A-0613

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MAT-601

M.Sc. MATH (MSCMT)

(Advanced Complex Analysis)

3rd Semester Examination, Session December 2024

Time : 2:00 Hrs. Max. Marks : 70

Note* :–** This paper is of Seventy (70) marks divided into Two (02) Sections 'A' and 'B'. Attempt the questions contained in these Sections according to the detailed instructions given therein. ***Candidates should limit their answers to the questions on the given answer sheet. No additional (B) answer sheet will be issued.

Section–A

(Long Answer Type Questions) $2 \times 19 = 38$

***Note* :–** Section 'A' contains Five (05) Long-answer type questions of Nineteen (19) marks each. Learners are required to answer any *two* (02) questions only.

1. State and prove the Cauchy integral formula. Furthermore, evaluate the following integral :

$$\int_{\gamma_1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$$

where $\gamma_1(t) = \frac{3}{2} e^{it}, 0 \leq t \leq 2\pi.$

2. State the Schwarz Lemma. Let D : = $\{z \in C : |z| < 1\}$ be the open unit disc. Answer the following questions :

   (i) Does there exists an analytic function $f : D \to D$ with $f(1/4) = 0$ and $f(-3/4) = 17/20$?

   (ii) Does there exists an analytic function $f : D \to D$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

3. Classify all the singularities of that for an entire function $f : C \to C$, ∞ is a pole if and only if f is a polynomial.

4. Evaluate :

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$$

5. Show that any Möbius transformation which maps the unit disc D onto itself is of the form :

$$\varphi(z) = e^{i\theta} \frac{z+\lambda}{1+\bar{\lambda} z}, \quad \lambda \in D, \theta \in R$$

Section–B

Short Answer Type Questions 4×8=32

***Note* :–** Section 'B' contains Eight (08) Short-answer type questions of Eight (08) marks each. Learners are required to answer any *four* (04) questions only.

1. State and prove Lioville's theorem.

2. Let $\in > 0$ and f be holomorphic function in :

$$\{z \in \mathrm{C} : |z| < 1 + \in\}$$

such that whenever $|z| = 1$ we have $|f(z)| > 1$ and $f(0) = i/2$ Show that f has a zero in the unit disc $\{z \in \mathrm{C} : |z| < 1\}$.

3. Find a Möbius transformations that maps 0, 1, ∞ to 1, i, –1 respectively.

4. Show that zeros of a non-constant holomorphic function are isolated.

5. State and prove the fundamental theorem of algebra.

6. Evaluate the following integral :

$$\int_{\mathrm{C}} \frac{dz}{z - p}$$

where C is a closed curve as shown in the following figure, and p is a point inside C.

7. Suppose that $\gamma : [a, b] \to C$ is a smooth curve and f is continuous on the trace γ^*. If $|f(z)| \leq M$ for all $z \in \gamma^*$ and L is the length of γ then show that :

$$\left|\int_{\gamma} f(z)\, dz\right| \leq ML$$

8. Show that the function $u(x, y) = e^x \cos y$ is a harmonic function. Find a harmonic function $v(x, y)$ such that $f = u + iv$ is a holomorphic function. Write f in terms of z.

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