rightarrow \quad \log\frac{y}{x}xc = \frac{1}{3\left(\frac{y}{x}\right)^3} \quad \Rightarrow \log yc = \frac{x^3}{3y^3}$ is required solution.

EXAMPLE3. Solve $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

SOLUTION: The given equation is $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad$ ... (1)

$\Rightarrow$ put $y = ux$, then $\frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow \quad u + x\frac{du}{dx} = \frac{ux+\sqrt{x^2+u^2x^2}}{x} \quad \Rightarrow \quad u + x\frac{du}{dx} = \frac{ux+x\sqrt{1+u^2}}{x}$

$\Rightarrow \quad x\frac{du}{dx} = \sqrt{1+u^2} \quad \Rightarrow \quad \frac{du}{\sqrt{1+u^2}} = \frac{1}{x}dx$

$\Rightarrow$ integrating, we get

$\Rightarrow \quad \sinh^{-1}u = \log x + \log c \quad \Rightarrow \quad \log\left(u + \sqrt{u^2+1}\right) = \log cx$

$\therefore \quad \sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow \quad \left(u + \sqrt{u^2+1}\right) = cx \quad \Rightarrow \quad \left(\frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1}\right) = cx$

$\Rightarrow \quad \frac{y+\sqrt{y^2+x^2}}{x} = cx \quad \Rightarrow \quad y + \sqrt{y^2+x^2} = cx^2$ is required solution.

EXAMPLE4. Solve $xdy - ydx = \sqrt{x^2+y^2}$

SOLUTION: The given equation is $xdy - ydx = \sqrt{x^2+y^2}dx$

$\Rightarrow \quad xdy = \left(y + \sqrt{x^2+y^2}\right)dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(y+\sqrt{x^2+y^2}\right)}{x} = \frac{y}{x} + \{1 + (y/x)^2\}^{1/2}$

$\Rightarrow \quad$ take $\frac{y}{x} = u, \quad$ ie., $\quad y = ux, \quad \frac{dy}{dx} = u + x\frac{du}{dx}$

So that

$\Rightarrow \quad u + x\frac{du}{dx} = u + \sqrt{1+u^2} \quad \Rightarrow \quad \frac{dx}{x} = \frac{du}{\sqrt{1+u^2}}$

$\Rightarrow$ Integrating, $\log x + \log c = \log\left[u + \sqrt{1+u^2}\right] \quad \Rightarrow$

$xc = u + \sqrt{1+u^2}$

$\Rightarrow$ putting the value $u = \frac{y}{x} \quad \Rightarrow \quad xc = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}$

$\Rightarrow \quad x^2c = y + \sqrt{x^2+y^2}$ is required result.

2.5.1 *EQUATION REDUCIBLE TO HOMOGENEOUS FORM:-*

The differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}, \qquad \text{where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

Can be reduced to homogeneous form by taking variables X and Y such that

$$x = X + h \qquad y = Y + k \qquad \dots(1)$$

Where h and k are constants, then $dx = dX, dy = dY$

Now given equation becomes, $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$

$\Rightarrow \quad \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1} = \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1x+b_1y+c_1)} \qquad \dots(2)$

Solving by cross multiplication

$\Rightarrow \quad \frac{h}{bc_1-b_1c} = \frac{k}{ca_1-c_1a} = \frac{1}{ab_1-a_1b}$

$\Rightarrow \quad h = \frac{bc_1-b_1c}{ab_1-a_1b}, \qquad k = \frac{ca_1-c_1a}{ab_1-a_1b}$

Now equation (2) becomes $\frac{dy}{dx} = \frac{aX+bY}{a_1X+b_1Y}$

Which is homogeneous equation and can be solve $y = ux$. In solution putting $X = x - h \ , Y = y - k$, then we get the required solution.

SOLVED EXAMPLES

EXAMPLE1. Solve $\frac{dy}{dx} = \frac{y-x+1}{y+x-5}$

SOLUTION: The given equation

$$\frac{dy}{dx} = \frac{y-x+1}{y+x-5} \qquad \dots(1)$$

$$\left[Here \quad a = -1, b = 1, a_1 = 1, b_1 = 1, \frac{a}{a_1} \neq \frac{b}{b_1}\right]$$

Now we put $x = X + h, y = Y + k$, then $dx = dX, dy = dY$

Form(1)

$$\frac{dY}{dX}=\frac{Y+k-X-h+1}{Y+k+X+h-5}=\frac{(-X+Y)+(-h+k+1)}{(X+Y)+(h+k-5)} \quad ....(2)$$

Choose h and k so that

$$\left.\begin{array}{r}-h+k+1=0\\ h+k-5=0\end{array}\right\} \quad ...(3)$$

solving equation (3), we obtain $h=3, k=2$

from(2) $\frac{dY}{dX}=\frac{-X+Y}{X+Y}$ ...(4)

Put put $Y=uX$, then $\frac{dY}{dX}=u+X\frac{du}{dX}$

From (4) $u+x\frac{du}{dX}=\frac{-X+uX}{X+uX}=\frac{-1+u}{1+u}$

$\Rightarrow$ $X\frac{du}{dX}=\frac{-1+u}{1+u}-u$

$\Rightarrow$ $X\frac{du}{dX}=\frac{-1+u-u-u^2}{1+u}$

$\Rightarrow$ $X\frac{du}{dX}=-\frac{1+u^2}{1+u}$

Separating variables $\int\frac{1+u}{1+u^2}du=-\int\frac{1}{X}dX$

Integrating $\int\frac{1}{1+u^2}du+\frac{1}{2}\int\frac{2u}{1+u^2}du=-\log X+c$

$\Rightarrow$ $\tan^{-1}u+\frac{1}{2}\log(1+u^2)=-\log X+c$

$\Rightarrow$ Putting $u=\frac{Y}{X}$, $\tan^{-1}\frac{Y}{X}+\frac{1}{2}\log\left(1+\frac{Y^2}{X^2}\right)=-\log X+c$

$\Rightarrow$ $\tan^{-1}\frac{Y}{X}+\frac{1}{2}\log\left(\frac{X^2+Y^2}{X^2}\right)=-\log X+c$

$\Rightarrow$ $\tan^{-1}\frac{Y}{X}+\frac{1}{2}[\log(X^2+Y^2)-\log X^2]=-\log X+c$

$\Rightarrow$ $\tan^{-1}\frac{Y}{X}+\frac{1}{2}[\log(X^2+Y^2)-2\log X]=\log X+c$

$\Rightarrow$ $\tan^{-1}\frac{Y}{X}+\frac{1}{2}[\log(X^2+Y^2)]=c$

$\therefore$ $X=x-h=x-3,\ Y=y-k=y-3$

$\Rightarrow$ $\tan^{-1}\frac{x-3}{y-2}+\frac{1}{2}[\log((x-3)^2+(y-2)^2)]=c$ is required solution.

EXAMPLE2. Solve $(x-y)dy=(x+y+1)dx$

SOLUTION: The given equation $\frac{dy}{dx}=\frac{x+y+1}{x-y}$ ...(1)

$\Rightarrow$ $\left[Here\ \ a=1, b=1, a_1=1, b_1=-1, \frac{a}{a_1}\neq\frac{b}{b_1}\right]$

Put $x=X+h, y=Y+k$, then $dx=dX, dy=dY$

From (1) $\frac{dY}{dX}=\frac{X+h+Y+k+1}{X+h-Y-k}=\frac{(X+Y)+(h+k+1)}{(X+Y)-(h-k)}$ ...(2)

$\therefore$ choose h and k such that

$\Rightarrow$ $(h+k+1)=0,\quad (h-k)=0$

Since $h = -\frac{1}{2} = k$

Putting the value of h and k in (2)

$\frac{dY}{dX} = \frac{(X+Y)+0}{(X+Y)-0} = \frac{(X+Y)}{(X+Y)}$ is required solution.

2.6 *PFAFFIAN DIFFERENTIAL EQUATION:-*

The Pfaffian differential equation is a type of first-order partial differential equation. It is an expression of the form:

$$\sum_{i=1}^{n} f_i(x_1, x_2, x_3 \ldots\ldots, x_n)dx_i = 0$$

where f_i is a function of n variables $x_1, x_2, x_3 \ldots\ldots, x_n$.

This equation is called Pfaffian because it can be expressed as the exterior derivative of a differential form, which is said to be the Pfaffian form.

$M(x,y)dx + N(x,y)dy = 0$ and $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ are examples of Pfaffian differential equations in two and three variables.

2.7 *EXACT DIFFERENTIAL EQUATION:-*

The equation $M(x,y) + N(x,y) = 0$ is said to be an exact differential equation when ∃ a function $f(x,y)$ of two variables x and y having continuous partial derivatives such that

$$d[f(x,y)] = Mdx + Ndy,$$
$$(\partial f/\partial x)dx + (\partial f/\partial y)dy = Mdx + Ndy$$

Remarks. The equation $y^2dx + 2xydy = 0$ is an exact differential equation, ∃ a functionxy^2, such that

$d(xy^2) = \frac{\partial}{\partial x}(xy^2)dx + \frac{\partial}{\partial y}(xy^2)dy$ or $d(xy^2) = y^2dx + 2xydy$

So the equation $y^2dx + 2xydy = 0$ may be written as $d(xy^2) = 0$. This on integration yields $xy^2 = 0$, where c as arbitrary constant. And the general solution of $xy^2 = c$.

The exact differential equation have the following important property: *An exact differential equation can always be derived from its general solution directly by differentiating without any subsequent multiplication, elimination, etc.*

THEOREM: To determine the necessary and sufficient condition for a differential equation of first order and first degree to be exact.

Proof:

Statement: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \qquad ...(1)$$

to be exact $\partial M/\partial x = \partial N/\partial y$ ... (2)

Necessary condition: Let us consider the equation $Mdx + Ndy = 0$ be exact. Hence by the definition, ∃ a function $f(x, y)$ of x and y, such that

$$\Rightarrow \qquad d[f(x,y)] = (\partial f/\partial x)dx + (\partial f/\partial y)dy = Mdx + Ndy$$

Comparing the equation, we get

$$\Rightarrow \qquad M = (\partial f/\partial x) \qquad ... (4)$$

$$\Rightarrow \qquad N = (\partial f/\partial y) \qquad ... (5)$$

Now differentiating equation (4) and (5) with respect to y and$x$, respectively obtaining

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Since$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Hence, if equation (1) is exact, M and N satisfy condition(2).

Sufficient condition: Suppose that (2) holds and proof that (1) is an exact. For the function of $f(x, y)$, such that $d[f(x, y)] = Mdx + Ndy$

Let us consider $g(x, y) = \int Mdx$ ... (6)

Be the partial integral of M, the integral defined by keeping y fixed. We first show that $(N - \partial g/\partial y)$ is a function of y only, so

$$\frac{\partial}{\partial y}(N - \partial g/\partial y) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x\, \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y\, \partial x} \text{ as } \qquad \frac{\partial^2 g}{\partial x\, \partial y} = \frac{\partial^2 g}{\partial y\, \partial x}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}, \text{ using (6)}$$

$$= 0, \text{ using(2)}$$

Now we take

$$f(x, y) = g(x, y) + \int\{N - (\partial g/\partial y)\}\, dy \qquad ... (7)$$

From (9)

$$df = dg + \left(N - \frac{\partial g}{\partial y}\right) dy = \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) + Ndy - \frac{\partial g}{\partial y} dy$$

$$= (\partial g/\partial\, x)dx + Ndy = Mdx + Ndy, \text{ using (6)}$$

Hence if equation (2) is satisfied, (1) is an exact equation.

WORKING RULE: To solve the given equation$Mdx + Ndy = 0$, find out M and N. Then first ascertain with the help of $\partial M/\partial x = \partial N/\partial y$, whether then the equation is an exact or not. If the equation is exact then

i. Integrate M w.r.t. x treating y as constant.

ii. Integrate with respect to y only those terms of N which do not contain x.

iii. Equates the sum of these two integrals [i and ii] to an arbitrary constant and we express the required solution. If the given equation $Mdx + Ndy = 0$ is an exact, then

$$\underset{\text{treating } y \text{ as constant}}{\int M dx} + \underset{\substack{\text{taking only those term in N} \\ \text{which do not contain } x}}{\int N dx} = c$$

SOLVED EXAMPLES

EXAMPLE1: Solve $(ax + hy + g)dx + (hx + by + f)dy = 0$

SOLUTION: Let comparing the equation with $Mdx + Ndy = 0$, we obtain

$\Rightarrow \quad \frac{\partial M}{\partial y} = h, \frac{\partial N}{\partial x} = h$ so $\partial M/\partial x = \partial N/\partial y$ the given equation is exact.

Hence

$$\Rightarrow \underset{\text{treating } y \text{ as constant}}{\int M dx} + \underset{\substack{\text{taking only those term in N} \\ \text{which do not contain } x}}{\int N dx} = c$$

$\Rightarrow \quad \int (ax + hy + g)dx \; + \; \int (hx + by + f)dy \; = \; 0$

$\Rightarrow \quad \frac{1}{2}ax^2 + hxy + gx + \frac{1}{2}bx^2 + fy = c$

$\Rightarrow \quad ax^2 + 2hxy + 2gx + bx^2 + 2fy + c = 0$

where c is constant and replaced $-2c = c$.

EXAMPLE2: Solve $xdx + ydy + \frac{xdy - ydx}{x^2+y^2} = 0$.

SOLUTION: The given differential equation can be defined as

$$\Rightarrow \quad \left[x - \frac{y}{x^2+y^2}\right]dx + \left[y + \frac{x}{x^2+y^2}\right]dy = 0$$

$\Rightarrow$ Here, $M = x - \frac{y}{x^2+y^2}, \qquad N = y + \frac{x}{x^2+y^2}$.

Then

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \& \qquad \frac{\partial N}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

So $\partial M/\partial x = \partial N/\partial y$ hence the given equation is exact. Therefore

$$\Rightarrow \underset{\text{treating } y \text{ as constant}}{\int M dx} + \underset{\substack{\text{taking only those term in N} \\ \text{which do not contain } x}}{\int N dx} = c$$

$\Rightarrow \quad \frac{x^2}{2} - y.\frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c.$

$\Rightarrow \quad \frac{x^2}{2} - 2\tan^{-1}\frac{x}{y} + y^2 = 2c = k.$

EXAMPLE3: Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}(1 - x/y)dy = 0$

SOLUTION: Comparing the given equation can be written as

$\Rightarrow \quad \left(1 + e^{x/y}\right)dx + e^{x/y}(1 - x/y)dy = 0$

$\Rightarrow \quad M = 1 + e^{x/y}, \qquad N = e^{x/y}(1 - x/y)$

$\Rightarrow \quad \frac{\partial M}{\partial y} = e^{x/y}(-x/y^2) \qquad \& \qquad \frac{\partial N}{\partial y} = e^{x/y}(-x/y^2)$

So its solution is $\partial M/\partial x = \partial N/\partial y$. Hence

$\Rightarrow \quad \int M dx \quad + \quad \int N dx \quad = c$

treating y as constant   taking only those term in N which do not contain x

$\Rightarrow \quad \int\left(1 + e^{x/y}\right)dx = c$ or $x + ye^{x/y} = c.$

EXAMPLE3: Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

SOLUTION: The given equation can be written as

$\Rightarrow \quad (x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$\Rightarrow \quad M = (x^2 - 4xy - 2y^2), \qquad N = (y^2 - 4xy - 2x^2)$

$\therefore \quad \frac{\partial M}{\partial y} = -4x - 4y \qquad \& \qquad \frac{\partial N}{\partial y} = -4y - 4x$

So that $\partial M/\partial x = \partial N/\partial y$.Hence

$\Rightarrow \quad \int M dx \quad + \quad \int N dx \quad = c$

treating y as constant   taking only those term in N which do not contain x

$\Rightarrow \quad \int(x^2 - 4xy - 2y^2)dx + \int y^2\, dy = c_1$

$\Rightarrow \quad x^3/3 - 4y \times (x^2/2) - 2y^2x + y^3/3 = c/3, \qquad \therefore [c_1 = c/3]$

$\Rightarrow \quad x^3 + y^3 - 6xy(x + y) = c$, c being an arbitrary constant.

2.8 *INTEGRATING FACTOR:-*

The equation$Mdx + Ndy = 0$, is not exact can sometimes be made exact by multiplying by some suitable function of x and y. Such a function is said to be an ***Integrating Factor.***

Theorem: The differential equation $Mdx + Ndy = 0$ possess an infinite number of integrating factor.

Proof: Let the given equation $\quad Mdx + Ndy = 0 \quad \ldots(1)$

Suppose $\mu(x,y)$be an I.F. of (1), then by definition\

$$\mu(Mdx+Ndy)=0$$

Must be an exact differential equation and $\exists$ a function $V(x,y)$, such that

$$dV=\mu(Mdx+Ndy)$$

where V =constant is a solution of (1)

Since $f(V)$ be any function of V. So

$\Rightarrow \quad f(V)dV=\mu f(V)(Mdx+Ndy)$

Since the expression on L.H.S. of (3) is an exact differential equation, it follows that the expression on R.H.S. of (3) must also be an exact differential.

In this section, the following list of exact differential equation is

(i). $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$ (ii) $\left(\frac{y}{x}\right)=\frac{xdy-ydx}{y^2}$

(iii) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$ (iv) $d\left(\frac{x^2}{y}\right)=\frac{2yxdy-x^2dx}{y^2}$

(v) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$ (vi) $d\left(\frac{x^2}{y^2}\right)=\frac{2y^2xdy-2yx^2dx}{y^4}$

(vii) $d[\log(xy)]=\frac{xdy+ydx}{xy}$ (viii) $d(xy)=xdy+ydx$

(ix) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$ (x) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydy-xdx}{x^2+y^2}$

(xi) $d\left[\log\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$ (xii) $d\left[\log\left(\frac{x}{y}\right)\right]=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\log(x^2+y^2)\right]=\frac{xdx+ydy}{x^2+y^2}$ (xiv) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

(xv) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$ (xvi) $d(\sin^{-1}xy)=\frac{xdy+ydx}{(1-x^2y^2)^{1/2}}$

Rule I: The integrating factor of given equation $\boldsymbol{Mdx+Ndy=0}$ can be explore by inspection as explained below.

SOLVED EXAMPLES

EXAMPLE1. Solve $y(2xy+e^x)dx=e^xdy$.

SOLUTION: Given equation $y(2xy+e^x)dx=e^xdy$

$\Rightarrow \quad 2xy^2dx+ye^xdx=e^xdy$

$\Rightarrow \quad 2xdx+\frac{ye^xdx-e^xdy}{y^2}=0$ or $2xdx+d\left(\frac{e^x}{y}\right)=0$

$\Rightarrow$ Now integrating, $x^2+\frac{e^x}{y}=c$ or $x^2+e^x=cy$

EXAMPLE2. Solve $(x^3+xy^2+a^2y)dx+(y^3+yx^2-a^2x)dy=0$.

SOLUTION: Given equation $(x^3+xy^2+a^2y)dx+(y^3+yx^2-a^2x)dy=0$

$\Rightarrow \quad x(x^2+y^2)dx+y(x^2+y^2)dy+a^2(ydx-xdy)=0$

$\Rightarrow \quad xdx+ydy+a^2\frac{(ydx-xdy)}{(x^2+y^2)}=0$ or $xdx+ydy+a^2\tan^{-1}\frac{x}{y}=0$

⇒ By integrating, $\frac{x^2}{2}+\frac{y^2}{2}+a^2\tan^{-1}\frac{x}{y}=\frac{c}{2}$ or $x^2+y^2+a^2\tan^{-1}\frac{x}{y}=c$

Rule II: If the given equation $Mdx+Ndy=0$ is homogeneous and $Mx+Ny\neq 0$, then show that the integrating factor is $1/(Mx+Ny)$.

Proof: Let the given equation $Mdx+Ndy=0$, we get

⇒ $Mdx+Ndy=\frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$

⇒ $\frac{Mdx+Ndy}{(Mx-Ny)}=\frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{(Mx-Ny)}{(Mx+Ny)}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$ ... (1)

⇒ Since $Mdx+Ndy=0$ is a homogeneous, M and N must be same degree in variables x and y and hence

⇒ $\frac{(Mx-Ny)}{(Mx+Ny)}=f\left(\frac{x}{y}\right)$ ... (2)

Now putting the value of (2) in (1)

⇒ $\frac{Mdx+Ndy}{(Mx-Ny)}=\frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$ ... (3)

⇒ $\frac{1}{2}\left\{\log(xy)+f\left(e^{\log(x/y)}\right)d\left(\log\frac{x}{y}\right)\right\}=\frac{1}{2}\left\{\log(xy)+g\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right\}$ $\left[\because f\left(e^{\log(x/y)}\right)=g\log(x/y)\right]$

⇒ $d[(1/2)\times\log(xy)+(1/2)\times\int g\log(x/y)\,d\log(x/y)]$

⇒ displaying that the $1/(Mx+Ny)$ is an integrating factor for a given equation $Mdx+Ndy=0$.

SOLVED EXAMPLES

EXAMPLE: Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

SOLUTION: The given equation $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

⇒ The given equation is homogeneous differential equation and comparing $Mdx+Ndy=0$, $M=(x^2y-2xy^2)$, $N=(x^3-3x^2y)$

⇒ $Mx+Ny=x(x^2y-2xy^2)-y(x^3-3x^2y)=x^2y^2\neq 0$,

⇒ Then the integrating factor, $1/(Mx+Ny)=\frac{1}{x^2y^2}$. on multiplying factor by $\frac{1}{x^2y^2}$,

⇒ $(y/2-2/x)dx-(x/y^2-3/y)dy=0$,

⇒ $\int\{(y/2-2/x)dx\}+\int(3/y)dy=0$ or $x/y-2\log x+3\log y=\log c$

⇒ $\log y^2-\log x^2-\log c=-x/y$ or $\log(y^2/cx^2)=-x/y$

⇒ $y^2=cx^2e^{-x/y}$, where c is an arbitrary constant.

Rule III: If the given equation $\boldsymbol{Mdx+Ndy=0}$ is of the form $\boldsymbol{f_1(xy)ydx+f_2(xy)xdy=0}$, then prove that $\boldsymbol{1/(Mx+Ny)}$ is an integrating factor of $\boldsymbol{Mdx+Ndy=0}$ provided $\boldsymbol{(Mx-Ny)\neq 0}$.

Proof: Suppose $$Mdx+Ndy=0 \quad ...(1)$$
is of the form $$f_1(xy)ydx+f_2(xy)xdy=0 \quad ...(2)$$
Comparing both equations

$$\Rightarrow \quad \frac{M}{yf_1(xy)}=\frac{N}{xf_2(xy)}=\mu$$

$$\Rightarrow \quad M=\mu y f_1(xy) \quad \text{or} \quad N=\mu x f_2(xy) \quad ...(3)$$

⇒ Now

$$\Rightarrow \quad Mdx+Ndy=\frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \quad \frac{Mdx+Ndy}{Mx-Ny}=\frac{1}{2}\left\{\frac{(Mx+Ny)}{(Mx-Ny)}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \quad =\frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \text{ from (3}$$

$$\Rightarrow \quad =\frac{1}{2}\left\{f(xy)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \quad \text{where } \frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}=f(xy)$$

$$\Rightarrow \quad =\frac{1}{2}\left\{f\left(e^{\log xy}\right)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}=\frac{1}{2}\left\{g(\log xy)d(\log xy)+d\left(\log\frac{x}{y}\right)\right\} \quad \left[\therefore f\left(e^{\log(xy)}\right)=g\log(xy)\right]$$

$$\Rightarrow \quad d[(1/2)\times\log(x/y)+(1/2)\times\int g\log(xy)\,d\log(xy)]$$

⇒ Hence prove that $Mx-Ny$ is an integrating factor of $Mdx+Ndy=0$.

SOLVED EXAMPLES

EXAMPLE. Solve $(xy\sin xy+\cos xy)ydx+(xy\sin xy-\cos xy)xdy=0$

SOLUTION: Suppose

$$(xy\sin xy+\cos xy)ydx+(xy\sin xy-\cos xy)xdy=0 \quad ...(1)$$

The equation (1) Comparing $Mdx+Ndy=0$, we get,

$M=(xy\sin xy+\cos xy)y$ and $N=(xy\sin xy-\cos xy)x$

The equation (1) is the form $f_1(xy)ydx+f_2(xy)xdy=0$

Again,

$$Mx-Ny=xy(xy\sin xy+\cos xy)-xy(xy\sin xy-\cos xy)$$

$$\therefore \quad Mx-Ny=2xy\cos xy\neq 0.$$

Since the integrating factor of (1)

$$=1/(Mx+Ny)=1/(2xy\cos xy)$$

On multiplying (1) by $1/(2xy\cos xy)$, we obtain

$$\Rightarrow [(1/2)\times(y\tan xy+1/x)dx+(1/2)\times(x\tan xy-1/y)dy] \quad ...(2)$$

From (2)

$$\Rightarrow \quad \int(1/2)\times(y\tan xy+1/x)dx+\int(-1/2y)dy=(1/2)\log c$$

$$\Rightarrow \quad (1/2)\times(\log\sec xy+\log x)dx-(1/2)\times\log y=(1/2)\log c$$

$\Rightarrow \quad (\log secxy + \log x/y) = \log c \quad$ or $\quad (y/x)secxy = c$.

Rule IV: If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function x alone $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx + Ndy = 0$.

Proof: The given equation $\quad Mdx + Ndy = 0 \qquad \dots(1)$

and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ so that $\quad Nf(x) = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \quad \dots(2)$

Multiplying both sides of equation (1) by $e^{\int f(x)dx}$, we have

$M_1dx + N_1dy = 0$,

where

$\Rightarrow \quad M_1 = Me^{\int f(x)dx}$ and $N_1 = Ne^{\int f(x)dx} \qquad \dots(3)$

From (3) $\qquad \frac{\partial M_1}{\partial y} = \frac{\partial M}{\partial y} e^{\int f(x)dx} \qquad \dots(4)$

and $\quad \frac{\partial N_1}{\partial y} = \frac{\partial N}{\partial x} e^{\int f(x)dx} + Ne^{\int f(x)dx} f(x) = e^{\int f(x)dx}\left\{\frac{\partial N}{\partial x} + Nf(x)\right\}$

$\Rightarrow \quad e^{\int f(x)dx}(\partial N/\partial x + \partial M/\partial y - \partial N/\partial x)$, $\quad$ from(2)

So that $\qquad \frac{\partial N_1}{\partial y} = \frac{\partial M}{\partial y} e^{\int f(x)dx}$

$\Rightarrow \quad$ Now from (5) and (6), $\qquad \partial M_1/\partial y = \partial N_1/\partial x$

Hence $\quad M_1dx + N_1dy = 0$ must be exact and $e^{\int f(x)dx}$ is integrating factor.

SOLVED EXAMPLES

EXAMPLE. Solve $(x^2 + y^2 + x)dx + xydy = 0$

SOLUTION. Let $\quad (x^2 + y^2 + x)dx + xydy = 0 \qquad \dots(1)$

Now the equation (1) comparing with $Mdx + Ndy = 0$, we have

$M = (x^2 + y^2 + x), \qquad N = xy$

$\Rightarrow \quad \partial M/\partial y = 2y, \qquad \partial N/\partial x = y.$

S $\qquad \partial M/\partial y \neq \partial N/\partial y.$

Then we obtain

$\Rightarrow \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - 1) = \frac{1}{x}$, which is a function of x.

$\Rightarrow \quad$ Since the integration factor is

$\Rightarrow \qquad e^{\int (1/x)dx} = e^{\log x} = x.$

$\Rightarrow \qquad$ multiplying (1) by x, we get

$\Rightarrow \qquad (x^3 + xy^2 + x^2)dx + x^2ydy = 0$ is an exact, so

$\Rightarrow \quad \int (x^3 + xy^2 + x^2)dx = (1/6) \times c \quad$ or $\quad (1/4) \times x^4 + (1/2) \times x^2y^2 + (1/3) \times x^3 = c/6.$

$\Rightarrow \quad 3x^4 + 6x^2y^2 + 4x^3 = c, \quad$ where c is an arbitrary constant.

Rule V: If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function y alone $f(y)$, then $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

Proof: Proceed exactly as for Rule IV.

SOLVED EXAMPLES

EXAMPLE. Solve

$$(2xy^4e^y+2xy^3+y)dx+(x^2y^4e^y-x^2y^2-3x)dy=0 \qquad \dots(1)$$

SOLUTION. From (1) compare with $Mdx+Ndy=0$, we have

$\Rightarrow \quad M=(2xy^4e^y+2xy^3+y), \qquad N=(x^2y^4e^y-x^2y^2-3x) \qquad \dots(2)$

Here

$\Rightarrow \partial M/\partial y=8xy^3e^y+2xy^4e^y+6xy^2+1,\ \partial N/\partial x=2xy^4e^y-x^2y^2-3x.$

$\Rightarrow \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=-4(2xy^4e^y+2xy^2+1)=-\frac{4}{y}(2xy^4e^y+2xy^2+y)=-\frac{4M}{y}$

$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-\frac{4}{y}$

$\Rightarrow$ Since the integrating factor of equation (1) is

$$=e^{\int -4/y\,dy}=e^{-4\log y}=(1/y^4).$$

$\Rightarrow$ Multiplying (1) by $1/y^4$, we get

$\Rightarrow \{2xe^y+(2x/y)+(1/y^3)\}dx+\{x^2e^y-(x^2/y^2)-3(x/y^4)\}dy=0$

$\Rightarrow \int\{2xe^y+(2x/y)+(1/y^3)\}dx=c$ or $x^2e^y+(x^2/y)+(x/y^3)=c.$

Rule VI: If the given equation $Mdx+Ndy=0$, is of the form $x^\alpha y^\beta(mydx+nxdy)=0$, then its integrating factor is $x^{km-1-\alpha}y^{kn-1-\beta}$, where k have any value.

Proof. By assumption, the given equation can be defined as

$\Rightarrow \qquad x^\alpha y^\beta(mydx+nxdy)=0 \qquad \dots(1)$

$\Rightarrow$ Multiplying (1) by$x^{km-1-\alpha}y^{kn-1-\beta}$, we get

$$x^{km-1-\alpha}y^{kn-1}(mydx+nxdy)=0$$

$\Rightarrow \quad km\,x^{km-1}y^{kn}dx+kny^{kn-1}x^{km}dy=0$ or $d(x^{km},y^{kn})=0$

$\Rightarrow$ so that $x^{km-1-\alpha}y^{kn-1-\beta}$integrating factor of given equation

$$x^\alpha y^\beta(mydx+nxdy)=0.$$

SOLVED EXAMPLES

EXAMPLE. Solve $(y^2+2x^2y)dx+(2x^3-xy)dy=0 \qquad \dots(1)$

SOLUTION. The given equation (1) in standard form

$\Rightarrow \qquad x^\alpha y^\beta(mydx+nxdy)+x^{\alpha\prime}y^{\beta\prime}(m'ydx+n'xdy)=0 \quad \dots(2)$

$\Rightarrow$ we have

$$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0 \qquad ...(3)$$

⇒ From (2) and (3), we get

⇒ $\alpha = 0,\ \beta = 1,\ m = 1,\ n = -1,\ \alpha' = 2,\ \beta' = 0,\ m' = 2,\ n' = 2$

⇒ Since, the integrating factor for first term on L.H.S. of (3) is

⇒ $x^{k-1}y^{-k-1-1}$, i.e., $x^{k-1}y^{-k-2}$ ...(4)

⇒ The second term on L.H.S.of (3) is

⇒ $2^{2k'-1-2}y^{2k'-1}$ $i.e.,$ $2^{2k'-3}y^{2k'-1}$ ...(5)

⇒ from (4) and (5), $k - 1 = 2k' - 3$ and $-k - 2 = 2k' - 1$

⇒ $k - 2k' = -2$ and $k + 2k' = -1 \Rightarrow k = -3/2$ and $k' = 1/4$

Putting the value of k in (4) or k' in (5), then the integrating factor of (3) or (1) is $x^{-5/2}y^{-1/2}$. Multiplying (1) by $x^{-5/2}y^{-1/2}$, we obtain

⇒ $\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \left(x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right)dy = 0$

⇒ $\frac{x^{-3/2}y^{3/2}}{-(3/2)} + \frac{2x^{1/2}y^{1/2}}{(1/2)} = \frac{2C}{3}$ or $6x^{1/2}y^{1/2} - x^{-3/2}y^{3/2} = C.$

2.9 *LINEAR DIFFERENTIAL EQUATION:-*

A differential equation is called linear if it can be obtained in the form

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

where P and Q are constants and are the function of x is called Linear differential equation of first order with y as dependent variable. So to solve the equation, multiply both sides by $e^{\int Pdx}$, then

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}Py = Qe^{\int Pdx}$$

Or
$$\frac{d}{dx}\left\{ye^{\int Pdx}\right\} = Qe^{\int Pdx}$$

Integrating both sides

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$$

Which is the required solution of differential equation.

Working Rule:

1. The given equation in the form $\frac{dy}{dx} + Py = Q$ and $\frac{dy}{dx} + Px = Q$ as may be.
2. Find integrating factor $e^{\int Pdx}$ or $e^{\int Pdy}$.
3. The solution of Differential equation either

$$y.(I.F.) = \int \{Q.(I.F)\}dx + c$$

Or

$$x.(I.F.) = \int\{Q.(I.F)\}dy + c \text{ as may be.}$$

SOLVED EXAMPLES

EXAMPLE1. Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$.

SOLUTION: The given equation

$$\frac{dy}{dx} + 2xy = e^{-x^2} \qquad ...(1)$$

where y is dependent variable

$\Rightarrow$ $P = 2x$ and $Q = e^{-x^2}$, then $\int Pdx = \int 2xdx = 2.\frac{1}{2}x^2 = x^2$.

$\Rightarrow$ Therefore $I.F. = e^{\int Pdx} = e^{x^2}$.

Hence

$\Rightarrow$ $y.(I.F.) = \int\{Q.(I.F)\}dx + c$

$\Rightarrow$ $y.e^{x^2} = \int\left(e^{-x^2}.e^{x^2}\right)dx + c$

$\Rightarrow$ $ye^{x^2} = \int dx + c$ or $ye^{x^2} = x + c$.

EXAMPLE2. Solve $\frac{dy}{dx}(x + 2y^3) = y$.

SOLUTION. Let $\frac{dy}{dx}(x + 2y^3) = y \qquad ...(1)$

where x is dependent.

$\Rightarrow$ Thus, we have $\frac{dx}{dy} = \frac{x+2y^3}{y}$, or $\frac{dx}{dy} - \frac{1}{y}x = 2y^2 \qquad ...(2)$

$\Rightarrow$ from (2)

$\Rightarrow$ $\int Pdy = -\int(1/y)dy = -\log y$ so IF.of (2)= $e^{-\log y} = \frac{1}{y}$

$\Rightarrow$ Hence $x/y = \int 2y^2.(1/y)dx + c$

$\Rightarrow$ $x/y = y^2 + c$, where c is an arbitrary constant.

2.10 *EQUATION REDUCIBLE TO THE LINEAR FORM:-*

An differential equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \qquad ...(1)$$

where P and Q are constants. Putting $f(y) = v$ so that $f'(y)(dy/dx) = dv/dx$, (1) becomes

$$dv/dx + Pv = Q \qquad \ldots(2)$$

Which is linear in v and x and its solution can be defined by Linear differential equation. Thus we get,

$I.F = e^{\int P dx}$ and $v.e^{\int P dx} = \int Q e^{\int P dx} dx + c$

Finally, replace v by $f(v)$ to solution in terms of x and y alone.

2.11 *BERNOULLI'S EQUATION:-*

Particular Case of Linear differential equation:-

An equation of the form $\frac{dy}{dx} + Py = Q\, y^n \qquad \ldots(1)$

Where P and Q are constants or function of x and n is constant except 0 and 1, is known as *Bernoulli's Equation*.

From (1) $\qquad y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad \ldots(2)$

Suppose $\qquad y^{1-n} = v \qquad \ldots(3)$

Differentiating (3) w.r.t. x $\quad \frac{1}{(1-n)}\frac{dv}{dx} y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, or $\quad y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx} \qquad \ldots(4)$

Putting the value of (3) and(4) in (1)

$\frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q \qquad$ or $\qquad \frac{dv}{dx} + P(1-n)v = Q(1-n)$

Which is linear in v and x. Its $I.F. = e^{\int P(1-n)dx} = e^{(1-n)\int P dx}$

Hence $\quad v.e^{(1-n)\int P dx} = \int Q.e^{(1-n)\int P dx} dx + c$, c being arbitrary constant.

$y^{1-n}e^{(1-n)\int P dx} = \int Q.e^{(1-n)\int P dx} dx + c$, from(3)

SOLVED EXAMPLES

EXAMPLE. Solve $(dy/dx) + x\, sin2y = x^3 cos^2 y$.

SOLUTION: Given equation $(dy/dx) + x\, sin2y = x^3 cos^2 y \qquad \ldots(1)$

Now dividing by $cos^2 y$ in equation (1)

$$sec^2 y(dy/dx) + 2x\, tany = x^3 \quad \ldots(2)$$

Putting $tany = v$ so that $sec^2 y(dy/dx) = dv/dx$.

Hence $\quad dv/dx + 2xv = x^3$, which is linear in v and x and its solution

$$e^{\int 2x dx} = e^{x^2}.$$

$\Rightarrow \qquad \boldsymbol{v}.e^{x^2} = \int x^3 . e^{x^2} dx + c$, $\quad c$ being an arbitrary constant.

$\Rightarrow \qquad \boldsymbol{v}.e^{x^2} = (1/2) \times \int t.e^t dt + c$, Now $x^2 = t$ and $2xdx = dt$

$\Rightarrow \quad = (1/2) \times [t \times e^t - \int (1 \times e^t) dt] + c = (1/2) \times (te^t - e^t) + c$

$\Rightarrow \quad tany\ e^{x^2} = \left(\frac{1}{2}\right) \times e^{x^2}(x^2 - 1) + c$ as $v = tany$ & $t = x^2$

$\Rightarrow \quad tany = (1/2) \times (x^2 - 1) + ce^{-x^2}$, dividing by e^{x^2}

2.12 *DIFFERENTIAL EQUATION OF FIRST ORDER BUT NOT A FIRST DEGREE:-*

A differential equation of first order but not a first degree is defined as

$$P_0 p^n + P_1 p^{n-1} + \cdots\cdots\cdots + P_{n-1} p + p_n = 0 \qquad \ldots(1)$$

where $\frac{dy}{dx} = p$ and $P_0, P_1, P_2 \ldots P_n$ are the function of x &y.

Differential equation of first order but not a first degree can be solved by four method given below.

Method I : Equation Solvable for p

Suppose

$$P_0 p^n + P_1 p^{n-1} + \cdots\cdots\cdots + P_{n-1} p + p_n = 0 \qquad \ldots(1)$$

be given differential equation o first order but not a first degree$n > 1$.

From (1) solvable for p, it can be put in the form

$$[p - f_1(x,y)][p - f_2(x,y)] \ldots \ldots \ldots . [p - f_n(x,y)] = 0 \quad \ldots(2)$$

From (2)

$p = dy/dx = f_1(x,y), \quad p = dy/dx = f_2(x,y) \ldots \ldots . . p = dy/dx = f_n(x,y)$

Suppose the n components equations are

$F_1(x,y,c_1) = 0, \quad F_2(x,y,c_2) = 0, \ldots \ldots \ldots F_n(x,y,c_n) = 0$

Which $c_1, c_2, \ldots \ldots c_n$ are arbitrary constants of integration.

If we replace $c_1, c_2, \ldots \ldots c_n = c$, then

$$F_1(x,y,c) = 0, \quad F_2(x,y,c) = 0, \ldots \ldots \ldots F_n(x,y,c) = 0$$

$$\therefore \quad F_1(x,y,c), \quad F_2(x,y,c), \ldots \ldots \ldots F_n(x,y,c) = 0$$

SOLVED EXAMPLES

EXAMPLE1: Solve $\mathrm{p}^2 - 7\mathrm{p} + 12 = 0$

SOLUTION: Let $\mathrm{p}^2 - 7\mathrm{p} + 12 = 0 \qquad \ldots(1)$

$\Rightarrow \quad \mathrm{p}^2 - 4\mathrm{p} - 3\mathrm{p} + 12 = 0 \Rightarrow \qquad (\mathrm{p} - 3)(\mathrm{p} - 4) = 0$

$\Rightarrow$ Its components are $\mathrm{p} = 3, 4$

$\Rightarrow$ Solving the equation $\mathrm{p} = 3$ i.e., $\mathrm{dy/dx} = 3$, we have

$$\mathrm{y} = 3\mathrm{x} + \mathrm{c}$$

$\Rightarrow$ also $\mathrm{p} = 4$ i.e., $\mathrm{dy/dx} = 4$, is $\mathrm{y} = 4\mathrm{x} + \mathrm{c}$

So the solution of differential equation are $y = 3x + c$, $y = 4x + c$.
The single solution are $(y - 3x - c)(y - 4x - c) = 0$.
EXAMPLE2: Solve $p^2 + 2pycotx = y^2$.
SOLUTION: The given equation is $p^2 + 2pycotx = y^2$ (1)
Solving for p, we have

$$\Rightarrow \quad p = \frac{dy}{dx} = \frac{-2ycotx \pm \sqrt{4y^2cot^2x + 4y^2}}{2} = -y\,cotx \pm ycosecx$$

$$= y(-cotx \pm cosecx)$$

$$\Rightarrow \quad \frac{dy}{dx} = y(-cotx + cosecx) \quad \ldots (1)$$

$$\Rightarrow \quad \frac{dy}{dx} = -y(cotx + cosecx) \quad \ldots (2)$$

$\Rightarrow$ from (1), we get

$$\Rightarrow \quad \frac{dy}{y} = (-cotx + cosecx)dx \quad \Rightarrow \quad \int \frac{dy}{y} = \int (-cotx + cosecx)dx$$

$$\Rightarrow \quad \log y - \log c = -\log sinx + \log tan\frac{1}{2}x$$

$$\Rightarrow \quad \log\left(\frac{y}{c}\right) = \log\left(\frac{tan\frac{1}{2}x}{sinx}\right) = \log\left\{\frac{\frac{\frac{sinx}{2}}{\frac{cosx}{2}}}{2sinx/2\ \ cosx/2}\right\}$$

$$= \log\left\{\frac{1}{2cos^2\frac{x}{2}}\right\} = \log\left\{\frac{1}{1 + cosx}\right\}$$

$$\Rightarrow \quad y/c = \frac{1}{1+cosx} \quad \Rightarrow \quad y = \frac{c}{1+cosx} \quad \ldots (3)$$

$\Rightarrow$ similarly, from (2), we get

$$\Rightarrow \quad \frac{dy}{y} = -(cotx + cosecx)dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int (cotx + cosecx)dx$$

$$\Rightarrow \quad \log y - \log c = -\left(\log sinx + \log tan\frac{1}{2}x\right)$$

$$\Rightarrow \quad \log\left(\frac{y}{c}\right) = -\log\left\{(sinx)\left(tan\frac{1}{2}x\right)\right\} = -\log 2sin\frac{x}{2}cos\frac{x}{2}.\left(\frac{sin\frac{x}{2}}{cos\frac{x}{2}}\right)$$

$$= -\log\left\{2sin^2\frac{x}{2}\right\}$$

$$\Rightarrow \quad \log\left(\frac{y}{c}\right) = -\log(1 - cosx) = -\log(1 - cosx)^{-1} = \log\left(\frac{1}{1-cosx}\right)$$

$$\Rightarrow \quad y/c = \frac{1}{1-cosx} \quad \Rightarrow \quad y = \frac{c}{1-cosx} \quad \ldots (4)$$

From (3) and (4) the combined solution is

$$\Rightarrow \quad \left(y - \frac{c}{1+cosx}\right)\left(y - \frac{c}{1-cosx}\right) = 0$$

Method II : Equation Solvable for y

If the given differential equation is solvable for y, then we can express y explicitly in terms of x and p. Thus an equation solvable for y can put in the form $\quad y = f(x,p) \quad \ldots(1)$

Differentiating (1) w.r.t. x and obtaining p for $\frac{dy}{dx}$, we have

$$p = \phi(x, p, dp/dx) \quad \ldots(2)$$

Which is an differential equation assuming two variables x and p. Let its solution be

$\Rightarrow \ \psi(x,p,c) = 0,$ c being an arbitrary constant. $\quad \ldots(3)$

Eliminating p between (1) and (3), the solution of (1) is

$$g(x,y,c) = 0 \quad \ldots(4)$$

Eliminating p between (1) and (3) is not possible, then we have

$\Rightarrow \quad x = f_1(p,c) \qquad y = f_2(p,c)$

where p being parameter.

Special Case:

- **Equation that do not contain x**

in this case the equation has in the form $f(y,p) = 0$. If it is solvable for p, it will obtain $\quad p = \phi(y) \ \ i.e., dy/dx = \phi(y)$

If it is solvable for y, it will obtain $y = \psi(p)$, which can be solved by the method just explained.

- **Lagrange 's Equation**

  The equation of the form

  $$y = xF(p) + f(p) \quad \ldots(1)$$ is called Lagrange's equation.

  Differentiating (1) w.r.t. x, we get

  $$p = F(p) + x\,F'(p)(dp/dx) + \ f'(p)(dp/dx)$$

  $\Rightarrow \quad p - F(p) = \frac{dp}{dx}[xF'(p) + f'(p)] \quad$ or $\quad \frac{dx}{dp} = \frac{xF'(p)+f'(p)}{p-F(p)}$

  $\Rightarrow \quad \frac{dx}{dp} - \frac{F'(p)}{p-F(p)}x = \frac{f'(p)}{p-F(p)}$

Which is linear equation in x and p and can be resolved by usual method in the form

$$x = \phi(p,c) \quad \ldots(2)$$

Since eliminate p between (1) and (2) to obtain the required solution.

If p cannot be eliminated, then

$$y = \phi(p,c)F(p) + f(p) \quad \ldots(3)$$

Hence the required solution in parametric form, p being parameter.

- **Clairaut form**

An equation of the form $y = px + f(p)$ is called Clairaut equation.

To prove that the general solution of Clairaut's equation $y = px + f(p)$ is $y = cx + f(c)$ which is obtained by replacing p by c, where c is an arbitrary constant.

Proof. The given equation $y = px + f(p)$ ... (1)

Differentiating (1) w.r.t. x and assuming p for dy/dx, we get
$p = p + x(dp/dx) + f'(p)(dp/dx)$ or $[x + f'(p)](dp/dx) = 0$ ... (2)
The factor $x + f'(p)$ which does not involve dp/dx, from (2) given as
$dp/dx = 0$ so that $p = c$, c being constant.
Hence substituting the value of p in (1)

$$y = cx + f(c).$$

Method III: Equations Reducible to Clairaut's form by transformation

Form I: To solve $y^2 = (py/x)x^2 + f(py/x)$, put $x^2 = u$ and $y^2 = v$.

Now $x^2 = u$ and $y^2 = v$ $\Rightarrow$ $2xdx = du$ and $2ydy = dv$

$\therefore$ $\frac{2ydy}{2xdx} = \frac{dv}{du}$ or $\frac{py}{x} = P$, where $P = \frac{dv}{du}$

Hence the given differential equation to appropriate $v = Pu + f(P)$
This is in Clairaut's form and so
$v = cu + f(c)$ or $y^2 = cx^2 + f(c)$,
c being an arbitrary constant.

Form II: To solve equation of the form $e^{by}(a - bp) = f(pe^{by-ax})$, we use the transformation $e^{ax} = u$ and $e^{by} = v$.

EXAMPLE: Solve $e^{3x}(p - 1) + p^3e^{2y} = 0$

SOLUTION: Given $e^{3x}(p - 1) + p^3e^{2y} = 0$ ... (1)

From (1)

$\Rightarrow$ $1 - p = p^3e^{2y-3x}$ or $e^y(1 - p) = (pe^{y-x})^3$,

Now **formII**$a = 1, b = 1$.

Substituting $e^x = u$ and $e^y = v$ so that$e^x dx = udu$ and $e^y dy = dv$, we obtain

$\Rightarrow$ $\frac{e^y dy}{e^x du} = \frac{dv}{du}$ or $\frac{v}{u}p = P$ or $p = \frac{uP}{v}$ where $P = \frac{dv}{du}$.

Which is in Clairaut's form. So

$\Rightarrow$ $v = uc + c^3$ or $e^y = ce^x + c^3$,

where c being constant.

Form III: Sometimes the substitution $y^2 = v$ will transform the given equation to Clairaut's form.

EXAMPLE: Solve $y = 2px + y^2p^3$.

SOLUTION: given $y = 2px + y^2p^3$ ... (1)

Multiplying equation (1) both sides by$y$, we have

$\Rightarrow \quad y = 2px + y^2p^3 \quad$ or $\quad y = x(2py) + (1/8)(2yp)^3 \quad \ldots (2)$

$\Rightarrow$ Substituting $y^2 = v$ so that

$$2y(dy/dx) = dv/dx \quad \text{or} \quad 2yp = P$$

where $P = dv/dx$

$\Rightarrow$ From (2), $\quad v = xP + P^3/8$, Which is in Clairaut's form..

So replacing $p = c$ in (1)

$\Rightarrow v = xc + c^3/8 \quad$ or $\quad y = cx + c^3/8$

SOLVED EXAMPLES

EXAMPLE1. Solve$y + px = x^4p^2$.

SOLUTION: The given equation $y + px = x^4p^2$,

where $p = dy/dx \quad \ldots (1)$

$\Rightarrow \quad y = x^4p^2 - px \quad \ldots (2)$

Differentiating (2) w.r.t. x $\; p = 4x^3p^2 + 2x^4p(dp/dx) - [p + x(dp/dx)]$

$\Rightarrow \quad 2p - 4x^3p^2 + (dp/dx)(x - 2x^4p) = 0$

or

$$2p(1 - 2x^3p^2) + x(dp/dx)(1 - 2x^3p) = 0$$

$\Rightarrow \quad (1 - 2x^3p^2)[2p + x(dp/dx)] = 0 \quad \ldots (3)$

$\Rightarrow$ Now from (3)

$\Rightarrow \quad 2p + x(dp/dx) = 0 \quad$ or $\quad 1/p\, dp + 2(1/x)dx = 0$

$\Rightarrow$ Now we integrating, $\log p + 2\log x = \log c \quad$ or $\quad px^2 = c \quad$ or $p = c/x^2$

Substituting the value of p in (1), then

$\Rightarrow \quad y + x(c/x^2) = x^4(c^2/x^4) \quad$ or $\quad xy + c = c^2x$

EXAMPLE2. Solve $y = ptanp + \log cosp$.

SOLUTION: Given $\quad y = ptanp + \log cosp \quad \ldots (1)$

Differentiating (1) w.r.t. x and assuming p for dy/dx, we obtain

$\Rightarrow \quad p = [\tan p + p\, sec^2p + \{1/cosp\}(-sinp)](dp/dx)$

$\Rightarrow \quad p = psec^2p(dp/dx) \quad$ or $\quad dx = sec^2pdp \quad \ldots (2)$

$\Rightarrow$ Integrating, (2), $\quad x = tanp + c \quad \ldots (3)$

c being an arbitrary constant. Hence (1) and (3) form the solution in parametric form, p being the parameter.

Method IV: Equation Solvable for x

If the given differential equation $f(x, y, p) = 0$ is solvable for x. Then it can be written in the form

$\Rightarrow \quad f(y, p) = 0 \quad \ldots (1)$

Now differentiating (1) with respect to y and writing $1/p$ for dy/dx, we have

$$\Rightarrow \quad \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \quad \dots (2)$$

Which is the equation assuming y and p. Let us suppose its solution

$$\Rightarrow \quad \psi(x, p, c) = 0 \quad \dots (3)$$

where c being arbitrary constant.

Substituting p between (1) and (3), we get the desired solution of (1) in the form

$$\Rightarrow \quad \psi(x, y, c) = 0 \quad \dots (4)$$

Now we solve equation (1) and (3) to explore x and y in term of p and c in the form

$$\Rightarrow \quad x = f_1(p, c), \qquad y = f_2(p, c) \quad \dots (5)$$

Which obtain us the solution of (1) in the form of parametric equations, p being parameter.

SOLVED EXAMPLES

EXAMPLE. Solve $y = 2px + y^2p^3$.

SOLUTION: Given $\quad y = 2px + y^2p^3 \quad \dots (1)$

Solving, $2px = y - y^2p^3 \quad \Rightarrow \quad x = y(1/2p) - y^2p^2/2 \quad \dots (2)$

Differentiating (2) w.r.t.y and writing $1/p$ for dy/dx, we explained

$$\Rightarrow \quad \frac{1}{p} = (1/2p) - y/2p^2\, dp/dy - 2yp^2/2 - y^2/2 \times 2p \times dp/dy$$

$$\Rightarrow \quad \frac{1}{2p} + yp^2 + \left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \quad p\left(\frac{1}{2p^2} + py\right) + y\left(\frac{1}{2p^2} + py\right)\frac{dp}{dy} = 0$$

$$\Rightarrow \quad \left(\frac{1}{2p^2} + py\right)\left(p + y\frac{dp}{dy}\right) = 0$$

$$\Rightarrow \quad p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{1}{2p^2} + py = 0$$

$\Rightarrow \quad p + y\frac{dp}{dy} = 0$ will obtain the solution of (2) . From $p + y\frac{dp}{dy} = 0$,

we get $\frac{dp}{dy} = -\frac{p}{y}$ Integrating, $\log p = -\log y + \log c \quad$ or

$\log py = \log c$

$\Rightarrow \quad py = c \quad$ or $\quad p = c/y$

Putting the value of p in equation (1), we obtain

$\Rightarrow \quad y = 2(c/y)x + y^2(c/y)^3 \quad$ or $\quad y = 2cx/y + c^3/y$ is required solution.

2.13 *PRINCIPLE OD DUALITY:-*

The principle of duality in differential equations refers to the fact that certain differential equations can be transformed into a dual form by interchanging certain variables or operators. In other words, the dual form of a differential equation is obtained by making a particular transformation that switches the roles of certain variables or operators in the original equation.

Formally, let us consider a linear differential equation of the form:

$$L[y] = f(x)$$

where L is a linear differential operator, y is the dependent variable, and $f(x)$ is a given function. The principle of duality states that if we apply a certain transformation to the differential equation, such as interchanging certain variables or operators, we can obtain a dual equation of the form:

$$L*[z] = g(x)$$

where $L*$ is the dual operator, z is the dual variable, and $g(x)$ is a new function related to $f(x)$ by the transformation.

The principle of duality has many applications in mathematics and physics, particularly in the study of partial differential equations and their solutions. Dual equations often provide a simpler or more intuitive way to understand the properties of a system, and can also lead to new insights or techniques for solving differential equations.

The principle of duality has numerous applications in mathematics and physics. Here are some examples:

1. Electromagnetism: In electromagnetism, the principle of duality is used to relate electric and magnetic fields. Specifically, the electric and magnetic fields are related by a duality transformation that interchanges the electric and magnetic field vectors. This transformation is useful in understanding the symmetries of Maxwell's equations and in solving certain problems in electromagnetism.
2. Laplace transform: The Laplace transform is a mathematical tool used to solve differential equations. The principle of duality can be applied to the Laplace transform by interchanging the roles of time and frequency. This leads to a dual transform, known as the Fourier transform, which is useful in signal processing and other applications.
3. Partial differential equations: The principle of duality can be used to transform certain partial differential equations into dual equations, which can provide a simpler way to understand the properties of the system being studied. For example, the heat equation can be transformed into the wave equation by a duality transformation that interchanges the roles of time and space variables.
4. Quantum mechanics: In quantum mechanics, the principle of duality is used to relate particles and waves. Specifically, the wave-particle duality principle states that particles can exhibit wave-like behavior and waves can exhibit particle-like behavior. This principle is essential to the

understanding of the behavior of quantum systems, such as atoms and subatomic particles.

Overall, the principle of duality is a powerful tool for understanding the symmetries and properties of mathematical and physical systems, and has many applications in diverse areas of science and engineering.

2.14 *SUMMARY:-*

The first-order, first-degree differential equations are linear and can be solved using a variety of methods, including separation of variables, integrating factors, and homogeneous equations. The first-order differential equation that is not a first-degree differential equation can be more challenging to solve than a simple first-degree equation, and may require the use of specific techniques to obtain a solution.

2.15 *GLOSSARY:-*

- Exact Differential Equation.
- Integrating factor.
- Linear Differential equation.
- Equations reducible to linear form
- Bernoulli's Equation (particular case)

2.16 *REFERENCES:-*

- Earl A. Coddington **(**1961). An Introduction to Ordinary Differential Equations, Dover Publications.
- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.

2.17 *SUGGESTED READING:-*

- Erwin Kreyszig (2011). Advanced Engineering Mathematics (10th edition). Wiley.
- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.

- N.P.Bali (2006). Goldan Differential Equations.
- B. Rai, D. P. Choudhury & H. I. Freedman (2013). A Course in Ordinary Differential Equations (2nd edition). Narosa.
- George F. Simmons (2017). Differential Equations with Applications and Historical Notes (3rd edition). CRC Press. Taylor & Francis.

2.18 *TERMINAL QUESTION:-*

(TQ-1) Solve the following differential equations:

a. $(1+x)ydx + (1-y)xdy = 0.$
b. $(1-x^2)(1-y) = xy(1+y)dy.$
c. $\frac{dy}{dx} = \frac{x(2\,log\,x+1)}{.siny+ycosy}.$
d. $dy/dx = e^x + x^2e^{-y}.$
e. $x + y(dy/dx) = 2y.$
f. $(ds/dx) + x^2 = x^2e^{3s}.$
g. $y - x(dy/dx) = x + y(dy/dx).$
h. $x\frac{dy}{dx} + \frac{y^2}{x} = y.$
i. $2\frac{dy}{dx} = \frac{x}{y} - 1.$
j. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$
k. $(x-y)dy = (x+y+1)dx$
l. $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$
m. $\frac{dy}{dx} = \frac{x+y+7}{2x+2y+3}$
n. $(dy/dx) + (1/x)y = x^n$
o. $(dy/dx) + y = e^{-x}$
p. $p^2 + 2pycptx = y^2$
q. $p^2 - 5p + 6 = 0$
r. $y = px + a/p$
s. $y = px + logp$

SELF CHECK QUESTIONS

Choose the Correct Option:

1. The solution of differential equation $p^2 - 8p + 15 = 0$ is
   (a) $p = 5, p = 3$ (b) $(y-5x-c)(y-3x-c) = 0$
   (c) $(y+5x)(y+3x+c) = 0$ (d) None

2. Solution of the equation $y^2logy = xyp + p^2$ is

(a) $\log y = cx + x^2$ (b) $\log y = cx^2 + e^x$
(c) $(y+5x)(y+3x+c) = 0$ (d) None

3. Solution of the equation $y = px + \log p$ is
(a) $y = e^x + c$ (b) $y = cx + \log c$
(c) $y = logcx$ (d) $x = e^y + c$

4. The differential equation $Mdx + Ndy = 0$, where M and N are the functions of x and y is exact if
(a) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (b) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$
(c) $M + N = 0$ (d) $M = N$

2.19 *ANSWERS:-*

TERMINAL ANSWERS

(a) $xy = ce^{y-x}$,

(b) $\log\{x(1-y)^2\} = \frac{1}{2}x^2 - \frac{1}{2}x^2 - 2y + c$,

(c) $y\, siny = x^2 \log x + c$

(d) $e^y = e^x + \frac{1}{3}x^3 + c$,

(e) $(e^{3s} - 1) = c_1 e^{(3s+x^3)}$, where $c_1 = e^{3s}$

(f) $\log(y-x) = c + x/(y-x)$,

(g) $\frac{1}{2}\log(x^2+y^2) + \tan^{-1}(y/x) = logc$,

(h) $cx = e^{x/y}$, **(i)** $(x-2y)(x+y)^2 = c$,

(j) $sin(y/x) = cx$

(k) $2\tan^{-1}\{(2y+1)/(2x+1)\} = \log\left\{c^2\left(x^2+y^2+x+y+\frac{1}{2}\right)\right\}$,

(i) $x - 2y + \log(x-y+2) = c$,

(m) $(2/3)(x+y) - (11/9)\log(3x+3y+10) = x + c$

(n) $xy = x^{n+2}/(n+2) + c$

(o) $ye^x = x + c$ **(p)** $\left(y - \frac{c}{1+cosx}\right)\left(y - \frac{c}{1-cosx}\right) = 0$,

(q) $(y-2x-c)(y-3x-c) = 0$

(r) $y = cx + a/x$, **(s)** $y = cx + \log c$

SELF CHECK ANSWERS

1. (b), **2**. (a), **3**. (b), **4**. (a)

BLOCKII
GENERAL THEORY OF LINEAR DIFFERENTIAL EQUATION

UNIT 3:- LINEAR DIFFERENTIAL EQUATIONS

CONTENT:

3.1 *INTRODUCTION:-*

The course is devoted to the solution of the linear differential equations of higher order with constant or variable coefficients. In this course, learners also learn method of successive approximations, the existence and uniqueness of initial value problem and their solution. The course matter has many applications in several fields. This course develops the problem-solving skills of learners.

3.2 *OBJECTIVES:-*

On completion of the course, learners will be able to-

- Identify the type of a given differential equation and select and apply the appropriate analytical technique for finding the solution.
- Learner will be able to solve first order first degree differential equations utilizing the standard techniques.
- Determine the complete solution of a differential equation with constant coefficients.
- Solve linear differential equations of higher order with variable coefficients.
- Understand method of successive approximations, the existence and uniqueness of IVPs and their solution.

3.3 *BASIC CONCEPTS:-*

Linear differential equations are a fundamental concept in mathematics and physics. A linear differential equation is an equation that involves a function and its derivatives, where the highest power of the function and its derivatives is one. The basic concept of linear differential equations are given below

3.3.1 *INDEPENDENT AND DEPENDENT VARIABLES:-*

The variable whose value is assigned is called independent variables. In another words, the variables whose domain is known is called independent variables and the variable whose value is obtained corresponding to the assigned value is called dependent variable.

If f be a function defined from A to $B, i.e., f: A \to B$ then $\forall\ x \in A\ \exists!\ y \in B$ such that $f(x) = y$.Here, the variable x is called independent variable and the variable y is called dependent variable.

REMARK: Independent variable causes a change in dependent variable but it is not possible that dependent variable could cause a change in independent variable.

3.3.2 *DERIVATIVES:-*

The rate of change of one variable with respect to another variable is called derivative.

Consider a function $y = f(x)$ then the derivative of y at a point $P(x, y)$ is the slope of tangent to the curve $y = f(x)$ at a point $P(x, y)$ and it is denoted by $\frac{dy}{dx}$ and called total derivative or ordinary derivative.

If $z = z(x, y)$ then at any point $P(x, y, z)$ on the surface, the slope of tangent in x −direction is denoted by $\frac{\partial z}{\partial x}$ and it is called partial derivative of z with respect to x and the slope of tangent in y −direction is denoted by $\frac{\partial z}{\partial y}$ and it is called partial derivative of z with respect to y.

3.3.3 *DIFFERENTIAL EQUATIONS:-*

An equation which expressed the relationship between dependent variables, independent variables and derivatives of dependent variable with respect to independent variable is called differential equation.

3.3.4 *CLASSIFICATION OF DIFFERENTIAL EQUATIONS:-*

3.3.4.1 *ORDINARY DIFFERENTIAL EQUATION (O.D.E.):-*

A differential equation which involves derivatives of one or more than one dependent variables with respect to single independent variable, $i.e.$, differential equation involves only ordinary derivatives, is called ordinary differential equation.

3.3.4.2 *PARTIAL DIFFERENTIAL EQUATION (P. D. E.):-*

A differential equation which involves derivatives of one or more than one dependent variables with respect to more than one independent variable, $i.e.$, differential equation involves partial derivatives, is called partial differential equation.

3.3.5 *CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATION:-*

3.3.5.1 *SIMPLE ORDINARY DIFFERENTIAL EQUATION:-*

An ordinary differential equation which contains only one dependent variable.

EXAMPLE1: The differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}=\tan x$ contains only one dependent variable.

3.3.5.2 *SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS:-*

An ordinary differential equation which contains more than one dependent variable.

EXAMPLE2: The differential equations $\frac{dz}{dx}+x\frac{dy}{dx}=\sin x$ and $\frac{dz}{dx}+\frac{dy}{dx}=\cos x$ contains two dependent variables.

3.3.6 *CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATION:-*

Partial Differential Equations (PDEs) can be classified based on various criteria. Here are some common ways to classify them:

3.3.6.1 *SIMPLE PARTIAL DIFFERENTIAL EQUATION:-*

A partial differential equation which contains only one dependent variable.

EXAMPLE1: The differential equation $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ contains only one dependent variable.

3.3.6.2 *SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS:-*

A partial differential equation which contains more than one dependent variable.

EXAMPLE2: The differential equations $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ and $\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} = 0$ contains two dependent variables.

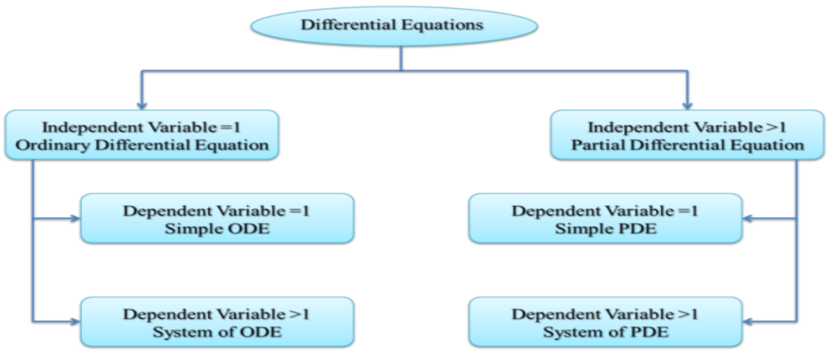


Fig.1: Classifications of Differential Equations

3.3.7 *ORDER AND DEGREE OF A DIFFERENTIAL EQUATION:-*

The highest order derivative occurs in a differential equation is called the **order of the differential equation.**
The highest power of the highest order derivative occurring in the differential equation is called **degree of the differential equation**, after making it free from radicals, fractions and transcendental functions as per the derivatives are concerned.

In another words, the highest exponent of the highest order derivative in differential equation is called **degree of differential equation** provided all the derivatives are in natural power.

REMARK- The order of the differential equation is always defined but the degree of differential equation may or may not define.

EXAMPLE1: The differential equation $v = \cos\frac{dv}{du}$ is of first order but degree does not exist.

EXAMPLE2: The differential equation $e^{y'''} - y'' + xy = 0$ is of order three but degree does not exist.

EXAMPLE3: The differential equation $\left(\frac{d^3y}{dx^3}\right)^{3/2} + \left(\frac{d^3y}{dx^3}\right)^{2/3} = 0$ is of order three and degree nine.

SELF CHECK QUESTIONS-1

(SCQ-1) Find the order and degree of the following differential equations:-

i. $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right) + y = 3e^x$

ii. $\left(\frac{d^2y}{dx^2}\right)^3 + 7\left(\frac{dy}{dx}\right)^4 = 5\sin x$

iii. $\frac{d^2y}{dx^2} + a^2y = 0$

iv. $\left(\frac{dy}{dx}\right)^2 - 3\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - \log x = 0$

v. $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 4x$

vi. $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{2/3} = \frac{d^2y}{dx^2}$

vii. $\frac{d^2y}{dx^2} - \sqrt{\frac{dy}{dx}} = 0$

3.4 *LINEAR AND NON-LINEAR DIFFERENTIAL EQUATION:-*

A differential equation of n-th order is denoted as $f(x, y, y', y'', \dots, y^n) = 0$ is said to be linear if:

i. All the derivatives and dependent variables are of degree one only, and

ii. There does not exist any term containing product of two derivatives or product of derivative and/or dependent variables.

A differential equation which is not linear is called a **non-linear differential equation**.

EXAMPLE1: The differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin x$ is a linear differential equation of second order.

SELF CHECK QUESTIONS-2

(SCQ-1) Which of the following differential equations is linear?

i. $(y + x)\frac{dy}{dx} + y = 0$

ii. $3\frac{dy}{dx} + (x + 4)y = x^2 + \frac{d^2y}{dx^2}$

iii. $\frac{d^3y}{dx^3} = \cos(2ty)$

iv. $y^{(4)} + \sqrt{x}y''' + \cos(x) = e^y$

(SCQ-2) Let $f(x, y, y', y'', \dots, y^n) = 0$ be differential equation of order n. Then choose the incorrect statement.

i. If $\deg f = 1 \Rightarrow f(x, y, y', y'', \ldots, y^n) = 0$ is linear differential equation.
ii. If $f(x, y, y', y'', \ldots, y^n) = 0$ is linear differential equation $\Rightarrow \deg f = 1$.
iii. If $\deg f > 1 \Rightarrow f(x, y, y', y'', \ldots, y^n) = 0$ is non-linear differential equation.
iv. If $f(x, y, y', y'', \ldots, y^n) = 0$ is non-linear differential equation $\Rightarrow \deg f > 1$.

(SCQ-3) If $y(x) = x|x|$ is solution of n-thorder differential equation defined $\forall\ x \in \mathbb{R}$, then possible value of n is?

3.4.1 *FIRST ORDER FIRST DEGREE LINEAR DIFFERENTIAL EQUATION:-*

A linear differential equation of first order first degree is of the form
$a_0(x)\frac{dy}{dx} + a_1(x)y = f(x)$ where $a_0(x) \neq 0$
The most general form of first order first degree linear differential equation is

$$\frac{dy}{dx} + Py = Q$$

Where P and Q are constants or functions of x only.

REMARK: In general, the differential equation $\frac{dy}{dx} + Py = Q$ is non-exact.

3.4.2 *SOLUTION OF FIRST ORDER FIRST DEGREE LINEAR DIFFERENTIAL EQUATION* $\boldsymbol{\frac{dy}{dx} + Py = Q}$*:-*

The given differential equation is

$$\frac{dy}{dx} + Py = Q \qquad \ldots(1)$$

where P and Q are constants or functions of x only.
To solve such type of differential equation we multiplied both side by its integrating factor, $i.e.$, first we find out its integrating factor.
The differential equation $\frac{dy}{dx} + Py = Q$ can be re-written as

$$(Py - Q)dx + dy = 0 \qquad \ldots(2)$$

Compare equation (2) with $M(x, y)dx + N(x, y)dy = 0$, we get

$$M = Py - Q \text{ and } N = 1$$

So, $$\frac{\partial M}{\partial y} = P \text{ and } \frac{\partial N}{\partial x} = 0$$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Therefore, the given differential equation is non-exact.

So, if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \phi(x)$ (function of x only)

Then integrating factor is given by
I.F. $= e^{\int \phi(x)dx}$
Here, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{P-0}{1} = P$ (function of x only)
So, I.F. $= e^{\int \phi(x)dx} = e^{\int P dx}$
$e^{\int P dx}$ is an integrating factor. Hence the differential equation $\frac{dy}{dx} + Py = Q$ always reducible into exact differential equation by multiplying both side by $e^{\int P dx}$. We get

$$e^{\int P dx} \cdot \left\{\frac{dy}{dx} + Py\right\} = e^{\int P dx} \cdot Q$$

$$\Rightarrow e^{\int P dx} \cdot \frac{dy}{dx} + Py \cdot e^{\int P dx} = Q \cdot e^{\int P dx}$$

$$\Rightarrow e^{\int P dx} \cdot dy + Py \cdot e^{\int P dx} dx = Q \cdot e^{\int P dx} dx$$

$$\Rightarrow d(y \cdot e^{\int P dx}) = Q \cdot e^{\int P dx} dx$$

Integrating, $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$
Which is required solution of given differential equation.

WORKING RULE:-

Change the linear differential equation in standard form $\frac{dy}{dx} + Py = Q$
Find an integrating factor by using formula, I.F.= $e^{\int P dx}$
The required solution is obtained by using formula

$$y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot \ dx + c$$

Where c is arbitrary constant.

EXAMPLE: Solve $\frac{dy}{dx} + \frac{y}{x} = x^2$ if $y = 1$ when $x = 1$.

SOLUTION:-Since the given differential equation is linear differential equation of first order first degree. Compare the given differential equation with standard form $\frac{dy}{dx} + Py = Q$, we get
$P = \frac{1}{x}$and$Q = x^2$
So, integrating factor is $e^{\int P dx} = e^{\int (\frac{1}{x}) dx} = x$
Therefore, the general solution of the given differential equation is

$$y \cdot (I.F.) = \int Q \cdot (I.F.) \cdot \ dx + c$$

$$\Rightarrow y \cdot x = \int x^2 \cdot x \cdot dx + c$$

$\Rightarrow xy = \frac{1}{4}x^4 + c$ …(1)
where$c$ is an arbitrary constant.
Now, the given condition is $y = 1$ when $x = 1$
So, from (1) $c = \frac{3}{4}$
Hence, the required solution is $xy = \frac{1}{4}x^4 + \frac{3}{4}$

3.4.3 *EQUATION REDUCIBLE TO LINEAR FORM:-*

CASE I:

An equation of the form

$f'(y)\frac{dy}{dx} + P \cdot f(y) = Q$ …(1)

where P and Q are constants or functions of x only.

The given differential equation can be reduced to linear form by putting $f(y) = t$

Now differentiating both sides with respect to x

$f'(y)\frac{dy}{dx} = \frac{dt}{dx}$

So, from (1),

$\frac{dt}{dx} + P \cdot t = Q$ …(2)

Which is linear in t and x.

So, its solution can be obtained by using the working rule defined above.

Hence the solution is $t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$

Replace t by $f(y)$ we get solution in terms of x and y

$$f(y) \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

CASE II:

An equation of the form $f'(x)\frac{dx}{dy} + P \cdot f(x) = Q$ …(1)

where P and Q are constants or functions of y only.

The given differential equation can be reduced to linear form by putting $f(x) = t$

Now differentiating both sides with respect to y

$f'(x)\frac{dx}{dy} = \frac{dt}{dy}$

So, from (1),

$\frac{dt}{dy} + P \cdot t = Q$ …(2)

Which is linear in t and y.

Integrating factor of equation (2) is $e^{\int P dy}$. So, its solution is

$$t \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + c$$

Replace t by $f(x)$ we get solution in terms of x and y

$$f(x) \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + c$$

SELF CHECK QUESTIONS-3

(SCQ-1) Solve the following differential equations:-

i. $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\left(\frac{dy}{dx}\right) = 0$

ii. $x\left(\frac{dy}{dx}\right) - y = 2x^2 \csc x$

iii. $x\log x\left(\frac{dy}{dx}\right)+y=2\log x$

iv. $(2x-10y^3)\left(\frac{dy}{dx}\right)+y=0$

v. $\left(\frac{dy}{dx}\right)+\left(\frac{1}{x}\right)=\frac{e^y}{x^2}$

vi. $\left(\frac{dy}{dx}\right)+y\sec x=\tan x$

vii. $(x^2+y^2+2x)dx+2ydy=0$

viii. $\left(\frac{dy}{dx}\right)-{(\tan y)}\Big/{(1+x)}=(1+x)e^x\sec y$

3.5 *GENERAL THEORY OF LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER:-*

We have already discussed that linear differential equation are those in which the dependent variable and its derivatives occurs only in the first degree and there is no term containing their product. Thus, the general form of linear differential equation of nth order is

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=Q(x) \qquad \ldots(1)$$

Where $x\in[a,b]$

$a_0(x),a_1(x),a_2(x),\ldots,a_n(x)$ and $Q(x)$all are continuous function of x and$a_0(x)\neq 0\ \forall\ x\in[a,b]$

In terms of operator D equation (1) can be rewritten as

$$[a_0(x)D^n+a_1(x)D^{n-1}+a_2(x)D^{n-2}+\cdots+a_{n-1}(x)D+a_n(x)]y=Q(x)$$

Where $D=\frac{d}{dx}\Longrightarrow L[D]y=\ Q(x)$

Where $L[D]=a_0(x)D^n+a_1(x)D^{n-1}+a_2(x)D^{n-2}+\cdots+a_{n-1}(x)D+a_n(x)$

3.5.1 *CLASSIFICATION OF LINEAR DIFFERENTIAL EQUATION:-*

- **HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION:-**

The linear differential equation (1) is said to be **homogeneous linear differential equation** of order $n$if $Q(x)\equiv 0$ $i.e.,$ the n-th order homogeneous linear differential equation can be written as

$$[a_0(x)D^n+a_1(x)D^{n-1}+a_2(x)D^{n-2}+\cdots+a_{n-1}(x)D+a_n(x)]y=0\ \forall\ x\in[a,b]$$

Particular Case:

The second order homogeneous linear differential equation is

$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$
where $a_0(x) \neq 0, a_0(x), a_1(x), a_2(x)$ all are continuous on the given domain.

- **NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION:-**

The linear differential equation (1) is said to be **non-homogeneous linear differential equation** of order n if $Q(x)$is not identically zero in the given domain.

- **LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENT:-**

The linear differential equation (1) said to be **linear differential equation** with constant coefficient of order n if all the coefficient $a_0(x), a_1(x), a_2(x), \dots, a_n(x)$ of the given differential equation are constant.

- **LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENT:-**

The linear differential equation (1) is said to be linear differential equation with variable coefficient if atleast one of the coefficients of the differential equation is not constant.

EXAMPLE1:$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$ is second order linear homogenous differential equation with constant coefficients.

EXAMPLE2:$x^2\frac{d^4y}{dx^4} + x^2(x-2)\frac{dy}{dx} + (x-2)y = 0$ is second order linear homogenous differential equation with variable coefficients.

EXAMPLE3:$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (1-x^2)y = \sin x$ is second order linear non-homogenous differential equation with variable coefficients.

3.5.2 *SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENT:-*

The linear differential equation with constant coefficient of order n is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = Q(x)$$

or,

$$[a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n]y = Q(x) \quad \dots(1)$$

where $a_0, \neq 0, a_0, a_1, a_2, \dots, a_n$ all are constant.
The general form of (1) can be written as

$$[D^n + k_1D^{n-1} + \cdots + k_n]y = Q(x)$$

Where $f(D) = D^n + k_1D^{n-1} + \cdots k_n$ is a polynomial in D.
Thus, the operator D stands for the operations of differential and can be treated much the same as an algebraic quantity. $i.e., f(D)$ can be factorized by ordinary roots of algebra and the factors may be taken in any order.

3.5.3 *COMPLEMENTARY FUNCTION OF HOMOGENOUS LINEAR DIFFERENTIAL EQUATION:-*

Consider a homogenous linear differential equation of order n is
$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = 0 \qquad \ldots(1)$$
Where $k_1, k_2, \ldots, k_n$ are constant.
In terms of operator D equation (1) can be re-written as
$$[D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n] y = 0 \qquad \ldots(2)$$
Its symbolic coefficient equal to zero
$$i.e., D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n = 0 \qquad \ldots(3)$$
is called the auxiliary equation.
Since the auxiliary equation is of degree n so, by fundamental theorem of algebra $m_1, m_2, \ldots, m_n$ be its n roots.
Now, we have different cases arise:

CASE I:

If all the roots of the auxiliary equation be real and distinct, then equation (3) is equivalent to
$$[(D - m_1)(D - m_2) \ldots (D - m_n)] y = 0 \qquad \ldots(4)$$
Since the factors in equation (4) can be taken in any order, so it will be satisfied by the solution of $(D - m_r)y = 0$ where $1 \le r \le n$.
Now for $(D - m_r)y = 0$
$$\Rightarrow \frac{dy}{dx} - m_r y = 0$$
$$\Rightarrow \frac{dy}{y} = m_r dx$$
$\Rightarrow y = c_r e^{m_r x}$ where $1 \le r \le n$.
$\Rightarrow y = c_r e^{m_r x}$ satisfies $(D - m_r)y = 0$.
$\Rightarrow y = c_r e^{m_r x}$ satisfies $[(D - m_1)(D - m_2) \ldots (D - m_n)] y = 0$.
So, $y = c_r e^{m_r x}$ where $1 \le r \le n$is solution of equation (4).
Thus, the complete solution of equation (4) is
$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \quad \text{or} \quad y = \sum_{r=1}^{n} c_r e^{m_r x}$$

CASE II:

If two roots of equation (4) are equal $[i.e., m_1 = m_2 = m\,(\text{say})]$ then equation (4) can be rewritten as
$$(D - m)^2 (D - m_3)(D - m_4) \ldots (D - m_n) = 0$$
Now, for $\qquad (D - m)^2 y = 0 \qquad \ldots(5)$
Put $(D - m)y = z$
So, from equation (5) $(D - m)z = 0$
$$\Rightarrow \frac{dz}{dx} - mz = 0$$
$$\Rightarrow \frac{dz}{z} = m dx$$
$$\Rightarrow z = c_1 e^{mx}$$
$$\Rightarrow (D - m)y = c_1 e^{mx}$$

$\Rightarrow \frac{dy}{dx} - my = c_1 e^{mx}$ …(6)

Equation (6) is linear differential equation of first order first degree. Its integrating factoris e^{-mx}

So, the solution of equation (6) is

$$y \cdot e^{-mx} = \int [c_1 e^{mx} \cdot e^{-mx}] dx + C$$

$$\Rightarrow y \cdot e^{-mx} = c_1 x + c_2$$

$$\Rightarrow y = (c_1 x + c_2) e^{mx}$$

Therefore, with the help of case I the complete solution of equation (4) is given by

$$y = (c_1 x + c_2) e^{mx} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Particular Case:

If the given auxiliary equation has three equal roots $[i.e., m_1 = m_2 = m_3 = m$ (say)] then the complete solution is given as

$$y = (c_1 x^2 + c_2 x + c_3) e^{mx} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

<u>CASE III:</u>

When the auxiliary equation has complex roots.

If one pair of roots of equation (4) be imaginary. $i.e., m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$

Then with the help of case I the complete solution is given as

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$\Rightarrow y = e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$\Rightarrow y = e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$\Rightarrow y = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and$C_2 = i(c_1 - c_2)$

Particular Case:

If the given auxiliary equation has two pairs of imaginary roots be equal.

$i.e., m_1 = m_2 = \alpha + i\beta$and $m_3 = m_4 = \alpha - i\beta$

Then the complete solution is given as

$$y = e^{\alpha x}[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$$

<u>CASE IV:</u>

When the auxiliary equation has surd roots. If one pair of roots of auxiliary equation be surds.$i.e., m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta}$

Then the complete solution is given as

$y = c_1 e^{(\alpha + \sqrt{\beta})x} + c_2 e^{(\alpha - \sqrt{\beta})x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

$$\Rightarrow y = e^{\alpha x}\left[c_1 e^{\sqrt{\beta} x} + c_1 e^{-\sqrt{\beta} x}\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

$$\Rightarrow y = e^{\alpha x}\left[c_1\left(\cosh x\sqrt{\beta} + \sinh x\sqrt{\beta}\right) + c_2\left(\cosh x\sqrt{\beta} - \sinh x\sqrt{\beta}\right)\right] + \cdots + c_n e^{m_n x}$$

$$y = e^{\alpha x}\left[C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$

3.5.4 *WORKING RULE FOR FINDING COMPLETE SOLUTION OF THE GIVEN HOMOGENOUS LINEAR DIFFERENTIAL EQUATION:-*

Consider the homogenous linear differential equation of order n with constant coefficients is

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+k_{n-1}\frac{dy}{dx}+k_n y=0 \qquad \ldots(1)$$

Step I: Re-write the equation (1) in the symbolic form as

$$[D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n]y=0$$

Step II: The auxiliary equation is

$$D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n=0$$

Step III: Find the roots of auxiliary equation.

Step IV: Write down the complete solution with the help of the following table.

S. No.	Nature of roots of auxiliary equation	Complementary Function
1.	If all the roots of auxiliary equation are real and distinct say, $m_1, m_2, m_3, \ldots$	$c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\cdots$
2.	If all the roots of auxiliary equation are real and two equal roots say, $m_1, m_2, m_3, \ldots$ where $m_1=m_2=m$	$(c_1x+c_2)e^{mx}+c_3e^{m_3x}+\cdots$
3.	If all the roots of auxiliary equation are real and three equal roots say, $m_1, m_2, m_3, m_4 \ldots$ where $m_1=m_2=m_3=m$	$(c_1x^2+c_2x+c_3)e^{mx}+c_4e^{m_4x}+\cdots$
4.	If auxiliary equation has one pair of imaginary roots say $\alpha+i\beta, \alpha-i\beta,\ m_3, \ldots$	$e^{\alpha x}[C_1\cos\beta x+C_2\sin\beta x]+c_3e^{m_3x}+\cdots$
5.	If auxiliary equation have two pair of imaginary roots say, $\alpha\pm i\beta, \alpha\pm i\beta,\ m_5, \ldots$	$e^{\alpha x}[(C_1x+C_2)\cos\beta x+(C_3x+C_4)\sin\beta x]+c_5e^{m_5x}+\cdots$
6.	If auxiliary equation has one pair of surd roots say, $m_1=\alpha+\sqrt{\beta}, m_2=\alpha-\sqrt{\beta},\ m_3, \ldots$	$e^{\alpha x}\left[C_1\cosh x\sqrt{\beta}+C_2\sinh x\sqrt{\beta}\right]+c_3e^{m_3x}+\cdots$

EXAMPLE1: Solve$(D^3 + D^2 + 4D + 4)y = 0$

SOLUTION: Here the given differential equation is $(D^3 + D^2 + 4D + 4)y = 0$

Its corresponding auxiliary equation is$D^3 + D^2 + 4D + 4 = 0$

$$i.e., (D^2 + 4)(D + 1) = 0$$

$\Rightarrow D = -1, \ \pm 2i$

Hence the complete solution is

$$y = c_1 e^{-x} + e^{0x}(c_2 cos2x + c_3 sin2x)$$

$$\Rightarrow \qquad y = c_1 e^{-x} + c_2 cos2x + c_3 sin2x$$

EXAMPLE2: Solve $\frac{d^4x}{dt^4} + 4x = 0$

SOLUTION: Given equation in symbolic form is $(D^4 + 4)x = 0$

Therefore, Auxiliary equation is $D^4 + 4 = 0$

Or,$(D^4 + 4D^2 + 4) - 4D^2 = 0$

$$\Rightarrow (D^2 + 2)^2 - (2D)^2 = 0$$

$$\Rightarrow (D^2 + 2D + 2)(D^2 - 2D + 2) = 0$$

Therefore, either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

$\Rightarrow D = \frac{-2\pm\sqrt{(-4)}}{2}$ and $D = \frac{2\pm\sqrt{(-4)}}{2}$

$\Rightarrow \ D = -1 \pm iandD = 1 \pm i$

Hence the required solution is $x = e^{-t}(c_1 cost + c_2 sint) + e^t(c_3 cost + c_4 sint)$

EXAMPLE3: Solve $\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$

SOLUTION: Here the given differential equation is $(D^2 + (a + b)D + ab)y = 0$

The corresponding auxiliary equation is$D^2 + (a + b)D + ab = 0$

$$\Rightarrow (D + a)(D + b) = 0$$

$$\Rightarrow D = -a, \qquad -b$$

Hence the required solution is $y = c_1 e^{-ax} + c_2 e^{-bx}$

EXAMPLE 4: Solve $(D^2 - 4D + 1)y = 0$

SOLUTION: Here the given differential equation is $(D^2 - 4D + 1)y = 0$

The corresponding auxiliary equation is$D^2 - 4D + 1 = 0$

$$\Rightarrow D = \frac{4 \pm \surd(16 - 4)}{2}$$

$$\Rightarrow D = 2 \pm \surd 3$$

Hence the required solution is $y = 2c_1 e^{(2+\surd 3)x} + c_2 e^{(2-\surd 3)x}$

$$\Rightarrow y = e^{2x}\{c_1 e^{x\surd 3} + c_2 e^{-x\surd 3}\}$$

EXAMPLE 5: Solve $(D^3 - 2D^2 - 4D + 8)y = 0$

SOLUTION: Here the given differential equation is $(D^3 - 2D^2 - 4D + 8)y = 0$

The corresponding auxiliary equation is$D^3 - 2D^2 - 4D + 8 = 0$

$$\Rightarrow D^2(D - 2) - 4(D - 2) = 0$$

$$\Rightarrow (D-2)(D^2-4)=0$$
$$\Rightarrow (D-2)(D-2)(D+2)=0$$
$$\Rightarrow D=2, \quad 2, \quad -2$$

Therefore, the required solution is $y=(c_1+c_2x)e^{2x}+c_3e^{-2x}$

EXAMPLE 6: Solve $(D^4-7D^3+18D^2-20D+8)y=0$

SOLUTION: Here given differential equation is $(D^4-7D^3+18D^2-20D+8)y=0$

The corresponding auxiliary equation is$D^4-7D^3+18D^2-20D+8=0$

$$\Rightarrow D^3(D-1)-6D^2(D-1)+12D(D-1)-8(D-1)=0$$
$$\Rightarrow (D-1)(D^3-6D^2+12D-8)=0$$
$$\Rightarrow (D-1)[D^2(D-2)-4D(D-2)+4(D-2)]=0$$
$$\Rightarrow (D-1)(D-2)(D^2-4D+4)=0$$
$$\Rightarrow (D-1)(D-2)(D-2)^2=0$$
$$\Rightarrow D=1, \quad 2 \text{ (Thrice)}$$

Therefore, the required solution is $y=c_1e^x+(c_2x^2+c_3x+c_4)e^{2x}$

EXAMPLE 7: Solve $(D^4+4)y=0$

SOLUTION: Here the given differential equation is $(D^4+4)y=0$

The corresponding auxiliary equation is$D^4+4=0$

$$\Rightarrow D^4=-4$$
$$\Rightarrow D^2=\pm 2i$$

$\Rightarrow D^2=2i$and$-2i$ (1)

Or,$D=\pm\sqrt{(2i)}\ and\ \pm\sqrt{(-2i)}$

Let $\sqrt{(2i)}=a+ib$

Squaring both sides, we get

$$2i=(a^2-b^2)+(2ab)i$$

Equating real and imaginary parts on both sides, we get

$a^2-b^2=0$and$2ab=2\ or\ ab=1$

Therefore $a^2-\left(\frac{1}{a^2}\right)=0$ since $b=\frac{1}{a}$

Or $a^4-1=0$ or $a^4=1$

$$\Rightarrow a=\pm 1, \quad \pm i$$

If $a=1,$ we have from $ab=1,\ b=1$

Hence $\sqrt{(2i)}=1+i$

Similarly, we can prove that$\sqrt{(-2i)}=1-i$

Therefore from (1), the roots of the auxiliary equation are

$\pm(1+i)$and$\pm(1-i)$

$$i.e., 1\pm i\ and\ -1\pm i$$

Therefore, the required solution is

$$y=e^x[c_1cosx+c_2sinx]+e^{-x}[c_3cosx+c_4sinx]$$

EXAMPLE 8: Solve $(D^4+D^2+1)y=0$

SOLUTION: Here the given differential equation is

$$(D^4+D^2+1)y=0$$

The corresponding auxiliary equation is

$$D^4 + D^2 + 1 = 0 \qquad \ldots(1)$$

$$\Rightarrow (D^4 + D^2 + 1) - D^2 = 0$$

$$\Rightarrow (D^2 + 1)^2 - D^2 = 0$$

$$\Rightarrow (D^2 + 1 + D)(D^2 + 1 - D) = 0$$

Now $D^2 + D + 1 = 0$ gives $D = \frac{1}{2}\left[-1 \pm \sqrt{(1-4)}\right]$

$$\Rightarrow D = \frac{1}{2}\left[-1 \pm i\sqrt{3}\right]$$

Similarly, $D^2 - D + 1 = 0$ gives $D = \frac{1}{2}\left[1 \pm i\sqrt{3}\right]$

Therefore, the solution of auxiliary equation (1) is $\frac{1}{2}\left[-1 \pm i\sqrt{3}\right], \frac{1}{2}\left[1 \pm i\sqrt{3}\right]$

Therefore, the required solution is

$$y = e^{-x/2}\left[c_1 cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 sin\left(\frac{x\sqrt{3}}{2}\right)\right] + e^{x/2}\left[c_3 cos\left(\frac{x\sqrt{3}}{2}\right) + c_4 sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

EXAMPLE 9: Solve $(D^6 - 1)y = 0$

SOLUTION: Here the given differential equation is $(D^6 - 1)y = 0$

The corresponding auxiliary equation is $D^6 - 1 = 0$

$$\Rightarrow (D^6 - 1)(D^4 + D^2 + 1) = 0$$

$$\Rightarrow (D - 1)(D + 1)(D^2 - D + 1)(D^2 + D + 1) = 0$$

Its roots are 1, -1, $\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$ and $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$

Therefore, the required solution is

$$y = c_1e^x + c_2e^{-x} + e^{x/2}\left[c_3 \cos\left(\frac{1}{2}x\sqrt{3}\right) + c_4 \sin\left(\frac{1}{2}x\sqrt{3}\right)\right] + e^{-x/2}\left[c_5 \cos\left(\frac{1}{2}x\sqrt{3}\right) + c_6 \sin\left(\frac{1}{2}x\sqrt{3}\right)\right]$$

SELF CHECK QUESTIONS-4

(SCQ-1) Solve the following differential equations:

- **i.** $(D^3 - 13D + 12)y = 0$
- **ii.** $(D^2 + 7D + 10)y = 0$
- **iii.** $(D^3 - 4D^2 + 5D - 2)y = 0$

3.5.5 *INVERSE OPERATOR:-*

$\frac{1}{f(D)}Q(x)$ is that function of x, not containing any arbitrary constant which when operated upon by $f(D)$ gives $Q(x)$. $i.e., f(D)\left[\frac{1}{f(D)}Q(x)\right] = Q(x)$.

Hence $\frac{1}{f(D)}Q(x)$ satisfies the equation $f(D)y = Q(x)$ and is therefore its particular integral.

REMARK: $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

3.5.6 *SOME IMPORTANT RESULTS:-*

- $\boldsymbol{\frac{1}{D}Q(x) = \int Q(x)\,dx}$

PROOF: Let $\frac{1}{D}Q(x) = y$ ...(1)

Operating both sides by D

$$D.\frac{1}{D}Q(x) = D \cdot y$$
$$\Rightarrow Q(x) = D \cdot y$$
$$\Rightarrow Q(x) = \frac{dy}{dx}$$

Integrating both side with respect to x, we get

$y = \int Q(x)\,dx$, Since equation (1) does not contain any arbitrary constant. So, no constant of integration be added.

Hence, $\frac{1}{D}Q(x) = \int Q(x)\,dx$

- $\boldsymbol{\frac{1}{(D-a)}Q(x) = e^{ax}\int Q(x)\cdot\ e^{-ax}dx}$

PROOF: Let $\frac{1}{(D-a)}Q(x) = y$ ...(1)

Operating both sides by $(D - a)$

$$(D-a)\cdot\frac{1}{(D-a)}Q(x) = (D-a)\cdot y$$
$$\Rightarrow Q(x) = (D-a)\cdot y$$
$$\Rightarrow Q(x) = \frac{dy}{dx} - ay$$

Which is first order, first degree linear differential equation. Its integrating factor is e^{-ax}.

So, its solution is

$ye^{-ax} = \int Q(x)\ \cdot e^{-ax}dx$, Since equation (1) does not contain any arbitrary constant. So, no constant of integration be added.

$$y = e^{ax}\int Q(x)\ \cdot e^{-ax}dx$$

Hence, $\frac{1}{(D-a)}Q(x) = e^{ax}\int Q(x)\cdot e^{-ax}dx$

3.5.7 *RULES FOR FINDING THE PARTICULAR INTEGRAL OF NON-HOMOGENOUS LINEAR*

DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS:-

General method for finding particular integral with constant coefficient:

Consider the non-homogenous linear differential equation of order n.

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = Q(x) \qquad \ldots(1)$$

In terms of operator D equation (1) can be rewritten as

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = Q(x)$$

Therefore, particular integral is

$$\frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n} Q(x)$$

or, $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)} Q(x)$

Where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be n roots of $D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n$

Therefore $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)} Q(x) = \frac{1}{(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)} \left[\frac{1}{(D-\alpha_1)} . Q(x)\right]$

$$= \frac{1}{(D-\alpha_2)(D-\alpha_3)\ldots(D-\alpha_n)} \left\{\frac{e^{\alpha_1 x}}{Q(x)e^{-\alpha_1 x}}\, dx\right\}$$

Repeat this process for each factor in same manner, we get the required particular integral.

Some Particular Cases:

<u>**CASE I:**</u>

Consider $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = Q(x)$

When R.H.S. of equation (1) is of the form e^{ax}

$i.e., Q(x) = e^{ax}$ provided $f(a) \neq 0$

Since $De^{ax} = ae^{ax}$

$$D^2 e^{ax} = a^2 e^{ax}$$

In general, $D^n e^{ax} = a^n e^{ax}$

$$\Rightarrow (D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)e^{ax}$$
$$= (a^n + k_1 a^{n-1} + k_2 a^{n-2} + \cdots + k_n)e^{ax}$$
$$i.e., [f(D)e^{ax}] = [f(a)e^{ax}]$$

Now, operating on both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}[f(D)]e^{ax} = \frac{1}{f(D)}[f(a)]e^{ax}$$
$$\Rightarrow e^{ax} = f(a).\frac{1}{f(D)} e^{ax}$$

$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ Provided $f(a) \neq 0$ $\qquad \ldots(1)$

Particular Case:

In the above case if a is simple root of auxiliary equation.

$i.e., a$ is root of the auxiliary equation $f(D) = 0$

$\Rightarrow (D-a)$is factor of $f(D)$

$$\Rightarrow f(D) = (D-a)\emptyset(D)$$

Where$\emptyset(a) \neq 0$

Now, P.I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)} \frac{1}{\emptyset(D)} e^{ax}$

$= \frac{1}{(D-a)} \frac{1}{\emptyset(a)} e^{ax}$

{by (1)}

$= \frac{1}{\emptyset(a)} \frac{1}{(D-a)} e^{ax}$

$= \frac{1}{\emptyset(a)} e^{ax} \int e^{ax} e^{-ax} dx$

$= \frac{1}{\emptyset(a)} e^{ax} \int dx$

$= x \frac{1}{\emptyset(a)} e^{ax}$

Therefore $\frac{1}{f(D)} e^{ax} = x \frac{1}{\emptyset(a)} e^{ax}$

Or,$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ provided $f'(a) \neq 0$

Similarly, if a is root of auxiliary equation of order two, then

$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$ provided $f''(a) \neq 0$

and so on.

<u>CASE II:</u>

When the R.H.S. of auxiliary equation is of the form $\sin(ax+b)$ or $\cos(ax+b)$,provided $f(-a^2) \neq 0$.

Since $D \sin(ax+b) = a\cos(ax+b)$

$$\Rightarrow D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$\Rightarrow D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$

$$\Rightarrow D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

In general,

$$(D^2)^k \sin(ax+b) = (-a^2)^k \sin(ax+b)$$

$$\Rightarrow f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$$

Operating both sides by $\frac{1}{f(D^2)}$

$$\frac{1}{f(D^2)} f(D^2) \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2) \sin(ax+b)$$

$$\Rightarrow \sin(ax+b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax+b)$$

$\Rightarrow \frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)$ Provided $f(-a^2) \neq 0$

(4)

Particular Case:

If $f(-a^2) = 0$

By Euler's formula

$\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$

Therefore,

$\frac{1}{f(D^2)} \sin(ax+b) =$ Imaginary Part of $\frac{1}{f(D^2)} e^{i(ax+b)}$ since $f(-a^2) = 0$

$\Rightarrow \frac{1}{f(D^2)}\sin(ax+b)$ =Imaginary Part of of $x\frac{1}{f'(D^2)}e^{i(ax+b)}$ where $D^2=-a^2$

Therefore, $\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)$ provided $f'(-a^2) \neq 0$

If $f'(-a^2)=0$

Then $\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$ provided $f''(-a^2) \neq 0$ and so on.

Similarly, $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$ provided $f(-a^2) \neq 0$

If $f(-a^2)=0$, $\frac{1}{f(D^2)}\cos(ax+b) = x\frac{1}{f'(-a^2)}\cos(ax+b)$ provided $f'(-a^2) \neq 0$

If $f'(-a^2)=0$,

Then $\frac{1}{f(D^2)}\cos(ax+b) = x^2\frac{1}{f''(-a^2)}\cos(ax+b)$ provided $f''(-a^2) \neq 0$ and so on.

CASE III: When $Q(x)=x^m$

P.I. $=\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$

Now expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m or operate on x^m term by term.

Since the $(m+1)^{th}$ are higher order derivatives of x^m are zero. We need not be considered terms beyond D^m.

CASE IV: When $Q(x)=e^{\alpha x}W$ when W being a function of x.

Let V be any function of x.

Since $D(e^{\alpha x}V) = e^{\alpha x}DV + \alpha e^{\alpha x}V$

$= e^{\alpha x}(D+\alpha)V$

Again, $D^2(e^{\alpha x}V) = e^{\alpha x}D^2V + 2\alpha e^{\alpha x}DV + \alpha^2 e^{\alpha x}V$

$= e^{\alpha x}(D+\alpha)^2V$

In general, $D^n(e^{\alpha x}V) = e^{\alpha x}(D+\alpha)^nV$

Therefore $f(D)(e^{\alpha x}V) = e^{\alpha x}f(D+\alpha)V$

Operating both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)}f(D)(e^{\alpha x}V) = \frac{1}{f(D)}[e^{\alpha x}f(D+\alpha)]V$$

$$\Rightarrow e^{\alpha x}V = \frac{1}{f(D)}[e^{\alpha x}f(D+\alpha)]V$$

Now put $f(D+\alpha)V = W$

$V = \frac{1}{f(D+\alpha)}W$, so that

$$e^{\alpha x}\frac{1}{f(D+\alpha)}W = \frac{1}{f(D)}(e^{\alpha x}W)$$

$$\Rightarrow \frac{1}{f(D)}(e^{\alpha x}W) = e^{\alpha x}\frac{1}{f(D+\alpha)}W$$

CASE V: When $Q(x)$ is any other function of x of the above form.

Then P.I. $= \frac{1}{f(D)} Q(x) = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)} Q(x)$

Resolving into partial fractions, we get

$$\frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)} Q(x)$$
$$= \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \cdots \frac{A_n}{(D-m_n)}\right] Q(x)$$

or, $\frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)} Q(x) = A_1 \frac{1}{(D-m_1)} Q(x) + A_2 \frac{1}{(D-m_2)} Q(x) + \cdots + A_n \frac{1}{(D-m_n)} Q(x)$

Hence particular integral is given by

P.I.$= A_1 e^{m_1 x} \int Q(x) e^{-m_1 x} dx + A_2 e^{m_2 x} \int Q(x) e^{-m_2 x} dx + \cdots + A_n e^{m_n x} \int Q(x) e^{-m_n x} dx$

3.5.8 *WORKING RULE TO SOLVE THE NON-HOMOGENOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS:-*

Consider the non-homogenous linear differential equation with constant coefficient of order n is

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = Q(x) \qquad \ldots(1)$$

Equation (1) can be rewritten in terms of operator D as

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n) y = Q(x)$$

StepI: Find the complementary function for its homogenous part$(\mathrm{D}^{\mathrm{n}} + \mathrm{k}_1 \mathrm{D}^{\mathrm{n}-1} + \mathrm{k}_2 \mathrm{D}^{\mathrm{n}-2} + \cdots + \mathrm{k}_{\mathrm{n}}) y = 0$with the help of following table.

S. No.	Nature of roots of auxiliary equation	Complementary Function
1.	If all the roots of auxiliary equation are real and distinct say, $m_1, m_2, m_3, \ldots$	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots$
2.	If all the roots of auxiliary equation are real and two equal roots say, $m_1, m_2, m_3, \ldots$ where $m_1 = m_2 = m$	$(c_1 x + c_2) e^{mx} + c_3 e^{m_3 x} + \cdots$
3.	If all the roots of auxiliary equation are real and three equal roots say, $m_1, m_2, m_3, m_4 \ldots$ where $m_1 = m_2 = m_3 = m$	$(c_1 x^2 + c_2 x + c_3) e^{mx} + c_4 e^{m_4 x} + \cdots$
4.	If auxiliary equation has one pair of imaginary roots say $\alpha + i\beta, \alpha - i\beta,\ m_3, \ldots$	$e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x] + c_3 e^{m_3 x} + \cdots$

5.	If auxiliary equation have two pair of imaginary roots say, $\alpha \pm i\beta, \alpha \pm i\beta,\ m_5, \ldots$	$e^{\alpha x}[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x] + c_5 e^{m_5 x} + \cdots$
6.	If auxiliary equation has one pair of surd roots say, $m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta},\ m_3, \ldots$	$e^{\alpha x}\left[C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}\right] + c_3 e^{m_3 x} + \cdots$

StepII: Find the particular integral by using any of the above form.
StepIII: The general solution of equation (1) is given by C.F.+P.I

SOLVED EXAMPLES

EXAMPLE1: Solve $(D^2 - 7D + 6)y = e^{2x}$, given that $y = 0$ when $x = 0$

SOLUTION:-The given differential equation is $(D^2 - 7D + 6)y = e^{2x}$
Its auxiliary equation is $D^2 - 7D + 6 = 0$

$$\Rightarrow D = 1, \quad 6$$

Therefore,$C.F. = c_1 e^x + c_2 e^{6x}$, where c_1 and c_2 are arbitrary constants.
Now $P.I. = \frac{1}{D^2 - 7D + 6} e^{2x}$

$\Rightarrow P.I. = \frac{1}{(2)^2 - 7(2) + 6} e^{2x} = -\frac{1}{4} e^{2x}$ …(1)

Given that $y = 0$ when $x = 0$
Therefore from (1), $0 = c_1 e^0 + c_2 e^0 - \frac{1}{4} e^0$
Or,$c_2 = \frac{1}{4} - c_1$
Hence from (1) the required solution is

$$y = c_1 e^x + \left(\frac{1}{4} - c_1\right) e^{6x} - \frac{1}{4} e^{2x}$$

$$\Rightarrow y = c_1(e^x - e^{6x}) + \frac{1}{4}(e^{6x} - e^{2x})$$

$$\Rightarrow y = c_1(e^x - e^{6x}) + \frac{1}{4} e^{2x}(e^{4x} - 1)$$

EXAMPLE2: Solve $D^2(D+1)^2(D^2 + D + 1)^2 y = e^x$

SOLUTION:- Here the given differential equation is

$$D^2(D+1)^2(D^2 + D + 1)^2 y = e^x$$

Its auxiliary equation is$D^2(D+1)^2(D^2 + D + 1)^2 y = 0$
The roots are$0,\ 0,\ -1,\ -1,\ \frac{1}{2}\left[-1 \pm i\sqrt{3}\right],\ \frac{1}{2}\left[-1 \pm i\sqrt{3}\right]$
$i.e.$, $0,\ -1,\ \frac{1}{2}\left[-1 \pm i\sqrt{3}\right]$twice each.
Therefore,

$$C.F. = (c_1x + c_2)e^{0x} + (c_3x + c_4)e^{-x} + e^{-x/2}\left[(c_5x + c_6)cos\left(\frac{1}{2}\sqrt{3}x\right) + (c_7x + c_8)sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

And, $P.I. = \frac{1}{D^2(D+1)^2(D^2+D+1)^2}e^x$

$$P.I. = \frac{1}{1^2(1+1)^2(1^2+1+1)^2}e^x = \frac{1}{36}e^x$$

Therefore, required solution is

$y = C.F. + P.I.$, where $C.F.$ and $P.I.$ are given above.

EXAMPLE3: Solve $(D^2 - D - 2)y = sin2x$

SOLUTION:- Here the given differential equation is

$$(D^2 - D - 2)y = sin2x$$

Its auxiliary equation is $D^2 - D - 2 = 0$, which gives

$$D = \frac{1}{2}\left[1 \pm \sqrt{(1+8)}\right]$$

$$\Rightarrow D = \frac{1}{2}[1 \pm 3]$$

$$\Rightarrow D = 2, \quad -1$$

Therefore, $C.F. = c_1e^{2x} + c_2e^{-x}$

And, $P.I. = \frac{1}{D^2-D-2}sin2x$

$$\Rightarrow P.I. = \frac{1}{-2^2 - D - 2}sin2x$$

$$\Rightarrow P.I. = -\frac{1}{(D+6)}sin2x$$

$$\Rightarrow P.I. = -\frac{(D-6)}{(D-6)(D+6)}sin2x$$

$$\Rightarrow P.I. = -\frac{(D-6)}{(D^2-36)}sin2x$$

$$\Rightarrow P.I. = -\frac{(D-6)}{(2^2-36)}sin2x$$

$$\Rightarrow P.I. = \frac{1}{40}[D(sin2x) - 6(sin2x)]$$

$$\Rightarrow P.I. = \frac{1}{40}[2cos2x - 6sin2x]$$

$$\Rightarrow P.I. = \frac{1}{20}[cos2x - 3sin2x]$$

Hence the required solution is

$$y = C.F. + P.I.$$

$$y = c_1e^{2x} + c_2e^{-x} + \frac{1}{20}[cos2x - 3sin2x]$$

EXAMPLE4: Solve $(D^2 - D - 8)y = 2sin^2x$

SOLUTION:- Here the given differential equation is

$$(D^2 - D - 8)y = 2sin^2x$$

Its auxiliary equation is $D^2 - D - 8 = 0$

$$\Rightarrow D = \frac{1}{2}\left[1 \pm \sqrt{(1+32)}\right]$$

$$\Rightarrow D = \frac{1}{2}\left[1 \pm \sqrt{(33)}\right]$$

Therefore $C.F. = e^{x/2}\left[c_1 e^{-\sqrt{33}x/2} + c_2 e^{\sqrt{33}x/2}\right]$

(1)

And, $P.I. = \frac{1}{D^2-D-8} 2sin^2 x$

$$\Rightarrow P.I. = \frac{1}{D^2 - D - 8}(1 - cos2x)$$

$$\Rightarrow P.I. = \frac{1}{D^2 - D - 8}(e^{0x}) - \frac{1}{D^2 - D - 8}cos2x$$

$$\Rightarrow P.I. = \frac{1}{0^2 - 0 - 8}(e^{0x}) - \frac{1}{-2^2 - D - 8}cos2x$$

$$\Rightarrow P.I. = -\frac{1}{8}(1) - \frac{1}{(D+12)}.\frac{(D-12)}{(D-12)}cos2x$$

$$\Rightarrow P.I. = -\frac{1}{8} - \frac{(D-12)}{(D^2-144)}cos2x$$

$$\Rightarrow P.I. = -\frac{1}{8} - \frac{(D-12)}{(-2^2-144)}cos2x$$

$\Rightarrow P.I. = -\frac{1}{8} + \frac{1}{148}(-2sin2x - 12cos2x)$ ...(2)

Hence the required solution is

$y = C.F. + P.I.$, where $C.F.$ and $P.I.$ are given by (1) and (2) above.

EXAMPLE5: Solve $(D^2 - 4)y = x^2$

SOLUTION:- Here the given differential equation is

$$(D^2 - 4)y = x^2$$

Its auxiliary equation is $D^2 - 4 = 0$

$$\Rightarrow D = \pm 2$$

Therefore $C.F. = c_1 e^{2x} + c_2 e^{-2x}$

And, $P.I. = \frac{1}{D^2-4}x^2$

$$\Rightarrow P.I. = \frac{1}{-4\left(1 - \frac{1}{4}D^2\right)}x^2$$

$$\Rightarrow P.I. = \frac{-1}{4}\left(1 - \frac{1}{4}D^2\right)^{-1} x^2$$

$$\Rightarrow P.I. = \frac{-1}{4}\left(1 + \frac{1}{4}D^2 + \cdots\right)x^2$$

$$\Rightarrow P.I. = \frac{-1}{4}\left[x^2 + \frac{1}{4}D^2(x^2)\right]$$

$$\Rightarrow P.I. = \frac{-1}{4}\left[x^2 + \frac{1}{4}(2)\right]$$

$$\Rightarrow P.I. = \frac{-1}{4}\left[x^2 + \frac{1}{2}\right]$$

Therefore the required solution is $y = C.F. + P.I.$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left[x^2 + \frac{1}{2}\right]$$

EXAMPLE6: Solve $(D^2 - 4D + 4)y = x^2 + e^x + cos2x$

SOLUTION:- Here the given differential equation is

$$(D^2 - 4D + 4)y = x^2 + e^x + cos2x$$

Its auxiliary equation is $D^2 - 4D + 4 = 0$

$$\Rightarrow (D-2)^2 = 0$$

$$\Rightarrow D = 2, \quad 2$$

Therefore $C.F. = (c_1x + c_2)e^{2x}$

Now, $P.I. = \frac{1}{D^2-4D+4}(x^2 + e^x + cos2x)$

$$\Rightarrow P.I. = \frac{1}{(D-2)^2}x^2 + \frac{1}{(D-2)^2}e^x + \frac{1}{(D^2-4D+4)}cos2x$$

$$\Rightarrow P.I. = \frac{1}{4\left(1-\frac{1}{2}D\right)^2}x^2 + \frac{1}{(1-2)^2}e^x + \frac{1}{(-2^2-4D+4)}cos2x$$

$$\Rightarrow P.I. = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2 + \frac{e^x}{1} - \frac{1}{4D}cos2x$$

$$\Rightarrow P.I. = \frac{1}{4}\left(1 + D + \frac{3}{4}D^2 + \cdots\right)x^2 + e^x - \frac{1}{4}\int cos\ 2x\ dx$$

$$\Rightarrow P.I. = \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2) + e^x - \frac{1}{4}.\frac{1}{2}sin2x\right]$$

$$\Rightarrow P.I. = \frac{1}{4}\{x^2 + 2x + (3/2)\} + e^x - (1/8)sin2x$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = (c_1x + c_2)e^{2x} + \frac{1}{4}\{x^2 + 2x + (3/2)\} + e^x - (1/8)sin2x$$

EXAMPLE7: Solve $(D^2 - 2D + 5)y = e^{2x}sinx$

SOLUTION:- Here the given differential equation is

$$(D^2 - 2D + 5)y = e^{2x}sinx$$

Its auxiliary equation is $D^2 - 2D + 5 = 0$

$$\Rightarrow D = \frac{1}{2}\left[2 \pm \sqrt{(4-20)}\right]$$

$$\Rightarrow D = 1 \pm 2i$$

$$\Rightarrow C.F. = e^x(c_1cos2x + c_2sin2x)$$

And, $P.I. = \frac{1}{D^2-2D+5}e^{2x}sinx$

$$\Rightarrow P.I. = e^{2x}\frac{1}{\{(D+2)^2 - 2(D+2) + 5\}}.sinx$$

$$\Rightarrow P.I. = e^{2x}\frac{1}{D^2 + 2D + 5}sinx$$

$$\Rightarrow P.I. = e^{2x}\frac{1}{(-1 + 2D + 5)}sinx$$

$$\Rightarrow P.I. = e^{2x}\frac{1}{(2D+4)}sinx$$

$$\Rightarrow P.I. = e^{2x}\frac{1}{(2D+4)}\cdot\frac{(2D-4)}{(2D-4)}sinx$$
$$\Rightarrow P.I. = e^{2x}\frac{1}{4D^2-16}(2D-4)sinx$$
$$\Rightarrow P.I. = e^{2x}\frac{1}{4(-1)^2-16}(2D-4)sinx$$
$$\Rightarrow P.I. = -e^{2x}\left(\frac{1}{20}\right)(2cosx-4sinx)$$
$$\Rightarrow P.I. = -e^{2x}\left(\frac{1}{10}\right)(cosx-2sinx)$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = e^x(c_1cos2x + c_2sin2x) - \left(\frac{1}{10}\right)e^{2x}(cosx-2sinx)$$

EXAMPLE8: Solve $(D^2-5D+6)y = xe^{4x}$

SOLUTION:-Here the given differential equation is

$$(D^2-5D+6)y = xe^{4x}$$

Its auxiliary equation is $D^2-5D+6=0$, which gives $D = 2,\ 3$

Therefore,$C.F. = c_1e^{2x} + c_2e^{3x}$

And,$P.I. = \frac{1}{D^2-5D+6}\ xe^{4x}$

$$\Rightarrow P.I. = \ e^{4x}\frac{1}{(D+4)^2-5(D+4)+6}\ x$$
$$\Rightarrow P.I. = \ e^{4x}\frac{1}{D^2-3D+2}\ x$$
$$\Rightarrow P.I. = \ e^{4x}\frac{1}{2\{1+({}^3/_2)D+({}^1/_2)D^2\}}\ x$$
$$\Rightarrow P.I. = \frac{1}{2}e^{4x}\left[1+\left\{\left(({}^3/_2)D+({}^1/_2)D^2\right)\right\}\right]^{-1}x$$
$$\Rightarrow P.I. = \frac{1}{2}e^{4x}\left[1-\left\{\left(({}^3/_2)D+({}^1/_2)D^2\right)\right\}+\cdots\right]x$$
$$\Rightarrow P.I. = \frac{1}{2}e^{4x}\left[x-\left(({}^3/_2)D(x)\right)\right]$$
$$\Rightarrow P.I. = \frac{1}{2}e^{4x}[x-({}^3/_2)]$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2}e^{4x}[x-({}^3/_2)]$$

EXAMPLE9: Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$

SOLUTION:- Here the given differential equation is

$$(D^2-3D+2)y = e^x$$

Its auxiliary equation is $D^2-3D+2=0$

$$\Rightarrow D = 1,\qquad 2$$

Therefore,$C.F. = c_1e^x + c_2e^{2x}$

And,$P.I. = \frac{1}{D^2-3D+2}e^x$

$$\Rightarrow P.I. = e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} 1$$
$$\Rightarrow P.I. = e^x \frac{1}{D^2 - D} 1$$
$$\Rightarrow P.I. = -e^x \frac{1}{D(1-D)} 1$$
$$\Rightarrow P.I. = -e^x \frac{1}{D}(1-D)^{-1} 1$$
$$\Rightarrow P.I. = -e^x \frac{1}{D}(1 + D + \cdots) 1$$
$$\Rightarrow P.I. = -e^x \frac{1}{D}(1)$$
$$\Rightarrow P.I. = -e^x (x)$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = c_1 e^x + c_2 e^{2x} - e^x (x)$$

EXAMPLE10:$(4D^2 - 12D + 9)y = 144e^{3x/2}$

SOLUTION:- Here the given differential equation is

$$(4D^2 - 12D + 9)y = 144e^{3x/2}$$

Its auxiliary equation is $4D^2 - 12D + 9 = 0$

$$i.e., (2D-3)^2 = 0$$
$$\Rightarrow D = 3/2 \text{ (twice)}$$

Therefore,$C.F. = (c_1 x + c_2)e^{3x/2}$

And,$P.I. = \frac{1}{4D^2 - 12D + 9}.144e^{3x/2}$

$$\Rightarrow P.I. = 144e^{3x/2} \frac{1}{4\left[D + \left(\frac{3}{2}\right)\right]^2 - 12\left[D + \left(\frac{3}{2}\right)\right] + 9} 1$$
$$\Rightarrow P.I. = 144e^{3x/2} \frac{1}{4D^2} 1$$
$$\Rightarrow P.I. = 36e^{3x/2}\left(\frac{1}{2}x^2\right)$$
$$\Rightarrow P.I. = 18x^2 e^{3x/2}$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = (c_1 x + c_2)e^{3x/2} + 18x^2 e^{3x/2}$$

EXAMPLE11: Solve $(D^3 - D)y = e^x + e^{-x}$

SOLUTION:- Here the given differential equation is

$$(D^3 - D)y = e^x + e^{-x}$$

Its auxiliary equation is $D^3 - D = 0$

$$\Rightarrow (\ D^2 + 1) = 0$$
$$\Rightarrow D = 0, \quad -1, \quad 1$$

Therefore,$C.F. = c_1 e^{0x} + c_2 e^{-x} + c_3 e^x$

Or, $C.F. = c_1 + c_2 e^{-x} + c_3 e^x$

And,$P.I. = \frac{1}{D^3 - D}(e^x + e^{-x})$

$$\Rightarrow P.I. = \frac{1}{D^3 - D}(e^x) + \frac{1}{D^3 - D}(e^{-x})$$

$$\Rightarrow P.I. = e^{x}\frac{1}{(D+1)^3-(D+1)}(1) + e^{-x}\frac{1}{(D-1)^3-(D-1)}(1)$$

$$\Rightarrow P.I. = e^{x}\frac{1}{D^3+3D^2+2D}(1) + e^{-x}\frac{1}{D^3-3D^2+2D}(1)$$

$$\Rightarrow P.I. = e^{x}\frac{1}{2D}\left(1+\frac{3}{2}D+\frac{1}{2}D^2\right)^{-1}(1)$$

$$+ e^{-x}\frac{1}{2D}\left(1-\frac{3}{2}D+\frac{1}{2}D^2\right)^{-1}(1)$$

$$\Rightarrow P.I. = e^{x}\frac{1}{2D}(1) + e^{-x}\frac{1}{2D}(1)$$

$$\Rightarrow P.I. = \frac{1}{2}e^{x}x + \frac{1}{2}e^{-x}x$$

$$\Rightarrow P.I. = \frac{1}{2}x(e^{x}+e^{-x})$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = c_1 + c_2e^{-x} + c_3e^{x} + \frac{1}{2}x(e^{x}+e^{-x})$$

SELF CHECK QUESTIONS-5

(SCQ-1) Solve the following differential equations:

i. $(D^2-6D+7)y = e^x + e^{-x}$
ii. $(D^2-5D+6)y = e^{4x}$
iii. $(D+a)y = e^{mx}$
iv. $(D^2+9) = cos2x + sin2x$
v. $(D^4-2D^2+1)y = cosx$
vi. $(D^3+D^2-D-1) = sin2x$
vii. $(D^2-4)y = e^x + sin3x$
viii. $(D^2+2D+1)y = (x-1)$
ix. $(D^2+D-6)y = 2x + x^2$
x. $(D-1)(D^2+1)^2(D^2+D+1)^3y = 2$
xi. $(D^2+1)y = e^x cosx$
xii. $(D^4-2D^3-3D^2+4D+5)y = x^2e^{2x}$
xiii. $(D^2+4D)y = 5xe^{-2x}$
xiv. $(D-2)^2y = 8(e^{2x}+x^2)$
xv. $(D^2-4D+4)y = x^2e^{2x}$
xvi. $(D^2+D-2)y = e^x$

3.5.9 *LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS:-*

- **CAUCHY'S HOMOGENOUS LINEAR DIFFERENTIAL EQUATION:-**

A differential equation is of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \qquad \ldots(1)$$

Where $a_1, a_2, \ldots, a_n$ are constants, $Q(x)$ a function of x, is called Cauchy's homogenous linear differential equation.

To solve such types of differential equations first we convert into linear differential equation with constant coefficients by putting $x = e^t$ or $t = \log x$. Then if $D = \frac{d}{dt}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dt}$$

or, $x \frac{dy}{dx} = Dy$

Similarly, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

In general, $x^n \frac{d^n y}{dx^n} = D(D-1)(D-2) \ldots (D-(n-1))y$

Put these values in equation (1), equation (1) converts into linear differential equation with constant coefficients, which can be solved as before.

- **LEGENDRE'S LINEAR DIFFERENTIAL EQUATION:-**

A differential equation is of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b) \frac{dy}{dx} + a_n y = Q(x) \qquad \ldots(1)$$

Where $a_1, a_2, \ldots, a_n$ are constants, $Q(x)$ a function of x, is called Legendre's linear differential equation.

To solve such types of differential equations first we convert into linear differential equation with constant coefficients by putting $ax + b = e^t$ or $t = \log(ax+b)$. Then if $D = \frac{d}{dt}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{a}{ax+b}$$

$$\Rightarrow (ax+b) \frac{dy}{dx} = a \frac{dy}{dt}$$

or, $(ax+b) \frac{dy}{dx} = aDy$

Similarly, $(ax+b)^2\frac{d^2y}{dx^2}=a^2D(D-1)y$

$$(ax+b)^3\frac{d^3y}{dx^3}=a^3D(D-1)(D-2)y$$

In general, $(ax+b)^n\frac{d^ny}{dx^n}=a^nD(D-1)(D-2)\dots(D-(n-1))y$

Put these values in equation (1), equation (1) converts into linear differential equation with constant coefficients, which can be solved as before.

EXAMPLE1: Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^m$

SOLUTION: Here the given differential equation is $(x^2D^2+xD-1)y=x^m$

Putting $x=e^z$ or $z=logx$ and $D\equiv d/dz$ in the given equation, we get

$$[D(D-1)+D-1]y=e^{mz}$$

$\Longrightarrow (D^2-1)y=e^{mz}$, which is a linear equation in y.

Therefore, the auxiliary equation is $D^2-1=0$

$$\Longrightarrow D=-1,\qquad 1$$

Therefore $C.F.=c_1e^z+c_2e^{-z}$

Or, $C.F.=c_1x+c_2x^{-1}$

And, $P.I.=\frac{1}{D^2-1}e^{mz}$

$$\Longrightarrow P.I.=\frac{1}{m^2-1}x^m$$

Therefore, the required solution is $y=C.F.+P.I.$

$$\Longrightarrow y=c_1x+c_2x^{-1}+\frac{1}{m^2-1}x^m$$

EXAMPLE 2: Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=x^2+x.$

SOLUTION: Here the given differential equation is

$$(x^3D^3+2x^2D^2+3xD-3)y=x^2+x$$

Putting $x=e^z$ or $z=logx$ and $D\equiv d/dz$ in the given equation, we get

$[D(D-1)(D-2)+2D(D-1)+3D-3]y=e^{2z}+e^z$

$\Longrightarrow (D^3-D^2+3D-3)=e^{2z}+e^z$, which is a linear equation in y with constant coefficients.

Its auxiliary equation is $D^3-D^2+3D-3=0$

$$\Longrightarrow D^2(D-1)+3(D-1)=0$$

$$\Longrightarrow (D^2+3)(D-1)=0$$

$$\Longrightarrow D=1,\qquad \pm i\sqrt{3}$$

Therefore $C.F.=c_1e^z+c_2\cos(z\sqrt{3})$, where $z=\log x$

Or, $C.F.=c_1x+c_2\cos(\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)$

And, $P.I.=\frac{1}{D^3-D^2+3D-3}(e^{2z}+e^z)$

$$\Longrightarrow P.I.=\frac{1}{D^3-D^2+3D-3}e^{2z}+\frac{1}{D^3-D^2+3D-3}e^z$$

$$\Rightarrow P.I. = \frac{1}{8-4+6-3}e^{2z} + e^{z}\frac{1}{(D+1)^3-(D+1)^2+3(D+1)-3}(1)$$

$\Rightarrow P.I. = \frac{1}{7}e^{2z} + e^{z}\ \frac{1}{D^3+2D^2+4D}(1)$

$$\Rightarrow P.I. = \frac{1}{7}e^{2z} + e^{z}\frac{1}{4D\left(1+\frac{1}{2}D+\frac{1}{4}D^2\right)}(1)$$

$$\Rightarrow P.I. = \frac{1}{7}e^{2z} + e^{z}\frac{1}{4D}\left(1+\frac{1}{2}D+\frac{1}{4}D^2\right)^{-1}(1)$$

$\Rightarrow P.I. = \frac{1}{7}e^{2z} + e^{z}\frac{1}{4D}(1)$,after expansion and differentiation.

$$\Rightarrow P.I. = \frac{1}{7}e^{2z} + e^{z}\left(\frac{1}{4}z\right)$$

$$\Rightarrow P.I. = \left(\frac{1}{7}\right)x^2 + x\left(\frac{1}{4}\log x\right)$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = c_1x + c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x) + \left(\frac{1}{7}\right)x^2 + x\left(\frac{1}{4}\log x\right)$$

$$\Rightarrow y = c_1x + c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x) + \left(\frac{1}{7}\right)x^2 + \frac{1}{4}x\log x$$

EXAMPLE3: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$

SOLUTION: Here the given differential equation is $(x^2D^2 - xD - 3)y = x^2\log x$

Putting $x = e^z$ or $z = log x$ and $D \equiv d/dz$ in the given equation, we get $[D(D-1) - D - 3]y = ze^{2z}$,which is a linear equation in y with constant coefficients.

Its auxiliary equation is $D^2 - 2D - 3 = 0$

$$\Rightarrow D = -1, \quad 3$$

Therefore, $C.F. = c_1e^{-z} + c_2e^{3z}$

$\Rightarrow C.F. = c_1x^{-1} + c_2x^3$,

And, $P.I. = \frac{1}{D^2-2D-3}ze^{2z}$

$$\Rightarrow P.I. = e^{2z}\frac{1}{D^2-2D-3}z$$

$$\Rightarrow P.I. = e^{2z}\frac{1}{(D+2)^2-2(D+2)-3}z$$

$$\Rightarrow P.I. = e^{2z}\frac{-1}{3}\left[1-\left(\frac{2}{3}\right)D-\left(\frac{2}{3}\right)D^2\right]^{-1}z$$

$$\Rightarrow P.I. = \frac{-1}{3}e^{2z}\left[1+\left(\frac{2}{3}\right)D+\cdots\right]z$$

$$\Rightarrow P.I. = \frac{-1}{3}e^{2z}\left[z+\left(\frac{2}{3}\right)\right]$$

$$\Rightarrow P.I. = \frac{-1}{3}x^2\left[\frac{2}{3} + \log x\right]$$

Therefore, the required solution is $y = C.F. + P.I.$

$$\Rightarrow y = c_1x^{-1} + c_2x^3 - \frac{1}{3}x^2\left[\frac{2}{3} + \log x\right]$$

SELF CHECK QUESTIONS-6

(SCQ-1) Solve the following differential equations:

i. $(x^2D^2 + xD + 1)y = \log x$

ii. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$

iii. $(x^3D^3 + 3x^2D^2 + xD + 1)y = x\log x$

iv. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2\log x$

3.6 *PICARD'S METHOD OF SUCCESSIVE APPROXIMATION FOR FIRST ORDER FIRST DEGREE INITIAL VALUE PROBLEM:-*

Consider an initial value problem $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ …(1)

Integrate equation (1) over the range x_0 to x, we get

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)dx$$

or, $y(x) - y_0 = \int_{x_0}^{x} f(x,y)dx$

or, $y(x) = y_0 + \int_{x_0}^{x} f(x,y)dx$ …(2)

Therefore, solution of initial value problem (1) is same as finding a function $y(x)$ which satisfies equation (2). Since the information concerning the expression of y in terms of x is absent in the integral on the right-hand side of (2). So, the exact value of y cannot be obtained. Therefore, we determine a sequence of approximate solution of (2) as follows.

For the first approximation, we put $y = y_0$ in the integral on the right-hand side of (2), we get

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x,y_0)dx \qquad \text{…(3)}$$

Where y_1 denotes the corresponding value of y and is said to be first approximation.

To determine second approximation, we put $y = y_1$ in the integral on the right-hand side of (2), we get

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x,y_1)dx \qquad \text{…(4)}$$

Proceeding in the similar fashion, we get a sequence of approximate solution $y_1(x), y_2(x), \ldots, y_n(x), \ldots$ where

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \qquad \ldots(5)$$

The above method is known as Picard's iteration method or Picard's method of successive approximation.

EXAMPLE: Find the third approximation of the solution of equation $\frac{dy}{dx} = x^2 - y$, where $y = 0$ when $x = 0$ by Picard's method of successive approximations.

SOLUTION: Given problem is $\frac{dy}{dx} = x^2 - y$, where $y = 0$ when $x = 0$ …(1)

By Picard's method of successive approximations, we know the n th approximation y_n of the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is

$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$ …(2)

Comparing equation (2) with equation (1), we get

$f(x, y) = x^2 - y, x_0 = 0$ and $y_0 = 0$ …(3)

Therefore from (2) $y_n = y_0 + \int_{x_0}^{x} (x^2 - y_{n-1})dx$ …(4)

For first approximation putting $n = 1$ in equation (4) and using equation (3), we get

$$y_1 = y_0 + \int_{x_0}^{x} (x^2 - y_0)dx$$

$$\Rightarrow \; y_1 = 0 + \int_{0}^{x} (x^2 - 0)dx$$

$$\Rightarrow \; y_1 = \int_{0}^{x} x^2 dx = \left[\frac{1}{3}x^3\right]_0^x$$

$\Rightarrow \; y_1 = \frac{1}{3}x^3$ …(5)

For second approximation putting $n = 2$ in equation (4) and using equation (5), we get

$$y_2 = y_0 + \int_{x_0}^{x} (x^2 - y_1)dx$$

$$\Rightarrow \; y_2 = 0 + \int_{0}^{x} \left(x^2 - \frac{1}{3}x^3\right) dx$$

$$\Rightarrow \; y_2 = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4\right]_0^x$$

$\Rightarrow \; y_2 = \frac{1}{3}x^3 - \frac{1}{12}x^4$ …(6)

For third approximation putting $n = 3$ in equation (4) and using equation (6), we get

$$y_3 = y_0 + \int_{x_0}^{x} (x^2 - y_2)dx$$

$$\Rightarrow y_3 = 0 + \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4\right)\right] dx$$
$$\Rightarrow y_3 = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right]_0^x$$
$$\Rightarrow y_3 = \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5$$

SELF CHECK QUESTIONS-7

(SCQ-1) Using the Picard's method of successive approximations, find the third approximation of the solution of the equation $\frac{dy}{dx} = x + y^2$, where $y = 0$ when $x = 0$.

(SCQ-2) Find the third approximation of the solution of the equation $\frac{dy}{dx} = z, \frac{dz}{dx} = x^2 z + x^4 y$ by Picard's method, $y = 5, z = 1$ when $x = 0$.

3.7 *LIPSCHITZ CONDITION:-*

A function $f(x, y)$ is said to satisfy Lipschitz condition in a domain D in $\mathbb{R}^2$ if there exists a positive integer K such that

$$|f(x, y_2) - f(x, y_1)| \leq K|y_2 - y_1| \text{ where } (x, y_1), (x, y_2) \in D$$

The constant K is known as Lipschitz constant.

3.7.1 *SUFFICIENT CONDITION FOR LIPSCHITZ CONDITION:-*

Consider a function $f(x, y)$ is defined on a convex set D in $\mathbb{R}^2$. If there exists a constant $K > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq K \text{ for all } (x, y) \in D$$

Then function $f(x, y)$ satisfies Lipschitz condition on D with Lipschitz constant K.

3.8 *EXISTENCE AND UNIQUENESS THEOREM:-*

- **<u>PICARD'S THEOREM FOR EXISTENCE OF SOLUTION OF INITIAL VALUE PROBLEM:</u>**

  <u>STATEMENT:</u> Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

  Let us consider a rectangular region R which is defined as $|x - x_0| < a, |y - y_0| < b$

  If

i. $f(x, y)$ is continuous in region R.

ii. $f(x, y)$ is bounded in $R. i.e., |f(x, y)| < M \ \forall (x, y) \in R$, for some positive real number M.

Then there exists a solution of given initial value problem in
$|x - x_0| < h$ where $h = \min\left\{a, \frac{b}{M}\right\}$.

- **PICARD'S THEOREM FOR EXISTENCE AND UNIQUENESS OF SOLUTION OF INITIAL VALUE PROBLEM:**

  STATEMENT: Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. Let us consider a rectangular region R which is defined as $|x - x_0| < a, |y - y_0| < b$ If

i. $f(x, y)$ is continuous in region R.

ii. $f(x, y)$ is bounded in $R. i.e., |f(x, y)| < M \ \forall (x, y) \in R,$ for some positive real number M.

iii. $f(x, y)$ satisfies Lipschitz condition in region R, then there exists a unique solution of given initial value problem in $|x - x_0| < h$ where $h = \min\left\{a, \frac{b}{M}\right\}$.

EXAMPLE1: Test the existence and uniqueness of the solutions of the initial value problem $\frac{dy}{dx} = \sqrt{y},\ y(1) = 0$ in the suitable rectangle R.If more than one solution exists, then find all solutions.

SOLUTION: The given initial value problem is $\frac{dy}{dx} = \sqrt{y},\ y(1) = 0$

Here $f(x, y) = \sqrt{y}, x_0 = 1, y_0 = 0$.

Since $f(x, y)$ is continuous and bounded in a rectangular region R which containing point $(1,0)$.

Hence, by Picard's existence theorem, there exists at least one solution in R.

Let us now test the Lipschitz condition:

For any two points $(x, y_1), (x, y_2) \in R$, we have

$$|f(x, y_2) - f(x, y_1)| = \left|\sqrt{y_2} - \sqrt{y_1}\right|$$
$$= \left|\frac{(\sqrt{y_2} - \sqrt{y_1})(\sqrt{y_2} + \sqrt{y_1})}{(\sqrt{y_2} + \sqrt{y_1})}\right|$$
$$= \left|\frac{y_2 - y_1}{\sqrt{y_2} + \sqrt{y_1}}\right|$$

Or, $\frac{|f(x,y_2) - f(x,y_1)|}{|y_2 - y_1|} = \frac{1}{\sqrt{y_2} + \sqrt{y_1}}$

The above equation can be made as large as possible by choosing y_1 and y_2 sufficiently small, $i.e.$, a finite value for the Lipschitz constant K cannot be determined.

Since $\sqrt{y_1} + \sqrt{y_2} < 2\sqrt{y}$, if $y = max\{y_1, y_2\}$

$\Rightarrow \left|\frac{1}{\sqrt{y_1} + \sqrt{y_2}}\right| > \frac{1}{2\sqrt{y}} > M$ if $\sqrt{y} < \frac{1}{2M}$

So, in the neighbourhood of $y = 0$, the above inequality is satisfied for every $M > 0$.

Therefore, the initial value problem does not have a unique solution.
Now we can find the solution of given initial value problem $\frac{dy}{dx} = \sqrt{y}$, $y(1) = 0$, where $y \neq 0$, by using variable separable method.
The given differential equation is $\frac{dy}{dx} = \sqrt{y}$

Or, $\frac{dy}{\sqrt{y}} = dx$ …(1)

Integrating equation (1), we get

$$2\sqrt{y} = x + c$$

Or, $y = \left(\frac{x+c}{2}\right)^2$ …(2)

Put $x = 1$ and $y = 0$, we get

$$c = -1$$

Hence, one solution of the given problem is $y = \left(\frac{x-1}{2}\right)^2$

Also $y \equiv 0$ also satisfies the given initial value problem.
Hence, the solutions of the given initial value problem are $y \equiv 0$ and $y = \left(\frac{x-1}{2}\right)^2$.

Remark: If arbitrary constant c can be determined uniquely then we cannot say the initial value problem has a unique solution.

3.9 *SUMMARY:-*

In this unit we have studied the linear differential equation typically involves finding the general solution, which includes an arbitrary constant, or initial conditions can be used to determine a particular solution.

3.10 *GLOSSARY:-*

- Classification of ordinary differential equation.
- Equation reducible to linear form.
- Picard's Methods.
- Lipschitz Conditions

3.11 *REFERENCES:-*

- E.L.INCE (2012) Ordinary Differential Equations.
- Willaim A.Adkins, Mark G.Davidson (2012) Ordinary Differential Equations.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.

3.12 *SUGGESTED READING:-*

- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- A.K.Nandakumarsan, P.S.Datti &Raju K.George (2017) Ordinary Differential Equations (Principles and Applications)
- Nita H.Sah (2010) Ordinary and Partial Differential Equations: Theory and applications.

3.13 *TERMINAL QUESTIONS:-*

(TQ-1) Examine the existence and uniqueness of solution of the initial value problem $\frac{dy}{dx} = y^{1/3},\ y(0) = 0.$

(TQ-2) Discuss the existence and uniqueness of solution of the initial value problem $\frac{dy}{dx} = y^{4/3},\ y(x_0) = y_0.$

(TQ-3) Examine the existence and uniqueness of solution of the initial value problem $\frac{dy}{dx} = y^2,\ y(1) = -1.$

(TQ-4) Examine whether the following differential equation possesses unique solution. Justify your answer. $\frac{dy}{dx} = \begin{cases} y(1-2x),\ x > 0 \\ y(2x-1),\ x < 0 \end{cases}$ subject to the condition: $y = 1$ at $x = 1$.

(TQ-5) Discuss the existence and uniqueness of solution of the initial value problem

$$\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0.$$

3.14 *ANSWERS:-*

SELF CHECK ANSWERS-1

i. order-3, degree-3, **ii.**order-2, degree-3, **iii**.Order-2,degree-1
iv. order-3,degree-2, **v.** order-2,degree-1, **vi.** order-2,degree-3
vii. order-2,degree-2

SELF CHECK ANSWERS-2

(SCQ-1) i. Non-linear differential equation
ii. Linear differential equation
iii. Non-linear differential equation
iv. Non-linear differential equation

(SCQ-2) i. Incorrect statement, ii. Correct statement,
iii. Correct statement, iv. Incorrect statement,

(SCQ-3) 2

SELF CHECK ANSWERS-3

i. $xe^{\tan^{-1} y} = \frac{e^{\tan^{-1} y}}{2} + c,$

ii. $y = 2x\log|cosx - cotx| + c,$
iii. $y = logx + c(logx)^{-1},$
iv. $xy^2 - 2y^5 = c,$
v. $y(secx + tanx) = secx + tanx - xtanx,$
vi. $\log(x^2 + y^2) + x = c,$
vii. $siny\,(1 + x) = (1 + x)^2e^x - 2(1 + x)e^x + e^x + c,$
viii. $cosecy = \frac{1}{2x} + c.$

SELF CHECK ANSWERS-4

i. $y = c_1e^x + c_2e^{2\sqrt{3}x} + c_2e^{-2\sqrt{3}x}$, **ii.** $c_1e^{-2x} + c_2e^{-5x},$
iii. $y = (c_1 + c_2x) + c_3e^{2x}$

SELF CHECK ANSWERS-5

i. $y = e^{3x}\left(c_1\,cosh\,\sqrt{2}x + c_2\,sinh\,\sqrt{2}x\right) + \frac{e^x}{2} + \frac{e^{-x}}{14}$
ii. $y = c_1e^{2x} + c_2e^{3x} + \frac{e^{4x}}{2}$
iii. $y = c_1e^{-ax} + \frac{e^{mx}}{m+a}$
iv. $y = c_1cos3x + c_2sin3x + \frac{1}{5}cos2x + \frac{1}{5}sin2x$
v. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)xe^x + \frac{1}{4}cosx$
vi. $y = c_1e^x + (c_2 + c_3x)e^{-x} + \frac{1}{101}\left[\frac{1}{2}cos2x - sin2x\right]$
vii. $y = c_1e^{2x} + c_2e^{-2x} - \frac{e^x}{3} - \frac{sin3x}{13}$
viii. $y = (c_1 + c_2x)e^{-x} + x - 3$
ix. $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{216}[36x^2 + 84x + 26]$
x. $y = c_1e^x + (c_2 + c_3x)cosx + (c_4 + c_5x)sinx + (c_6 + c_7x)cosx + (c_8 + c_9x)sinx + (c_{10} + c_{11}x + x^2c_{12})cos\frac{\sqrt{3}}{2}x + (c_{13} + c_{14}x + x^2c_{15})sin\frac{\sqrt{3}}{2}x$
xi. $y = c_1cosx + c_2sinx + \frac{1}{2}e^xsinx$
xii. $y = c_1e^x + c_2xe^x + c_3e^{-x} + c_4e^{-5x} + \frac{1}{12}x^2e^{2x}$
xiii. $y = c_1 + c_2e^{-4x} - \frac{5}{4}xe^{-2x}$
xiv. $y = (c_1 + c_2x)e^{2x} + 4x^2e^{2x} + 4x + 8$
xv. $y = (c_1 + c_2x)e^{2x} + \frac{x^4}{2}e^{2x}$
xvi. $y = c_1e^x + c_2e^{2x} + \frac{1}{3}xe^x$

SELF CHECK ANSWERS-6

i. $y = c_1 coslogx + c_2 sinlogx + logx$

ii. $y = c_1 coslogx + c_2 sinlogx + \frac{x}{2}$

iii. $y = (c_1 + c_2 logx)x^2 + \frac{x^2}{6}(logx)^3$

SELF CHECK ANSWERS-7

i. $y_1(x) = \frac{x^2}{2} + c_1,\ \ y_2(x) = \frac{x^3}{3} + xc_1 + c_2, y_3(x) = \frac{x^4}{4} + \frac{1}{3}xc_1 + c_2x + c_3$

ii. $\frac{dy}{dx} = z_1(x) = \frac{x^3}{3} + x^5 + c_1, y_1(x) = \frac{x^4}{12} + \frac{1}{6}x^6 + c_1x + c_2$

TERMINAL ANSWERS

(TQ-1) $y_1 = x^2, y_2 = \frac{x^2}{2} + \frac{x^5}{5}, y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^{11}}{275} + \frac{x^8}{40}$

(TQ-2) $y_1 = 1 + x^2, y_2 = 1 + x^2 + \frac{x^4}{2}, y_3 = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

(TQ-3) $y_1 = 2 + x + x^2, y_2 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4},$

$y_3 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24}$

(TQ-4) $y_1 = 1 + x + \frac{x^2}{2},\ y_2 = 1 + x + x^2 + \frac{x^3}{6},$

$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$

(TQ-5) $y_1 = 3(e^x - 1), y_2 = 9(e^x - 1) - 6x,$

$y_3 = -18x - 6x^2 + 21e^x - 21,$

UNIT-4: VARIATION OF PARAMETERS

CONTENT:

4.1 *INTRODUCTION:-*

The course is devoted to the solution of the linear differential equations of second order with variable coefficients. In this course, learners also learn Wronskian, the existence and uniqueness of initial value problem and their solution. The course matter has many applications in several fields. This course develops the problem-solving skills of learners.

4.2 *OBJECTIVES:-*

On completion of the course, learners will be able to-

- Identify the type of a given differential equation and select and apply the appropriate analytical technique for finding the solution.
- Learners will be able to solve first order first degree differential equations utilizing the standard techniques.
- Determine the complete solution of a differential equation with constant coefficients.
- Solve linear differential equations of higher order with variable coefficients.
- Understand method of successive approximations, the existence and uniqueness of IVPs and their solution.

4.3 *WRONSKIAN OF HOMOGENOUS LINEAR DIFFERENTIAL EQUATION OF ORDER n:-*

Consider the homogenous linear differential equation of order n is of the form

$$k_0(x)\frac{d^n y}{dx^n} + k_1(x)\frac{d^{n-1}y}{dx^{n-1}} + k_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + k_{n-1}(x)\frac{dy}{dx} + k_n(x)y = 0, x \in [a,b] \qquad \ldots(1)$$

Where $k_0(x) \neq 0\ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x), \ldots, k_n(x)$ all are continuous functions of x.

Let $y_1, y_2, \ldots, y_n$ be any n solutions of the differential equation (1), then Wronskian of the solutions $y_1, y_2, \ldots, y_n$ is defined as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & \cdots & \cdots & y_n(x) \\ y_1{}'(x) & y_2{}'(x) & \cdots & \cdots & \cdots & y_n{}'(x) \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1{}^{(n-1)}(x) & y_2{}^{(n-1)}(x) & \cdots & \cdots & \cdots & y_n{}^{(n-1)}(x) \end{vmatrix}$$ and is

called Wronskian of n solutions.

Remark: If $y_1, y_2, \ldots, y_n$ be any n solutions of nth order homogenous linear differential equation then Wronskian of these solutions is always continuous and differentiable function but higher order derivative of Wronskian may or may not exists.

Particular Case:

Wronskian of Second Order Homogenous Linear Differential Equation:

Consider the second order homogenous linear differential equation is of the form

$$k_0(x)\frac{d^2 y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0, x \in [a,b] \qquad \ldots(1)$$

Where $k_0(x) \neq 0 \ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x)$ all are continuous functions of x.

Let y_1, y_2 be any two solutions of the differential equation (2), then Wronskian of the solutions y_1, y_2 is defined as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1{}'(x) & y_2{}'(x) \end{vmatrix}$$

$$\Rightarrow W(x) = y_1(x).y_2{}'(x) - y_2(x).y_1{}'(x)$$

4.4 *ABEL'S FORMULA:-*

Consider the second order homogenous linear differential equation is of the form

$$k_0(x)\frac{d^2y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0, x \in [a,b] \qquad \ldots(1)$$

Where $k_0(x) \neq 0 \ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x)$ all are continuous functions of x.

Let y_1, y_2 be any two solutions of the differential equation (1), then we have

$$k_0(x)\frac{d^2y_1}{dx^2} + k_1(x)\frac{dy_1}{dx} + k_2(x)y_1 = 0 \qquad \ldots(2)$$

and$$k_0(x)\frac{d^2y_2}{dx^2} + k_1(x)\frac{dy_2}{dx} + k_2(x)y_2 = 0 \qquad \ldots(3)$$

Multiply equation (2) by y_2 and equation (3) by y_1 and subtract we get

$$k_0(x)\left[y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2}\right] + k_1(x)\left[y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}\right] = 0$$

$$\Rightarrow k_0(x)\big(-W'(x)\big) + k_1(x)\big(-W(x)\big) = 0$$

$$\Rightarrow k_0(x)W'(x) + k_1(x)W(x) = 0$$

Therefore $W(x)$ is a solution of first order first degree linear differential equation and is given by

$$W(x) = Ae^{-\int\frac{k_1(x)}{k_0(x)}dx}$$

Where A be any arbitrary constant. The above formula is called Abel's Formula. In short, if y_1, y_2 be any two solutions of the second order homogenous linear differential equation

$$k_0(x)\frac{d^2y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0, x \in [a,b]$$

Where $k_0(x) \neq 0 \ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x)$ all are continuous functions of x.

Then Wronskian of the solutions y_1, y_2 is given by

$W(x) = Ae^{-\int \frac{k_1(x)}{k_0(x)}dx}$

4.4.1 *OBSERVATIONS (USING ABEL'S FORMULA):-*

Let y_1, y_2 be any two solutions of the second order homogenous linear differential equation

$$k_0(x)\frac{d^2y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0, x \in [a,b]$$

Where $k_0(x) \neq 0\ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x)$ all are continuous functions of x.

Then Wronskian $W(x)$ of the solutions y_1, y_2 is given by

$$W(x) = Ae^{-\int \frac{k_1(x)}{k_0(x)}dx}$$

1. $W(x)$ Wronskian of y_1, y_2, is either identically zero or never zero. $i.e.$ if $W(x_0) = 0$ for some $x_0 \in [a,b]$ then $W(x) = 0\ \forall x \in [a,b]$ and if $W(x_0) \neq 0$ for some $x_0 \in [a,b]$ then $W(x) \neq 0\ \forall x \in [a,b]$
2. If $W(x_0) > 0$ for some $x_0 \in [a,b]$ then $W(x) > 0\ \forall x \in [a,b]$
3. If $W(x_0) < 0$ for some $x_0 \in [a,b]$ then $W(x) < 0\ \forall x \in [a,b]$
4. y_1 and y_2 are linearly dependent on $[a,b]$ if and only if $W(x)$ Wronskian of y_1, y_2 is identically zero on $[a,b]$ $i.e. W(x) = 0\ \forall x \in [a,b]$
5. y_1 and y_2 are linearly independent on $[a,b]$ if and only if $W(x)$ Wronskian of y_1, y_2 is never zero on $[a,b]$ $i.e. W(x) \neq 0\ \forall x \in [a,b]$
6. y_1 and y_2 are linearly dependent on $[a,b]$ if and only if there exist $x_0 \in [a,b]$ such that $W(x_0) = 0$
7. y_1 and y_2 are linearly independent on $[a,b]$ if and only if there exist $x_0 \in [a,b]$ such that $W(x_0) \neq 0$

4.4.2 *RESULTS AND PROPERTIES:-*

Let y_1, y_2 be any two solutions of the second order homogenous linear differential equation

$$k_0(x)\frac{d^2y}{dx^2} + k_1(x)\frac{dy}{dx} + k_2(x)y = 0, x \in [a,b]$$

Where $k_0(x) \neq 0\ \forall x \in [a,b]$ and $k_0(x), k_1(x), k_2(x)$ all are continuous functions of x.

Then Wronskian $W(x)$ of the solutions y_1, y_2 is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x).y_2'(x) - y_2(x).y_1'(x)$$

1. If x_0 be a common zero of y_1 and y_2 then y_1 and y_2 are linearly dependent.

Proof: Since x_0 be common zero of y_1 and y_2,then$y_1(x_0) = y_2(x_0) = 0$

So, $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$

$\Rightarrow y_1$ and y_2 are linearly dependent.

Remark: If y_1 and y_2 are linearly independent then they never have common zero.

1. If x_0 be common point of extrema of y_1 and y_2 then y_1 and y_2 are linearly dependent.

Proof: Since x_0 be common point of extrema of y_1 and y_2,then$y_1'(x_0) = y_2'(x_0) = 0$

So, $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix} = 0$

$\Rightarrow y_1$ and y_2 are linearly dependent.

Remark: If y_1 and y_2 are linearly independent then they never have common point of extrema.

1. If x_0 be repeated zero of y_1 *i.e.* $y_1(x_0) = y_1'(x_0) = 0$ then y_1 and y_2 are linearly dependent.

Proof: Since x_0 be repeated zero of y_1, then$y_1(x_0) = y_1'(x_0) = 0$

So, $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & y_2(x_0) \\ 0 & y_2'(x_0) \end{vmatrix} = 0$

$\Rightarrow y_1$ and y_2 are linearly dependent.

Remark: If either y_1 or y_2 has repeated zero then y_1 and y_2 are linearly dependent. So, therefore y_1 and y_2 are linearly independent, then neither y_1 nor y_2 have repeated zero.

SOLVED EXAMPLES

EXAMPLE1: Consider the differential equation $\frac{d^2y}{dx^2} + (\sin x)\frac{dy}{dx} + 2y = 0$. Let y_1 and y_2 be two linear independent solution on $(-\infty, \infty)$. If $y_1(0) = 0,\ y_2(0) = 1$ then $y_1{}'(0)$ may takes the value –

- **i.** 1
- **ii.** 0
- **iii.** π/2
- **iv.** All of them

SOLUTION: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 y_2{}' - y_2 y_1{}'$

Since y_1 and y_2 be two linear independent solution on $(-\infty, \infty)$.

Therefore $W(y_1, y_2) \neq 0\ \forall\ x \in (-\infty, \infty)$.

$\Rightarrow y_1 y_2{}' - y_2 y_1{}' \neq 0 \forall\ x \in (-\infty, \infty)$

Since $y_1(0) = 0,\ y_2(0) = 1$

$\Rightarrow W(y_1, y_2)(0) \neq 0$

$\Rightarrow y_1(0) y_2{}'(0) - y_2(0) y_1{}'(0) \neq 0$

$\Rightarrow -y_1{}'(0) \neq 0$

$\Rightarrow y_1{}'(0)$ may takes value 1 or $\frac{\pi}{2}$

EXAMPLE2: Let y_1 and y_2 be two solutions of differential equation $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sec x)y = 0 . y_1(0) = 1,\ y_1{}'(0) = 0,\ W\left(\frac{1}{2}\right) = \frac{1}{3}$.

then $y_2{}'(0)$ is

- **i.** $\frac{3}{4}$
- **ii.** $\frac{1}{3}$
- **iii.** $\frac{1}{2}$
- **iv.** $\frac{1}{4}$

SOLUTION: Here, $W(x) = ce^{-\int \frac{-2x}{1-x^2}dx} = ce^{-\log(1-x^2)} = \frac{c}{1-x^2}$

Therefore, $W\left(\frac{1}{2}\right) = \frac{c}{1-\frac{1}{4}}$

$$\Rightarrow W\left(\frac{1}{2}\right) = \frac{4c}{3}$$

Since $W\left(\frac{1}{2}\right) = \frac{1}{3}$

$$\Rightarrow \frac{1}{3} = \frac{4c}{3}$$

$$\Rightarrow c = \frac{1}{3}$$

$\Rightarrow W(x) = \dfrac{1}{4(1-x^2)}$

Therefore $W(0) = \frac{1}{4(1-0^2)} = \frac{1}{4}$

Since $W(y_{1,}y_2)(0) = y_1(0)y_2{}'(0) - y_1{}'(0)y_2(0)$

$\Rightarrow \dfrac{1}{4} = 1 \cdot y_2{}'(0) - 0 \cdot\ y_2(0)$

$\Rightarrow y_2{}'(0) = \dfrac{1}{4}$

EXAMPLE3: Let y_1 and y_2 be two linear independent solutions of differential equation $\frac{d^2y}{dx^2} + (\sin x)y = 0$ where $0 \le x \le 1.$ Let $g(x) = W(y_1, y_2)(x)$. Then

- **i.** $g'(x) > 0\ \forall\ x \in [0,1]$
- **ii.** $g'(x) < 0\ \forall\ x \in [0,1]$
- **iii.** g' vanishes at only one point of $[0,1]$
- **iv.** g' vanishes at all point of $[0,1]$

SOLUTION: Here, $W(x) = ce^{-\int \frac{0}{1} dx} = c$

Therefore $g(x) = c$

$\Rightarrow g'(x) = 0\ \ \forall\ x \in [0,1]$

EXAMPLE4: Consider the differential equation

$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sin x^2)y = 0$.Let$\emptyset_1$,$\emptyset_2$be two solutions of the given differential equation such that

$\emptyset_1\ (0) = 1,\ \ \emptyset_2\ (0) = 1,\ \ \emptyset_1'(0) = 1,\ \ \emptyset_2'(0) =\ 2$.Then$W(x = 1)$ is

- **i.** e^2
- **ii.** e
- **iii.** $2e$
- **iv.** $2e^2$

SOLUTION: Since, $W(x) = ce^{-\int \frac{a_1(x)}{a_0(x)} dx}$

So, $W(x) = ce^{-\int \frac{-2x}{1} dx}$

$\Rightarrow \quad W(x) = ce^{x^2}$

$\Rightarrow \quad W(0) = c$

Also, $\ W(0) = \emptyset_1\ (0) \cdot \emptyset_2\ '(0) - \emptyset_1\ '(0) \cdot \emptyset_2\ (0) = 2 - 1 = 1$

$\Rightarrow \quad c = 1$

Therefore $W(x) = e^{x^2}$

Hence, $W(x = 1) = e$

EXAMPLE5: Let y_1 and y_2 be two distinct solutions of equation $a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$ and suppose that $y_2(x)\neq 0\ \forall\ x\in I.$ Then prove that $\frac{y_1}{y_2}$ is monotonic function.

SOLUTION: Since $\frac{d}{dx}\left(\frac{y_1}{y_2}\right)=\frac{y_2y_1{}'-y_1y_2{}'}{y_2{}^2}=\frac{-(y_1y_2{}'-y_2y_1{}')}{y_2{}^2}$

$$\Rightarrow \frac{d}{dx}\left(\frac{y_1}{y_2}\right)=\frac{-W(y_1,y_2)}{y_2{}^2}$$

Since denominator is always positive. Now we have following possibilities:

$\Rightarrow$ If numerator is $0 \Rightarrow \frac{y_1}{y_2}$ is constant function.

$i.e., W(y_1,y_2)=0 \Rightarrow \frac{y_1}{y_2}$is monotonic function.

$\Rightarrow$If numerator $>0 \Rightarrow \frac{d}{dx}\left(\frac{y_1}{y_2}\right)$ is positive.

$i.e., W(y_1,y_2)<0\ \forall\ x\in I \Rightarrow \frac{y_1}{y_2}$is strictly monotonic increasing function.

$\Rightarrow$If numerator $<0 \Rightarrow \frac{d}{dx}\left(\frac{y_1}{y_2}\right)$ is negative.

$i.e., W(y_1,y_2)>0\ \forall\ x\in I \Rightarrow \frac{y_1}{y_2}$Strictly monotonic decreasing function.

Therefore, in all the cases $\frac{y_1}{y_2}$ is monotonic function.

SELF CHECK QUESTIONS

(SCQ-1) Consider the functions$\mathrm{f(x)} = \mathrm{x}|x|$and $\mathrm{g(x)} = x^2$. Then

- **i.** $\{f,g\}$is a linearly independent pair of functions on $(-\infty,0)$
- **ii.** $\{f,g\}$is a linearly independent pair of functions on $(0,\infty)$
- **iii.** $\{f,g\}$is a linearly dependent pair of functions on R
- **iv.** $\{f,g\}$is a linearly independent pair of functions on R

(SCQ-2) Consider the two functions$\mathrm{f(x)} = \mathrm{x}|x|\sin x$and $\mathrm{g(x)} = x\sin x$. Then, $\{f,g\}$ is

- **i.** Linearly independent on $(-\infty,0)$
- **ii.** Linearly independent on $(0,\infty)$
- **iii.** Linearly dependent on R
- **iv.** Linearly independent on R

(SCQ-3) Consider $\mathrm{f(x)} = |x|e^{ax}$and $\mathrm{g(x)} = xe^{ax}$. Then, the pair $\{f,g\}$ is

- **i.** Linearly independent on R
- **ii.** Linearly dependent on $(\varepsilon,\varepsilon)$ for some $0<\varepsilon<1$
- **iii.** Linearly independent on $(0,\infty)$

iv. Linearly independent on $(-\infty, 0)$

(SCQ-4) Let $Y_1(x)$ and $Y_2(x)$defined on $[0,1]$ be twice continuously differentiable functions satisfying$\frac{d^2y(x)}{dx^2}+\frac{dy(x)}{dx}+y(x)=0$. Let $W(x)$ be the Wronskian of Y_1 and Y_2 and satisfy $W\left(\frac{1}{2}\right)=0$. Then

i. $W(x)=0$ for $x\in[0,1]$

ii. $W(x)\neq 0$ for $x\in\left[0,\frac{1}{2}\right]\cup\left[\frac{1}{2},1\right]$

iii. $W(x)>0$ for $x\in\left[\frac{1}{2},1\right]$

iv. $W(x)<0$ for $x\in\left[0,\frac{1}{2}\right]$

(SCQ-5) Let $\frac{d^2y}{dx^2}-q(x)y=0, 0\leq x\leq\infty, y(0)=1, \frac{dy}{dx}(0)=1$ where $q(x)$a positive monotonically increasing continuous function is. Then

i. $y(x)\to\infty$ as $x\to\infty$

ii. $\frac{dy}{dx}\to\infty$ as $x\to\infty$

iii. $y(x)$has finitely many zeros in $[0,\infty)$

iv. All

(SCQ-6) Consider the differential equation
$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(\sin x^2)y=0$.Let$\emptyset_1, \emptyset_2$be two solutions of the given differential equation such that
$\emptyset_1(0)=1,\ \emptyset_2(0)=1,\ \emptyset_1'(0)=1,\ \emptyset_2'(0)=2$.Then$W(x=1)$ is
$2e^{-1}$

i. $2e^{-2}$

ii. $2e^{4}$

iii. $2e^{-4}$

iv. e

(SCQ-7) Let y_1 and y_2 be two solutions of differential equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(\sec x)y=0.y_1(0)=1,\ y_1'(0)=0,\ W\left(\frac{1}{2}\right)=\frac{1}{3}$. then $y_2'(0)$ is

i. $\frac{3}{4}$

ii. $\frac{1}{3}$

iii. $\frac{1}{2}$

iv. $\frac{1}{4}$

4.5 *SECOND ORDERS LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS:-*

An equation of the form $k_0(x)\frac{d^2y}{dx^2}+k_1(x)\frac{dy}{dx}+k_2(x)y=f(x)$ where $k_0(x)\neq 0$…(1)

is called linear differential equation of second order with variable coefficients.

The standard form of equation (1) is $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$

NOTE: $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$ be the corresponding homogenous linear differential equation of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$.

4.5.1 *SOLUTION OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS:-*

4.5.1.1 *METHOD I: REDUCTION OF ORDER:-*

Consider the linear differential equation of second order with variable coefficients is

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x) \qquad \ldots(1)$$

Let $z(x)$ be the non-zero solution of corresponding homogenous equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \qquad \ldots(2)$$

Let $y=v(x)\cdot z(x)$ be the general solution of (1). So from (1), we have

$$\left(v\frac{d^2z}{dx^2}+2\frac{dv}{dx}\frac{dz}{dx}+z\frac{d^2v}{dx^2}\right)+P(x)\left(v\frac{dz}{dx}+z\frac{dv}{dx}\right)+Q(x)(vz)=R(x)$$

$$\Rightarrow z\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{dz}{dx}+P(x)z\right)+v\left(\frac{d^2z}{dx^2}+P(x)\frac{dz}{dx}+Q(x)z\right)=R(x)$$

$\Rightarrow z\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{dz}{dx}+P(x)z\right)=R(x)$ as $z(x)$ be solution of equation (2).

Put $\frac{dv}{dx}=t$ in above equation, we get

$$z\frac{dt}{dx}+t\left(2\frac{dz}{dx}+P(x)z\right)=R(x)$$

Or, $\frac{dt}{dx}+t\left(\frac{2\frac{dz}{dx}}{z}+P(x)\right)=\frac{R(x)}{z}$ …(3)

which is linear differential equation of first order.

To solve equation (3) we get the value of t.

Now put $t = \frac{dv}{dx}$ and integrating we find the value of v.

So the general solution of equation (1) is $y = v(x) \cdot z(x)$.

Limitation: This method is applicable only when a non- zero solution of corresponding homogenous equation is given.

EXAMPLE: Find the solution of the differential equation $\frac{d^2y}{dx^2} + 16y = sec4x$ by using reduction of order method.

SOLUTION: Here the given differential equation is

$$\frac{d^2y}{dx^2} + 16y = sec4x \qquad \ldots(1)$$

linear differential equation of second order with variable coefficients. Its corresponding homogenous part is $\frac{d^2y}{dx^2} + 16y = 0$.

So, the auxiliary equation is $D^2 + 16 = 0$

$\Longrightarrow D = \pm 4i$

Therefore, Complementary Function is $y_c = c_1 cos4x + c_2 sin4x$

Let us take one non-zero solution. (say $z(x) = cos4x$)

Therefore, by using method of reduction of order its general solution is of the form

$$y = vz \qquad \ldots(2)$$

Now we have $\frac{dt}{dx} + \left(\frac{2z'}{z} + p\right)t = \frac{R}{Z}$

In the given differential equation (1) we have $P(x) = 0,\ Q(x) = 16,\ R(x) = sec4x$

Therefore $t' + \left(\frac{2z'}{z} + p\right)t = \frac{R}{Z}$

$$\Longrightarrow \frac{dt}{dx} + \left(\frac{-8sin4x}{cos4x} + 0\right)t = \frac{sec4x}{cos4x}$$

$$\Longrightarrow \frac{dt}{dx} - \frac{8sin4x}{cos4x}t = \frac{sec4x}{cos4x}$$

$$\Longrightarrow \frac{dt}{dx} - 8tan4x\, t = sec^2 4x \qquad \ldots(3)$$

Now, integrating factor is $e^{-\int 8tan4x}dx = e^{\frac{-8\log sec4x}{4}} = cos^2 4x$

Therefore, solution is:

$$t \cdot cos^2 4x = \int sec^2 4x \cdot cos^2 4x \cdot dx + c_1$$

$$\Longrightarrow t \cdot cos^2 4x = x + c_1$$

$$\Longrightarrow t = (x + c_1)sec^2 4x \qquad \ldots(4)$$

Since, $\frac{dv}{dx} = t$

$$\Longrightarrow \frac{dv}{dx} = (x + c_1)sec^2 4x$$

$$\Longrightarrow v = \int (x + c_1)sec^2 4x dx + c_2$$

$$\Rightarrow v = \int xsec^2 4x + c_1 \int sec^2 4x\, dx + c_2$$

$$\Rightarrow v = x.\frac{tan4x}{4} - \int \frac{tan4x}{4} dx + c_1 \frac{tan4x}{4} + c_2$$

$$\Rightarrow v = (x + c_1)\frac{tan4x}{4} + \frac{1}{16}\log cos4x + c_2$$

Therefore, the general solution of equation (1) is $y = vz$

$$\Rightarrow y = \left((x + c_1)\frac{tan4x}{4} + \frac{1}{16}\log cos4x + c_2\right) cos4x$$

$$\Rightarrow y = (x + c_1)\frac{sin4x}{4} + \frac{1}{16} cos4x \cdot \log cos4x + c_2 \cdot cos4x$$

$$\Rightarrow y = \left[c_1 \frac{sin4x}{4} + c_2 cos4x\right] + \left[x\frac{sin4x}{4} + \frac{1}{16} cos4x \cdot \log cos4x\right]$$

4.5.1.2 *METHOD II: CHANGE OF INDEPENDENT VARIABLE:-*

Consider the linear differential equation of second order with variable coefficients is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \qquad \ldots(1)$$

We want to change the independent variable by some transformation $z = f(x)$

So, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\frac{dy}{dz} + \frac{dz}{dx}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\frac{dy}{dz} + \frac{dz}{dx}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\frac{dy}{dz} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2}$$

So from equation (1)

$$\left(\frac{d^2z}{dx^2}\frac{dy}{dz} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2}\right) + P(x)\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right) + Q(x)y = R(x)$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P(x) \cdot \frac{dz}{dx}\right)\frac{dy}{dz} + Q(x)y = R(x)$$

$$\Rightarrow \frac{d^2y}{dz^2} + \left(\frac{\frac{d^2z}{dx^2} + P(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right)\frac{dy}{dz} + \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}y = \frac{R(x)}{\left(\frac{dz}{dx}\right)^2}$$

We choose z so that $\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}$ = constant $\Rightarrow z = f(x)$

Limitation: This method is applicable if $\frac{\frac{d^2z}{dx^2}+P(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ is constant.

EXAMPLE1: Solve the differential equation $cosx\frac{d^2y}{dx^2} + sinx\frac{dy}{dx} - 2cos^3xy = 2cos^5x$ by using method of change of independent variable.

SOLUTION: Here the given differential equation is

$$cosx\frac{d^2y}{dx^2} + sinx\frac{dy}{dx} - 2cos^3xy = 2cos^5x$$

Or, $\frac{d^2y}{dx^2} + tanx\frac{dy}{dx} - 2cos^2xy = 2cos^4x$ …(1)

linear differential equation of second order with variable coefficients.

Compare equation (1) with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$, we get

$P(x) = tanx, \quad Q(x) = -2cos^2x, \quad R(x) = 2cos^4x$

By using method of change of independent variable, equation (1) reduces to

$$\frac{d^2y}{dx^2} + \left[\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2} \quad \ldots(2)$$

Where z is given by $\frac{Q}{\left(\frac{dz}{dx}\right)^2}$ = constant

$$\Rightarrow \frac{-2cos^2x}{\left(\frac{dz}{dx}\right)^2} = -2 \text{ (say)}$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = cos^2x$$

$$\Rightarrow z = \text{sinx}$$

In this case, $\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\text{sinx}+\text{tanx}\cdot\text{cosx}}{(\text{cosx})^2} = 0(\text{constant})$

Therefore, from equation (2)

$$\frac{d^2y}{dz^2} + 0\cdot\frac{dy}{dz} - 2y = 2cos^2x \quad \ldots(3)$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1-z^2) \quad \ldots(4)$$

The corresponding auxiliary equation is $D^2 - 2 = 0$

$$\Rightarrow D = \pm\sqrt{2}$$

Therefore, complementary function is $y_c = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$

Now, the particular integral is $y_p = \frac{1}{(D^2-2)}(2-2z^2)$

$$\Rightarrow y_p = \frac{2}{(D^2-2)}e^0 - \frac{2}{(D^2-2)}z^2$$

$$\Rightarrow y_p = -1 + \frac{1}{\left(1-\frac{D^2}{2}\right)}z^2$$

$$\Rightarrow y_p = -1 + \left(1+\frac{D^2}{2}+\cdots\right)z^2$$

$$\Rightarrow y_p = -1 + z^2 + 1$$

$$\Rightarrow y_p = z^2$$

Therefore, general solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

$$\Rightarrow y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x$$

EXAMPLE2: The general solution of differential equation $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0$ is-

i. $y = c_1\cos(2\tan^{-1}x) + c_2\sin(2\cot^{-1}x)$

ii. $y = c_1\cos(2\tan^{-1}x) + c_2\sin(2\tan^{-1}x)$

iii. $y = c_1\cos(2\cot^{-1}x) + c_2\sin(2\cot^{-1}x)$

iv. All of the above

SOLUTION: Here the given differential equation is

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0 \qquad \ldots(1)$$

Compare equation (1) with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$, we get

$$P(x) = \frac{2x}{1+x^2}, \qquad Q(x) = \frac{4}{(1+x^2)^2}, \qquad R(x) = 0$$

Therefore, by using method of change of independent variable equation (1) reduces to

$$\frac{d^2y}{dx^2} + \left[\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Where z is given by $\frac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{constant}$

$$\Rightarrow \frac{4}{(1+x^2)^2}\frac{1}{\left(\frac{dz}{dx}\right)^2} = 4 \text{ (say)}$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = \left(\frac{1}{1+x^2}\right)^2$$

$\Rightarrow \frac{dz}{dx} = \frac{1}{1+x^2}$

$\Rightarrow z = \tan^{-1} x$

Now, $\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{-2x}{(1+x^2)^2}+\frac{2x}{(1+x^2)^2}}{\left(\frac{dz}{dx}\right)^2} = 0(\text{constant})$

Therefore $\frac{d^2y}{dz^2} + 4y = 0$

The corresponding auxiliary equation is $D^2 + 4 = 0$

$\Rightarrow D^2 = -4$

$\Rightarrow D = \pm 2i$

Therefore, general solution is $y = c_1 \cos 2z + c_2 \sin 2z$

$\Rightarrow y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(2\tan^{-1} x)$

$\Rightarrow y = c_1 \cos(2\tan^{-1} x) + c_2 \sin\left(2\left(\frac{\pi}{2} - \cot^{-1} x\right)\right)$

$\Rightarrow y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(\pi - 2\cot^{-1} x)$

$\Rightarrow y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(2\cot^{-1} x)$

Similarly, $y = c_1 \cos\left(2\left(\frac{\pi}{2} - \cot^{-1} x\right)\right) + c_2 \sin(2\cot^{-1} x)$

$\Rightarrow y = c_1 \cos(\pi - 2\cot^{-1} x) + c_2 \sin(2\cot^{-1} x)$

$\Rightarrow y = c_1 \cos(2\cot^{-1} x) + c_2 \sin(2\cot^{-1} x)$

EXAMPLE3: The particular integral of $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y = x^5$ is-

i. $\frac{x^2}{2}$
ii. $\frac{x}{4}$
iii. $\frac{x^2}{4}$
iv. $\frac{x}{2}$

SOLUTION: Here the given differential equation is

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y = x^5$$

Or, $\frac{d^2y}{dx^2} - \frac{1}{x} + 4x^2 y = x^4$

(1)

Compare equation (1) with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$, we get

$$\mathrm{P}(x) = -\frac{1}{x}, \qquad \mathrm{Q}(x) = 4x^2, \qquad R(x) = x^4$$

Therefore, by using method of change of independent variable equation (1) reduces to

$$\frac{d^2y}{dx^2}+\left[\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz}+\frac{Q}{\left(\frac{dz}{dx}\right)^2}y=\frac{R}{\left(\frac{dz}{dx}\right)^2} \qquad \ldots(2)$$

Where z is given by $\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\text{constant}$

$$\Rightarrow \frac{4x^2}{\left(\frac{dz}{dx}\right)^2}=4\ (\text{say})$$

$$\Rightarrow \frac{dz}{dx}=x$$

Now, $\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{1-1}{x^2}=0$

Therefore, from equation (2), we have

$$\frac{d^2y}{dz^2}+4y=\ x^2$$

Hence the particular integral is $y_p=\frac{1}{D^2+4}x^2$

$$\Rightarrow y_p=\frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1}x^2$$

$$\Rightarrow y_p=\frac{1}{4}x^2$$

4.5.1.3 *METHOD III: CHANGE OF DEPENDENT VARIABLE OR NORMAL FORM OR REMOVAL OF SECOND TERM:-*

Consider the linear differential equation of second order with variable coefficients is

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x) \qquad \ldots(1)$$

Let $y=v\cdot z$ be the general solution of (1). So from (1)

$$\left(v\frac{d^2z}{dx^2}+2\frac{dv}{dx}\frac{dz}{dx}+z\frac{d^2v}{dx^2}\right)+P(x)\left(v\frac{dz}{dx}+z\frac{dv}{dx}\right)+Q(x)(vz)=R(x)$$

$$\Rightarrow z\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{dz}{dx}+P(x)z\right)+v\left(\frac{d^2z}{dx^2}+P(x)\frac{dz}{dx}+Q(x)z\right)=R(x) \ldots(2)$$

We choose z so that coefficient of $\frac{dv}{dx}$ is zero.

$$i.e.,\qquad 2\frac{dz}{dx}+P(x)z=0$$

$$\Rightarrow \frac{dz}{z}=-\frac{P(x)}{2}dx$$

$$\Rightarrow z=e^{-\int\frac{P(x)}{2}dx}$$

Therefore from (2)

$$z\frac{d^2v}{dx^2}+v\left(\frac{d^2z}{dx^2}+P(x)\frac{dz}{dx}+Q(x)z\right)=R(x)$$

$$\Rightarrow\frac{d^2v}{dx^2}+v\left(\frac{\frac{d^2z}{dx^2}+P(x)\frac{dz}{dx}+Q(x)z}{z}\right)=\frac{R(x)}{z}$$

which is the required normal form.

Since $\frac{dz}{dx}=-\frac{P(x)}{2}\cdot z$

$$\Rightarrow\frac{d^2z}{dx^2}=-\frac{1}{2}\left(P(x)\cdot\frac{dz}{dx}+z\cdot\frac{dP}{dx}\right)$$

$$\Rightarrow\frac{d^2z}{dx^2}=-\frac{1}{2}\left(P(x)\cdot\left(-\frac{P(x)}{2}\cdot z\right)+z\cdot\frac{dP}{dx}\right)$$

$$\Rightarrow\frac{d^2z}{dx^2}=-\frac{1}{2}\left(-\frac{1}{2}(P(x))^2\cdot z+z\cdot\frac{dP}{dx}\right)$$

So, $\frac{d^2v}{dx^2}+\frac{1}{z}\left(\frac{1}{4}(P(x))^2\cdot z-\frac{1}{2}\frac{dP}{dx}\cdot z+P\left(-\frac{Pz}{2}\right)+Q\cdot z\right)v=\frac{R}{z}$

$$\Rightarrow\frac{d^2v}{dx^2}+\left(\frac{1}{4}(P(x))^2-\frac{1}{2}\frac{dP}{dx}-\frac{1}{2}(P(x))^2+Q\right)v=\frac{R}{z}$$

$$\Rightarrow\frac{d^2v}{dx^2}+\left(Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}(P(x))^2\right)v=\frac{R}{z}$$

Limitation: This method is applicable if $Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}(P(x))^2$ is either constant or $\frac{k}{dx^2}$ where k be any constant. In short, if the given differential equation is $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$,then general solution is $y=v\cdot z$

Where $z=e^{-\int\frac{P(x)}{2}dx}$ and v is given by solving the differential equation $\frac{d^2v}{dx^2}+\left(Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}(P(x))^2\right)v=\frac{R}{z}$

EXAMPLE: Solve the differential equation $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-8)y=x^2e^{-x^2/2}$ by using normal form.

SOLUTION: Here the given differential equation is

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-8)y=x^2e^{-x^2/2} \qquad \ldots(1)$$

Compare equation (1) with $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, we get

$\mathrm{P(x)}=2\mathrm{x}$, $\quad \mathrm{Q(x)}=x^2-8$, $\quad R(x)=x^2e^{-x^2/2}$

Therefore, $Q-\frac{\mathrm{P}'(\mathrm{x})}{2}-\frac{(\mathrm{P}'(\mathrm{x}))^2}{4}=x^2-8-1-x^2=-9$ (constant)

Therefore, by using normal form, let $y=v\cdot z$ be the general solution of (1).

Where z is given by$z = e^{-\int \frac{P}{2} dx}$

$\Longrightarrow z = e^{-\int x dx}$

$\Longrightarrow z = e^{-x^2/2}$

Also v is given by solving the differential equation $\frac{d^2 v}{dx^2} + v\left(\frac{\frac{d^2 z}{dx^2} + P(x)\frac{dz}{dx} + Q(x)z}{z}\right) = \frac{R(x)}{z}$

$$\Longrightarrow \frac{d^2 v}{dx^2} - 9v = x^2$$

The corresponding auxiliary equation is $D^2 - 9 = 0$

$\Longrightarrow D = \pm 3$

Therefore auxiliary equation is $v_c = c_1 e^{3x} + c_2 e^{-3x}$

Now, particular integral is $v_p = \frac{1}{(D^2-9)} x^2$

$$\Longrightarrow \; v_p = \frac{1}{-9\left(1 - \frac{D^2}{9}\right)} x^2$$

$$\Longrightarrow v_p = \frac{-1}{9}\left(1 + \frac{D^2}{9}\right) x^2$$

$$\Longrightarrow v_p = \frac{-1}{9}\left(x^2 + \frac{2}{9}\right)$$

Therefore, $v = v_c + v_p = c_1 e^{3x} + c_2 e^{-3x} - \frac{x^2}{9} - \frac{2}{81}$

Hence the general solution is $y = \left(c_1 e^{3x} + c_2 e^{-3x} - \frac{x^2}{9} - \frac{2}{81}\right) e^{-x^2/2}$

4.5.1.4 *METHOD IV: METHOD OF VARIATION OF PARAMETER:-*

Consider the linear differential equation of second order with variable coefficients is

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \qquad \ldots(1)$$

Let the solution of corresponding homogenous differential equation be

$y_c = c_1 u(x) + c_2 v(x)$

Let the particular integral of equation (1) be

$$y_p = A(x)u(x) + B(x)v(x) \qquad \ldots(2)$$

Now, $\frac{dy_p}{dx} = A\frac{du}{dx} + \frac{dA}{dx}u + B\frac{dv}{dx} + \frac{dB}{dx}v$

We choose A and B so that $\frac{dA}{dx}u + \frac{dB}{dx}v = 0 \qquad \ldots(3)$

Therefore $\frac{dy_p}{dx} = \frac{dA}{dx}u + \frac{dB}{dx}v$

Also, $\frac{d^2 y_p}{dx^2} = \frac{dA}{dx}\frac{du}{dx} + A\frac{d^2 u}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} + B\frac{d^2 v}{dz^2}$

Since y_p is always a solution of (1). So, from equation (1) we have

$$\frac{d^2y_p}{dx^2}+P(x)\frac{dy_p}{dx}+P(x)y_p=R(x)$$

$$\Rightarrow \frac{dA}{dx}\frac{du}{dx}+A\frac{d^2u}{dz^2}+\frac{dB}{dx}\frac{dv}{dx}+B\frac{d^2v}{dz^2}+P\left(A\frac{du}{dx}+B\frac{dv}{dx}\right)+Q(Au+Bv)=R$$

$$\Rightarrow A\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+B\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv\right)+\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}=R..(4)$$

Since u and v are solutions of corresponding homogeneous differential equation.

Therefore, $\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0$ and $\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv=0$

So, from equation (4), we get

$$\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}=R \qquad \ldots(5)$$

Solving (3) and (5),

$$\frac{dA}{dx}u+\frac{dB}{dx}v=0$$

$$\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}=R$$

$$\Rightarrow \frac{dA/dx}{vR}=\frac{dB/dx}{-uR}=\frac{-1}{u\frac{dv}{dx}-\frac{du}{dx}v}$$

Therefore $\frac{dA}{dx}=\frac{-vR}{u\frac{dv}{dx}-\frac{du}{dx}v}, \frac{dB}{dx}=\frac{uR}{u\frac{dv}{dx}-\frac{du}{dx}v}$

Therefore $A=-\int\frac{vR}{W}dx, B=\int\frac{uR}{W}dx$

Where $W=\begin{vmatrix} u & v \\ du/dx & dv/dx \end{vmatrix}=u\frac{dv}{dx}-\frac{du}{dx}v$

Put A, B in equation (2)

$$\Rightarrow y_p=-\int\frac{vR}{W}dx\cdot u(x)+\int\frac{uR}{W}dx\cdot v(x)$$

Then the general solution of equation (1) is $y=y_c+y_p$

$$\Rightarrow y=c_1u(x)+c_2v(x)=-\int\frac{vR}{W}dx\cdot u(x)+\int\frac{uR}{W}dx\cdot v(x)$$

Limitation: This method is applicable if y_c is known.

EXAMPLE1: Solve by method by variation of parameters $\frac{d^2y}{dx^2}+n^2y=secnx$

SOLUTION: Here the given differential equation is

$$\frac{d^2y}{dx^2}+n^2y=secnx \qquad \ldots(1)$$

Its complementary function is $y_c=c_1cosnx+c_2sinnx$

Let $y_p=Acosnx+Bsinnx$ be particular solution of equation (1).

Therefore, $\frac{dy_p}{dx} = \frac{dA}{dx}cosnx + \frac{dB}{dx}sinnx - nAsinnx + nBcosnx$

We choose A and B such that

$\frac{dA}{dx}cosnx + \frac{dB}{dx}sinnx = 0$ …(2)

$\Longrightarrow \frac{dy_p}{dx} = -nAsinnx + nBcosnx$

$\Longrightarrow \frac{d^2y_p}{dx^2} = -n\frac{dA}{dx}sinnx + n\frac{dB}{dx}cosnx - n^2Acosnx - n^2Bsinnx$

Substituting these values in (1), we get

$$-n\frac{dA}{dx}sinnx + n\frac{dB}{dx}cosnx - n^2Acosnx - n^2Bsinnx + n^2Acosnx + n^2Bcosnx = secnx$$

$\Longrightarrow -\frac{dA}{dx}sinnx + \frac{dB}{dx}cosnx = \frac{1}{n}secnx$ …(3)

Solving these two equations (2) and (3), we have

$\frac{dB}{dx} = \frac{1}{n}$ and $\frac{dA}{dx}$= -$\frac{1}{n}tannx$

$\Longrightarrow B = \frac{x}{n}$ and $A = \frac{1}{n^2}logcosnx$

Therefore, $y_p = \frac{1}{n^2}cosnxlogcosnx + \frac{x}{n}sinnx$

Hence general solution is $y = y_c + y_p$

$$y = c_1cosnx + c_2sinnx + \frac{1}{n^2}cosnxlogcosnx + \frac{x}{n}sinnx$$

EXAMPLE2: Solve$L_y = xe^xlogx,\ x > 0$. Given that $xe^x,\ e^x$are solutions of $L_y = 0$.

SOLUTION: Since $xe^x,\ e^x$are solutions of $L_y = 0$. Therefore complementary function of given differential equation is

$y_c = c_1xe^x + c_2e^x$

Here $u(x) = xe^x,\ v(x) = e^x$

Let $y_p = A(x)u(x) + B(x)v(x)$ be particular integral of given differential equation, where A and B be functions of x given by $A = -\int\frac{vR}{W}dx\ ,\ B = \int\frac{uR}{W}dx$ and W is Wronskian of xe^x and e^x is given by

$W = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x & e^x \end{vmatrix}$

$\Longrightarrow W = xe^{2x} - xe^{2x} - e^{2x}$

$\Longrightarrow W = -e^{2x}$

Therefore $A = -\int\frac{vR}{W}dx$

$$\Longrightarrow A = -\int\frac{e^x.xe^x.logx}{-e^{2x}}dx$$

$$\Rightarrow A = \int x \cdot logx \cdot dx$$

$$\Rightarrow A = logx \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$\Rightarrow A = \frac{x^2}{2} logx - \frac{x^2}{4}$$

Also, $B = \int \frac{uR}{W} dx$

$$\Rightarrow B = -\int \frac{x.e^x \cdot xe^x \cdot logx}{e^{2x}} dx$$

$$\Rightarrow B = -\int x^2 \cdot logx \cdot dx$$

$$\Rightarrow B = -\left[logx \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} \cdot dx\right]$$

$$\Rightarrow B = -\frac{x^3}{3} logx + \frac{x^3}{9}$$

Therefore, general solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 xe^x + c_2 e^x + \left(\frac{x^2}{2} logx - \frac{x^2}{4}\right) xe^x + \left(\frac{x^3}{3} logx - \frac{x^3}{9}\right) e^x$$

EXAMPLE3: Solve $\frac{d^2y}{dx^2} + 4y = tan2x$ by using variation of parameter method.

SOLUTION: Here the given differential equation is

$$\frac{d^2y}{dx^2} + 4y = tan2x \qquad \dots(1)$$

Compare equation (1) with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$, we get

$P = 0, Q = 4, R = tan2x$

The corresponding homogenous part of equation (1) is $\frac{d^2y}{dx^2} + 4y = 0$

So, auxiliary equation is $D^2 + 4 = 0$

$$\Rightarrow D^2 = 4i^2$$

$$\Rightarrow D = \pm 2i$$

Therefore complementary function is $y_c = c_1 cos2x + c_2 sin2x$

Here, $u(x) = cos2x, v(x) = sin2x$

$$\Rightarrow \frac{du}{dx} = -2sin2x, \frac{dv}{dx} = 2cos2x$$

Therefore, $W = \begin{vmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{vmatrix} = 2$

Now, $A = -\int \frac{sin2x.tan2x}{2} dx$

$$\Rightarrow A = \frac{-1}{2} \int \frac{1 - cos^2 2x}{cos2x} dx$$

$$\Rightarrow A = \frac{-1}{2}\int (sec2x - cos2x)dx$$

$$\Rightarrow A = \frac{-1}{4}\log(sec2x + tan2x) + \frac{1}{4}sin2x$$

And, $B(x) = \int \frac{cos2x.tan2x}{2}dx$

$$\Rightarrow B(x) = \frac{-cos2x}{4}$$

Therefore,

$$y_p = \left[\frac{-1}{4}\log(sec2x + tan2x) + \frac{1}{4}sin2x\right]cos2x + \left(\frac{-cos2x}{4}\right)sin2x$$

Hence the general solution of equation (1) is $y = y_c + y_p$

$$\Rightarrow y = c_1cos2x + c_2sin2x + \left[\frac{-1}{4}\log(sec2x + tan2x) + \frac{1}{4}sin2x\right]cos2x + \left(\frac{-cos2x}{4}\right)sin2x$$

4.6*VARIATION OF PARAMETER FOR LINEAR DIFFERENTIAL EQUATION OF ANY ORDER:-*

Consider $L_y = R(x)$ be a linear differential equation of order n.

Let $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ be complementary function of the given differential equation which is given. Then its particular integral is defined as

$$y_p = \sum_{k=1}^{n}\int \frac{w_k(x)}{w(x)}R(x)dx.$$

Where $W(x)$ is the Wronskian of $y_1, \ y_2, \dots, y_n$

$$i.e., \qquad W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & \cdots & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & \cdots & \cdots & y_n'(x) \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & \cdots & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

and $w_k(x)$where $k = 1, 2, \dots, n$is the determinant given by replacing the k-the column of $W(x)$ by $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \vdots \\ 1 \end{bmatrix}$

4.7 *SUMMARY:-*

In this unit we have studied the wronskian of homogeneous linear differential equation of order N, Abel's Formula, second order differential equation with variable coefficients, variation of parameter for linear differential equation of any order.

4.8 *GLOSSARY:-*

- Observation(Abel Formula)
- Normal form

4.9 *REFERENCES:-*

- H.S.Bear (2013) Differential Equations: A Consice Course.
- Micheal E. Taylor (2021) Introduction to Differential Equations: Second Edition.

4.10 *SUGGESTED READING:-*

- Bernd S.W. Schroder (2009) A workbook for differential equations.
- Michael E. Taylor (2021) Introduction to Differential equations: Second Addition
- A.K.Sharma (2010)Text of Differential Equations.

4.11 *TERMINAL QUESTIONS:-*

(TQ-1) Solve the following equations by the method of variation of parameter:

i. $\frac{d^2y}{dx^2}+y=cosecx$

ii. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=2xe^x$

iii. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=6xe^{2x}$

iv. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=\frac{e^x}{x},\ x>0$

v. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=15\sqrt{x}e^{2x}$

4.12 *ANSWERS:-*

SELF CHECJK ANSWERS

(SCQ-1)- iii,**(SCQ-2)**-iii, **(SCQ-3)**-i, **(SCQ-4)**-i, **(SCQ-5)**-iv
(SCQ-6)-iv, **(SCQ-7)**-iv

TERMINAL ANSWERS (TQ'S)

i. $y = A\,cosx + Bsinx - xcosx - sinxIn|sinx|$

ii. $y = Ae^{x} + Bxe^{x} + \frac{1}{3}x^{3}e^{x}$

iii. $y = Ae^{2x} + Bxe^{2x} + x^{3}$

iv. $y = Ae^{x} + xe^{x}(B + Inx)$

v. $y = Ae^{2x} + Bxe^{2x} + 4e^{2x}\frac{5}{x^{2}}$

UNIT 5:- ORDINARY, REGULAR AND SINGULAR POINTS

CONTENTS:-

5.1 *INTRODUCTION:-*

In this unit we study of differential equations, the concepts of ordinary points, singular points, and regular singular points play a significant role. These terms are used to classify points in the domain of a differential equation based on their behavior and properties.

5.2 *OBJECTIVES:-*

After studying this unit you will be able to understanding the nature of ordinary points, singular points, and regular singular points helps mathematicians and scientists analyze and solve differential equations in different contexts. By classifying points in the domain based on their behavior, it becomes possible to develop appropriate methods and techniques for finding solutions and studying the properties of these equations.

5.3 *ORDINARY POINTS:-*

Since the process for non-homogeneous equations is so similar, we only take into account homogeneous equations. We focus on the second-order linear example to keep things simple. The aim is to solve them locally around $x = x_0$

Definition: Consider the homogeneous second order linear ordinary differential equation (ODE)

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If around $x = x_0$ the function $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic, then x_0 is an ordinary point. Otherwise it is called a singular point. At an ordinary point, we can rewrite the ODE as

$$y'' + p(x)y' + q(x)y = 0,\ p(x) = \frac{Q(x)}{P(x)}, q(x) = \frac{R(x)}{P(x)}.$$

To find a series solution, it suffices to plug the general form $\left(\sum_{n=0}^{\infty} a_n (x - x_0)^n\right)$ into the ODE and solve for the coefficients $\{a_n\}$. These coefficients will satisfy some recurrence relation, which relate a_n to a_m for $m < n$.

Example: Consider the first-order ODE

$$y' - y = 0$$

We already know the solution $y = ce^x$ to this, but instead let's find a series solution around $x = 0$. By uniqueness, the series we get must be equal to $y = ce^x$ around $x = 0$, for some c. We have

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Hence the coefficient of x^n is zero, for each n. Rewrite the first sum so that we can more easily extract this coefficient

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

It follows that

$$a_{n+1}(n+1) - a_n = 0, \quad n = 0,1,2,...$$

This is a recurrence relation for the coefficient $\{a_n\}$. For example, the first few coefficient are

$$a_1 = \frac{a_0}{1} = a_0, a_2 = \frac{a_0}{2} = \frac{a_0}{2!}, a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, ...$$

In general, we can use the recurrence relation to write all the coefficient in terms of a_0

$$a_k = \frac{a_0}{k!}$$

There are no constraints on what a_0 is; it plays the role of the constant c. The series solution is

$$y = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 e^x, \text{ as expected.}$$

There is a deep relationship between ODEs and recurrence relations. A second order constant coefficient ODE will produce a recurrence involving a_{n+2} (coming from y'') and a_{n+1} (coming from y') and a_n (from y). In general, an *n*-th order constant coefficient ODE produces an *n*-th order recurrence. Then we are free to choose the first *n* coefficients $a_0, a_1, a_2, ..., a_{n-1}$ these act as initial conditions and are analogous to the constants forming the general solution. An easy way to get the *i*-th solution in a fundamental system of solutions is to set $a_i = 1$ and all other $a_k = 0$.

5.4 *REGULAR SINGULAR PONTS*

Series solutions that revolve around regular points perform better than those that revolve around irregular points. Series solutions might not exist at the singular points of ODEs because such points may have non-analytic solutions. This might happen with very harmless ODEs. However, we may still apply the series techniques from the preceding section, appropriately modified, to a class of mild singularities.

Definition: Suppose the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ has a singular point $x = x_0$. This means that $Q(x)/P(x)$ and $R(x)/P(x)$ are not analytic at $x = x_0$. However if both

$$\lim_{x\to x_0} \frac{Q(x)}{P(x)}(x - x_0), \lim_{x\to x_0} \frac{R(x)}{P(x)}(x - x_0)^2 \qquad \dots (1)$$

exist, then we say $x = x_0$ is a regular singular point and we can still find series solutions. Otherwise it is an irregular singular point.

Equation (1) means that the function $Q(x)/P(x)$ has a pole of order at most one at $x = x_0$, and $R(x)/P(x)$ has pole of order at most two. For *e.g.*, a rational function having a pole of order at most *n* at $x = x_0$ means it is of the form

$$\frac{g(x)}{(x - x_0)^n}$$

For some function $g(x)$ which is well-defined, *i.e.*, has no pole at $x = x_0$. (Equivalently, it means that in a series expansion around $x = x_0$, we must include terms of negative order up to $(x = x_0)^{-n}$.)

Example 1: Find the regular-singular points of the differential equation

$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, where α is a real constant.

Solution: Since,

$$\frac{(x-1)Q(x)}{P(x)} = \frac{2x}{1+x}, \frac{(x-1)^2 R(x)}{P(x)} = \frac{(x-1)[\alpha(\alpha+1)]}{1+x}.$$

Furthermore, the following limits are finite,

$$\lim_{x\to 1}\frac{(x-1)Q(x)}{P(x)}=1,\lim_{x\to 1}\frac{(x-1)^2R(x)}{P(x)}=0$$

We conclude that $x_0=1$ is a regular singular point.

5.5 *AUTONOMOUS SYSTEMS:-*

If F (t, X) is a continuous function, satisfying Lipschitz condition in some domain D of (n+1)-dimensional (t,X) space ,then the initial value problem,

$$\frac{dX}{dt}=\mathbf{F\ (t,X)}\quad \textbf{with } \mathbf{x(t_0)=X_0}$$

has a unique solution in D. This unique solution depends continuously on the initial values $\mathbf{t_0}$ and $\mathbf{X_0}$. It means, if we perturb the initial value $\mathbf{X_0}$ by an infinitesimal amount or by a small amount, the solution **X(t)** is changed also by a small amount in a very small interval about $\mathbf{t_0}$**.** Here a question arises-

"whether a small change in the initial data leads to a small change in the solution for large values of t."?

Study of solution of above problem is known as '**stability theory'**. This theory has been applied successfully in various areas and '**automatic controls'**, is one of them. Historically, stability theory is related to non-linear differential equations. To get exact (or explicit) solution of such differential equations is very difficult. So we focus on qualitative behavior of solutions, without actually solving the equations.

In this chapter, we shall study only **time independent** systems. So their general form will be

$$\dot{X} = F(X) \qquad ...(1.1)$$

Here dot(.) represents differentiation with respect to time. Such a system defines a time independent vector field in a region of n- space. A good example of it, is steady fluid flow in three dimensional space. Here F(X) represents the velocity of the fluid at the point X. The solution X (t,c) describes a streamline of a moving fluid particle.

We shall focus on a special case viz. where F vanishes at some value c. In such situation, the function X(t)=c is the solution of equation (1.1) and the streamline becomes a point at c where the velocity F vanishes. This point is called **stagnation point**.

5.6 *CRITICAL POINT:-*

A point c in x^n, at which F(c)=0 is called an equilibrium point or critical point of an autonomous system $\dot{X}$ =F(X).

Let us restrict ourselves to the two dimensional system

$$\frac{dx}{dt}=f(x,y) \ \& \ \frac{dy}{dt}=g(x,y) \qquad \ldots(1.2)$$

We observe that every solution x=x(t) , y=y(t) of the system (1.2) defines a curve in the x-y plane. This curve is called **orbit** or **trajectory** of the system and the x-y plane is called **phase plane** of the system. We shall define stability of the equilibrium point after this mathematical discussion→

Let us consider the motion of a simple pendulum consisting of a concentrated mass m, suspended by a weightless rod of length *l*. Let s be the arc length subtended by angle θ. Then

$S=l\theta$

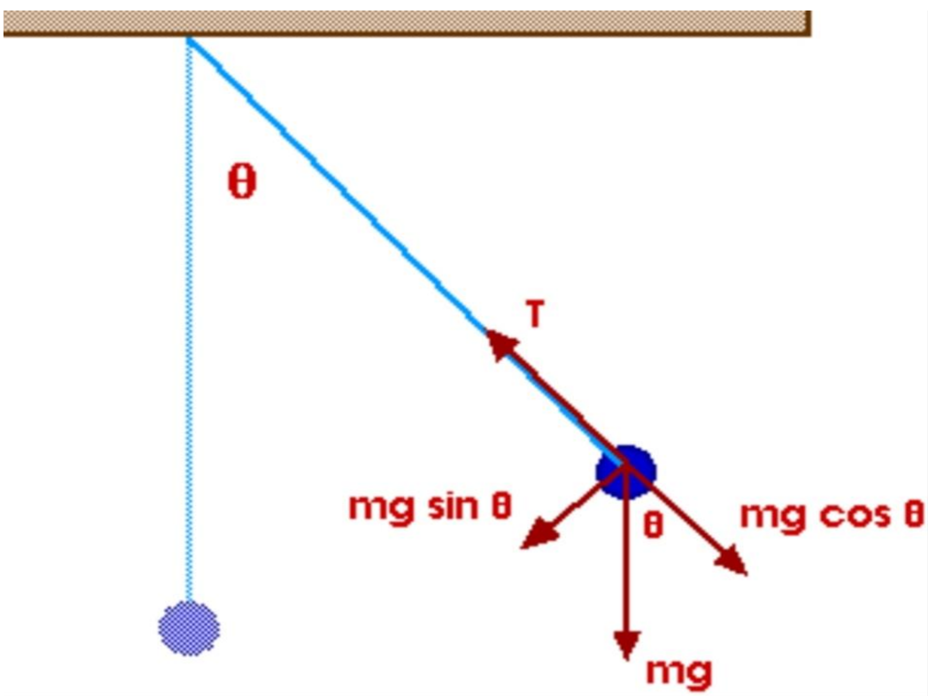


Fig.1

The tangential component of the gravitation force is (-mgsinθ).So by Newton's second law of motion, the equation of motion is

$m(d^2s/dt^2)$ =-mgsinθ

⇨ $(d^{2}(l\theta) / dt^2) = -g\sin\theta$

⇨ $(d^2\theta / dt^2) = -\frac{g}{l}\sin\theta$

Let $g/l=k$.

So , $(d^2\theta / dt^2) + k\sin\theta = 0$. ...(1.3)

From the knowledge of mechanics, we know that $\omega = \dot{\theta}$

⇨ $\dot{\omega} = \ddot{\theta}$

So, we have $\dot{\omega} + k\sin\theta = 0$.

Hence, we have an autonomous system

$\dot{\theta} = \omega$,

$\dot{\omega} = -k\sin\theta$.

Since $d\omega/dt = (d\omega/d\theta)(d\theta/dt) = \omega(d\omega/d\theta)$.

So, the above equation becomes

$\omega(d\omega/d\theta) + k\sin\theta = 0$.

$\omega d\omega + k\sin\theta d\theta = 0$.

On integration ,we get

$$\omega^2 = 2k\cos\theta + h \qquad \ldots(1.4)$$

Maximum value of RHS is h+ 2K , which must always be non negative as $\omega^2 \geq 0$.

So, $h \geq -2k$.

These curves on phase plane are shown as below:

From the figure, it is evident that there are infinite many critical points at $\omega=0$ and $\theta = n\pi\ ; n = 0, \pm 1, \pm 2, \ldots \ldots$

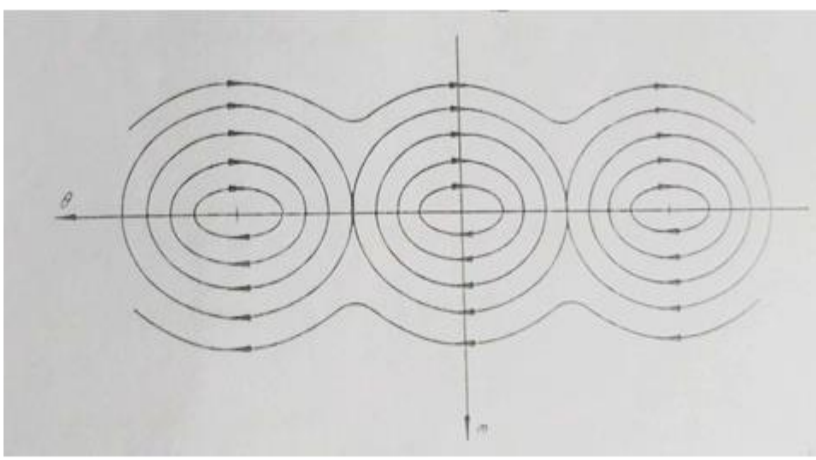


Fig.2

The pendulum will be in **stable equilibrium** if n is even. It means, in such cases, the pendulum will be in a vertically downward position.Pendulum will be in **unstable equilibrium** when n is odd (i.e. when the pendulum is in a vertically upward position) From equation (1.4) we observe that for $|h| < 2k$; curves are closed, surrounding the points $\omega=0$ and $\theta = 2n\pi$. For h>

2k, curves are open curve (figure 2). when h=2k, curves have transition i.e. when $\omega^2 = 4k\cos^2(\theta/2)$.

5.7 *PHYSICAL SIGNIFICANCE OF STABILITY:-*

If $\omega^2 < 4k\cos^2(\theta/2)$, The pendulum oscillates about its equilibrium position ($\theta = 2n\pi$).If the initial velocity is such that $\omega^2 > 4k\cos^2(\theta/2)$, then the pendulum always turns in the same direction, about the point of suspension. After the study of this discussion, we are in a situation to define stability in a formal way →

5.8 *DEFINITONS:-*

Definitions: Let c be a critical point for the system $\dot{X}$ =F(X).

The point c , is said to be →

(i) **stable** : If for given $\epsilon > 0$,there exists a $\delta > 0$ such that whenever

$$|| X(0) - c|| < \delta \; ; ||X(t) - c|| < \epsilon, \; \forall \, t > 0 \; ;$$

(ii) **Asymptotically stable,** if there exist a $\delta > 0$ such that whenever

$$|| X(0) - c|| < \delta \; , \lim_{t\to\infty} ||X(t) - \; c\,|| = 0:$$

(iii)**Strictly stable**, if it is stable and asymptotically stable,

(iv) **unstable**, if it is NOT stable.

5.9 *GEOMETRICAL INTERPRETATION OF STABILITY:-*

Let us discuss the geometric meaning of stability. Let R_δ be a spherical region of radius δ and R_ϵ be a spherical region of radius ϵ.

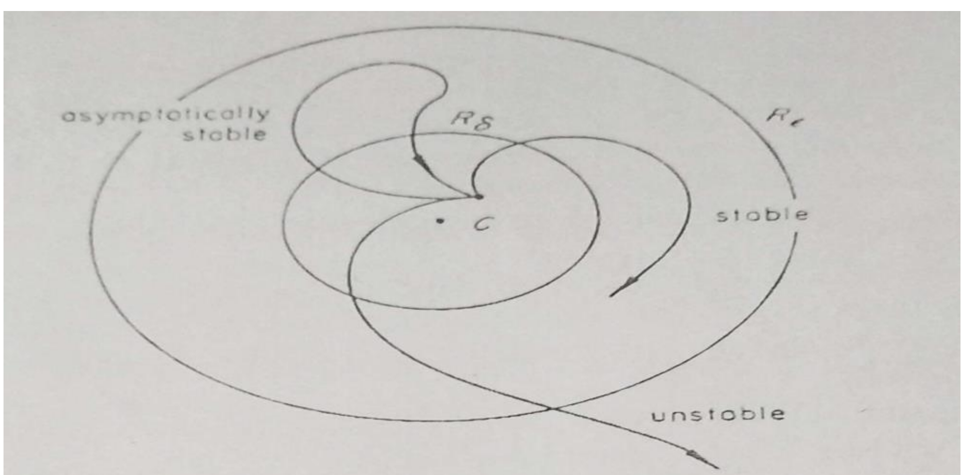


Fig.3

The equilibrium point c will be called **stable** if each trajectory in R_δ , at time t = 0, remains inside R_ϵ for all t > 0. The equilibrium point c is **asymptotically stable** if every trajectory which is sufficiently near c at t = 0 approaches c as $t \to \infty$. The equilibrium point c is **unstable** if every trajectory in R_δ at t=0, escapes from the region R_ϵ. It should be noted that asymptotic stability does NOT imply strict stability. Actually we can construct an asymptotically stable autonomous system which is unstable.

5.10 *STABILITY FOR LINEAR SYSTEM WITH CONSTANT COEFFICIENTS:-*

Let us consider a linear autonomous system with Constant Coefficients

$$\dot{X} = AX \qquad \ldots (2.1)$$

Here A is a nonsingular real $n \times n$ matrix. Also we suppose that origin is a Critical point for this system.

Theorem 1: Suppose $\dot{X}$ =AX is a linear autonomous system with $n \times n$ real non-singular constant coefficient matrix A. Also we suppose critical points at the origin of R^n. Then the critical point is →
(i) Strictly stable, if real parts of Eigen values of A are negative.
(ii)Stable if A has at least one pair of purely imaginary eigenvalues of multiplicity one;
(iii) Unstable otherwise.

Proof:

As we know, fundamental solutions of linear systems can be expressed in terms of the eigenvalues. Also fundamental solutions are of the form $Pt^k e^{\alpha t}\cos\beta t$, $Qt^k e^{\alpha t}\sin\beta t$. Here P and & Q are constant vectors while α , β are real and imaginary parts of eigenvalues of A respectively. Also k is a non-negative integer, which depends on the multiplicity of the eigenvalues.

(**i**) As the origin is the critical point, so we have

$$| Pt^k e^{\alpha t}\cos\beta t | \le |P| t^k e^{\alpha t}$$

Since k is finite while α is negative and P depends on the initial condition in such a way that $|P| < \delta$. So we have $| Pt^k e^{\alpha t}\cos\beta t | < \epsilon$ whenever $|P| < \delta$

In a similar manner, we can prove that

$| Qt^k e^{\alpha t}\sin\beta t | < \epsilon$ whenever $|Q| < \delta$.

This proves the stability.

Now, for the asymptotic stability, We observe that if , α <0.

$\lim_{t\to\infty} | Ptke^{\alpha t}cos\beta t | = 0$, by L' hospital rule.

Similarly, $\lim_{t\to\infty} | Qtke^{\alpha t}sin\beta t | = 0$

Hence, the origin is strictly stable, provided α <0.

(**ii**) Suppose eigenbalues of A be $\pm i\beta$. it means $\alpha = 0$. So, fundamental solution will be P$\cos\beta t$ and Q$\sin\beta t$. Whenever |P| < δ and |Q| < δ ,we observe that |P$\cos\beta t$| < δ and |Q$\sin\beta t$| < δ.

Here, one thing is interesting. Since $\cos\beta t$ *and* $\sin\beta t$ do not tend to zero as $t \to \infty$. So, the origin is stable, but not asymptotically. Other pairs of pure imaginary eigenvalues of multiplicity one can be treated in a similar way.

(iii) If $\alpha \geq 0$, then both $|Pt^k e^{\alpha t}\cos\beta t|$ and $|Qt^k e^{\alpha t}\sin\beta t|$ are unbounded. Which means origin is unstable.

EXAMPLE: Discuss the stability of damped harmonic motion given by $\ddot{x} + 2\dot{x} + 2x = 0$.

Solution: Let us take $\dot{x} = y$

So, given equation becomes $\dot{y} + 2y + 2x = 0$.

Or $\dot{x} = y$ & $\dot{y} = -2y - 2x$ …(2.2)

The characteristic equation is given by

$$\begin{vmatrix} -K & 1 \\ -2 & -2-K \end{vmatrix} = 0$$

Or $k^2 + 2k + 2 = 0$

Or $(k + 1)^2 + 1 = 0$

$\Rightarrow$ $k = -1 \pm i$

According to above theorem, the origin is strictly stable.

The solution of equation (2.2) is:

$x = e^{-t}(a\cos t + b\sin t)$,

$x + y = e^{-t}(-a\sin t + b\cos t)$.

The trajectories associated with equation (2.2) can be obtained by introducing polar coordinates.

So, the system (2.2) takes the form

$x = r\cos\theta = ce^{-t}\cos(t-a)$, $c = (a^2 + b^2)^{1/2}$

$x + y = r\sin\theta = ce^{-t}\sin(t-a)$; $\alpha = \tan^{-1}(b/a)$

by eliminating 't' we get

$r = ce^{-(\theta+\alpha)}$

This describes a family of spirals.

5.11 *LINEAR PLANE AUTONOMOUS SYSTEM:-*

In this section, we shall discuss the linear autonomous system

$$\dot{X} = AX \qquad ...(3.1)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a non-singular constant matrix. The characteristic equation for the system is

$$|A - kI| = 0$$

$$\Rightarrow \begin{vmatrix} a-k & b \\ c & d-k \end{vmatrix} = 0$$

$$k2-(a+d)k+(ad-bc)=0 \qquad \ldots(3.2)$$

Suppose we introduce p = a+d,

q = ad – bc

The equation (3.2) becomes $k^2 - pk + q = 0$. If k_1 and k_2 are roots of above equation, then,

$$k_1=\frac{1}{2}[p+\sqrt{p^2-4q}] \text{ and } k_2=\frac{1}{2}[p-\sqrt{p^2-4q}] \qquad \ldots(3.3)$$

Obviously, stability of this linear system of equations depends upon the discriminant $\Delta = p^2 - 4q$ →

(1) If $\Delta > 0$ *and* $q > 0$, then k_1 and k_2 have same sign and both are +ve or –ve according as p>0 or p< 0. If Δ >0 and q<0, then k_1, k_2 have different signs.
(2) If Δ =0, then k_1 and k_2 are equal and positive or negative according as p>0 or p<0.
(3) If $\Delta < 0$, then k_1 and k_2 are complex numbers where the real part is +ve ,zero or –ve according as p>0, p=0, p<0.

If we discuss these situations with the help of theorem (2.1), we **conclude** that origin for the system is→

(1) Strictly stable ,if

$\Delta > 0, q > 0$ *and* $p < 0,$

$\Delta = 0, \quad p < 0,$

$\Delta < 0, \quad p < 0,$

(2) Stable if

$\Delta < 0, p = 0,$

(3) Unstable if

$\Delta > 0, q < 0$ *and* $p > 0,$

$\Delta = 0, \quad p > 0,$

$\Delta < 0, \quad p > 0,$

To discuss the behavior of the trajectories near a critical point, we apply a linear transformation

$$Y = BX \text{ with } |B| \neq 0 \qquad \ldots(3.4)$$

We choose this transformation in such a way that the essential behavior near the critical point remains unchanged.

(1)Real and distinct root:

If we apply the transformation with B given by

$B = \begin{pmatrix} c & k1-a \\ c & k2-a \end{pmatrix},$

The system (3.1) transforms into the system

$\dot{x} = k_1 x, \quad \dot{y} = k_2 y$ …(3.5)

Where (for simplicity) x and y are used again as the new coordinates.
The new solutions are
$x(t) = c_1 e^{k_1 t}$, $y(t) = c_2 e^{k_2 t}$ . ...(3.6)
where c_1 and c_2 are arbitrary real constants.
If we eliminate 't' from above equations, we get

$$y = cx^{k_2/k_1} \quad \ldots(3.7)$$

Where c is an arbitrary constant.
If k_1 and k_2 have same sign, then equation (3.7) represents parabolic curves tangent at the origin as shown in figure. This critical point is called a 'proper node' for the system. When k_1 and k_2 are negative, the origin is stable and is called stable node. It is also asymptotically stable.

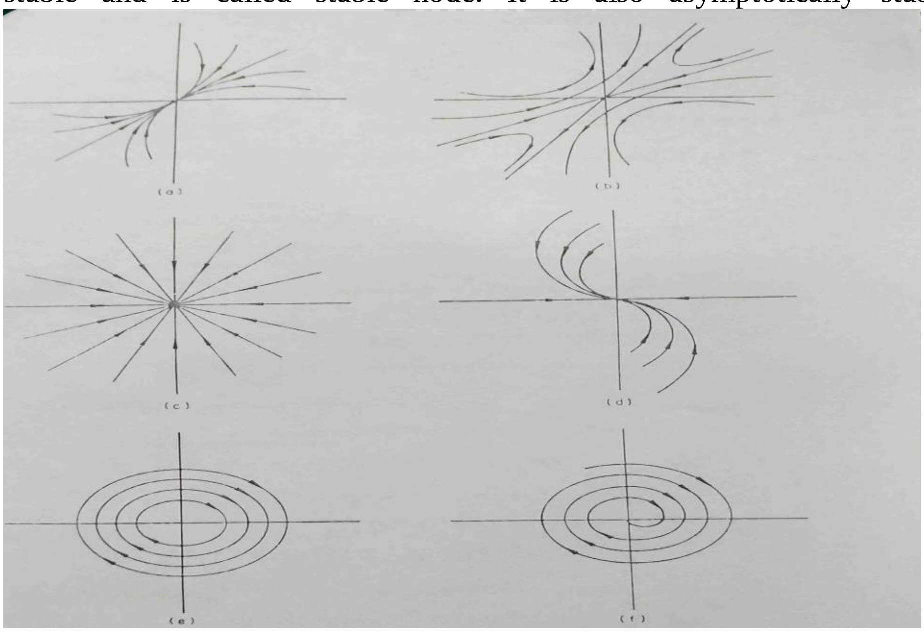


Fig.4

When k_1 and k_2 are positive, the origin is unstable and is called unstable node. If k_1 and k_2 have opposite signs, equation (3.7) represents hyperbolic curves as shown in fig 4-1(b). In this case, the origin is called a saddle point and is unstable.

(2) Real And Equal Roots: In this case, $\Delta = (a-d)^2 + 4bc = 0$, and hence, $k_1 = k_2 = (a+d)/2 = k$ (say)
The first simpler case arises when b=0 or c=0 and a=d. Then the system (3.1) becomes

$$\dot{x} = kx \text{ and } \dot{y} = ky.$$

On solving these equations, we obtain

$$x = c_1 e^{kt} \text{ and } y = c_2 e^{kt}$$

where c_1 and c_2 are arbitrary constants.

If we eliminate 't' from above equations, we obtain

$$y = (c_2/c_1)x$$

It means trajectories are straight lines.

If $k<0$, the origin is asymptotically stable and is a proper node. If $k>0$, it is unstable.

Now, we discuss **a more general** situation i.e. we consider other possibilities which are more complicated. In general case, we may choose

$$B = \begin{pmatrix} \frac{a-d}{2b} & 1 \\ \frac{1}{b} & 0 \end{pmatrix}, b \neq 0$$

Now, the equation system (3.1) is transformed into

$\dot{x} = kx\ \dot{y} = x + ky$

If we solve these equations, we obtain

$x = c_1 e^{kt}$ and $y = (c_1 t + c_2)\, e^{kt}$

These trajectories are shown in figure (3-1d)

The critical point is an improper node. It is asymptotically stable if $k<0$ and unstable if $k>0$.

Complex conjugate roots:

Let $k_1 = \alpha + i\beta \text{ and } k_2 = \alpha - i\beta \text{ where } \beta > 0.$

In this situation, we choose

$$B = \begin{pmatrix} c & \alpha - a \\ 0 & \beta \end{pmatrix}$$

Now, the equation system (3.1) is transformed as

$$\dot{x} = \alpha x - \beta y$$

$$\text{and } \dot{y} = \beta x + \alpha y \quad \ldots(3.8)$$

(a) if $\alpha = 0, \text{then system } (3.8) \text{ becomes}$

$\dot{x} = \beta y$ and $\dot{y} = \beta x$

General solution of the above equation system is

$$x = c_1 \cos\beta t + c_2 \sin\beta t$$

$$y = c_1 \sin\beta t - c_2 \cos\beta t$$

where c_1 and c_2 are arbitrary constants. On squaring and adding above equations we get

$$x^2 + y^2 = c_1^2 + c_2^2$$

now, trajectories are circles , as shown by figure(3-1(e)) the critical point is a centre which is obviously stable. However, it is NOT asymptotically stable.

(b) If $\alpha \neq 0$, the solution of the system (3.8) is given by

$x = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

$y = e^{\alpha t}(c_1 \sin\beta t - c_2 \cos\beta t)$

On squaring and adding, we get

$$x^2 + y^2 = (c_1^2 + c_2^2)^2\, e^{2\alpha t}$$

so, the trajectories are the family of a spirals. The critical point is a focal point, shown by figure 3-1(f). It is asymptotically stable $\alpha < 0$ *and unstable if* $\alpha > 0.$

5.12 *PERTURBED SYSTEM:-*

Suppose we consider an autonomous non-linear system

$$\dot{x} = f(x,y) \text{ and } \dot{y} = g(x,y) \qquad \text{.....(4.1)}$$

With f(0,0) = g(0,0) = 0, so that origin is a critical point.
Let f and g be real analytic functions of x and y. So, by expanding f(x,y) and g(xy) with the help of Taylor's theorem , we get
$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + f_{xx}(0,0)\frac{x^2}{2} + f_{xy}(0,0)xy + f_{yy}(0,0)\frac{y^2}{2} +$ ....and $g(x,y) = g(0,0) + g_x(0,0)x + g_y(0,0)y + fg_{xx}(0,0)\frac{x^2}{2} + g_{xy}(0,0)xy + g_{yy}(0,0)\frac{y^2}{2} + \ldots\ldots.$

Let us denote $f_x(0,0)$,$f_y(0,0)$, $g_x(0,0)$ and $g_y(0,0)$ by a,b,c and d respectively and remaining higher order terms by $f_1(x,y)$ and $g_1(x,y)$, we get

$$\dot{x} = ax + by + f_1(x,y) \text{ And } \dot{y} = cx + dy + g_1(x,y) \qquad \text{...(4.2)}$$

We should note that f(0,0) = g(0,0) = 0.
Also we assume that ad-bc $\neq 0$
Both the functions f_1 and g_1 are called perturbations and they satisfy
$f_1 = 0(r)$, $g_1 = 0(r)$, $r = \sqrt{x^2 + y^2}$
This condition ensures that f_1 and $g_1 \rightarrow 0$, faster than the linear terms in equation (4.2). Hence, it would seem that the nature of critical point of the non-linear system (4.2) is similar to that of the associated linear system.

$$\dot{x} = ax + by \text{ and } \dot{y} = cx + dy \qquad \text{...(4.3)}$$

In general the nature o the critical point of the non-linear system will be same as that of the associated linear system.
However, there are some exceptional cases→
when the roots of (4.3) are purely imaginary, the origin is the centre of the linear system, whereas it may be centre of spiral point of non-linear system. When the roots real and equal and b=c=0 and a=d, then origin is a node of the linear system, whereas it may be centre of spiral point of non-linear system.

All the discussion can be summarized as →
"If the critical point (0,0) of the associated linear system is strictly stable, then critical point of the non-linear system

$$\dot{x} = ax + by + f_1(x,y) \text{ and } \dot{y} = cx + dy + g_1(x,y)$$

is also strictly stable provided that $f_1 = 0(r)$ and $g_1 = 0(r)$."

5.13 *METHOD OF LYPUNOV FOR NON-LINEAR SYSTEM:-*

Lyapunov was a leading Russian Mathematician, who investigated the stability of non-linear autonomous systems of differential equation without actually determining the solutions. The basis of this concept method is the concept that the potential energy of a Conservative dynamical system has a relative minimum at a stable equilibrium point.

Suppose V(x) be a potential function. Let us consider a trajectory, on which V(x(t)) decreases to zero, then any trajectory of the system which crosses the surface V(x(t)) = Constant, surrounding the origin remains in that region. This confirms that origin is stable and indeed asymptotically stable.
Let us define the function V(x) formally→

Definition:
Let V(x) be a real-valued function of class c^1 in some open region Ω about the origin. The function V(x) is said to be positive definite if →

(i)V(x) > 0 for all $x \neq 0$ *in Ω*
(ii) V(x) =0 if and only if x=0

Definition: If the function V(X) is positive definite and satisfies

$$\dot{V}(t) = \frac{d}{dt}V(x(t)) = \nabla V(X).\,\dot{X} = \nabla V(X).\,F(X) \leq 0; \text{ in } \Omega,$$

where ∇ is the vector operator $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \ldots)$;

then V is called a Lyapunov function for the autonomous non-linear system $\dot{X}$ = F(X)

Theorem 1: If there exists a Lyapunov function V (X) in Ω, then the origin is stable.

Proof: Suppose $S_\in$ be a sphere of radius ∈, with contre at the origin in Ω. Since V is continuous on the compact set $S_\in$, it assumes its minimum value m on $S_\in$. Since, V is positive definite, so it assumes a positive minimum on $S_\in$. Since V(0)= 0 and is continuous at the origin, there exists $\delta < \in$ such that V(X)<m for |X| $<\delta$. Suppose this sphere be S_δ Let X(t, X_0) be a trajectory of the system initially at X_0 in S_δ.

Then $V(X_0)$ <m. By hypothesis $\dot{V} \leq 0$ for X in Ω. Thus $V(X(t)) \leq V(X(0)) < m$. But $V(X(t)) \geq m$ on S_δ. We conclude that X(t) must remain in S_δ for all t>0.Hence the origin is stable.

Theorem 2: If V is a Lyapunov function such that $-\nabla V(X).F(X)$ is positive definite in Ω, then the origin is asymptotically stable.

Proof: As the origin is stable by the previous theorem, V(X) decreases along a trajectory of the system to V_0 as t tends to infinity. Now, we shall prove that $V_0=0$. but us assume that $V_0>0$. Then there exists $\alpha < \in$Lt such that $V(X) < V_0$ for all X in S_α. Now, let -V assumes a minimum value m in the region $\alpha \leq |X| \leq \in$.

Since –V>0, we have V$\leq -m$ for all t≥ 0.

Thus, $V(X(t,X_0)) - V(X_0) = \int_0^t \frac{\partial V}{\partial t} dt \leq -mt.$

Consequently, as t→∞, V(X (t, X_0) tends to negative infinity.

But, this contradicts the assumption that Vis positive definite in Ω. And equals V_0 when t→∞. Hence V_0 must vanish.

=> the origin is asymptotically stable.

Theorem3: Let V be a real valued function of class C^1 in Ω with V(0) = 0, and let $V(X_0)$ >0 for all X in |X|< δ. If$\nabla V(X).F(X)$ is positive definite in Ω. then the origin is unstable

Proof: Let X_0 be the initial point in of the trajectory of the system. By hypothesis V(0) = 0 and $V(X_0)$>0 for all X in S_δ. Since V>0, V is increasing, and thus along the trajectory we have

$$\dot{V} \geq m > 0$$

where m is the positive minimum value of $\dot{V}$.

in the region 0< |X|≤ δ.

thus, $V(X(t,X_0)) - V(X_0) = \int_0^t \frac{\partial V}{\partial t} dt \geq mt.$

Consequently, as t →∞, V(X(t, X_0) approaches infinity.

=> origin is unstable.

5.14 *DISCUSSION:-*

If we can construct Lyapunov functions, we can determine by the application of preceding theorems the stability or instability of critical points for autonomous systems. Actually there is NO general method of constructing Lyapunov function. There are very few exceptions of methods applicable to certain classes of systems.

Example 1: Let us consider the system

$$\dot{x} = -y + xy$$
$$\dot{y} = x - x^2$$

which has a critical point at the origin. Suppose the Lyapunov function be $V= ½(x^2 +y^2)$. Then,

$$\dot{V} = x(-y+xy) + y(x-x^2) = 0,$$

As V is positive definite and $\dot{V} = 0$, the Lyapunov function V exists.
Hence with the help of Theorem 5.1, the origin is stable.

Example 2: Consider the system

$$\dot{x} = y$$
$$\dot{y} = -x - y - x^3$$, having a critical point at the origin.

Solution: Suppose $V=\frac{1}{4}(2x^2+2y^2 +x^4)$. Then

$\dot{V}=y(x+x^3)+(-x-y-x^3)y=-y^2$

As V is positive definite and $\dot{V} < 0$(y=0 is NOT a trajectory of the system), the Lyapunov function V exists. Also with the help of Theorem 5.2, we conclude that the origin is asymptotically stable.

5.15 *LIMIT CYCLE:-*

Already we have observed that an autonomous system sometimes possesses periodic solutions whose trajectories are represented by closed curves in the phase plane. Autonomous system, viz. negatively damped non-linear oscillator, admit solutions which generally tend to a limiting finite periodic solution. Such limiting closed curve in the phase plane is called a limit cycle. A limit cycle is a closed curve. No other solution which is a closed curve exists in its neighborhood. It is an isolated, closed curve.

Every neighboring trajectory spirals and tends to limit cycle from the inside or from the outside as $t \to +\infty$ or $t \to -\infty$.

If all the neighboring trajectories approach a limit cycle, as $t \to +\infty$ or $t \to -\infty$. then the limit cycle is said to be stable.

Note:

It should be noted that limit cycles arise physically only in non-linear, non-conservative systems. Now we illustrate a well known example and discuss limit cycle.

Example 1→ Let us consider the system

$$\dot{x} = y + \frac{x(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = -x + \frac{y(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}} \qquad \text{.....(1)}$$

Let x = r$\cos\theta$ and y = r$\sin\theta$. Then, we get

$$\dot{x} = y + \frac{x}{r}(1 - r^2)$$

$$\dot{y} = -x + \frac{y}{r}(1 - r^2) \qquad \text{....(2)}$$

As we know

$$x\dot{x} + y\dot{y} = \frac{1}{2}\frac{d}{dt}r^2$$

$$y\dot{x} + x\dot{y} = -r^2\frac{d\theta}{dt}$$

Putting these in equation (2) and solving, we obtain

$$\dot{r} = 1 - r^2$$

$$\dot{\theta} = -1$$

From the second equation, we get

$\theta = -t + a,$ where a is an arbitrary constant.

First equation can be solved by using the method of separation of variables.

$$r = \left(\frac{ce^{2t} - 1}{ce^{2t} + 1}\right)cost$$

Where c is an arbitrary constant.

Suppose $\theta(0) = 0,$

This implies a = 0 and hence $\theta = -t.$

Hence, the solution of system may be written as

$$x = \left(\frac{ce^{2t} - 1}{ce^{2t} + 1}\right)cost$$

$$y = -\left(\frac{ce^{2t} - 1}{ce^{2t} + 1}\right)sint \qquad \text{....(3)}$$

If c =0, then the solution will given by $x^2 + y^2 = 1$.

If c > 0, then trajectories are spirals inside the circle $x^2 + y^2 = 1$, approaching the circle as t→∞

If c < 0, then trajectories approach the circle spirally from outside as t→∞.

In this way conclude that this circle is a limit cycle of the system (1)

Note- In the above example, we showed how a limit cycle was determined. Generally it is very difficult (sometimes almost impossible) to

find a limit cycle of a system.
Now in the next theorem, we shall discuss of closed trajectories of the system.

$\dot{x} = f(x,y)$ & $\dot{y} = g(x,y)$ .....(4)

Theorem1: Suppose f(x,y) and g(x,y) have continuous first partial derivatives in a simply connected domain D in R^2. If $f_x + g_y$ has the same sign in D, then the system given by equation (4) has no closed trajectory in D.

Proof :Suppose C be a closed curve in D. Then using the concept of Green's theorem, we get

$$\int_C (f(x,y)dy - g(x,y)dx) = \iint_R (f_x + g_y)\, dxdy \quad ...(1)$$

Suppose C is represented parametrically by x= x(t), y=y(t). then

$$\int_C (f(x,y)dy - g(x,y)dx) = \int_0^T (f\frac{dy}{dt} - g\frac{dx}{dt})dt$$

Where T is the period of C. If we use equation 6.4, we have

$$\int_C (f(x,y)dy - g(x,y)dx) = \int_0^T (fg - gf)dt = 0.$$

So, by using equation (6.5) ,we get

$$\iint_R (f_x + g_y)\, dxdy \quad = 0.$$

Above result is true only if $f_x + g_y$ changes sign. But, this is a contradiction. Hence, C is not a closed trajectory in D.

5.16 *EXERCISES:-*

1. Describe the nature of the critical point of each system and sketch the trajectories

   (a) $\dot{x} = x$,
   $\dot{y} = 2x + 2y$.

   (b) $\dot{x} = -x + 2y$,
   $\dot{y} = x - y$.

   (c) $\dot{x} = 2x - 8y$,
   $\dot{y} = x - 2y$.

   (d) $\dot{x} = -x$,
   $\dot{y} = x - y$

   (e) $\dot{x} = -x + y$,
   $\dot{y} = 2x$

(f) $\dot{x}$ = -3x+2y,

$\dot{y}$ = - 2x

2. Determine the asymptotic behavior of the solution of each system near the critical point. Sketch the trajectories of the associated linear system.

(a) $\dot{x}$=2sin x + y,

$\dot{y}$=sin x-3y.

(b) $\dot{x}$= -x- x^2 + xy,

$\dot{y}$= - y + xy-y^2

(c) $\dot{x}$=x+e^{-y}-1,

$\dot{y}$=-y-e^{-y}+1.

3. The equation of motion of a mass-spring system with damping is given by

$$m\ddot{x} + c\dot{x} + kx = 0,$$

where m, c and k are positive constants. By changing this equation into a system, discuss the nature and stability of the critical point.

4. Determine the type of the critical point (0,0) depending on a real parameter **μ** of the nonlinear system

$\dot{x}$= - 2x - y + x^2

$\dot{y}$ = 4x+**μ**y – y^2,

where **μ** $\neq$2.

5. Prove that if x(t), y(t), $t_1 < t < t_2$, is a solution of $\dot{x}$ = f(x,y), $\dot{y}$ = g(x,y), then x(t+ c), y(t + c) for any real constant c is also a solution. This property does not hold in general for non autonomous systems. Illustrate with the example $\dot{x}$= x, $\dot{y}$= tx.

6. Using the Lyapunov function V(x,y) = $\frac{1}{2}(x^2 + y^2)$, determine the stability of the critical point (0,0) for each system.

(a) $\dot{x}$ = -x-$\frac{x^3}{3}$Cosy,

$\dot{y}$ = -y-y^3

(b) $\dot{x}$= -y-x$\sin^2$x,

$\dot{y}$=x-y$\sin^2$x.

(c) $\dot{x}$=x- y^2,

$\dot{y}$= y + xy.

7. Consider the system

$$\dot{x}=y- xf(x,y),$$
$$\dot{y} = -x- yf(x,y),$$

where f(x,y) is analytic at the origin and f(0,0)=0. Describe the relation between f(x,y) and the type of stability.

5.17 *OBJECTIVE QUESTIONS:-*

Q1 If a continuous function satisfies … condition in some domain D, then the initial value problem $\frac{dX}{dt}$=F(t,X) with x(t_0)=X_0 has a unique solution in D-

i. Lipschitz
ii. Riemann
iii. Lagrange
iv. Gauss

Q2 *" whether a small change in the initial data leads to a small change in the solution for large values of t."?* Study of solution of above problem is known as

i. Initial value problem
ii. Existence theory
iii. Stability theory
iv. None of these.

Q3 A point c in x^n , at which F(c)=0 is called a …of an autonomous system $\dot{X}$ =F(X). Initial value problem

i. Null point
ii. Critical point
iii. Extraordinary point
iv. None of these.

Q4 If we define a system $\dot{X}$ =F(X) , then it is

i. Always time dependent
ii. Always time independent
iii. Occasionally time dependent
iv. None of these.

Q5 If a continuous function satisfies Lipschitz condition in some domain D, then the initial value problem $\frac{dX}{dt}$=F(t,X) with x(t_0)=X_0 has … solution in D-

i. At least two
ii. Infinite
iii. Unique

iv. Nothing can be said

Q6 Let c be a critical point for the system $\dot{X}$ =F(X). If there exist a $\delta > 0$ such that whenever || X(0) – c|| <δ , $\lim_{t \to \infty} ||X(t) - c||$ =0, then the point is called-

i. Asymptotically stable
ii. Asymptotically unstable
iii. Stable
iv. None of these.

Q7 Let c be a critical point for the system $\dot{X}$ =F(X), then the point c , is said to be **…** If for given $\in > 0$,there exists a $\delta > 0$ such that whenever || X(0) – c|| <δ ; ||X(t) – c||<∈, $\forall\, t > 0$;

i. Asymptotically stable
ii. Asymptotically unstable
iii. Stable
iv. None of these.

Q8 Let c be a critical point for the system $\dot{X}$ =F(X), then the point c, is said to be… if it is stable and asymptotically stable,

i. Strictly unstable
ii. Strictly stable
iii. Unstable
iv. None of these

Q9 Suppose $\dot{X}$ =AX is a linear autonomous system with $n \times n$ real non-singular constant coefficient matrix A. Also we suppose critical points at the origin of R^n. Then the critical point is … if real parts of eigen values of A are negative.

i. Strictly stable
ii. Unstable
iii. Stable but not strictly
iv. None of these

Q10 Suppose $\dot{X}$ =AX is a linear autonomous system with $n \times n$ real non-singular constant coefficient matrix A. Also we suppose critical points at the origin of R^n. Then the critical point is …if A has at least one pair of purely imaginary eigenvalues of multiplicity one

i. Stable but not strictly
ii. Stable
iii. Unstable
iv. None of these.

Q11 Let V(x) be a real-valued function of class c^1 in some open region Ω about the origin. The function V(x) is said to be …if V(x) > 0 for all x $\neq 0 \; in \; \Omega$ and V(x) =0 if and only if x=0

i. Neither positive definite nor negative definite
ii. Negative definite
iii. Positive definite
iv. None of these.

Q12 If there exists a Lyapunov function V(x) in Ω, then the origin is-
i. Unstable but not strictly
ii. Stable
iii. Always unstable
iv. None of these.

Q13 If V is a Lyapunov function such that $-\nabla V(X).F(X)$ is positive definite in Ω, then the origin is -
i. asymptotically stable
ii. always unstable
iii. always stable
iv. None of these.

Q14 Let V be a real valued function of class C^1 in Ω with V(0) = 0, and let $V(X_0)$ >0 for all X in $|X|<\delta$. If $\nabla V(X).F(X)$ is positive definite in Ω. then the origin is -
i. asymptotically stable
ii. occasionally unstable
iii. unstable
iv. None of these

Q15 A limit cycle is a -
i. closed curve
ii. never a closed curve
iii. may be a closed curve
iv. None of these.

5.18 *SELF CHECK QUESTIONS:-*

EXAMPLE 1. Let us consider the system

$$\dot{x} = -y + xy$$
$$\dot{y} = x - x^2$$

which has a critical point at the origin.

SOLUTION: Suppose the Lyapunov function be

$V = \frac{1}{2}(x^2 + y^2)$

Then, $\dot{V} = x(-y+xy) + y(x-x^2) = 0$,

As V is positive definite and $\dot{V} = 0$, the Lyapunov function V exists.

Hence with the help of Theorem 6.9.1, the origin is stable.

EXAMPLE 2 . Discuss the stability of damped harmonic motion given by $\ddot{x}$ + 2$\dot{x}$ +2x = 0.

SOLUTION: Let us take $\dot{x}$ = y

So, given equation becomes $\dot{y}$ +2y +2x = 0.

Or $\dot{x}$ = y & $\dot{y}$ = -2y -2x ………(2.2)

The characteristic equation is given by

$$\begin{vmatrix} -K & 1 \\ -2 & -2-K \end{vmatrix} = 0$$

Or $k^2 + 2k + 2 = 0$

Or $(k + 1)^2 + 1 = 0$

⇨ $k = -1 \pm i$

According to above theorem , the origin is strictly stable.The solution of equation (2.2) is:

$$x = e^{-t}(a\cos t + b\sin t),$$

$$x + y = e^{-t}(-a\sin t + b\cos t).$$

The trajectories associated with equation (2.2) can be obtained by introducing polar coordinates. So, the system (2.2) takes the form

$$x = r\cos\theta = ce^{-t}\cos(t-a) \ , \quad c = (a^2 + b^2)^{1/2}$$

$$x + y = r\sin\theta = ce^{-t}\sin(t-a) \ ; \ \alpha = \tan^{-1}(b/a)$$

by eliminating 't' we get

$$r = ce^{-(\theta+\alpha)}$$

Which describes a family of spirals.

Example 3: Using the Lyapunov function V(x,y) = $\frac{1}{2}(x^2 + y^2)$, determine the stability of the critical point (0,0) for each system.

(a) $\dot{x} = -x-\frac{x^3}{3}\text{Cos}y$

$\dot{y} = -y-y^3$

(b) $\dot{x}= -y-x\sin^2 x,$

$\dot{y}=x-y\sin^2 x.$

(c) $\dot{x}=x- y^2,$

$\dot{y}= y + xy.$

Example 4: Find the regular-singular points of the differential equation

$$(x+2)^2(x-1)y'' + 3x(x-1)y' + 2y = 0.$$

Solution: Since,

$$\frac{(x-1)Q(x)}{P(x)} = -\frac{3(x-1)}{(x+2)^2}, \frac{(x-1)^2 R(x)}{P(x)} = \frac{2(x-1)}{(2+x)^2}.$$

Furthermore, the following limits are finite,

$$\lim_{x\to 1}\frac{(x-1)Q(x)}{P(x)}=0, \lim_{x\to 1}\frac{(x-1)^2 R(x)}{P(x)}=0$$

We conclude that $x_1 = -1$ is a regular singular point.

5.19 *SUMMARY:-*

In this unit, we understood the process of generalization of dynamical systems. , If we perturb the initial value $\mathbf{X_0}$ by an infinitesimal amount or by a small amount, the solution **X**(**t**) is changed also by a small amount in a very small interval about $\mathbf{t_0}$**.** Here a question arises- ***"whether a small change in the initial data leads to a small change in the solution for large values of t."?***

Study of solution of above problem is known as '**stability theory**'. This theory has been applied successfully in various areas and '**automatic controls'**, is one of them. Autonomous system, viz. negatively damped non-linear oscillator, admit solutions which generally tend to a limiting finite periodic solution. Such limiting closed curve in the phase plane is called a limit cycle.

5.20 *GLOSSARY:-*

CRITICAL POINT:
A point c in x^n , at which F(c)=0 is called an equilibrium point or critical point of an autonomous system $\dot{X}$ =F(X).

Limit Cycle: Autonomous system, viz. negatively damped non-linear oscillator, admit solutions which generally tend to a limiting finite periodic solution. Such limiting closed curve in the phase plane is called a limit cycle.

Lyapunov : A Lyapunov was a leading Russian Mathematician, who investigated the stability of non-linear autonomous systems of differential equation without actually determining the solutions. The basis of this concept method is the concept that the potential energy of a Conservative dynamical system has a relative minimum at a stable equilibrium point.

5.21 *REFERENCES:-*

- G F Simmons (1991) Differential Equations with Historical Notes.

5.22 *SUGGESTED READING:-*

- NPTEL videos.
- Schaum series.
- Advanced mathematical Methods for Scientists and Engineers

5.23 *TERMINAL QUESTIONS:-*

(TQ-1) Prove that a limit cycle is a closed curve.
(TQ-2) Discuss the various types of critical points.
(TQ-3) Discuss the importance of Lyapunov constant.

5.24 *ANSWERS:-*

OBJECTIVE ANSWERS

1a 2c 3b 4b 5c 6a 7c 8b 9a 10b 11c 12b 13a 14c 15a

UNIT6:- SECOND ORDER DIFFERENTIAL EQUATIONS

CONTENT:-

6.1 *INTRODUCTION:-*

In this unit, we will study the second-order differential equation of a mathematical equation which involves the second derivative of an unknown function. It is widely used in various fields of science and engineering to describe physical phenomena and model dynamic systems. Second-order differential equations are of great importance because they capture more complex behaviors and dynamics than first-order equations.

6.2 *OBJECTIVES:-*

After studying this unit you will be able to mathematicians and scientists can gain a deeper understanding of the behavior and properties of second-order differential equations, enabling them to solve complex problems and make accurate predictions in diverse fields of study.

6.3 *LINEARLY INDEPENDENT FUNCTIONS:-*

Suppose two functions f(x) and g(x) are defined on interval [a, b]. If one of the functions, say f(x), be a constant multiple of other (here g(x)), then both are said to be linearly dependent functions on [a, b].

If neither of them is a constant multiple of other then both are said to be linearly independent. Here we should note that if f(x) is identically zero then f(x) and g(x) are linearly dependent, for each and every g(x), because

$$f(x) = 0.g(x).$$

THEOREM-

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogenous equation.

$$y'' + P(x)y' + Q(x)y = 0 \qquad ..(1)$$

on the interval [a,b]. Then

$$c_1y_1(x) + c_2y_2(x) \qquad ...(2)$$

is the general solution of the equation (1) on [a, b], in the sense that every solution of equation (1) over the interval [a, b] can be obtained from (2) by a suitable choice of the arbitrary constants c_1 and c_2.

PROOF-

Suppose y(x) is the solution of equation (1) on [a, b]. We already know that solution of equation (1) on [a, b] is completely determined by its value and the value of its derivative at a single point.

If $y_1(x)$ and $y_2(x)$ be the solution of equation (1) then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (1) Where c_1 and $c_2 \in \mathbf{R}$.

As $c_1y_1(x) + c_2y_2(x)$ and y(x) are both solution of equation (1) on [a, b], it is sufficient to prove that for some point $x_0 \in [a, b]$, we can find c_1 and c_2 so that

$c_1y_1(x_0) + c_2y_2(x_0) = y(x_0)$

$c_1y'_1(x_0) + c_2y'_2(x_0) = y'(x_0)$.

This system of equations is solvable for c_1 and c_2 if

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = y_1(x_0)y'_2(x_0) - y_2(x_0)y'_1(x_0) \neq 0$$

It means that we have to discuss **Wronskian**, given as

$W(y_1, y_2) = y_1(y_2)' - y_2(y_1)'$

Now, we prove a Lemma which simplifies the problem of the Showing that the location of the point x_0 is of no consequence.

LEMMA 1:

If $y_1(x)$ and $y_2(x)$ are two solutions of (1) on [a, b], then the Wronskian of $y_1(x)$ *and* $y_2(x)$ either identical to zero or never zero on [a, b].

PROOF:

Here it is given that $W = W(y_1, y_2) = y_1(y_2)' - y_2(y_1)'$

$W'(y_1, y_2) = y_1(y_2)'' + y'_1(y_2)' - y'_2(y_1)' - y_2(y_1)''$

$W'(y_1, y_2) = y_1(y_2)'' - y_2(y_1)'$

Also y_1 and y_2 are both solutions of (1), we have

$y_1'' + Py_1' + Qy_1 = 0$ …(1)

$y_2'' + Py_2' + Qy_2 = 0$ …(2)

If we multiply (1) by y_2 and (2) by y_1 and then subtracting, we get

$y_1 y_2'' - y_2 y_1'' + P(y_1y_2' - y_2y_1') = 0.$

- ⇨ $W' + PW = 0$
- ⇨ $\frac{dW}{dx} + PW = 0$
- ⇨ $\frac{dW}{dx} = - PW$
- ⇨ $\frac{dW}{W} = - P\,dx$

On integrating, we get $W = C\,e^{-\int P dx}$ …(3)

From calculus we know that exponential function is never zero .

So W is zero only if C=0.

- ⇨ W is either identically zero or never zero in [a, b].

Now, more than half work is done. To prove the theorem, now we have to show that the Wronskian of two linearly independent solutions of (1) is not identically zero.

LEMMA 2:

If $y_1(x)$ and $y_2(x)$ are two solutions of equation (1) on [a, b], then $y_1(x)$ and $y_2(x)$ are linearly independent on [a, b] if and only if Wronskian

$$W = W(y_1, y_2) = y_1(y_2)' - y_2(y_1)'$$

is identically zero.

PROOF:

Suppose $y_1(x)$ and $y_2(x)$ be linearly dependent.

If either function is identically zero on [a, b] , then $W(y_1, y_2)$ is obviously zero. If none of them is identically zero, then each one is a constant multiple of other (due to linear dependence).

Let $y_1 = cy_2$ for some constant c. So $y_1' = cy_2'$

Now very easily it can be shown, $W(y_1, y_2) = y_1(y_2)' - y_2(y_1)' = 0.$

Conversely, suppose the Wronskian be identically zero.

If y_1 (or y_2) is identically zero in [a, b]. Then obviously y_1 and y_2 are linearly dependent.

Now, suppose y_1 is not identically zero in on [a,b].

Then by continuity, there exist a sub-interval [c, d] on [a, b] where y_1 is not vanishing at every point.

So, we can write

$\frac{W(Y1,Y2)}{Y_1^2} = \frac{y_1(y_2)' - y_2(y_1)'}{Y_1^2} = 0$

⇨ $(\frac{Y_2}{Y_1})' = 0$

On integration, $(\frac{Y_2}{Y_1})$ = constant on [c,d] and they have equal deviation also.

Hence, $y_2(x) = c.y_1(x)$.

⇨ y_1 and y_2 are linearly independent.

Note - From above theorem, we have two tests to determine linear dependence of y_1 and y_2 →

A) If it is convenient, show $y_1(x)/y_2(x)$ = constant.
B) Otherwise, show that $W(y_1,y_2) = 0$.

EXAMPLE: Prove that $y = c_1\sin x + c_2\cos x$ is general solution of $y'' + y = 0$ on any interval. Also obtain the particular solution for which $y(0) = 2$ and $Y'(0) =3$.

SOLUTION: We observe that $y_1(x) =\sin x$ and $y_2(x) = \cos x$ satisfy $y'' + y = 0$. So $y_1(x)$ and $y_2(x)$ are the solution of given differential equation.

Now, we find $W(y_1,y_2)$ on [a, b].

$$W(y_1,y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0.$$

Also $P(x) = 0$ and $Q(x) = 1$ are naturally continuous on [a, b].

⇨ $y = c_1\sin x + c_2\cos x$ is the general solution of $y'' + y = 0$, on [a, b].

We can extend [a, b] to **R** as it does not affects the continuity of p(x) and Q(x). So the general solution is valid for every x.

For particular solution, $c_1\sin 0 + c_2\cos 0 = 2$ and $c_1\cos 0 - c_2\sin 0 =3$.

For particular solution, $c_1 =3$, $c_2 =2$

So, $y = 3\sin x +2\cos x$ is the general solution with given conditions.

6.4 *QUALITATIVE PROPERTIES OF SOLUTONS:-*

Sometimes it is really very difficult to get the solution of

$$y'' + P(x)y' + Q(x)y = 0 \qquad \dots(1)$$

in terms of known elementary functions. In such situations, we try to understand essential characteristic of the solution of (1) by direct analysis of equation itself, in the absence of formal expressions.

To make our work easy (there is no loss of generality), let us discuss

$$y'' + y = 0 \qquad \dots(2)$$

we have already discussed its general solution thoroughly.

Suppose we don't know all this.

Let's start from the scratch. Our purpose is to observe how their properties can be determined by (2) and initial conditions they satisfy.

Suppose y= s(x) be the solution of (2) with initial conditions s(0) = 0, and s'(0) = 1.

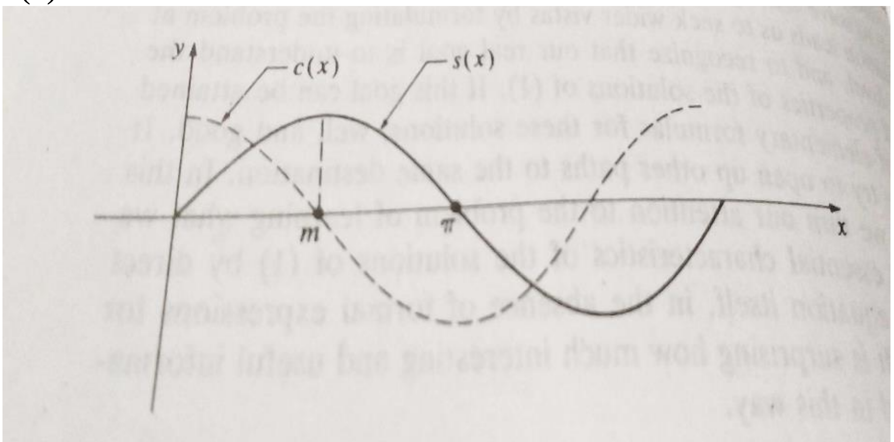


Fig.1

If we draw the curve, we observe that the graph of s(x) by letting x increase from zero, the initial conditions inform us to start the curve from the origin & let it rise with slope beginning at 1

Since s''(x) = -s(x). So above x-axis, s''(x) < 0 and increase in magnitude. We know that s''(x) is the rate of change of s'(x) i.e. slope ; which decrease at an increasing rate as the curve lifts. Suppose it reaches at zero at, say x= m. After that curve falls towards x-axis & the curve cuts x-axis at a point, say x= π. As s''(x) depends only on s(x) , we observe that graph between x = 0 and x = π is symmetric about x = m = π/2 and s'(π) = -1. A similar argument can be made for the next portion and so on indefinitely.

Now, we introduce y = c(x) as the solution of (2) , determined by the initial condition c(0) = 1 and c'(0) = 0. With the same reasoning, as discussed earlier we shall show that

s'(x) = c(x) and c'(x) = -s'(x) … (3)

PROOF:

From (2), we observe that

y''' + y' = 0

Or (y')'' + y' = 0.

Implies that derivative of solution of (2) is again a solution. It also implies s(x) and c(x) are both solutions of equation (2). From previously discussed theorem, it is sufficient to prove that both have the same derivative and same value at x = 0. This is obvious as s'(0) = - s'(x) , c(0) = 1 and s''(0) = - s(x)= 0 , c'(0) = 0. The second formula in (3) is a natural consequence of the first.

for c'(x) = s''(x) = -s(x) .

Claim:

We now use (3) to prove that

$s(x)^2 + c(x)^2 = 1$ …(4)

On differentiating, with respect to x, we get

$2s(x).c(x) - 2c(x)\, s(x)$ is the derivative of LHS of (4),

Which is zero also.

=> $s(x)^2 + c(x)^2 =$ constant

But $s(0)^2 + c(0)^2 = 1$

=> constant = 1

Hence , $s(x)^2 + c(x)^2 = 1$.

Claim : s(x) and c(x) are linearly independent.

$W[s(x), c(x)] = s(x)c'(x) - c(x).s'(x) = -s(x)^2 - c(x)^2 = -1 \neq 0$.

So, s(x) and c(x) are linearly independent.

Note - From above discussion, we concluded two **major** things:

1) We have squeezed almost every significant property of sinx and cosx from (2) by the method of differential equation alone (without using trigonometry).

2) Mainly we used convexity arguments (which involves sign and magnitude of the second derivative)

Generalization of above discussion is called Sturm separation theorem.

6.5 *STURM SERARATION THEOREM:-*

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then the zeros of these functions are distinct and occur alternatively , in the sense that $y_1(x)$ vanishes exactly once between two successive zeroes of $y_2(x)$, and conversely.

PROOF : Since y_1 and y_2 are linearly independent. So, $W(y_1, y_2) \neq 0$.

Since it is continuous, so must have constant sign.

Now we show that y_1 and y_2 have a common zero.

Otherwise, $W(y_1, y_2) = 0$, which is not possible.

So we now assume that x_1 and x_2 are successive zeroes of y_2 and we shall prove that y_1 vanishes between these two points.

Obviously, in this situation $W(y_1, y_2) = y_1(x)y'_2(x) - 0$

⇨ both factors $y_1(x)$ and $y'_2(x) \neq 0$.

Also $y'_2(x_1)$ and $y'_2(x_2)$ must have opposite sign, because if y_2 is increasing at x_1, it must be decreasing at x_2 and vice -versa.

As the wronskian has constant sign, $y_1(x_1)$ and $y_2(x_2)$ must also have opposite signs.

But $y_1(x)$ is continuous.

⇨ $y_1(x)$ must be zero at some point between x_1 and x_2.

Now we show that $y_1(x)$ can't be zero more than once between x_1 and x_2.

For if, it vanish more than once between x_1 and x_2, then the same argument shows that y_2 must vanish between these zeroes of y. But this is a contradiction to the initial assumption that x_1 and x_2 are successive zeroes of y_2.

EXAMPLE – Reduce $y'' + P(x)y' + Q(x)y = 0$ …(1)

into $u'' + q(x)u = 0$ …(2)

by suitable choice of dependent variable.

SOLUTION – Generally form (1) is termed as '**standard form'** while (2) is known as '**normal form'.**

Let us put $y(x) = u(x)v(x)$ in (1),

$$y' = uv' + u'v$$

and

$$y'' = uv'' + 2u'v' + u''v$$

From (1) ,

$$vu'' + (2v' + Pv)u' + (v'' + Pv' + Qv)u = 0 \quad \dots(3)$$

If we make coefficient of u′ as zero, then

$$2v' + Pv = 0 \text{ or } v = e^{-\frac{1}{2}\int P dx} \quad \dots(4)$$

Putting this in equation (3) and after some manipulation, we get (2) with

$$q(x) = Q(x) - (1/4)P(x^2) - (1/2)P'(x) \quad \dots(5)$$

From(4), we observe that $v \neq 0$, for any point, so the above transformation of (1) into (2) has no effect whatever on the zeroes of solution.

NOTE →

Now we shall observe that if q(x) in (2) is negative function, then the solutions of this equation don't oscillate.

THEOREM – If $q(x) < 0$, and if u(x) is non-trivial solution of $u'' + q(x)u = 0$, then u(x) has at most one zero.

PROOF: Here we assume that u(x) is not identically zero i.e. u(x) is non-trivial. Let x_0 be a zero of u(x) i.e. $u(x_0) = 0$. So $u'(x_0) \neq 0$.

Let us assume $u'(x_0) > 0$.

- ⇨ u(x) is positive over some interval to the right of x_0.

Given that $q(x) < 0$. So $u''(x) = -q(x)u(x)$ is positive function on the same interval.

- ⇨ u′(x) is an increasing function.
- ⇨ u(x) can't have zero to the right of x_0.

In the similar way we can prove that u(x) has no root to the left of x_0.

A similar argument hold when $u'(x) < 0$.

Hence u(x) has either no zeroes at all or only one.

THEOREM: Let u(x) be a non-trivial solution of $u'' + q(x)u = 0$, where $q(x) > 0$, for all $x > 0$. If

$$\int_1^\infty q(x)dx = \infty \quad \dots(6)$$

Then u(x) has infinitely many zeroes on the +ve x-axis.

PROOF: We prove the result by the method of contradiction.

Let us assume that $u(x) = 0$, at most finite number of times for $0 < x < \infty$, So that a point $x_0 > 1$ exist, with property that $u(x) \neq 0$ for all $x \geq 0$. As $u(x)$ can be replaced by $-u(x)$, if necessary, so without loss of generality, we can assume that $u(x) > 0$, for every $x \geq x_0$.

Claim: $u'(x)$ will be negative somewhere to right of x_0, so that there will be one or more zero after x_0.

Let $v(x) = \frac{-u'(x)}{u(x)}$ for some $x \geq 0$. A simple calculation shows $v'(x) = q(x) + v(x)^2$

⇨ on integrating from x_0 to x, $x > x_0$,

$v(x) - v(x_0) = \int_{x_0}^{x} q(x)dx + \int_{x_0}^{x} v(x)^2 dx$

From (6), we conclude that $v(x)$ is positive if x is large enough.

⇨ $u(x)$ and $u'(x)$ have opposite signs if x is very large.

⇨ $u'(x)$ is –ve.

⇨ $u(x)$ has one more root.

So the proof is complete.

6.6 *SECOND ORDER DIFFERENTIAL EQUATIONS:-*

Ordinary and singular point -

All of us know that the general homogeneous second order differential equation is of the form

$$y'' + P(x)y' + Q(x) = 0 \qquad \ldots(1)$$

Sometimes it is not possible to solve a above equation in terms of familiar elementary functions. On some other occasion viz when $P(x)$ and $Q(x)$ are constant and in a few other cases, it is possible to solve equation (1) explicitly.

These types of equations have a huge significance in pure and applied mathematics. But in general, power series solutions are only choices as solutions.

The main concept behind the solution of equation (1) is that the behavior of its solution near the point x_0 depends on the behavior of its coefficient functions $P(x)$ and $Q(x)$ near x_0. Here we restrict ourselves to the case when $P(x)$ and $Q(x)$ are "**well behaved**" in the sense of being analytic at x_0. It means each $P(x)$ and $Q(x)$ has a power series expansion valid in some neighborhood of x_0.

In such cases x_0 is called **'ordinary point'** of equation (1). Consequently every solution of equation (1) is also analytic at this point. It means, we can say that the analyticity of $P(x)$ and $Q(x)$ at a certain point say x_0 imply that solutions of equation (1) are also analytic at there.

Now we can define **'singular point'**. A point which is not ordinary point is called singular point.

Before proving the claims in the above paragraph let us discuss some elementary concepts, which will be use frequently here.

6.7 *A REVIEW OF POWER SERIES:-*

When we study **'elementary analysis'**, we generally encounter with some specific functions known as **'elementary functions'**. There are two types of 'elementary functions' viz-

Algebraic functions - Such functions y = f(x) satisfy an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots\ldots\ldots\ldots\ldots + P_1(x)y + P_0(x) = 0$$

where each $P_i(x)$ is a polynomial. The functions may be polynomial or rational functions.

Transcendental/Non-algebraic Functions- All non algebraic functions viz trigonometric, inverse trigonometric, exponential and logarithmic functions with all possible combinations viz adding, subtracting, multiplying and dividing or forming a 'function of a functions' are transcendental functions.

If you move ahead that is beyond elementary functions, there are higher 'transcendental functions' which are generally called 'special functions'. Some important special functions are Gamma and Beta Function, Reimann- zeta function, elliptical functions and some other used especially in 'Mathematical Physics'.

In 18th and 19th centuries various mathematician's developed the field of special functions viz Eular, Gauss, Abel, Jacobi, Weirstrass, Reimann, Hermite, Poincare etc. But with time the taste of mathematical community changed to broader class of functions i.e. class of continuous functions, class of integrable functions etc.

So instead of studying biography (that is a particular type of special function), we preferred sociology (i.e. a class of particular type of functions). For balance treatment of analysis we need both of them.

Special functions have a huge variety on their origin, nature and applications. However one large group with considerable degree of unity consists of those which arise as solutions of second order linear differential equations.

DISCUSSION- Now we try to understand (in general way), how these functions arise?

Let us take $$y'' + y' = 0 \qquad \ldots(1)$$

Then from the knowledge of elementary calculus,

y = sinx and y = cosx

Satisfy equation (1). So their linear combination will also be a solution.

Now we discuss the equation,

$$x\,y'' + y' + xy = 0 \qquad \ldots(2)$$

Here situation is very typical. We can't solve this equation in terms of elementary functions.
We know that we can solve second order linear equations with constant coefficients by change of independent variable in terms of elementary functions.
But other second order linear differential equations can't be solved in this way.
In this chapter we solve these equations in terms of power series & define special functions →

POWER SERIES –
An infinite series of the form
$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots..$ …(3)
is called a power series in x.
If we generalize above series, we may write
$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$ …(4)
as a generalized power series in $(x-x_0)$.
By the translation of coordinates, equation(4) can be reduced to equation(3).
The series (3) is said to converge at a point x, if $\lim_{n\to\infty} \sum_{n=0}^{\infty} a_n x^n$ exists and in this case the sum of series is the value of this limit.
Obviously this power series is convergent at x = 0. If we discuss the convergence of power series, we observe three pattern which can be illustrated by these examples →

$$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!x^2 + 3!x^3 \ldots\ldots.. \quad \ldots(5)$$
$$\sum_{n=0}^{\infty} (x^n/n!) = 1 + x + x^2/2! + x^3/3! + \ldots\ldots. \quad \ldots(6)$$
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \ldots\ldots.. \quad \ldots(7)$$

Equation (5) converges only for x = 0. If $x \neq 0$, it diverges.
Equation (6) converges for each $x \in$ **R.**
Equation (7) converges if $|x| < 1$ and diverges if $|x| > 1$.
Power series of type (5) have no much practical uses; while of type (6) are easiest to handle.
Generally, majority power series of type (7), with a "**radius of convergence**" defined as →
'Each power series in x has a radius of convergence, R where $0 \leq R \leq \infty$, with the property that the series converges if $|x| < R$ and diverge if $|x| > R$.'
We observe that R always exists. If R is finite and non-zero, then we can determine 'radius of convergence' (-R, R) s.t. power series converges within the interval while diverges outside the interval. At the end points of interval of convergence, a power series may or may not converge.
Let us suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$, $R > 0$.
We denote its sum by f(x). So

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.. \quad \ldots(8)$$

Then for $|x| < R$, f(x) is continuous and derivatives of all order exists. Also the series can be differentiated term by term as :

$$f'(x) = \sum_{n=0}^{\infty} na_n x^n = a_1 + 2a_2x + 3a_3x^2 + \ldots$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^n = 2a_2 + 3.2a_3x + \ldots$$

and so on.

Each of these resulting series is convergent for $|x| < R$.

These successive differentiated series produce the following basic formula

$$a_n = \frac{f^n(x)}{n!} \qquad \ldots(9)$$

Also we know that the series (8) can be integrated term by term, provided the limits of integration lie inside the interval (-R, R).

Suppose we have another power series in x, which converges to function g(x) for $|x| <R$, i.e.

$$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1x + b_2x^2 + \ldots\ldots.. \qquad \ldots(10)$$

then we can define

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)\, x^n = a_0 \pm b_0 + (a_1 \pm b_1)x + \ldots.. \ldots\ldots..$$

i.e. term by term addition/ subtraction is possible. Also f(x) & g(x) can be multiplied as these are polynomials.

i.e. $f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = a_0b_n + a_1b_{n-1} + \ldots\ldots a_nb_0$.

Explicitly, $f(x)g(x) = (a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots\ldots..)$

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_0b_1 + a_2b_0)x^2 + \ldots\ldots$$
$$(a_0b_n + a_1b_{n-1} + \ldots\ldots.. a_nb_0)x^n + \ldots..$$

If both series converge to the same function i.e. $f(x) = g(x)$ for $|x| < R$, then equation (9) reflects that $a_0 = b_0$, $a_1 = b_1, \ldots\ldots$

Suppose f(x) is a continuous function with derivates of all order for $|x| < R$ and $R > 0$.

Is it possible to represent f(x) by a power series?

If we use equation (9) to a_n, the naturally we'll hope

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots\ldots \qquad \ldots(11)$$

will hold for $|x| < R$.

Generally above equation is true, but unfortunately there are some counter examples to disprove it (we shall study this in real analysis).

If a function is analytic at x_0, then definitely we can obtain its power series expansion.

A function f(x) with the property that power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \ldots(12)$$

is valid for some x_0 is said to be analytic at x_0.

This is one of the advantages (there are many more) of complex analysis over real analysis.

NOTE –

Though according to syllabus, we have to discuss differential equations of second order. But before jumping there, it would be better to discuss series solution of first order differential equations.

First order differential equations can also be solved with the help of elementary functions.

Let's discuss $y' = y$ …(1)

Let us consider that this equation has a power series solution.

$$y = a_0 + a_1x + a_2x^2 + \dots\dots\dots\dots + a_nx^n + \dots. \quad \dots(2)$$

which converges for $|x| < R$, $R > 0$.

It means we assume that equation (1) has a solution, which is analytic at the origin.

So, equation (2) can be differentiated term by term for $|x| < R$.

⇨ $y' = a_1 + 2a_2x + 3a_3x^2 + \dots\dots\dots\dots + (n+1)a_{n+1}x^n + \dots. \quad \dots(3)$

As $y' = y$, so the series (2) and (3) must have the same coefficients.

⇨ $a_1 = a_0,\ 2a_2 = a_1,\ 3a_3 = a_2, \dots\dots\dots\ (n+1)a_{n+1} = a_n, \dots..$

⇨ $a_1 = a_0,\ a_2 = a_1/2,\ a_3 = a_0/2.3, \dots\dots.\ ,\ a_n = a_0/n!, \dots\dots$

So $y = a_0(1 + x + x^2/2! + x^3/3! + \dots.. x^n/n! + \dots)$ …(4)

Here, $a_0 \in$ **R.**

This example suggests a useful method for obtaining the power series expansion of a given function.

Note – Now we return to the study of second order linear differential equations :

First of all we discuss some illustrations →

EXAMPLE : Let us discuss $y'' + y = 0$ …(1)

We know that general homogenous second order differential equation is of the form

$$y'' + P(x)y' + Q(x)y = 0 \quad \dots(2)$$

here $P(x) = 0$, $Q(x) = 1$

As these are analytic at all points, so we can think of a solution as

$$y = a_0 + a_1x + a_2x^2 + \dots\dots\dots\dots + a_nx^n + \dots. \quad \dots(3)$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dots\dots\dots\dots + (n+1)a_{n+1}x^n + \dots. \quad \dots(4)$$

$$y'' = 2a_2 + 3.2a_3x + 3.4a_4x^2 + \dots\dots + (n+1)(n+2)a_{n+2}x^n + \dots. \quad \dots(5)$$

Putting equation (3) & (5) in equation (1) and adding term by term, we get

$$(2a_2 + a_0) + (2.3a_3 + a_1)x + (3.4a_4 + a_2)x^2 + \dots\dots\dots\dots\dots\dots$$
$$[(n+1)(n+2)a_{n+2} + a_n]x^n + \dots. = 0.$$

Equating to zero, the coefficients of successive power of x, we get

$2a_2 + a_0 = 0,\ 2.3a_3 + a_1 = 0,\ \ 3.4a_4 + a_2 = 0,\ (n+1)(n+2)a_{n+2} + a_n = 0, \dots..$

On solving

$a_2 = -a_0/2,\ \ a_3 = -a_1/2.3,\ \ \ a_4 = a_0/2.3.4,\ \ \ a_5 = a_1/2.3.4.5, \dots..$

So from (3),

$$y = a_0(1 - x^2/2! + x^4/4! + \dots..) + a_1(x - x^3/3! + x^5/5! - \dots.) \quad \dots(6)$$

So, $y_1(x) = (1 - x^2/2! + x^4/4! + \dots..)$

And $y_2(x) = (x - x^3/3! + x^5/5! - \dots\dots)$

With the help of ratio test, it can easily shown that both series are convergent for every $x \in$ **R.**

So their addition is justified. Also from calculus

$$y_1(x) = \cos x \text{ and } y_2(x) = \sin x$$

⇨ $$y = a_0 \cos x + a_1 \sin x : a_0, a_1 \in \mathbf{R}.$$

NOTE –

The problem was simple. So we easily got two familiar elementary functions to make its solution very easy. But in general, we are unable to get a familiar elementary function. Let's see →

6.8 *LEGENDRE'S EQUATIONS:-*

Let us discuss the series solution of **Legendre's equation**

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0 \qquad \ldots(7)$$

Here p is a constant. Obviously the coefficient functions

$$P(x) = \frac{-2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{p(p+1)}{1-x^2} \qquad \ldots(8)$$

are analytic at the origin.

⇨ Origin is an ordinary point & so we can think about a solution

$$y = \sum_{n=0}^{\infty} a_n (x)^n \qquad \ldots(\text{i})$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x)^n$$

and $$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x)^n \qquad \ldots(\text{ii})$$

Replacing n+1 by n in equation (i), we get

$y' = \sum n a_n x^{n-1}$ or $xy' = \sum n a_n x^n$

$$-2xy' = \sum (-2n) a_n x^n \qquad \ldots(\text{iii})$$

Now replacing n+2 by n in equation (ii), we get

$y'' = \sum (n-1) n a_n x^{n-2}$

$$-xy'' = \sum -(n-1) n a_n x^n \qquad \ldots(\text{iv})$$

And $$p(p+1)y = \sum p(p+1) a_n x^n \qquad \ldots(\text{v})$$

From equation (7), the sum of these series is required to be zero, so the coefficient of x^n must be zero for each n.

⇨ $(n+1)(n+2)a_{n-2} - (n-1)na_n - 2na_n + p(p+1)a_n = 0.$

On solving we get,

$$a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n \qquad \ldots(9)$$

RECURSION FORMULA: enables us to express a_n in terms of a_0 and a_1 according as n is even or odd

$$a_2 = \frac{-p(p+1)}{1.2} a_0$$

$$a_3 = \frac{-(p-1)(p+2)}{2.3} a_1$$

$$a_4 = \frac{-(p-2)(p+3)}{3.4} a_2 = \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$$a_5 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1$$

$$a_6 = \frac{-p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}a_0$$

$$a_7 = \frac{-(p-1)(p-3)(p-5)(p+2)(p+4)(p+4)}{7!}a_1$$

…..and so on.

Putting all these in y = $\sum a_n x^n$, we obtain

y = a_0[1- $\frac{p(p+1)}{2!}x^2$ + $\frac{p(p-2)(p+1)(p+3)}{4!}x^4$ - $\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6$ + …….]

+ [x - $\frac{(p-1)(p+2)}{3!}x^3$ + $\frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5$ - $\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+4)}{7!}x^7$

+ ……….] …(10)

This is the formal solution of equation (10) with a_0, a_1 ∈ **R**

Both bracket series are called **Legendre's functions**.

Note – Whatever we have learnt from the examples, can be generalized as following theorem, which can tell us about the nature of solutions near ordinary points.

THEOREM: Let x_0 be an ordinary point of the differential equation

$y'' + P(x)y' + Q(x)y = 0$ …(11)

with a_0 and a_1 as arbitrary constants.

Then there exists a unique function y(x) which is analytic at x_0, is the solution of equation (11) in a certain neighborhood of this point and satisfy the initial condition $y_1(x_0) = a_0$ and $y'(x_0) = a_1$

Furthermore, if the power series expansion of P(x) and Q(x) are valid on interval

$|x- x_0| < R$, $R > 0$; then the power series expansion of this solution is also on the same interval.

PROOF: There is no loss of generality, if we restrict ourselves at $x_0 = 0$. With this slight simplification, the hypothesis of theorem is that P(x) and Q(x) are analytic at x_0 and therefore has a power series expansion.

$P(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1x + p_2x^2 + \ldots$ …(12)

$Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1x + q_2x^2 + \ldots$ …(13)

which converge on $|x| < R$, $R > 0$.

Keeping in view the specified initial conditions, we try to find out the solution of equation for

for (11) in the form a power series.

$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \ldots$ …(14)

with radius of convergence at least R.

Now $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + 2a_2x + 3a_3x^2 + \ldots$ …(15)

And

$y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n = 2a_2 + 2.3a_3x + 3.4a_4x^2 + \ldots$ … (16)

As we now know the rule of multiplication of power series, so

$p(x)y' = (\sum_{n=0}^{\infty} p_n x^n)[\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n]$

$= \sum_{n=0}^{\infty}[\sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1}]x^n$ …(17)

$Q(x)y = (\sum_{n=0}^{\infty} q_n x^n)(\sum_{n=0}^{\infty} q_n x^n)$

$= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} q_{n-k}a_k\, x^n)$ …(18)

If we substitute all these in equation (11) and add series term by term, we obtain

$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1} + \sum_{k=0}^{n} q_{n-k}a_k]x^n$ $=0$

Hence we the following recursion formula for a_n :

$(n+1)(n+2)a_{n+2} = - \sum_{k=0}^{n}[\, p_{n-k}(k+1)a_{k+1} + q_{n-k}a_k\,]$ … (19)

If we put n = 0,1,2,3,….in equation (19), we get

$2a_2 = -(p_0a_1 + q_0a_0)$,

$2.3a_3 = -(p_1a_1 + 2p_0a_2 + q_1a_0 + q_0a_1)$,

$3.4a_4 = -(p_2a_1 + 2p_1a_2 + 3p_0a_3 + q_2a_0 + q_1a_1 + q_0a_2)$ and so on.

Thus we have determined $a_2, a_3, a_4, \ldots\ldots$ in terms of a_0 and a_1.

So the resulting series (14), which formally satisfy (11) and the given initial conditions, is uniquely determined by these requirements.

As we discussed some examples, we observed very simple 'two term recursion formulae 'for the coefficients of the unknown series solutions. These are very simple expressions which makes very easy to determine the general term & precise information about their radii of convergence.

But if we observe equation (19), it is clear that it may not be possible in general. In many cases, the best we can do to find the radii of convergence of the series expansion of P(x) and Q(x) & to conclude from the theorem that the radius of convergence for the solution must be at least as large as the smaller of these numbers.

Thus for Legendre's equation, it is clear from (8) and the familiar expansion

$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \ldots\ldots\ldots;\ R = 1.$

That R = 1, for both P(x) and Q(x).

⇨ Any solution of the form y $\sum a_n x^n$ must be valid at least on the $|x| < 1$.

EXAMPLE1: Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$, in terms of power series in x. Can you express this solution by means of elementary functions?

SOLUTION: $y = a_0(1 + x^2 - 1/3\,(x^4) + 1/5(x^6) + \ldots..) + a_1x$

$= a_0(1 + x\tan^{-1}x) + a_1x$

EXAMPLE2: Now discuss the solution of $y'' + xy' + y = 0$……(1)
Obtain the general solution $y = \sum a_n x^n$ in the form of $y = a_0y_1(0) + a_1y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

(1) By ratio test, determine that $y_1(x)$ and $y_2(x)$ converges for every $x \in$ **R.**

(2) Prove that $y_1(x)$ is the series expansion of $e^{\frac{-x^2}{2}}$.

SOLUTION (1): $y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \ldots.$

$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3.5} - \ldots\ldots$

EXAMPLE3: Verify that the equation $y'' + y' - xy = 0$ has three term recursion formula. Also find its series solution $y_1(x)$ and $y_2(x)$ such that –
SOLUTION: $y_1(0) = 1$, $y_1'(0) = 0$.

$y_2(0) = 0$, $y_2'(0) = 1$

$$a_{n+2} = \frac{-(n+1)a_{n+1} - a_{n-1}}{(n+1)(n+2)}$$

$$y_1(x) = 1 + \frac{x^2}{2.3} - \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \ldots.. ,$$

$$y_2(x) = x - \frac{x^2}{2.} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} - \frac{4x^5}{2.3.4.5} + \ldots.$$

6.9 *REGULAR SINGULAR POINTS:-*

As we already know that a point x_0 is a singular point of differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad \ldots(1)$$

If one or other (or both) of the coefficient functions P(x) and Q(x) fails to be analytic at x_0.
Now in such situations, if we want to study the solution of equation (1) near x_0, methods and theorems discussed in previous section are not applicable.
There are many physical situations, where we come across such singularities; and we need appropriate method to study the behavior of solution near x_0. So such singularities demand particular attention. As a simple example, $x = 0$ is a singular point of
$y'' + (2/x)y' + (-2/x^2)y = 0$.
It can be easily verified that $y_1 = x$ and $y_2 = x^{-2}$ are independent solutions provided $x > 0$.

⇨ $y = c_1x + c_2x^{-2}$ is the general solution for $x > 0$.

⇨ If we want to discuss the behavior of solution near x = 0, it is possible by taking

c_2 =0; otherwise very little information can be obtained near the singular point x_0.

To overcome (up to great extent) this difficulty, let us define:

Regular Singular Point →

A singular point x_0 of equation (1) is said to a regular point if the function if $(x - x_0)P(x) - (x - x_0)^2Q(x)$ are analytic. Roughly speaking, it means the singularity in P(x) cannot be worse than $(1/x-x_0)$ and that in Q(x) cannot be worse than $(1/x-x_0)^{2,}$ otherwise x_0 will be called 'irregular point'.e.g. : Legendre's Equation is:

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

⇨ $$y'' - \left(\frac{2x}{1-x^2}\right)y' + \frac{p(P+1)}{1-x^2}y = 0$$

Obviously x = 1 and x = -1 are singular points.

Now for x = 1,

$$(x-1)P(x) = \frac{(x-1)2x}{(x-1)(x+1)} = \frac{2x}{(x+1)}$$

$$(x-1)^2Q(x) = \frac{(x-1)^2p(p-1)}{-(x^2-1)} = \frac{-(x-1)p(p+1)}{(x+1)}$$

Since $(x-1)P(x)$ and $(x-1)^2Q(x)$ are analytic at x = 1.

⇨ x = 1 is a regular singular point.

Similarly x = - 1 is a regular singular point.

6.10 *BESSEL'S EQUATION OF ORDER 'p':-*

Let us take **Bessel's equation of order p**, where p is a non-negative constant.

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

⇨ $$y'' + (1/x)y' + [(x^2 - p^2)/x^2]y = 0$$

Obviously x = 0 is a singular point.

Now , $xP(x) = x.1/x$ and $x^2Q(x) = x^2 - p^2$

Clearly $xP(x)$ and $x^2Q(x)$ are analytic.

So x = 0 is a regular singular point.

SOLVED EXAMPLE

QUESTION 1: For each of the following differential equation, locate and classify its singular point on the x-axis →

(i) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$

(ii) $x^2(x^2-1)^2y'' - x(x-1)y' + 2y = 0$

(iii) $x^2y'' + (2-x)y' = 0$
(iv) $(3x + 1)xy'' - (x+1)y' + 2y = 0.$

SOLUTION:

(i) x = 0 is irregular , x = 1 is regular
(ii) x = 0,1 are regular ; x = -1 is irregular.
(iii) x =0 is irregular.
(iv) x = 0, -1/3 are regulars.

QUESTION2: Discuss the nature of the point x = 0, for each of the following equation →

(i) $y'' + (\sin x)y = 0$
(ii) $xy'' + (\sin x)y = 0$
(iii) $x^2y'' + (\sin x)y = 0$
(iv) $x^3y'' + (\sin x)y = 0$
(v) $x^4y'' + (\sin x)y = 0.$

SOLUTION:

(i) ordinary point
(ii) ordinary point
(iii) regular singular point
(iv) regular singular point
(v) iregular singular point.

6.11 *PICARD'S THEOREM:-*

Let f(x,y) and $\partial f/\partial y$ be continuous functions of x and yon a closed rectangle R with sides parallel to x-axis. If x_0, y_0 be an interior point of R, then there exist a number h > 0 with the property that the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0 \qquad \ldots (1)$$

has one and only solution y =y(x) on the interval $|x - x_0| \leq h$.

Proof- Proof of the theorem is very long and intricate. So we will do it in various steps as →

Step 1 – We already know that every solution of (1) is also a continuous solution of the integral solution.

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\,dt \qquad \ldots (2)$$

and vice versa.

This helps us to conclude that (1) has a unique solution on an interval $|x - x_0| \leq h$, if and only if (2) has unique continuous solution in the same interval.

Now these sequence of functions y_n is defined as

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0(t)]dt \qquad \ldots(3)$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]dt \qquad \ldots(4)$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]dt$$

which converges to the solution of equation (2). Also we observe that $y_n(x)$ is n^{th} partial sum of series of the functions

$y_0 + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] = y_0 + [\, y_1(x) - y_0(x)\,] + \ldots\ldots\ldots [y_n(x) - y_{n-1}(x)\,] + \ldots\ldots$

⇨ Convergence of the sequence (3) is equivalent to the convergence of of this series.

To prove it, let us take $h > 0$ s.t. $|x - x_0| \leq h$.

Claim: Now we show that on this interval, following statements are true.

(i) The series (4) converges to a function y(x),

(ii) y(x) is a continuous solution of (2)

(iii) y(x) is the only continuous solution of (2).

Now we prove these one by one →

(I) As again is in the statement, f(x,y) and $\partial f/\partial y$ are continuous functions on the rectangle R. As R includes its boundaries so it is closed and bounded.

⇨ f(x,y) and $\partial f/\partial y$ are also bounded on R.

⇨ there exist constant M & k such that

$$|f(x,y)| \leq M \qquad \ldots (5)$$

$$\left|\frac{\partial}{\partial y} f(x,y)\right| \leq k \qquad \ldots(6)$$

For every $(x,y) \in R$.

Suppose (x,y_1) and (x,y_2) be distinct points in R with same x-coordinates. We can use mean value theorem such that

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \left|\frac{\partial}{\partial y} f(x,y^*)\right|$$

Or $$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x,y^*)\right| . |y_1 - y_2| \quad \ldots(7)$$

For some $y_1 < y_2$

Using equation (6), we can write

$$|f(x,y_1) - f(x,y_2)| \leq k|y_1 - y_2| \qquad \ldots (8)$$

For any points (x,y_1) and $(x,y_2) \in$ **R.**

Using Archimedean property, we can choose $h > 0$ s.t.

$$Kh < 1 \qquad \ldots(9)$$

Observe that the rectangle R′ defined by $|x - x_0| \leq h$ and $|y - y_0| \leq mh$ is contained in **R**.

Now, we restrict ourselves on $|x - x_0| \leq h$.

To prove (1), it is sufficient to prove that the series

$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \ldots\ldots |y_n(x) - y_{n-1}(x)|$ …(10)

Converges.

For that purpose first we estimate the terms $|y_n(x) - y_{n-1}(x)|$.

We essentially observe that each of the function $y_n(x)$ has a graph, which lies in rectangle R′ and consequently in R.This is obvious for $y_0(x) = y_0$.

⇨ Points $[t, y_0(t)]$ are in R′.

From (5), we have $f[t,y_0(t)] \leq M$ and

$|y_1(x) - y_0| = |\int_{x_0}^{x} f[t, y_0(t)dt]| \leq Mh$

Which proves the statement for $y_1(x)$.

⇨ Points $[t, y_1(t)]$ are in R′.

So, $f[t,y_1(t)] \leq M$ and $|y_2(x) - y_0| = |\int_{x_0}^{x} f[t, y_1(t)dt]| \leq Mh$.

Similarly, $|y_3(x) - y_0| = |\int_{x_0}^{x} f[t, y_2(t)dt]| \leq Mh$, and so on.

We know that continuous function on closed interval have maximum and minimum value. Here $y_1(x)$ is continuous. Let us define

$a = \max|y_1(x) - y_0|$

⇨ $|y_1(x) - y_0| \leq a$.

⇨ Also the points $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R′.

So, from (8), we have

$|f[t,y_1(t)] - f[t,y_0+(t)]| \leq k|y_1(t) - y_2(t)| \leq ka$;

and we have

$|y_2(x) - y_1(x)| = |\int_{x_0}^{x} [f[t, y_1(t)] - f[t, y_0(t)]dt]]| \leq kah = a(kh)$.

In the similar way,

$|y_3(x) - y_2(x)| = |\int_{x_0}^{x} [f[t, y_2(t)] - f[t, y_1(t)]dt]]| \leq k^2ah = a(kh)^2$,

If we continue in this way, we obtain

$|y_n(x) - y_{n-1}(x)| \leq a(kh)^{n-1}$, $n=1,2,3\ldots\ldots\ldots\ldots$

Hence each term of series (10) is less than or equal to the corresponding term of the series of the constants.

$|y_0| + a + a(kh) + a(kh)^2 + \ldots\ldots\ldots + a(kh)^{n-1} + \ldots$

From (9), $kh < 1$

⇨ This is a G.P. with common ratio less than 1. Hence it is convergent.

⇨ By comparison test, (10) is also convergent.

⇨ (4) converges to a sum, say $y(x)$.

Hence $y_n(x) \rightarrow y(x)$

Since the graph of each $y_n(x)$ less in R′, then obviously y(x) will follow this tradition.

(ii) Now we show that y(x) is continuous.

From (i), we observe that convergence of $y_n(x)$ to y(x) is uniform i.e. by choosing sufficient large n, we can make $y_n(x)$ as close as we please to y(x) forch every x in the interval .

⇨ If $\in > 0$ is given, $\exists\ n_0 \in \mathbf{N}$ such that if $n \geq n_0,$ we have

$| y(x_0) - y_n(x) | < t$, for every x in the interval.

Hence each $y_n(x)$ is continuous sequence & sequence is uniformly convergent. Hence the limit function y(x) is also continuous.

Now we show that y(x) is actually the solution of equation (2).

i.e. we have to show that

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt = 0 \qquad \ldots(11)$$

we already know that

$$y_n(x) - y_0 - \int_{x_0}^{x} f[t,(t)]y_{n-1}(t)]\ dt = 0 \qquad \ldots(12)$$

if we subtract left side of equation (12) from the left side of (11) , we get

$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt = y(x) - y_n(x) -$

$\int_{x_0}^{x} f[t,(t)]y_{n-1}(t) - f[t, y(t)]dt$

so we obtain

$| y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt | = | y(x) - y_n(x) -$

$\int_{x_0}^{x} f[t,(t)]y_{n-1}(t) - f[t, y(t)]dt |$

As the graph of y(x) lies in R′.

⇨ Graph of y(x) lies in R. So from (8),

⇨ $| y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt | \leq | y(x) - y_n(x)| + kh.\max|y_{n-1}(x) - y(x)|$ …(13)

Now uniform convergence of $y_n(x)$ implies that RHS of (13) can be made as small as we please by taking large n.

⇨ LHS of (13) must be zero.

⇨ y(x) is the solution of (2).

(iii) **Uniqueness of the solution :**

Suppose **Y(x)** is another solution of equation (2), on the interval $|x - x_0| \leq h$. We will prove that y(x) = **Y**(x) on this interval. For that, it is necessary to know that graph of **Y(x)** lies in R′ and consequently in R.
Let us establish this.

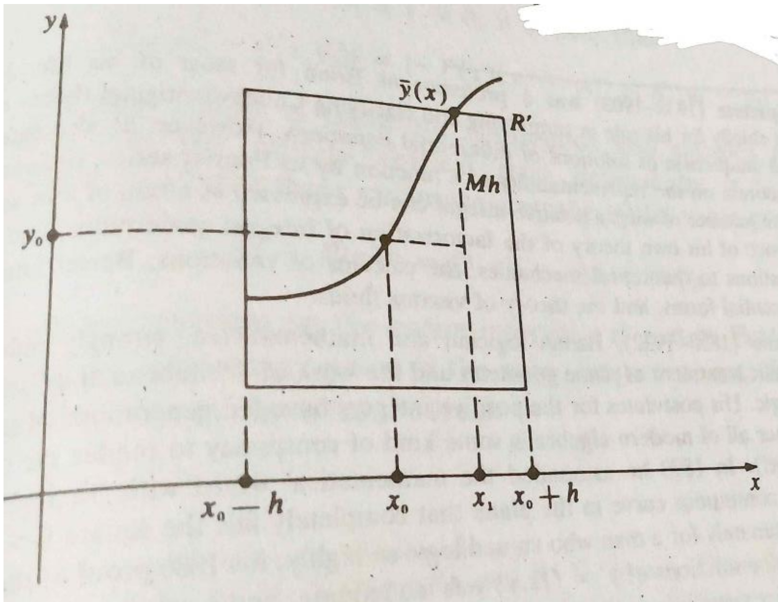


Fig.2

Suppose the graph **Y(x)** leaves R′. Then the properties of this function (i.e. continuity and he fact $\mathbf{Y(x_0)} = (y_0)$.

Imply that there exist an x_1 such that $|x_1 - x_0| < k$, $|\mathbf{Y(x_1)} - y_0| < Mh$, and $|\mathbf{Y(x)} - y_0| < Mh$ if $|x - x_0| < |x_1 - x_0|$

So $\frac{|Y(x_1) - y_0|}{|x_1 - x_0|} = \frac{M\square}{(x_1 - x_0)} > \frac{M\square}{\square} = M$.

Now by using mean value theorem, there exist a number x^* with $x_0 < x^* < x_1$ such that

$\frac{|Y(x_1) - y_0|}{|x_1 - x_0|} = |\mathbf{Y}(x^*)| = |f[x^*, \mathbf{Y}(x^*)]| \leq M$

Because the point $[x^*, \mathbf{Y}(x^*)] \in R'$. But this is contradiction.

So the graph of **Y**(x) lies in R′.

Now we prove the uniqueness part.

Since y(x) and **Y**(x) both are solution of (2).

$|\mathbf{Y(x)} - y(x)| = |\int_{x_0}^{x} (f[t, Y(t)] - f[t, y(t)])| \leq kh.\max|\mathbf{Y(x)} - y(x)|$

As graph y(x) and **Y**(x) lies in R′, so we used equation (8).

$\max|\mathbf{Y(x)} - y(x)| \leq kh.\max|\mathbf{Y(x)} - y(x)|$.

- ⇨ $\max|\mathbf{Y(x)} - y(x)| = 0$, otherwise $1 \leq kh$, contradicts !
- ⇨ $\mathbf{Y}(x) = y(x)$

  i.e. solution of (2) is unique.

NOTE-

This theorem is called **local existence and uniqueness theorem**, because it guarantees the existence of a unique solution in some interval $|x - x_0| \leq h$, where h may be very small.

6.12 *METHOD OF SUCCESSIVE APPROXIMATION:-*

Some simple types of differential equations can be solved explicitly in terms of elementary functions. Some other can be solved with the help of power series.

However many differential equations falls outside these categories.

Now we discuss another method say successive approximation to solve initial value problems.

Let us take $y' = f(x,y)$, $y(x_0) = y_0$ …(1)

Here $f(x,y)$ is an arbitrary function which is continuous in some neighborhood of (x_0,y_0).

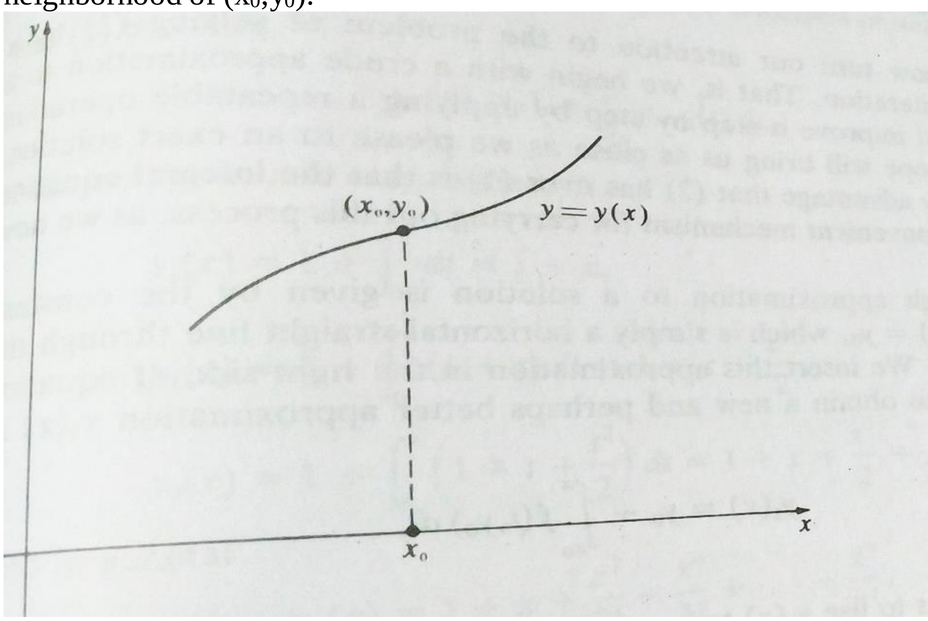


Fig.3

Geometrically speaking, we are devise a method for constructing a function $y = f(x)$, whose graph passes through the point (x_0,y_0) & also satisfy $y' = f(x,y)$ in some neighborhood of x_0.

The key to this method lies in replacing the IVP (1) by the equivalent integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\,dt \qquad \ldots(2)$$

Since the known function occurs under the integral sign, so it is called integral equation.

Suppose y(x) be the solution of equation (1).

- ⇨ y(x) is continuous and RHS of $y'(x) = f[x,y(x)]$ is a continuous function of x.

Now we can integrate from x_0 to x & by using $y(x_0) = y_0$, we get the result (2). Since upper limit is also x, so to avoid confusion, dummy suffix x can be replace by t and we get exactly (2).

Thus any solution of (1) is continuous solution of (2).

Conversely suppose y(x) be continuous solution of (2).

Integral will vanish when $x = x_0$ and so $y(x_0) = y_0$.If we differentiate (2) we get

$$y'(x) = f[x, y(x)].$$

- ⇨ (1) and (2) are equivalent in the sense that the solutions of (1) (if any exist) are precisely the continuous solution of (2).

Now we try to solve (2) by a process of integration. We start with a rough approximation to a solution and prove it in every next step by applying a repeatable process.

Let us start with $y_0(x) = y_0$.

Actually it is a horizontal straight line through the point (x_0,y_0).

We put in RHS of (2) to obtain possibly a better approximation $y_1(x)$ as

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t, y_0]\,dt$$

To make it further a better approximation $y_2(x)$, we do

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\,dt$$

After n steps, we get

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt \qquad \ldots(3)$$

This method is known as **Picard's method of successive approximation.**

EXAMPLE: Discuss the solution of initial value problem

$$y' = -y,\ \ y(0) = 1 \qquad \ldots(1)$$

if we use ordinary calculus, we have

$$\frac{dy}{dx} = y$$

- ⇨ $\int \frac{dy}{y} = \int dx$
- ⇨ $\log_e y = x + c \qquad \ldots(i)$

given x =0 and y = 1

- ⇨ $\log_e y = 0 + c$
- ⇨ c = 0.

So, $\log_e y = x$ or $y = e^x \qquad \ldots(ii)$

We again solve this IVP by successive approximations.

Let $y_0(x) = 1$ and $y_n(x) = 1 + \int_0^x y_{n-1}(t)\,dt \qquad \ldots(iii)$

So $y_1(x) = 1 + \int_0^x dt = 1 + x$

$y_2(x) = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2}$

Similarly $y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

And in general

$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots. \quad + + \frac{x^n}{n!}.$

We observe that when $n \to \infty$, RHS $\to e^x$

This is the solution.

EXAMPLE: Solve the initial value problems by Picard's method :

(i) $y' = x + y$, $y(0) = 1$.

(ii) $y' = y^2$, $y(0) = 1$

(iii) $y' = 2x(1+y)$, $y(0) = 0$.

SOLUTION:(i) $y_1(x) = 1 + x + \frac{x^2}{2}$

$y_2(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

and

Finally $y(x) = 2e^x - x - 1$.

SOLUTION: (ii) $y_1(x) = 1 + x$

$y_2(x) = 1 + x + x^2 + \frac{x^3}{3}$

$y_3(x) = 1 + x + x^2 + x^3 + (2/3)x^4 + (1/5)x^5 + (1/4)x^6 + (1/63)x^7$

SOLUTION: (iii) $y_1(x) = x^2$

$y_2(x) = x^2 + \frac{x^4}{2}$

$y_3(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$, and so on.

$y(x) = e^{x^2} - 1$

Note – Now one may have doubt whether solution of an initial value problem by approximation method always exist? For that reason we already have proved Picard's Theorem. This theorem emphasizes on existence of unique solution of given IVP. That's why this theorem is called 'existence and uniqueness theorem'. There are some other versions of this theorem, you can explore them now.

6.13 *EIGENVALUES, EIGENFUNCTIONS & THE VIBRATING STRING:-*

Let us assume a non-trivial solution y(x) of the equation

$$y'' + \lambda y = 0 \qquad \ldots(1)$$

satisfying two boundary conditions

$$y(0) = 0 \text{ and } y(\pi) = 0. \qquad \ldots(2)$$

Erstwhile we solve the initial value problems. Actually we try to solve second order differential equations with conditions at a single point x_0.
But now we are discussing boundary value problems.i.e. a second order differential equation with two different conditions at two different points. This types of problems are more difficult and deep (in both theory and applications) than initial value problems.
If $\lambda = 0$, in equation (1), $y(x) = c_1x + c_2$
If $\lambda < 0$, then by theorem discussed initial part of this chapter, only the trivial solution of (1) can satisfy.
If $\lambda > 0$, then solution of equation (1) is
$y(x) = c_1\sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x$.

Using y(0) = 0, we get $y(x) = c_1\sin\sqrt{\lambda}x$ …(3)

For $y(\pi) = 0$, $\sqrt{\lambda}.\pi = n\pi$, n is a positive integer.

So, $\lambda = n^2$ i.e. , $\lambda = 1,4,9,16,\ldots\ldots.$

These values of , λ are called '**eigenvalues**' of the problem.

Corresponding solutions of sinx, sin2x, sin3x, …… …(4)
Are called respective '**eigenfunction**'.
Obviously eigenvalues are unique while eigenfunctions are not. Before going ahead, we remember two points →

(i) Eigenvalues from an increasing sequence of natural numbers, diverging to infinity.

(ii) n^{th} eigen function i.e. sin(nx), becomes zero at end point of $[0, \pi]$ & has exactly (n-1) zeroes inside the interval.

We use this concept thoroughly in various physical phenomena viz vibrating strings, harmonic oscillator, heat equations etc.

NOTE : Observing huge size of this unit, we are giving only statement of following theorem.

6.14 *FORIER CONVERGENCE THEOREM:-*

The Fourier series for f(x) converges to f(x) at all values of x where f(x) is continuous.

If f(x) has a discontinuity at x = a, then Fourier series converges to $\frac{1}{2}[f(a^+) + f(a^-)]$. It means at the point of discontinuity, the value of function is redefined as the average of its two one sided limits there,

$$f(a) = \frac{1}{2}[f(a^+) + f(a^-)]$$

Then the Fourier series represents the function everywhere.

6.15 *OBJECTIVE QUESTIONS:-*

Q1 If f(x) is identically zero and g(x) is non-zero over a common domain, then f(x) and g(x) are -

i. Always linearly dependent
ii. Always linearly independent
iii. Never linearly dependent
iv. Nothing can be said.

Q2 If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogenous equation $y'' + P(x)y' + Q(x)y = 0$, on [a,b], then for any constants c_1 and c_2, the term $c_1y_1(x)+c_2y_2(x)$

i. Can't be a solution
ii. Will always be a solution
iii. May be a solution for finitely many values only
iv. Nothing can be said.

Q3 If $y_1(x)$ and $y_2(x)$ are two solutions of $y'' + P(x)y' + Q(x)y = 0$, on [a,b], then the Wronskian of $y_1(x)$ and $y_2(x)$ is

i. Always zero
ii. Never zero
iii. Identical to zero or never zero on [a,b]
iv. Nothing can be said.

Q4 $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, on [a,b], if and only if the Wronskian $W(y_1, y_2)$-

i. Never zero
ii. Zero only once
iii. always zero
iv. Nothing can be said.

Q5 If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then zeros of these function

i. Are always distinct
ii. Are always identical
iii. Have at least one common value

iv. Nothing can be said.

Q6 If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then zeros of these function

i. Occur at same points
ii. Occur alternatively
iii. Are all zero
iv. Nothing can be said.

Q7 If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then the zeros of these functions are distinct and occur alternatively. This theorem is called

i. Gauss Separation Theorem
ii. Leibnitz Separation Theorem
iii. Euler Separation Theorem
iv. Sturm Separation Theorem

Q8 If we reduce $y''+ P(x)y'+ Q(x)y = 0$, into $u''+ q(x)u = 0$, by suitable choice of dependent variable, then reduced form is called

i. Normal form
ii. Echelon form
iii. General form
iv. None of these.

Q9 If $q(x) < 0$, and if $u(x)$ is non-trivial solution of $u''+ q(x)u = 0$, then $u(x)$ has

i. at least one zero
ii. exactly one zero
iii. at most one zero
iv. Nothing can be said.

Q10 Let $u(x)$ be a non-trivial solution of $u''+q(x)u = 0$, where $q(x) > 0$, for all $x > 0$. If $\int_1^\infty q(x)dx = \infty$, then $u(x)$ has

i. infinitely many zeroes on the -ve x-axis
ii. infinitely many zeroes on the +ve x-axis
iii. finitely many zeroes on the +ve x-axis
iv. Nothing can be said.

6.16 *SELF CHECK QUESTIONS:-*

EXAMPLE 1: Prove that $y = c_1\sin x + c_2\cos x$ is general solution of $y'' + y = 0$ on any interval. Also obtain the particular solution for which $y(0) = 2$ and $Y'(0) = 3$.

SOLUTION : We observe that $Y_1(x)$ =sinx and $Y_2(x)$ = cosx satisfy $y'' + y = 0$. So $Y_1(x)$ and $Y_2(x)$ are the solution of given differential equation. Now, we find $W(y_1,y_2)$ on [a, b].

$$W(y_1,y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0.$$

Also $P(x) = 0$ and $Q(x) = 1$ are naturally continuous on [a, b].

⇨ $Y = c_1\sin x + c_2\cos x$ is the general solution of $y'' + y = 0$, on [a, b].

We can extend [a, b] to **R** as it does not affects the continuity of p(x) and Q(x). So the general solution is valid for every x.

For particular solution, $c_1\sin 0 + c_2\cos 0 = 2$ and $c_1\cos 0 - c_2\sin 0 = 3$.

For particular solution, $c_1 = 3$, $c_2 = 2$

So, $y = 3\sin x + 2\cos x$ is the general solution with given conditions.

EXAMPLE 2: Let us discuss $y'' + y = 0$ …(1)

We know that general homogenous second order differential equation is of the form

$$y'' + P(x)y' + Q(x)y = 0$$

here $P(x) = 0,\ Q(x) = 1$ …(2)

As these are analytic at all points, so we can think of a solution as

$$y = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n + \ldots. \quad \ldots(3)$$

⇨ $y' = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots\ldots + (n+1)a_{n+1}x^n + \ldots \quad \ldots(4)$

⇨ $y'' = 2a_2 + 3.2a_3x + 3.4a_4x^2 + \ldots\ldots + (n+1)(n+2)a_{n+2}x^n + \ldots \quad (5)$

Putting equation (3) & (5) in equation (1) and adding term by term, we get

$(2a_2 + a_0) + (2.3a_3 + a_1)x + (3.4a_4 + a_2)x^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$$[(n+1)(n+2)a_{n+2} + a_n]x^n + \ldots. = 0.$$

Equating to zero, the coefficients of successive power of x, we get

$2a_2 + a_0 = 0,\ 2.3a_3 + a_1 = 0,\ 3.4a_4 + a_2 = 0,\ (n+1)(n+2)a_{n+2} + a_n = 0, \ldots..$

On solving

$$a_2 = -a_0/2,\quad a_3 = -a_1/2.3,\quad a_4 = a_0/2.3.4,\quad a_5 = a_1/2.3.4.5,\ \ldots..$$

So from (3),

$$y = a_0(1 - x^2/2! + x^4/4! + \ldots..) + a_1(x - x^3/3! + x^5/5! - \ldots.) \quad \ldots(6)$$

So, $y_1(x) = (1 - x^2/2! + x^4/4! + \ldots..)$

And $y_2(x) = (x - x^3/3! + x^5/5! - \ldots\ldots)$

With the help of ratio test, it can easily shown that both series are convergent for every $x \in$ **R.** So their addition is justified. Also from calculus

$$y_1(x) = \cos x \text{ and } y_2(x) = \sin x$$

⇨ $y = a_0\cos x + a_1\sin x : a_0, a_1 \in$ **R**.

6.17 *SUMMARY:-*

Sometimes it is really very difficult to solve a differential equation explicitly. Then we discuss QUALITATIVE MEHODS to understand the nature of solutions. Whatever we have done in this unit is actually this qualitative discussion.

6.18 *GLOSSARY:-*

- Ordinary point
- Singular point
- Elementary Function
- Regular Singular point
- Eigenvalues
- Eigenfunction

6.19 *REFERENCES:-*

- William E. Boyce and Richard C. DiPrima (2009) 9th edition Elementary Differential Equations and Boundary Value Problems by William.
- William F. Trench, Trinity University (2013) Elementary Differential Equations with Boundary Value Problems.
- Erwin Kreyszig (2010) Advanced Engineering Mathematics.

6.20 *SUGGESTED READING:-*

- Wolfgang Walter (2012) Ordinary Differential Equations.
- Dennis G.Zill (2012) A First Course in Differential Equations: With Modeling Applications.

6.21 *TERMINAL QUESTIONS:-*

Example 1: Discuss the linear independence of two functions.
Example 2 : What are transcendental functions ?
Example 3 : What are algebraic functions ?
Example 4 : Discuss the importance of singular points.

6.22 *ANSWERS:-*

OBJECTIVE ANSWERS

1-i ,2- ii , 3- iii, 4- iii, 5- I, 6- ii, 7- iv, 8- i , 9- iii, 10- ii

BLOCKIII
INTEGRAL CURVES AND DAMPED LINEAR OSCILLATOR

UNIT 7: - TRAJECTORIES

CONTENTS:

7.1 *INTRODUCTION:-*

In this previous unit, you have already studied

- About the Variation of parameter.
- About the Second Order differential equations with suitable examples.
- About the Linear differential equation with examples.

In this unit, we discuss about the trajectories, orthogonal trajectories in Cartesian coordinates, Orthogonal of trajectories in polar coordinates, Oblique trajectories in Cartesian coordinates.

7.2 *OBJECTIVES:-*

After studying this unit you will be able to

- Understanding the trajectories.
- Understanding the orthogonal trajectories in Cartesian coordinates.

- Analyzing the use of trajectories in this context is important for studying these systems.

7.3 *TRAJECTORIES:-*

Definition:

Trajectory: A curve which cuts every number of a given family of curves in accordance with some given law is known as a ***Trajectory*** of the family of curves.

Orthogonal Trajectory*:** If a curve cuts every member of given family of curves at right angles, it is called an ***Orthogonal Trajectories of the family of the curve.

Oblique Trajectory*:** If a curve cuts every member of given family of curves at constant angles$\alpha(\neq 90^0)$, it is called an ***Oblique Trajectories of the family of the curve.

7.4 *SELF ORTHOGONAL FAMILY OF CURVES:-*

Definition: If each member of a given family of curves intersects all other members orthogonally, then the given family of curves is said to be self orthogonal.

From self orthogonal family of the curves, if the differential equation of the family of the curves is identical with the differential equation of orthogonal trajectories, then the family of curves must be self orthogonal.

7.5 *ORTHOGONAL TRAJECTORIES IN CARTESIAN COORDINATES:-*

Let the equation of the given family of the curves be

$$f(x, y, c) = 0 \qquad \dots(1)$$

Where c is parameter

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(x, y, dy/dx) = 0 \qquad \dots (2)$$

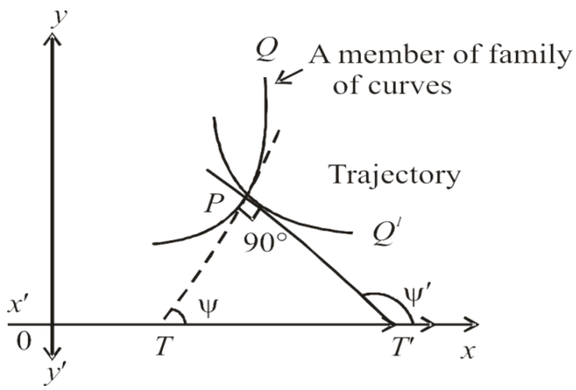


Fig.1

Let ψ be the angle between the tangents PT to the member PQ and $x-axis$ at any point $p(x, y)$, then we have

$$tan\psi = \frac{dy}{dx} \qquad \dots (3)$$

Let (X, Y) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ψ' be the angle which the tangent PT' to the trajectories makes with $x - axis$.

$$tan\psi' = \frac{dY}{dX} \qquad \dots (4)$$

Hence from (3) and (4), we get

$$tan\psi tan\psi' = -1 \qquad or \qquad \frac{dy}{dx} \times \frac{dY}{dX} = -1$$

$$\frac{dy}{dx} \times \frac{dY}{dX} = -1$$

$$\frac{dy}{dx} = -\frac{1}{dY/dX} = -\frac{dX}{dY}$$

Now the point of intersection of (2) with trajectory, we obtain

$$x = X, \qquad y = Y$$

Eliminating x, y and dy/dx from above equations, we have

$$F(X, Y, dX/dY) = 0$$

Hence, which is the differential equation of required family of trajectories.

Now

$$F(x, y, dy/dx) = 0$$

$$F(x, y, -dx/dy) = 0$$

Showing that it can be obtained by replacing $dr/d\theta$ by $(-dx/dy)$.

SOLVED EXAMPLES

EXAMPLE1: Find the orthogonal trajectories of family of curves $y = ax^2$, a being parameter.

SOLUTION: Given family of curves is

$$y = ax^2 \qquad \dots (1)$$

where a being parameter. Differentiating w.r.t. x, we obtain

$$dy/dx = 2ax \qquad \dots (2)$$

From (1), $a = y/x^2$

Putting the value of a in (2), we get

$$dy/dx = 2x\ y/x^2$$

$$dy/dx = 2y/x$$

Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is

$$-dx/dy = 2y/x \qquad or \qquad xdx + ydy = 0$$

Integrating, $x^2/2 + y^2 = b^2$ or $x^2/2b^2 + y^2/b^2 = 1$

Which is required the orthogonal trajectories, b being parameter.

EXAMPLE2: Find the orthogonal trajectories of parabolas whose equation is $y^2 = 4ax$.

SOLUTION: The equation of parabolas is

$$y^2 = 4ax \qquad \dots (1)$$

Differentiating (1) $2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a$

From (1), $a = y^2/4x$

Putting the value of a in above equation

$$y\frac{dy}{dx} = 2\,y^2/4x \Rightarrow \frac{dy}{dx} = y/2x$$

Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is

$$\frac{dx}{dy} = -y/2x \Rightarrow ydy = -2xdx$$

Integrating above equation

$$\frac{y^2}{2} = -x^2 + c \Rightarrow y^2 = -2x^2 + c$$

EXAMPLE3: Find the orthogonal trajectories of the system of curves $(dy/dx)^2 = a/x$.

SOLUTION: The given curve is

$$(dy/dx)^2 = a/x \qquad \dots(1)$$

Where a is constant. Replacing dy/dx by $-dx/dy$, the differential equation of orthogonal trajectories is given as below

$$-(dx/dy)^2 = a/x \qquad or \qquad dy = \pm x^{1/2}a^{1/2}dx$$

Integrating above equation

$$y + c = 1/a^{1/2} \times 2/3 \times x^{3/2}$$

$$3\sqrt{a}(y + c) = \pm 2x^{3/2}$$

Squaring both sides

$$9a(y + c)^2 = 4x^3$$

Which is required orthogonal trajectories, c being parameter.

7.6 *ORTHOGONAL TRAJECTORIES IN POLAR COORDINATES:-*

Let the equation of the given family of the curves be

$$f(r, \theta, c) = 0 \qquad \dots(1)$$

Where c is parameter.

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(r, \theta, dr/d\theta) = 0 \qquad \dots(2)$$

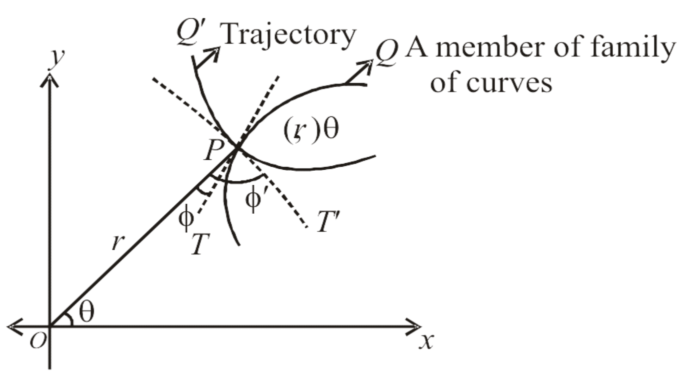


Fig.2

Let ϕ be the angle between the tangents PT to the member PQ and $x - axis$ at any point $p(r, \theta)$, then we have

$$tan\phi = \frac{dy}{dx} \qquad \dots(3)$$

Let (R,Θ) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ϕ' be the angle which the tangent PT' to the trajectories makes with $x-axis$.

$$tan\phi' = R\frac{d\Theta}{dR} \qquad ...(4)$$

Hence from (3) and (4), we get

$$\phi' - \phi = 90^0 \qquad so\ much \qquad \phi' = 90^0 + \phi$$

$$\therefore \qquad tan\phi' = tan(90^0 + \phi\,) =$$

$$-cot\phi \quad or \quad tan\phi tan\phi' = -1$$

Putting the value of (3) and (4) in above equation

$$\left(r\frac{d\theta}{dr}\right)\left(R\frac{d\Theta}{dR}\right) = 1 \qquad or \qquad \frac{dr}{d\theta} = -rR\frac{d\Theta}{dR}$$

Now the point of intersection of (2) with trajectory, we obtain

$$r = R, \qquad \theta = \Theta$$

Eliminating r,θ and $dr/d\theta$ from above equations, we have

$$F(R,\Theta,-R^2 d\Theta/dR) = 0$$

Hence, which is the differential equation of required family of trajectories.
Now

$$F(r,\theta,dr/d\theta) = 0$$

$$F(r,\theta,-r^2 d\theta/dr) = 0$$

Showing that it can be obtained by replacing $dr/d\theta$ by $-r^2 d\theta/dr$.

SOLVED EXAMPLES

EXAMPLE1: Find the orthogonal trajectories of cardioids$r = a(1+cos\theta)$.

SOLUTION: The given curve is $\quad r = a(1+cos\theta)$

Take both sides logarithm

$$\log r = \log a + \log(1+cos\theta)$$

Differentiating both sides w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{sin\theta}{(1+cos\theta)}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, the differential equation of orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{sin\theta}{(1+cos\theta)}$$

$$= -\frac{2sin\,\theta/2\,cos\,\theta/2}{(1+2cos^2\,\theta/2-1)} = -tan\,\theta/2$$

$$r\frac{d\theta}{dr} = tan\,\theta/2 \quad or\ \frac{1}{r}\frac{dr}{d\theta} = cot\frac{\theta}{2}$$

Now integrating factor $\int \frac{1}{r}\,dr = \int cot\frac{\theta}{2}\,d\theta$

$$\log r = 2\log sin\left(\frac{\theta}{2}\right) + \log c$$

$$\log r = \log sin^2\left(\frac{\theta}{2}\right) + \log c$$

$$r = csin^2\left(\frac{\theta}{2}\right)$$

$$r = c\frac{(1 - cos\theta)}{2} = b(1 - cos\theta)\ taking\ b = c/2$$

EXAMPLE2: Find the orthogonal trajectories of the series logarithmic spirals$r = a^{\theta}$.

SOLUTION: The given curve is

$$r = a^{\theta} \Rightarrow \log r = \theta\log a \qquad \dots(1)$$

Differentiating both sides w.r.t.θ

$$\frac{dr}{d\theta} = a^{\theta}\log a = r\log a = r\frac{\log r}{\theta} \qquad from(1)$$

Replacing $dr/d\theta$ by $-r^2\,d\theta/dr$, the differential equation of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = r\frac{\log r}{\theta} \Rightarrow -\theta d\theta = \frac{1}{r}\log r\,dr$$

Integrating both sides

$$\int \frac{1}{r}\log r\,dr = \int -\theta d\theta + c_2$$

$$\frac{(\log r)^2}{2} = -\frac{\theta^2}{2} + c_2, \qquad \therefore\ c_2 = \frac{c^2}{2}$$

$$\frac{(\log r)^2}{2} = -\frac{\theta^2}{2} + \frac{c^2}{2}$$

$$(\log r)^2 = c^2 - \theta^2$$

$$\log r = \sqrt{c^2 - \theta^2}$$

$$r = e^{\sqrt{c^2 - \theta^2}}$$

Which is required equation.

EXAMPLE3: Find the orthogonal trajectories of $r^n cosn\theta = a^{\theta}$ is $r^n sinn\theta = c^n$.

SOLUTION: Given $r^n cosn\theta = a^n$, where a is a parameter.

Since taking both sides logarithm

$$n\log r + \log cosn\theta = n\log a$$

Differentiating both sides w.r..t. θ

$$\frac{n}{r}\frac{dr}{d\theta} - tann\theta = 0 \qquad or \qquad \left(\frac{1}{r}\right)\frac{dr}{d\theta} - tann\theta = 0$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, the differential equation of orthogonal trajectories is

$$\left(\frac{1}{r}\right)(-r^2)\frac{d\theta}{dr} - tann\theta = 0$$

$$\left(\frac{1}{r}\right)dr + cotn\theta d\theta = 0$$

Now integrating factor

$$\log r + \frac{1}{n}\log sinn\theta = \log c$$

Where c being constant.

$$n\log r + \log sinn\theta = n\log c$$

$$r^n sinn\theta = c^n$$

Which is the required equation of orthogonal trajectories.

7.7 *OBLIQUE TRAJECTORIES IN CARTESIAN COORDINATES:-*

Let the equation of the given family of the curv

$$f(x, y, c) = 0 \qquad \dots (1)$$

Where c is parameter.

Differentiating (1)w.r.t.x and eliminating c, between (1) and given curves (1), we have

$$F(x, y, dy/dx) = 0 \qquad \dots (2)$$

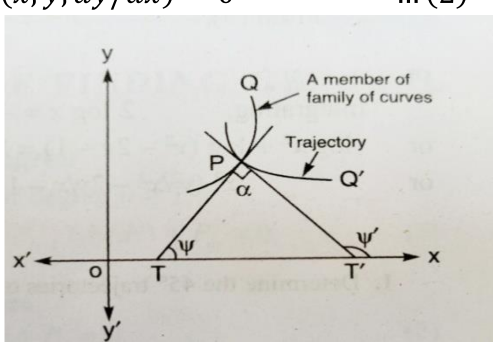


Fig.3

Let ψ be the angle between the tangents PT to the member PQ and $x - axis$ at any point $P(x, y)$, then we have

$$tan\psi = \frac{dy}{dx} \qquad ...(3)$$

Let (X, Y) be the current coordinates of any point of trajectories. At any point of intersection P of (2) with PQ', let ψ' be the angle which the tangent PT' to the trajectories makes with $x - axis$.

$$tan\psi' = \frac{dY}{dX} \qquad ...(4)$$

Suppose PT and PT' intersect at angle α, then we get

$\therefore \qquad tan\alpha = \frac{(dy/dx)-(dY/dX)}{1+(dy/dx)(dY/dX)}$

so that $\qquad \frac{dy}{dx} = \frac{(dy/dx)+(dY/dX)}{1-(dy/dx)(dY/dX)}$

Now from (2) with trajectory, we get

$$x = X, \qquad y = Y$$

Eliminating x, y and dy/dx from above equations, we have

$$F\left(X, Y, \frac{(dy/dx) + tan\alpha}{1 - (dy/dx)\, tan\alpha}\right) = 0$$

Hence, which is the differential equation of required family of trajectories. Now

$$F(x, y, dy/dx) = 0$$
$$F\left(x, y, \frac{(dy/dx)+tan\alpha}{1-(dy/dx)\, tan\alpha}\right) = 0$$

Showing that it can be obtained by replacing dy/dx by $\left[\frac{(dy/dx)+tan\alpha}{1-(dy/dx)\, tan\alpha}\right]$,

$i.e.,\ \ (p + tan\alpha)/(1 - ptan\alpha)$where $p = dy/dx$.

EXAMPLE: Find the family of the curves whose tangents form the angle of $\frac{\pi}{4}$ with the hyperbola $xy = c$.

SOLUTION: Let the given curve

$$xy = c \qquad ...(1)$$

where c is parameter

Differentiating (1),

$$y + x(dy/dx) = 0 \quad or \quad y + px = 0, \quad where\ p = dy/dx$$

Replacing p by $\frac{p+tan\left(\frac{\pi}{4}\right)}{1-\left(\frac{\pi}{4}\right) tan\left(\frac{\pi}{4}\right)}$ $\quad i.e., \frac{p+1}{1-p}$ the differential equation of desired family of curves is

$$\boldsymbol{y} + \frac{p+1}{1-p}x = 0 \quad or \quad p = \frac{y+x}{y-x} \quad or \quad \frac{dy}{dx} = \frac{(y/x)+1}{(y/x)-1}$$

Suppose $\frac{y}{x} = v, \ i.e., \ y = vx$ so that $\frac{dy}{dx} = v + (dv/dx)$

From above equations $v + \frac{dv}{dx} = \frac{v+1}{v-1}$ or $x\frac{dv}{dx} = -\frac{v^2-2v-1}{v-1}$

$$\left(\frac{2}{x}\right)dx = -\left\{\frac{2(v-1)}{(v^2-2v-1)}\right\}dv$$

Integrating, $2\log x = -\log(v^2 - 2v - 1) + \log c,$

c being an arbitrary constant.

$\log x^2 + \log(v^2 - 2v - 1) = \log c$ or $x^2 \ (v^2 - 2v - 1) = c$

Putting the value of $\frac{y}{x} = v$ in above equation

$$x^2 \ \left(\left(\frac{y}{x}\right)^2 - 2\frac{y}{x} - 1\right) = c$$

$$x^2 - 2xy - y^2 = c$$

SELF CHECK QUESTIONS

1. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.
2. Find the orthogonal trajectories of the system of the curve $\left(\frac{dy}{dx}\right)^2 = \frac{a}{x}$.
3. Which among the following is true for the curve $r^n = a\ sinn\theta$
   a. Given family of a curve is self orthogonal.
   b. Orthogonal trajectories is $r^n = kcosn\theta$. Where k is constant.
   c. Orthogonal trajectories is $r^n = kcosecn\theta$. Where k is constant.
   d. Orthogonal trajectories is $r^n = ksinn\theta$. Where k is constant.
4. What is oblique trajectories?

7.8 *SUMMARY:-*

In this unit we studied the trajectories of the family of the curve, orthogonal trajectories and oblique trajectories with example.

7.9 *GLOSSARY:-*

- **Trajectory:** The path followed by an object as it moves through space, often influenced by forces such as gravity, friction, or other interactions.

- Cartesian Coordinates
- Oblique trajectories
- Polar Coordinates

7.10 *REFERENCES:-*

- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- B. Rai, D. P. Choudhury & H. I. Freedman (2013). A Course in Ordinary Differential Equations (2nd edition). Narosa.

7.11 *SUGGESTED READING:-*

- N.P.Bali (2006). Goldan Differential Equations.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.

7.12 *TERMINAL QUESTIONS:-*

(TQ-1) Find the orthogonal trajectories of the family of curves$y = ax^2$, a being a parameter.

(TQ-2) Find the orthogonal trajectories of the family of curves$3xy = x^3 - a^3$, a being a parameter.

(TQ-3) Find the orthogonal trajectories of $x^2 + y^2 = 2ax$.

(TQ-4) Find the orthogonal trajectories of the family of curves:

a. $\frac{x^2}{a^2} + \frac{y^2}{(b^2+\lambda)} = 1, \lambda$ being the parameter.

b. $\frac{x^2}{a^2} + \frac{y^2}{(a^2+\lambda)} = 1, \lambda$ being the parameter.

(TQ-5) Find the orthogonal trajectories of the family of parabolas $y^2 = 4a(x + a),$ where a being a parameter.

(TQ-6) Find the orthogonal trajectories of the family of cardioids $r = a(1 - cos\theta)$, where a being a parameter.

(TQ-7) Find the orthogonal trajectories of the family of cardioids $r = a(1 + cos\theta)$, where a being a parameter.

(TQ-8) Find the orthogonal trajectories of $r = a(1 + cosn\theta)$.

(TQ-9) Find the orthogonal trajectories of $r^n sinn\theta = a^n$.

(TQ-10) Find the orthogonal trajectories of the family of parabolas $r = \frac{2a}{(1+cos\theta)}$, where a being a parameter.

(TQ-11) Find the orthogonal trajectories of the family of curves:

i. $y = ax^n$.
ii. $y = ax^3$.
iii. $y = 4ax$.
iv. $x^2 + y^2 = a^2$.
v.

7.13 *ANSWERS:-*

SELF CHECK ANSWERS

1. $2x^2 + y^2 = k$,
2. $9a(y + c)^2 = 4x^3$**,**
3. b,
4. A curve which intersects the curves of the given family at a constant angle α is called an oblique trajectory of the given family.

TERMINAL ANSWERS

(TQ-1) $\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$
(TQ-2) $x^2 = y - (1/2) + ce^{-2y}$
(TQ-3) $x^2 + y^2 = cy$
(TQ-4)
a. $x^2 + y^2 - 2a^2 \log x = c$
b. $x^2 + y^2 - 2a^2 \log x = c$
(TQ-5) $y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$
(TQ-6) $r = b(1 + cos\theta)$
(TQ-7) $r = b(1 - cos\theta)$
(TQ-8) $r^{n^2} = b(1 - cosn\theta)$
(TQ-9) $r^n cosn\theta = c^n$
(TQ-10) $r = \frac{2c}{(1-cos\theta)}$
(TQ-11)
i. $x^2 + ny^2 = c$
ii. $x^2 + 3y^2 = c$
iii. $2x^2 + y^2 = c^2$
iv. $y = cx$

UNIT 8:- INTEGRAL CURVES AND DAMPED OSCILLATION

CONTENTS:

8.1 *INTRODUCTION:-*

In this previous class, you have already studied

- About Trajectories.
- About orthogonal trajectories in Cartesian coordinates.
- About Orthogonal of trajectories in polar coordinates.
- About Oblique trajectories in Cartesian coordinates.

In this unit, we will study the integral curves and damped oscillation and concepts that are closely related in the study of differential equations particularly in the context of damped harmonic oscillators. Understanding the behavior of integral curves and damped oscillation is essential for analyzing and modeling various physical systems.

8.2 *OBJECTIVES:-*

After studying this unit you will be able to

- Understanding the Integral Curves and ODE in Local Chart with theorems.
- To visualize and understand the behavior of solutions to the ODE.
- To discuss about damped Oscillation.
- To analyze and understand the behavior of the system undergoing damped oscillation.

8.3 *INTEGRAL CURVES:-*

An integral Curve is a parametric curve that represents a specific solution to the ordinary differential equation represented by the vector field. Geometrically, they are curves so that the given vector field is the tangent vector to the curves everywhere.

Here is an example of vector fields with many integral curves drawn:

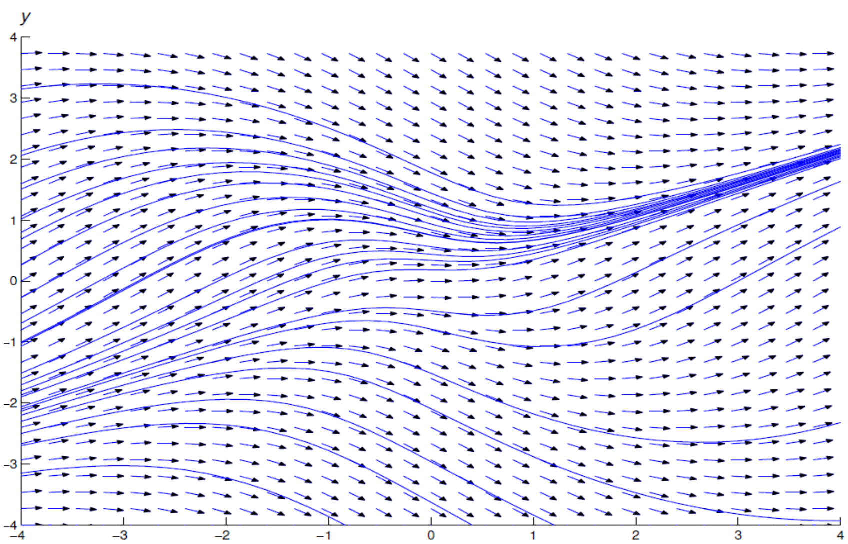


Fig.1

Above conception of integral curves can be generalized to smooth manifolds easily. Recall that a smooth curve in a smooth manifold M is smooth map

$$\gamma : I \to M$$

where I is an interval in $\mathbb{R}. \forall\ a \epsilon I$, the tangent vector of γ at the point $\gamma(a)$ is given below

$$\dot{\gamma}(a) = \frac{d\gamma}{dt}(a) := d\gamma_a\left(\frac{d}{dt}\right)$$

Where $\frac{d}{dt}$ is the standard coordinate tangent vector of $\mathbb{R}$.

Definition: Let $X \in \Gamma^{\infty}(TM)$ be a smooth vector field on M. A smooth curve $\gamma: I \to M$ is called an integral curve of X if for any $t \in I$,

$$\dot{\gamma}(t) = X_{\gamma(t)}$$

EXAMPLES:

Lemma 1.2. If $\gamma: I \to M$ is an integral curve of a vector field X, then

1. **Let $I_a = \left\{\frac{t}{t+a} \in I\right\}, then$**

$$\gamma_a: I_a \to M, \qquad \gamma_a(t) := \gamma(t+a)$$

is an integral curve of X.

Solution: Suppose the vector field $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}$, then the integral curves of X are the straight lines parallel ti $x^1 - axis$, parameterized given as below

$$\gamma(t) = (c_1 + t, c_2, .., c_n).$$

Now we note that $\forall$ the smooth function $f\ on\ \mathbb{R}$, we get

$$d\gamma\left(\frac{d}{dt}\right)f = \frac{d}{dt}(fo\gamma) = \nabla f . \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1}$$

Note: The curve

$$\check{\gamma}(t) = (c_1 + 2t, c_2, .., c_n)$$

Has the same picture as γ, it is not an integral curve of X, but an integral curve of $2X$, since $\dot{\check{\gamma}} = 2\frac{\partial}{\partial x^1}$.

2. **Let $I^a = \left\{\frac{t}{at} \in I\right\},\ (a \neq 0)\ then$**

$$\gamma^a: I^a \to M, \qquad \gamma^a(t) := \gamma(at)$$

is an integral curve of $X^a = aX$.

Solution: Suppose $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then if $\gamma(t) = \big(x(t), y(t)\big)$ is an integral curve of X, $\forall\ f \in C^{\infty}(\mathbb{R}^2)$, then we get

$$x'(t)\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} = \nabla f . \frac{d\gamma}{dt} = X_{\gamma(t)}f = x(t)\frac{\partial f}{\partial y} - y(t)\frac{\partial f}{\partial x}$$

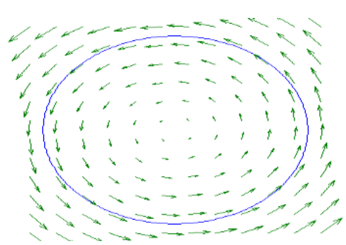


Fig.2.

Which is equivalent to this system is

$$x'(t) = -y(t), \ \ y'(t) = x(t)$$

Since the solution of the system is given by

$$x(t) = acost - bsint, \qquad y(t) = asint + bcost$$

These are circles centre at the origin in the plane parameterized by the angle.

8.4 *ODE (LOCAL CHARTS): EXISTENCE, UNIQUENESS AND SMOOTHNESS:-*

To study further properties of integral curves, we need to convert the equation $\dot{\gamma}(t) = X_{\gamma(t)}$ in to ODEs on function defined on Euclidian region. The following nice local formula for a vector field, whose proof is left as an exercise:

Lemma1.3. Let X be a smooth vector field on M. Then in Local chart (φ, U, V) we have $X = \sum X(x^i)\,\partial_i$, where $x^i: U \to \mathbb{R}$ is the i^{th} coordinate function defined by φ.

Proof: Suppose $\gamma: I \to M$ be an integral curve of X.Since $\dot{\gamma}(t) = X_{\gamma(t)}$ at a given point $\gamma(t)$, assume $\gamma(t) \in U$ and $(\varphi, U, V)0$ is coordinate chart.

By using the Local Chart map φ, one can convert the point $\gamma(t) \in U$ to

$$\varphi\big(\gamma(t)\big) = \Big(x^1\big(\gamma(t)\big), \dots, x^n\big(\gamma(t)\big)\Big) \in \mathbb{R}^n$$

If we denote $y^i = x^i o\gamma: I \to \mathbb{R}$, then convert the equation defining integral curves into equations on these one-variable functions $y^{i,}s$. According to previous lemma, we obtain

$$\dot{\gamma}(t) = d\gamma_t\left(\frac{d}{dt}\right) = \sum_i d\gamma_t\left(\frac{d}{dt}\right)(x^i)\partial_i = \sum_i \left(x^i o\gamma\right)'(t)\partial_i$$

$$= \sum_i \left(y^i\right)'(t)\,\partial_i$$

So that $\dot{\gamma}(t) = X_{\gamma(t)}$ becomes

$$\sum_i \left(y^i\right)'(t)\,\partial_i = \sum_i X^i\left(\gamma(t)\right)\partial_i = \sum_i X^i\, o\varphi^{-1}\big(y^1(t), \dots y^n(t)\big)\partial_i \quad \forall \quad t \in I.$$

Now we convert the following system of ODEs on the one-variable functions $(y^i)'$s.

$$\left(y^i\right)'(t) = \sum_i X^i\, o\varphi^{-1}(y^1, \dots y^n), \qquad \forall\, t \in I, \qquad \forall 1 \le i \le n.$$

Hence this is a system of first order ODEs on the (one-variable) functions $y^i = x^i o\gamma : \mathbb{R} \to \mathbb{R}$. conversely, any solution of a system of ODEs defines an integral curve of the vector field X inside the open set U.

8.5 *DAMMPED OSCILLATION:-*

The oscillation which takes place in the presence of dissipative force are known as damped oscillation.

- Here amplitude of oscillation decreases w.r.t.time
- Dampingforcealwaysactsinaoppositedirectiontothatofmotionandisvelocitydependence.
- Forsmallvelocitythedampingforceisdirectlyproportionaltothevelocity Mathematically

$$F_d \propto v$$

$$F_d = -bv \qquad \dots(1)$$

8.6 *DAMMPED HARMONIC OSCILLATION:-*

Suppose a body of mass m oscillating under a spring force of constant$k$. Let x be a displacement of a body from equilibrium position at any instant and instantaneous velocity is $\frac{dx}{dt}$.The force acting on a body at this instant are:

i. A restoring force proportional to displacement, but acting in the opposite direction, which can be written as

$$-kx$$

Where k is constant.

ii. A frictional (Damping) force proportional to the velocity, but opposite to the direction of motion, which can be written as

$$-b\frac{dx}{dt}$$

where b is a positive constant.

So the net force acting on a body is

$$F = -kx - b\frac{dx}{dt}$$

But by Newton's Law $F = m(d^2x/dt^2)$, where m is mass of the body and d^2x/dt^2 is the instantaneous acceleration, then we get

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

Substituting $\frac{b}{m} = 2r$ and $\frac{k}{m} = \omega^2$, we have

$$m\frac{d^2x}{dt^2} + \omega^2 x + 2r\frac{dx}{dt} = 0 \qquad \dots(1)$$

Hence this is the homogeneous linear differential equation of second order.

Now let $$x = Ce^{\alpha t}$$

where C and α are arbitrary constants.

Differentiating w.r.t. to t, we obtain

$$\frac{dx}{dt} = C\alpha e^{\alpha t}$$

And

$$\frac{d^2x}{dt^2} = C\alpha^2 e^{\alpha t}$$

Putting the value of $\frac{d^2x}{dt^2}, \frac{dx}{dt}$ and x in above equation, we have

$$C\alpha^2 e^{\alpha t} + 2rC\alpha e^{\alpha t} + \omega^C e^{\alpha t} = 0$$

$$Ce^{\alpha t}(\alpha^2 + 2r\alpha + \omega^2) = 0$$

$$\alpha^2 + 2r\alpha + \omega^2 = 0$$

$$\alpha = r \pm \sqrt{r^2 - \omega^2}$$

The general solution of (1) for $r \neq \omega$ is

$$x = C_1 e^{\{-r+\sqrt{r^2-\omega^2}\}t} + C_2 e^{\{-r-\sqrt{r^2-\omega^2}\}t} \qquad \dots(2)$$

Where C_1 and C_2 are arbitrary constants depends on the initial position and velocity of oscillator. Depending on the values of r and ω, three types of motion are possible. Such as

- Under Damping $(r^2 < \omega^2)$
- Over Damping$(r^2 > \omega^2)$
- Critical Damping $(r^2 = \omega^2)$

I. <u>Under Damped $r^2 < \omega^2$:</u>

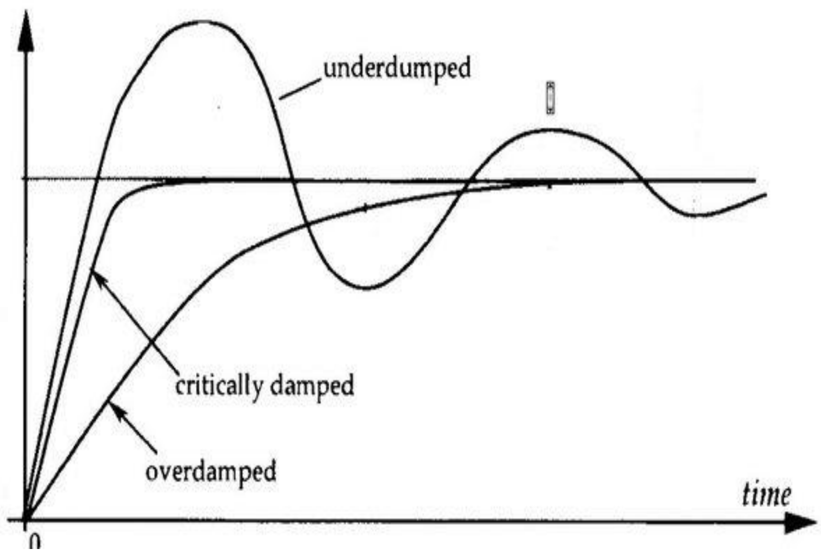


Fig.3.

So $\quad r^2 - \omega^2 = -ve$

Hence $\quad \sqrt{r^2 - \omega^2} = \sqrt{-(\omega^2 - r^2)} = i\omega'$

Where $\omega' = \sqrt{\omega^2 - r^2}$

Hence the solution becomes

$$x = C_1 e^{\{-r+i\omega'\}t} + C_2 e^{\{-r-i\omega'\}t}$$

$$x = e^{-rt}\left[C_1 e^{\{i\omega'\}t} + C_2 e^{\{-i\omega'\}t}\right]$$

$$x = e^{-rt}[C_1(cos\omega' t + isin\omega' t) + C_2(cos\omega' t - isin\omega' t)]$$

$$x = e^{-rt}[(C_1 + C_2)cos\omega' t + i(C_1 - C_2)isin\omega' t]$$

Substituting $C_1 + C_2 = a\ sin\phi$and $i(C_1 - C_2) = acos\phi,$ where a and ϕ are arbitrary constants, then we have

$$x = e^{-rt}[a\ sin\phi\ cos\omega' t + iacos\phi\ sin\omega' t]$$

$$x = e^{-rt}sin(\omega' t + \phi) = e^{-rt}sin\left(\sqrt{\omega^2 - r^2}t + \phi\right) \qquad \dots (3)$$

Equation (3) represents damped harmonic oscillation with amplitude ae^{-rt} which decreases exponentially with time and the time period of vibration is $T = \frac{2\pi}{\sqrt{\omega^2 - r^2}}$ which is greater than that in absence of damping.

Example: Motion of simple pendulum.

II. <u>Over Damped $r^2 > \omega^2$:</u>

In this case $\sqrt{r^2 - \omega^2}$ is real and less than r. Therefore from (2), both $\left\{-r \pm \sqrt{r^2 - \omega^2}\right\}$ are negative. Its means that the displacement of x of the particle continuously decreases with time and when once displaced returns to its equilibrium position quite slowy without performing any oscillation. This motion is called **over damped** or a **periodic** motion and it shown by figure 3.

III. Critical Damped $r^2 = \omega^2$:

If we substituting $r^2 = \omega^2$ in equation (2), then this solution does not satisfy equation (1).

Hence

$$\sqrt{r^2 - \omega^2} = h$$

$$x = C_1 e^{\{-r+h\}t} + C_2 e^{\{-r-h\}t}$$

$$= e^{-rt}\left(C_1 e^{\{h\}t} + C_2 e^{\{-h\}t}\right)$$

$$= e^{-rt}\left(C_1(1 + ht + \cdots) + C_2(1 - ht + \cdots)\right)$$

$$= e^{-rt}[(C_1 + C_2) + ht(C - C_2)]$$

$$= e^{-rt}[D + Et]$$

Where $D = (C_1 + C_2)$ and $E = h(C - C_2)$

Above Fig.3. is clear that in case the particle tends to the position of equilibrium much rapidly than when $r^2 > \omega^2$(Case II). Hence the motion is called critically damped motion.

8.7 *LOGARITHMIC DECREMENT:-*

This measures the rate at which the amplitude dies away. Let

$$x = ae^{-rt}\cos\omega' t$$

So $x = a_0$ at $t = 0$ and a_1, a_2, a_3 ….. be the amplitudes at time $t = T, 2T, 3T$ ... where T is period oscillation, then we get

$$a_1 = a_0 e^{-rT}$$

$$a_2 = a_0 e^{-2rT}$$

$$a_3 = a_0 e^{-3rT}$$

$$\dots\dots\dots\dots\dots\dots$$

Now

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots e^{rT} = e^{\lambda}$$

Where $\lambda(= rT = bT/2m)$ is Logarithmic decrement. Hence

$$\lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} = \cdots$$

Hence it is clear that the Logarithmic decrement is the logarithm of the ratio of two amplitudes of oscillation which are separated by one period.

8.8 *POWER DISSIPATION IN DHO:-*

Whenever a system is set into oscillations, it is subject to frictional (damping) forces arising from air resistance or from within the system itself. These forces oppose the motion of the system. The work done against these forces is dissipated out of the system as heat. Therefore the mechanical energy of the system continuously decreases with time, and the amplitude of oscillation gradually decays to zero. Suppose obtain an expression for this power dissipation from the oscillator.

Now the damped harmonic oscillator is given by

$$x = ae^{-rt} cos\omega' t \qquad \ldots (1)$$

Where a and ϕ are arbitrary constants, r is damping constant and $\omega' = \sqrt{\omega^2 - r^2}$ is the angular frequency of the (damped) oscillator. ω is the angular frequency of undamped oscillator, k being the force-constant. Hence

$$u = \frac{dx}{dt} == e^{-rt}[-r\sin(\omega' t + \phi) + \omega' \cos(\omega' t + \phi)]$$

Now the damping is very small so that $r \ll \omega$, then the term $-r\sin(\omega' t + \phi)$ in the equation for u can be neglected and we can write

$$u = ae^{-rt}\omega' \cos(\omega' t + \phi) \qquad \ldots (2)$$

Now the total energy can be written as

$$E = kinetic\ energy + potential\ energy$$

$$= \frac{1}{2}m\mu^2 + \frac{1}{2}kx^2$$

Putting the values of x and u from (1) and (2) , we have

$$E = \frac{1}{2}ma^2e^{-2rt}\omega' \cos^2(\omega' t + \phi) + \frac{1}{2}ka^2e^{-2at}\omega' \sin^2(\omega' t + \phi)$$

Again, since $r \ll \omega$, then we have

$$\omega'^2 = \omega^2 - r^2 = \omega^2 = \frac{k}{m}$$

$$E = \frac{1}{2}ma^2e^{-2rt}\left(\frac{k}{m}\right)\cos^2(\omega' t + \phi) + \frac{1}{2}ka^2e^{-2rt}\sin^2(\omega' t + \phi)$$

$$= \frac{1}{2}ka^2e^{-2rt}(\cos^2(\omega' t + \phi) + \sin^2(\omega' t + \phi))$$

$$E = \frac{1}{2}ka^2e^{-2rt}$$

This is that the energy of the oscillator decreases with time. **The rate at which the energy is lost is the *power dissipation* P.**

$$P = -\frac{dE}{dt} = -\frac{1}{2}ka^2e^{-2rt}(-2r)$$

$$= \frac{1}{2}ka^2e^{-2rt}(2r)$$

From above equation becomes

$$\boldsymbol{P = 2rE} \qquad \ldots (3)$$

Relaxation Time: The relaxation time is the time taken for the total mechanical energy to decay to $1/e$ of its original value. If the energy is E_0 at $t = 0$, then

$$E_0 = \frac{1}{2}ka^2$$

Since $$E = E_0 e^{-2rt}$$

Now if τ be the relaxation time, then at $t = \tau$, and $E = \frac{E_0}{e}$, we get

$$\frac{E_0}{e} = E_0 e^{-2rt}$$
$$e^{-1} = e^{-2rt}$$
$$-1 = -2rt$$
$$\tau = \frac{1}{2r}$$

Putting the value E and τ in (3), we have

$$P = 2rE$$
$$P = \frac{E}{\tau}$$
$$E = E_0 e^{-t/\pi}$$

The dissipated energy appears as heat in the following oscillating system.

8.9 *QUALITY FACTOR:-*

The quality factor Q of an oscillating system is a measure of damping, or the rate of energy decay, of the system. It is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi \frac{energy\ stored\ in\ system}{energy\ loss\ per\ period}$$

Mean Life-time: The mean life time of damped oscillator is the time taken for the amplitude of oscillation to decay to $1/e$ of the initial value.

Relation between Quality Factor and Relaxation Time:

The quality factor is

$$Q = 2\pi \frac{energy\ stored\ in\ system}{energy\ loss\ per\ period}$$

If E is total energy of oscillator and P is the rate of energy decay, then

$$Q = 2\pi \frac{E}{PT}$$

Where T is the *period*. Now if τ be the relaxation time, then

$$P = \frac{E}{\tau}$$

So that

$$Q = 2\pi \frac{E}{(E/\tau)T} = \frac{2\pi\tau}{T}$$

But $\frac{2\pi}{T} = \omega$, the angular frequency of oscillator, therefore

$$Q = \omega\tau$$

Which is required solution.

SOLVED EXAMPLES

EXAMPLE1: The differential equation of oscillating system is

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = 0$$

If $\omega \gg r$, then find the time in which

i. Amplitude becomes $1/e$ of its initial value.

ii. Energy becomes $1/e$ of its initial value.

iii. Energy becomes $1/e^4$ of its initial value.

SOLUTION: The given equation with condition $\omega \gg r$, is the equation of Harmonic oscillator is

$$x = ae^{-rt}\sin(\omega' t + \phi)$$

Where a and ϕ are arbitrary constants and $\omega' = \sqrt{(\omega^2 - r^2)}$and amplitude is ae^{-rt}.

i. Suppose a_0 be the amplitude at $t = 0$ and a_0/e at t. Then we obtain

$$a_0 = a$$

and

$$\frac{a_0}{e} = ae^{-rt} = a_0 e^{-rt}$$

Now $e^{-1} = e^{-rt} \Rightarrow -1 = -rt \Rightarrow t = \frac{1}{r} sec.$

ii. The energy of damped oscillation is given by

$$E = E_0 e^{-2rt}$$

When the energy falls to E_0/e, we obtain

$$\frac{E_0}{e} = E_0 e^{-2rt} \Rightarrow e^{-1} = e^{-2rt} \Rightarrow -1 = -2rt \Rightarrow t = \frac{1}{2r} sec.$$

iii. When the energy falls to E_0/e^4, we obtain

$$\frac{E_0}{e^4} = E_0 e^{-2rt}$$

$$e^{-4} = e^{-2rt} \Rightarrow -4 = -2rt \Rightarrow t = \frac{2}{r} sec.$$

EXAMPLE2: The quality factor Q of a tuning fork is 5×10^4. Find the value of time-interval after which its energy becomes $1/10$ of its initial value.

SOLUTION: The quality factor of a damped oscillator is given below

$$Q = \omega\tau$$

Where ω is angular frequency and τ is relaxation time. Then

$$\tau = \frac{Q}{\omega}$$

Here $Q = 5 \times 10^4$ and $\omega = 2\pi n = 600\,\pi/sec$.

$$\tau = \frac{5 \times 10^4}{600\pi} sec.$$

Now, the energy of damped is

$$E = E_0 e^{-2rt} = E_0 e^{-t/\tau} \qquad [\because\ \tau = 1/2r]$$

Let the time-interval t' which the energy becomes$1/10$ of its initial value. Now substituting $\frac{E}{E_0} = \frac{1}{10}$ and $t' = t$ in the last expression, we obtain

$$\frac{1}{10}\frac{1}{10} = e^{-t\prime/\tau}$$

$$10 = e^{-t\prime/\tau}$$

$$\log_e 10 = \frac{t'}{\tau}$$

$$t' = \tau \log_e 10$$

$$= \frac{5 \times 10^4}{600 \times 3.14} \times 2.3 = 61 sec.$$

EXAMPLE3: A body of mass $0.2kg$ is hung from a spring of constant $80\,N/m$. The body is subjected to a resistive force given by bv, where v is the velocity in m/s. Calculate the value of the undamped frequency and the value of τ if the damped frequency is $\sqrt{3/2}$ of the undamped frequency.

SOLUTION: The undamped frequency of mass m suspended by a spring of force-constant k is given below

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Here $m = 0.2kg$ and $k = 80N/m$

$$n = \frac{1}{2\pi}\sqrt{\frac{80}{0.2}} = \frac{10}{\pi} = 3.18s^{-1}$$

The damped frequency n' is $n' = \frac{1}{2\pi}\sqrt{\omega^2 - r^2}$

Where $\omega = \sqrt{k/m}$ and $r = b/2m$.

$$n' = \frac{1}{2\pi}\sqrt{\frac{k}{m} - r^2}$$

But $n' = \frac{\sqrt{3}}{2}n$.

$$\frac{1}{2\pi}\sqrt{\frac{k}{m} - r^2} = \frac{\sqrt{3}}{2}\frac{10}{\pi}$$

$$\frac{k}{m} - r^2 = 300 \qquad \left[\because \frac{k}{m} = \frac{80}{0.2} = 400\right]$$

$\therefore$
$$400 - r^2 = 300 \Rightarrow r = 10$$

Therefore, the relaxation time is

$$\tau = \frac{1}{2r} = \frac{1}{20} = 0.05 sec.$$

EXAMPLE4: Q is a sonometer wire is 2×10^3. On plucking, it executes 240 vibrations per second. Calculate time in which the amplitude decreases to $1/e^2$ of the initial value.

SOLUTION: The quality factor is

$$Q = \omega\tau$$

Here $Q = 2 \times 10^3$ and $\omega = 2\pi n = 2 \times 3.14 \times 240 s^{-1}$

$$\tau = \frac{Q}{\omega} = \frac{2 \times 10^3}{2 \times 3.14 \times 240} = 1.327 s$$

Now

If a_0 be initial amplitude and $\frac{a_0}{e^2}$ the amplitude after time l then, we get

$$\frac{a_0}{e^2} = a_0 e^{-rt}$$

$$e^{-2} = e^{-rt} \Rightarrow 2 = rt \Rightarrow t = \frac{2}{r} = 4\tau \qquad [\tau = 1/2r]$$

$$= 4 \times 1.327 = 5.2 s$$

SELF CHECK QUESTIONS

1. Due to damping, the period of an oscillator slightly increases. (True/False)

2. The relation between quality factor Q and relaxation time τ of an oscillator is $q = \omega\tau$. (True/False)

8.10 *SUMMARY:-*

In this unit we studied the Integral Curves, ODE (in Local Charts): Existence, Uniqueness and Smoothness and Damped Oscillation with suitable example.

8.11 *GLOSSARY:-*

- **Integral Curve:** A curve in a vector field that represents the path traced by a particle moving according to the field's vector values at each point.
- **Damped Oscillation**: A type of oscillatory motion in which the amplitude of the oscillation gradually decreases over time due to the presence of damping forces or resistances.
- **Oscillation:** A repetitive and periodic motion around an equilibrium position.
- **Quality Factor (Q):** A measure of the sharpness of resonance in a damped system, calculated as the ratio of the natural frequency to the damping rate.
- **Underdamped, Overdamped, and Critically Damped**: Different classifications of damped systems based on the value of the damping ratio with respect to certain thresholds.

8.12 *REFERENCES:-*

- George F. Simmons(2017) 2nd edition Differential Equations with Applications and Historical Notes
- Morris Tenenbaum and Harry Pollard (1985) Ordinary Differential Equations.

8.13 *SUGGESTED READING:-*

- John R. Taylor (2004) Classical Mechanics.
- Shepely L.Ross (2007) 3rd edition Differential Equation.

8.14 *TERMINAL QUESTIONS:-*

(TQ-1) Find the equation of motion for damped harmonic oscillator and discuss the cases of under, over and critical Damping's.

(TQ-2) Discuss logarithmic decrement for a damped harmonic oscillator.
(TQ-3)A particle is oscillating under a damping force. Show that the power dissipation is $P = E/\tau$, where E is the average energy and τ the relaxation time. What happens to the dissipated energy?
(TQ-4) Define quality factor, mean life-time and relaxation time for a damped harmonic oscillator.
(TQ-5) Obtain a relation between quality factor and relaxation time.
(TQ-6) Explain the effect of damping on oscillatory motion.
(TQ-7) If the relaxation time of a damped harmonic oscillator is 50 second, find the time in which

- **i.** The amplitude falls to $1/e$ the initial value.
- **ii.** Energy of the system falls to $1/e$ times the initial value.
- **iii.** Energy falls to $1/e^4$ of the initial value.

(TQ-8)The oscillations of a tuning fork of frequency 200 cps die away to $1/e$ times their amplitude in I second. Show that the reduction in frequency due to air damping is exceedingly small.
(TQ-9)A damped vibrating system starting from rest has an initial amplitude of 20 cm which reduces to 2cm after 100 complete oscillations, each of period 2.3 second. Find the logarithmic decrement of a system.

8.15 *ANSWERS:-*

SELF CHECK ANSWERS

1. True
2. True

TERMINAL ANSWERS

(TQ-7) **i.** 100sec. **ii.** 50 sec. **iii.** 200 sec.
(TQ-9) 0.023

UNIT 9:- FUNDAMENTAL EXISTENCE THEOREM

CONTENTS:

9.1 *INTRODUCTION:-*

In the previous units you should have already studied

An Autonomous system:- A system of two first order differential equations of the form

$$\frac{dx}{dt} = f(x,y), \frac{dy}{dt} = g(x,y) \qquad ....(1)$$

is said to be autonomous, when the independent variable t does not appear explicitly.

That system (1) gives the slope of a path passing through a point $P(x,y)$ as

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \qquad ...(2)$$

If $f(x,y) = 0$ but $g(x,y) \neq 0$at P, we can take $\frac{dx}{dy} = \frac{f(x,y)}{g(x,y)}$instead of (2) and conclude from $\frac{dx}{dy} = 0$ that the tangent at P is vertical.

Phase Plane:- If $f\,(x,y)$ and $g\,(x,y)$ be continuously differentiable functions in some region R in the xy- plane, then xy – plane is called the phase plane of (1)

Critical Point:- A critical point of the system (1) is a point (x_0,y_0) at which both $f\,(x,y)$ and $g(x,y)$ are zero.The nature of the critical point (0,0) i.e., node, saddle points, spiral, centre of the system (1) is determined by the nature of the eigenvalues.

9.2 *OBJECTIVES:-*

After studying this unit you will be able to

- Describe the Fundamental Existence Theorem.
- Understand and explain the concept of stability.
- Investigate the stability of the trivial solution $x = 0, y = 0$ of an autonomous system.
- Understand the Liapunov function.

9.3 *FUNDAMENTAL EXISTENCE THEOREM:-*

For the first order differential equation

$$y' = f(x,y) \qquad \text{..... (1)}$$
$$y(x_0) = y_0$$

Consider the rectangular region T defined by $|x - x_0| \leq c\, and |y - y_0| \leq d$ in this region, centre is the point (x_0, y_0).Let the function f and $\frac{\delta f}{\delta y}$ is continuous at each point in T.Then there exists an interval,$|x - x_0| \leq h$ and a function $\phi(x)$that has the following properties:

(a) $y = \phi(x)$ is a solution of equation(1) in the interval$|x - x_0| \leq h$
(b) $\phi(x)$satisfies the inequality $|\phi(x) - y_0| \leq d$ in the interval $|x - x_0| \leq h$
(c) $\phi(x_0) = y_0$
(d) $\phi(x)$is unique in the interval $|x - x_0| \leq h$

PROOF: We shall prove this theorem by the method of successive approximations. Let us define a sequence of functions $y_0(x), y_1(x), y_2(x), \dots y_n(x), \dots$ as follows:

$y_0(x) = y_0$,$y_1(x) = y_0 + \int_{x_0}^{x} f\big(t, y_0(t)\big)dt$,

$$y_2(x) = y_0 + \int_{x_0}^{x} f\big(t, y_1(t)\big)dt\,, \qquad \text{.... (2)}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt \ ,$$

We shall divide the proof into next three sections. Since f is continuous in the rectangle T. It follows that f must be bounded in T. Let $M > 0$ be a number such that $|f(x,y)| \leq M$ for every point in T. We now take h to be the smaller of the two numbers c and $\frac{d}{M}$, and define the rectangle R to be the set of points (x,y) for which $|x - x_0| \leq h, |y - y_0| \leq d$.

LEMMA 1: If $|x - x_0| \leq h$ then

$$|y_n(x) - y_0| \leq d \quad \text{for n=1, 2, 3,..........}$$

The proof of this lemma will be accomplished by induction on n.

If $|x - x_0| \leq h$, Then

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f\big(t, y_0(t)\big)dt\right| \quad \text{.......by (2)}$$

$$\leq M \left|\int_{x_0}^{x} dt\right|$$

$$\leq M|x - x_0|$$

$$\leq Mh$$

$$\leq d$$

This proves the desired result for n=1. Now suppose that it is true for k i.e., for $|x - x_0| \leq h$,$|y_k(x) - y_0| \leq d$, it follows that the point $\big(x, y_k(x)\big)$ is in R so that $\left|f\big(x, y_k(x)\big)\right| \leq M$. Thus

$$|y_{k+1}(x) - y_0| = \left|\int_{x_0}^{x} f\big(t, y_k(t)\big)dt\right|$$

$$\leq M|x - x_0|$$

$$\leq Mh$$

$$\leq d$$

Which shows that $\big(x, y_{k+1}(x)\big)$ lies in R. Thus we can say if $|x - x_0| \leq h$, then the point $\big(x, y_n(x)\big), n = 1, 2, 3 \ldots\ldots\ldots$. are in R which is a slightly different way of saying Lemma1. The Lipschitz condition may now be used to deduce the following lemma.

LEMMA 2: If $|x - x_0| \leq h$ then

$$\left|f\big(x, y_n(x)\big) - f\big(x, y_{n-1}(x)\big)\right| \leq K|y_n(x) - y_{n-1}(x)|$$

for n=1, 2, 3.......

We are now in a position to give an inductive proof of still another lemma.

LEMMA 3: If $|x - x_0| \leq h$ then

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}|x - x_0|^n}{n!} \leq \frac{MK^{n-1}h^n}{n!} for n = 1,2,3 \ldots \quad \text{...(3)}$$

We have already verified (3) for $n = 1$ in Lemma 1 where we have shown that

$$|y_1(x) - y_0| \leq M|x - x_0|$$

Assuming that

$$|y_{n-1}(x) - y_{n-2}(x)| \leq \frac{MK^{n-2}|x-x_0|^{n-1}}{(n-1)!} \qquad \text{.... (4)}$$

We now show that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}|x - x_0|^n}{n!}$$

We will prove this for the case $x_0 \leq x \leq x_0 + h$.

$|y_n(x) - y_{n-1}(x)| = \left|\int_{x_0}^{x}[f(t, y_{n-1}(t)) - f(t, y_{n-2}(x))]dt\right|$ using (2)

$$\leq \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y_{n-2}(x))| \, |dt|$$

Using Lemma 2 we conclude that

$$|y_n(x) - y_{n-1}(x)| \leq K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)| |dt|$$

$\leq \frac{MK^{n-1}}{(n-1)!} \int_{x_0}^{x} |x - x_0|^{n-1} \, dt$using (4)

Or $|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}}{(n)!} |x - x_0|^n$ ...(5)

For the case $x_0 - h \leq x \leq x_0$, the same type of argument will yield the same result which completes the proof of the Lemma.

To utilize the results of Lemma 3 we now compare the two infinite series

$\sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)]and\sum_{n=1}^{\infty} \frac{MK^{n-1}h^n}{(n)!}$

The second of these series is an absolutely convergent series. Moreover, by Lemma 3 the second series dominates the first series. Hence by the Weierstrass's M test the series

$$\sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] \qquad \text{... (6)}$$

Converges absolutely and uniformly on the interval$|x - x_0| \leq h$. If we consider the k^{th} partial sum of the series (4)

$$\sum_{n=1}^{k}[y_n(x) - y_{n-1}(x)]$$

$$= [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] \ldots \ldots + [y_k(x) - y_{k-1}(x)],$$

We see that $\sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] = y_k(x)$.

That is the statement that the series (6) converges absolutely and uniformly is equivalent to the statement that the sequence $y_n(x)$ converges uniformly on the interval $|x - x_0| \leq h$.

If we now define$f(x) = \lim_{n\to\infty} y_n(x)$

And recall from the definition of the sequence $y_n(x)$ that each $y_n(x)$ is continuous on$|x - x_0| \leq h$, it follows (since the convergence is uniform) that $\emptyset(x)$ is also continuous and $\emptyset(x) = \lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt$

Because of the continuity of f and the uniform convergence of the sequence$y_n(x)$, we may interchange the order of the two limiting processes to show that $\emptyset(x)$ is a solution of the integral equation.

$$\emptyset(x) = y_n + \int_{x_0}^{x} f\big(t, \emptyset(t)\big)dt \qquad ....(7)$$

It follows immediately upon differentiation of equation (7) that $\emptyset(x)$is a solution of the differential equation $y' = f(x, y)$ on the interval$|x - x_0| \leq h$. Furthermore, it is clear from equation (7)$\phi(x_0) = y_0$.

Finally, since it have shown in Lemma 1 that $|y_n(x) - y_0| \leq d$for each n and for $|x - x_0| \leq h$, it follows that the same inequality must hold for $\emptyset(x) = \lim_{n\to\infty} y_n(x)$ that is if ,$|x - x_0| \leq h$ then $|\phi(x) - y_0| \leq d$.

This completes the proof of parts (a), (b) and (c) of the existence theorem.

9.4 *ALMOST LINEAR SYSTEM:-*

Consider the non –linear system of the form$\frac{dx}{dt} = a_1x + b_1y + f_1(x, y)$

$$\frac{dy}{dt} = a_2x + b_2y + f_2(x, y) \qquad ...(1)$$

The system (1) can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \qquad ...(2)$$

where$a_1, b_1, a_2, b_2,$ are constants.

By dropping the non linear terms$f_1(x, y)$and$f_2(x, y)$, the related linear system is

$$\frac{dx}{dt} = a_1x + b_1y$$

$$\frac{dy}{dt} = a_2x + b_2y \qquad ...(3)$$

For system (2) let us assume that

(a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

then the related linear system (3) has (0,0) as a critical point.

(b) $f_1(x, y)$and$f_2(x, y)$ are continuous and have continuous partial derivatives for all (x,y).

(c) $\lim_{(x,y)\to(0,0)} \frac{f_1(x,y)}{\sqrt{x^2+y^2}} = 0$

$$\lim_{(x,y)\to(0,0)} \frac{f_2(x,y)}{\sqrt{x^2+y^2}} = 0 \qquad \ldots(4)$$

Then (0,0) is said to be simple critical point of the system (2) and system (2) is called almost linear system.

9.5 *STABLE AND UNSTABLE CRITICAL POINT:-*

DEFINITION1-(Stable and Unstable): If $X(t) = (x(t), y(t))$, $X_0 = (x_0, y_0)$ and $X^* = (x^*, y^*)$ then the critical point (x^*, y^*) is said to be stable provided for given $\varepsilon > 0$ there exists a$\delta > 0$ such that $|(X(t) - X^*| < \varepsilon$whenever$|X_0 - X^*| < \delta, \forall\, t > 0$. The critical point$(x^*, y^*)$is said to be unstable if it is not stable.

DEFINITION 2-(Asymptotically stable): The critical point (x^*, y^*) is asymptotically stable if it is stable and every trajectory that begins sufficiently close to (x^*, y^*) also approaches (x^*, y^*)as $t \to \infty$i.e., for $\delta >$ 0, (x^*, y^*) also approaches (x^*, y^*)as $t \to \infty$i.e., for $\delta > 0$,

$$|(X\ (t) - X^*| < \varepsilon$$

$$\Longrightarrow \lim_{t\to\infty} X(t) = X^*$$

9.6 *STABILITY PROPERTIES OF CRITICAL POINT (0, 0):-*

Consider the linear autonomous system of the form

$$\frac{dx}{dt} = a_1x + b_1y, \qquad \frac{dy}{dt} = a_2x + b_2y \qquad \ldots(1)$$

we may assume that$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

Clearly the system (1) has origin (0,0) as a critical point.

System (1) can be written in matrix form as $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$

where the coefficient matrix $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$.

Then,eigen values of A are the roots of the characteristic equation

$$|A-\lambda I| = 0 \text{ or } \lambda^2 - (a_1 + b_2)\lambda + (a_1b_2 - a_2b_1) = 0 \;\ldots.(2)$$

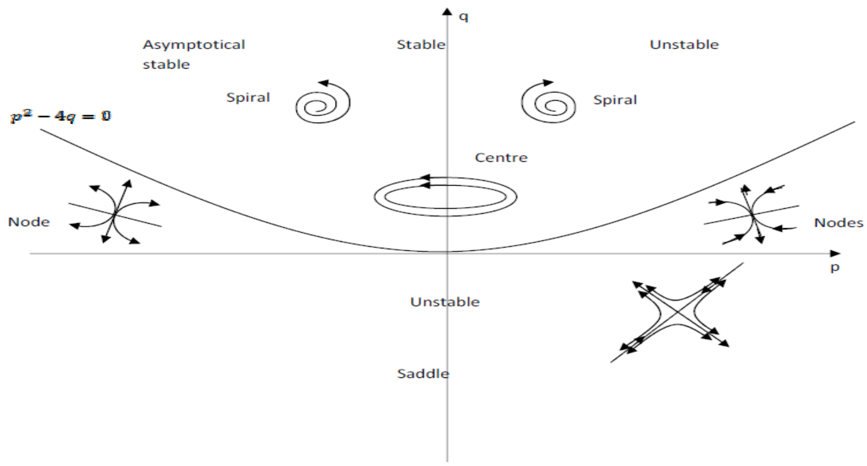


Fig.1

Now if $\lambda_1$and $\lambda_2$are the eigen values, then the equation (2) can be written in the form

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 + p\lambda + q = 0 \qquad \ldots(3)$$

where$p = -(\lambda_1 + \lambda_2) = -(a_1 + b_2)$ and $q = \lambda_1\lambda_2 = (a_1 b_2 - a_2 b_1)$

Equation (3) gives $\lambda = \frac{-p\pm\sqrt{p^2-4q}}{2}$

i.e.,$\lambda_1,\ \lambda_2 = \frac{-p\pm\sqrt{p^2-4q}}{2}$

Above the parabola$p^2 - 4q = 0$, we have$p^2 - 4q < 0$,so λ_1 and $\lambda_2$are conjugate complex numbers.If bothλ_1 and$\lambda_2$have non-negative real parts,then the critical point, which is a spiral point, is unstable and if both$\lambda_1$and$\lambda_2$have non positive real parts then the critical point, which is a spiral point,is asymptotically stable.Again if $p = 0$ then bothλ_1 and$\lambda_2$are pure imaginary. In this case the critical point, which is a centre, is stable.

Below the p–axis we have $q < 0$ which means that λ_1 and $\lambda_2$are real, distinct and have opposite signs. In this case the critical point, which is a saddle point, is unstable.

In the region between $p^2 - 4q \geq 0$ and $q > 0$, $\lambda_1$and λ_2 are real and of the same sign so in this case the critical point, which is a node, is asymptotically stable. On the basis of the theory given above, a stability criterion is given in table 1 and figure1, which at a glance shows the nature and stability properties of the critical point$(0, 0)$.

Table1

r_1, r_2	Linear system		Almost linear system	
	Type	Stability	Type	Stability

$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable	Node	Asymptotically stable
$\lambda_1 > \lambda_2 > 0$	Node	Unstable	Node	Unstable
$\lambda_1 > 0 > \lambda_2$	Saddle point	Unstable	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or S_{p^P}	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically stable	Node or S_{p^P}	Asymptotically stable
$\lambda_1, \lambda_2 = \lambda \pm i\mu$				
$\lambda > 0$	S_{p^P}	Unstable	S_{p^P}	Unstable
$\lambda < 0$	S_{p^P}	Asymptotically stable	S_{p^P}	Asymptotically stable
$\lambda = 0$	Centre	Stable	Centre or S_{p^P}	Indeterminate

S_{p^P} = Spiral Point

SOLVED EXAMPLES

EXAMPLE1: For the system of equations

$$\frac{dx}{dt} = 2x + y + xy^2, \frac{dy}{dt} = x - 2y - xy \quad ....(1)$$

Verify that $(0,0)$ is a critical point. Show that the system is almost linear and discuss the type and stability of the critical point (0,0).

SOLUTION: For critical points we must have

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0$$

This gives $\quad 2x + y + xy^2 = 0$

and $\quad x - 2y - xy = 0$

Solving these equations we obtain $x = 0$ and $y = 0$, Thus (0,0) is a critical point.

The given system can be written as $\frac{dx}{dt} = 2x + y + f(x, y)$

$$\frac{dy}{dt} = x - 2y + g(x, y)$$

where $f(x, y) = xy^2$ and $g(x, y) = -xy$.

Using the polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$,

We get

$$\frac{f(x,y)}{r} = \frac{r^3 \cos\theta \sin^2\theta}{r} = r^2 \cos\theta \sin^2\theta$$

which tends to 0 as r tends to 0.

Similarly, $\frac{g(x,y)}{r} = -\frac{r^2 \cos\theta \sin\theta}{r} = -r\cos\theta\sin\theta$

which again tends to 0 as r tends 0.

Therefore, system (1) is almost linear.

Also, the related linear system in the neighbourhood of (0,0) is

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = x - 2y$$

Its matrix form is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad ..(2)$$

Theeigen values of (2) are the roots of the equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda^2 - 5 = 0 \Rightarrow \lambda = \pm\sqrt{5}$$

Therefore $\lambda_1 = +\sqrt{5}$ and $\lambda_2 = -\sqrt{5}$

The eigen values are real, distinct and of opposite sign. Therefore the critical point (0,0) is an unstable saddle point of the system (2).

EXAMPLE 2: For the set of non–linear differential Equations

$$\frac{dx}{dt} = x - xy, \frac{dy}{dt} = xy - y \qquad ...(1)$$

(i) Show that the point (0,0) and (1,1) are equilibrium points of the above system.

(ii) Show that the point (0,0) is a saddle point and (1,1) is a centre of above system.

SOLUTION :(i) For the equilibrium points, we have

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0$$

This gives$x - xy=0 \Rightarrow x\,(1-y) = 0 \Rightarrow x = 0\,, y = 1$

$$-y + xy = 0 \Rightarrow y\,(x-1) = 0 \Rightarrow y = 0\,, x = 1$$

Hence $(0,0)$ and $(1,1)$ are the equilibrium points or critical points.

(ii) In the neighborhood of (0,0), the above given system reduces to related linear system

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = -y$$

This system can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad ...(2)$$

Eigen values of (2) are the roots of the equation

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (1-\lambda)(-1-\lambda) = 0$$
$$\Rightarrow \lambda = 1, \quad -1$$

Therefore $\lambda_1=1$, $\lambda_2=-1$ are the eigenvalues,both are real, distinct and of opposite sign.Therefore$(0,0)$ is a saddle point.

In the neighbored of (1,1),the given system can be reduced to the new system by putting

$$x = u + 1, \qquad y = v + 1$$

$$\frac{dx}{dt} = \frac{du}{dt}, \frac{dy}{dt} = \frac{dv}{dt}$$

Putting these values in (1), we find

$$\frac{du}{dt} = (u + 1)(-v) = -v - uv$$

$$\frac{dv}{dt} = u(v + 1) = uv + u$$

The auxiliary equation of the associated linear system

$$\frac{du}{dt} = -v$$

$$\frac{dv}{dt} = u$$

Its matrix form is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \ldots(3)$$

The Eigen values of (3) are the roots of the equation

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$\Rightarrow \quad \lambda = 0 \ \pm i$

i. e.,$\lambda_1 = i,\ \lambda_2 = -i$

These are of the form $\lambda + i\mu$ with $\lambda = 0$.Thus $u(t)$ and $v(t)$ oscillate with constant amplitudes as t increases in closed curve surrounding the equilibrium point (1,1) and hence (1,1) is the centre.

EXAMPLE 3: Find the critical points of the System

$$\frac{dx}{dt} = y^2 - 5x + 6$$

$$\frac{dy}{dt} = x - y \qquad \ldots(1)$$

SOLUTION: For Critical Points, we must have

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0$$

This gives $y^2 - 5x + 6 = 0$ and $x - y = 0$

Solving these we get $y^2 - 5y + 6 = 0 \Rightarrow y = 3, 2$

Since $x = y$ so (3,3) and(2,2) are the critical points of the system (1).

EXAMPLE 4: Find the critical point of the system

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -x + 2y$$

Discuss the type and stability of the critical point and find the general solution of the system.

SOLUTION: For critical Points we must have

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0$$

$$\Rightarrow x = 0 \text{ and} -x + 2y = 0$$

On solving these equations we obtain $x = 0$ and $y = 0$ thus $(0,0)$ is a critical point.

The given system can be written in the matrix from as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of given system are the characteristic roots of the equation

$$\begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 1, 2.$

Since eigenvalues are real distinct and of the same sign, the critical point is a node. Also, since $\lambda_1 > 0, \lambda_2 > 0$, it is unstable.

To find the general solution of given system, we find the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1, \lambda_2 = 2$.

Eigen vector corresponding to the eigenvalue $\lambda_1 = 1$ is given by

$$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \lambda_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -x + y = 0$

$\Rightarrow \binom{1}{1}$ is one possible eigen vector.

Eigen vector corresponding to the eigen value $\lambda_2 = 2$ is given by

$$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \lambda_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -x = 0$

$\Rightarrow \binom{0}{1}$ is the other possible eigenvector.

Then the general solution of given system can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t}$$

$$\Rightarrow x = c_1 e^{t}$$

$$y = c_1 e^t + c_2 e^{2t}$$

where c_1 and c_2 are arbitrary constants.

9.7 *LIAPUNOV'S FUNCTION:-*

Let $V(x) = V(x_1, x_2, \ldots \ldots \ldots . . x_n)$ be a function of class C in an open region H containing the origin. Suppose $V(0) = 0$ and that V is positive at all other points of H. Then V has a minimum at the origin and we say that V is positive definite in H. Obviously the origin is a critical point of V i.e. a point at which all the partial derivatives $\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots \ldots, \frac{\partial V}{\partial x_n}$ vanish. The origin will said to be an isolated critical point, if there is a circular disk about the origin, such that the origin is the only critical point of V inside the circular disk. The derivative V along trajectories of an autonomous system is defined by the equation

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x_1} X_1(x) + \frac{\partial V}{\partial x_2} X_2(x) + \cdots \frac{\partial V}{\partial x_n} X_n(x).$$

If $V(x)$ is positive definite in H and if $\dot{V} \leq 0$ throughout H then V(x) is said to be a Liapunov`s function for the equilibrium point at the origin of the system $\frac{dx}{dt} = X(x)$.

9.8 *THEOREMS ON STABILITY AND UNSTABILITY BY LIAPUNOV'S FUNCTION:-*

THEOREM 1:(Liapunov's Stability Theorem): If for a system of differential equation $\frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots x_n\right)$, i =1,2,…......,n there exists a Liapunov function $V\left(x_1, x_2, \ldots x_n\right)$ of fixed sign whose total derivative $\frac{dV}{dt}$ with respect to time composed by virtue of above system is a function of constant signs, of sign opposite to that of V, or identically equal to zero, then the stationary point $x_i = 0$, $i = 1, 2, \ldots \ldots \ldots . n$ of the above system is stable.

THEOREM2 :(Liapunov`sAsymptotic–Stability Theorem): If for a system of differential equations $\frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots x_n\right)$, í = 1, 2, …….., n there exists a function of fixed sign $V\left(x_1, x_2, \ldots x_n\right)$ (a Liapunov function) whose total derivative $\frac{dV}{dt}$ with respect to time composed by virtue of above system is a function of constant signs ,of sign opposite to that of

V ,then the stationary point$x_i = 0$, $i = 1, 2, \ldots \ldots \ldots n$of the above system is asymptotically stable.

THEOREM3 :(Liapunov`s Asymptotic–Stability Theorem):

If for a system of differential equations $\frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots x_n\right)$, i = 1, 2,……., n there exists a function $V\left(x_1, x_2, \ldots x_n\right)$ differentiable in the neighbourhood of the origin of coordinates such that $V\left(0, 0, \ldots, 0\right) = 0$. If the total derivative$\frac{dV}{dt}$ composed by virtue of above system is a positive definite function and arbitrarily close to the origin of coordinates there are points in which the function $V\left(x_1, x_2, \ldots x_n\right)$, takes positive values, then the stationary point$x_i = 0$, $i = 1, 2, \ldots \ldots \ldots n$of the above system is unstable.

NOTE: There is no general method for constructing Liapunov functions. Simply a Liapunov function may be sought in the form

$$V(x, y) = ax^2 + by^2$$
$$V(x, y) = ax^4 + by^2$$
$$V(x, y) = ax^4 + by^4, \ (a > 0, b > 0).$$

SOLVED EXAMPLES

EXAMPLE1: Using a Liapunov function investigate for stability, the trivial solution $x = 0, y = 0$ of the system

$$\frac{dx}{dt} = y - x^3, \frac{dy}{dt} = -x - 3y^3$$

SOLUTION: We choose $x^2 + y^2$ as the function $V(x, y)$. It is positive definite. The derivative of the function$V$ is$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$

$$\frac{dV}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$= 2x(y - x^3) + 2y(-x - 3y^3)$$
$$= -2x^4 - 6y^4$$
$$= -2\,(x^4 + 3y^4)$$

Thus$\frac{dV}{dt}$ is negative definite function. It follows by Liapunov's asymptotic–stability theorem that the stationary point (0,0) of the given system is asymptotically–stable.

EXAMPLE2: Using a Liapunov function investigate for stability, the trivial solution $x = 0, y = 0$ of the system

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x$$

SOLUTION: Let us take $x^2 + y^2$ as the function $V(x, y)$. It is positive definite. The derivative of the function V is $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$

$$\frac{dV}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$= 2x(y) + 2y(-x)$$
$$= 2xy - 2yx$$
$$= 0$$

It follows by Liapunov's stability theorem that the stationary point (0,0) of the given system is stable. It is not asymptotically–stable.

EXAMPLE3: Investigate the stationary point $x = 0, y = 0$ of the system

$\frac{dx}{dt} = x, \frac{dy}{dt} = -y$ for stability.

SOLUTION: Let us consider the function $V(x, y) = x^2 - y^2$.

The derivative of the function V is $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$

$$\frac{dV}{dt} = 2x\frac{dx}{dt} - 2y\frac{dy}{dt}$$
$$= 2x(x) - 2y(-y)$$
$$= 2x^2 + 2y^2$$

$\frac{dV}{dt}$ is a positive definite function. Since arbitrarily close to the origin of coordinate there are points in which $V > 0$. It follows by Liapunov's in stability theorem that the stationary point (0,0) is unstable.

SELF CHECK QUESTIONS

Choose the Correct Option:

(SCQ-1) The nature and stability of the critical point (0,0) of the linear system of

$\frac{dx}{dt} = -3x + 4y,\ \ \frac{dy}{dt} = -2x + 3y$ is

(a) Centre , stable
(b) Centre , unstable
(c) Node , unstable
(d) None of these

(SCQ-2) The type and stability of the critical point (0 ,0) of the system

$\frac{dx}{dt} = 3x + 5y,\ \ \frac{dy}{dt} = -5x - 3y$

(a) Node, unstable
(b) Centre, stable
(c) Centre, unstable
(d) Saddle point, stable

(SCQ -3) For the system $\dot{x} = -x + hy,\ \dot{y} = x - y$

(i) The nature and stability of the critical point(0 ,0) if h=0 is

(a) Node, unstable
(b) Centre, stable
(c) Node, asymptotically stable
(d) None of these

(ii) For the same system the nature and stability of the critical point (0,0) if h<0 is

(a) Spiral point ,unstable
(b) Spiral point , asymptotically stable
(c) Saddle point ,unstable
(d) Node , asymptotically stable

(iii) For the same system the nature and stability of the critical point (0 ,0), if 0<h <1is

(a) Node, asymptotically stable
(b) Saddle point, stable
(c) Node, unstable
(d) None of these

(SCQ-4) The type and Stability of the Critical point (0,0) of the system

$$\dot{x} = y + x(1 - x^2 - y^2)$$
$$\dot{y} = -x + y(1 - x^2 - y^2) \text{ is}$$

(a) Spiral point ,unstable
(b) Node, unstable
(c) Saddle point, unstable
(d) None of these

(SCQ-5) The type and nature of critical point (0, 0) of $\dot{x} = x,\ \dot{y} = 2y$ is

(a) Node, unstable
(b) Saddle point, stable
(c) Spiral point, stable
(d) None of these

(SCQ-6) The nature of critical point (0,0) of $\dot{x} = x + 3y,\ \dot{y} = 3x + y$ is

(a) Unstable

(b) Stable
(c) None of these

(SCQ-7) The stability of the trivial solution $x = 0, y = 0$ of the system$\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -y$ using $x^2 + y^2$ as aLiapunov function is

(a) Unstable
(b) Asymptotically stable
(c) None of these

(SCQ-8) The stability of the trivial solution $x = 0, y = 0$ of the system$\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$ using $x^2 + y^2$ as aLiapunov function is

(a) Stable
(b) Asymptotically stable
(c) None of these

(SCQ-9) The stability of the trivial solution $x = 0, y = 0$ of the system $\frac{dx}{dt} = -y - \frac{x}{2} - \frac{x^3}{4}$, $\frac{dy}{dt} = x - \frac{y}{2} - \frac{y^3}{4}$ using $x^2 + y^2$ as aLiapunov function is

(a) Unstable
(b) Asymptotically stable
(c) None of these

9.9 *SUMMARY:-*

In this unit, first of all you are explained the fundamental existence theorem. Then linear system and almost linear system has been explained. After that critical points and their stability for these systems has been discussed. That is a discussion on how the critical points are checked for their stability by finding the Eigen values for the system has been done. Another method for checking whether the system is stable, asymptotically stable or instable by using Liapunov.

9.10 *GLOSSARY:-*

- Autonomous System
- Sable Critical and Unstable Critical points.
- Liapunov Functions.

9.11 *REFERENCES:-*

- E.L.Ince (2012) Ordinary Differential Equation.
- Shepley L.Ross (2007) Differential Equations 3rd edition.
- M.D.Raisinghania (2013) Ordinary and Partial Differential Equation18th edition.

9.12 *SUGGESTED READING:-*

- Suman Kumar Tumuluri (2021) A first course in Ordinary Differential Equations.
- Stanley J. Farlow (2012) An introduction to Differential Equations and their application.

.

9.13*TERMINAL QUESTIONS:-*

(TQ-1) Write a note on the stability of critical points of the plane autonomous system

$$\frac{dx}{dt} = F(x,y), \frac{dy}{dt} = G(x,y)$$

(TQ-2) Determine the type and stability of the critical point (0, 0) of the almost linear system

$$\dot{x} = 4x + 2y + 2x^2 - 3y^2, \qquad \dot{y} = 4x - 3y + 7xy.$$

Also, find the general solution of the corresponding linear System.

(TQ-3) For the system of equations

$\dot{x} = x - y + xy$, $\dot{y} = 3x - 2y - xy$, verify that (0,0)is a critical point. Show that the system is almost linear and discuss the type and stability of the critical point (0,0).

(TQ-4) Define Liapunov function.

(TQ-5) Write a note on Liapunov's theorem on stability.

(TQ-6) Investigate the trivial solution $x = 0, y = 0$ of the system below for stability

(a) $\frac{dx}{dt} = -xy^4 \frac{dy}{dt} = yx^4$

Hint:$V(x,y) = x^4 + y^4$

(b) $\frac{dx}{dt} = y + x^3 \frac{dy}{dt} = -x + y^3$

Hint:$V(x,y) = x^2 + y^2$

9.14 *ANSWERS:-*

SELF CHECK ANSWERS

(SCQ-1) a	(SCQ-2) b	(SCQ-3(i)) c
(SCQ-3(ii)) b	(SCQ-3(iii)) a	(SCQ-4) a
(SCQ-5) a	(SCQ-6) a	(SCQ-7) b
(SCQ-8) a	(SCQ-9) b	

TERMINAL ANSWERS

(TQ-2) $\lambda_1 = -4$, $\lambda_2 = 5$ are real , unequal and have opposite sign. Critical point (0,0) is an unstable saddle point. The general solution is

$$x = c_1 e^{-4t} + 2c_2 e^{5t}$$
$$y = -4c_1 e^{-4t} + c_2 e^{5t}$$

(TQ-3) Spiral Point, asymptotically stable.

(TQ-(6a)) Stable

(TQ-(6b)) Unstable

UNIT 10:- DIFFERENTIAL EQUATION WITH PERIODIC SOLUTION

CONTENTS:

10.1 *INTRODUCTION:-*

In the previous classes, you have already studied

- About an autonomous system
- About phase plane
- About critical points

10.2 *OBJECTIVES:-*

After studying this unit, you will be able

- To define and explain the periodic solution
- To understand the Poincare-Bendixson theorem
- To understand the Lienard's theorem

10.3 *PERIODIC SOLUTIONS:-*

Let us consider a nonlinear autonomous system

$$\frac{dx}{dt} = F(x,y) \qquad \ldots(1)$$
$$\frac{dy}{dt} = G(x,y)$$

where the functions $F(x,y)$ and $G(x,y)$ are continuous and have continuous first order partial derivatives throughout the phase plane. So far we have studied practically nothing about paths of (1) except in the neighborhood of certain types of critical points. However, sometimes it looks more interesting to know the global properties of paths in comparison to local properties. The properties that describe their behaviour over large regions of the phase plane are known as the global properties of paths.They are in general very difficult to establish.

Now, the problem is to determine whether (1) has closed paths. This problem has very close connection with the issue of whether (1)has periodic solutions. A solution $x(t)$ and $y(t)$ of (1) is said to be periodic if neither function is constant, if both are defined for all t, and if there exists a number $T > 0$ such that $x(t+T) = x(t)$ and $y(t+T) = y(t)$ for all t. The smallest T with this property is called the period of the solution.

Evidently each periodic solution of (1) defines a closed path that is traversed once as t increases from t_0 to $t_0 + t$ for any t_0. Conversely, if $C = [x(t), y(t)]$ is a closed path of (1), then $x(t), y(t)$ is a periodic solution. So the problem of searching for periodic solutions of (1) reduces to a problem of searching for closed paths. A nonlinear system can perfectly have a closed path that is isolated, in the sense that no other closed paths are near to it.

SOLVED EXAMPLE

EXAMPLE 1: Show that the system

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2) \quad .... (1)$$

$$\frac{dy}{dt} = x + y(1 - x^2 - y^2) \quad ....(2)$$

has a periodic solution.

SOLUTION: Using the polar coordinates r and θ as $x = r\cos\theta$, $y = r\sin\theta$, we have

$$x^2 + y^2 = r^2 \quad .... (3)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad .... (4)$$

Differentiating (3) and (4), we get

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \quad ... (5)$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt} \quad .... (6)$$

On multiplying (1) by x and (2) by y and adding we get

$$r\frac{dr}{dt} = r^2(1 - r^2) \quad .... (7)$$

On multiplying (1) by y and (2) by x and subtracting we get

$$r^2\frac{d\theta}{dt} = r^2 \quad .... (8)$$

The given system has a single critical point at $r = 0$. For finding the paths let us consider $r > 0$.

From (7) and (8) we have

$$\frac{dr}{dt} = r(1 - r^2) \quad .... (9)$$

$$\frac{d\theta}{dt} = 1 \quad .... (10)$$

Integrating (9)

$$\int \frac{dr}{r(1-r^2)} = \int dt$$

$$\int \left[\frac{1}{r} + \frac{1}{2(1-r)} - \frac{1}{2(1+r)}\right] dr = \int dt$$

$\Rightarrow \log\left(\frac{1-r^2}{r^2 c}\right) = -2t$

$\Rightarrow \quad \frac{1-r^2}{r^2 c} = e^{-2t}$

$\Rightarrow \quad r^2(1+ce^{-2t}) = 1$

$\Rightarrow \quad r = \frac{1}{\sqrt{1+ce^{-2t}}}$ .... (11)

Integrating (10) we get

$$\theta = t + t_0 \qquad .... (12)$$

The corresponding general solution of given system is

$$x = \frac{\cos(t+t_0)}{\sqrt{1+ce^{-2t}}}$$

$$y = \frac{\sin(t+t_0)}{\sqrt{1+ce^{-2t}}}$$

Analyzing (11) and (12) geometrically, we find that if $c = 0$, then $r = 1$ and $\theta = t + t_0$ which trace out the closed circular path $x^2 + y^2 = 1$ in the counter-clockwise direction. If $c < 0$ it is clear that $r > 1$ and that$r \to 1$as $t \to \infty$.

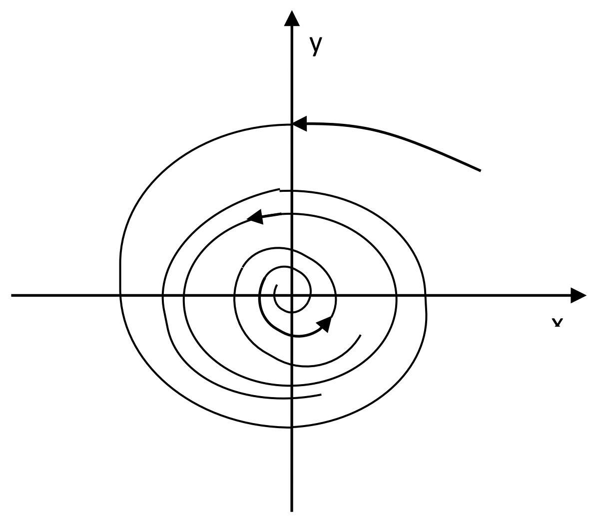


Fig.1

Also if $c > 0$ we see that$r < 1$, and again $r \to 1$ as $t \to \infty$. These observations show that there exists a single closed path $(r = 1)$which all other paths approach spirally from the outside or inside as $t \to \infty$. This proves that the given system has a closed path(periodic solution).

Note: In the given system a closed path is shown by actually finding such a path. Now here is a test based theorem that can make possible to conclude that certain regions of the phase plane do or do not contain closed paths.

THEOREM: A closed path of the system

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y) \qquad \text{.... (1)}$$

necessarily surrounds at least one critical point of this system.

PROOF: Let C be a closed curve in the phase plane, and assume that C does not pass through any critical point of the system (1). If $P = (x,y)$is a point on C, then

$V(x,y) = F(x,y)i + G(x,y)j$is a nonzero vector and therefore has a definite direction given by the angle θ.

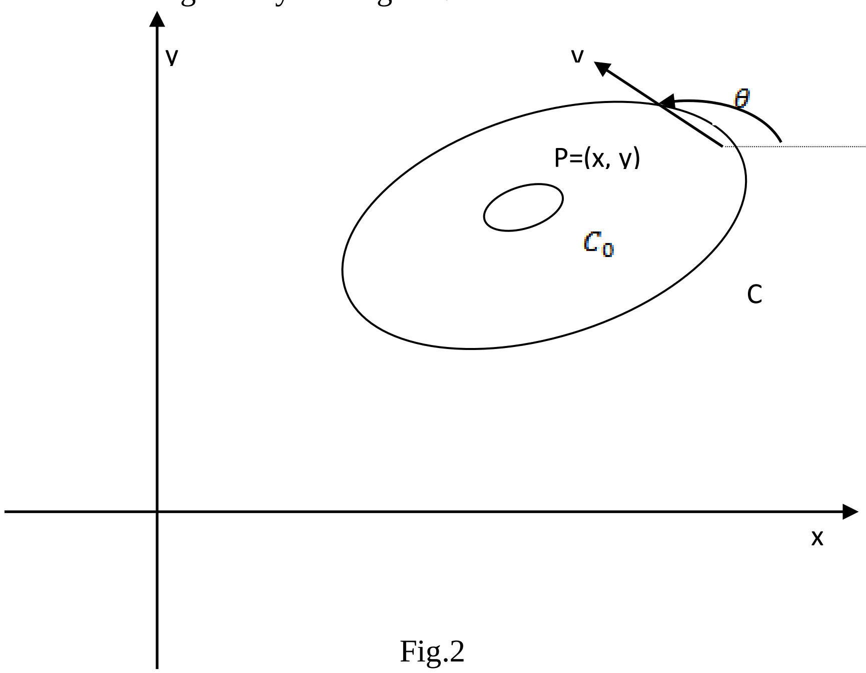


Fig.2

If P moves once around C in the counter-clockwise direction, the angle θ changes by an amount$\Delta\theta = 2\pi n$, where n is positive integer, zero or a negative integer .This integer n is called the index of C. If Cshrinks continuously to a smaller simple closed curve C_0 without passing over any critical point then its index varies continuously and since the index is an integer, it cannot change.

(a) If C is a path of (1), show that its index is 1.
(b) If C is a path of (1) that contains no critical points, show that a small C_0 has index 0, and from this inference theorem 1 is complete.

THEOREM: If $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$ is always positive or always negative in a certain region of the phase plane, then the system

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

can not have closed paths in that region.

PROOF: Let us assume that the region contains a closed path $\mathrm{C} = [\mathrm{x}(t), \mathrm{y}(t)]$ with interior R. Then Green's theorem and our hypothesis give $\int(F\,dy - G\,dx) = \iint\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right)dx\,dy \neq 0$.

However, along C we have $dx = F\,dt$ and $dy = Gdt$, so

$$\int(F\,dy - G\,dx) = \int_0^T (FG - GF)\,dt = 0$$

This contradiction shows that our initial assumption is false, so the region under consideration can not contain any closed path.

10.4 *POINCARE-BENDIXSON THEOREM:*

Let R be a bounded region of the phase plane together with its boundary and assume that R does not contain any critical points of the system

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

If $\mathrm{C} = [\mathrm{x}(t), \mathrm{y}(t)]$ is a path of (1) that lies in R for some t_0 and remains in R for all $t \geq t_0$, then $\mathbf{C}$ is either itself a closed path or it spirals toward a closed path as $t \to \infty$. Thus in either case the system (1) has a closed path in R.

ILLUSTRATION WITH THE HELP OF AN EXAMPLE:-

Because a closed path like C_0 must surround a critical point P and R must exclude all critical points.

The system $$\frac{dx}{dt} = -y + x(1 - x^2 - y^2)$$
$$\frac{dy}{dt} = x + y(1 - x^2 - y^2)$$

provides a simple application of these ideas. It is clear that (2) has a critical point at $(0,0)$, and also that the region R between the circles $r = \frac{1}{2}$ and $r = 2$ contains no critical points.Taking $x = r\cos\theta$, $y = r\sin\theta$, we find that $\frac{dr}{dt} = r(1 - r^2)$ for $r > 0$. This shows that $\frac{dr}{dt} > 0$on the inner circle and $\frac{dr}{dt} < 0$ on the outer circle, so the vector $V(x, y) = F(x, y)i + G(x, y)j$ points into R at all boundary points.

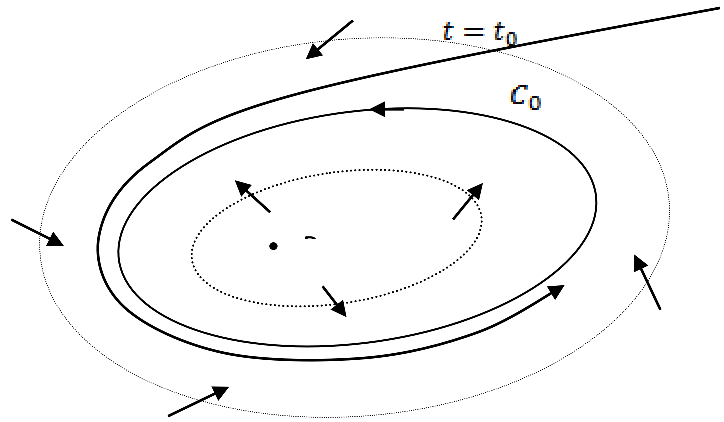


Fig.3

Thus any path through a boundary point will enter R and remain in R as $t \to \infty$, and by the Poincare- Bendixson theorem we know that R contains a closed path C_0 and we have already seen that the circle $r = 1$ is the closed path.

When we speak about the existence of closed paths for equations of the form $\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$,...(1)which is called Lienard's equation ,we of course mean a closed path of the equivalent system $\frac{dx}{dt} = y, \frac{dy}{dt} = -g(x) - f(x)y$;

A more practical criterion has been developed for the system of the form (1) in the form of the Lienard's theorem which is as follows:-

10.5 *LIENARD'S THEOREM:-*

Let the function $f(x)$ and $g(x)$satisfy the following conditions:

(i) Both are continuous and have continuous derivatives for all x;

(ii) $g(x)$ is an odd function such that $g(x) > 0$ for $(x) > 0$, and $f(x)$ is an even function

(iii) The odd function $F(x) = \int_0^x f(x)\,dx$ has exactly one positive zero at $x = a$, and $F(x) \to \infty$ as $x \to \infty$.

Then equation (1) has a unique closed path surrounding the origin i the phase plane, and this path is approached spirally by every other path as $t \to \infty$.

SOLVED EXAMPLE

EXAMPLE:-Show that the differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

has a periodic solution,μ is assumed to be a positive constant.

SOLUTION:- Here $f(x) = \mu(x^2 - 1)$

$$g(x) = x$$

Obviously condition (1) and (2) are satisfied. Now since

$$F(x) = \int_0^x f(x)\,dx = \int_0^x \mu(x^2 - 1)\,dx$$

$$F(x) = \frac{\mu x^3}{3} - \mu x$$

$= \frac{1}{3}\mu x(x^2 - 3)$,

We see that $F(x)$ has a single positive zero at $x = \sqrt{3}$

is a negative for $0 < x < \sqrt{3}$

is positive for $x > \sqrt{3}$

$F(x)$tends to ∞ as $x \to \infty$.

And $F'(x) = \mu(x^2 - 1)$ is positive for $x > 1$, so $F(x)$ is certainly non-decreasing (in-fact increasing) for $x > \sqrt{3}$. Since all the conditions of the theorem are true, we can conclude that equation $\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$ has a unique closed path (periodic solution) that is approached spirally (asymptotically) by every other path(nontrivial solution).

SELF CHECK QUESTIONS

(SCQ-1) Explain the periodic solutions.

(SCQ-2) Write the statements of Poincare-Bendixson theorem and Lienard theorem.

(SCQ-3) Transform the system

$$\frac{dx}{dt} = 4x + 4y - x(x^2 + y^2)$$

$\frac{dy}{dt} = -4x + 4y - y(x^2 + y^2)$ into polar co-ordinate form.

10.6 *SUMMARY:-*

In this unit, you have learnt about the periodic solutions. Poincare Bendixson's theorem and Lienard's theorem are also explained in this unit. Now you are able to check whether the given system has a periodic solution or not with the help of Poincare Bendixson's theorem and Lienard's theorem.

10.7 *GLOSSARY:-*

- Period solution
- Path
- Bounded region
- Critical point

10.8 *REFERENCES:-*

- Vladimir I. Arnold (1992) Ordinary Differential Equation.
- Dominic Jordan and Peter Smith(2007)Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers .

10.9 *SUGGESTED READING:-*

- Gerald Teschl (2012) Ordinary Differential Equations and Dynamical Systems.
- M.D.Raisinghania(2021)Ordinary and Partial Differential equation (20th Edition), S. Chand.
- Lawrence Perko (2001) Differential Equation and Dynamical System.

10.10 *TERMINAL QUESTIONS:-*

(TQ-1) In each of the following questions, determine whether or not given differential equation has a periodic solution

(a) $\frac{d^2x}{dt^2} - (x^2+1)\frac{dx}{dt} + x^5 = 0$

Hint: Use Poincare-Bendixson theorem.

(b)

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \left(\frac{dx}{dt}\right)^5 - 3x^3 = 0$$

Hint: Use Poincare-Bendixson theorem.

(c)

$$\frac{d^2x}{dt^2} + x^6\frac{dx}{dt} - x^2\frac{dx}{dt} + x = 0$$

Hint: Use Lienard's theorem.

(TQ-2) Show that the differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

has a periodic solution,μ is assumed to be a positive constant.

10.11 *ANSWERS:-*

SELF CHECK ANSWERS

(SCQ-3) $\frac{dr}{dt} = r(1 - r^2), \frac{d\theta}{dt} = -4.$

TERMINAL ANSWERS

(TQ-1(a)) No periodic solution
(TQ-1(b)) No periodic solution
(TQ-1(c)) A periodic solution

UNIT 11:- METHOD OF BOGOLIUBOV AND KRYLOV

CONTENTS:

11.1 *INTRODUCTION:-*

In the previous classes, you should have studied and learnt

- To solve the linear differential equation with constant coefficients.
- The methods of finding integration and differentiation.
- About the periodic functions.

In this unit, a method of treating weakly nonlinear differential equations of the form

$$\frac{d^2y}{dx^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right) \qquad 0 < \epsilon << 1$$

originally developed by Krylov and Bogoliobov is being discussed. In 13.3 we will show how to determine the solution to first order of approximation using the method of Krylov and Bogoliobov.

11.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- Find the solution of a non linear equation by Bogoliubov and Krylovmethod.
- Understand the Bogoliubov and Krylov method.

11.3 *FIRST APPROXIMATION OF KRYLOV AND BOGOLIOBOV:-*

Let us consider a nonlinear differential equation having the form

$$\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right) \qquad 0 < \epsilon << 1 \qquad \ldots(1)$$

If $\epsilon = 0$, then equation (1) reduces to the linear equation

$$\frac{d^2y}{dt^2} + y = 0 \qquad \ldots(2)$$

Auxiliary equation $(D^2 + 1) = 0$ gives the roots

$$D = \pm i,$$

so the solution of equation (2) may be written as

$$y = a\cos(t + \phi) \qquad \ldots.(3)$$

where a and ϕ are constants.

The derivative of the solution given by equation (3) is

$$\frac{dy}{dt} = -a\sin(t + \phi) \qquad \ldots(4)$$

If $\epsilon \neq 0$, but is sufficiently small, one can assume that the nonlinear equation (1) also has a solution of the form of equation (3) with derivative of the form of equation (4), provided that a and ϕ are functions of t rather than being constants.

That is, we assume a solution of equation (1) of the form

$$y = a(t)\cos[t + \phi(t)] \qquad ...(5)$$

where a and ϕ are functions of t to be determined such that the derivative of the solution (5) is of the form

$$\frac{dy}{dt} = -a(t)\sin[t + \phi(t)] \qquad ...(6)$$

Differentiating this assumed solution (5), we obtain

$$\frac{dy}{dt} = \frac{da}{dt}\cos[t + \phi(t)] - a\sin[t + \phi(t)]\left(1 + \frac{d\phi}{dt}\right)$$

$$\frac{dy}{dt} = \frac{da}{dt}\cos[t + \phi(t)] - a\sin[t + \phi(t)] - a\frac{d\phi}{dt}\sin[t + \phi(t)] \ldots(7)$$

In order for $\frac{dy}{dt}$ to have the form given by equation (6), we must require

$$\frac{da}{dt}\cos[t + \phi(t)] - a\frac{d\phi}{dt}\sin[t + \phi(t)] = 0 \qquad ...(8)$$

Differentiating the assumed derivative, equation(6), we obtain

$$\frac{d^2y}{dt^2} = -\frac{da}{dt}\sin(t + \phi) - a\cos(t + \phi) - a\frac{d\phi}{dt}\cos(t + \phi) \ldots(9)$$

Substituting the assumed solution, its derivative and the second derivative from equation (5), (6), and (9) in to the differential equation (1), we obtain

$$-\frac{da}{dt}\sin(t+\phi) - a\cos(t+\phi) - a\frac{d\phi}{dt}\cos(t+\phi) + a\cos(t+\phi)$$
$$= \epsilon F\,[a\cos(t+\phi), -a\sin(t+\phi)] \qquad ...(10)$$

Or

$$\frac{da}{dt}\sin(t+\phi) + a\frac{d\phi}{dt}\cos(t+\phi) = -\epsilon F[a\cos(t+\phi), -a\sin(t+\phi)] \ \ldots(11)$$

If we let $\psi(t)$ denote$t + \phi(t)$, then equation (8) and (11) may be written

$$\frac{da}{dt}\cos\psi(t) - a\,\frac{d\phi}{dt}\sin\psi(t) = 0 \qquad \ldots(12a)$$
$$\frac{da}{dt}\sin\psi(t) + a\frac{d\phi}{dt}\cos\psi(t) = -\epsilon\, F\,(a\cos\psi, -a\sin\psi) \ ....(12b)$$

Solving equations (12a) and (12b)for $\frac{da}{dt}$and $\frac{d\phi}{dt}$, we obtain the following equations

$$\frac{da}{dt} = -\epsilon F\,[a(t)\cos\psi(t), -a(t)\sin\psi(t)]\sin\psi(t) \ldots\ (13a)$$
$$\frac{d\phi}{dt} = -\left[\frac{\epsilon}{a(t)}\right] F[a(t)\cos\psi(t), -a(t)\sin\psi(t)]\cos\psi(t) \ldots (13b)$$

These are the exact equations for the functions a andϕ when the solution for y and its derivative take the forms given by equations (5) and (6). These equations are coupled first order nonlinear differential equation.

Since $F\sin\psi$and $F\cos\psi$are the periodic functions of$\psi$with period 2π, so the Fourier expansion of both of these functions is possible. Therefore,

$$F\sin\psi = K_0(a) + \sum_{m=1}^{\infty}[K_m(a)\cos(m\psi) + L_m(a)\sin(m\psi)], \ \ldots\ (14a)$$
$$F\cos\psi = P_0(a) + \sum_{m=1}^{\infty}[P_m(a)\cos(m\psi) + Q_m(a)\sin(m\psi)], \ldots\ (14b)$$

Where
$$K_0(a) = \frac{1}{2\pi}\int_0^{2\pi} F\sin\psi\, d\psi, \ldots (15a)$$
$$K_m(a) = \frac{1}{\pi}\int_0^{2\pi} F\sin\psi\cos m\psi\, d\psi\ , \ldots\ (15b)$$
$$L_m(a) = \frac{1}{\pi}\int_0^{2\pi} F\sin\psi\sin m\psi\, d\psi, \ ....\ (15c)$$
$$P_o(a) = \frac{1}{2\pi}\int_o^{2\pi} F\cos\psi\, d\psi, \qquad \ldots\ (15d)$$
$$P_m(a) = \frac{1}{\pi}\int_0^{2\pi} F\cos\psi\cos m\psi\, d\psi\ , \ldots\ (15e)$$
$$Q_m(a) = \frac{1}{\pi}\int_0^{2\pi} F\cos\psi\sin m\psi\, d\psi\ , \ldots\ (15f)$$

Thus equations (13a) and(13b) can be written as

$$\frac{da}{dt} = -\epsilon K_0(a) - \epsilon\sum_{m=1}^{\infty}[K_m(a)\cos(m\psi) + L_m(a)\sin(m\psi)], \ldots (16a)$$
$$\frac{d\phi}{dt} = -\left(\frac{\epsilon}{a}\right)P_0(a) - \left(\frac{\epsilon}{a}\right)\sum_{m=1}^{\infty}[P_m(a)\cos(m\psi) + Q_m(a)\sin(m\psi)], ...(16b)$$

The first approximation of Krylov and Bogoliubov consists of neglecting all the terms on the right side of equations (16a) and (16b)except for the first; that is

$$\frac{da}{dt} = -\epsilon K_0(a) \ = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} F\,(a\cos\psi, -a\sin\psi)\sin\psi\, d\psi, \quad \ldots\ (17a)$$

$\frac{d\phi}{dt} = -\left(\frac{\epsilon}{a}\right) P_0(a) = -\left(\frac{\epsilon}{2\pi a}\right) \int_0^{2\pi} F\,(a\cos\psi, -a\sin\psi)\cos\psi\, d\psi$,....(17b)

These two equations can be written as

$$\frac{da}{dt} = \epsilon A_1(a), \ldots (18a)$$

$$\frac{d\phi}{dt} = \epsilon B_1(a), \qquad \ldots (18b)$$

The general procedure consists of solving equation (18a) for a and substituting this result into equation (18b) and solving for ϕ.

Case I: Let F be a function only of y, i.e.

$$F\left(y, \frac{dy}{dt}\right) = F_1\,(y), \qquad \ldots (19)$$

For this case, the equation for $a(t)$ is

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_0^{2\pi} F_1(a\cos\psi)\sin\psi\, d\psi\,, \quad \ldots (20)$$

The integrand is an odd function of ψ and thus the integral is zero i.e.,

$$\frac{da}{dt} = 0.$$

Consequently$a\,(t) = A, \ldots (21)$

where A is a constant, and $\phi(t)$ is given by the expression

$\phi(t) = \epsilon\Omega(A)t + \phi_0, \ldots (22)$

where ϕ_0 is a constant and$\Omega(A) = -\left(\frac{1}{2\pi A}\right) \int_0^{2\pi} F_1(A\cos\psi)\cos\psi\, d\psi$, ... (23)

Thus, for the case where F depends only on y,the first order approximation will have the form

$$y = A\cos\{[1 + \epsilon\Omega(A)]t + \phi_0\} \ldots (24)$$

This case corresponds to a conservative oscillator. The effect of the non linearity is seen in the fact that the frequency of the oscillation, $\omega = 1 + \epsilon\Omega(A)$, depends on the amplitude A of the motion.

Case II: Let F depends only on$\frac{dy}{dt}$, i.e.

$$F\left(y, \frac{dy}{dt}\right) = F_2\left(\frac{dy}{dt}\right), \qquad \ldots (25)$$

For this case, the equation for $\phi(t)$ is

$$\frac{d\phi}{dt} = -\left(\frac{\epsilon}{2\pi a}\right) \int_0^{2\pi} F_2(-a\sin\psi)\cos\psi\, d\psi. \qquad \ldots (26)$$

If $F_2(v)$ is an even function of v, then this case reduces to that of the previous situation given above. If $F_2(v)$ is an odd function of v, then

$$\frac{d\phi}{dt} = 0 \Longrightarrow \phi(t) = \phi_0\,, \ldots (27)$$

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right) \int_0^{2\pi} F_2(-a\sin\psi)\sin\psi\, d\psi = \epsilon A_1(a) \ldots (28)$$

Thus in the first approximation, where the function F is a function of $\frac{dy}{dt}$ only, the solution is

$$y = \mathrm{a(t)}\cos\left(\mathrm{t} + \phi_0\right) \qquad \text{….(29)}$$

The oscillation has a variable amplitude and a frequency $\omega = 1$.

SOLVED EXAMPLES

EXAMPLE1: Using Bogoliubov and Krylov method solve the differential equation

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} + \mathrm{y} + \epsilon\mathrm{y}^2 = 0 \text{ …. (1)}$$

SOLUTION: Comparing equation (1) with $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} + \mathrm{y} = \epsilon\mathrm{F}\left(\mathrm{y}, \frac{\mathrm{dy}}{\mathrm{dt}}\right)$,

we see that $F = -\mathrm{y}^2$ (i.e., F depends only on y)

Using the results of case I, we obtain

$\frac{\mathrm{da}}{\mathrm{dt}} = 0 \Rightarrow \mathrm{a(t)} = \mathrm{A} = \text{constant}$….. (2)

The equation for $\phi(\mathrm{t})$ is

$\frac{\mathrm{d}\phi}{\mathrm{dt}} = -\left(\frac{\epsilon}{2\pi\mathrm{A}}\right)\int_0^{2\pi} -(\mathrm{A}^2\cos^2\psi)\cos\psi\,\mathrm{d}\psi$.... (3)

Using the relation $\cos^3\psi = \frac{1}{4}(3\cos\psi + \cos3\psi)$in(3) and solving we get

$\frac{\mathrm{d}\phi}{\mathrm{dt}} = 0 \Rightarrow \phi(\mathrm{t}) = \phi_0$…. (4)

where A and ϕ_0 are constants . Consequently, the solution of Equation (1), using the first approximation of Krylov and Bogoliubov is

$y = \mathrm{A}\cos\left(\mathrm{t} + \phi_0\right)$…. (5)

Thus the first approximation gives exactly the same solution as the linear equation obtained by letting $\epsilon = 0$.The amplitude is constant and the frequency is 1 that is $\omega = 1$.

EXAMPLE 2: Using the method by Krylov and Bogoliubov, solve the conservative differential equation

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} + \mathrm{y} + \epsilon\mathrm{y}^3 = 0$$

SOLUTION: The conservative differential equation

$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dt}^2} + y + \epsilon y^3 = 0$…. (1)

has $\mathrm{F} = -\mathrm{y}^3$.

For this case, we have $\frac{\mathrm{da}}{\mathrm{dt}} = 0$

$\Rightarrow \mathrm{a(t)} = \mathrm{A} = \text{constant}$…. (2)

and $\phi(\mathrm{t})$ is determined by the differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = -\left(\frac{\epsilon}{2\pi\mathrm{A}}\right)\int_0^{2\pi} -(\mathrm{A}^3\cos^3\psi)\cos\psi\,\mathrm{d}\psi$$

$$= \frac{\epsilon A^3}{2\pi A}\int_0^{2\pi} \cos^4\psi \, d\psi$$

$$= \frac{\epsilon A^3}{2\pi A}\left(\frac{3\pi}{4}\right)$$

$\frac{d\phi}{dt} = \frac{3\epsilon A^2}{8}$ …. (3)

Solving for $\phi(t)$,we obtain

$$\phi(t) = \left(\frac{3\epsilon A^2}{8}\right)t + \phi_0 \qquad \text{…. (4)}$$

where ϕ_0 is a constant. Therefore, to the first approximation, the solution of equation (1) is

$$y = A\cos\{\left[1 + \left(\frac{3\epsilon A^2}{8}\right)\right]t + \phi_0\} \text{…. (5)}$$

EXAMPLE 3: Using the method by Krylov and Bogoliubov, solve the differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon\left(\frac{dy}{dt}\right)^2 = 0$$

SOLUTION: The differential equation

$\frac{d^2y}{dt^2} + y + \epsilon\left(\frac{dy}{dt}\right)^2 = 0$…. (1)

has $F = -\left(\frac{dy}{dt}\right)^2$.

In the first approximation we have

$$\frac{d\phi}{dt} = 0$$

or $\phi(t) = \phi_0 = \text{constant}$, and

$$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} (-a^2\sin^2\psi)\sin\psi \, d\psi$$

$= \frac{\epsilon a^2}{2\pi}\int_0^{2\pi} \sin^3\psi \, d\psi$ …. (2)

Substituting the result $\sin^3\psi = \frac{1}{4}(3\sin\psi - \sin 3\psi)$ into equation (2) and integrating, we obtain

$\frac{da}{dt} = 0$ …. (3)

or $a(t) = A = \text{constant}$. Consequently, the first approximation of Krylov –Bogoliubov gives the following solution for equation (1)

$$y = A\cos(t + \phi_0)$$

This is exactly the same result as obtained in example 1.

NOTE: When the function$F$ is equal to either y^2or $\left(\frac{dy}{dt}\right)^2$ or a linear combination of them, then the solution in the first approximation is the same as in the linearcase$(\text{i.e.}, \epsilon = 0)$. This means that the effect of the non-linearity show up only in the higher–order approximations to the solution.

EXAMPLE 4: Using the method by Krylov and Bogoliubov, solve the differential equation

$$\frac{d^2y}{dt^2} + y + 2\epsilon\frac{dy}{dt} = 0\epsilon > 0$$

SOLUTION: The differential equation

$$\frac{d^2y}{dt^2} + y + 2\epsilon\frac{dy}{dt} = 0 \qquad ...(1)$$

has$F = -\frac{2dy}{dt}$.

In the first approximation we have

$\frac{d\phi}{dt} = 0$....(2)

or$\phi = \phi_0 = \text{constant}$and

$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi}(2a\sin\psi)\sin\psi\, d\psi$.... (3)

Since,$\sin^2\psi = \frac{1}{2}(1 - \cos 2\psi)$ ....(4)

Substituting equation (4) into equation (3) and integrating gives

$$\frac{da}{dt} = -\frac{\epsilon a}{2\pi}(2\pi)$$

$\Rightarrow \frac{da}{dt} = -\epsilon a$.... (5)

$\Rightarrow a(t) = Ae^{-\epsilon t}$.... (6)

where$A$ is an arbitrary constant.

Thus the first approximation of Krylov and Bogoliubovyields the following solution to equation (1)

$y = Ae^{-\epsilon t}\cos(t + \phi_0)$.

EXAMPLE 5: Using the method by Krylov and Bogoliubov, solve the differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon\frac{dy}{dt} = 0\epsilon > 0$$

SOLUTION: The differential equation

$\frac{d^2y}{dt^2} + y + \epsilon\frac{dy}{dt} = 0$.....(1)

has$F = -\frac{dy}{dt}$.

In the first approximation we have

$\frac{d\phi}{dt} = 0$….(2)

or$\phi = \phi_0 =$ constant and

$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi}(a\sin\psi)\sin\psi\, d\psi$ …. (3)

Since,$\sin^2\psi = \frac{1}{2}(1 - \cos 2\psi)$ …. (4)

Substituting equation (4) into equation (3) and integrating gives

$$\frac{da}{dt} = -\frac{\epsilon a}{2\pi}(\pi)$$

$\Rightarrow \frac{da}{dt} = -\frac{\epsilon a}{2}$…. (5)

The solution to equation (5) is

$a(t) = A\, e^{-\frac{\epsilon t}{2}}$…. (6)

where A is an arbitrary constant.

Thus the first approximation of Krylov and Bogoliubovyields the following solution to equation (1)

$y = Ae^{-\frac{\epsilon t}{2}}\cos(t + \phi_0)$….(7)

This may be compared with the exact solution of equation (1) which is

$y = Ae^{\frac{-\epsilon t}{2}}\cos\left[\left(1 - \frac{\epsilon^2}{4}\right)^{\frac{1}{2}} t + \phi_0\right]$.

Thus the Krylov-Bogoliubov technique gives the correct frequency to terms of orderϵ^2.

SELF CHECK QUESTIONS

(SCQ-1) Using the method by Krylov and Bogoliubov, solve the differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon\left|\frac{dy}{dt}\right|\frac{dy}{dt} = 0$$

11.4 *SUMMARY:-*

In this unit, you have studied that the first approximation of Krylov and Bogoliubovto the oscillatory solution of

$$\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right), \qquad 0 < \epsilon << 1$$

is$y(t) = a(t)\cos\{t + \phi(t)\}$

where a(t) and $\phi(t)$ are solutions to the following system of first order differential equations

$\frac{da}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\sin\psi\, d\psi$.

11.5 *GLOSSARY:-*

- First approximation of Krylov and Bogoliubov

11.6 *REFERENCES:-*

1. Introduction to Non- linear Mechanics , N. Krylov and Bogoliubov, Princeton Univ. Press, Princeton NJ.1943.

2. Non- linear Differential Equations , R.A . Struble ,Mc-grawhill ,New-York,1962.

3. An Introduction to Non-linear Oscillations, Ronald E. Mickens , Cambridge Univ. Press. 1981.

11.7 *SUGGEDTED READING:-*

1. Introduction to Non- linear Mechanics, N. Krylov and Bogoliubov, Princeton Univ. Press, Princeton NJ.1943.

2. Non- linear Differential Equations, R. A. Struble ,Mcgrawhill,New-York,1962.

3. An Introduction to Non-linear Oscillations, Ronald E. Mickens, Cambridge Univ. Press. 1981.

11.8 *TERMINAL QUESTIONS:-*

(TQ-1) Explain Krylov and Bogoliubov method for finding the solution of a differential equation of the form $\frac{d^2y}{dt^2} + y = \epsilon F\left(y, \frac{dy}{dt}\right)$ when

(i) F depends only on y

(ii) F depends only on $\frac{dy}{dt}$.

(TQ-2) Using the method by Krylov and Bogoliubov, solve the differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon\frac{dy}{dt} = 0 \qquad \epsilon > 0$$

11.9 *ANSWERS*

SELF CHECK ANSWERS

(SCQ-1) $\phi = \phi_0,\ a(t) = \frac{a_0}{[1+(4\epsilon a_0/3\pi)t]}$

$$y = \frac{a_0 \cos(t + \phi_0)}{1 + (4\epsilon a_0/3\pi)t}$$

TERMINAL ANSWERS

(TQ-2) $\phi = \phi_0$, $a\,(t) = Ae^{-\epsilon t/2}$

$$y = Ae^{-\epsilon t/2}\cos(t + \phi_0)$$

BLOCK IV
SPECIAL FUNCTIONS

UNIT 12:- Chebyshev Polynomials and Legendre Polynomials

CONTENTS:

12.1 *INTRODUCTION:-*

Legendre's polynomials are used to represent a wide range of physical phenomena, such as the wave function of the hydrogen atom, the spherical harmonics, and the Legendre functions of the second kind, which are used in the solution of Laplace's equation in spherical coordinates. The Legendre polynomials also appear in the solution of boundary value problems for linear differential equations, and in numerical analysis as a basis for approximating functions on the interval [-1, 1].

Chebyshev polynomials are used in a variety of applications, such as signal processing, data compression, and numerical analysis. They are particularly important in the field of approximation theory, where they are used as a basis for approximating functions on the interval [-1, 1] or [0, 1].

Chebyshev polynomials also have applications in the study of orthogonal polynomials, where they are used to construct other families of orthogonal polynomials.

12.2 *OBJECTIVES:-*

After studying this unit you will be able to

- To discuss about Chebyshev polynomial, Legendre's polynomials and its equation and generating function.
- To study the recurrence formulae of Chebyshev polynomial and Legendre's polynomials.
- To study the important properties for this polynomials.
- To study the orthogonal properties of Chebyshev polynomial and Legendre's polynomials.

The main objectives of Legendre's polynomials and Chebyshev polynomials are provide to the numerous applications in mathematics, physics, and engineering. They are used to represent a wide range of physical phenomena and to solve a variety of mathematical problems.

12.3 *CHEBYSHEV POLYNOMIALS:-*

The Chebyshev polynomials of first and second kind are described by

$$T_n(x) = cos(ncos^{-1}x)$$
$$U_n(x) = sin(ncos^{-1}x)$$

Where $T_n(x)$ and $U_n(x)$ are first and second kind, n is a non-negative integer.

Theorem: $T_n(x)$ and $U_n(x)$ are independent solutions of Chebyshev equation.

$$(1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2y = 0$$

Proof: The Chebyshev equation is

$$(1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2y = 0 \qquad \dots(1)$$

By the definition of Chebyshev polynomials, we get

$$y = T_n(x) = cos(ncos^{-1}x)$$

$\therefore$

$$\frac{dy}{dx} = \frac{d}{dx}T_n(x) = \frac{d}{dx}cos(ncos^{-1}x) = -sin(ncos^{-1}x).n.\frac{-1}{(1-x^2)^{1/2}}$$

$$= sin(ncos^{-1}x).\frac{n}{(1-x^2)^{1/2}}$$

Again

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}T_n(x) = \frac{d}{dx}\left(\frac{d}{dx}T_n(x)\right)$$

$$= n\,\frac{d}{dx}\left(sin(ncos^{-1}x).\frac{n}{(1-x^2)^{1/2}}\right)$$

$$= n\left[-\frac{1}{2}(1-x^2)^{-3/2}(-2x).sin(ncos^{-1}x) + (1-x^2)^{1/2}cos(ncos^{-1}x).\frac{n}{(1-x^2)^{1/2}}\right]$$

$$\frac{d^2}{dx^2}T_n(x) = \frac{nx}{(1-x^2)^{3/2}}sin(ncos^{-1}x) - \frac{n^2}{1-x^2}cos(ncos^{-1}x)$$

Using above equations, we obtain

$$(1-x^2)\frac{d^2}{dx^2}T_n(x) - x\frac{d}{dx}T_n(x) + n^2T_n(x) = 0$$

$\frac{nx}{(1-x^2)^{3/2}}sin(ncos^{-1}x) - \frac{n^2}{1-x^2}cos(ncos^{-1}x) -$
$sin(ncos^{-1}x).\frac{nx}{(1-x^2)^{1/2}} + ncos(ncos^{-1}x)$=0

Showing that $T_n(x)$ is a solution of (1).

Similarly, to show that $U_n(x)$ is a solution of (1): Proceed as above

To show that $T_n(x)$ and $U_n(x)$ are independent solution of (1): we given by the definition

$T_n(x) = cos(ncos^{-1}x)$ and $U_n(x) = sin(ncos^{-1}x)$

∴ $T_n(1) = cos(ncos^{-1}1) = cos(n \times 0) = 1$

and $U_n(1) = sin(ncos^{-1}1) = sin(n \times 0) = 0$

Finally $U_n(x)$ cannot intimated as a constant multiple of $T_n(x)$. This is prove that $T_n(x)$ and $U_n(x)$ are independent solution of (1)

12.4 *ORTHOGONAL PROPERTIES:-*

To show that

i. $\int_{-1}^{1}\frac{T_r(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & r \neq n \\ \pi/2, & r = n \neq 0 \\ \pi, & r = n = 0 \end{cases}$

ii. $\int_{-1}^{1}\frac{U_r(x)U_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & m \neq n \\ \pi/2, & r = n \neq 0 \\ \pi, & r = n = 0 \end{cases}$

Proof: We given, by the definition

i. $T_r(x) = cos(mcos^{-1}x)$ & $T_n(x) = cos(ncos^{-1}x)$ ... (1)

∴ Putting $x = cos\theta,\ dx = -sin\theta d\theta$ in (1)

$T_r(cos\theta) = cos(mcos^{-1}cos\theta)$ & $T_n(x) = cos(ncos^{-1}cos\theta)$

$$T_r(cos\theta) = cos(m\theta) \qquad \& \qquad T_n(x) = cos(n\theta)$$

Let

$$\int_{-1}^{1} \frac{T_r(x)T_n(x)}{\sqrt{1-x^2}} dx = \int_{\pi}^{0} \frac{cosm\theta cosn\theta}{sin\theta}(-sin\theta)d\theta = \int_0^{\pi} cosm\theta cosn\theta\, d\theta$$

⇒ **If $\boldsymbol{r \neq n}$so that $\boldsymbol{(r-n) \neq 0}$, then**

$$I = \frac{1}{2}\int_0^{\pi} 2cosr\theta cosn\theta\, d\theta = \frac{1}{2}\int_0^{\pi} [cos(r+n)\theta + cos(r-n)\theta]\, d\theta$$

$$= \frac{1}{2}\left[\frac{cos(r+n)\theta}{r+n} + \frac{cos(r-n)\theta}{r-n}\right]_0^{\pi} = 0$$

⇒ **If $\boldsymbol{r = n \neq 0}$, then**

$$I = \int_0^{\pi} cosr\theta cosn\theta\, d\theta = \int_0^{\pi} cos^2 r\theta\, d\theta = \int_0^{\pi} \frac{1+cos2r\theta}{2} d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{sin2r\theta}{2r}\right]_0^{\pi} = \frac{\pi}{2}$$

⇒ **If $\boldsymbol{r = n = 0}$, then $\boldsymbol{cosm\theta = cosn\theta = 1}$**

$$I = \int_0^{\pi} cosr\theta cosn\theta\, d\theta = \int_0^{\pi} (1)(1)\, d\theta = [\theta]_0^{\pi} = \pi$$

ii. $\int_{-1}^{1} \frac{U_r(x)U_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & r \neq n \\ \pi/2, & r = n \neq 0 \\ \pi, & r = n = 0 \end{cases}$

Proof: $U_r(x) = sin(rcos^{-1}x) \qquad \& \qquad U_n(x) = sin(ncos^{-1}x) \qquad \ldots(1)$

∴ Putting $x = cos\theta,\ dx = -sin\theta d\theta$ in (1)

$U_r(cos\theta) = sin(rcos^{-1}cos\theta) \qquad \& \qquad U_n(x) = sin(ncos^{-1}cos\theta)$

$$U_r(cos\theta) = sin(r\theta) \qquad \& \qquad U_n(x) = sin(n\theta)$$

Let

$$\int_{-1}^{1} \frac{U_r(x)U_n(x)}{\sqrt{1-x^2}} dx = \int_{\pi}^{0} \frac{sinr\theta sinn\theta}{sin\theta}(-sin\theta)d\theta = \int_0^{\pi} sinr\theta sinn\theta\, d\theta$$

⇒ **If $\boldsymbol{r \neq n}$so that $\boldsymbol{(r-n) \neq 0}$, then**

$$J = \frac{1}{2}\int_0^{\pi} 2sinr\theta sinn\theta\, d\theta = \frac{1}{2}\int_0^{\pi} [cos(r-n)\theta + cos(r+n)\theta]\, d\theta$$

$$= \frac{1}{2}\left[\frac{sin(r-n)\theta}{r-n} + \frac{cos(r+n)\theta}{r+n}\right]_0^{\pi} = 0$$

⇒ **If $\boldsymbol{r = n \neq 0}$, then**

$$I=\int_0^{\pi} sinr\theta sinn\theta\, d\theta=\int_0^{\pi} sin^2 r\theta\, d\theta=\int_0^{\pi}\frac{1-cos2r\theta}{2}d\theta$$
$$=\frac{1}{2}\left[\theta-\frac{sin2r\theta}{2r}\right]_0^{\pi}=\frac{\pi}{2}$$

⇒ **If $r=n=0$, then $sinr\theta=sinn\theta=1$**

$$I=\int_0^{\pi} sinr\theta sinn\theta\, d\theta=\int_0^{\pi}(0)(0)\, d\theta=0$$

12.5 *RECURRENCE RELATION:-*

i. $T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0$
ii. $(1-x^2)T'_n(x)=-nxT_n(x)+nT_{n-1}(x)$
iii. $U_{n+1}(x)-2xU_n(x)+U_{n-1}(x)=0$
iv. $(1-x^2)U'_n(x)=-nxU_n(x)+nU_{n-1}(x)$

Proof : We given, by the definition of Chebyshev polynomials

i.

$$\boldsymbol{T_n(x)=cos(ncos^{-1}x)} \qquad \dots(\mathbf{1})$$

So ∴ Putting $x=cos\theta,\ dx=-sin\theta d\theta$ in (1)

$$T_n(x)=cos(ncos^{-1}cos\theta)=cosn\theta$$
$$T_{n+1}(x)=cos(n+1)\theta, \qquad T_{n-1}(x)=cos(n-1)$$

We prove that $\boldsymbol{T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0}$

Now we take L.H.S

⇒ $T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=cos(n+1)\theta-2xcosn\theta+cos(n-1)\theta$

⇒ $=[cos(n+1)\theta+cos(n-1)\theta]-2cos\theta cosn\theta$

⇒ $=[cos(n+1)\theta+cos(n-1)\theta]-2cos\theta cosn\theta$

⇒ $\boldsymbol{=2cos\theta cosn\theta-2cos\theta cosn\theta=0}$

ii. $\boldsymbol{(1-x^2)T'_n(x)=-nxT_n(x)+nT_{n-1}(x)} \quad \dots(\mathbf{2})$

Proof: differentiating (1)

$$T'_n(x)=-sin(ncos^{-1}x).\frac{-n}{\sqrt{1-x^2}}$$
$$=-sin(ncos^{-1}cos\theta).\frac{-n}{\sqrt{1-cos^2\theta}}$$
$$=\frac{nsinn\theta}{sin\theta}$$

Putting the value of $T'_n(x),T_n(x),T_{n-1}(x)$ and Putting $x=cos\theta$, $dx=-sin\theta d\theta$ in (2)

$$(1-x^2)T'_n(x)=-nxT_n(x)+nT_{n-1}(x)$$

$$(1-cos^2\theta)\frac{nsinn\theta}{sin\theta} = -ncos\theta cosn\theta + ncos(n-1)\theta$$

$$sin^2\theta\frac{nsinn\theta}{sin\theta} = -ncos\theta cosn\theta + ncos(n-1)\theta$$

$$nsin\theta sinn\theta = -ncos\theta cosn\theta + ncos(n-1)\theta \quad ...(3)$$

Now we take R.H.S of above equation(3)

$$= -ncos\theta cosn\theta + ncos(n\theta-\theta)$$

$$= -ncos\theta cosn\theta + ncos(n-1)\theta$$

$$= -ncos\theta cosn\theta + ncosn\theta cos\theta + nsinn\theta sin\theta$$

$$= nsinn\theta sin\theta$$

$$\boldsymbol{L.H.S = R.H.S}$$

iii. $\boldsymbol{U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0}$

Proof: $U_n(x) = sin(ncos^{-1}x) \quad ...(1)$

$\therefore$ Putting $x = cos\theta,\ dx = -sin\theta d\theta$ in (1)

$$U_n(x) = sin(ncos^{-1}cos\theta)$$

$$U_n(x) = sin(n\theta)$$

We prove that $\boldsymbol{U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0}$

Similarly we take L.H.S.

$\Rightarrow$ $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = sin(n+1)\theta - 2cos\theta sin(ncos^{-1}cos\theta) + sin(n-1)\theta$

$\Rightarrow$ $= [sin(n+1)\theta + sin(n-1)\theta] - 2cos\theta sinn\theta$

$\Rightarrow$ $= 2sinn\theta cos\theta - 2cos\theta sinn\theta$

$\Rightarrow$ $\boldsymbol{= 2cos\theta sinn\theta - 2cos\theta sinn\theta = 0}$

iv. $\boldsymbol{(1-x^2)U'_n(x) = -nxU_n(x) + nU_{n-1}(x)} \quad ...(2)$

Proof: differentiating (2)

$$U'_n(x) = cos(ncos^{-1}x).\frac{-n}{\sqrt{1-x^2}}$$

$$= cos(ncos^{-1}cos\theta).\frac{-n}{\sqrt{1-cos^2\theta}}$$

$$= -\frac{ncosn\theta}{sin\theta}$$

Putting the value of $U'_n(x), U_n(x), U_{n-1}(x)$ and Putting $x = cos\theta$, $dx = -sin\theta d\theta$ in (2)

$$(1-x^2)U'_n(x) = -nxU_n(x) + nU_{n-1}(x)$$

$$-(1-cos^2\theta)\frac{ncosn\theta}{sin\theta} = -ncos\theta sinn\theta + nsin(n-1)\theta$$

$$-sin^2\theta\frac{ncosn\theta}{sin\theta} = -ncos\theta sinn\theta + nsin(n-1)\theta$$

$$-nsin\theta cosn\theta = -ncos\theta sinn\theta + nsin(n-1)\theta \qquad ...(3)$$

Again we take R.H.S of above equation(3)

$$= -ncos\theta sinn\theta + nsin(n\theta - \theta)$$

$$= -ncos\theta sinn\theta + nsin(n-1)\theta$$

$$= -ncos\theta sinn\theta + nsinn\theta cos\theta - ncosn\theta sin\theta$$

$$= -ncosn\theta sin\theta$$

$$\boldsymbol{L.H.S = R.H.S}$$

Theorem I: To prove that $T_n(x) = (1/2) \times \left[\left\{x + i(1-x^2)^{1/2}\right\}^n + i\left\{x - i(1-x^2)^{1/2}\right\}^n\right]$

Proof: we take

$$T_n(x) = cos(ncos^{-1}x)$$

Now we put $x = cos\theta$

$$T_n(x) = cos(ncos^{-1}cos\theta) = cosn\theta = \frac{\left(e^{in\theta} + e^{-in\theta}\right)}{2}$$

$$= \frac{\left[\left(e^{i\theta}\right)^n + \left(e^{-i\theta}\right)^n\right]}{2}$$

$$= \frac{[(cos\theta + isin\theta)^n + (cos\theta - isin\theta)^n]}{2}$$

$$= \frac{[(cos\theta + isin\theta)^n + (cos\theta - isin\theta)^n]}{2}$$

$$\boldsymbol{T_n(x) = \frac{[(x + i\{1-x^2\})^n + (x - i\{1-x^2\})^n]}{2}}$$

TheoremII: To prove that $U_n(x) = -(i/2) \times \left[\left\{x + i(1-x^2)^{1/2}\right\}^n - i\left\{x - i(1-x^2)^{1/2}\right\}^n\right]$

Proof: we take

$$U_n(x) = sin(ncos^{-1}x)$$

Now we put $x = cos\theta$

$$T_n(x) = sin(ncos^{-1}cos\theta) = sinn\theta = \frac{\left(e^{in\theta} - e^{-in\theta}\right)}{2}$$

$$= \frac{\left[\left(e^{i\theta}\right)^n - \left(e^{-i\theta}\right)^n\right]}{2i}$$

$$= \frac{[(cos\theta + isin\theta)^n - (cos\theta - isin\theta)^n]}{2i}$$

$$= \frac{[(cos\theta + isin\theta)^n - (cos\theta - isin\theta)^n]}{2i}$$

$$\boldsymbol{T_n(x) = -\frac{i}{2}[(x + i\{1-x^2\})^n + (x - i\{1-x^2\})^n]}$$

Theorem III: To prove that $T_n(x) = \sum_{s=0}^{n/2}(-1)^s \frac{n!}{(2s)!(n-2s)!}(1-x^2)^s x^{n-2s}$

Proof: As in theorem I, we obtain

$$T_n(x) = \frac{[(x + i\{1 - x^2\})^n + (x - i\{1 - x^2\})^n]}{2}$$
$$= \frac{1}{2}\left[\sum_{r=0}^{n} n_{C_r} x^{n-r}\left\{i(\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{1/2}\right\}^r + \sum_{r=0}^{n} n_{C_r} x^{n-r}\left\{-i(\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{1/2}\right\}^r\right]$$

Since by binomial theorem, we obtain

$$(a + b)^n = a^n + n_{C_1} a^{n-1} b + n_{C_1} a^{n-1} b^2 + \cdots + n_n b^n = \sum_{r=0}^{n} n_{C_r} a^{n-r} b^r$$

$$T_n(x) = \sum_{r=0}^{n} n_{C_r} x^{n-r} i^r (\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{r/2}\,[1 + (-1)^r]$$

But $$1 + (-1)^r = \begin{cases} 0, if\ r\ is\ odd \\ 2, if\ r\ is\ even \end{cases}$$

$$T_n(x) = \sum_{reven,\le n} n_{C_r} x^{n-r} i^r (\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{r/2} \quad \dots (1)$$

Since r is even, so $r = 2s$, where s be an integer. $r \le n \Rightarrow \quad 2s \le n \Rightarrow$ $s \le n/2$

$\Rightarrow$ Now if n is even r goes $0\ to\ n/2$, if n is odd r goes $0\ to\ (n-1)/2$, then

$$(n/2) = \begin{cases} n/2, if\ n\ is\ even \\ \dfrac{(n-1)}{2}, if\ n\ is\ odd \end{cases}$$

Hence $r = 2s$ from (1)

$$\boldsymbol{T_n(x)} = \sum_{s=0}^{n/2} \boldsymbol{n_{C_{2s}} x^{n-2s} i^{2s} (1 - x^2)^s}$$
$$= \sum_{s=0}^{n/2} (-\mathbf{1})^s \frac{\boldsymbol{n}!}{(\mathbf{2}\boldsymbol{s})!\,(\boldsymbol{n} - \mathbf{2}\boldsymbol{s})!} (\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^s \boldsymbol{x}^{\boldsymbol{n-2s}}$$

TheoremIV: To prove that $\boldsymbol{U_n}(\boldsymbol{x}) = \sum_{\boldsymbol{s=0}}^{(\boldsymbol{n-1})/2} (-\mathbf{1})^{\boldsymbol{s}} \frac{\boldsymbol{n}!}{(\mathbf{2}\boldsymbol{s}+\mathbf{1})!(\boldsymbol{n}-\mathbf{2}\boldsymbol{s}-\mathbf{1})!} (\mathbf{1} - \boldsymbol{x}^{2})^{\boldsymbol{s}+\mathbf{1/2}} \boldsymbol{x}^{\boldsymbol{n-2s-1}}$

Proof: As in theorem II, we obtain

$$U_n(x) = -\frac{\boldsymbol{i}}{\mathbf{2}}[(\boldsymbol{x} + \boldsymbol{i}\{\mathbf{1} - \boldsymbol{x}^{\mathbf{2}}\})^{\boldsymbol{n}} + (\boldsymbol{x} - \boldsymbol{i}\{\mathbf{1} - \boldsymbol{x}^{\mathbf{2}}\})^{\boldsymbol{n}}]$$
$$= -\frac{i}{2}\left[\sum_{r=0}^{n} n_{C_r} x^{n-r}\left\{i(\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{1/2}\right\}^r - \sum_{r=0}^{n} n_{C_r} x^{n-r}\left\{-i(\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{1/2}\right\}^r\right]$$

Since by binomial theorem, we obtain

$$(a+b)^n = a^n + n_{C_1} a^{n-1} b + n_{C_1} a^{n-1} b^2 + \cdots + n_n b^n = \sum_{r=0}^{n} n_{C_r} a^{n-r} b^r$$

$$U_n(x) = -\frac{i}{2}\sum_{r=0}^{n} n_{C_r} x^{n-r} i^r (\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{r/2}\,[1-(-1)^r]$$

But $$1+(-1)^r = \begin{cases} 0, if\ r\ is\ even \\ 2, if\ r\ is\ odd \end{cases}$$

$$T_n(x) = -i \sum_{r odd, r\le n} n_{C_r} x^{n-r} i^r (\mathbf{1}-\boldsymbol{x}^{\mathbf{2}})^{r/2} \quad \ldots(1)$$

Since r is odd, so $r = 2s+1$, where s be an integer. $r \le n \Rightarrow \quad 2s+1 \le n \Rightarrow \quad s \le (n-1)/2$

$\Rightarrow$ Now if n is odd r goes $0\ to\ (n-1)/2$, if n is odd r goes $0\ to\ (n-2)/2$, then

$$\left[\frac{(n-2)}{2}\right] = \begin{cases} \frac{(n-1)}{2}, if\ n\ is\ odd \\ \frac{(n-2)}{2}, if\ n\ is\ even \end{cases}$$

Hence $r = 2s+1$ from (1)

$$\boldsymbol{U_n(x)} = \sum_{s=0}^{(n-1)/2} \boldsymbol{n_{C_{2s+1}} x^{n-2s-1} i^{2s+1} (1-x^2)^{s+1/2}}$$

$$= \sum_{s=0}^{(n-1)/2} (-\mathbf{1})^s \frac{\boldsymbol{n}!}{(\boldsymbol{2s}+\mathbf{1})!\,(\boldsymbol{n}-\boldsymbol{2s}-\mathbf{1})!} (\mathbf{1} - \boldsymbol{x}^{\mathbf{2}})^{\boldsymbol{s}+\mathbf{1}/\mathbf{2}} \boldsymbol{x}^{\boldsymbol{n}-\boldsymbol{2s}-\mathbf{1}}$$

12.6 *GENERATING FUNCTION FOR CHEBYSHEV PLOYNOMIALS:-*

i. $\boldsymbol{\frac{1-k^2}{1-2kx-k^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, k^n}$

Proof: Let for $n = 0$,putting $x = cos\theta = \left(e^{i\theta} + e^{-i\theta}\right)/2$

L.H.S. $\Rightarrow$ $\frac{1-k^2}{1-2kx-k^2} = \frac{1-k^2}{1-2k\left(e^{i\theta}+e^{-i\theta}\right)/2-k^2} = \frac{1-k^2}{1-k\left(e^{i\theta}+e^{-i\theta}\right)-k^2} = \frac{1-k^2}{1-k\left(e^{i\theta}+e^{-i\theta}\right)-k^2}$

$$= \frac{1-k^2}{1-ke^{i\theta}-ke^{-i\theta}-k^2}$$

$$= \frac{1-k^2}{(1-ke^{i\theta}) - ke^{-i\theta}(1-ke^{i\theta})} = \frac{1-k^2}{(1-ke^{-i\theta})(1-ke^{i\theta})}$$

$$= (1-k^2)\left(1-ke^{-i\theta}\right)^{-1}\left(1-ke^{i\theta}\right)^{-1}$$

$$= (1-k^2)\sum_{a=0}^{\infty}\left(ke^{i\theta}\right)^a \sum_{b=0}^{\infty}\left(ke^{-i\theta}\right)^b$$

$$= (1-k^2)\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} e^{i(a-b)\theta}\, k^{a+b}$$

$$= \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} e^{i(a-b)\theta}\, k^{a+b} + \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} e^{i(a-b)\theta}\, k^{a+b+2} \quad ...(1)$$

Now we taking $a = 0, b = 0$ in (1), we have $= e^{i(0-0)\theta} = e^0 = 1 = T_0(x)$

For $n \geq 1$, we obtain $k^n$by taking

$\Rightarrow \quad a + b = n, (i.e., b = n - a\; so\; s \geq 0 \Rightarrow n - a \geq 0 \Rightarrow a \leq n\; for\; k^n)$ in (1) and

$\Rightarrow a + b + 2 = n (i.e., b = n - a - 2\; so\; s \geq 0 \Rightarrow n - a - 2 \geq 0 \Rightarrow a \leq n - 2\; for\; k^n)$ in (1)

Hence the total coefficient of k^n in (1)

$$= \sum_{a=0}^{n} e^{i(a-\{n-a\})\theta} - \sum_{b=0}^{n-2} e^{i(a-\{n-a-2\})\theta}$$

$$= e^{-in\theta}\sum_{a=0}^{n} e^{2ia\theta} - e^{-i(n-2)\theta}\sum_{a=0}^{n} e^{2ia\theta}$$

$$= e^{-in\theta}\left[1 + e^{2i\theta} + e^{4i\theta} + \cdots to\ (n+1) terms\right] - e^{-i(n-2)\theta}\left[1 + e^{2i\theta} + e^{4i\theta} + \cdots to\ (n+1) terms\right]$$

$$= e^{-in\theta}\frac{-\left(e^{2i\theta}\right)^{n+1}}{1-e^{2i\theta}} - e^{-i(n-2)\theta}\frac{1-\left(e^{2i\theta}\right)^{n-1}}{1-e^{2i\theta}}$$

$$= \frac{e^{-in\theta} - e^{i(n+2)\theta}}{1-e^{2i\theta}} - \frac{e^{-i(n-2)\theta} - e^{in\theta}}{1-e^{2i\theta}}$$

$$= \frac{e^{-in\theta} - e^{i(n+2)\theta} - e^{-i(n-2)\theta} + e^{in\theta}}{1-e^{2i\theta}}$$

$$= \frac{e^{-in\theta}\left(1-e^{2i\theta}\right) + e^{in\theta}\left(1-e^{2i\theta}\right)}{\left(1-e^{2i\theta}\right)} = e^{-in\theta} + e^{in\theta} = 2cosn\theta$$

$$= cosn\theta = \left(e^{in\theta} + e^{-in\theta}\right)/2$$

$$= 2T_n(x), \quad [T_n(x) = cos(ncos^{-1}x) = cos(ncos^{-1}cos\theta) = cosn\theta]$$

ii. $\boldsymbol{\frac{\sqrt{1-x^2}}{1-2kx-k^2} = \sum_{n=1}^{\infty} U_{n+1}(x)\, k^n}$

Proof: Now we take L.H.S

$$= \frac{\sqrt{1-x^2}}{1-2kx-k^2}$$

Putting $x = cos\theta$

$$=\frac{\sqrt{1-cos^2\theta}}{1-2kcos\theta-k^2}=\frac{sin\theta}{1-k(e^{i\theta}+e^{-i\theta})-k^2}$$
$$=\frac{sin\theta}{1-k(e^{i\theta}+e^{-i\theta})-k^2}=\frac{sin\theta}{1-k(e^{i\theta}+e^{-i\theta})-k^2}$$
$$=\frac{sin\theta}{1-ke^{i\theta}-ke^{-i\theta}-k^2}=\frac{sin\theta}{(1-ke^{-i\theta})(1-ke^{i\theta})}$$
$$=\frac{sin\theta}{(1-ke^{-i\theta})(1-ke^{i\theta})}=sin\theta\sum_{a=0}^{\infty}\left(ke^{i\theta}\right)^a\sum_{b=0}^{\infty}\left(ke^{-i\theta}\right)^b$$
$$=sin\theta\sum_{a=0}^{\infty}\left(ke^{i\theta}\right)^a\sum_{b=0}^{\infty}\left(ke^{-i\theta}\right)^b=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}k^{a+b}e^{i(a-b)\theta}$$

For $n\geq 1$, we obtain $k^n$by taking
$\Rightarrow\quad a+b=n,(i.e.,b=n-a\ so\ s\geq 0\Rightarrow n-a\geq 0\Rightarrow a\leq n\ for\ k^n)$ in (1) and
$\Rightarrow a+b+2=n(i.e.,b=n-a-2\ so\ s\geq 0\Rightarrow n-a-2\geq 0\Rightarrow a\leq n-2\ for\ k^n),$
Hence the total coefficient of k^n in (1)

$$=sin\theta e^{-in\theta}\sum_{a=0}^{\infty}e^{2ai\theta}$$
$$=sin\theta e^{-in\theta}\left[1+e^{2i\theta}+e^{4i\theta}+\cdots to\ (n+1)terms\right]$$
$$=sin\theta\ e^{-in\theta}\,\frac{1-\left(e^{2i\theta}\right)^{n+1}}{1-e^{2i\theta}}=\frac{e^{i\theta}-e^{-i\theta}}{2i}\times\ e^{-in\theta}\times\frac{1-\left(e^{2i\theta}\right)^{n+1}}{-e^{i\theta}(e^{i\theta}-e^{-i\theta})}$$
$$=\ e^{-i(n+1)\theta}\times\frac{\left[\left(e^{2i\theta}\right)^{n+1}-1\right]}{(e^{i\theta}-e^{-i\theta})}=sin(n+1)\theta$$
$$=\boldsymbol{sin\{(n+1)cos^{-1}x\}=U_{n+1}(x),as\ x=cos\theta}\ \&\quad\theta=\boldsymbol{cos^{-1}x}.$$

SOLVED EXAMPLES

EXAMPLE1: To show that

- **i.** $T_n(1)=1,T_n(-1)=(-1)^n,\ T_{2n}(0)=(-1)^n,T_{2n+1}(0)=0$
- **ii.** $U_n(1)=0,U_n(-1)=0,\ T_{2n}(0)=0,\ T_{2n+1}(0)=(-1)^n$

PROOF:

- **i.** We given $\quad T_n(x)=cos(ncos^{-1}x)\qquad\qquad ...(1),$

Then putting $x=1$, we obtain
$$T_n(1)=cos(ncos^{-1}1)=cos(n\times 0)=1$$
Now $T_n(x)=cos(ncos^{-1}x)$, then again putting $x=-1$
$$T_n(-1)=cos(ncos^{-1}-1)=cos(n\times\pi)=(-1)^n$$
Since again replacing $x$by 0 and n by $2n$ in (1), we get
$$T_{2n}(0)=cos(2ncos^{-1}0)=cos(2n\times\pi/2)=cosn\pi=(-1)^n$$
Since again replacing $x$by 0 and n by $2n+1$ in (1), we given below

$T_{2n+1}(0) = cos\big((2n+1)cos^{-1}0\big) = cos\big((2n+1) \times \pi/2\big) = 0.$

ii. Proceed as above yourself.

EXAMPLE2: To show that $T_m\{T_n(x)\} = T_n\{T_m(x)\} = T_{nm}(x)$

SOLUTION: We have $T_n(x) = cos(ncos^{-1}x)$

$$T_m\{T_n(x)\} = T_m\{cos(ncos^{-1}x)\} = cos[mcos^{-1}\{cos(ncos^{-1}x)\}]$$
$$= cos[nmcos^{-1}x]$$

Again $T_m(x) = cos(ncos^{-1}x)$

$$T_n\{T_m(x)\} = n\{cos(mcos^{-1}x)\} = cos[ncos^{-1}\{cos(mcos^{-1}x)\}]$$
$$= cos[mncos^{-1}x]$$

Hence

$$\boldsymbol{T_m\{T_n(x)\} = T_n\{T_m(x)\} = T_{nm}(x)}$$

EXAMPLE3: To show that $(1-x^2)^{1/2}\{T_n(x)\} = \{U_{n+1}(x)\} - xU_n(x)$

SOLUTION: We have $T_n(x) = cos(ncos^{-1}x)$

$$U_n(x) = sin(ncos^{-1}x)$$

putting $x = cos\theta$ in above equation

$T_n(cos\theta\,) = cos(ncos^{-1}cos\theta\,) = cosn\theta$, $U_n(cos\theta\,) = sin(ncos^{-1}cos\theta\,) = sinn\theta$

$$(1-cos^2\theta)^{1/2}\{T_n(cos\theta)\} = \{U_{n+1}(cos\theta)\} - cos\theta\, U_n(cos\theta)$$
$$sin\theta\{T_n(cos\theta)\} = \{U_{n+1}(cos\theta)\} - cos\theta\, U_n(cos\theta)$$
$$sin\theta cosn\theta = sin(n+1)\theta - cos\theta sinn\theta$$
$$sin\theta cosn\theta = sin(n\theta+\theta) - cos\theta sinn\theta \qquad \dots(1)$$

Now we take **R.H.S.**

$$= sinn\theta cos\theta + cosn\theta sin\theta - cos\theta sinn\theta$$
$$= cosn\theta sin\theta = \boldsymbol{L.H.S.}$$

12.7 *LEGENDRE'S EQUATION AND ITS SOLUTION:-*

The differential equation of the form

$$\left.\begin{array}{c}(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0 \\ or \\ (1-x^2)y'' - 2xy' + n(n+1)y = 0\end{array}\right\} \qquad \dots(1)$$

is called Legendre's equation, where n is a positive integer. Now solve equation (1) in series of descending power of x. Let the solution of equation (1) is

$$y = \sum_{l=0}^{\infty} a_l x^{k-l}, a_0 \neq 0 \qquad \dots(2)$$

$$\frac{dy}{dx} = \sum_{l=0}^{\infty} a_l x^{k-l-1}(k-l)$$

$$\frac{d^2y}{dx^2} = \sum_{l=0}^{\infty} a_l x^{k-l-2}(k-l)(k-l-1)$$

Substituting the value of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we obtain

$$(1-x^2)\sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2} - 2x\sum_{l=0}^{\infty} a_l(k-l)x^{k-l-1} + n(n+1)\sum_{l=0}^{\infty} a_l x^{k-l} = 0$$

or

$$\left(\sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2} - \sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2+2}\right) - 2x\sum_{l=0}^{\infty} a_l(k-l)x^{k-l-1} + n(n+1)\sum_{l=0}^{\infty} a_l x^{k-l} = 0$$

or

$$\sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2} - \sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l} - 2\sum_{l=0}^{\infty} a_l(k-l)x^{k-l-1+1} + n(n+1)\sum_{l=0}^{\infty} a_l x^{k-l} = 0$$

or

$$\sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2} - \sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l} - 2\sum_{l=0}^{\infty} a_l(k-l)x^{k-l} + n(n+1)\sum_{l=0}^{\infty} a_l x^{k-l} = 0$$

or

$$\sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2} + \sum_{l=0}^{\infty}\{n(n+1) - 2(k-l) - (k-l)(k-l-1)\}\, a_l x^{k-l} = 0$$

$$or \sum_{l=0}^{\infty} a_l(k-l)(k-l-1)x^{k-l-2}$$
$$+\sum_{l=0}^{\infty}\{n(n+1)-2(k-l)-(k-l)(k-l-1)\}\,a_l x^{k-l}$$
$$=0$$

Now the coefficient of $a_l x^{k-l}$ is

$$\begin{aligned}\{n(n+1)-2(k-l)-(k-l)(k-l-1)\}\\ &= n^2+n-2(k-l)-(k-l)^2+(k-l)\\ &= \{n^2-(k-l)^2+n-2(k-l)\}+(k-l)\\ &= \{n-k+l\}\{n+k-l\}+n-2k+2l+k-l\\ &= \{n-k+l\}\{n+k-l\}+n-k+l\\ &= \{n-k+l\}\{n+k-l\}+\{n-k+l\}\\ &= \{n-k+l\}\{n-k+l+1\}\end{aligned}$$

Hence the equation (2) may be written as

$$\sum_{l=0}^{\infty} a_l\{k-l\}\{k-l+1\}x^{k-l-2}+\sum_{l=0}^{\infty}\{n-k+l\}\{n-k+l+1\}\,a_l x^{k-l}$$
$$=0 \qquad \dots (3)$$

Equating to zero coefficient of x namely x^l in above equation, we obtain

$$a_0(n-k)(n+k-1)=0$$
$$or$$
$$k=n,-(n+1) \qquad (\because \quad a_0\neq 0)$$

Now the next power of x is $k-1$, so

$$(n-k+1)(n+k)a_1=0$$

For $k=n$ and $-(n+1)$, neither $(n-k+1)$ nor $(n+k)$ in zero. Therefore $a_1=0$

From (3)

$$\{k-l+2\}\{k-l+1\}a_{l-2}+\{n-k+l\}\{n-k+l+1\}a_l=0$$
$$a_l = \frac{\{k-l+2\}\{k-l+1\}}{\{n-k+l\}\{n-k+l+1\}}a_{l-2} \qquad \dots (4)$$

Substituting $n=3,5,7$ .... in above equation and nothing $a_1=0$, we obtain $a_1=a_3=a_5=a_7=\cdots\cdots=0$ to obtain $a_2=a_4=a_6\cdots\cdots etc,$ we consider two cases

CaseI: When $k=n$ then (4) becomes

$$a_l=\frac{\{n-l+2\}\{n-l+1\}}{\{l\}\{2n-l+1\}}a_{l-2}$$

Substituting $l=2,4,6$ .... in (4)

$$a_2=-\frac{n\{n-1\}}{2\{2n-1\}}a_0,\quad a_4=-\frac{n\{n-2\}\{n-3\}}{4\{2n-3\}}a_2$$
$$=\frac{n\{n-1\}\{n-2\}\{n-3\}}{2.4\{2n-1\}\{2n-3\}}a_0 \quad \dots \dots$$

From (2)

$$y = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots\cdots\cdots\cdots$$

$$y = a_0\left[x^n - \frac{n\{n-1\}}{2\{2n-1\}}x^{n-2} + \frac{n\{n-1\}\{n-2\}\{n-3\}}{2.4\{2n-1\}\{2n-3\}}x^{n-4} - \cdots\right] \quad \ldots(5)$$

CaseII: When $k = -(n+1)$ then (4) becomes

$$a_l = \frac{\{n+l-1\}\{n+l\}}{\{l\}\{2n+l+1\}}a_{l-2}$$

Substituting $= 2,4,6$ .... , we obtain

$$a_2 = -\frac{\{n+1\}\{n+2\}}{2\{2n+3\}}a_0,\ \ a_4 = -\frac{\{n+2\}\{n+3\}}{4\{2n+5\}}a_2$$
$$= \frac{\{n+1\}\{n+2\}\{n+3\}\{n+4\}}{2.4\{2n+3\}\{2n+5\}}a_0 \quad \ldots\ldots$$

From (2)

$$y = a_0 x^{-n-1} + a_1 x^{-n-2} + a_2 x^{-n-3} + \cdots\cdots\cdots\cdots$$

$$y = a_0\left[x^{-n-1} - \frac{\{n+1\}\{n+2\}}{2\{2n+3\}}x^{-n-3} + \frac{\{n+1\}\{n+2\}\{n+3\}\{n+4\}}{2.4\{2n+3\}\{2n+5\}}x^{-n-5} - \cdots\right] \quad \ldots(6)$$

If we take$a_0 = \frac{1.3.5\ldots(2n+1)}{n!}$, then the solution (5) is denoted by $P_n(x)$ and is called **Legendre polynomial of first kind or Legendre polynomial of degree** n.If we take $a_0 = \frac{n!}{1.3.5\ldots(2n+1)}$ then the solution (6) is denoted by $Q_n(x)$ and is called **Legendre polynomial of second kind.**

Hence the general solutions of (1) is

$$y = AP_n(x) + BQ_n(x)$$

Where A and B are constants.

Definition: Legendre polynomial of first kind or Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1.3.5\ldots(2n+1)}{n!}\left[x^n - \frac{n\{n-1\}}{2\{2n-1\}}x^{n-2} + \frac{n\{n-1\}\{n-2\}\{n-3\}}{2.4\{2n-1\}\{2n-3\}}x^{n-4} - \cdots\right] \quad \ldots(1)$$

$$= \sum_{l=0}^{n/2}(-1)^l\frac{(2n-2l)}{2^n l!\,(n-l)!\,(n-2l)!}x^{n-2l}$$

Where

$$\left[\frac{n}{2}\right] = \begin{cases}\frac{n}{2}, & if\ n\ is\ even\\ \frac{(n-1)}{2}, & if\ n\ is\ odd\end{cases}$$

Definition: Legendre polynomial of second kind is denoted and defined by

$$Q_n(x) = \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1} - \frac{\{n+1\}\{n+2\}}{2\{2n+3\}}x^{-n-3} + \frac{\{n+1\}\{n+2\}\{n+3\}\{n+4\}}{2.4\{2n+3\}\{2n+5\}}x^{-n-5} + \cdots\right]$$

Putting $n = 0,1,2,3,4 \ldots \ldots$ in (1), we obtain

$$P_0(x) = \frac{1}{0!}x^0 = 1,$$

$$P_1(x) = \frac{1}{1!}x^1 = x,$$

$$P_2(x) = \frac{1.3}{2!}\left[x^2 - \frac{2.1}{2.3}x^0\right] = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^2 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

and

$$P_5(x) = \frac{1}{8}(63x^4 - 70x^2 + 15x)$$

And so on

EXAMPLE: Express $2 - 3x + 4x^2$ in terms of Legendre polynomial.

SOLUTION: $\Rightarrow$ $1 = P_0(x), x = P_1(x), (3x^2 - 1)/2 = P_2(x)$

$\Rightarrow$ Now $2 - 3x + 4x^2 = 2P_0(x) - 3P_1(x) + (4/3)[2P_2(x) + 1]$, by (1)

$= 2P_0(x) - 3P_1(x) + (8/3)P_2(x) + (4/3)P_0(x)$

$= (10/3)P_0(x) - 3P_1(x) + (8/3)P_0(x)$.

12.8 *GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS:-*

THEOREM: To show that $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x), |x| \leq 1, |z| \leq 1$

OR

To show that $P_n(x)$ is the coefficient of z^n in the expansion of $(1 - 2xz + z^2)^{-1/2}$ in ascending powers of z.

Note: $\boldsymbol{(1 - 2xz + z^2)^{-1/2}}$ ***is called the generating function of Legendre polynomial*** $\boldsymbol{P_n(x)}$.

PROOF: Since $|x| \leq 1, |z| \leq 1$, we obtain

$\Rightarrow$ $(1 - 2xz + z^2)^{-1/2} = [1 - h(2x - z)]^{-1/2}$

$$\Rightarrow \ = 1 + \frac{1}{2}h(2x - z) + \frac{1.3}{2.4}z^2(2x - z)^2 + \cdots\cdots + \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1} + \frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}z^n(2x-z)^n + \cdots \qquad (1)$$

Now the coefficient of z^n in

$$\Rightarrow \quad \frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}z^n(2x-z)^n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}(2x)^n = \frac{1.3.5\ldots(2n-1)}{n!}(x)^n \quad \ldots(2)$$

Again the coefficient of z^n in

$$\Rightarrow \quad \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1} = \frac{1.3\ldots(2n-3)}{2^{n-1}.1.2.4\ldots(n-1)}z^{n-1}[-(n-1)2^{n-2}x^{n-2}]$$

$$\Rightarrow = -\frac{1.3\ldots(2n-3)}{n!} \times \frac{n(n-1)}{2(2n-1)}x^{n-2} \qquad (3)$$

And so on. Using (2), (3),…., we see the coefficient of z^n in expansion of $(1-2xz+z^2)^{-1/2}$, form (1) is obtained by

$$\frac{1.3.5\ldots(2n-1)}{n!}(x)^n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right] = P_n(x)$$

Thus we can say that $P_1(x), P_2(x), \ldots\ldots\ldots$. Will be coefficients of z, z^2, ... in the expansion of$(1-2xz+z^2)^{-1/2}$. Hence we obtain

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \cdots + z^nP_n(x)$$

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^nP_n(x)$$

SOLVED EXAMPLES

EXAMPLE1: Prove that: $1 + \frac{1}{2}P_1(cos\theta) + \frac{1}{3}P_2(cos\theta) + \cdots = \log\left[\left(1 + sin\frac{\theta}{2}\right)/sin\frac{\theta}{2}\right]$

SOLUTION: From generating function, we obtain

$$\sum_{n=0}^{\infty} z^nP_n(x) = (1-2xz+z^2)^{-1/2} \qquad \ldots(1)$$

Integrating (1) w.r.t. z from 0 to 1, we have

$$\sum_{n=0}^{\infty}\int_0^1 z^n P_n(x)\,dz = \int_0^1 \frac{dz}{\sqrt{(1-2xz+z^2)}} \qquad \ldots(2)$$

Now replacing x by $cos\theta$ on both sides, (2) obtain

$$\sum_{n=0}^{\infty} P_n(cos\theta)\int_0^1 z^n\,dz = \int_0^1 \frac{dz}{\sqrt{(1-2cos\theta z+z^2)}}$$

$$\sum_{n=0}^{\infty} P_n(cos\theta)\left[\frac{z^{n+1}}{n+1}\right]_0^1 = \int_0^1 \frac{dz}{\sqrt{(z-cos\theta)^2+sin^2\theta}}$$

$$\sum_{n=0}^{\infty}\frac{P_n(cos\theta)}{n+1} = \left[\log(z-cos\theta)+\sqrt{(z-cos\theta)^2+sin^2\theta}\right]_0^1$$

$$= \log\left\{(1-cos\theta)+\sqrt{(1-cos\theta)^2+sin^2\theta}\right\} - \log(1-cos\theta)$$

$$= \log\frac{(1-cos\theta)+\sqrt{2}\sqrt{(1-cos\theta)}}{1-cos\theta}$$

$$= \log\frac{\sqrt{(1-cos\theta)}\sqrt{(1-cos\theta)}+\sqrt{2}\sqrt{(1-cos\theta)}}{\sqrt{(1-cos\theta)}\sqrt{(1-cos\theta)}}$$

$$= \log\frac{\sqrt{(1-cos\theta)}+\sqrt{2}}{\sqrt{(1-cos\theta)}}$$

$$= \log\frac{\sqrt{\left(2sin^2\frac{1}{2}\theta\right)}+\sqrt{2}}{\sqrt{\left(2sin^2\frac{1}{2}\theta\right)}} = \log\frac{1+sin\frac{1}{2}\theta}{sin\frac{1}{2}\theta}$$

$$= \frac{P_0(cos\theta)}{1}+\frac{1}{2}P_1(cos\theta)+\frac{1}{3}P_2(cos\theta)+\ldots = \log\frac{1+sin\frac{1}{2}\theta}{sin\frac{1}{2}\theta}$$

$$or \quad 1+\frac{1}{2}P_1(cos\theta)+\frac{1}{3}P_2(cos\theta)+\ldots$$

$$= \log\frac{1+sin\frac{1}{2}\theta}{sin\frac{1}{2}\theta}. \qquad [\because P_0(cos\theta)=1]$$

EXAMPLE2: Prove that $\frac{1+z}{z\sqrt{1-2xz+z^2}}-\frac{1}{z}=\sum_{n=0}^{\infty}(P_n+P_{n+1})z^n$.

SOLUTION: we have

$$\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)^{-1/2} \qquad \ldots(1)$$

$\therefore$ We take L.H.S.

$$= (1/z)(1-2xz+z^2)^{-1/2}+(1-2xz+z^2)^{-1/2}-(1/z)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z}, \qquad by\,(1) \qquad \ldots(2)$$

But

$$\sum_{n=0}^{\infty} z^n P_n = P_0 + zP_1 + z^2P_2 + \cdots + z^2P_n + z^{n+1}P_{n+1}$$

$$= 1 + z(P_1 + zP_2 + \cdots + z^n P_{n+1} + \cdots) \qquad (\because P_0 = 1)$$

$$= 1 + \sum_{n=0}^{\infty} z^n P_{n+1}$$

Using above equation in (2), we obtain

$$= \frac{1}{z}\left[1 + \sum_{n=0}^{\infty} z^n P_{n+1}\right] + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z} = \sum_{n=0}^{\infty} z^n P_{n+1} + 1 + \sum_{n=0}^{\infty} z^n P_n$$

$$= \sum_{n=0}^{\infty} z^n (P_n + P_{n+1}) = R.H.S.$$

EXAMPLE3: Prove that

$$\frac{1 - z^2}{(1 - 2xz + z^2)^{-1/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n$$

SOLUTION: We have

$$\sum_{n=0}^{\infty} z^n P_n = (1 - 2xz + z^2)^{-1/2} \qquad \ldots (1)$$

Differentiating w.r.t. z we obtain

$$-\frac{\mathbf{1}}{\mathbf{2}}(1 - 2xz + z^2)^{-3/2}(-2x + 2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n$$

$$(1 - 2xz + z^2)^{-3/2}(-x + z) = \sum_{n=0}^{\infty} n z^{n-1} P_n \qquad \ldots (2)$$

Multiplying both sides of (1) by $2z$, we have

$$2z(1 - 2xz + z^2)^{-3/2}(-x + z) = 2\sum_{n=0}^{\infty} n z^n P_n$$

Adding (1) and (3), we get

$$\frac{1}{(1 - 2xz + z^2)^{1/2}} + \frac{2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} 2n z^n P_n$$

$$\frac{1 - 2xz + z^2 + 2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n.$$

12.9 *ORTHOGONAL PROPERTIES OF LEGENDRE POLYNOMIALS:-*

Prove that

i. $\int_{-1}^{1} P_m(x)\, P_n(x) dx = 0 \quad if\ m \neq n.$

ii. $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1} \quad or \quad \int_{-1}^{1} P_m(x)\, P_n(x) dx = \begin{cases} 0 \ \ if\ m \neq n \\ \frac{2}{(2n+1)} if\ m = n \end{cases} \quad or\ \int_{-1}^{1} P_m(x)\, P_n(x) dx = \frac{2}{(2n+1)} \delta_{mn},$

$where\ \delta_{mn} = \begin{cases} 0\ \ if\ m \neq n \\ 1\ \ if\ m = n \end{cases}$

Where δ_{mn} is called Kronecker delta.

Proof:

i. When $m \neq n$.

The Legendre equation is

$$(1 - x^2) P''_m - 2x P'_m + m(m+1) P_m = 0 \qquad \ldots (1)$$

And

$$(1 - x^2) P''_n - 2x P'_n + n(n+1) P_n = 0 \qquad \ldots . (2)$$

Multiplying (1) by P_n and (2) by P_m and the subtracting

$$(1 - x^2)(P_n P''_m - P_m P''_n) - 2x(P_n P'_m - P_m P'_n) + [m(m+1) - n(n+1)] P_m P_n = 0$$

$$(1 - x^2) \frac{d}{dx} (P_n P'_m - P_m P'_n) - 2x(P_n P'_m - P_m P'_n) = (n^2 - m^2 + n - m) P_m P_n$$

$$\frac{d}{dx} (P_n P'_m - P_m P'_n)(1 - x^2) = (n - m)(n + m + 1) P_m P_n$$

Integrating both sides w.r.t. x from$(-1\ to\ 1)$, we obtain

$$(n - m)(n + m + 1) \int_{-1}^{1} P_m(x)\, P_n(x) = [(1 - x^2)(P_n P'_m - P_m P'_n)]_{x=-1}^{x=1}$$

$$\int_{-1}^{1} P_m(x)\, P_n(x) = 0, \quad as\ m \neq n \qquad \ldots (3)$$

ii. When $m = n$, we take the form

$$\int_{-1}^{1} [P_n(x)]^2\, dx = 2/(2n+1)$$

From the generating function

$$\sum_{n=0}^{\infty} z^n P_n = (1 - 2xz + z^2)^{-1/2} \qquad \ldots (4)$$

Also

$$\sum_{n=0}^{\infty} z^m P_m = (1 - 2xz + z^2)^{-1/2} \qquad \ldots (5)$$

Multiplying the corresponding sides of (4) and (5), we obtain

$$(1 - 2xz + z^2)^{-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_m(x) P_n(x) z^{m+n}$$

Integrating both sides of above equation w.r.t. x, we have

$$\int_{-1}^{1}(1-2xz+z^2)^{-1}\,dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_m(x)P_n(x)\,dx\right\}z^{m+n}$$

Use of (3), in above equation reduces to

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}[P_n(x)]^2\,dx\right]z^{2n} = \int_{-1}^{1}\frac{dx}{1+z^2-2xz}$$

$$=\left[\frac{\log(1+z^2-2xz)}{-2z}\right]_{-1}^{1} = -\frac{1}{2z}[\log(1-z)^2-\log(1+z)^2]$$

$$=-\frac{1}{2z}[2\log(1-z)-2\log(1+z)] = -\frac{1}{z}[\log(1-z)-\log(1+z)]$$

$$=-\frac{1}{z}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots\cdots\right)-\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots\cdots\right)\right]$$

$$=\frac{2}{z}\left(z-\frac{z^3}{3}+\frac{z^5}{5}-\cdots\cdots\right)=\frac{2}{z}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}$$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}[P_n(x)]^2\,dx\right]=\sum_{n=0}^{\infty}\frac{2}{2n+1}z^{2n}$$

Equating coefficients of z^{2n} from both sides, (7) gives

$$\int_{-1}^{1}[P_n(x)]^2\,dx = 2/(2n+1).$$

12.10 *RECURRENCE RELATIONS FOR LEGENDRE POLYNOMIALS:-*

i. $$\boldsymbol{nP_n=(2n-1)xP_{n-1}-(n-1)P_{n-2},\ \ n\geq 2}$$

or

$$\boldsymbol{(n+1)P_{n+1}=(2n+1)xP_x-nP_{n-1}, n\geq 1}$$

Proof: We know that the generating function, we get

$$\sum_{n=0}^{\infty}z^nP_n=(1-2xz+z^2)^{-1/2} \qquad \ldots(1)$$

Differentiating both sides of (1) w.r.t.z, we obtain

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z)=\sum_{n=0}^{\infty}n\,z^{n-1}P_n$$

$$(1-2xz+z^2)^{-3/2}(x-z)=\sum_{n=0}^{\infty}n\,z^{n-1}P_n$$

Multiplying both sides by $1-2xz+z^2$, (2) gives

$$(1-2xz+z^2)^{-1/2}(x-z) = (1-2xz+z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n$$

$$(x-z)\sum_{n=0}^{\infty} z^n P_n = (1-2xz+z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n, \quad by\ (1)$$

$$\left(x\sum_{n=0}^{\infty} z^n P_n - z\sum_{n=0}^{\infty} z^n P_n\,(x)\right)$$
$$= \left(\sum_{n=0}^{\infty} n\,z^{n-1}P_n - 2x\sum_{n=0}^{\infty} n\,z^n P_n + \sum_{n=0}^{\infty} n\,z^{n+1}P_n\right)$$

Equating coefficients of z^n from both sides, we obtain

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1}$$
$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \qquad \dots(3)$$
$$xP_n = \frac{n+1}{2n+1}P_{n+1} + \frac{n}{2n+1}P_{n-1}$$

Replacing n by $n-1$ in (3), we obtain

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$$

ii. $\boldsymbol{nP_n = xP'_n - P'_{n-1}}$

Proof: From the generating function

$$\sum_{n=0}^{\infty} z^n P_n = (1-2xz+z^2)^{-1/2} \qquad \dots(1)$$

Differentiating both sides of (1) w.r.t. z, we have

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n$$

$$(1-2xz+z^2)^{-3/2}(-x+z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n$$

Again , differentiating both sides of (1) w.r.t. x, we get

$$(1-2xz+z^2)^{-3/2}(z) = \sum_{n=0}^{\infty} z^n\,P'_n$$

$$z(1-2xz+z^2)^{-3/2}(-x+z) = (-x+z)\sum_{n=0}^{\infty} z^n\,P'_n$$

$$\sum_{n=0}^{\infty} n\,z^{n-1}P_n = (-x+z)\sum_{n=0}^{\infty} z^n\,P'_n, \qquad by\ (2)$$

$$\sum_{n=0}^{\infty} n\,z^{n-1}P_n = \left(-x\sum_{n=0}^{\infty} z^n\,P'_n + z\sum_{n=0}^{\infty} z^n\,P'_n\right)$$

Equating coefficient of z^n on both sides, we obtain

$$\boldsymbol{nP_n = xP'_n - P'_{n-1}}$$

iii. $\boldsymbol{(2n+1)P_n = P'_n - P'_{n-1}}$

Proof: From recurrence relation (i), we obtain

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Differentiating w.r.t. x, we get

$$(2n+1)xP'_n + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$$

$$or$$

$$(2n+1)(nP_n + xP'_{n-1}) + (2n+1)P_n = (n+1)P'_{n+1} - nP'_{n-1}$$

[$\because$ from recurrence(ii), $xP'_n = nP_n + P'_{n-1}$]

$$(2n+1)P_n(n+1) = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad \dots (1)$$

Replacing n by $n-1$ in (1), we obtain

$$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

$$\frac{dP_n(x)}{dx} = \frac{dP_{n-2}(x)}{dx} + (2n+1)P_{n-1}(x) \qquad \dots (2)$$

The equation (1) and (2) are the required form of the results.

iv. $\boldsymbol{(n+1)P_n = P'_{n+1} - xP'_n}$ ***or*** $\boldsymbol{P'_n - x\,P'_{n-1} = nP_{n-1}}$

Proof: From recurrence relation (ii) and (iii), we obtain

$$nP_n = xP'_n - P'_{n-1}$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1}$$

Subtracting above equations, we have

$$(2n+1)P_n - nP_n = P'_{n+1} - P'_{n-1} - xP'_n + P'_{n-1}$$

$$2nP_n + P_n - nP_n = P'_{n+1} - P'_{n-1} - xP'_n + P'_{n-1}$$

$$\boldsymbol{(n+1)P_n = P'_{n+1} - xP'_n}$$

v. $(1-x^2)P'_n = n(P_{n-1} - xP_n)$ or $(1-x^2)P'_n = = nxP_n - nP_{n-1}$

Proof: From recurrence relation (ii) and (iv), we obtain

$$nP_n = xP'_n - P'_{n-1}$$

$$(n+1)P_n = P'_{n+1} - xP'_n$$

Replacing n by $n-1$ in above equation, $nP_{n-1} = P'_n - xP'_{n-1}$ ... (1)

Multiplying both sides by , $xnP_{n-1} = x^2P'_n - xP'_{n-1}$ ... (2)

Subtracting (2) from (1), we get

$$\boldsymbol{n(P_{n-1} - xP_n) = (1-x^2)P'_n \quad or \quad (1-x^2)P'_n = nxP_n - nP_{n-1}}$$

vi. $\boldsymbol{(1-x^2)P'_n = (n+1)(xP_n - P_{n+1})}$

Proof: From recurrence relation (ii) and (v), we obtain

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \Rightarrow [(n+1)+n]\,xP_n$$

$$= (n+1)P_{n+1} + nP_{n-1}$$

And $$(1-x^2)P'_n = nP_{n-1} - nxP_n$$

Now from above equation

$$(1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$$

12.11 *BELTRAMI'S RESULT:-*

Prove that

$$\boldsymbol{(2n+1)(x^2-1)P'_n = n(n+1)(P_{n+1}-P_{n-1})}$$

Proof:

$$(x^2-1)P'_n = n(P_{n-1} - xP_n) \qquad \dots(1)$$
$$(x^2-1)P'_n = (n+1)(P_{n-1} - xP_n) \qquad \dots(2)$$

Multiplying (1) n by $n+1$ and (2) by n and adding, we obtain

$$(n+1)(1-x^2)P'_n + n(1-x^2)P'_n = (n(n+1)P_{n-1} - n(n+1)P_n)$$
$$(2n+1)(1-x^2)P'_n = n(n+1)(P_{n-1} - P_{n+1})$$

$(2n+1)(x^2-1)P'_n = n(n+1)(P_{n+1} - P_{n-1})$ is required solution.

12.12 *CHRISTOFFEL'S SUMMATION FORMULA:-*

Prove that

$$\boldsymbol{\sum_{k=0}^{m}(2k+1)P_k(x)\,P_k(y) = \frac{m+1}{x-y}[P_{m+1}(x)P_1(y) - P_m(x)P_{m+1}(y)]}$$

Deduce that

$$\sum_{k=0}^{m}(2k+1)P_k(x) = \frac{m+1}{x-y}[P_{m+1}(x) - P_m(x)]$$

Proof: From recurrence relation I, we get

$$(2k+1)xP_k(x) = (k+1)P_{k+1}(x) + kP_{k-1}(x) \qquad \dots(1)$$

And

$$(2k+1)yP_k(y) = (k+1)P_{k+1}(y) + kP_{k-1}(y) \qquad \dots(2)$$

Multiplying (1) by $P_k(y)$ and (2) by $P_k(x)$ and then subtracting, we have

$$(2k+1)(x-y)P_k(x)P_k(y)$$
$$= (k+1)[P_{k+1}(x)P_k(y) - P_{k+1}(y)P_k(x)]$$
$$- k[P_{k-1}(x)P_k(y) - P_{k-1}(y)P_k(x)]$$

Replacing k by $0, \dots\dots\dots .2,3, \dots\dots..m-1, m$ successfully in (3) and adding equation, we obtain

$$(x-y)\sum_{k=0}^{m}(2k+1)\,P_k(x)P_k(y)$$
$$= (m+1)[P_{m+1}(x)P_m(y) - P_{m-1}(y)P_m(x)]$$

$$\sum_{k=0}^{m}(2k+1)\,P_k(x)P_k(y) = \frac{m+1}{x-y}[P_{m+1}(x)P_1(y) - P_m(x)P_{m+1}(y)]$$

To show that

$P'_n = (2n-1)P_{n-1} - (2n-1)P_{n-3} + (2n-9)P_{n-5} + \cdots\cdots\cdots$, the last terms of the series being $3P_1$ or P_0 according as n is even or odd.

$$\boldsymbol{P'_n(x) = \sum_{r=0}^{\left[\frac{1}{2}(n-1)\right]} (2n-4r-1)P_{n-2r-1}(x),}$$

Where

$$\left[\frac{1}{2}(n-1)\right] = \begin{cases} (n-1)/2\,, if\ n\ is\ even \\ (n-2)/2\,, if\ n\ is\ odd \end{cases}$$

Proof: Replacing n by $n-1$ in recurrence relation III, we obtain

$$P'_n = (2n-1)P_{n-1} + P'_{n-2} \qquad \dots (1)$$

CaseI: Let n be even, Replacing n by $n, n-2, n-4, \dots 4, 2$ successively in (1) and using $P_0 = 1$ and $P'_0 = 0$, we have

$$P'_n = (2n-1)P_{n-1} + P'_{n-2}$$
$$P'_{n-2} = (2n-5)P_{n-3} + P'_{n-4}$$
$$P'_{n-1} = (2n-9)P_{n-5} + P'_{n-6}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$P'_4 = 7P_3 + P'_2$$
$$P'_2 = 3P_1 + P'_0$$

Adding these and simplifying, we have

$$P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + \dots\dots\dots + 3P_1 \qquad \dots (2)$$

CaseII: Let n is odd. Replacing n by $n, n-2, n-4, \dots 5, 3$ successively in (1) and using $P_1(x) = x$, and $P_0 = 1$ so that $P_0 = 1 = P'_1$, we have

$$P'_n = (2n-1)P_{n-1} + P'_{n-2}$$
$$P'_{n-2} = (2n-5)P_{n-3} + P'_{n-4}$$
$$P'_{n-1} = (2n-9)P_{n-5} + P'_{n-6}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$P'_4 = 9P_3 + P'_3$$
$$P'_2 = 5P_2 + P'_1 = 5P_2 + P_0$$

Adding these and simplifying, we have

$$P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + \dots\dots\dots + 5P_1 + P_0 \qquad \dots (4)$$

Comparing (2) and (3), we get

$$P'_n(x) = \sum_{r=0}^{\left[\frac{1}{2}(n-1)\right]} (2n-4r-1)P_{n-2r-1}(x)$$

12.13 *RODRIGUE'S FORMULA:-*

To show that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}$$

Proof: By the definition of Legendre polynomials, we obtain

$$P_n(x) = \sum_{n=0}^{[n/2]} (-1)^r \frac{(2n-2r)!\,x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!} \qquad \dots (1)$$

Where $$[n/2] = \begin{cases} \frac{n}{2}, if\ n\ is\ even \\ \frac{(n-1)}{2}, if\ n\ is\ odd \end{cases}$$

Now, binomial theorem, we have

$$(x^2-1)^n = \sum_{r=0}^{n} n_{C_r}(x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} n_{C_r} x^{2n-2r}(-1)^r$$

$\therefore$ $$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{r=0}^{n} n_{C_r}(-1)^r \frac{d^n}{dx^n}x^{2n-2r}$$

But $\frac{d^n}{dx^n}x^m = 0$ if $m < n$; $\frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n}$, if $m \geq n$

$\therefore$ $\frac{d^n}{dx^n}x^{2n-2r} = 0$, if $2n-2r < n\ i.e. r > \frac{n}{2}$.

Making use of above equation, we see that we must replace $\sum_{r=0}^{n} by\ \sum_{r=0}^{n/2} if\ \ n$ is even and by $\sum_{r=0}^{(n-1)/2} if\ \ n$ is odd. Hence

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{r=0}^{n/2} n_{C_r}(-1)^r \frac{d^n}{dx^n}x^{2n-2r}$$

$$= \frac{1}{2^n n!}\sum_{r=0}^{n/2} n_{C_r}(-1)^r \frac{(2n-2r)!}{(2n-2r-n)!}x^{2n-2r-n}$$

$$= \sum_{r=0}^{n/2}(-1)^r \frac{n!}{r!\,(n-r)!}\frac{1}{2^n n!}\frac{(2n-2r)!}{(n-2r)!}x^{2-2r} = P_n(x)$$

12.14 *LAPLACE'S DEFINITE INTEGRALS FOR $P_n(x)$:-*

(I) Laplace's first integral $P_n(x)$. when is $+ve$ integer, then

$$\boldsymbol{P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}cos\phi\right]^n d\phi}$$

Proof: We Know that

$$\int_0^{\pi}\frac{d\phi}{a \pm bcos\phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}\ when\ a^2 > b^2. \quad ...(1)$$

Let $a = 1 - zx$ and $b = z\sqrt{x^2-1}$

$\therefore$ $a^2 - b^2 = (1-zx)^2 - z^2(x^2-1) = 1 - 2zx + z^2$

Using these values of a, b and $a^2 - b^2$,in (1)

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[1 - zx \pm z\sqrt{(x^2-1)}cos\phi\right]^{-1} d\phi$$

$$\pi\sum_{n=0}^{\infty} z^n P_n(x) = \int_0^{\pi}(1-zt)^{-1}\,d\phi\ \ if\ \ t = x \pm \sqrt{(x^2-1)}\ cos\phi$$

$$= \int_0^{\pi} (1 + zt + z^2t^2 + \cdots\cdots\cdots)\, d\phi = \int_0^{\pi} \sum_{n=0}^{\infty} (zt)^n\, d\phi$$

$$= \sum_{n=0}^{\infty} z^n \int_0^{\pi} t^n d\phi$$

∴

$$\pi \sum_{n=0}^{\infty} z^n P_n(x) = \sum_{n=0}^{\infty} z^n \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)}cos\phi\right]^n d\phi \quad \ldots(2)$$

From (2),[Equating coefficient of z^n]

$$\pi P_n(x) = \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)}cos\phi\right]^n d\phi$$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[x \pm \sqrt{(x^2-1)}cos\phi\right]^n d\phi \quad \ldots(3)$$

Deductions: Prove that

i. $P_n(cos\theta) = \frac{1}{\pi}\int_0^{\pi} [cos\theta \pm isin\theta cos\phi]^n d\phi$

ii. $P_1(x) = \frac{1}{\pi}\int_0^{\pi} \left[x \pm \sqrt{(x^2-1)}cos\theta\right]^n d\phi$

Solution:

i. Suppose $x = cos\theta$,then we obtain

$$\sqrt{(x^2-1)} = \sqrt{cos^2\theta - 1} = \sqrt{\{(-1)(1-cos^2\theta)\}} = \sqrt{\{i^2 sin^2\theta\}}$$
$$= isin\theta$$

From (3), we obtain

$$P_n(cos\theta) = \frac{1}{\pi}\int_0^{\pi} [cos\theta \pm isin\theta cos\phi]^n d\phi$$

ii. Let $n = 1$ in (3)

$$P_1(x) = \frac{1}{\pi}\int_0^{\pi} \left[x \pm \sqrt{(x^2-1)}cos\theta\right]^n d\phi$$

(II) Laplace's second integral $P_n(x)$. when is $+ve$ integer, then

$$\boldsymbol{P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{1}{\left[x \pm \sqrt{(x^2-1)}cos\phi\right]^{n+1}} d\phi}$$

Proof: From integral calulus,we obtain

$$\int_0^{\pi} \frac{d\phi}{a \pm bcos\phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}, where\ a^2 > b^2 \quad \ldots(1)$$

Suppose $a = 1 - zx$ and $b = z\sqrt{x^2-1}$

∴ $a^2 - b^2 = (1-zx)^2 - z^2(x^2-1) = 1 - 2zx + z^2$

Using these values of a, b and $a^2 - b^2$,in (1)

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[1-zx \pm z\sqrt{(x^2-1)}cos\phi\right]^{-1} d\phi$$

$$\frac{\pi}{z}\left(1-2x\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \int_0^{\pi}\left[1-z\left\{x \pm \sqrt{(x^2-1)}cos\phi\right\}\right]^{-1} d\phi \quad ...(2)$$

Let $t = x \pm \sqrt{(x^2-1)}cos\phi$ ...(3)

We know that $(1-2zx+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ ...(4)

Replacing z by $1/z$ in (4), we have

$$\left(1-2x\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x) \quad ...(5)$$

Now using (3) and (5), we get

$$\frac{\pi}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x) = \int_0^{\pi}(-1+zt)^{-1}\,d\phi = \int_0^{\pi}(zt)^{-1}\left(1-\frac{1}{zt}\right)^{-1} d\phi$$

$$\int_0^{\pi}\frac{1}{zt}\sum_{n=0}^{\infty}\left(\frac{1}{zt}\right)^n d\phi = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{t^{n+1}}$$

$$\therefore \quad \sum_{n=0}^{\infty}\frac{\pi}{z^{n+1}}P_n(x) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{(x^2-1)}cos\phi\right]^{n+1}} \quad ...(6)$$

Equating the coefficients of $1/z^{n+1}$ from sides, (6) obtain

$$\pi P_n(x) = \int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{(x^2-1)}cos\phi\right]^{n+1}}$$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{(x^2-1)}cos\phi\right]^{n+1}} \quad ...(7)$$

Deductions: Replacing n by $-(n+1)$ in (7), we get

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{(x^2-1)}cos\phi\right]^{-n}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[x\pm\sqrt{(x^2-1)}cos\theta\right]^n d\phi = P_n(x)$$

Thus

$$P_n(x) = P_{-(n+1)}(x)$$

SOLVED EXAMPLES

EXAMPLE1: Prove that $P'_{n+1} + P'_n = P_0 + 3P_1 + \cdots + (2n+1)P_n$.

or

$$\sum_{r=0}^{n}(2r+1)\,P_r(x) = P'_{n+1}(x) + P'_n(x)$$

SOLUTION: From recurrence relation III, we obtain

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad \dots(1)$$

Replacing n by $1,2,\dots n-1, n$ successively in (1), we have

$$3P_1 = P'_2 - P'_0$$
$$5P_2 = P'_3 - P'_1$$
$$7P_3 = P'_4 - P'_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1}$$

Adding these terms

$$3P_1 + 5P_2 + 7P_3 + \cdots\cdots + (2n+1)P_n$$
$$= -P'_0 - P'_1 + P'_n + P'_{n+1} \quad \dots.(2)$$

Since

$P_0 = 1$ and $P_1 = x$, we have $P'_0 = 0$ and $P'_1 = 0 = P_0$.

From (2), we obtain

$$3P_1 + 5P_2 + 7P_3 + \cdots\cdots + (2n+1)P_n = -0 - P_0 + P'_n + P'_{n+1}$$
$$or$$
$$3P_1 + 5P_2 + 7P_3 + \cdots\cdots + (2n+1)P_n = P'_n + P'_{n+1}$$
$$or$$
$$\sum_{r=0}^{n}(2r+1)P_r(x) = P'_n(x) + P'_{n+1}(x)$$

EXAMPLE2: Prove that

i. $c + \int P_n\,dx = (P_{n+1} - P_{n-1})/(2n+1)$

ii. $\int_x^1 P_n\,dx = (P_{n+1} - P_{n-1})(2n+1)$

Proof: From recurrence relation III, we obtain

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \;\; or \;\; P_n = \frac{1}{2n+1}\frac{d}{dx}(P_{n+1} - P_{n-1}) \quad \dots(1)$$

i. Now integrating above equation

$$\int P_n\,dx + c = (P_{n+1} - P_{n-1})/(2n+1)$$

ii. Integrating both sides of (1) w.r.t.x between limits $x\ to\ 1$, we get

$$\int_x^1 P_n\,dx = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]_x^1$$
$$= \frac{1}{2n+1}[P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(x) + P_{n-1}(x)]$$

$[P_{n-1}(x) - P_{n+1}(x)]/(2n+1)$, as $P_{n+1}(1) = P_{n-1}(1) = 1$

EXAMPLE3: Using Rodrigue's formula, find values of $P_0(x), P_1(x), P_2(x)$ and $P_3(x)$.

SOLUTION: Rodrigue's formula is given by

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad ...(1)$$

Putting $n = 0$ in (1), $P_0(x) = \frac{1}{2^0.0!}(x^2-1)^0 = 1$

Putting $n = 1$ in (1), $P_1(x) = \frac{1}{2^1.1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}2x = x$

Putting $n = 2$ in (1),

$P_2(x) = \frac{1}{2^2.2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}[2(x^2-1).2x] = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1)$

Putting $n = 3$ in (1), we get

$$P_3(x) = \frac{1}{2^2.3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^2-1)^3\right]$$

$$= \frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2.2x] = \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}x(x^2-1)^2\right]$$

$$= \frac{1}{8}\frac{d}{dx}[(x^2-1)^2 + x.2(x^2-1).2x] = \frac{1}{8}\frac{d}{dx}(5x^4-6x^2+1)$$

$$= \frac{1}{8}\frac{d}{dx}(20x^3-12x) = \frac{1}{2}(5x^3-3x)$$

EXAMPLE4: Prove that

- **i.** $\int_{-1}^{1} P_n(x)\,dx = 2 \; if \; n = 0.$
- **ii.** $\int_{-1}^{1} P_n(x)\,dx = 0 \; if \; n \geq 1.$

SOLUTION:

- **i.** When $n = 0, P_n(x) = P_0(x) = 1$

  $\therefore \quad \int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} dx = 2$

- **ii.** Using Rodrigue's formula, we obtain

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n.n!}\int_{-1}^{1} D^n\,(x^2-1)^n dx, where\; D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n.n!}[D^{n-1}(x^2-1)^n]_{-1}^{1} = \frac{1}{2^n.n!}[D^{n-1}(x^2-1)^n]_{-1}^{1}$$

$$= \frac{1}{2^n.n!}\Big[D^{n-1}(x-1)^n(x+1)^n + n-1_{C_1}D^{n-1}(x-1)^n D(x+1)^n + \cdots$$
$$+ D^{n-1}(x-1)^n(x+1)^n\Big]_{-1}^{1}$$

$$\left[D^n(uv) = D^n u.v + n_{C_1}D^{n-1}u.Dv + \cdots + u.D^n v\right]$$

$$= \frac{1}{2^n.n!}[n!\,(x-1)(x+1)^n + \cdots + n!\,(x-1)(x+1)^n]_{-1}^{1} = 0$$

$$\left[\because D^n(ax+b)^m = a^n\frac{m!}{(m-n)!}(ax+b)^{m-n}\right]$$

12.15 *RECURRENCE RELATIONS FOR* $Q_n(x)$:-

We have already defined that

$$Q_n = \frac{n!}{1.3.5 \ldots (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \cdots\cdots\right]$$

$$= \frac{2^n (n!)^2}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+1)}{2(2n+3)}x^{-(n+3)} + \cdots\cdots\right]$$

$$= \frac{2^n n!}{(2n+1)!}\left[x^{-(n+1)}n! + \frac{(n+2)!}{2(2n+3)}x^{-(n+2+1)} + \cdots\cdots\right]$$

$$= \frac{2^n n!}{(2n+1)!}\left[x^{-(n+1)}n! + \frac{(n+2)!}{2(2n+3)}x^{-(n+2+1)} + \cdots\cdots\right]$$

$$= \frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2.4 \ldots 2r(2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$\therefore \boldsymbol{Q_n = \frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}}$$

Differentiating w.r.t. x, we have

$$\boldsymbol{Q'_n = -\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}} \quad \ldots(\mathbf{1})$$

Substituting $n-1$ for n in (3), we obtain

$$Q'_{n-1}(x) = -\frac{2^{n-1}n!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r-1)}$$

$$Q'_{n-1}(x) = -\frac{2n.\,2^{n-1}n!}{(2n-1)!\,2n}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r-1)}$$

$$\boldsymbol{Q'_{n-1}(x) = -\frac{2^n.\,n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r-1)}}$$

Again, Substituting $n-1$ for n in (3), we obtain

$$Q'_{n+1}(x)$$
$$= -\frac{2^{n+1}(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r r!\,(2n+5)\ldots(2n+2r+1)(2n+2r+3)}$$

$$Q'_{n+1}(x)$$
$$= -\frac{2^{n+1}n!\,(2n+2)!}{(2n+3)(2n+2)(2n+1)(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r r!\,(2n+5)\ldots(2n+2r+3)}$$

$$\boldsymbol{Q'_{n+1}(x) = -\frac{2^n.\,n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r+3)}}$$

I. $\boldsymbol{Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n}$

Proof: Given

$$\boldsymbol{Q'_n - Q'_{n-1} = (2n+1)Q_n \Rightarrow Q'_n = Q'_{n-1} + (2n+1)Q_n}$$

We take R.H.S

$$Q'_{n-1} + (2n+1)Q_n$$

$$= -\frac{2^n.n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$+ (2n+1)\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

$$= \frac{2^n.n!}{2n!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)} - \sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r-1)}\right]$$

$$= \frac{2^n.n!}{2n!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r+1)} \times \{(2n+3)\dots(2n+2r+1)\}\right]$$

$$= \frac{2^n.n!}{2n!}\left[\sum_{r=0}^{\infty}\frac{-2r(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r+1)}\right]$$

$$= -\frac{2^n.n!}{2n!}\left[0 + \sum_{r=1}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)(2n+3)\dots(2n+2r+1)}\right]$$

$$= -\frac{2^n.n!}{2n!}\left[\sum_{s=0}^{\infty}\frac{(n-2s+2)!\,x^{-(n+2s+3)}}{2^s(s)!\,(2n+1)(2n+3)\dots(2n+2s+3)}\right]$$

$$[putting\ r = s+1\ so\ that\ s = r-1]$$

$$= \boldsymbol{Q'_{n+1}(x) = L.H.S}$$

II. $\boldsymbol{Q'_{n+1} + (n+1)Q'_{n-1} = (2n+1)xQ'_n}$

Proof: Given

$$\boldsymbol{Q'_{n+1} + (n+1)Q'_{n-1} = (2n+1)Q'_n}$$

$$\Rightarrow \quad \boldsymbol{Q'_{n+1} = (2n+1)Q'_n - (n+1)Q'_{n-1}}$$

We take R.H.S

$$(2n+1)Q'_n - (n+1)Q'_{n-1}$$

$$= -(2n+1)\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

$$-(n+1)(-)\frac{2^n.n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$= -\frac{2^n n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}(2n+1)}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$+\frac{2^n.n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}(n+1)(2n+2r+1)}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$= -\frac{2^n n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$\times[(2n+1)(2n+2r+1)-(n+1)(2n+2r+1)]$$

$$= -\frac{n.2^n n!}{(2n)!}\left[\left\{\sum_{r=1}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)(2n+3)\dots(2n+2r+1)}\right\}+0\right]$$

$$= -\frac{n.2^n n!}{(2n)!}\left\{\sum_{s=0}^{\infty}\frac{(n+2s+2)!\,x^{-(n+2r+3)}}{2^s s!\,(2n+1)(2n+3)\dots(2n+2s+3)}\right\}$$

$$[putting\ r=s+1\ so\ that\ s=r-1]$$

$$= \boldsymbol{nQ'_{n+1}}$$

III. $\boldsymbol{(2n+1)xQ_n = (n+1)Q_{n+1} + nQ_{n-1}}$ ***or*** $\boldsymbol{xQ_n = \frac{n+1}{2n+1}Q_{n+1} + \frac{n}{2n+1}Q_{n-1}}$ ***or*** $\boldsymbol{(n+1)Q_{n+1} - (2n+1)xQ_n + nQ_{n-1} = 0}$

Proof: We have $nQ_{n-1}-(2n+1)xQ_n$

$$= n.\frac{2^{n-1}(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r-1)}$$

$$-(2n+1)x\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

$$[putting\ the\ values\ of\ Q_{n-1} and\ Q_n]$$

$$= \frac{2^{n-1}(n)!.2n}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}(2n+2r+1)}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r-1)(2n+2r+1)}$$

$$-\frac{2^n(2n+1)\,n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

$$= \frac{2^n(n)!}{(2n+1)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}\right.$$

$$\left.\times\{n(2n+2r+1)-(2n+1)(n+2r)\}\right]$$

$$= \frac{2^n(n)!}{(2n+1)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}(-2r)(n+1)}{2^r r!\,(2n+3)(2n+5)\dots(2n+2r+1)}\right.$$

$$\left.\times\{n(2n+2r+1)-(2n+1)(n+2r)\}\right]$$

$$= \frac{2^n(n+1)!}{(2n+1)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^{r-1}(r-1)!\,(2n+3)(2n+5)\dots(2n+2r+1)}\right]$$

$$= -(n+1)\frac{2^n(n)!}{(2n+1)!}\left[\sum_{r=1}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^{r-1}(r-1)!\,(2n+3)(2n+5)\dots(2n+2r+1)} + 0\right]$$

$$= -(n+1)\frac{2^n(n)!}{(2n+1)!}\left[\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s)!\,(2n+3)(2n+5)\dots(2n+2s+3)}\right],$$

$$taking\ r = s+1$$

$$= -(n+1)\frac{2^n(n)!\,(2n+2)}{(2n+2)!}\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s)!\,(2n+3)(2n+5)\dots(2n+2s+3)}$$

$$= -(n+1)\frac{2^n(n+1)!}{(2n+3)!}\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s(s)!\,(2n+5)\dots(2n+2s+3)}$$

$$= \boldsymbol{-(n+1)Q_{n+1}}$$

IV. $\boldsymbol{(2n+1)(1-x^2)Q'_n = n(n+1)(Q_{n-1} - Q'_{n+1})}$

Proof: Let Q_n is a solution of Legendre's equation, given as below

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

$$\therefore \quad \frac{d}{dx}\left[(1-x^2)Q'_n\right] = -n(n+1)Q_n \qquad \dots(1)$$

Integrating both sides of (1) between the limits$(\infty\ to\ x)$, we have

$$\left[(1-x^2)Q'_n\right]_{\infty}^{x} = -n(n+1)\int_{\infty}^{x}[Q_n]\,dx$$

$$or$$

$$(1-x^2)Q'_n(x) = -n(n+1)\int_{\infty}^{x}[Q_n]\,dx$$

$$\left[\because (Q'_n)_{x=\infty} = 0\ and\ (x^2Q'_n)_{x=\infty} = 0\right]$$

But by recurrence relation I, we have

$$Q'_{n-1} - Q'_{n+1} = (2n+1)Q_n \quad \dots (2)$$

Integrating both sides of (3) between the limits$(\infty\ to\ x)$, we have

$$[Q_{n-1} - Q_{n+1}]_{\infty}^{x} = \int_{\infty}^{x} (2n+1)Q_n dx$$

$$or$$

$$Q_{n+1}(x) - Q_{n-1}(x) = \int_{\infty}^{x}(2n+1)Q_n dx \quad \dots (3)$$

$$\left[\because \left(Q'_{n+1}\right)_{x=\infty} = 0 = (Q_{n-1})_{x=\infty}\right]$$

Hence

$$(1-x^2)Q'_n(x) = -n(n+1)\frac{Q_{n+1}(x) - Q_{n-1}(x)}{2n+1}$$

$$or$$

$$\boldsymbol{(2n+1)(1-x^2)Q'_n(x) = -n(n+1)[Q_{n+1}(x) - Q_{n-1}(x)]}$$

12.16 *CRISTOFFEL'S SECOND SUMMATION FORMULA:-*

Results. $\boldsymbol{(y-x)\sum_{r=1}^{n}(2r+1)P_r(x)Q_r(y) = 1-(n+1)[P_{n+1}(x)Q_n(y) - P_n(x)Q_{n+1}(y)]}$

Proof: From recurrence formulas for $P_n(x)$ and $Q_n(x)$, we obtain

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad \dots (1)$$

$$(2n+1)yQ_n(y) = (n+1)Q_{n+1}(y) + nQ_{n-1}(y) \quad \dots (2)$$

Multiplying (2) by $Q_n(y)$ and (3) by $P_n(x)$ and subtracting, we obtain

$$(2n+1)(x-y)P_n(x)Q_n(y) + n\{P_{n-1}(x)Q_n(y) - Q_{n-1}(y)P_n(x)\}$$
$$= (n+1)\{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)\}$$

Taking $n = 1,2,3 \dots \dots n$ in above equation and adding, we have

$$(y-x)\sum_{r=1}^{n}(2r+1)P_r(x)Q_r(y) + \{Q_1(x)P_0(y) - Q_0(y)P_1(x)\} = -(n+1)\{P_{n+1}(x)Q_n(y) - Q_{n+1}(y)P_n(x)\}$$

Since

$$Q_1(y) = y, Q_0(y) = 1, P_1(x) = x, P_0(x) = 1$$

Hence

$$\boldsymbol{(y-x)\sum_{r=1}^{n}(2r+1)P_r(x)Q_r(y)}$$
$$\boldsymbol{= 1-(n+1)[P_{n+1}(x)Q_n(y) - P_n(x)Q_{n+1}(y)]}$$

The above equation gives the required results

12.17 *A RELATION CONNECTING $P_n(x)$ AND$Q_n(x)$:-*

Prove that

$$\frac{\mathbf{1}}{\boldsymbol{y-x}} = \sum_{\boldsymbol{m=0}}^{\infty} (\mathbf{2}\boldsymbol{m}+\mathbf{1})\, \boldsymbol{P_m(x)Q_m(y)}$$

And hence deduce that

$$\boldsymbol{Q_m(y)} = \int_{-1}^{1} \frac{\boldsymbol{P_m(x)}}{\boldsymbol{y-x}}\boldsymbol{dx}, \quad (\boldsymbol{y} > 1)$$

Proof: Let $f(x) = \frac{1}{y-x}$

$$f(x) = \frac{1}{y(1-x/y)} = \frac{1}{y}\left(1-\frac{x}{y}\right)^{-1} = y^{-1}\left(1+\frac{x}{y}+\frac{x^2}{y}+\cdots+\frac{x^m}{y^m}+\cdots\right)$$

$$= y^{-1} + x.y^{-2} + x^2y^{-3} + \cdots + x^m.y^{-m-1} + \cdots \quad \ldots(1)$$

$$= A_0 + xA_1 + A_2x^2 + \cdots\cdots \quad \ldots(2)$$

Let

$$f(x) = \sum_{m=0}^{\infty} B_m P_m(x)$$

Then we know that $B_m = \frac{1.2.3.....m}{1.3.5.......(2m-1)}\left[A_m + \frac{(m+1)(m+2)}{2(2m+3)}A_{m+2} + \cdots\cdots\right]$

From (1) and (2), we obtain

$$A_0 = y^{-1}, \qquad A_1 = y^{-2}, \ldots., A_m = y^{-(m+1)}, \ldots$$

$\therefore$

$$B_m = \frac{m!}{1.3.5\ldots\ldots(2m-1)}\left[y^{-(m+1)} + \frac{(m+1)(m+2)}{2(2m+3)}.y^{-(m+3)} + \cdots\cdots\right]$$

$$= (2m+1)Q_m(y)$$

$$\frac{\mathbf{1}}{\boldsymbol{y-x}} = \sum_{\boldsymbol{m=0}}^{\infty} (\mathbf{2}\boldsymbol{m}+\mathbf{1})\, \boldsymbol{Q_m(y)P_m(x)} \quad \ldots(3)$$

Now multiplying (3) by $P_m(x)$ and integrating w.r.t. x in the interval $(-1,1)$, we have

$$\int_{-1}^{1} P_m(x).\frac{1}{y-x}dx = \int_{-1}^{1} P_m(x)\left[\sum_{m=0}^{\infty}(2m+1)\, Q_m(y)P_m(x)\right]dx$$

$$= Q_m(y)\int_{-1}^{1}[P_m(x)]^2\,(2m+1)dx$$

$$\left[\because \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, m \neq n\right]$$

$$= Q_n(y).(2m+1).\frac{2}{2m+1}$$

$\therefore \qquad \frac{\mathbf{1}}{\mathbf{2}}\int_{\mathbf{-1}}^{\mathbf{1}} P_m(x)\frac{1}{y-x}dx = Q_n(y)$

This is known as Neumann's integral for $\boldsymbol{Q_n(y)}$.

SOLVED EXAMPLES

Example1: Prove that

i. $(x^2-1)\left(Q_n P'_n - P_n Q'_n\right) = c$

ii. $\frac{Q_n}{P_n} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2}$

iii. From ii deduce that

a. $Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1}$.

b. $Q_1(x) = \frac{1}{2}\log\frac{x+1}{x-1} - 1$.

Solution:

i. Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \dots(1)$$

Since

$$(1-x^2){P_n}'' - 2x{P_n}' + n(n+1)P_n = 0 \qquad \dots(2)$$

and $$(1-x^2){Q_n}'' - 2x{Q_n}' + n(n+1)Q_n = 0 \qquad \dots(3)$$

Multiplying (2) by Q_n and (3) by P_n and then subtracting, we obtain

$$(1-x^2)({P_n}''Q_n - {Q_n}''P_n) - 2x({P_n}'Q_n - {Q_n}'P_n) = 0$$

$$or$$

$$(1-x^2)\frac{d}{dx}({P_n}'Q_n - {Q_n}'P_n) - 2x({P_n}'Q_n - {Q_n}'P_n) = 0$$

$$or$$

$$\frac{d}{dx}\{(1-x^2)({P_n}'Q_n - {Q_n}'P_n)\} = 0 \qquad \dots(4)$$

Integrating w.r.t. x, (4) gives $(1-x^2)({P_n}'Q_n - {Q_n}'P_n) = -c$

$$\boldsymbol{(x^2-1)({P_n}'Q_n - {Q_n}'P_n) = c} \qquad \dots(\mathbf{5})$$

ii. From i , we obtain

$${P_n}'Q_n - {Q_n}'P_n = \frac{c}{x^2-1} = \frac{c}{x^2}\left(1-\frac{1}{x^2}\right)^{-1}$$

$${P_n}'Q_n - {Q_n}'P_n = \frac{c}{x^2-1} = \frac{c}{x^2}\left(1+\frac{1}{x^2}+\frac{1}{x^4}+\cdots\right) \qquad \dots(6)$$

We know that

$$Q_n = \frac{n!}{1.3.5\dots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-(n+3)} + \cdots\right]$$

and

$$P_n = \frac{1.3.5\dots(2n+1)}{n!}\left[x^n - \frac{n(n-1)}{2.(2n-1)}x^{n-2} + \cdots\right]$$

Using above equation, L.H.S. of (6)

$$= \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-(n+3)} + \cdots\right] \times \frac{1.3.5\ldots(2n+1)}{n!}\left[x^n - \frac{n(n-1)}{2.(2n-1)}x^{n-2} + \cdots\right] - \frac{1.3.5\ldots(2n+1)}{n!}\left[x^n - \frac{n(n-1)}{2.(2n-1)}x^{n-2} + \cdots\right] \times \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-(n+3)} + \cdots\right]$$

Since the coefficient of $1/x^2$ in L.H.S. of (6)

$$= \frac{n!}{1.3.5\ldots(2n+1)}.\frac{1.3.5\ldots(2n+1)}{n!}n - \frac{n!}{1.3.5\ldots(2n+1)}.\frac{1.3.5\ldots(2n+1)}{n!}(-n-1)$$

$$= \frac{n}{2n+1} + \frac{n+1}{2n+1} = \frac{2n+1}{2n+1} = 1$$

Hence from (5), we obtain

$(x^2-1)(P_n{}'Q_n - Q_n{}'P_n) = 1$ *or* -$(x^2-1)(Q_n{}'P_n - P_n{}'Q_n) = 1$

$$\frac{(P_n{}'Q_n - Q_n{}'P_n)}{P_n^2} = -\frac{1}{(x^2-1)P_n^2} \quad or \quad \frac{d}{dx}\left(\frac{Q_n}{P_n}\right) = -\frac{1}{(x^2-1)P_n^2}$$

Integrating both sides w.r.t.x from(∞ to x), we have

$$\left[\frac{Q_n}{P_n}\right]_\infty^x = -\int_\infty^x \frac{dx}{(x^2-1)P_n^2} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2}$$

$$\frac{Q_n(x)}{P_n(x)} - \lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2}$$

Now,

$$\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \lim_{x\to\infty}\frac{\frac{d^n}{dx}nQ_n(x)}{\frac{d^n}{dx}nP_n(x)}$$

[By L' Hospital Rule]

$$= \lim_{x\to\infty}\frac{\frac{n!}{1.3.5\ldots(2n+1)}\{(-1)^n(n+1)(n+2)\ldots 2nx^{-(2n+1)} + \cdots\}}{\frac{1.3.5\ldots(2n+1)}{n!}n!} = 0$$

Hence

$$\boldsymbol{\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2}} \qquad \ldots(\mathbf{7})$$

iii. Deductions from (ii)

a. Replacing n by 0 in (7), we obtain

$$\frac{\boldsymbol{Q_0(x)}}{\boldsymbol{P_0(x)}} = \int_x^{\infty} \frac{\boldsymbol{dx}}{\boldsymbol{(x^2-1)P_0^2(x)}}$$

$$\boldsymbol{Q_0(x)} = \int_x^{\infty} \frac{\boldsymbol{dx}}{\boldsymbol{(x^2-1)}} \qquad (P_0(x) = 1)$$

$$= \frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_x^{\infty} = -\frac{1}{2}\log\frac{x-1}{x+1} = \frac{1}{2}\log\frac{x+1}{x-1}$$

$$\because \lim_{x\to\infty} \log\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty} \log\left(\frac{1-1/x}{1+1/x}\right) = 0$$

b. Replacing n by 1 in (7), we obtain

$$\frac{\boldsymbol{Q_1(x)}}{\boldsymbol{P_1(x)}} = \int_x^{\infty} \frac{\boldsymbol{dx}}{\boldsymbol{(x^2-1)P_1^2(x)}}$$

$$\boldsymbol{Q_1(x) = x\int_x^{\infty} \frac{dx}{(x^2-1)}} \qquad (P_1(x) = x)$$

$$= x\int_x^{\infty} \left[\frac{1}{(x^2-1)} - \frac{1}{x^2}\right] dx = x\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right]_x^{\infty}$$

$$= -x\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right]$$

$$\left[\because \log\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty} \log\left(\frac{1-1/x}{1+1/x}\right) = 0\right]$$

$$= -\frac{x}{2}\log\left(\frac{x-1}{x+1}\right) - 1 = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) - 1$$

EXAMPLE2: Prove that $Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1}x - \frac{3}{2}x$

SOLUTION: From recurrence relation III for $Q_n(x)$, we have

$$(n+1)Q_{n-1} = (2n+1)xQ_n - nQ_{n-1} \quad \dots (1)$$

Replacing n by 1 in (1), we obtain

$$2Q_2 = 3xQ_1 - Q_0$$

$$= 3x\left[\frac{x}{2}\log\frac{x+1}{x-1} - 1\right] - \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$$

$$= \frac{3x^2-1}{2}\log\frac{x+1}{x-1} - 3x$$

$$= P_2(x)\log\frac{x+1}{x-1} - 3x$$

$$\left[P_2(x) = \frac{3x^2-1}{2}\right]$$

$$Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1}x - \frac{3}{2}x$$

SELF CHECK QUESTIONS

1. Prove that $x^4 = (8/35)P_n(x) + (4/7)P_2(x) + (1/5)P_0(x)$.

2. Show that $\int_{-1}^{1} x^4 P_6(x)dx = 0$.

12.18 *SUMMARY:-*

In this unit we studied the Chebyshev polynomials, Legendre equation and its solution as Legendre function of first and second kind. We have also studied the recurrence relation, generating function, orthogonal properties of Chebyshev polynomials and Legendre polynomials, Rodrigues formulae and other important formulas for these functions.

12.19 *GLOSSARY:-*

- Orthogonal properties
- Recurrence Relation
- Laplace's definite integrals
- Christoffel's expansion

12.20 *REFERENCES:-*

- Daniel A. Murray (2003). Introductory Course in Differential Equations, Orient.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.
- G F Simmons (1991) Differential Equations with Historical Notes.

12.21 *SUGGESTED READING:-*

- B. Rai, D. P. Choudhury & H. I. Freedman (2013). A Course in Ordinary Differential Equations (2nd edition). Narosa.
- Erwin Kreyszig (2010) Advanced Engineering Mathematics.

12.22 *TERMINAL QUESTIONS:-*

(TQ-1) Prove that $\frac{1+z}{z\sqrt{1-xz+z^2}} - \frac{1}{z} = \sum_{n=0}^{\infty}(P_n + P_{n+1})\, z^n$

(TQ-2) Prove that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$

(TQ-3) Prove that $(2n+1)(x^2-1)P'_n = n(n+1)(P_{n+1} - P_{n-1})$ and deduce that

$$\int_{-1}^{1}(x^2-1)\, P_{n+1}(x)P'_n(x)dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

(TQ-4) show that $P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{(x^2-1)}cos\theta\right]^n d\theta$ where n is a positive integer.

(TQ-5) show that $\int_0^{\pi} x^2 P_{n+1}P_{n-1}dx = \frac{2n(n+1)}{(2n-1)(2n+3)(2n+1)}$.

(TQ-6) Show that $\int_{-1}^{+1} xP_nP_{n-1}dx = \frac{2n}{4n^2-1}$.

(TQ-7) Show that $n[P_nQ_{n-1} - Q_nP_{n-1}] = 1$.

(TQ-8) Show that $P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left[x\pm\sqrt{(x^2-1)}cos\phi\right]^{n+1}}d\phi$

(TQ-9) Show that $\frac{d^{n+1}}{dx^{n+1}}[Q_n(x)] = \frac{(-2)^n n!}{(x^2-1)^{n+1}}$

(TQ-10) Prove that

- **i.** $n(Q_nP_{n-1} - P_{n-1}Q_n) = (n-1)(Q_{n-1}P_{n-2} - P_{n-1}Q_{n-2})$ and deduce that
- **ii.** $n(Q_nP_{n-1} - P_{n-1}Q_n) = -1\ or\ \ P_nQ_{n-1} - Q_nP_{n-1} = 1/n$

(TQ-11) Using Rodrigue's formula, prove that $P'_{n+1} - P'_{n-1} = (2n+1)P_n$.

(TQ-12) If $x > 1$, show that $P_n(x) < P_{n+1}(x)$

UNIT 13:- Bessel Functions and Hermite Polynomials

CONTENTS:

13.1 *INTRODUCTION:-*

Bessel's equation arises in many areas of physics and engineering, such as the theory of vibrations of circular membranes, the study of electric and magnetic fields in cylindrical coordinates, and the analysis of wave propagation in cylindrical or spherical geometries. It has important applications in acoustics, optics, signal processing, and quantum mechanics.

In this unit, we discuss about the Bessel function through the generating function, recurrence formulae, orthogonal property and Integral of representation of Bessel Function.

13.2 *OBJECTIVES:-*

After studying this unit you will be able to

- To discuss about Bessel functions and its equation and generating function.
- To study the recurrence formulae of Bessel functions.
- To study the important properties for this function.
- The main objective of Hermite polynomials is to provide a set of orthogonal polynomials, which means that they satisfy a particular inner product property.
- To provide the Hermite polynomials also have a generating function that allows them to be expressed in terms of other mathematical functions.

13.3 *BESSEL'S DEFERENTIAL EQUATION:-*

The differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \qquad \dots(1)$$

is called the Bessel Differential equation, where n is positive constant.

13.4 *SOLUTION OF BESSEL'S EQUATION:-*

Let the solution (1) be

$$y=\sum_{r=0}^{\infty} a_r x^{m+r} \qquad \dots(2)$$

Then

$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\,(m+r)x^{m+r-1} \Rightarrow \frac{d^2y}{dx^2}$$

$$=\sum_{r=0}^{\infty} a_r\,(m+r)(m+r-1)x^{m+r-2}$$

Putting the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we obtain

$$x^2\sum_{r=0}^{\infty} a_r\,(m+r)(m+r-1)x^{m+r-2}+\frac{1}{x}\sum_{r=0}^{\infty} a_r\,(m+r)x^{m+r-1}$$

$$+(x^2-n^2)\sum_{r=0}^{\infty} a_r\,x^{m+r}=0$$

$$\sum_{r=0}^{\infty} a_r\,(m+r)(m+r-1)x^{m+r} + \sum_{r=0}^{\infty} a_r\,(m+r)x^{m+r} + (x^2-n^2)\sum_{r=0}^{\infty} a_r\,x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r\,(m+r)(m+r-1)x^{m+r} + \sum_{r=0}^{\infty} a_r\,(m+r)x^{m+r} + \sum_{r=0}^{\infty} a_r\,x^{m+r+2} - n^2\sum_{r=0}^{\infty} a_r\,x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r\,x^{m+r}\{(m+r)(m+r-1)+(m+r)-n^2\} + \sum_{r=0}^{\infty} a_r\,x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r\,x^{m+r}\{m^2+mr+n^2+rm-m-n+m+r-n^2\} + \sum_{r=0}^{\infty} a_r\,x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r\,x^{m+r}\{(m+r)^2-n^2\} + \sum_{r=0}^{\infty} a_r\,x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r\,x^{m+r}\{(m+r+n)(m+r-n)\} + \sum_{r=0}^{\infty} a_r\,x^{m+r+2} = 0$$

Hence equating to zero the coefficients of lowest power of x, we obtain

$$(m+n)(m-n)a_0 = 0 m = n, -n\ as\ \ a_0 \neq 0 \quad \ldots (3)$$

Also equating to zero coefficient of x^{m+1}, we have

$$(m+r+n)(m+r-n)a_1 = 0$$

But

$(m+r+n)(m+r-n) \neq 0$ by equation (3), so we have $a_1 = 0$

Similarly equating to zero the coefficients of x^{m+r},we get

$$(m+r+n)(m+r-n)a_r + a_{r-2} = 0 \Rightarrow a_{r-2} = -\frac{a_r}{(m+r+n)(m+r-n)} \quad \ldots (4)$$

Putting $r = 3,5,7,\ldots..$ in (4) it follows that $a_3 = a_5 = a_7 = \cdots = 0$

Case1: If $m = +n$

Also putting $r = 2,4,6,\ldots..$in (4), we obtain

$$a_2 = -\frac{a_0}{(m+2+n)(m+2-n)} \Rightarrow a_4 = -\frac{a_0}{(m+2+n)(m+2-n)(m+4+n)(m+4-n)}$$

Putting these value in (2), as$a_3 = a_5 = a_7 = \cdots = 0$, we have

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} = a_0 x^m + a_1 x^{m+2} + a_4 x^{m+4} + \cdots\cdots\cdots\cdots$$

$$y = a_0 x^m \left[1 - \frac{x^2}{(m+2+n)(m+2-n)} + \frac{x^2}{(m+2+n)(m+2-n)(m+4+n)(m+4-n)} - \cdots\right]$$

Replacing m by n and $-n$, we have

$$y = a_0 x^n \left[1 - \frac{x^2}{2.(2+2n)} + \frac{x^2}{2.4.(4+2n)(2n+2)} - \cdots\right] \quad \ldots(5)$$

$$y = a_0 x^{-n} \left[1 - \frac{x^2}{2.(2-2n)} + \frac{x^2}{2.4.(4-2n)(-2n+2)} - \cdots\right] \quad \ldots(6)$$

The particular solution of (1) obtained from (5) above by taking arbitrary constant $a_0 = \frac{1}{2^n \Gamma(n+1)}$ is **known as Bessel function of first kind of order $\boldsymbol{n}$.** It is denoted by$\boldsymbol{J_n(x)}$.

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(1+n)} + \frac{x^2}{4.8(2+n)(n+1)} - \cdots\right]$$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

Similarly

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

It is known as Bessel function of second kind of order $\boldsymbol{n}$. It is denoted by$\boldsymbol{J_{-n}(x)}$.

13.5 *GENERAL SOLUTION OF BESSEL'S EQUATION:-*

The general solution of Bessel's equation (1) is

$$y = AJ_n(x) + BJ_{-n}(x).$$

Where A and B are arbitrary constant.

13.6 *RECURRENCE FORMULA FOR $J_n(x)$:-*

Prove that

I. $\boldsymbol{\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)}$ ***or*** $\boldsymbol{xJ'_n = nJ_n - xJ_{n+1}}$.

Proof. We know that

$$J_n = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

Differentiating with respect to x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} . \frac{1}{2} . \left(\frac{x}{2}\right)^{2r+n-1}$$

$$xJ_n' = n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . \left(\frac{x}{2}\right)^{2r+n-1+1}$$
$$+ x\sum_{r=1}^{\infty} \frac{(-1)^r (2r)}{r(r-1)!\,\Gamma(n+r+1)} . \frac{1}{2} . \left(\frac{x}{2}\right)^{2r+n-1}$$

$$J_n' = n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . \left(\frac{x}{2}\right)^{2r+n}$$
$$+ x\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} . \left(\frac{x}{2}\right)^{2r+n-1}$$
$$= nJ_n(x) + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+r+2)} . \left(\frac{x}{2}\right)^{2s+n+1}, \quad [s = r-1]$$
$$= nJ_n(x) - x\sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma(n+r+2)} . \left(\frac{x}{2}\right)^{2s+n+1}$$

$$\boldsymbol{xJ'_n = nJ_n - xJ_{n+1}} \quad \ldots\ldots(\mathbf{1}).$$

II. $\boldsymbol{xJ'_n = -nJ_n + xJ_{n-1}}$.

Proof.

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} . \frac{1}{2} . \left(\frac{x}{2}\right)^{2r+n-1}$$

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{r!\,\Gamma(n+r+1)} . \frac{1}{2} . \left(\frac{x}{2}\right)^{2r+n-1}$$

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\Gamma(n+r+1)} . \left(\frac{x}{2}\right)^{2r+n-1}$$
$$- n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . \frac{1}{2} . \frac{x}{x} . \left(\frac{x}{2}\right)^{2r+n-1}$$

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\Gamma(n+r+1)} . \left(\frac{x}{2}\right)^{2r+n-1} - \frac{n}{x}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . \frac{1}{2} . \left(\frac{x}{2}\right)^{2r+n}$$

$$J'_n = \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{2r+n-1} - \frac{n}{x} J_n(x)$$

$$= J_{n-1}(x) - -\frac{n}{x} J_n(x)$$

$$\boldsymbol{xJ'_n = -nJ_n + xJ_{n-1}} \qquad \cdots (\mathbf{2})$$

III. $\boldsymbol{2J'_n = J_{n-1} - J_{n+1}}$

Proof. Adding (1) & (2), we obtain

$$2xJ'_n = nJ_n - xJ_{n+1} + -nJ_n + xJ_{n-1}$$

$$2xJ'_n = x[J_{n-1} - J_{n+1}]$$

IV. $\boldsymbol{2nJ_n = x(J_{n-1} - J_{n+1})}$

Proof. Subtracting (1) & (2), we get

$$0 = nJ_n - xJ_{n+1} + nJ_n - xJ_{n-1}$$

$$0 = 2nJ_n - xJ_{n+1} - xJ_{n-1}$$

$$\boldsymbol{2nJ_n = xJ_{n+1} + xJ_{n-1}}$$

V. $\boldsymbol{\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}}$

Proof. Multiplying (1) by x^{-n-1}

$$x^{-n}J'_n = nx^{-n-1}J_n - x^{-n}J_{n+1}$$

$$x^{-n}J'_n - nx^{-n-1}J_n = -x^{-n}J_{n+1}$$

$$\boldsymbol{\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}}$$

VI. $\boldsymbol{\frac{d}{dx}(x^{n}J_n) = x^{n}J_{n-1}}$

Proof. Multiplying (2) by x^{-n-1}

$$x^{n}J'_n = -nx^{n-1}J_n + x^{n}J_{n-1}$$

$$x^{n}J'_n + nx^{n-1}J_n = x^{n}J_{n-1}$$

$$\boldsymbol{\frac{d}{dx}(x^{n}J_n) = x^{n}J_{n-1}}$$

SOLVED EXAMPLES

EXAMPLE1: Show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}.\,sinx$

SOLUTION: We know that

$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \cdots\cdots\cdots\cdots\cdots\right]\backslash$

Substituting $n = \frac{1}{2}$ and using $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, we obtain

$$\boldsymbol{J_{1/2}(x)} = \sqrt{\frac{2x}{\pi}}\left[1 - \frac{x^2}{2.3} + \frac{x^4}{3.5.2.4} - \cdots\cdots\cdots\right]$$

$$= \sqrt{\frac{2x}{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2x}{\pi}}[sinx]$$

EXAMPLE2: Show that $J_n(x)$ is even and odd function for even n and for odd n respectively

SOLUTION: The definition of Bessel function

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$$

Replacing x by $-x$, we obtain

$$J_n(-x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(-\frac{x}{2}\right)^{2r+n} = (-1)^n J_n(x)$$

i. If n is even $J_n(-x) = J_n(x)$, therefore $J_n(x)$is even.
ii. If n is odd $J_n(-x) = -J_n(x)$, therefore $J_n(x)$is odd.

EXAMPLE3: Prove that $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$

SOLUTION: we know that $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$ ... (1)

Now we take L.H.S of (1)

$$\frac{d}{dx}[xJ_n(x)J_{n+1}(x)]$$

$$= xJ_n(x)J'_{n+1}(x) + xJ'_n(x)J_{n+1}(x) + J_n(x)J_{n+1}(x) \quad \dots (2)$$

From recurrence relation I and II, we get

$$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$$

$$xJ'_n(x) = -nJ_n(x) - xJ_{n-1}(x) \quad \dots (3)$$

Substituting n as $(n+1)$ in (3), we have

$$xJ'_{n+1}(x) = -(n+1)xJ_{n+1}(x) + xJ_n(x)$$

Putting the value of $xJ'_n(x)$ and $xJ'_{n+1}(x)$from in (3), we obtain

$$= J_n(x)[-(n+1)xJ_{n+1}(x) + xJ_n(x)] + J_{n+1}(x)[nJ_n(x) - xJ_{n+1}(x)] + J_n(x)J_{n+1}(x)$$

$$= x[J_n^2(x) - J_{n+1}^2(x)] = R.H.S$$

13.7 *GENERATING FUNCTION FOR BESSEL'S EQUATION:-*

Theorem: Show that when n is a positive integer $J_n(x)$ is the coefficient of z^n in the expansion of $exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\}$ in ascending and descending power of z.

Proof: We given, $exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\} = exp\left(\frac{xz}{2}\right).exp\left(-\frac{x}{2z}\right)$

$$= \left[1 + \left(\frac{x}{2}\right)z + \left(\frac{x}{2}\right)^2 \frac{z^2}{2!} + \cdots \left(\frac{x}{2}\right)^n . \frac{z^n}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{z^{n+1}}{(n+1)!} + \cdots\right]$$
$$\times \left[1 - \left(\frac{x}{2}\right)z^{-1} + \left(\frac{x}{2}\right)^2 \frac{z^{-2}}{2!} + \cdots + \left(\frac{x}{2}\right)^n \frac{(-1)^n z^{-n}}{n!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} z^{-(n+1)} + \cdots\right] \quad \ldots (1)$$

The coefficient of z^n in product (1) is derived by multiplying the coefficient of $z^n, z^{n+1}, z^{n+2}, \ldots$ in the 1st bracket with the coefficient of $z^0, z^{-1}, z^{-2}, \ldots$ in the 2nd bracket

$\Rightarrow \quad = \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+1} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{2!(n+2)!} - \cdots$

$\Rightarrow \quad \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r} = J_n(x), \quad [(n+r)! = \Gamma(n+r+1)]$

Similarly the coefficient of z^{-n} in the expansion (1) is

$$= (-1)^n \left[\left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{2!\,(n+2)!}\right]$$
$$= (-1)^n J_n(x)$$

Finally, the term of z is

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \cdots = J_0(x)$$

Thus the equation (1) gives,

$$exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\} = J_0(x) + \left(z - \frac{1}{z}\right)J_1(x) + \left(z^2 + \frac{1}{z^2}\right)J_1(x) + \cdots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$, therefore

$$exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

SOLVED EXAMPLES

EXAMPLE1: Show that

I. $cos(xsin\alpha) = J_0 + 2\cos 2\alpha . J_2 + 4\cos 4\alpha . J_4 + \cdots$
II. $sin(xsin\alpha) = 2sin\alpha . J_1 + 2sin3\alpha . J_2 + \cdots$
III. $cos(xcos\alpha) = J_0 - 2\cos 2\alpha . J_2 + 4\cos 4\alpha . J_4 - \cdots$
IV. $sin(xcos\alpha) = 2\cos 2\alpha . J_1 - 2cos3\alpha . J_3 + 2\cos 5\alpha . J_5 - \cdots$
V. $cosx = J_0 - 2J_2 + 4J_4 - \cdots = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$
VI. $sinx = 2J_1 - 2J_3 + 2J_5 - \cdots = 2\sum_{n=1}^{\infty}(-1)^n J_{2n+1}(x)$

Proof: We have,

$$exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\} = J_0 + (z - z^{-1})J_1 + (z^2 - z^{-2})J_2 + \cdots \quad \ldots (1)$$

Suppose $z = e^{i\alpha}$ so that $z^n = e^{in\alpha}$ and $z^{-n} = e^{-in\alpha}$, then from (1), we obtain

$$exp\left\{\frac{x}{z}\left(z-\frac{1}{z}\right)\right\} = J_0 + \left(e^{i\alpha} - e^{-i\alpha}\right)J_1 + \left(e^{2i\alpha} + e^{-2i\alpha}\right)J_2 + \cdots \quad ...(2)$$

Hence $cosn\alpha = \left(e^{in\alpha} + e^{-in\beta}\right)/2$ and $sinn\alpha = \left(e^{in\alpha} - e^{-in\alpha}\right)/2i$, we obtain

$$e^{xisin\alpha} = J_0 + 2i\ sin\alpha.J_1 + 2cos2\alpha.J_2 + 2isin3\alpha.J_3 + \cdots$$

$$cos(xsin\alpha) + isin(xsin\alpha)$$
$$= (J_0 + 2cos2\alpha.J_2 + \cdots)$$
$$+ 2i\ (sin\alpha.J_1 + 2isin3\alpha.J_3 + \cdots) \quad ...(3)$$

Part I: Separating the real parts in equation (3), we get

$$cos(xsin\alpha) = (J_0 + 2cos2\alpha.J_2 + 2cos4\alpha.J_4 + \cdots) \quad (i)$$

Part II: Separating the imaginary parts in equation (3), we get

$$sin(xsin\alpha) = (sin\alpha.J_1 + 2isin3\alpha.J_3 + 2\ sin5\alpha.J_5 + \cdots) \quad (ii)$$

Part III: Putting α by $\pi/2 - \phi$ in (i), we obtain

$$cos(xcos\alpha) = J_0 - 2\cos 2\alpha\,.J_2 + 4\cos 4\alpha\,.J_4 - \cdots$$

Part IV: Putting α by $\pi/2 - \phi$ in (ii), we obtain

$$sin(xcos\alpha) = 2\cos 2\alpha\,.J_1 - 2cos3\alpha.J_3 + 2\cos 5\alpha\,.J_5 - \cdots$$

Part V &VI: Putting α by 0 in (i) and (ii), we obtain

$$cosx = J_0 - 2J_2 + 4J_4 - \cdots = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

$$sinx = 2J_1 - 2J_3 + 2J_5 - \cdots = 2\sum_{n=1}^{\infty}(-1)^n J_{2n+1}(x)$$

EXAMPLE2: Prove that $xsinx = 2(2^2J_n - 4^2J_4 + 6^2J_6 - \cdots)$

SOLUTION: we have,

$$cos(xsin\alpha) = (J_0 + 2cos2\alpha.J_2 + 2cos4\alpha.J_4 + \cdots) \quad ...(i)$$

Differentiating (i) w. r.t. α, we get

$$-sin(xsin\alpha).xcos\alpha = (0 - 2.2J_2\ sin2\alpha. -2.4J_4\ sin4\alpha. + \cdots)$$

Again differentiating (i) w. r.t. α, we have

$$-cos(xsin\alpha).(xcos\alpha)^2 + sin(xsin\alpha).(xsin\alpha)$$
$$= (-2.2^2J_2\ cos2\alpha. -2.4^2J_4\ cos4\alpha. + \cdots) \quad ...(ii)$$

Separating α by $\pi/2$ in (ii)

$$xsinx = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - \cdots)$$

13.8 *ORTHOGONALITY PROPERTY FOR BESSEL'S EQUATION:-*

If λ_i and λ_j are roots of the equation $J_n(\lambda a) = 0$, then

$$\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx = \begin{cases} 0, & if\ i \neq j \\ \frac{a^2}{2}J_{n+1}^2(\lambda_i a), & if\ i = j \end{cases}$$

Proof:

CaseI

Suppose $i \neq j$ $i.e.,$ suppose λ_i and λ_j are different roots of $J_n(\lambda a) = 0$

$$\therefore \qquad J_n(\lambda_i a) = 0 \text{ and } J_n(\lambda_j a) = 0 \qquad \ldots(1)$$

Let $$u(x) = J_n(\lambda_i x) \text{ and } v(x) = J_n(\lambda_j x) \qquad \ldots(2)$$

Then u and v are Bessel functions satisfying the modified Bessel equation

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \qquad \ldots(3)$$

$$x^2u'' + xu' + ({\lambda_i}^2x^2 - n^2)u = 0 \qquad \ldots(4)$$

$$x^2v'' + xv' + ({\lambda_j}^2x^2 - n^2)v = 0 \qquad \ldots(5)$$

Multiplying (4) by v and (5) by u and then subtracting, we obtain

$x^2(vu'' - uv'') + x(vu' - uv') + x^2(\lambda_i^2 - \lambda_j^2)uv = 0 \Rightarrow x(vu'' - uv'') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$

$$x\frac{d}{dx}(vu' - uv') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$$

$$x\frac{d}{dx}[x(vu' - uv')] = x(\lambda_j^2 - \lambda_i^2)uv$$

Integrating the above equation w.r.t. x from 0 to a, $(\lambda_j^2 - \lambda_i^2)\int_0^a xuv\, dx = [x(vu' - uv')]_0^a$

Using (2), the equation gives

$$(\lambda_j^2 - \lambda_i^2)\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)\, dx$$

$$= \left[x\{J_n(\lambda_j x)J'_n(\lambda_i x) - J_n(\lambda_i x)J'_n(\lambda_j x)\}\right]_0^a$$

$$= a\{J_n(\lambda_j a)J'_n(\lambda_i a) - J_n(\lambda_i a)J'_n(\lambda_j a)\} = 0$$

Since $\lambda_i \neq \lambda_j$ the above equation obtain

$$\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)\, dx = 0 \quad if \quad i \neq j \qquad \ldots.(6)$$

CaseII

Suppose $i = j$ $i.e.,$ multiplying (4) by $2u'$, we obtain

$$2x^2u''u' + 2x(u')^2 + 2({\lambda_i}^2x^2 - n^2)uu' = 0$$

$$\frac{d}{dx}\left[x^2(u')^2 - n^2u^2 + {\lambda_i}^2x^2u^2\right] - 2{\lambda_i}^2xu^2 = 0$$

$$2{\lambda_i}^2xu^2 = \frac{d}{dx}\left[x^2(u')^2 - n^2u^2 + {\lambda_i}^2x^2u^2\right]$$

Integrating the above equation w.r.t. x from 0 to a, we obtain

$$2{\lambda_i}^2\int_0^a xu^2dx = \left[x^2(u')^2 - n^2u^2 + {\lambda_i}^2x^2u^2\right]_0^a$$

Using $J_n(0) = 0$ the above equation, we have

$$2\lambda_i{}^2\int_0^a xJ_n^2(\lambda_i x)^2 dx$$
$$= \left[x^2\left(J'_n(\lambda_i x)\right)^2 - n^2\left(J_n(\lambda_i x)\right)^2 + \lambda_i{}^2x^2\left(J_n(\lambda_i x)\right)^2\right]_0^a$$
$$2\lambda_i{}^2\int_0^a xJ_n^2(\lambda_i x)^2 dx = a^2\left[\left(J'_n(\lambda_i x)\right)^2\right]_{at\ x=a} \quad ...(7)$$

From recurrence relation I , we get

$$\frac{d}{dx}[J_n(x)] = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Replacing x by $\lambda_i x$ in above equation, we get

$$\frac{d[J_n(\lambda_i x)]}{d(\lambda_i x)} = \frac{n}{(\lambda_i x)}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$
$$\frac{1}{\lambda_i}.\frac{d[J_n(\lambda_i x)]}{dx} = \frac{n}{(\lambda_i x)}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

$$J'_n(\lambda_i x) = \frac{n}{(\lambda_i x)}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$
$$[\{J'_n(\lambda_i x)\}^2]_{at\ x=a} = \left[\left\{\frac{n}{(\lambda_i x)}J_n(\lambda_i x) - J_{n+1}(\lambda_i x)\right\}^2\right]_{at\ x=a}$$
$$= \{0 - \lambda_i J_{n+1}(\lambda_i a)\}^2 = \lambda_i^2 J_{n+1}^2(\lambda_i a)$$

Using in equation (7), we have

$$\int_0^a xJ_n^2(\lambda_i x)^2 dx = \frac{a^2}{2}J_{n+1}^2(\lambda_i a) \qquad ....(8)$$

Combining equation (6) and (8), we can write

$$\boldsymbol{\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx = \frac{a^2}{2}J_{n+1}^2(\lambda_i a)\delta_{ij}}$$

Where $\delta_{ij} = (kroniker\ delta) = \begin{cases}0, i \neq j\\ 1, i = j\end{cases}$

13.9 *BESSEL INTEGRALS:-*

Show that

I. $J_n(x) = \frac{1}{\pi}\int_0^\pi cos(n\alpha - xsin\alpha)d\alpha$, where n is a positive integer.

II. $J_n(x) = \frac{1}{\pi}\int_0^\pi cos(n\alpha - xsin\alpha)d\alpha$, where n is any integer.

III. $J_0(x) = \frac{1}{\pi}\int_0^\pi cos(xsin\alpha)d\alpha = \frac{1}{\pi}\int_0^\pi cos(xcos\alpha)d\alpha$.

IV. Deduce that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \cdots = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r.r!)^2}$.

SOLUTION:

PartI: we shall use the following results:

$$\left.\begin{aligned}\int_0^{\pi} cosm\alpha\, cosn\alpha\, d\alpha = \int_0^{\pi} sinm\alpha\, sinn\alpha\, d\alpha &= \pi/2 \ when\, m = 0 \\ &= 0\, when \ \ m \neq n\end{aligned}\right\} \dots (1)$$

$$cos(xsin\alpha) = J_0 + 2J_1cos2\alpha + 2J_2cos4\alpha + \cdots \qquad \dots (2)$$

And

$$sin(xsin\alpha) = 2J_1sin\alpha + 2J_3sin3\alpha + 2J_5sin5\alpha + \cdots \quad \dots\ (3)$$

Now multiplying both sides of (2) by $cosn\alpha$, then integrating w.r.t.α, the limit $(0\ to\ \pi)$ and using (1), we obtain

$$\left.\begin{aligned}\textstyle\int_0^{\pi} cos(xsin\alpha)cosn\alpha\, d\alpha &= 0, if\ n\ is\ odd \\ &= \pi J_n, if\ n\ is\ even\end{aligned}\right\} \qquad \dots (4)$$

Now again multiplying both sides of (3) by $sinn\alpha$, then integrating w.r.t.α, the limit $(0\ to\ \pi)$ and using (1), we have

$$\left.\begin{aligned}\textstyle\int_0^{\pi} sin(xsin\alpha)sinn\alpha\, d\alpha &= \pi J_n, if\ n\ is\ odd \\ &= 0, if\ n\ is\ even\end{aligned}\right\} \qquad \dots (5)$$

Let us consider n be odd so adding above odd functions in equation (4) and (5), we get

$$\int_0^{\pi} [cos(xsin\alpha)cosn\alpha d\alpha + \ sin(xsin\alpha)sinn\alpha]\, d\alpha = \pi J_n$$

$\int_0^{\pi} cos(n\alpha - xsinn\alpha)\, d\alpha = \pi J_n$ or $J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\alpha - xsin\alpha)\, d\alpha$ ... (6)

Similarly, Let us consider n be even so adding above even functions in equation (4) and (5), we get (6). Thus (6) holds for each positive integer (even as well as odd).

PartII: Let n be any integer, then the part I, if n is positive integer, we obtain

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\alpha - xsin\alpha)\, d\alpha \qquad \dots (7)$$

Let n be negative integer, then $= -m$, where m is positive integer. To prove that the result for negative integer, we prove that

$$J_{-m}(x) = \frac{1}{\pi}\int_0^{\pi} cos(-m\alpha - xsin\alpha)\, d\alpha \qquad \dots (8)$$

Let $\alpha = \pi - \beta$ so that$d\alpha = -d\beta$, then we have the R.H.S.of (8)

$$= \frac{1}{\pi}\int_{\pi}^{0} cos\{-m(\pi - \beta) - sin(\pi - \beta)\}\,(-d\beta)$$

$$= \frac{1}{\pi}\int_0^{\pi} cos[(m\beta - xsin\beta) - m\pi]\,(d\beta)$$

$$= \frac{1}{\pi}\int_0^{\pi} [cos(m\beta - xsin\beta)cosm\pi + sin(m\beta - xsin\beta)sinm\pi]\, d\beta$$

$$= \frac{1}{\pi}\int_0^{\pi} (-1)^m cos(m\beta - xsin\beta)\, d\beta \quad [\because sinm\pi = 0\ \&\ cosm\pi = (-1)^m]$$

$= \frac{1}{\pi}(-1)^m \int_0^{\pi} cos(m\beta - xsin\beta)\, d\beta$ [using (7) as $m\ as +$ integer]

$= J_{-m}(x)$=L.H.S. of (8)

PartIII: Now integrating (2) w.r.t. α between the limit $(0\ to\ \pi)$, then

$$\int_0^{\pi} cosk\alpha\, d\alpha = 0 \qquad ...(9)$$

If k is positive integer, we get

$$\int_0^{\pi} cos(xsin\alpha)d\alpha = J_0(x)\int_0^{\pi} d\alpha + 0 + 0 + \cdots = J_0(x).\pi$$

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} cos(xsin\alpha)d\alpha$$

Substituting α by $\frac{\pi}{2} - \alpha$ in (2), we obtain

$$cos(xcos\alpha) = J_0 - 2\cos 2\alpha\,.J_2 + 4\cos 4\alpha\,.J_4 - \cdots \qquad ...(10)$$

Again integrating (10) w.r.t. α and using (9), we have

$\int_0^{\pi} cos(xcos\alpha)d\alpha = J_0(x).\pi - 0 - 0 ...$ or $J_0(x) = \frac{1}{\pi}\int_0^{\pi} cos(xcos\alpha)d\alpha$

PartIV: From (10), $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{x^2cos^2\alpha}{2!} + \frac{x^4cos^4\alpha}{4!} - \cdots\right)d\alpha$

But $\int_0^{\pi} cos^{2n}\alpha d\alpha = \frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{2\cdot4\cdot6\ldots.(2n)}\pi$

Using this equation, $J_0(x) = \frac{1}{\pi}\left[x - \frac{x^2}{2!}\cdot\frac{1}{2}\pi + \frac{x^2}{4!}\cdot\frac{1\cdot3}{2}\pi - \cdots\cdots\right] = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r\cdot r!)^2}$

13.10 *BESSEL SERIES:-*

If $f(x)$ is described in region $0 \le x \le a$ and has an expansion of the form

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \qquad ....(1)$$

Where the λ_i are the roots of the equation

$$J_n(\lambda_i x) = 0 \qquad ...(2)$$

Then

$$c_i = \frac{2\int_0^a xf(x)J_n(\lambda_i x)dx}{a^2 J_{n+1}^2(\lambda_i a)} \qquad ...(3)$$

Proof: Multiplying both sides $xJ_n(\lambda_j x)$, we have

$$xf(x)J_n(\lambda_j x) = \sum_{i=1}^{\infty} c_i x J_n(\lambda_i x)J_n(\lambda_j x) \qquad ...(4)$$

Integrating both sides of (4),w.r.t. x from $0\ to\ a$, we obtain

$$\int_0^a xf(x)J_n(\lambda_j x)dx = c_i \sum_{i=0}^{\infty} \int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx \quad \dots(5)$$

From the orthogonal property of Bessel functions, we get

$$\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx = \begin{cases} 0, & if\ i \neq j \\ \frac{a^2}{2}J_{n+1}^2(\lambda_i a), & if\ i = j \end{cases} \quad \dots(6)$$

Using (6), (5) reduce to

$$\int_0^a xf(x)J_n(\lambda_j x)dx = c_j \frac{a^2}{2}J_{n+1}^2(\lambda_i a)$$

Replacing j by i in above equation, we obtain

$$c_j \frac{a^2}{2}J_{n+1}^2(\lambda_i a) = \int_0^a xf(x)J_n(\lambda_i x)dx$$

$$c_i = \frac{2\int_0^a xf(x)J_n(\lambda_i x)dx}{a^2 J_{n+1}^2(\lambda_i a)}$$

SOLVED EXAMPLES

Example1: Prove that $J_n(x) = (-2)^n x^n \frac{d^n}{d(x^2)^n} J_0(x)$

Proof: Putting the values of $J_0(x)$ in series R.H.S., we obtain

$$R.H.S. = (-2x)^n \left[\frac{d^n}{d(x^2)^n}\left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{x}{2}\right)^{2r}\right)\right]$$

$$= (-2x)^n \left[\frac{d^n}{dt^n}\left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r\Gamma(r+1)}\left(\frac{t^r}{2^{2r}}\right)\right)\right]$$

$$= (-2x)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r-n)}\left(\frac{t^{r-n}}{2^{2r}}\right)$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$= (-1)^n J_{-n}(x) = J_n(x)$$

Example2: Prove that $\int_0^\infty e^{-ax}J_0(tx)dx = \frac{1}{\sqrt{a^2+t^2}}$

Proof: Let we take L.H.S.and using series representation for the Bessel function and changing the order of integration and summation, we given below

$$I = \int_0^\infty e^{-ax}J_0(tx)dx = \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{t}{2}\right)^{2r}}{(r!)^2} \int_0^\infty x^{2r}e^{-ax}dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{t}{2}\right)^{2r}\Gamma(2r+1)}{(r!)^2 a^{2r+1}} \qquad using\ the\ def.of\ gamma\ function$$

Now

$$I = \frac{1}{a}\sum_{r=0}^{\infty} \frac{(1/2)_r}{r!}\left(-\frac{t^2}{a^2}\right)^r$$
$$= \frac{1}{a}\left(1 + \frac{t^2}{a^2}\right)^{-1/2} = \frac{1}{\sqrt{t^2 + a^2}}$$

13.11 *HERMITE EQUATION AND ITS SOLUTION* $\boldsymbol{H_n(x)}$*:-*

The Hermite's equation is the form

$$\left(\frac{d^2y}{dx^2}\right) - 2x\frac{dy}{dx} + 2ny = 0 \qquad \dots(1)$$

where n is a constant. Now solve equation (1) in series by using Frobenius method.
Let

$$y = \sum_{l=0}^{\infty} A_l x^{k+l}, \quad A_l \neq 0 \qquad \dots(2)$$

Now differentiating (2), then we substituting the value of $y, \frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in (1), we obtain

$$\sum_{l=0}^{\infty} A_l(k+l)(k+l-1)x^{k+l-2} - 2xA_l(k+l)x^{k+l-1} + 2n\sum_{l=0}^{\infty} A_l x^{k+l} = 0$$

$$\sum_{l=0}^{\infty} A_l(k+l)(k+l-1)x^{k+l-2} - 2\left\{\sum_{l=0}^{\infty} A_l(k+l)x^{k+l} - \sum_{l=0}^{\infty} A_l n x^{k+l}\right\} = 0$$

$$\sum_{l=0}^{\infty} A_l(k+l)(k+l-1)x^{k+l-2} - 2\sum_{l=0}^{\infty} A_l(k+l-n)x^{k+l} = 0 \qquad \dots(3)$$

Since the equation is identity. We equate to zero the coefficient of smallest power of x, namely x^{k-2}, in equation (3) and we get

$A_0 k(k-1) = 0$ or $k(k-1) = 0$ as $A_0 \neq 0$ $\qquad \dots(4)$

So the roots of indicial equation (4) and $k = 0,1$. They are distinct and differ by an integer.
So again equating to zero the next smallest power of x is $k-1$ in (3), we obtain

$$A_1(k+1)k = 0 \qquad \dots(5)$$

when$k = 0$, (5) shows that $A_1$is indeterminate. Hence A_0 and A_1 can be taken as constants,
equating to zero the coefficient of x^{k+l-2}, (3) gives

$$A_l(k+l)(k+l-1) - 2A_{l-2}(k+l-2-n) = 0$$

$$A_l = \frac{2(k+l-2-n)}{(k+l)(k+l-1)} A_{l-2} \qquad \dots (6)$$

Putting $k = 0$ in (6), we obtain

$$A_l = \frac{2(l-2-n)}{(l)(l-1)} A_{l-2} \qquad \dots (7)$$

Putting $l = 2,4,6, \dots 2l$ in (7) we get

$$A_2 = -\frac{2n}{2.1} A_0 = -\frac{2n}{2!} A_0 = -\frac{(-1)^1 . 2^1 . n}{2!} A_0$$

$$A_4 = \frac{2(2-n)}{4.3} A_2 = \frac{(-1)^2 . 2(2-n)}{4.3} . \frac{2n}{2!} A_0 = \frac{(-1)^2 . 2^2 . n(n-2)}{2!} A_0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_{2l} = \frac{(-1)^n . 2^n . (n-1)(n-2) \dots (n-2l+1)}{(2l+1)!} A_0$$

Putting $l = 3,5, \dots 2l+1$ in (7) we get

$$A_3 = -\frac{(-1)^1 . 2^1 . (n-1)}{3!} A_1$$

$$A_5 = \frac{(-1)^2 . 2^2 . (n-1)(n-3)}{5!} A_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_{2l+1} = \frac{(-1)^n . 2^n . (n-1)(n-3) \dots (n-2l+1)}{(2l+1)!} A_1$$

Substituting the above value in (2) with $= 0$, we obtain

$$y = A_0 \left[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \dots + \frac{(-2)^l . (n)(n-2) \dots (n-2l+1)}{(2l+1)!} x^{2l} + \dots \right] + A_1 \left[x - \frac{2(n-1)}{3!} x^3 + \frac{2^2(n-1)(n-3)}{5!} x^5 + \dots + \frac{(-2)^l . (n-1)(n-3) \dots (n-2l+1)}{(2l+1)!} x^{2l+1} + \dots \right] \qquad \dots (8)$$

$$y = A_0 U + A_1 V \qquad \dots (9)$$

Since U and V are not constant, U and V form a fundamental set (linearly Independent) of (1). Hence (8) and (9) is most general solution of (1) with A_0 and A_1 as two arbitrary constant.

Remarks. In practice we require the solution of equation (1) such that

i. It is finite for all value of x and
ii. As $x \to \infty$, $exp. (1/2x^2) y(x) \to 0$

The solution (8) does not satisfy the condition as $x \to \infty$, $exp. (1/2x^2) y(x) \to 0$.However, if the series terminate then this condition will be satisfied. Replacing l by $l+2$ in (7)

$$A_{l+2} = \frac{2(l-n)}{(l+1)(l+2)} A_l \qquad \dots (10)$$

If l is a positive integer, then for $l = n, A_{l+2} = 0$ $i.e.$, the series terminates.We shall now define the series of (1) in descending powers of x by considering n to be non-negative integer.
For $l = 0$ equation(2), we obtain

$$y = A_n x^n + A_{n-2}x^{n-2} + A_{n-4}x^{n-4} + \dots\dots\dots \qquad \dots(11)$$

From (10), we get $A_l = -\frac{2(l+1)(l+2)}{2(n-l)} A_{l+2}$ ... (12)

Putting $l = n-2, n-4, \dots\dots\dots.$ in (12), we have
$A_n = -\frac{(n)(n-2)}{2.2} A_n, A_{n-4} = -\frac{(n)(n-1)(n-2)(n-3)}{2^2.2.4} A_n$ and so on
Putting these value in (11)

$$y = \left[x^n - \frac{(n)(n-2)}{2.2}x^{n-2} - \frac{(n)(n-1)(n-2)(n-3)}{2^2.2.4}x^{n-4} + \dots\dots\dots + \frac{(-1)^l.n(n-1)..(n-2l+1)}{2^l.2.4\dots.2l}x^{n-2l} + \cdots\right]$$

$$= \sum_{l=0}^{[n/2]} \frac{(-1)^l.n(n-1)..(n-2l+1)}{2^l.2.4\dots.2l}x^{n-2l} \quad as \quad [n/2]$$

$$= \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{(n-1)}{2} & if\ n\ is\ odd \end{cases}$$

Taking $A_n = 2^n$, then the Hermite polynomial of order n is defined by

$$y = H_n(x) \sum_{l=0}^{[n/2]} (-1)^l \frac{n!}{l!\,(n-2l)!}(2x)^{n-2l}$$

where $H_n(x)$ is called the Hermite polynomial of order n.

13.12 *GENERATING FUNCTION FOR $H_n(x)$:-*

Theorem: Prove that $e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \boldsymbol{H_n(x)}$.

Proof: we given $e^{2xt-t^2} = e^{2xt}.e^{-t^2} = \sum_{l=0}^{\infty} \frac{(2xt)^l}{l!} \sum_{m=0}^{\infty} \frac{(-t^2)^m}{m!} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2x)^l(-1)^m}{l!m!} t^{l+2m}$

Let $l + 2m = n$ so $l = n - 2m$
Hence the coefficient of t^n is defined by

$$(-1)^l \frac{(2x)^{2-2l}}{l!\,(n-2l)!}$$

Which gives all value of$l$ for which equation (2) is the coefficient of t^n.
If n is even , $l \le n/2$ shows that l varies from 0 to $n/2$.
Again n is odd , $l \le n/2$ shows that l varies from 0 to $(n-1)/2$.
So the total coefficient of t^n in expansion of e^{2tx-t^2} is obtained by

$$\sum_{i=0}^{\infty}(-1)^l \frac{(2x)^{n-2l}}{l!\,(n-2l)!} = \frac{H_n(x)}{n!}$$

13.13 *ORTHOGONALITY PROPERTIES FOR $H_n(x)$:-*

Theorem: Prove that $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{nm}$

or

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0, & if\ m \neq n \\ \sqrt{\pi} 2^n n!, & if\ m = n \end{cases}$$

Or

Prove that the Hermite polynomials are orthogonal over $(-\infty, \infty)$ with respect to the weight function e^{-x^2}.

Proof: Using the generating function of Hermite polynomials, we obtain

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx-t^2} \text{ and } \sum_{m=0}^{\infty} H_m(x) \frac{s^n}{m!} = e^{2sx-s^2}$$

$$\therefore \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_n(x) H_m(x)}{n!\, m!} t^n s^n = e^{2tx-t^2+2sx-s^2}$$

Multiplying both sides by e^{-x^2} and then integrating both sides w.r.t. x from $[-\infty, \infty]$, we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx \right] \frac{t^n s^n}{n!\, m!} = \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)-(t^2+s^2)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)-(t+s)^2} \times e^{(t+s)^2-(t^2+s^2)}\, dx$$

$= e^{2ts} \int_{-\infty}^{\infty} e^{-[x-(t+s)]^2} dx = e^{2ts} \int_{-\infty}^{\infty} e^{-y^2} dy$ putting $[x-(t+s) = y\ so\ that\ dx = dy]$

$$= e^{2ts} \sqrt{\pi}, \text{ as } \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2ts)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \sqrt{\pi}}{n!} t^n s^n \qquad \dots(1)$$

Hence the power of t and s are always equal in each term of R.H.S.of (1). So when $m \neq n$, then we obtain

$$\frac{1}{n!\, m!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0$$

$\therefore$

$$\frac{1}{n!\, m!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0\,, when\ m \neq n \quad \dots(2)$$

Again equating coefficient $t^n s^n$ on both sides in (1), we obtain

$$\frac{1}{n!\, n!} \int_{-\infty}^{\infty} e^{-x^2} \left(H_n(x)\right)^2 dx = \frac{2^n \sqrt{\pi}}{n!}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\left(H_n(x)\right)^2 dx = n!\,2^n\sqrt{\pi} \qquad \dots(3)$$

Let $$\delta_{mn} = \begin{cases} 0, & if\ m \neq n \\ 1, & if\ m = n \end{cases} \qquad \dots(4)$$

From (3) and (4), we have

$$\boldsymbol{\int_{-\infty}^{\infty} e^{-x^2}\left(H_n(x)\right)^2 dx = n!\,2^n\sqrt{\pi}\ \delta_{mn}}$$

13.14 *RECURRENCE RELATION FOR $H_n(x)$:-*

I. $\boldsymbol{H'_n(x) = 2nH_{n-1}(x)(n \geq 1);\ H'_0(x) = 0}$

II. $\boldsymbol{H_{n+1}(x) = 2nH_n(x) - 2xH_{n-1}(x)(n \geq 1);\ H_1(x) = 2xH_0(x)}$

III. $\boldsymbol{H'_n(x) = 2nH_n(x) - H_{n+1}(x)}$

IV. $\boldsymbol{H''_n(x) - 2nH'_n(x) + 2nH_{n+1}(x) = 0}$

Proof:

Part I: We have

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} \qquad \dots(1)$$

Differentiating both sides w.r.t. x, we obtain

$$\sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!} = 2te^{2tx-t^2} = 2t\sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty} H'_n(x)\frac{t^{n+1}}{n!} \qquad \dots(2)$$

The equating coefficient of t^n from both sides for $n = 0$, (2) obtain

$$H'_0(x) = 0$$

Again equating coefficient of t^n from both sides for $n \geq 1$, (2) obtain

$$\frac{H'_n(x)}{n!} = \frac{H'_{n-1}(x)}{(n-1)!}$$

So

$$H'_n(x) = 2nH_{n-1}(x) \qquad \dots(3)$$

$$[\because n! = (n-1)!]$$

Part II: We know that

$$\textstyle\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} \qquad \dots(4)$$

Differentiating both sides w.r.t. t, we have

$$(2x-2t)e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

$$(2x-2t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \frac{0.t^{0-1}}{0!}H_0(x) + \sum_{n=0}^{\infty} H_n(x)\frac{nt^{n-1}}{n!}$$

$$2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

$$[\because\ 0!, \qquad H_n(x) = 1 \ \ and\ n! = n(n-1)!]$$

The equating coefficient of t^n from both sides for $n = 0$, above equation gives

$$2xH_n(x) = H_1(x)$$

Again equating coefficient of t^n from both sides for $n \geq 1$, above equation obtain

$$2x\frac{H_n(x)}{n!} - 2\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{n!}$$

Multiplying both sides of above equation by $n!$and nothing $n! = n(n-1)!$ then we have

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x) \qquad \dots(5)$$

Part III: From equation (3) and (5)

$$H'_n(x) = 2nH_{n-1}(x)$$

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)\ ,\quad \text{adding both equations}$$

$$H'_n(x) + H_{n+1}(x) = 2nH_{n-1}(x) + 2xH_n(x) - 2nH_{n-1}(x)$$

$$\boldsymbol{H'_n(x) = 2nH_{n-1}(x) - H_{n+1}(x)}$$

Part IV: The $H_n(x)$ is a solution of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0$$

$$\therefore \qquad \boldsymbol{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0}$$

13.15 *RODRIGUES FORMULA FOR* $\boldsymbol{H_n(x)}$*:-*

To prove that $\boldsymbol{H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}}$.

SOLUTION: Using the generating function, we get

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} \qquad \dots(1)$$

Expanding RH.S.by Taylor's theorem, (1) obtain

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\frac{\partial^n}{\partial t^n}e^{2tx-t^2}\right]_{t=0}\frac{t^n}{n!}$$

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n}e^{2tx-t^2}\right]_{t=0} = \left[\frac{\partial^n}{\partial t^n}e^{x^2-(x-t)^2}\right]_{t=0}$$

$$e^{x^2}\left[\frac{\partial^n}{\partial t^n}e^{-(x-t)^2}\right]_{t=0} = e^{x^2}\left[(-1)^n\frac{\partial^n}{\partial t^n}e^{-(x-t)^2}\right]_{t=0}$$

$$\left[\because \frac{\partial^n}{\partial t^n} f(x-t) = (-1)^n \frac{\partial^n}{\partial t^n} f(x-t)\right]$$
$$e^{x^2}\left[\frac{d^n}{dt^n}e^{-x^2}\right]_{t=0} = (-1)^n e^{x^2}\frac{d^n}{dt^n}e^{-x^2}$$

SOLVED EXAMPLES

EXAMPLE1: Prove that $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}; H_{2n+1}(0) = 0.$

SOLUTION: Using generating function, we obtain

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} \qquad \dots(1)$$

Replacing x by 0 in (1), we get

$$\sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!} = e^{-t^2} = \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}$$

From (2) equating coefficient of t^{2n} on both sides, we have

$$\frac{H_{2n}(0)}{(2n)!} = \frac{(-1)^n}{n!} \quad \text{or} \quad H_{2n}(0) = (-1)^n\frac{(2n)!}{n!}$$

Since R.H.S.of above equation does not contain odd powers of t equating coefficients of t^{2n+1} on both sides in above equation define

$$\frac{H_{2n+1}(0)}{(2n+1)!} = 0.$$

So

$$\boldsymbol{H_{2n+1}(0) = 0}.$$

EXAMPLE2: Prove that$H''_n(x) = 4n(n-1)H_{n-2}(x)$

SOLUTION: From recurrence relation, we obtain

$$H'_n(x) = 2nH_{n-1}(x)$$

Differentiating w.r.t. x, we obtain

$$H''_n(x) = 2nH'_{n-1}(x)$$
$$2n \times 2(n-1)H_{n-2}(x)$$
$$\boldsymbol{H''_n(x) = 4n(n-1)H_{n-2}(x)}$$

EXAMPLE3: $\boldsymbol{\int_0^x e^{-y^2} H_n(y)dy = H_{n-1}(0) - e^{-x^2}H_{n-1}(x)}$

SOLUTION: Using Rodrigue's formula in the left hand side of above equation

$$\int_0^x e^{-y^2} H_n(y)dy = \int_0^x (-1)^n \frac{d^n}{dy^n}\left(e^{-y^2}\right)dy = (-1)^n\left[\frac{d^{n-1}}{dy^{n-1}}\left(e^{-y^2}\right)\right]_0^x$$
$$= -\left[e^{-y^2}H_{n-1}(y)\right]_0^x$$
$$= \boldsymbol{H_{n-1}(0)} - e^{-x^2}\boldsymbol{H_{n-1}(x)}$$

SELF CHECK QUESTION

(SCQ-1) Prove that $\sum_{n=0}^{\infty} \frac{(c)_n H_n(x) t^n}{n!} = (1-2xt)^{-c} \;\; {}_2F_0\left(\frac{c}{2}, \frac{c}{2} + \frac{1}{2}; -; -\frac{4t^2}{(1-2xt)^2}\right)$

(SCQ-2) Prove that $\int_0^x e^{-y^2} H_n(y)dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$

13.16 *SUMMARY:-*

In this unit we studied the Hermite differential equation and Hermite polynomials and we also explained the recurrence relation, generating function, Rodrigue formula and orthogonality property for Hermite polynomials.

13.17 *GLOSSARY:-*

- Bessel Series.
- Bessel integrals.
- Hermite polynomial.
- Hermite differential equation.

13.18 *REFERENCES:-*

- G. N. Watson (2020) A Treatise on the theory of Bessel's Function.
- Carlo Viola (2016) An introduction to Special Function.

13.19 *SUGGESTED READING:-*

- M.D. Raisinghania,(2018). Ordinary and Partial Differential equation (18th Edition), S. Chand.
- M.D. Raisinghania,(2021). Ordinary and Partial Differential equation (20th Edition), S. Chand.
- **Math World (Wolfram):** URL link (https://mathworld.wolfram.com/BesselFunction.html) and **Hermite polynomials**: URL link (https://mathworld.wolfram.com/HermitePolynomial.html) on the Math World website.

- **Wikipedia:** URL link (https://en.wikipedia.org/wiki/Bessel_function) and Hermite polynomials (https://en.wikipedia.org/wiki/Hermite_polynomials)

13.20 *TERMINAL QUESTIONS:-*

(TQ-1) Prove that

i. $J_{-1/2}(x) = \sqrt{(2/\pi x)}\ cosx.$

ii. $J_{1/2}(x) = \sqrt{(2/\pi x)}\ sinx.$

iii. $\left[J_{-1/2}(x)\right]^2 + \left[J_{1/2}(x)\right]^2 = 2/\pi x.$

(TQ-2) Prove that $\lim_{z\to 0} \frac{J_n(x)}{z^n} = \frac{1}{2^n\Gamma(n+1)^n},$

(TQ-3) Prove that $\int_0^1 \frac{uJ_0(xu)}{(1-u^2)^{1/2}} du = \frac{sinx}{x}$

(TQ-4) **(a)** Prove that $x^nJ_n(x)$ is a solution of $x\frac{d^2y}{dx^2} + (1-2n)\times\frac{dy}{dx} + xy = 0.$

(b) Prove that $x^nJ_{-n}(x)$ is a solution of

$$x\frac{d^2y}{dx^2} + (1+2n)\times\frac{dy}{dx} + xy = 0.$$

(TQ-5) Prove that

$$\int_0^1 t\,\{J_n(t)\}^2 dt = \frac{1}{2}x^2\{J_n^2(x) - J_{n-1}(x)J_{n+1}(x)\}.$$

(TQ-6) Show that $\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$

(TQ-7) Show that $J_{n+1}(x) = x\int_0^1 J_n(xy)y^{n+1}\,dy.$

(TQ-8) Show that $xsinx = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - \cdots\cdots)$

(TQ-9) Show that $J_n(-x) = (-1)^nJ_n(x).$

(TQ-10) Expand the function $f(x) = 1, 0 \le x \le a)$ in series of $\sum_{i=1}^{\infty} c_iJ_0(\lambda_i x)$, where λ_i are the roots of the equation

$$J_0(\lambda a) = 0.$$

(TQ-11) Prove that $H_n(x) = 2^n\left\{exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}x^n$

(TQ-12) Prove that $P_n(x) = \frac{2}{n!\sqrt{\pi}}\int_0^\infty e^{-t^2}t^2H_n(xt)dt$

(TQ-13) Prove that $\sum_{n=0}^{\infty}\frac{H_{n+s}(x)t^n}{n!} = exp(2xt - t^2)H_s(x-t)$

UNIT 14:- LAGUERRE POLYNOMIALS

CONTENTS:

14.1 *INTRODUCTION:-*

Laguerre polynomials are a family of orthogonal polynomials named after the French mathematician Edmond Laguerre. They are described as solutions to the Laguerre differential equation, which arises in various physical and mathematical problems, including quantum mechanics, statistical mechanics, and probability theory.

14.2 *OBJECTIVES:-*

After studying this unit you will be able to

- Understanding the properties of Laguerre polynomials.
- Understanding how to solve this equation using Laguerre polynomials.
- Analyzing the use of Laguerre polynomials in this context is important for studying these systems.

14.3 *LAGUERRE EQUATIONS.:-*

Laguerre's equation of order n is

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + (1-x)\left(\frac{dy}{dx}\right) + ny = 0 \qquad \dots(1)$$

where n is a positive integer. We get the equation (1) which is finite for all values of x and which tends to infinity no faster than $e^{x/2} as\ x \to \infty$.

Suppose

$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad C_0 \neq 0 \qquad \ldots (2)$

Differentiating (2) with respect to x and substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (1)

$$x\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} + (1-x)\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} + n\sum_{m=0}^{\infty} C_m\, x^{k+m} = 0$$

Or

$$\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-1} + \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} - \sum_{m=0}^{\infty} C_m(k+m)x^{k+m} + n\sum_{m=0}^{\infty} C_m\, x^{k+m-1} = 0$$

Or

$$\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1}(k+m-1+1) - \sum_{m=0}^{\infty} C_m x^{k+m}(k+m-n) = 0$$

$$\sum_{m=0}^{\infty} C_m(k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m x^{k+m}(k+m-n) = 0 \quad \ldots (3)$$

Now we have indicial equation, the coefficient of smallest power of x, in equation (3) and describe

$$C_0 k^2 = 0, \quad \forall k^2 = 0 \quad (\because C_0 \neq 0) \qquad \ldots (4)$$

From equation (4), next equating coefficient ofx^{k+m-1}, we get

$C_m(k+m)^2 - C_{m-1}(k+m-1-n) = 0$ or

$$C_m = \frac{k+m-1-n}{(k+m)^2} C_{m-1} \qquad \ldots (5)$$

Since the two independent solution in this case are $y_{k=0}$ and $\left(\frac{\partial y}{\partial k}\right)_{k=0}$. But $\left(\frac{\partial y}{\partial k}\right)_{k=0}$ assumes a term of $\log x$, so we infinite when $x = 0$.

putting $k = 0$ in equation (5) and (2), we get

$$C_m = \frac{m-1-n}{(k+m)^2} C_{m-1} \qquad \ldots (6)$$

$$y = \sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + \cdots \qquad \ldots (7)$$

Substituting $m = 1,2,3, \ldots \ldots$ in (6), we get

$C_1 = -\frac{-n}{1^2} C_0 = \frac{(-1)}{(1!)^2} nC_0$, $C_2 = \frac{1-n}{2^2} C_1 = \frac{(n-1)}{(2)^2} \times (-1)nC_0 = (-1)^2 \frac{n(n-1)}{(2!)^2} C_0$

$$C_3 = \frac{2-n}{3^2} C_2 = -\frac{(n-2)}{(3)^2} \times (-1)^2 \frac{n(n-1)}{(3!)^2} C_0$$
$$= (-1)^3 \frac{n(n-1)(n-2)}{(3!)^2} C_0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Since

$$C_r = (-1)^2 \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} C_0, \quad \text{for} r \le n.$$

Also, $C_{n+1} = C_{n+2} = C_{n+3} = \cdots = 0.$

Putting the values of $C_1, C_{2,} C_3 \cdots\cdots\cdots$ in equation (7)

$$y = C_0 \left[1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} + \cdots + (-1)^r \frac{n(n-1)\cdots\cdots(n-r+1)}{(r!)^2} x^r + \cdots\cdots\right]$$

$$= C_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1)\cdots\cdots(n-r+1)}{(r!)^2} x^r$$

$$= C_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1)\cdots\cdots(n-r+1)(n-r)(n-r-1)\cdots 3.2.1}{(r!)^2} x^r$$

Hence ,

$$y = C_0 \sum_{r=0}^{\infty} (-1)^r \frac{n!}{(n-r)(r!)^2} x^r$$

Taking $C_0 = 1$, we express the corresponding solution as the **Laguerre polynomial** and it is denoted by $L_n(x)$.

$$L_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n!}{(n-r)(r!)^2} x^r$$

Laguerre polynomial of order (or degree) $\boldsymbol{n}$ is denoted and defined by

$$L_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n!}{(n-r)(r!)^2} x^r$$

14.4 *GENERATING FUNCTION*:-

THEOREM I: show that the generating function for Laguerre polynomials is$\frac{exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$.

Proof: Now, we take L.H.S.

$\Rightarrow \quad \frac{exp\{-xt/(1-t)\}}{1-t} = \frac{1}{1-t} \sum_{r=0}^{\infty} \left(\frac{-xt}{1-t}\right)^r . \frac{1}{r!}$ as $\exp x = e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r (1-t)^{-(r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!s!}$,
{By binomial theorem}

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^r \frac{(r+s)!}{r!\,s!} x^r t^{r+s}$$

Suppose r be fixed, then the coefficient of t^n can be given by setting $r + s = n$ i.e., $s = n - r$.
Hence the coefficient of t^n is obtained by
$\sum_{r=0}^{\infty} (-1)^r \frac{n!}{(n-r)(r!)^2} x^r$ i.e., $L_n(x)$.

14.5 *RODRIGUE'S FORMULA*.-

Expression (Rodrigue's Formula)for the **Laguerre polynomial**
Show that $\boldsymbol{L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})}$
Proof: By Leibnitz theorem, we get

$$D^n(uv) = d^n \frac{(uv)}{dx^n}$$
$$= D^n u.v + n_{C_1} D^{n-1} u.Dv + \cdots\cdots + n_{C_r} D^{n-r} u.D^r v + \cdots + uD^n v$$

i.e., $D^n(uv) = \sum_{r=0}^{n} n_{C_r} D^{n-r} u.D^r v$

$\therefore \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} n_{C_r} D^{n-r} x^n . D^r e^{-x}$, from (1)

$= \sum_{r=0}^{n} \frac{e^x}{n!} \times n_{C_r} \frac{n!}{\{n-(n-r)!\}} x^{n-(n-r)} . (-1)^r e^{-x}$, as $D^n x^m = \frac{m!}{(m-n)!} x^{m-n}$ and $D^n e^{ax} = a^n e^{ax}$

$$= \sum_{r=0}^{n} \frac{e^x}{n!} \times \frac{n!}{r!\,(n-r)!} \frac{n!}{r!} x^r . (-1)^r e^{-x} = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r = L_n(x)$$

If we use the definition of Laguerre polynomial of (or degree)n , we obtain

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) \qquad \ldots (1)$$

We know that the given equation

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \qquad \ldots (2)$$

Putting $n = 0,1,2,3 \ldots..$in equation (2)
$\Rightarrow L_0(x) = \frac{e^x}{0!} \frac{d^0}{dx^0} (x^0 e^{-x}) = 1$, $L_1(x) = \frac{e^x}{1!} \frac{d^1}{dx^1} (x^1 e^{-x}) = e^x(e^{-x} - xe^{-x}) = 1 - x \Rightarrow$ $L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2} (x^2 e^{-x}) = \frac{e^x}{2!} \frac{d}{dx} \left\{ \frac{d}{dx} (x^2 e^{-x}) \right\} = \frac{e^x}{2!} \frac{d}{dx} (2xe^{-x} - x^2 e^{-x}) = \frac{e^x}{2!} [2e^{-x} + 2x(-e^{-x}) - \{2xe^{-x} + x^2(-e^{-x})\}] = \frac{1}{2!} (2 - 4x + x^2)$,

$\Rightarrow \quad L_3(x) = \frac{e^x}{3!}\frac{d^3}{dx^3}(x^3e^{-x}) = \frac{e^x}{3!}\frac{d^2}{dx^2}\left\{\frac{d}{dx}(x^3e^{-x})\right\} = \frac{e^x}{3!}\frac{d^3}{dx^3}(3x^2e^{-x} - x^3e^{-x}) = \frac{e^x}{3!}\frac{d}{dx}\left[\frac{d}{dx}\{(3x^2 - x^3)e^{-x}\}\right] = \frac{e^x}{3!}\frac{d}{dx}[(6x - 3x^2)e^{-x} - (3x^2 - x^3)e^{-x}] = \frac{e^x}{3!}\frac{d}{dx}[(6x - 6x^2 + x^3)e^{-x}] = (6 - 18x + 9x^2 - x^3)/3!$

$\Rightarrow \quad L_4(x) = \frac{e^x}{4!}\frac{d^4}{dx^4}(x^4e^{-x}) = \frac{e^x}{4!}\frac{d^3}{dx^3}\left[\frac{d}{dx}(x^4e^{-x})\right] = \frac{e^x}{4!}\frac{d^3}{dx^3}[4x^3e^{-x} - x^4e^{-x}] = \frac{e^x}{4!}\frac{d^2}{dx^2}\left[\frac{d}{dx}\{(4x^3 - x^4)e^{-x}\}\right] = \frac{e^x}{4!}\frac{d^2}{dx^2}[(12x^2 - 4x^3)e^{-x} - (4x^3 - x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}\left[\frac{d}{dx}(12x^2 - 8x^3 + x^4)e^{-x}\right] = \frac{e^x}{4!}\frac{d}{dx}[(24x - 24x^2 + 4x^3)e^{-x} - (12x^2 - 8x^3 + x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}[(24x - 36x^2 + 12x^3 - x^4)e^{-x}] = \frac{e^x}{4!}[(24 - 72x + 36x^2 - x^4)e^{-x} - (24x - 36x^2 + 12x^3 - x^4)e^{-x}] = \frac{1}{4!}(24x - 96x + 72x^2 - 16x^3 + x^4)$

Similarly

$\Rightarrow \quad L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$

…………………………………………………………………………………………………

…………………………………………………………………………………………………

$\Rightarrow \quad L_n(x) = \frac{1}{n!}\{(-x)^n + n^2(-x)^{n-1} + \cdots + n(n!)(-x) + n!\}$

In table 1 we express the first few polynomials $L_n(x)$ and as shown as graph

$L_0(x) =$	1
$L_1(x) =$	$1 - x$
$L_2(x) =$	$\frac{1}{2!}(2 - 4x + x^2)$
$L_3(x) =$	$\frac{1}{3!}(6 - 18x + 9x^2 - x^3)$
$L_4(x) =$	$\frac{1}{4!}(24x - 96x + 72x^2 - 16x^3 + x^4)$
$L_5(x) =$	$\frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$
	………………………………………
$L_n(x) =$	$\frac{1}{n!}\{(-x)^n + n^2(-x)^{n-1} + \cdots + n(n!)(-x) + n!\}$

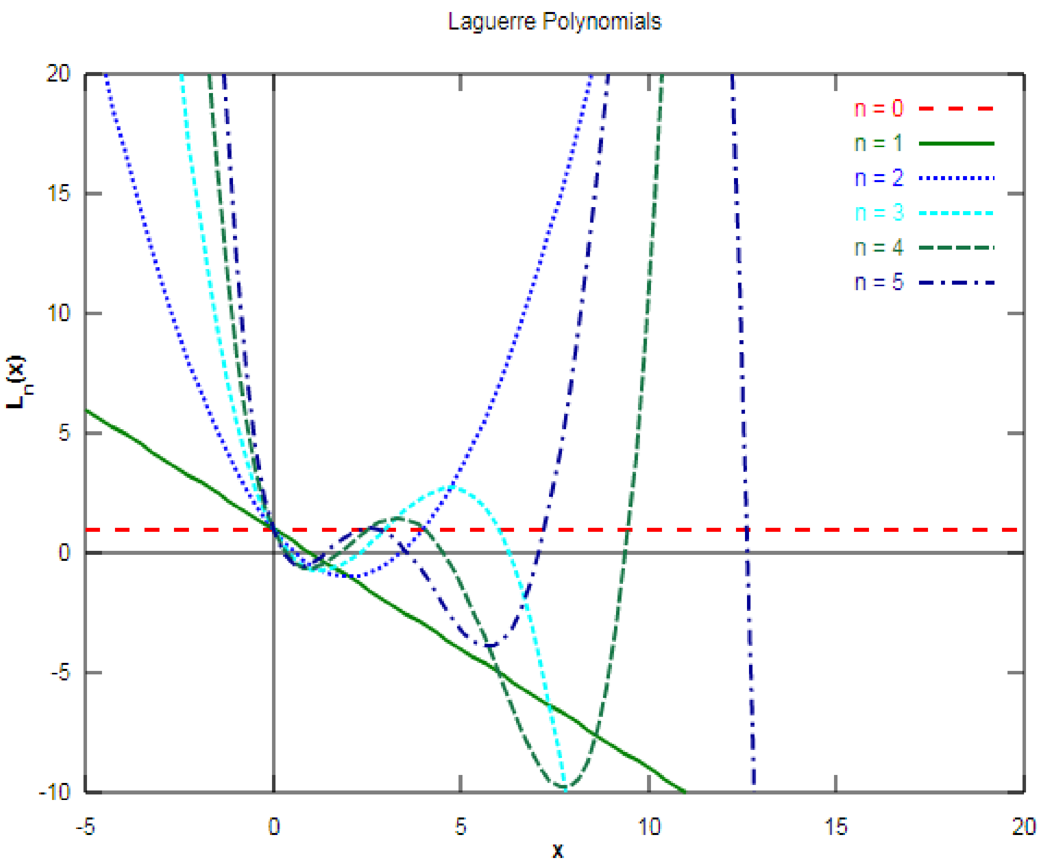


Fig.1

14.6 *ORTHOGONALITY PROPERTIES*.-

Laguerre polynomial is a kind of orthogonal polynomials whose inner product is zero. Let the differential equation satisfied by Laguerre **polynomials of** degree n and k.

$$x\frac{d^2L_n}{dx^2}+(1-x)\left(\frac{dL_n}{dx}\right)+nL_n=0 \qquad ...(1)$$

$$x\frac{d^2L_k}{dx^2}+(1-x)\left(\frac{dL_k}{dx}\right)+nL_k=0 \qquad ...(2)$$

Multiplying (1) by $e^{-x}L_k(x)$ and (2) by $e^{-x}L_n(x)$ and subtract.

$$\frac{d}{dx}\left[xe^{-x}\left\{L_k(x)\frac{dL_n}{dx}-L_n(x)\frac{dL_k}{dx}\right\}\right]+(n-k)e^{-x}L_k(x)L_n(x)=0$$

Integrating this expression $x\to 0\ to\infty$

$\Rightarrow \quad xe^{-x}\left[xe^{-x}\left\{L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right\}\right]_0^\infty + (n - k)\int_0^\infty e^{-x}L_k(x)L_n(x)\,dx = 0$

$\Rightarrow \quad (n-k)\int_0^\infty e^{-x}L_k(x)L_n(x) = 0$

If $n \neq k$, then

$\Rightarrow \quad \int_0^\infty e^{-x}L_k(x)L_n(x)dx = 0 \quad \ldots(3)$

This is the by Laguerre **polynomials of** different degree (n and k) are orthogonal to the interval$(0,\infty)$ with e^{-x}.

To define the orthogonality relation for $n = k$. Now we using the generating function, we get

$\frac{exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^\infty L_n(x)t^n$ and $\frac{exp\{-xt/(1-t)\}}{1-t} = \sum_{k=0}^\infty L_k(x)t^k$

we take the product of two sides

$\Rightarrow \quad \frac{exp\{-2xt/(1-t)\}}{(1-t)^2} = \sum_{n=0}^\infty L_n(x)t^n \sum_{n=0}^\infty L_k(x)t^k$

Multiplying both sides by e^{-x} and integrates from 0 to ∞

$\Rightarrow \quad \frac{1}{(1-t)^2}\int_0^\infty e^{-\frac{1-t}{1+t}x}\,dx = \sum_{n=0}^\infty \sum_{k=0}^\infty t^n t^k \int_0^\infty e^{-x}L_n(x)L_k(x)dx \quad \ldots(4)$

$\Rightarrow$ For $n = k$, the above equation reduce to

$\sum_{n=0}^\infty t^{2n}\int_0^\infty \int_0^\infty e^{-x}L_n^2(x)dx$

Now we use the formula $\int e^{-ax}dx = -\frac{e^{-ax}}{a}$, putting the value in equation (4)

$\Rightarrow \quad \left[\frac{1}{(1-t)^2}(-)\left(\frac{1-t}{1+t}\right)e^{-\frac{1-t}{1+t}x}\right]_0^\infty = \frac{1}{(1-t)(1+t)} = \frac{1}{1-t^2}$ for $t <<<< 1$, then we get

$\Rightarrow \quad \frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots\cdots\cdots\cdots\cdots = \sum_{n=0}^\infty t^{2n}$

$\Rightarrow \quad \sum_{n=0}^\infty t^{2n} = \sum_{n=0}^\infty t^{2n}\int_0^\infty e^{-x}L_n^2(x)dx$

On comparing$t^{2n}\forall\ n$, we get

$\Rightarrow \quad \int_0^\infty e^{-x}L_n^2(x)dx = 1 \quad \cdots(5)$

From (3) and (5) may now combined to obtain the orthogonality relation for Laguerre polynomials as

$$\Rightarrow \quad \int_0^\infty e^{-x}L_n(x)L_k(x)\,dx = \delta_{nk} = \begin{cases}0, if\ n \neq k\\ 1, if\ n = k\end{cases}$$

14.7 *LAGUERRE SERIES EXPENSION*:-

THEOREM. If $f(x)$ is polynomials of degree m, prove that $f(x)$ may be expressed in form

$$f(x) = \sum_{r=0}^m C_r L_r(x), \quad \text{where } C_r = \int_0^\infty e^{-x}\,L_r(x)f(x)dx$$

Proof: Let $f(x)$ be a polynomial of degree m, we get

$\Rightarrow \quad f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \qquad \cdots (1)$

$\Rightarrow$ Again, \

$$L_m(x) = k_m x^m + k_{m-1}x^{m-1} + \cdots + k_1 x + k_0 \quad \cdots (2)$$

$\Rightarrow$ Let $f(x) - (a_m/k_m)L_m(x)$. Two cases arise

Case (i) Let $f(x) - (a_m/k_m)L_m(x) = 0$

so that $f(x) = (a_m/k_m)L_m(x)$, Which is required result.

Case (ii) suppose $f(x) = (a_m/k_m)L_m(x) = g_{m-1}, g_{m-1}(x)$ being a polynomial of degree $m-1$. Taking $C_m = a_m/k_m$, then we get

$\Rightarrow \quad f(x) = C_m L_m(x) + g_{m-1}(x) \qquad \cdots (3)$

Taking $g_{m-1}(x)$ in place of $f(x)$, we obtain

$\Rightarrow \quad g_{m-1}(x) = C_{m-1}L_{m-1}(x) + g_{m-2}(x) \qquad \ldots (4)$

Putting the value of (4) in (3)

$\Rightarrow \quad f(x) = C_m L_m(x) + C_{m-1}L_{m-1}(x) + g_{m-2}(x) \qquad \cdots (5)$

$\Rightarrow \quad f(x) = C_m L_m(x) + C_{m-1}L_{m-1}(x) + \cdots + C_1 L_1(x) + C_0 L_0(x) = \sum_{r=0}^{m} C_r L_r(x) \qquad \cdots (6)$

$\Rightarrow \sum_{r=0}^{m} C_s L_s(x) \qquad \sum_{r=0}^{m} C_r L_r(x) = \sum_{r=0}^{m} C_s L_s(x),$ equation (6)obtain

$\Rightarrow \quad f(x) = \sum_{r=0}^{m} C_s L_s(x) \qquad \ldots (7)$

Multiplying both sides of equation (7) by $e^{-x}L_r(x)$ and integrating w.r.t. x then

$\Rightarrow \int_0^\infty e^{-x} L_r(x) f(x)dx = \sum_{s=0}^{m} C_s \left\{\int_0^\infty e^{-x} L_r(x)L_s(x)dx\right\} \ldots (8)$

$\Rightarrow$ but $\int_0^\infty e^{-x} L_r(x)L_s(x)dx = \begin{cases} 0, if\ r = s \\ 1, if\ r \neq s \end{cases} \qquad \ldots (9)$

From equations (8) and (9) obtain

$\Rightarrow \quad C_r = \int_0^\infty f(x)L_r(x)dx \qquad \ldots (10)$

which is required solution.

14.8 *RECURRENCE RELATION:-*

We will show that some important recurrence relations as given below

I. $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$
II. $xL'_n(x) = nL_n(x) - nL_{n-1}(x)$
III. $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$

Proof: Recurrence Relation I: Now we using the generating function

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{exp\{-xt/(1-t)\}}{1-t} \qquad \dots(1)$$

Differentiating both sides with respect to (1), we have

$$\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = \frac{1}{(1-t)^2} exp\left\{-\frac{xt}{1-t}\right\} - \frac{1}{1-t} \times exp\left\{-\frac{xt}{1-t}\right\} \times \frac{x}{(1-t)^2}$$

$$= \frac{1}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)t^n - \frac{x}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)t^n$$

Multiplying both sides by $(1-t)^2$

$$(1-t)^2\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = \sum_{n=0}^{\infty} L_n(x)t^n - x\sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1} \qquad \dots(2)$$

Solving equation (2), we get

$$(1+t^2-2t)\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = \sum_{n=0}^{\infty} L_n(x)t^n - x\sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1}$$

$$\left(\sum_{n=0}^{\infty} L_n(x).nt^{n-1} + \sum_{n=0}^{\infty} L_n(x).nt^{n-1+2} - 2\sum_{n=0}^{\infty} L_n(x).nt^{n-1+1}\right) = \sum_{n=0}^{\infty} L_n(x)t^n - x\sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1}$$

$$\left(\sum_{n=0}^{\infty} L_n(x).nt^{n-1} + \sum_{n=0}^{\infty} L_n(x).nt^{n+1} - 2\sum_{n=0}^{\infty} L_n(x).nt^{n}\right) = \sum_{n=0}^{\infty} L_n(x)t^n - x\sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1}$$

$\therefore$ We equating the coefficient of t^n from both sides, we obtain

$$(n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$

$$(n+1)L_{n+1}(x) - 2nL_n(x) + nL_{n-1}(x) - L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$

$$(n+1)L_{n+1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x) + 2nL_n(x) - nL_{n-1}(x) + L_{n-1}(x)$$

$$(n+1)L_{n+1}(x) = L_n(x)(2n - x + 1) - nL_{n-1}(x)$$

Hence $\boldsymbol{(n+1)L_{n+1}(x) = (2n+1-x)(n+1)L_n(x) - n(n+1)L_{n-1}(x)}$ **is required solution.**

Recurrence Relation II: $xL'_n(x) = nL_n(x) - nL_{n-1}(x)$

Proof: Again we using the generating function

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{exp\{-xt/(1-t)\}}{1-t} \quad \ldots (1)$$

Differentiating w. r.t.x, we obtain

$$\sum_{n=0}^{\infty} L'_n(x)t^n = \frac{1}{(1-t)} \times exp\left\{-\frac{xt}{1-t}\right\} \times \left\{\frac{-t}{1-t}\right\}$$

$$\sum_{n=0}^{\infty} L'_n(x)t^n = \frac{-t}{(1-t)}\sum_{n=0}^{\infty} L_n(x)t^n, \qquad from(1)$$

$$(1-t)\sum_{n=0}^{\infty} L'_n(x)t^n = -t\sum_{n=0}^{\infty} L_n(x)t^n$$

$$\left(\sum_{n=0}^{\infty} L'_n(x)t^n - t\sum_{n=0}^{\infty} L'_n(x)t^n\right) = -\sum_{n=0}^{\infty} L_n(x)t^{n+1}$$

$$\sum_{n=0}^{\infty} L'_n(x)t^n - \sum_{n=0}^{\infty} L'_n(x)t^{n+1} = -\sum_{n=0}^{\infty} L_n(x)t^{n+1}$$

We equating the coefficient of t^n from both sides, we give

$$L'_n(x) - L'_{n-1}(x) = -L_{n-1}(x)$$

$$\boldsymbol{L'_{n-1}(x) = L_{n-1}(x) + L'_n(x)}$$

Replacing n by $n+1$

$$\boldsymbol{L'_{n+1}(x) = L'_n(x) - L_n(x)}$$

From recurrence relation I

$$(n+1)L_{n+1}(x) = L_n(x)(2n - x + 1) - nL_{n-1}(x) \quad \ldots (2)$$

Again differentiating w.r.t. x, we obtain

$$(n+1)L'_{n+1}(x) = L'_n(x)(2n - x + 1) - L_n(x) - nL'_{n-1}(x) \quad \ldots (3)$$

Putting the values of L'_{n-1} and L'_{n+1} in (3)

$$(n+1)[L'_n(x) - L_n(x)] = L'_n(x)(2n - x + 1) - L_n(x) - n[L_{n-1}(x) + L'_n(x)]$$

$$L'_n(x)(n+1) - L_n(x)(n+1)$$
$$= 2nL'_n(x) - xL'_n(x) + L'_n(x) - L_n(x) - nL_{n-1}(x) + nL'_n(x)$$
$$L'_n(x)[n+1-2n+x-1+n] = L_n(x)[n+1-1] - nL_{n-1}(x)$$
$$\boldsymbol{xL'_n(x) = nL_n(x) - nL_{n-1}(x)}$$

Recurrence Relation III: $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$

Proof: The generating function of Laguerre polynomials is

$$\frac{exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \qquad \cdots(1)$$

Differentiating w. r.t.x, we have

$$\sum_{n=0}^{\infty} L'_n(x)t^n = \frac{1}{1-t} exp\left[-\frac{xt}{1-t}\right]\left[\frac{-t}{1-t}\right]$$

$$\left[\frac{-t}{1-t}\right]\sum_{n=0}^{\infty} L'_r(x)t^r = -t(1-t)^{-1}\sum_{n=0}^{\infty} L_r(x)t^r \ , \quad using\ (8)$$

$$= -t\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} L_r(x)\, t^r\ , by\ binomial\ expension$$

$$\sum_{n=0}^{\infty} L'_n(x)t^n == -t\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} L_r(x)\, t^{r+s+1} \quad ...(2)$$

It is clearly that t^n on L.H.S. of equation (2) is $L'_n(x)$ and now we obtain t^n on R.H.S. of (2). We substituting $r+s+1=n \Rightarrow s=n-r-1$ & $s \geq 0 \Rightarrow n-r-1 \geq 0 \Rightarrow r \leq n-1$, which obtain all value of r are $1,2,3,\cdots\cdots\cdots n-1$ and $\forall\, r$, $-L_r(x)$is coefficient of t^n.

Hence the total coefficient of t^n on R.H.S. of (2) is given below

$$-\sum_{n=0}^{n-1} L_n(x)$$

Thus, the equating the coefficients of t^n from both sides of (2), we obtain

$$L'_n(x) = -\sum_{n=0}^{n-1} L_n(x)$$

SOLVED EXAMPLES

EXAMPLE1. Show that (i) $L_n(0) = 1$ (ii) $L_n(0) = n!$

SOLUTION: We know that

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{1}{1-t}e^{-tx/(1-t)} \qquad ...(1)$$

Substituting $x = 0$ in (1), by binomial theorem

$$\sum_{n=0}^{\infty} t^n L_n(0) = (1-t)^{-1} = 1 + t + t^2 + \cdots,$$

$$\sum_{n=0}^{\infty} t^n L_n(0) = \sum_{n=0}^{\infty} t^n \qquad \ldots (2)$$

Equating the coefficient t^non both sides, we have $L_n(0) = 1$.

(ii) $L_n(0) = n!$

SOLUTION: We know that, the generating function is

$$\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-xt/(1-t)} \qquad \cdots (3)$$

Substituting $x = 0$ in equation (3), $\sum_{n=0}^{\infty} t^n L_n(0) = (1-t)^{-1} = 1 + t + t^2 + \cdots$,

Equating the coefficient t^n on both sides, we have $\frac{t^n L_n(0)}{n!} = 1$, $or\ L_n(0) = n!$

EXAMPLE2. Show that $xL''_n(x) + (1-x)L'_n + L_n(x) = 0$ and hence deduce that$L'_n = -n$.

SOLUTON: Now we use the Laguerre's equation

$$x(x^2y/dx^2) + (1-x)\,dy/dx + ny = 0,$$

Putting $x = 0$ in above equation

$L'_n(0) + nL_n(0)$ or $L_n(0) = -n,$ as $L_n(0) = 1.$

EXAMPLE3. Expand $x^3 + x^2 - 3x + 2$ in series of Laguerre polynomials.

SOLUTION: We know that the Laguerre polynomials are

$$L_0(x) = 1, L_1(x) = 1 - x, L_2(x) = \frac{(2 - 4x + x^2)}{2}\ and\ L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3)$$

Now

$$x^3 = 6 - 18x + 9x^2 - 6L_3(x) \qquad \ldots (1)$$

$$x^2 = 4x - 2 + 2L_2(x) \qquad \ldots (2)$$

$x = 1 - L_1(x)$ and $L_0(x) = 1$ ... (3)

Now

$x^3 + x^2 - 3x + 2 = 6 - 18x + 9x^2 - 6L_3(x) + x^2 - 3x + 2$, by (1)

$= 8 - 21x + 10x^2 - 6L_3(x) = 8 - 21x + 10[4x - 2 + 2L_2(x)] - 66L_3(x)$, by (2)

$= -12 + 19x + 20L_2(x) - 6L_3(x) = -12 + 19[1 - L_1(x)] + 20L_2(x) - 6L_3(x)$, by (3)

$= 7 - 19L_1(x) + 20L_2(x) - 6L_3(x) = -7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x)$, by (3)

SELF CHECK QUESTIONS

(SCQ-1) L_{n+k} is a Laguerre polynomial of degree……..

(SCQ-2) Associated Laguerre differential equation is …....

(SCQ-3) Express $10 - 23x + 10x^2 - x^3$ in terms of Laguerre polynomials.

14.9 *SUMMARY:-*

In this unit, first of all we are explained the definitions of Laguerre Equations and discussed about the Generating function for Laguerre polynomials, Rodrigue's Formula, Orthogonality Properties Laguerre Series Expansions, Recurrence Relation (Formulae). Finally, the Laguerre polynomials are an important tool in the study of differential equations and their solutions.

14.10 *GLOSSARY:-*

- Series Expansion
- Laguerre equation

14.11 *REFERENCES:-*

- Dunham Jackson (2004) Fourier series and Orthogonal Polynomials.
- Refaat El Attar (2006) Special Function and Orthogonal Polynomials.

14.12 *SUGGESTED READING:-*

- M.D. Raisinghania, (2018). Ordinary and Partial Differential equation (18^{th} Edition), S. Chand.
- Carlo Viola (2016) An introduction to Special Function.
- **MathWorld(Wolfram):**https://mathworld.wolfram.com/Laguerre Polynomial.html
- **Wikipedia:** https://en.wikipedia.org/wiki/Laguerre_polynomials
- **Digital Library of Mathematical Functions:** https://dlmf.nist.gov/3

14.13 *TERMINAL QUESTIONS:-*

(TQ-1) If $L_n(x)$ to be the coefficient of t^n in the expansion of $\frac{1}{1-t} exp\left(\frac{xt}{1-t}\right)$, prove that $\int_0^{\pi/2} \frac{e^{-tan\theta}}{cos^2\theta} L_n(tan\theta) L_m(tan\theta) d\theta = \delta_{mn}$.

(TQ-2) If $\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} exp\left\{\frac{-tx}{1-t}\right\}$, prove that

i. $L'_n(x) = n[L'_{n-1}(x) - L_{n-1}(x)]$
ii. $xL'_n(x) = nL_n(x) - n^2L_{n-1}(x)$

(TQ-3) Prove that $\int_0^\infty e^{-st}L_n(t)dt = (1/s) \times (1 - 1/s)^n$.

(TQ-4) Prove that $\int_0^\infty e^{-y}x^kL_n(x)dx = \begin{cases} 0, if\ k < n \\ (-1)^n n!, if\ k = n. \end{cases}$

(TQ-6) State and prove that generating function for Laguerre polynomial.

(TQ-7) Prove that the recurrence relations of the following

I. $(n + 1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x)$
II. $xL'_n(x) = nL_n(x) - nL_{n-1}(x)$
III. $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$

14.14 *ANSWERS:-*

SELF CHECK ANSWERS

(SCQ-1) $n + k$

(SCQ-2) $xy'' + (1 - x + k)y' + ny = 0$

(SCQ-3) $L_0(x) + L_1(x) + 2L_2(x) + 6L_3(x)$

Teen Pani Bypass Road, Transport Nagar
Uttarakhand Open University, Haldwani, Nainital-263139
Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232,
E-mail:info@uou.ac.in
Website: https://www.uou.ac.in/