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MT–606

M.A./M.Sc. MATHEMATICS (MAMT/MSCMT)

(Analysis and Advanced Calculus–II)

4th Semester Examination, 2024 (June)

Time : 2:00 Hrs. Max. Marks : 70

Note* :–** This paper is of Seventy (70) marks divided into Two (02) Sections 'A' and 'B'. Attempt the questions contained in these Sections according to the detailed instructions given therein. ***Candidates should limit their answers to the questions on the given answer sheet. No additional (B) answer sheet will be issued.

Section–A

(Long Answer Type Questions) 2×19=38

***Note* :–** Section 'A' contains Five (05) Long-answer type questions of Nineteen (19) marks each. Learners are required to answer any *two* (02) questions only.

1. Define the following operator :

   (i) Adjoint Operator.

   (ii) Self-Adjoint Operator.

   (iii) Normal Operator.

   (iv) Unitary Operator.

2. Let H be a given Hilbert space and T* be adjoint of the operator. Then prove that T* is a bounded linear transformation and T determines T* uniquely.

3. Prove that if T is a normal operator on a Hilbert space H, then x is an eigenvector of T with eigenvalue λ iff x is an eigenvector of T* with $\bar{\lambda}$ as eigenvalue.

4. Let X and X be any two Banach spaces over the same field K of scalars and V be an open subset of X. Let $f : \mathrm{V} \to \mathrm{Y}$ be continuous function. Let u, v be any *two* distinct points of V such that $[u, v] \subset \mathrm{V}$ and f is differentiable in $[u, v]$. Then prove that :

$$\| f(v) - f(u) \| \leq \| v - u \| \sup\{\| \mathrm{D}f(x) \| : x \in [u, v]\}$$

5. Let f be a regulated function on a compact interval $[a, b]$ of R into a Banach space X over K and g be a continuous linear map of X into a Banach space Y over K. Proof that gof is regulated and :

$$\int_a^b (gof) = g\left(\int_a^b f\right)$$

Section–B

(Short Answer Type Questions) 4×8=32

***Note* :–** Section 'B' contains Eight (08) Short-answer type questions of Eight (08) marks each. Learners are required to answer any *four* (04) questions only.

1. Define the following :

   (i) Banach space with example.

   (ii) Hilbert space with example.

2. Define the following :

   (i) Vector space.

   (ii) Eigen values and Eigen vectors.

3. Define the following :

   (i) Continuously differentiable maps (C^1-maps).

   (ii) Lipschitz's Property.

4. State the following theorem :

   (i) Taylor's theorem.

   (ii) Taylor' formula with Lagrange's Reminder.

5. Prove that an operator T on H is self-adjoint, then $(Tx,y) = (x, Ty\} \forall x, y \in$ H and conversely.

6. Prove that a closed linear subspace M of a Hilbert space H reduces an operator T if and only if M is invariant under both T and T*.

7. Let X be a Banach space over the field K of scalars, and let I be an open interval in R containing [0, 1]. If $\psi : I \to X$ is $(n + 1)$ times continuously differentiable function of a single variable $t \in I$. Then prove that :

$$\psi(1) = \psi(0) + \psi'(0) + \frac{\psi''(0)}{2!} + \frac{\psi n(0)}{n!} + \int_0^1 \frac{(1-t)^n}{n!} \psi_{(i)}^{n+1} dt$$

8. Prove that if T is a Normal Operator on H, then $\| T^2 \| = \| T \|^2$.

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