A-125

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MT–506

M.A./M.Sc. MATHEMATICS (MAMT/MSCMT)

(Advanced Algebra–II)

2nd Semester Examination, 2024 (June)

Time : 2:00 Hrs. Max. Marks : 70

Note* :–** This paper is of Seventy (70) marks divided into Two (02) Sections 'A' and 'B'. Attempt the questions contained in these Sections according to the detailed instructions given therein. ***Candidates should limit their answers to the questions on the given answer sheet. No additional (B) answer sheet will be issued.

Section–A

(Long Answer Type Questions) 2×19=38

***Note* :–** Section 'A' contains Five (05) Long-answer type questions of Nineteen (19) marks each. Learners are required to answer any *two* (02) questions only.

1. (a) Define :

    (i) Automorphism

    (ii) Splitting field

   (b) Let H be a subgroup of all automorphisms of a field K. Then prove that the fixed field of H is subfield of K.

2. (a) Show that Galois group of $x^4 + 1 \in Q[x]$ is the Klein four group.

   (b) Prove that the order of the Galois group G(K|F) is equal to the degree of K over E, i.e., :

$$o[G(K|F)] = [K : F]$$

3. Let V and V′ be n and m dimensional vector space over a field F. Then prove that for given basis B and B′ of V and V′ respectively, the function assigning to each linear transformation t : V → V′ its matrix $M_B^B(t)$ relative to basis B, B′ is an isomorphism between the vector space Hom(V, V′) and the space $F^{m \times n}$ of all $m \times n$ matrices over F, i.e., Hom(V, V′) $\cong F^{m \times n}$.

4. Prove that the following statements are equivalent for any matrix $A \in F^{n \times n}$.

(i) A has a left inverse in $F^{n \times n}$

(ii) nullity (A) = 0

(iii) rank (A) = n

(iv) A has a right inverse in $F^{n \times n}$

(v) A has two sided inverse in $F^{n \times n}$

5. (a) If W is a subspace of an inner product space R^3 spanned by $B_1 = \{(1, 0, 1), (1, 2, -2)\}$ then find a basis of orthogonal complement $W^{\perp}$.

(b) Define :

(i) Adjoint operator

(ii) Complete orthonormal set.

Section–B

(Short Answer Type Questions) 4×8=32

***Note* :–** Section 'B' contains Eight (08) Short-answer type questions of Eight (08) marks each. Learners are required to answer any *four* (04) questions only.

1. Let F be a field and let F(x) be a non zero polynomial in F[x]. Then splitting field of $f(x)$ is an algebraic extension of F.

2. Prove that the general polynomial equation of degree is not solvable by radicals for $n \geq 5$.

3. Let $t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by :

$$t(a, b) = (2a - 3b, a + b\}, \text{ for all } (a, b) \in \mathbb{R}^3$$

Then find the matrix of t relative to the basis :

$$B = \{(1, 0), (0, 1)\}, B' = \{(2, 3), (1, 2)\}$$

4. Prove that for any matrix A over field F, rank (A) = rank(A^T).

5. State and prove Cayley-Hamilton theorem.

6. Prove that if V is an inner product space and $v \in$ V, then :

   (i) $\| v \| \geq 0$ and $\| v \| = 0$ iff $v = 0$

   (ii) $\| av \| = | a | \| v \|$

7. Let t be a linear transformation on inner product space V. Show that :

$$t = 0 \text{ iff} < t(u), v > = 0 \text{ for all } u, v, \in V$$

8. Prove that :

(i) Orthogonal matrix is always non singular

(ii) Determinant of orthogonal matrix is ± 1.

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