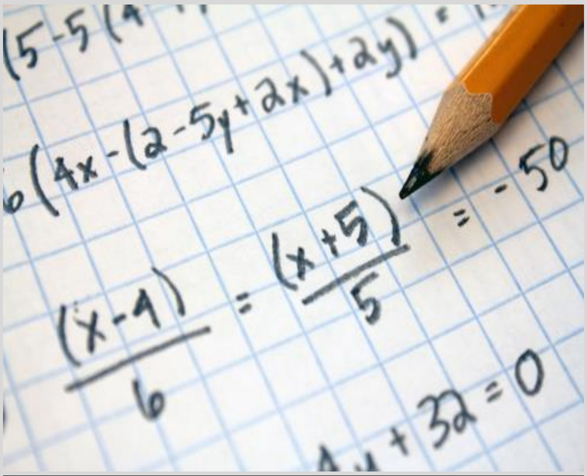
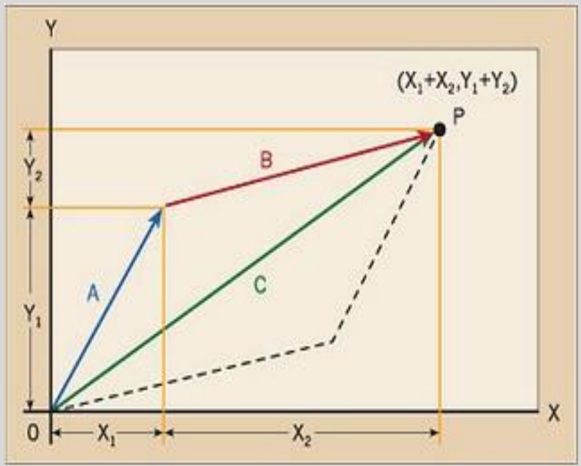
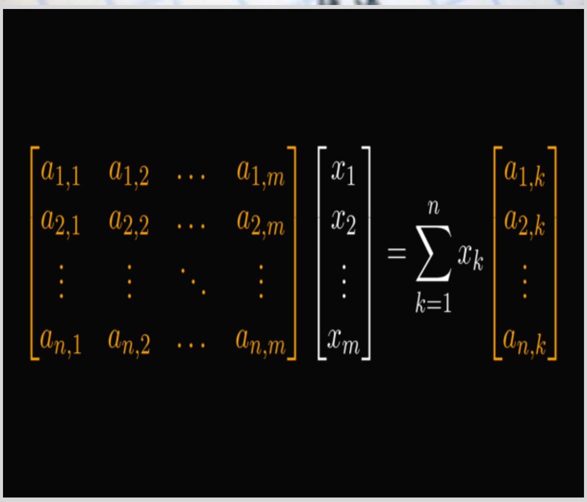
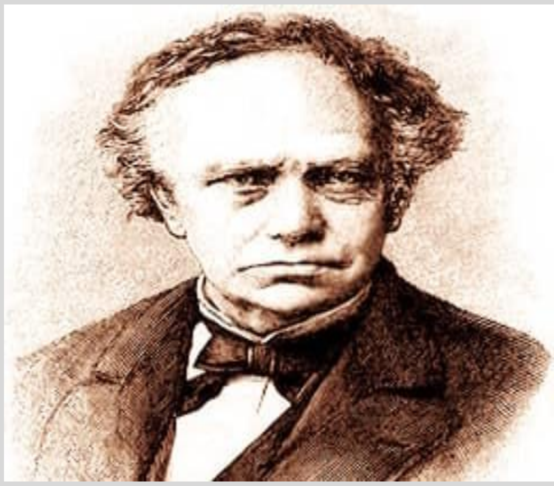
ALGEBRA,MATRICES AND VECTOR ANALYSIS

MT(N)121









DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES UTTARAKHAND OPEN UNIVERSITY HALDWANI, UTTARAKHAND
263139

<u>COURSE NAME:</u> ALGEBRA,MATRICES AND VECTOR ANALYSIS

<u>COURSE CODE: MT(N) 121</u>

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CONTENTS

ALGEBRA, MATRICES AND VECTOR ANALYSIS
MT(N) 121

COURSE INFORMATION

The present self learning material **“Algebra,Matrices and Vector Analysis”** has been designed for B.Sc. (Second Semester) learners of Uttarkhand Open University, Haldwani. This self-study material was created to increase learners access to excellent learning materials. There are 14 units in this course. Relations Between the Roots and The Coefficient of an Equation and Solutions Of Cubic and Biquadratic Equations are the focus of the first unit and second unit respectively. Algebra of Matrices, Determinants, Application of Matrices and Eigen Values and Eigen Vectors is covered in Unit 3,4,5 and Unit 6. Unit 7 explained Exponential and Trigonometrical variables. Units 8 and 9 each provided an explanation of Hyperbolic function and Inverse Hyperbolic and Trigonometric function and logarithm of complex number. Summation of Series is the topic of unit 10. The concepts of Infinite product and Gregory’s series is presented in Units 11. Discussion of Vectors Multiple products and Differentiation of vectors, Gradient, Divergence and Curl and Green’s, Gauss’s and Stoke’s theorems in the last three units. This subject matter is also employed in competitive exams. Simple, succinct, and clear explanations of the fundamental ideas and theories have been provided. The right amount of relevant examples and exercises have also been added to help learners to understand the material.

BLOCK-1: THEORY OF EQUATIONS

UNIT 1: RELATION BETWEEN THE ROOTS AND ITS COEFFICIENTS OF AN EQUATION

CONTENTS:

1.1 *INTRODUCTION:-*

In this unit we will discuss about the relationship between the roots (solutions) of a polynomial equation and its coefficients is a fundamental concept in algebra. Vieta's formulas are named after François Viète, a 16th-century French mathematician who made significant contributions to algebra. These formulas provide a bridge between the roots of a polynomial and the coefficients that define it. They are applicable to polynomial equations of various degrees, including both real and complex roots.

The core idea behind Vieta's formulas is that the roots of a polynomial equation hold valuable information about its coefficients, and vice versa.

By examining the coefficients, you can deduce properties of the roots, such as their sum and product, and by knowing the roots, you can determine relationships among the coefficients.

The main components of Vieta's formulas include:

Sum of Roots Formula: This formula relates the sum of the roots of a polynomial to its coefficients. It expresses how the coefficients of the polynomial's terms are connected to the negative of the sum of its roots.

Product of Roots Formula: The product of the roots is linked to the coefficients as well. This formula shows how the constant term and leading coefficient of the polynomial impact the product of its roots.

Special Cases for Quadratic Equations: Vieta's formulas are especially useful for quadratic equations, where they simplify to provide straightforward relationships between the roots and coefficients.

These formulas are not only essential for solving polynomial equations but also find applications in various mathematical and scientific disciplines, such as algebra, calculus, engineering, and physics. They serve as a powerful tool for simplifying polynomial expressions, analyzing polynomial properties, and understanding the behavior of functions defined by polynomial equations.

1.2 *OBJECTIVES:-*

After studying this unit, you will be able to

- To understand the Synthetic division.
- Lernear will be able to solve relation between roots and the coefficients of an equation.
- To understanding the Fundamental Theorem of Algebra.
- To Solve the Reciprocal roots.

1.3 *INTRODUCTION TO ALGEBRAIC EQUATIONS:-*

An algebraic equation is a mathematical statement that uses one or more variables and mathematical operations to express that two expressions are equal. An algebraic equation is also known as a polynomial equation because both sides of the equal sign contain polynomials. An algebraic equation is built up of variables, coefficients, constants as well as algebraic operations such as addition, subtraction, multiplication, division, exponentiation, etc. These equations are fundamental in algebra and are used to represent relationships between quantities or unknowns. Algebraic equations typically take the form:

Expression =Expression

In this equation, you have two sides, each containing algebraic expressions, and an equal sign between them, indicating that the two expressions are equal. The goal in solving algebraic equations is usually to find the values of the variables that satisfy the equation, making it true.

If there is a number or a set of numbers that satisfy the algebraic equation then they are known as **the roots or the solutions of that equation**. In this unit, we will discuss about algebraic equations, their types, examples, and how to solve algebraic equations.

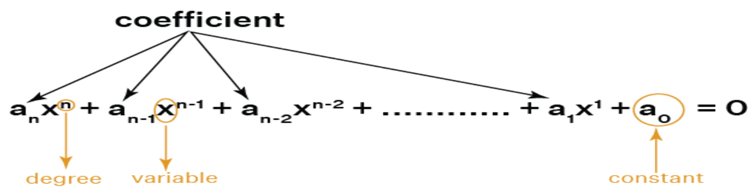

Fig.1 Algebraic Equations

The general form of the algebraic equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n = 0, (a_0 \neq 0)$$

where n is a positive integer.

The term a_n, which does not contain x, is called the constant term or absolute term.

An equation is knows as complete when all powers of x^n to x^0 are present in it, and incomplete when some of these powers are missing. An incomplete equation can be made complete by supplying the missing terms with zero coefficients. Thus the incomplete equation

$\Rightarrow 9x^5 + 7x^2 + 5 = 0$

$\Rightarrow 9x^5 + 0.x^4 + 0.x^3 + 7x^2 + 0.x + 5 = 0$

This form is complete equation.

An equation is said to be **numerical** if all roots coefficients are numbers while it is called **algebraically** if its coefficients are algebraic symbols. Hence $3x^2 + 2x + 7 = 0$ is numerical equation while $ax^2 + bx + c = 0$ is an algebraic equation.

Degree of equation: The degree of an algebraic equation refers to the exponent of the highest power of x occurring in the equation. Thus

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0.$$

where $a_0 \neq 0$ is of third degree. Here's how the degree is determined for different types of equations:

- **Linear Equation:** The degree of a linear equation is 1 because the highest power of the variable is 1. For example

$$2x + 3 = 7$$

In this equation, the degree is 1 because the variable "x" is raised to the power of 1.

- **Quadratic Equation:** The degree of a quadratic equation is 2 because the highest power of the variable is 2. For example
$$x^2 - 4x + 4 = 0$$
In this equation, the degree is 2 because the variable "x" is raised to the power of 2.
- **Cubic Equation:** The degree of a cubic equation is 3 because the highest power of the variable is 3. For example
$$2x^3 - 5x^2 + 3x - 1 = 0$$
In this equation, the degree is 3 because the variable "x" is raised to the power of 3.
- **Higher-Degree Equations:** Equations with degrees greater than 3 are called higher-degree equations. For example:

1. A quartic or biquadratic equation has a degree of 4.
2. A quintic equation has a degree of 5.
3. A polynomial equation of degree "n" has the highest power of "x"as "x^n".

Important Notes on Algebraic Equations:

1. An algebraic equation is an equation where two algebraic expressions are joined together using an equal sign.
2. Polynomial equations are algebra equations.
3. Algebraic equations can be one-step, two-step, or multi-step equations.
4. Algebra equations are classified as linear, quadratic, cubic, and higher-order equations based on the degree.

1.4 *SYNTHETIC DIVISION:-*

To find the quotient and the reminder when a polynomial is divided by a binomial.

This method was given by Horner.

Suppose $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$ be a polynomial of degree n and let it be divided by the binomial$x - h$. If $Q = b_0x^{n-1} + b_1x^{n-2} + \cdots\cdots + b_{n-1}$ be the quotient and R reminder, then the coefficient Q and R can be exhibited in the following manner.

$$\begin{array}{c|cccccc|c}
h & a_0 & a_1 & a_2 & a_3 & \cdots\cdots & a_{n-1} & a_n \\
 & & hb_0 & hb_1 & hb_2 & \cdots\cdots & hb_{n-2} & hb_{n-1} \\
\hline
 & b_0 & b_1 & b_2 & b_3 & \cdots\cdots & b_{n-1} & R
\end{array}$$

Synthetic division is a method used to perform polynomial long division for dividing a polynomial by a linear factor of the form $(x - c)$, where "c" is a constant. This method simplifies the division process and is often used when dividing polynomials quickly.

Rule for Synthetic division:

1. Write the polynomial in standard form, with all terms present, including any missing terms with coefficients of 0.
2. Identify the divisor, which should be in the form $(x - c)$, where "c" is a constant. For example, if you have $(x - 3)$ as the divisor, then "c" is 3.
3. Set up the synthetic division tableau. This is a compact form of writing the coefficients of the polynomial and simplifying the division process. Draw a long horizontal line and write the divisor's constant, "c" , to the left of the line. Above the line, write the coefficients of the polynomial in descending order of their degrees, leaving no gaps for missing terms. If there are missing terms, use 0 as the coefficient.
4. Start the synthetic division process:
   a. Bring down the first coefficient (the coefficient of the highest-degree term) below the line.
   b. Multiply this value by "c" (the divisor's constant), and write the result below the next coefficient.
   c. Add the result to the next coefficient above the line and write it below.
   d. Repeat this process for all coefficients, moving from left to right.

- The numbers below the line after completing the synthetic division represent the coefficients of the quotient polynomial. The last number is the remainder.
- Write the quotient polynomial, omitting any leading zero coefficients.
- The result of the synthetic division provides both the quotient polynomial and the remainder of the division. If the remainder is zero, it means that $(x - c)$ is a factor of the original polynomial. Otherwise, the remainder represents the remainder of the division.

<u>SOLVED EXAMPLE</u>

EXAMPLE1: Find the quotient and Reminder when

$x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x - 5$.

SOLUTION: The given polynomial is not complete. First we make it complete by supplying the missing terms with zero coefficients i.e. we obtain it as

$$x^5 - 4x^4 + 7x^3 - 11x - 13$$

Now

5	1	-4	7	0	-11	13
		5	5	60	300	1445
	$b_0 = 1$	$b_1 = 1$	$b_2 = 12$	$b_3 = 60$	$b_4 = 289$	$1432 = R$

$\therefore$ The quotient $Q = b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4$
$= x^4 + x^3 + 12x^2 + 60x + 289$ and Reminder $R = 1432$.

EXAMPLE2: Find the quotient and Reminder when
$2x^4 - 3x^3 + 4x^4 - 5x + 6$ is divided by $x - 2$.
SOLUTION: The given polynomial is not complete. First we make it complete by supplying the missing terms with zero coefficients i.e. we get

$$2x^4 - 3x^3 + 4x^4 - 5x + 6$$

Now

2	2	-3	4	-5	6
		4	2	12	14
	$b_0 = 2$	$b_1 = 1$	$b_2 = 6$	$b_3 = 7$	$20 = R$

$\therefore$ The quotient $Q = b_0x^3 + b_1x^2 + b_2x + b_3$
$= 2x^3 + x^2 + 6x + 7$ and Reminder $R = 1432$.

EXAMPLE3: Find the quotient and Reminder when
$2x^3 + 5x^2 + 9$ is divided by $x + 3$.
SOLUTION: The given polynomial is not complete. First we make it complete by supplying the missing terms with zero coefficients i.e. we write

$$2x^3 + 5x^2 + 9$$

Now

-3	2	5	0	9
		-6	3	-9
	$b_0 = 2$	$b_1 = -1$	$b_2 = 3$	$0 = R$

$\therefore$ The quotient $Q = b_0x^3 + b_1x^2 + b_2x + b_3$
$= 2x^2 - x + 3$ and Reminder $R = 0$.

EXAMPLE4: Find the quotient and Reminder when
$4x^3 - 8x^2 - x + 5$ is divided by $2x - 1$.
SOLUTION: The given polynomial is not complete. First we make it complete by supplying the missing terms with zero coefficients i.e. we write

$$4x^3 - 8x^2 - x + 5$$

Now

$\frac{1}{2}$	4	-8	-1	5
		2	-3	-2
	$b_0 = 4$	$b_1 = -6$	$b_2 = -4$	$3 = R$

$\therefore$ The quotient $\quad Q = b_0x^3 + b_1x^2 + b_2$
$= 4x^2 - 6x - 4$ and Reminder $R = 3$.

1.5 *FUNDAMENTAL THEOREM OF ALGEBRA:-*

*The fundamental theorem of algebra states that every non-constant single variable polynomial with complex coefficients has at least one complex root. This is true for polynomials with real coefficients, since every real number is a complex number with imaginary part equal to zero***.**

THEOREM1: Theorem n number of roots: Every equation of a degree n has n roots and no more.

Proof: Suppose

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$$
$$= 0, (a_0 \neq 0) \qquad \cdots(1)$$

be an equation of degree n.

Let α_1 be the root of equation, then from (1), we have

$f(x) = (x - \alpha_1)(a_0x^{n-1} + \cdots)$ or $f(x) = a_0(x - \alpha_1)f_{n-1}(x) \quad \cdots(2)$

where $f_{n-1}(x)$ is a function of x of degree $n - 1$, such that $f_{n-1}(\alpha_1) \neq 0$. Further let α_2 be a root of the equation $f_{n-1}(x) = 0$.Then we can write

$$f_{n-1}(x) = (x - \alpha_1)f_{n-2}(x)$$

Where $f_{n-2}(x)$ is a function of x of degree $n - 2$.

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)f_{n-2}(x) \qquad ...(3)$$

Continuing this process, we obtain

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \ldots \ldots (x - \alpha_n)$$

Thus $f(x) = 0$ has n roots $\alpha_1, \alpha_2 \ldots \ldots . . \alpha_n$.

If take x takes any different value from $\alpha_1, \alpha_2 \ldots \ldots . . \alpha_n$, then $f(x) \neq 0$.

Hence $f(x) = 0$ has n roots $\alpha_1, \alpha_2 \ldots \ldots . . \alpha_n$.

REMARKS:

a. All the n roots of $f(x) = 0$ are not necessarily real, some roots may be complex.
b. All the n roots of $f(x) = 0$ are not necessarily distinct, some roots may be repeated.
c. If more than n roots of $f(x) = 0$, then $f(x) = 0$

THEOREM2: Theorem on complex roots: In an equation with real coefficients complex roots occur in conjugate pairs.

Or

If $a+ib$ is a root of the equation $f(x)=0$, then $a-ib$ is also a root of the equation.

SOLUTION: Suppose

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots\cdots+a_{n-1}x+a_n=0 \quad \cdots(1)$$

Where $a_{1,}a_2 \ldots a_n$ are real, be an equation of degree n.

Suppose $a+ib$ be complex root of $f(x)=0$. Now

$$(x-a-ib)(x-a+ib)=(x-a)^2+b^2$$

Let $q(x)$ be the quotient and $Rx+R'$ be the remainder when $f(x)$ be divided by $(x-a)^2+b^2$. The reminder $Rx+R'$ must be first degree as divisor is of the second degree.

$\therefore \quad f(x)\equiv\{(x-a)^2+b^2\}q(x)+Rx+R' \quad \ldots(2)$

Substituting $x=a+ib$ in (2)

$$R(a+ib\,)+R'\equiv f(a+ib\,)=0$$

Since $a+ib$ is the roots of $f(x)=0$.Now equating real and imaginary parts

$\Rightarrow \quad Ra+R'=0$ and $bR=0$

$\Rightarrow \quad R=0$ and $R'=0$

$\therefore \quad f(x)=\{(x-a)^2+b^2\}q(x)=\{x-(a+ib)\}\{x-(a-ib)\}q(x)$

Hence $a-ib$ is also root of $f(x)=0$.

1.6 *REMAINDER AND FACTOR THEOREM:-*

REMAINDER THEOREM: If a polynomial $f(x)$ be divided by $x-c$ until a reminder independent of x is obtained, this remainder is equal to $f(c)$, which is the value of $f(x)$ when $x=c$.

The remainder is denoted by r and quotient by $q(x)$.

Hence

$$f(x)=(x-c)q(x)+r$$

Taking$x=c$, we obtain $f(c)=r$. If $r=0$ the division is exact.

FACTOR THEOREM: If $f(c)$ is zero, the polynomial $f(x)$ has the factor $x-c$. In other words, if c is the root of $f(x)=0, x-c$ is a factor of $f(x)$.

Example: 2 is the root of $x^3-8=0$, So $x-2$ is a factor of x^3-8.

<u>SELF CHECK QUESTIONS-1</u>

a. Find the quotient and remainder when (x^2-125) divided by $x-5$.
b. Find the quotient and remainder when$5x^4+2x^2-15x+10$ divided by $x+2$.

1.7 *RELATION BETWEEN THE ROOTS AND ITS COEFFICIENTS OF AN EQUATION:-*

Let us consider $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$ be a polynomial of degree n. Suppose $\alpha_1, \alpha_2, \alpha_3 \ldots.\alpha_n$ be the n roots. then the we have

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$$
$$\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots.(x-\alpha_n)$$

$$\equiv a_0\,[x^n - x^{n-1}(\alpha_1+\alpha_2+\cdots+\alpha_n) + x^{n-2}(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4\cdots)$$
$$- x^{n-3}(\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\cdots)+\cdots$$
$$+(-1)^r x^{n-r}(\alpha_1,\alpha_2,\cdots\alpha_r+\cdots)+\cdots$$
$$+(-1)^n\alpha_1,\alpha_2,\alpha_3\ldots.\alpha_n]$$

$$\equiv a_0\,\left[x^n - x^{n-1}\sum\alpha_1 + x^{n-2}\sum\alpha_1\alpha_2 - x^{n-3}\sum\alpha_1\alpha_2\alpha_3+\cdots\right.$$
$$+(-1)^r x^{n-r}\sum\alpha_1,\alpha_2,\cdots\alpha_r+\cdots$$
$$\left.+(-1)^n\alpha_1\,\alpha_2\,\alpha_3\ldots.\alpha_n\right] \qquad \cdots(1)$$

Now from (1), we get

$$a_1 = -a_0\sum\alpha_1$$

$$a_2 = a_0\sum\alpha_1\alpha_2$$

$$a_3 = -a_0\sum\alpha_1\alpha_2\alpha_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n = (-1)^n a_0\;\alpha_1\,\alpha_2\,\alpha_3\ldots.\alpha_n$$

Thus, we obtain

$\sum\alpha_1 =$ Sum of all roots $= -\frac{a_1}{a_0} = -\frac{Coefficient\ of\ x^{n-1}}{Coefficient\ of\ x^n}$

$\sum\alpha_1\alpha_2 =$ Sum of the products of the roots taken two at a time

$$= (-1)^2\frac{a_2}{a_0} = (-1)^2\frac{Coefficient\ of\ x^{n-2}}{Coefficient\ of\ x^n}$$

$\sum\alpha_1\alpha_2\alpha_3 =$ Sum of the products of the roots taken three at a time

$$= (-1)^2\frac{a_3}{a_0} = (-1)^3\frac{Coefficient\ of\ x^{n-3}}{Coefficient\ of\ x^n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\sum\alpha_1,\alpha_2,\cdots\alpha_r =$ Sum of the products of the roots taken r at a time

$$= (-1)^r\frac{a_r}{a_0} = (-1)^r\frac{Coefficient\ of\ x^{n-1}}{Coefficient\ of\ x^n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\alpha_1\ \alpha_2\ \alpha_3\ \dots.\alpha_n =$ Product of all the roots

$$= (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{constant\ term}{coefficient\ of\ x^n}$$

Note:

- **a.** If the equation is incomplete, we should first make it complete by supplying the missing terms with zero coefficients.
- **b.** The above relations between the roots and its coefficients of an equation do not enable us to solve the equation unless some other relations between the roots are given.

1.8 *PARTICULAR CASES:-*

. Relation between the roots and coefficients of a quadratic, cubic and biquadratic equation.

"Vieta's formula relates the coefficients of polynomials to the sums and products of their roots, as well as the products of the roots taken in groups"

- **Quadratic Equation:** Let α, β be the roots of quadratic $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a},\quad \alpha\beta = \frac{c}{a}$.

If p the sum and q be the product of roots of quadratic equation then the equation will be $ax^2 - px + q = 0$.

- **Cubic Equation:** Let α, β, γ be the roots of cubic $ax^3 + bx^2 + cx + d = 0$, then

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$$

$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \alpha(\beta + \gamma) + \beta\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$.

- **Bi-quadratic Equation:** Let $\alpha, \beta, \gamma, \delta$ be the roots of cubic $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$\sum \alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$= (\alpha - \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta)$$

$$= -\frac{d}{a}.$$

$$\alpha\beta\gamma\delta = \frac{e}{a}.$$

1.9 *CUBE ROOTS OF UNITY*

Suppose $x = (1)^{1/3}$

Now cubing both side, we obtain $x^3 = 1$ or $x^3 - 1 = 0$, then

$$x^3 - 1 \equiv (x-1)(x^2+x+1) = 0$$

Therefore the remaining two cube roots of unity are the roots of quadratic $x^2 + x + 1 = 0$.

These are $\frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$.

If one root denoted by ω, thus $1, \omega, \omega^2$ are the cubic roots of unity and therefore the roots of equation $x^3 - 1 = 0$ $i.e.$,

$$x^3 + 0.x^2 + 0.x - 1 = 0$$

We obtain

$1 + \omega, +\omega^2 = 0$ and $\omega^3 = 1$.

<u>SOLVED EXAMPLE</u>

EXAMPLE1: If α, β, γ be the roots of the cubic $x^3 + px^2 + qx + r = 0$.Find the value of $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$.

SOLUTION: If α, β, γ are the roots of the cubic $x^3 + px^2 + qx + r = 0$,then

$$\sum \alpha = \alpha + \beta + \gamma = -p$$

$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \alpha(\beta+\gamma) + \beta\gamma = q \text{ and } \alpha\beta\gamma = -r.$$

$$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = (\sum\alpha - \alpha)(\sum\alpha - \beta)(\sum\alpha - \gamma)$$

$$= (-p-\alpha)(-p-\beta)(-p-\gamma)$$

$$= -(p+\alpha)(p+\beta)(p+\gamma) = [p^3 + p^2\sum\alpha + p\sum\alpha\beta + \alpha\beta\gamma]$$

$$= [p^3 + p^2(-p) + pq - r] = r - pq.$$

EXAMPLE2: Solve the equation $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being the ratio$3:4$.

SOLUTION: The root of the given ratio is $3\alpha, 4\beta$ and β.

$$\sum \alpha = 3\alpha + 4\alpha + \beta = 7\alpha + \beta = \tfrac{1}{2} \qquad \cdots (1)$$

$$\sum \alpha\beta = 12\alpha^2 + 3\beta\alpha + 4\beta\alpha = \tfrac{22}{2} \Rightarrow 12\alpha^2 + 7\alpha\beta = -11 \cdots (2)$$

$$\alpha\beta\gamma = 3\alpha.4\alpha = \tfrac{24}{2} \text{ or } \alpha^2\beta = 1 \qquad \cdots (3)$$

Now from (1), we obtain

$$\beta = \frac{1}{2} - 7\alpha$$

Putting the value of β in (2), we obtain

$$12\alpha^2 + 7\alpha\left(\frac{1}{2} - 7\alpha\right) = -11$$

$$12\alpha^2 + \frac{7\alpha}{2} - 49\alpha^2 = -11$$

$$37\alpha^2 - \frac{7\alpha}{2} - 11 = 0$$

$$74\alpha^2 - 7\alpha - 22 = 0$$

$$74\alpha^2 - 44\alpha + 37\alpha - 22 = 0$$

$$(2\alpha + 1)(37\alpha - 22) = 0$$

$$\alpha = -\tfrac{1}{2} \text{ or } \alpha = \tfrac{22}{37}$$

When $\alpha = -\frac{1}{2}, \beta = \frac{1}{2} + \frac{7}{2} = 4.$

$\alpha = \frac{22}{37}, \beta = \frac{1}{2} - \frac{154}{37} = -\frac{271}{74}.$

Hence the required roots are

$-\frac{1}{2} \times 3, 4 \times -\frac{1}{2}, 4$ $i.e.,\left(-\frac{3}{2}, 2,4\right).$

EXAMPLE3: Solve the equation $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$, being given that it has a pair of roots whose sum is zero.

SOLUTION: Let α, β, γ be the root of the given equation.

$\therefore$

$$\sum \alpha = \alpha + \beta + \gamma + \delta = \frac{3}{6} = \frac{1}{2} \quad \cdots (1)$$

$$\sum \alpha\beta = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{8}{6} = \frac{4}{3} \cdots (2)$$

$$\sum \alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = \frac{1}{6} \cdots (3)$$

$$\alpha\beta\gamma\delta = \frac{2}{6} = \frac{1}{3} \qquad \cdots (4)$$

And also let $\alpha + \beta = 0 \qquad \cdots (5)$

Now from (1) and (5), we have

$$\gamma + \delta = \frac{1}{2} \qquad \cdots (6)$$

From (3) and (6), we obtain

$$\alpha\beta.\tfrac{1}{2} + 0 = \tfrac{1}{6}$$

$\therefore$

$$\alpha\beta = \tfrac{1}{3} \qquad \cdots (7)$$

From (4) and (7), we get

$$\tfrac{1}{3}.\gamma\delta = \tfrac{1}{3} \quad \text{or} \quad \gamma\delta = 1 \quad \cdots (8)$$

Now α and β are the roots of the quadratic

$t^2 - (\alpha + \beta)t + \alpha\beta = 0$ or $t^2 - 0 + \frac{1}{3} = 0$

$\therefore$

$$t = \pm\frac{i}{\sqrt{3}}$$

Again γ and δ are the roots of the quadratic

$t^2 - (\gamma + \delta)t + \gamma\delta = 0$ or $t^2 - \frac{1}{2}t + 1 = 0$ or $2t^2 - t + 2 = 0$

$$t = \frac{1 \pm \sqrt{1 - 4.2.2}}{2.2} = \frac{1 \pm i\sqrt{15}}{4}$$

Hence the required roots are $\pm\frac{i}{\sqrt{3}}, \frac{1 \pm i\sqrt{15}}{4}$.

EXAMPLE4: Find the condition that the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Should have two roots connected by the relation $\alpha + \beta = 0$

SOLUTION: Let $\alpha, \beta, \gamma, \delta$ be the root of the given equation.

$$\sum \alpha = \alpha + \beta + \gamma + \delta = -p \qquad \cdots (1)$$
$$\sum \alpha\beta = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = q \quad \cdots (2)$$
$$\sum \alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = r \quad \cdots (3)$$
$$\alpha\beta\gamma\delta = \frac{2}{6} = s \qquad \cdots (4)$$

And also let $\alpha + \beta = 0 \qquad \cdots (5)$

From (1) and (5), we have

$$\gamma + \delta = -p$$

From (3), (5) and (6), we obtain

$\alpha\beta(-p) + 0 = -r$ or $\alpha\beta = \frac{r}{p}$

Again from (4), we get

$\frac{r}{p}.\gamma\delta = s$ or $\gamma\delta = \frac{sp}{r}$

Now from (4), we get

$$0 + \frac{r}{p} + \frac{sp}{r} = q$$

$$r^2 + sp^2 - pqr = 0$$

Which is required condition.

SELF CHECK QUESTIONS-2

a. Find the condition which must be satisfied by coefficients of the equation

$$x^3 - px^2 + qx - r = 0$$

When two of its roots α, β are connected by the relation $\alpha + \beta = 0.$ ans

b. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$, the root being in geometrical progression.

1.10 *ROOTS WITH SIGN CHANGED:-*

To transform an equation into another whose roots are roots of the given equation with their sign changed.

Let us consider $\alpha_1, \alpha_2, \alpha_3 \ldots . \alpha_n$ be roots of given equation
$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$ ... (1)
Then

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$$
$$\equiv a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots \ldots . (x - \alpha_n) \quad \ldots (2)$$

Now Let y be a root of the required transformed equation. Then $y = -x$.

$$x = -y$$

Replacing x by $(-y)$ in (2)

$$a_0(-y)^n + a_1(-y)^{n-1} + a_2(-y)^{n-2} + \cdots\cdots + a_{n-1}(-y) + a_n$$
$$\equiv a_0(-y - \alpha_1)(-y - \alpha_2)(-y - \alpha_3) \ldots \ldots . (-y - \alpha_n)$$

$(-1)^n [a_0(y)^n - a_1(y)^{n-1} + a_2(y)^{n-2} - \cdots\cdots + (-1)^{n-1}a_{n-1}(y) + (-1)^n a_n] = (-1)^n[(y + \alpha_1)(y + \alpha_2)(y + \alpha_3) \ldots (y + \alpha_n)]$

$\Rightarrow \quad a_0(y)^n - a_1(y)^{n-1} + a_2(y)^{n-2} - \cdots\cdots + (-1)^{n-1}a_{n-1}(y) + (-1)^n a_n \equiv (y + \alpha_1)(y + \alpha_2)(y + \alpha_3) \ldots (y + \alpha_n)$ ... (3)

From (3), is clear that $-\alpha_1, -\alpha_2, \ldots \ldots \ldots -\alpha_n$ are the roots of the equation.

$$(y)^n - a_1(y)^{n-1} + a_2(y)^{n-2} - \cdots\cdots + (-1)^{n-1}a_{n-1}(y) + (-1)^n a_n = 0.$$

Which is required condition.

EXAMPLE: Change the sign of the roots of the equation

$$x^5 - 4x^3 + 3x^2 + 8x - 9 = 0$$

SOLUTION: The given equation is

$$x^5 - 4x^3 + 3x^2 + 8x - 9 = 0 \quad \ldots (1)$$

Changed the sign of every alternate term of (1), we have

$$x^5 - 0\,x^4 - 4x^3 - 3x^2 + 8x + 9 = 0$$

$$x^5 - 4x^3 - 3x^2 + 8x + 9 = 0.$$

1.11 *ROOTS MULTIPLIED BY A CONSTANT m:-*

To transform an equation into another whose roots are roots of the given equation multiplied by the constant m.

Let us consider $\alpha_1, \alpha_2, \alpha_3 \dots . \alpha_n$ be roots of given equation
$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n \quad \dots (1)$
Then

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n$$
$$\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots\dots.(x-\alpha_n) \quad \dots(2)$$

Now Let y be a root of the required transformed equation.
Then $\quad y = mx, \quad x = y/m$

Replacing x by (y/m) in (2), we obtain

$$a_0(y/m)^n + a_1(y/m)^{n-1} + a_2(y/m)^{n-2} + \cdots\cdots + a_{n-1}(y/m) + a_n$$
$$\equiv a_0(y/m-\alpha_1)(y/m-\alpha_2)(y/m-\alpha_3)\dots\dots.(y/m-\alpha_n)$$

Multiplying both sides by m^n, we obtain

$$a_0(y)^n + ma_1(y)^{n-1} + m^2a_2(y)^{n-2} - \cdots\cdots + m^{n-1}a_{n-1}(y) + m^na_n \equiv (y-m\alpha_1)(y-m\alpha_2)\dots(y-m\alpha_n)$$

where $m\alpha_1, m\alpha_2, \dots\dots..m\alpha_n$ are the roots of the equation

$$a_0(y)^n + ma_1(y)^{n-1} + m^2a_2(y)^{n-2} - \cdots\cdots + m^{n-1}a_{n-1}(y) + m^na_n = 0.$$

Which is required condition.

EXAMPLE: Transform the equation $72x^3 - 54x^2 + 45x - 7 = 0$ into another with integral coefficients and unity for the coefficient of the first term.

SOLUTION: The given equation is $72x^3 - 54x^2 + 45x - 7 = 0$

$$\Rightarrow \quad x^3 - \tfrac{3}{4}x^2 + \tfrac{5}{8}x - \tfrac{7}{72} = 0 \quad \dots (1)$$

The equation (1) multiplied by m.Then

$$x^3 - \frac{3}{2^2}mx^2 + \frac{5}{2^3}m^2x - \frac{7}{2^3.3^2}m^3 = 0$$

Since the least value of m to remove the fractional coefficients is

$$m = 2^2 \times 3 = 12$$

So substituting $m = 2^2.3$ in above equation

$$x^3 - \frac{3}{2^2}(2^2.3)x^2 + \frac{5}{2^3}(2^2.3)^2x - \frac{7}{2^3.3^2}(2^2.3)^3 = 0$$
$$x^3 - 9x^2 + 90x - 168 = 0$$

1.12 *RECIPROCAL ROOTS:-*

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let us consider $\alpha_1, \alpha_2, \alpha_3 \dots . \alpha_n$ be roots of given equation

$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \cdots + a_{n-1}x + a_n$ ... (1)

Then

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \cdots + a_{n-1}x + a_n \equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \dots \dots (x-\alpha_n) \quad \dots (2)$$

Now Let y be a root of the required transformed equation.

Then $y = 1/x, \quad x = 1/y$

Replacing x by $(1/y)$ in (2), we obtain

$$a_0(1/y)^n + a_1(1/y)^{n-1} + a_2(1/y)^{n-2} + \cdots \cdots + a_{n-1}(1/y) + a_n \equiv a_0(1/y-\alpha_1)(1/y-\alpha_2) \dots . (1/y-\alpha_n)$$

Multiplying both sides by y^n, we obtain

$$a_0 + a_1y + a_2y^2 + \cdots \cdots + a_{n-1}y^{n-1} + y^na_n \equiv a_0(1-y\alpha_1)(1-y\alpha_2) \dots (1-y\alpha_n)$$

Or

$$a_0 + a_1y + a_2y^2 + \cdots \cdots + a_{n-1}y^{n-1} + y^na_n \equiv a_0(-1)^n\alpha_1\,\alpha_2 \dots . \alpha_n(y-1/\alpha_1)(y-1/\alpha_2) \dots . (y-1/\alpha_n)$$

Which shows that $1/\alpha_1, 1/\alpha_2, \dots \dots , 1/\alpha_n$ are the roots of the equation

$$y^na_n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0$$

Which is required the solution.

EXAMPLE: Find the equation whose roots are the reciprocals of the roots of the equation

$$x^4 - 5x^3 + 7x^2 + 3x - 7 = 0.$$

SOLUTION: The given equation is

$$x^4 - 5x^3 + 7x^2 + 3x - 7 = 0.$$

Replacing x by $1/x$, the required equation is

$$\left(\frac{1}{x}\right)^4 - 5\left(\frac{1}{x}\right)^3 + 7\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) - 7 = 0$$

Or

$$1 - 5x + 7x^2 + 3x^3 - 7x^4 = 0$$

$$7x^4 - 3x^3 - 7x^2 + 5x - 1 = 0$$

1.13 *RECIPROCAL EQUATION:-*

"An equation which remains unaltered by changing x into $\frac{1}{x}$ is called a reciprocal equation."

Or

Let $f(x) = 0$ be an equation of roots $\alpha_1, \alpha_2, \alpha_3 \ldots . \alpha_n$. if $\frac{1}{\alpha_1}, \frac{1}{\alpha_2} \ldots \frac{1}{\alpha_n}$ are also roots of the same equation, then such equation are known as **reciprocal equations**.

Suppose the given equation be
$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \cdots + a_{n-1}x + a_n \quad \ldots (1)$

Then

$$x^n a_n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 \qquad \ldots (2)$$

Is equation whose roots are the reciprocal of equation (1)
Comparing (1) and (2)

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_2}{a_{n-2}} = \cdots = \frac{a_{n-1}}{a_1} = \frac{a_n}{a_0}$$

Hence $\qquad a_n^2 = a_0^2 \quad$ or $\quad a_n = \pm a_0$

Case1: If $a_n = a_0$, then, we get

$$a_1 = a_{n-1,}\, a_2 = a_{n-2}, \cdots \cdots \cdots$$

Which is known as reciprocal equations of the first type.

Case2: If $a_n = a_0$, then, we obtain

$$a_1 = -a_{n-1,}\, a_2 = -a_{n-2}, \cdots \cdots \cdots$$

Which is known as reciprocal equations of the second type.

A reciprocal equation of first type and even degree is called a **standard reciprocal equation**.

Note:

- **a.** If $f(x) = 0$ is a reciprocal equation of first type and odd degree, the $x = -1$ is always a root. If we remove the factor $x + 1$ corresponding to this root, we obtain a standard reciprocal equation.
- **b.** If $f(x) = 0$ is a reciprocal equation of second type and odd degree, then $x = 1$ is always a roots. If we remove the factor

$x-1$ corresponding to this root, we obtain a standard reciprocal equation.

c. If $f(x)=0$ is a reciprocal equation of second type and even degree, then $x=1$ and $x=-1$ are roots. If we remove the factor x^2-1 corresponding to these roots, we obtain a standard reciprocal equation.

SOLVED EXAMPLE

EXAMPLE1: Solve the equation $60x^4-736x^3+1433x^2-736x+60=0$.

SOLUTION: The given equation is

$$60x^4-736x^3+1433x^2-736x+60=0$$

Dividing by x^2, we have

$$60x^2-736x+1433-\frac{736}{x}+\frac{60}{x^2}=0$$

$$60\left(x^2+\frac{1}{x^2}\right)-736\left(x+\frac{1}{x}\right)+1433=0$$

Substituting $y=x+\frac{1}{x}$ in above equation and simplifying, we get

$$60y^2-736y+1433=0$$

On solving $y=\frac{101}{10}$ or $\frac{13}{6}$

When $y=\frac{101}{10} \Rightarrow x+\frac{1}{x}=\frac{101}{10} \Rightarrow 10x^2-101x+10=0, i.e.,$

$$(10x-1)(x-10)=0 \Rightarrow x=10,\frac{1}{10}$$

Similarly

$$y=\frac{13}{6} \Rightarrow x+\frac{1}{x}=\frac{13}{6} \Rightarrow 6x^2-101x+6=0,$$

$$(3x-2)(2x-3)=0 \Rightarrow x=\frac{2}{3},\frac{3}{2}$$

Hence the root of given equation are

$$10,\frac{1}{10},\frac{2}{3},\frac{3}{2}.$$

EXAMPLE2: Solve the equation $x^5-5x^4+9x^3-9x^2+5x-1=0$.

SOLUTION: The given equation is

$$x^5-5x^4+9x^3-9x^2+5x-1=0.$$

$$(x^5-1)-5x(x^3-1)+9x^2(x-1)=0$$

$$(x-1)(x^4+x^3+x^2+x+1)-5x(x-1)(x^2+x+1)+9x^2(x-1)=0$$

$$(x-1)[x^4+x^3+x^2+x+1-5x(x^2+x+1)+9x^2]=0$$

$$(x-1)[x^4-3x^3+5x^2-4x+1]=0$$

i.e., $(x-1)=0 \Rightarrow x=1$

$$x^4-3x^3+5x^2-4x+1=0 \qquad \text{... (1)}$$

From (1) can be written as

$$(x^4+1)-4(x^3+x)+5x^2=0$$

Dividing by x^2 we obtain

$$\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+5=0$$

$$\left\{\left(x+\frac{1}{x}\right)^2-2\right\}-4\left(x+\frac{1}{x}\right)+5=0$$

Substituting $x+\frac{1}{x}=y$,we get

$$y^2-2-4y+5=0$$
$$y^2-4y+3=0$$
$$(y-1)(y-3)=0\Rightarrow y=1,3$$

When $\qquad y=1\Rightarrow x+\frac{1}{x}=1\Rightarrow x^2-x+1=0$

$$x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\left(1\pm i\sqrt{3}\right)$$

Again when $y=3\Rightarrow x+\frac{1}{x}=3\Rightarrow x^2-3x+1=0$

$$x=\frac{3\pm\sqrt{9-4}}{2}=\frac{1}{2}\left(3\pm\sqrt{5}\right)$$

Hence the roots of the given equation are

$$1,3,\frac{1}{2}\left(1\pm i\sqrt{3}\right),\frac{1}{2}\left(3\pm\sqrt{5}\right).$$

1.14 *DESCARTE'S RULE OF SIGNS:-*

To determine the nature of some of the roots of a polynomial equation it is not always necessary to solve it; for instance, the truth of the following statements will be readily admitted.

1. If the coefficients of a polynomial equation are all positive, the equation has no positive root; for example, the equation
$$x^4+3x^2+3=0$$
cannot have a positive root.
2. If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the opposite sign, the equation has no negative root; thus for example, the equation
$$-x^8+x^7+x^5-2x^4+x^3-3x^2+7x-3=0$$
cannot have a negative root.
3. If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus for example, the equation
$$-x^8-2x^4-3x^2-3=0.$$
4. If the equation contains only odd powers of x and the coefficients are all of the same sign, the equation has no real root except $x=0$, thus the equation 7

$$x^7 + x^5 + 3x^3 + 8x = 0$$

has no real root except $x = 0$.

Suppose that the signs of the terms in a polynomial are $+ + - - + - - - + - + -$; here the number of changes of sign is 7. We shall show that if this polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+ -$, there will be at least one more change of sign in the product than in the original polynomial.

Writing down only the signs of the terms in the multiplication, we have the following:

$$\begin{array}{l} + + - - - + - + - \\ + - \\ \hline + + - - - + - + \\ \quad - - + + + - + - \\ \hline + \pm - \pm \pm + - + - \end{array}$$

Here in the last line the ambiguous sign $\pm$ is placed wherever there are two different signs to be added.

Here we see that in the product

1. an ambiguity replaces each continuation of sign in the original polynomial;
2. the signs before and after an ambiguity or set of ambiguities are unlike;
3. a change of sign is introduced at the end.

Let us take the most unfavourable case (i.e., the case where the number of changes of sign is less) and suppose that all the ambiguities are replaced by continuations; then the sign of the terms become

$$+ + - - + - - - + - - + - +$$

and the number of changes of sign is 8.

We conclude that if a polynomial is multiplied by a binomial (corresponding to a positive root) whose signs are $+ -$ there will be at least one more change of sign in the product than in the original polynomial.

If then we suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x - a$ corresponding to a positive root introduces at least one change of sign; therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x) = 0$ are equal to those of $f(x) = 0$ but opposite to them in sign; therefore the negative roots of

$f(x) = 0$ are the positive roots of $f(-x) = 0$; but the number of these positive roots cannot exceed the number of changes of sign in $f(-x)$; that is, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes in sign in of $f(-x)$.
All the above observations are included in the following result, called **Descarte's Rule of Signs.**

In any polynomial equation $f(x) = 0$, the number of real positive roots cannot exceed the number of changes in the signs of the coefficients of the terms in $f(x)$, and the number of real negative roots cannot exceed the number of changes in the signs of the coefficients of $f(-x)$.

SOLVED EXAMPLE

EXAMPLE1: Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ ha at least four imaginary roots.
SOLUTION: Suppose $2x^7 - x^4 + 4x^3 - 5 = 0$
We see that there are only three changes of sign in $f(x)$. then $f(x) = 0$ cannot have more than three positive roots.
Again

$$f(-x) = -2x^7 - x^4 - 4x^3 - 5$$
$$2x^7 + x^4 + 4x^3 + 5 = 0$$

We see that $f(-x)$ has no change of signs. Therefore the equation $f(x) = 0$ cannot have any negative root. Thus the maximum number of real roots of the equation $f(x) = 0$ is 3. But this equation is of degree 7.Hence at least $7 - 3 i.e., 4$ complex roots.

EXAMPLE2: Locate the situation of the roots of the equation $x^3 + x^2 - 2x - 1 = 0$.
SOLUTION: The equation is

$$x^3 + x^2 - 2x - 1 = 0.$$

The equation $f(-x) = 0$ is

$$-x^3 + x^2 + 2x - 1 = 0 \; i.e., x^3 - x^2 - 2x + 1 = 0$$

\The $f(x)$ has only one change of sign and so the equation $f(x) = 0$ Cannot have more than one positive root.
We have $f(0) = -1$, $f(1) = -1$, $f(2) = 7$
Since $f(1)$ and $f(2)$ are of opposite signs, therefore the positive root lies between 1 and 2.
Again $f(-x)$ has two changes of signs and so the equation $f(x) = 0$ can not have more than two negative

Again we have $f(0) = -1$, $f(-1) = 1$, $f(-2) = -1$
roots.

Since $f(-1)$ and $f(0)$ are positive signs, therefore one negative root lies between -1 and 0. Again $f(-2)$ and $f(-1)$ are also of opposite signs and so one negative root lies between -2 and -1.
Hence all the three roots of given equation are real and lie in the open intervals $(-2,-1), (-1,0)$ and $(1,2)$.

SELF CHECK QUESTIONS-3

1. If α, β, γ be the roots of the equation $x^4 - 9x^3 + 7x - 8 = 0$, then the value of expression $(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta$ is
   a. -9 b. 0 c. 7 d. -8

2. If α, β, γ be the roots of the equation $x^4 - 9x^3 + 23x - 15 = 0$ are in A.P then the mean root β is equal to
   a. 3 b. -3 c. 5 d. -5

3. Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + x^3 - 16x - 4x + 48 = 0$. If $\alpha\beta = 8$, then the value of $\gamma\delta$ is
   a. -8 b. -6 c. 6 d. 4

4. If two root of the equation $x^3 - 9x^2 + 14x + 24 = 0$ are in the ratio 3:2, then all the roots shall be
   a. $-2,4,5$ b. $6,4,2$ c. $6,4,-1$ d. $1,4,6$

5. If two root of the equation $4x^3 + 20x^2 - 23x + 6 = 0$ are equal, then all the roots shall be
   a. $\frac{1}{2}, -\frac{1}{2}, 6$ b. $-\frac{1}{2}, -\frac{1}{2}, -6$ c. $\frac{1}{2}, \frac{1}{2}, -6$ d. $-\frac{1}{2}, -\frac{1}{2}, 6$

6. If α, β are the roots of the equation $x^2 - x + 1 = 0$, then the equation whose roots are α^2 and β^2 is
   a. $x^4 - x^2 + 1 = 0$ b. $x^4 + x^2 + 1 = 0$ c. $x^2 + x + 1 = 0$
   d. $x^2 + x - 1 = 0$

7. The number of the positive roots of the equation $x^5 + 4x^4 + 9x^3 + 8x^2 + 7x - 3 = 0$ is
   a. 0 b. 1 c. 2 d. 3

8. The number of real roots of the equation $x^4 + x^2 - 1 = 0$ is
   a. 2 b. 0 c. 1 d. 4

9. Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + bx^3 + cx + dx + e = 0$, then $\sum \alpha = \cdots\cdots \left(-\frac{b}{a}\right)$

10. Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$, then $\alpha(\beta + \gamma) + \beta\gamma = \cdots\cdots (\frac{c}{a})$
11. The number of changes of signs in $2x^7 - x^4 + 4x^3 - 5 = 0$ is $\cdots\cdots$ (3)
12. The equation $x^4 + 15x^2 + 7x - 11 = 0$ can at most … …. positive roots. (one)
13. The equation $4x^6 + 17x^4 + 8x^2 - 9x - 11 = 0$ has … …. complex roots.(four)
14. The equation $3x^7 - 2x^4 + 4x^3 - 9 = 0$ cannot have more than … …. positive roots. (three)
15. The equation $3x^7 - x^4 + 5x^3 - 8 = 0$ cannot have any negative root.T
16. The equation $x^6 + x^4 - 3 = 0$cannot have any negative roots. F
17. The equation $x^4 + x^2 + 1 = 0$ have no real roots. T
18. The equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$ has at least six complex roots. T
19. The equation $x^{11} - x^7 + x^6 + x^2 + 1 = 0$ has two negative roots. F
20. The equation $x^3 + x^2 - 2x - 5 = 0$ has no positive roots. F

1.15 *SUMMARY:-*

In this unit we studied the relation between the roots and coefficients of a polynomial equation is described by sum of roots formula, product of roots formula, Introduction to algebra equation, reciprocal roots , reciprocal equation, remainder and factor theorem, Roots with sign changed and Descrate's rule of signs . These formulas provide a fundamental connection between the roots (solutions) of a polynomial equation and its coefficients.

1.16*GLOSSARY:-*

- Descrate's rule of signs
- Algebra equation
- Remainder and factor theorem
- Roots with sign changed

1.17 *REFERENCES:-*

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1.18 *SUGGESTED READING:-*

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- Kenneth Hoffman & Ray Kunze(2015)Linear Algebra (2nd edition). Prentice-Hall.
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1.19 *TERMINAL QUESTIONS:-*

(TQ-1) Show that the equation $x^5 + x^3 - 8x - 5 = 0$ cannot have more than three real roots and prove that it must have three real roots.

(TQ-2) Change the signs of the roots of the equation

$$2x^5 + 4x^3 - 13x^2 + 7x + 6 = 0.$$

(TQ-3) Find the equation whose roots are the reciprocals of the roots of the equation $x^4 - 5x^3 + 7x^2 + 3x - 7 = 0$.

(TQ-4) Change the sign of the root of equation

$$x^7 + 5x^5 - x^3 + x^2 + 7x + 3 = 0.$$

(TQ-5) Form the equation whose roots are the reciprocals of the roots of the equation $x^4 - 3x^3 + 7x^2 + 2x - 2 = 0$.

(TQ-6) Find the equation whose roots are twice the reciprocals of the roots of $x^4 + 3x^3 - 6x^2 + 2x - 4 = 0$.

(TQ-7) Solve the reciprocal roots

1. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.
2. $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$

(TQ-8) Solve the equation $x^3 - 12x^2 + 39x - 28 = 0$, whose roots are in the arithmetical progression.

(TQ-9) Solve the equation $x^3 - px^2 + qx - r = 0$, should have its roots in harmonic progression.

(TQ-10) The roots of the equation $2x^2 - 7x + 5 =$ are α and β. Without solving for the roots, find

1. $\frac{1}{\alpha} + \frac{1}{\beta}$ b. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ c. $\frac{\alpha+2}{\beta+2} + \frac{\beta+2}{\alpha+2}$

(TQ-11) Find the quotient and Remainder of the following when

1. $2x^2 + 3x - 4 = 0$ is divided by $(x + 2)$.

2. $x^4+5x^3-6x+3=0$ is divided by $(x-2)$.
3. $2x^3+4x^2-3x-6=0$ is divided by $(x+3)$.
4. $2x^3-11x^2+13x-44=0$ is divided by $(x-5)$.
5. $x^2+5x+1=0$ is divided by $(x-3)$.
6. $2x^3-11x^2+9x-20=0$ is divided by $(x-5)$.

1.20 *ANSWERS:-*

SELF CHECK ANSWERS-1

1. $Q=x^2+5x+25$
2. $Q=5x^3-10x^2+22x-59, R=128$

SELF CHECK ANSWERS-2

1. $pq=r$
2. $\left(\frac{2}{3},2,6\right)$

SELF CHECK ANSWERS-3

1. b	2. a	3. c	4. c	5. c
6.c	7. b	8.a	9. $-\frac{b}{a}$	10. $\frac{c}{a}$
11. 3	12. One	13. Four	14. Three	15. T
16. F	17. T	18.T	19. F	20. F

TERMINAL ANSWERS

(TQ-2) $2x^5+4x^4+13x^2+7x-6=0$
(TQ-3)$7x^4-3x^3-7x^2+5x-1=0$
(TQ-4) $x^7+5x^5-x^3-x^2+7x-3=0$
(TQ-5) $2x^4-5x^3-7x^2+3x-1=0$
(TQ-6) $x^4-x^3+6x^2-6x-4=0$
(TQ-7) a. $3\pm2\sqrt{2},\ 2\pm\sqrt{3}$ b. $\left(\pm1,2,\frac{1}{2}\right),\frac{5\pm i\sqrt{11}}{6}$
(TQ-8) 1,4,7
(TQ-9) $27r^2-9pqr+2q^3=0$
(TQ-10) a. $\frac{7}{5}$ b. $\frac{29}{10}$ c. $\frac{13}{6}$
(TQ-11)

a. $Q=2x+7, R=10$
b. $Q=x^3+2x^2-6x+12=0, R=33$
c. $Q=2x^2-2x+3, R=15$
d. $Q=2x^2-x+8, R=4$
e. $Q=x+2, R=5$

f. $Q = 2x^2 - x + 4$

UNIT 2: SOLUTIONS OF CUBIC AND BIQUADRATIC EQUATIONS

CONTENTS

2.1 *INTRODUCTION*

In the previous unit learners have learned about the root of the equation of two or more than two degree, synthetic divisions, solution of quadratic equations by different methods.

In this unit we will learned about the solution of cubic equation (polynomial of degree 3) and bi-quadratic equation (polynomial of degree 4). Solutions of cubic equation will be solved by using cardon's method and solutions of biquadratic equations are solved by using, Euler's method, Descarte's method and Ferrari's method.

2.2 *OBJECTIVES*

After reading this unit learners will be able to,

- Solve the cubic equation by using Cardan's method.
- Solve the biquadratic equation by using Euler's method
- Solve the biquadratic equation by Descarte's method
- Solve the biquadratic equation by Ferrari's method.

2.3 *CARDAN'S METHOD OF SOLVING THE CUBIC EQUATION*

Let the cubic equation be $ax^3 + 3bx^2 + 3cx + d = 0$ … (1)

Removing the second term and multiplying the roots by a, the equation (1) can be reduced to the form $z^3 + 3Hz + G = 0$, … (2)

Where $z = a\left(x + \frac{b}{a}\right) = ax + b$, G and H have their usual meanings i.e.,

$$G = a^2d - 3abc + 2b^3$$

And $H = ac - b^2$.

Let us assuming that $z = u^{1/3} + v^{1/3}$. … (3)

Cubing both sides of (3), we have $z^3 = u + v + 3u^{1/3}v^{1/3}\left(u^{1/3} + v^{1/3}\right)$

i.e., $z^3 - 3u^{1/3}v^{1/3}\,z - (u + v) = 0$ … (4)

Comparing the coefficient in (2) and (4), we get

$u^{1/3}v^{1/3} = -H$ or $uv = -H^3$ and $u + v = -G$

Hence u and v are the root of the quadratic $t^2 + Gt - H^3 = 0$. … (5)

Solving (5), we get $u = \frac{-G+\sqrt{(G^2+4H^3)}}{2}$, $v = \frac{-G-\sqrt{(G^2+4H^3)}}{2}$ ... (6)

Now by taking cube root we shall get three values of the cube root of u, namely, $u^{1/3}$, $\omega u^{1/3}$ and $\omega^2 u^{1/3}$. Similarly, we shall get three values of the cube root of v, namely $v^{1/3}$, $\omega v^{1/3}$ and $\omega^2 v^{1/3}$. If we take all possible combinations, we shall get nine values of the expression $u^{1/3} + v^{1/3}$ which is a root of the equation (2). But a cubic should have only three roots. Therefore, while combining the values of $u^{1/3}$ and $v^{1/3}$ we should not forget that they are also connected by the relation $u^{1/3}v^{1/3} = -H$. From this relation we get $v^{1/3} = -\frac{H}{u^{1/3}}$. Putting the value of $v^{1/3}$in (3), we get $z = u^{1/3} - \frac{H}{u^{1/3}}$, the other two values of z being obtained by replacing $u^{1/3}$, $\omega u^{1/3}$ and $\omega^2 u^{1/3}$ respectively.

Thus if z_1, z_2, z_3 are the roots of the cubic (2), we have

$$z_1 = u^{1/3} - \frac{H}{u^{1/3}} = u^{1/3} + v^{1/3},$$

$$z_2 = \omega u^{1/3} - \frac{H}{\omega u^{1/3}} = \omega u^{1/3} - \omega^2 \frac{H}{u^{1/3}} = \omega u^{1/3} + \omega^2 v^{1/3},$$

$$z_3 = \omega^2 u^{1/3} - \frac{H}{\omega^2 u^{1/3}} = \omega^2 u^{1/3} - \omega \frac{H}{u^{1/3}} = \omega^2 u^{1/3} + \omega v^{1/3},$$

Where, $\omega = \frac{1}{2}\left(-1 + \sqrt{3i}\right) and\ \omega^2 = \frac{1}{2}\left(-1 - \sqrt{3i}\right)$

Having found the values of *z*, we can find the values of *x* by the relation *z*= *ax*+*b*.

2.3.1 *APPLICATION OF CARDAN'S METHOD TO NUMERICAL EQUATIONS*

When $G^2 + 4H^3 > 0$ or = 0 i.e., when the cubic has two imaginary or two equal roots, then the values of u and v as found from the quadratic in t are real. Now by some suitable arithmetical process we can extract the cube roots of real quantities and thus we shall get the values of $u^{1/3}$ and $v^{1/3}$. But if $G^2 + 4H^3 < 0$ i.e., if all the roots of the cubic are real and different, the values of u and v are not real. Now there is no general arithmetical process for extracting the cube root of complex number and consequently Cardan's method fails to give the solution of a cubic all of whose roots are real and different. This was called by older mathematician **irreducible case of Cardon's solution**. However in this case we can make use of DeMoiver's theorem of Trigonometry to find the values of the cube roots of complex numbers.

Let $-G = A, G^2 + 4H^3 = -B^2$, so that $u = \frac{A}{2} + \frac{i}{2}B$ and $v = \frac{A}{2} - \frac{i}{2}B$

$$\therefore \quad z = \left(\frac{A}{2} + \frac{i}{2}B\right)^{1/3} + \left(\frac{A}{2} - \frac{i}{2}B\right)^{1/3}$$

Now put $\frac{A}{2} = r\cos\theta$ and $\frac{B}{2} = r\sin\theta$, so that

$$r^2 = \frac{A^2 + B^2}{4} = \frac{G^2 - G^2 - 4H^3}{4} = -H^3$$

And $\tan\theta = \frac{B}{A} = \frac{-\sqrt{\left[-\left(G^2 + 4H^3\right)\right]}}{G}$

$$\therefore \quad z = \left(r\cos\theta + ir\sin\theta\right)^{1/3} + \left(r\cos\theta - ir\sin\theta\right)^{1/3}$$

$$= r^{1/3}\left[\cos\left(\frac{2n\pi + \theta}{3}\right) + i\sin\left(\frac{2n\pi + \theta}{3}\right) + \cos\left(\frac{2n\pi + \theta}{3}\right) - i\sin\left(\frac{2n\pi + \theta}{3}\right)\right]$$

$$= 2r^{1/3}\left(\frac{2n\pi + \theta}{3}\right), where\, n = 0,1,2.$$

Hence the root of the equation in z are

$$2r^{1/3}\cos\frac{\theta}{3}, 2r^{1/3}\cos\left(\frac{2\pi + \theta}{3}\right) and\, 2r^{1/3}\cos\left(\frac{4\pi + \theta}{3}\right);$$

i.e., $2r^{1/3}\cos\frac{\theta}{3}, 2r^{1/3}\cos\left(\frac{2\pi + \theta}{3}\right) and\, 2r^{1/3}\cos\left(\frac{2\pi - \theta}{3}\right);$

i.e., $2(-H)^{1/2}\cos\frac{\theta}{3}, 2(-H)^{1/2}\cos\left(\frac{2\pi \pm \theta}{3}\right).$

Working rule to solve the cubic equation by Cardon's method

To solve the cubic equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ by Cardon's method, we generally follow the following steps.

Step 1: First, we did the coefficient of x^2 is equal to zero by diminishing its root by $h = -\frac{a_1}{na_0}$, where n = degree of equation.

Then we get the cubic equation of the form $z^3 + 3Hz + G = 0$
… (1)

Sometimes the given cubic is of the type $a_0x^3 + a_1x^2 + 0x + a_3 = 0$ i.e., the coefficient of *x* is equal to zero. In this case, we replace $x = \frac{1}{z}$ and construct the cubic equation of the type equation (1).

Step 2: In this step, we assume $z = u^{1/3} + v^{1/3}$ be the solution of cubic equation (1). After cubing both side we get the equation $z^3 - 3u^{1/3}v^{1/3}z - (u+v) = 0$. Then, we compare this equation by (1) and find the value of uv and $u+v$.

Step 3: Since we know that, if *u* and *v* are the roots of the quadratic equation then that equation can be written as $t^2 - (u+v)t + uv = 0$. Then, put the value of uv, $u+v$ and solve the quadratic equation to find the value of *u* and *v*.

Step 4: The following will be the roots of the given cubic equation $z^3 + 3Hz + G = 0$,

$u^{1/3} + v^{1/3}$, $\omega u^{1/3} + \omega^2 v^{1/3}$, $\omega^2 u^{1/3} + \omega v^{1/3}$ where $\omega = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \omega^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$.

So, the roots of the cubic equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ will be find out by the changes in the roots of $z^3 + 3Hz + G = 0$ as we have the relation between *x* and *z*.

SOLVED EXAMPLES

Example 1: Solve the cubic $x^3 - 18x - 35 = 0$ by Cardan's method.

Solution: The given equation is $x^3 - 18x - 35 = 0$
… (1)

Since, the given cubic equationis also in the form of $z^3 + 3Hz + G = 0$.

Let $z = u^{1/3} + v^{1/3}$ be the solution of the cubic (1).

Now, cubing both side we get

$z^3 = u + v + 3u^{1/3}v^{1/3}(u^{1/3} + v^{1/3})$

$z^3 - 3u^{1/3}v^{1/3}z - (u + v) = 0$
… (2)

Comparing equation (1) and (2) we get

$-3u^{1/3}v^{1/3} = -18$ and $-(u+v) = -35$

$\Rightarrow u^{1/3}v^{1/3} = 6$ and $u + v = 35$

$\Rightarrow uv = 6^3 = 216$ and $u + v = 35$

Let u and v are the roots of the quadratic equation then,

$t^2 - (u+v)t + uv = 0$ i.e., $t^2 - 35t + 216 = 0$

On solving the given quadratic, we get $t = 8, 27$

Let, $u = 8,$ then $u^{1/3} = 2$

Similarly, $v = 27$, then $v^{1/3} = 3$

So, the root of the cubic (1) are

$u^{1/3} + v^{1/3} = 3 + 2 = 5$

$$\omega u^{1/3} + \omega^2 v^{1/3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 3\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{5}{2} - i\frac{\sqrt{3}}{2}$$

$$\begin{bmatrix} \therefore \omega = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ \omega^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

And $\omega^2 u^{1/3} + \omega v^{1/3} = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$

Example 2: Solve the cubic equation $9x^3 + 6x^2 - 1 = 0$ by Cardan's method.

Solution: The given cubic equation is $9x^3+6x^2-1=0$
… (1)

Since in the given cubic equation the coefficient of x is zero. So, to transform the cubic equation of the form $z^3+3Hz+G=0$, we replace $x=\frac{1}{z}$.

The transformed cubic equation is,

$$9\left(\frac{1}{z}\right)^3+6\left(\frac{1}{z}\right)^2-1=0$$

i.e., $z^3-6z-9=0$
… (2)

Let $z=u^{1/3}+v^{1/3}$ be the solution of the cubic (2).

Now, cubing both side we get

$$z^3=u+v+3u^{1/3}v^{1/3}(u^{1/3}+v^{1/3})$$

$$z^3-3u^{1/3}v^{1/3}z-(u+v)=0 \quad \ldots (3)$$

Comparing equation (2) and (3) we get

$-3u^{1/3}v^{1/3}=-6$ and $-(u+v)=-9$

$\Rightarrow u^{1/3}v^{1/3}=2$ and $u+v=9$

$\Rightarrow uv=2^3=8$ and $u+v=9$

Let u and v are the roots of the quadratic equation then,

$t^2-(u+v)t+uv=0$ i.e., $t^2-8t+9=0$

On solving the given quadratic, we get $t=1,8$

Let, $u=1,$ then $u^{1/3}=1$

Similarly, $v=8,$ then $v^{1/3}=2$

The root of the cubic (2) are

$u^{1/3}+v^{1/3}=2+1=3$

$\omega u^{1/3}+\omega^2 v^{1/3}=1\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-\frac{3}{2}-i\frac{\sqrt{3}}{2}$

$$\begin{bmatrix} \therefore \omega=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ \omega^2=\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

And $\omega^2 u^{1/3}+\omega v^{1/3}=1\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)+2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\frac{3}{2}+i\frac{\sqrt{3}}{2}$

So, the root of the cubic (1) are

$\frac{1}{3},\frac{2}{-3+\sqrt{3}i},\frac{2}{-3-\sqrt{3}i}$

i.e., $\frac{1}{3},-\frac{(3+\sqrt{3}i)}{6},-\frac{(3-\sqrt{3}i)}{6}$

Example 3: Solve the cubic equation $x^3-15x^2-33x+847=0$ by Cardan's method.

Solution: The given cubic equation is $x^3-15x^2-33x+847=0$
… (1)

Since, the given cubic equation is not in the form of $z^3+3Hz+G=0$. So, first we remove the second term of the equation (1) by diminishing its root by

$$h=-\frac{a_1}{na_0}=-\frac{-15}{3.1}=5$$

The synthetic division procedure is given below:

5	1	-15	-33	847
	-	5	-50	-415
	1	-10	-83	432
	-	5	-25	
	1	-5	-108	
	-	0		

Now, the transformed equation is $z^3 - 108z + 432 = 0$, where $z = x - 5$
… (2)

The transformed equation is of the type $z^3 + 3Hz + G = 0$. So, let
$z = u^{1/3} + v^{1/3}$ be the solution of the cubic (2).

Now, cubing both side we get

$$z^3 = u + v + 3u^{1/3}v^{1/3}(u^{1/3} + v^{1/3})$$

$$z^3 - 3u^{1/3}v^{1/3}z - (u + v) = 0$$
… (3)

Comparing equation (2) and (3) we get

$$-3u^{1/3}v^{1/3} = -108 \text{ and } -(u + v) = 432$$

$$\Rightarrow u^{1/3}v^{1/3} = 36 \text{ and } u + v = -432$$

$$\Rightarrow uv = 36^3 \text{and } u + v = -432$$

Let u and v are the roots of the quadratic equation then,

$$t^2 - (u + v)t + uv = 0 \text{ i.e., } t^2 + 432t + 36^3 = 0$$

By *Sridharacharya method,* $t = \dfrac{-432 \pm \sqrt{(432)^2 - 4(36)^3}}{2}$

i.e., $t = -216$

$\therefore u = -216, \text{v} = -216$ then $u^{1/3} = -6$ and $v^{1/3} = -6$

So, the root of the cubic (1) are

$$u^{1/3} + v^{1/3} = -6 - 6 = -12$$

$$\omega u^{1/3} + \omega^2 v^{1/3} = -6\omega - 6\omega^2 = -6(\omega + \omega^2) = 6$$
$$\left[\therefore 1 + \omega + \omega^2 = 0\right]$$

And $\omega^2 u^{1/3} + \omega v^{1/3} = -6\omega^2 - 6\omega = -6(\omega^2 + \omega) = 6$

Hence roots of the given cubic (1) are

$-12+5, 6+5, 6+5$ i.e., -7, 11, 11.

Example 4: Solve the cubic equation $27x^3 + 54x^2 + 198x - 73 = 0$ by Cardan's method.

Solution: The given cubic equation is $27x^3 + 54x^2 + 198x - 73 = 0$ … (1)

Since, the given cubic equation is not in the form of $z^3 + 3Hz + G = 0$. So, first we remove the second term of the equation (1) by diminishing its root by

$$h = -\frac{a_1}{na_0} = -\frac{54}{3.27} = -\frac{2}{3}$$

The synthetic division procedure is given below:

-2/3	27	54	198	-73
	-	-18	-27	-116
	27	36	174	-189
	-	-18	-12	
	27	18	162	
	-	-18		
	27	0		

Now, the transformed equation is $27z^3 + 162z - 189 = 0$ or $z^3 + 6z - 7 = 0$ … (2)

Where, $z = x + \frac{2}{3}$

The transformed equation is of the type $z^3 + 3Hz + G = 0$. So, let $z = u^{1/3} + v^{1/3}$ be the solution of the cubic (2).

Now, cubing both side we get

$$z^3 = u + v + 3u^{1/3}v^{1/3}(u^{1/3} + v^{1/3})$$

$$z^3 - 3u^{1/3}v^{1/3}z - (u + v) = 0$$
… (3)

Comparing equation (2) and (3) we get

$-3u^{1/3}v^{1/3} = 6$ and $-(u + v) = -7$

$\Rightarrow u^{1/3}v^{1/3} = -2$ and $u + v = 7$

$\Rightarrow uv = -8$ and $u + v = 7$

Let u and v are the roots of the quadratic equation then,

$t^2 - (u + v)t + uv = 0$ i.e., $t^2 - 7t - 8 = 0$

By *simplifying we get*, $t = 8, -1$

Let, $u = 8$ and $v = -1$

So, the root of the cubic (1) are

$$u^{1/3} + v^{1/3} = 2 - 1 = 1$$

$$\omega u^{1/3} + \omega^2 v^{1/3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - 1\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + \frac{3\sqrt{3}}{2}i$$

$$\begin{bmatrix} \therefore \omega = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ \omega^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

And $\omega^2 u^{1/3} + \omega v^{1/3} = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} - \frac{3\sqrt{3}}{2}i$

Hence the root of the given cubic equation (1) are,

$$1 - \frac{2}{3}, -\frac{1}{2} + \frac{3\sqrt{3}}{2}i - -\frac{2}{3}, -\frac{1}{2} - \frac{3\sqrt{3}}{2}i - -\frac{2}{3}$$

$$\frac{1}{3}, -\frac{7}{6} + \frac{3\sqrt{3}}{2}i, -\frac{7}{6} - \frac{3\sqrt{3}}{2}i$$

Example 5: Solve the cubic equation $x^3 - 3x + 1 = 0$ by Cardan's method.

Solution: The given cubic equation is $x^3 - 3x + 1 = 0$
… (1)

Since, the given cubic equation is also in the form of $z^3 + 3Hz + G = 0$.

Let $z = u^{1/3} + v^{1/3}$ be the solution of the cubic (1).

Now, cubing both side we get

$$z^3 = u + v + 3u^{1/3}v^{1/3}(u^{1/3} + v^{1/3})$$

$$z^3 - 3u^{1/3}v^{1/3}z - (u + v) = 0$$
… (2)

Comparing equation (1) and (2) we get

$-3u^{1/3}v^{1/3} = -3$ and $-(u + v) = 1$

$\Rightarrow u^{1/3}v^{1/3} = 1$ and $u + v = -1$

$\Rightarrow uv = 1^3 = 1$ and $u + v = -1$

Let u and v are the roots of the quadratic equation then,

$t^2 - (u + v)t + uv = 0$ i.e., $t^2 + t + 1 = 0$

By *Sridharacharya method,* $t = \dfrac{-1 \pm \sqrt{(1)^2 - 4.1.1}}{2}$

i.e., $t = \dfrac{-1 \pm i\sqrt{3}}{2}$

Let, $u = \dfrac{-1+i\sqrt{3}}{2}$, then $u^{1/3} = \left(\dfrac{-1+i\sqrt{3}}{2}\right)^{\frac{1}{3}}$

Similarly, $v = \left(\dfrac{-1-i\sqrt{3}}{2}\right)$ then $v^{1/3} = \left(\dfrac{-1-i\sqrt{3}}{2}\right)^{\frac{1}{3}}$

So, the root of the cubic (1) are

$$x = u^{1/3} + v^{1/3} = \left(\frac{-1+i\sqrt{3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{\frac{1}{3}}$$

Let $r\cos\theta = -\dfrac{1}{2}$ and $r\sin\theta = \dfrac{\sqrt{3}}{2}$

Then, $r^2 = 1 \; or \, r = 1$ and $\tan\theta = -\sqrt{3} \; or \; \theta = \dfrac{2\pi}{3}$

$$x = r^{1/3}\left[\left(\cos\theta + i\sin\theta\right)^{1/3} + \left(\cos\theta - i\sin\theta\right)^{1/3}\right]$$

$$x = (1)^{1/3}\left[\left\{\cos(2\mathrm{n}\pi+\theta) + i\sin(2\mathrm{n}\pi+\theta)\right\}^{1/3} + \left\{\cos(2\mathrm{n}\pi+\theta) - i\sin(2\mathrm{n}\pi+\theta)\right\}^{1/3}\right]$$

By DeMoiver's theorem, $\left(\cos\theta + i\sin\theta\right)^n = \cos(\mathrm{n}\theta) + i\sin(\mathrm{n}\theta)$

$$x = \left[\left\{\cos\frac{(2\mathrm{n}\pi+\theta)}{3} + i\sin\frac{(2\mathrm{n}\pi+\theta)}{3}\right\} + \left\{\cos\frac{(2\mathrm{n}\pi+\theta)}{3} - i\sin\frac{(2\mathrm{n}\pi+\theta)}{3}\right\}\right]$$

$$x = 2\cos\frac{(2\mathrm{n}\pi+\theta)}{3}$$

Putting value of $\theta = \dfrac{2\pi}{3}$ and $n = 0,1,2$

Hence required roots are $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}$ and $2\cos\frac{14\pi}{9}$

2.4 *EULER'S SOLUTION OF THE BIQUADRATIC EQUATIONS*

Let the general form of the biquadratic equation is
$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ … (1)

be transformed in the form of $z^4 + 6Hz^3 + 4Gz + (a_0{}^2I - 3H^2) = 0$
… (2)

where $z = a_0x + a_1$

we assume $z = \sqrt{p} + \sqrt{q} + \sqrt{r}$

squaring both sides, we get

$[z^2 - (p+q+r)] = 2\left(\sqrt{p}\sqrt{q} + \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p}\right)$
… (3)

again,squaring both sides we get

$$\mathrm{z}^4 - 2z^2(p+q+r) + \left(p+q+r\right)^2 = 4\left[pq+qr+rp+2\sqrt{p}\sqrt{q}\sqrt{r}\left(\sqrt{p}+\sqrt{q}+\sqrt{r}\right)\right]$$

$\Rightarrow \mathrm{z}^4 - \left(2\sum p\right)z^2 - 8z\sqrt{p}\sqrt{q}\sqrt{r} + \left(\sum p\right)^2 - 4\left[\sum pq\right] = 0$
… (4)

Now, comparing the coefficient of *z* of equation (2) and (4), we get

$$\sum p = -3H, \sqrt{p}\sqrt{q}\sqrt{r} = -\frac{G}{2} \text{ or } pqr = \frac{G^4}{4}$$

and $\left(\sum p\right)^2 - 4\sum pq = a_0{}^2I - 3H^2$ or $\sum pq = 3H^2 - \frac{a_0{}^2I}{4}$

The cubic equation whose three roots are *p*, *q*, *r* be written as

$$t^3 - (p+q+r)t^2 + \left(\mathrm{pq+qr+rp}\right)\mathrm{t} - \mathrm{pqr} = 0$$

On putting the values of coefficient, equation will become

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a_0{}^2 I}{4}\right)t - \frac{G^2}{4} = 0$$

… (5)

This equation is known as **Euler's cubic**.

As we know that $G^2 + 4H^3 = a_0(HI - a_0 J)$

Dividing both side by 4 and simplifying the equation

$$-\frac{G^2}{4} = H^3 - \frac{a_0{}^2 HI}{4} + \frac{a_0{}^3 J}{4}$$

… (6)

From (5) and (6)

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a_0{}^2 I}{4}\right)t + H^3 - \frac{a_0{}^2 HI}{4} + \frac{a_0{}^3 J}{4} = 0$$

$$(t+H)^3 - \frac{a_0{}^2 I}{4}(t+H) + \frac{a_0{}^3 J}{4} = 0$$

Putting $t + H = a_0{}^2\theta$, we have

$$a_0{}^6\theta^3 - \frac{a_0{}^4 I}{4}\theta + \frac{a_0{}^3 J}{4} = 0 \text{ or } 4a_0{}^6\theta^3 - Ia_0\theta + J = 0 \quad \dots (7)$$

Equation (6) is called **reducing cubic** of the biquadratic equation.

Remark: In the Euler's cubic equation, G occurs in even powers, so if biquadratic equation occurs of the type

$$z^4 + 6Hz^2 - 4Gz + (a_0{}^2 I - 3H^2) = 0$$

Then its Euler's cubic would have been the same as that of the given quadratic and therefore the two will have the same reducing cubic.

2.5 *DESCARTE'S METHOD OF SOLVING A BIQUADRATIC EQUATION*

Let the general form of the biquadratic equation is
$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ … (1)

Removing the second term from (1) and multiplying the roots by a_0. Then thereduced equation can be written in the form

$f(z) = z^4 + 6Hz^2 + 4Gz + a_0{}^2I - 3H^2 = 0$ where $z = a_0x + a_1$
… (2)

Let z_1, z_2, z_3, z_4 are the roots of the equation (2). Since coefficient of z^3 is zero in (2), so sum of the roots i.e., $z_1 + z_2 + z_3 + z_4 = 0$

Or $z_1 + z_2 = -(z_3 + z_4) = p$ (say)

Also we assumed that $z_1z_2 = q$ and $z_3z_4 = q'$

$\Rightarrow f(z) \equiv (z^2 - pz + q)(z^2 + pz + q')$
… (3)

Comparing the coefficient of equation (2) and (3)

$q + q' - p^2 = 6H,$
… (4)

$p(q - q') = 4G$
…(5)

and $qq' = a_0{}^2I - 3H^2$
… (6)

Since we have, $(q - q')^2 = (q + q')^2 - 4qq'$.

$$\therefore \frac{16G^2}{p^2} = \left(p^2 + 6H\right)^2 - 4\left(a_0{}^2I - 3H^2\right)$$

Or $p^2(p^4 + 12Hp^2 + 36H^2) - 4p^2(a_0{}^2I - 3H^2) - 16G^2 = 0$

Or $p^6 + 12Hp^4 + 4p^2(12H^2 - a_0{}^2I) - 16G^2 = 0$

This being cubic equation in p^2 can be solved generally by trial method and having found p^2 we can find the values of q and q'. Thus, the quadratic factors of $f(z)$ are known. Hence, we can find the roots of $f(z)=0$ and consequently $f(\mathrm{x})=0$.

The above cubic equation in p^2 can be put as

$$(p^6+12Hp^4+48p^2H^2+48H^3)-4a_0{}^2I(p^2+4H)-16G^2-64H^3+16a_0{}^2H=0$$

Or $(p^6+4H)^3-4a_0{}^2I(p^2+4H)-16(-a_0{}^3J)=0$

$[\therefore G^2+4H^3=a_0{}^2HI-a_0{}^3J]$

Putting $p^2+4H=4a_0{}^2\theta$, we get $4a_0{}^3\theta^3-Ia_0\theta+J=0$.

This equation is called reducing cubic which is same as found earlier.

Note: We find the value of p^2 by trial i.e., we try to put those numbers which are whole squares like 1, 4, 9, 16 etc.

Working rule to solve the biquadratic equation by Descarte's method

To solve the biquadratic equation $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$ by Descarte'smethod, we generally follow the following steps.

Step 1: First, we did the coefficient of x^3 is equal to zero by diminishing its root by $h=-\dfrac{a_1}{na_0}$, where n = degree of equation.

we get the biquadratic equation of the form

$f(z)=b_0z^4+b_1z^2+b_2z+b_3\equiv(z^2+pz+q)(z^2-pz+q')$, where $z=x-h$ … (1)

Step 2: In this step, we equate the both side coefficient of equation (1) i.e.,

$q+q'-p^2=l$

$\Rightarrow q+q'=p^2-l$

$$p(q^{'}-\mathrm{q})=m$$

$$\Rightarrow q^{'}-q=\frac{m}{p}$$

And $qq^{'}=n$

Using the equation $(q^{'}-\mathrm{q})^2=(q+q^{'})^2-4qq^{'}$

We get a cubic equation in the variable p^2. For simplification we assume that $p^2=t$ (say) and we get a cubic equation in *t*.

Step 3: Using trial method, we put whole squares number like 1, 4, 9, 16 etc. to solve the cubic equation in *t*.

Step 4: Finding the value of $p,q,q^{'}$, we find the root of equation (1) by using the equation

$z^2+pz+q=0$ and $z^2-pz+q^{'}=0$.

After finding the roots in z, we find the roots in x by the relation $z=x-h$ and get the required roots of the biquadratic equation $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$.

Solved Examples

Example 6: Using Descarte's method solve the biquadratic equation $x^4-8x^3-12x^2+60x+63=0$.

Solution: The given biquadratic equation is $x^4-8x^3-12x^2+60x+63=0$ … (1)

In the given equation (1), first we remove the second term of the equationby diminishing its root by

$$h=-\frac{a_1}{na_0}=-\frac{-8}{4.1}=2$$

The synthetic division procedure is given below:

2	1	-8	-12	60	63
	-	2	-12	-48	24
	1	-6	-24	12	87
	-	2	-8	-64	

Now, the transformed equation in which coefficient of x^3 is zero,

$f(z) \equiv z^4 - 36z^2 - 52z + 87 = 0$, where $z = x - 2$
… (2)

Equation (2) can also be written as
$z^4 - 36z^2 - 52z + 87 \equiv (z^2 + pz + q)(z^2 - pz + q')$ … (3)

Equating both side coefficient we get,

$q + q' - p^2 = -36$
$\Rightarrow q + q' = p^2 - 36$

$p(q' - \text{q}) = -52$
$\Rightarrow q' - q = \dfrac{-52}{p}$

$qq' = 87$

Since it can be written that, $(q' - \text{q})^2 = (q + q')^2 - 4qq'$

$\Rightarrow \dfrac{52^2}{p^2} = (p^2 - 36)^2 - 348$

Let $t = p^2$ then, $t^3 - 72t^2 + 948t - 2704 = 0$

Since $t = 4$ (i.e., $p = 2$) satisfies the above cubic equation*t*.

So we get, $q+q' = 4-36 = -32$
$q' - q = -26$

Solving these we get, $q = -3$ and $q' = -29$

Now, from equation (2)

$(z^2+2z-3)(z^2-2z-29)=0$

$\Rightarrow z^2+2z-3=0$ or $z^2-2z-29=0$

$\Rightarrow z = -3, 1, 1\pm\sqrt{30}$

Hence, $x = z-2 = -1,\ 3,\ 3\pm\sqrt{30}$

Example 7: Using Descarte's method solve the biquadratic equation $x^4-3x^2-42x-40=0$.

Solution: The given biquadratic equation is $x^4-3x^2-42x-40=0$ … (1)

In the given equation (1), the coefficient of x^3 is already zero.

Now, $f(x) \equiv x^4-3x^2-42x-40 = (x^2+px+q)(x^2-px+q')$ … (2)

Equating both side coefficient of equation (2) we get,

$q+q'-p^2 = -3$
$\Rightarrow q+q' = p^2-3$

$p(q'-\text{q}) = -42$
$$\Rightarrow q'-q = \frac{-42}{p}$$

$qq' = -40$

Since it can be written that, $(q'-\text{q})^2 = (q+q')^2 - 4qq'$

$$\Rightarrow \frac{42^2}{p^2} = (p^2-3)^2+160$$

Let $t = p^2$ then, $t^3 - 6t^2 + 169t - 1764 = 0$

Since $t = 9$ (i.e., $p = 3$) satisfies the above cubic equation*t*.

So we get, $\begin{matrix} q + q' = 6 \\ q' - q = -14 \end{matrix}$

Solving these we get, $q = 10$ and $q' = -4$

Now, from equation (2)

$\text{f(x)} \equiv (\text{x}^2 + 3x + 10)(\text{x}^2 - 3x - 4) = 0$

$\Rightarrow x^2 + 3x + 10 = 0$ or $x^2 - 3x - 4 = 0$

$\Rightarrow x = -1, 4, \dfrac{-2 \pm i\sqrt{31}}{2}$, which are the required roots.

Example 8: Using Descarte's method solve the biquadratic equation $x^4 - 6x^3 - 9x^2 + 66x - 22 = 0$.

Solution: The given biquadratic equation is $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ … (1)

Here in (1), first we should remove the second term of the equation by diminishing its root by

$$h = -\frac{a_1}{na_0} = -\frac{-6}{4.1} = \frac{3}{2}$$

But in the synthetic division it will very difficult to diminish the root by 3/2. To remove this complication, we first multiply the root by 2.

The transformed equation is,

$y^4 - 12y^3 - 36y^2 + 528y - 352 = 0$ where $y = 2x$ … (2)

Now diminishing the roots by 3 using synthetic division.

3	1	-12	-36	528	-352	
	-	3	-27	-189	1017	
	1	-9	-63	339	665	
	-	3	-18	-243		
	1	-6	-81	96		
	-	3	-9			
	1	-3	-90			
	-	0				

Now, the transformed equation in which coefficient of x^3 is zero,

$f(z) \equiv z^4 - 90z^2 + 96z + 665 = 0$, where $z = y - 3$ … (3)

Equation (3) can also be written as

$z^4 - 90z^2 + 96z + 665 \equiv (z^2 + pz + q)(z^2 - pz + q')$

Equating both side coefficient we get,

$$q + q' - p^2 = -90$$
$$\Rightarrow q + q' = p^2 - 90$$

$$p(q' - q) = 96$$
$$\Rightarrow q' - q = \frac{96}{p}$$

$$qq' = 665$$

Since it can be written that, $(q' - q)^2 = (q + q')^2 - 4qq'$

$$\Rightarrow \frac{96^2}{p^2} = (p^2 - 90)^2 - 2660$$

Let $t = p^2$ then, $t^3 - 180t^2 + 5440t - 96^2 = 0$

Since $t = 36$ (i.e., $p = 6$) satisfies the above cubic equation *t*.

So we get, $$\begin{aligned} q + q' &= 36 - 90 = -54 \\ q' - q &= -16 \end{aligned}$$

Solving these we get, $q = -35$ and $q' = -19$

Now, from equation (3)

$(z^2 + 6z - 35)(z^2 - 6z - 19) = 0$

$\Rightarrow z^2 + 6z - 35 = 0$ or $z^2 - 6z - 19 = 0$

$\Rightarrow z = 3 \pm 2\sqrt{7}, -3 \pm 2\sqrt{11}$

$\Rightarrow y = 6 \pm 2\sqrt{7}, \pm 2\sqrt{11}$

Hence, roots of required equation (1) are $x = 3 \pm \sqrt{7}, \pm\sqrt{11}$

2.6 *FERRARI'S METHOD OF SOLVING BIQUADRATIC EQUATION*

Let $f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, be a biquadratic equation.

Let we consider

$$f(x) \equiv \frac{1}{a}\left\{\left(ax^2 + 2bx + c + 2a\theta\right)^2 - \left(2Mx + N\right)^2\right\}$$

Now compare both side coefficient of like power of *x*, we get

$M^2 = b^2 - ac + a^2\theta$ … (1)

$MN = bc - ad + 2ab\theta$ … (2)

$N^2 = (c + 2a\theta)^2 - ae$ … (3)

$\therefore (b^2 - ac + a^2\theta)\left\{(c + 2a\theta)^2 - ae\right\} = (bc - ad + 2ab\theta)^2 \left[\because M^2N^2 = (\mathrm{MN})^2\right]$

Or $4a^3\theta^3 - (ae - 4bd + 3c^3)a\theta + ace + 2bcd - ad^2 - eb^2 - c^3 = 0$

Or $4a^3\theta^3 - Ia\theta + J = 0$ … (4)

Here, equation (4) represents the equation of **reducing cubic** from which we can evaluate the value of θ. Then we find the value of M and N. Thus the given biquadratic can be written to the quadratic equation as,

$$ax^2 + 2(b - M)x + c + 2a\theta - N = 0 \quad \ldots (5)$$

And $ax^2 + 2(b + M)x + c + 2a\theta + N = 0$ … (6)

Let $\theta_1, \theta_2, \theta_3$ be the three values of θ which satisfies the equation (4) and corresponding values of M be M_1, M_2, M_3 and those of N be N_1, N_2, N_3.

Let α, δ be roots of (6) and β, γ be the roots of (5) when M, N, θ have the values M_1, N_1, θ_1 respectively. Then

$$\beta + \gamma = -\frac{2}{a}(b - M_1) \quad \text{and} \quad \beta\gamma = \frac{c + 2a\theta_1 - N_1}{a}$$

… (A)

$$\alpha + \delta = -\frac{2}{a}(b + M_1) \quad \text{and} \quad \alpha\delta = \frac{c + 2a\theta_1 + N_1}{a}$$

$$\left.\begin{aligned} \beta + \gamma - \alpha - \delta &= \frac{4}{a}M_1 \\ \text{similarly, } \gamma + \alpha - \beta - \delta &= \frac{4}{a}M_2 \\ \alpha + \beta - \gamma - \delta &= \frac{4}{a}M_3 \end{aligned}\right] \quad \ldots$$

(I)

$$\left.\begin{aligned} \beta\gamma + \alpha\delta &= \frac{2c}{a} + 4\theta_1 \\ \text{and } \gamma\alpha + \beta\delta &= \frac{2c}{a} + 4\theta_2 \\ \alpha\beta + \gamma\delta &= \frac{2c}{a} + 4\theta_3 \end{aligned}\right] \quad \ldots (II)$$

From (I) we get,

$$-\frac{4}{a}(M_2+M_3-M_1)=(2\alpha-2\delta-\beta-\gamma+\alpha+\delta)$$
$$=(3\alpha-\beta-\gamma-\delta)=(4\alpha-\sum\alpha)$$
$$=\left(4\alpha+\frac{4b}{a}\right)=\frac{4}{a}(a\alpha+b)$$

From (A) and From (I), we have

$$\left.\begin{aligned} a\alpha+b&=-M_1+M_2+M_3\\ a\beta+b&=M_1-M_2+M_3\\ a\gamma+b&=M_1+M_2-M_3\\ a\delta+b&=-M_1-M_2-M_3\end{aligned}\right] \qquad \ldots \text{(III)}$$

$$\left.\begin{aligned} \beta\gamma-\alpha\delta&=\frac{-2N_1}{a}\\ \text{and } \gamma\alpha-\beta\delta&=\frac{-2N_2}{a}\\ \alpha\beta-\gamma\delta&=\frac{-2N_3}{a}\end{aligned}\right] \qquad \ldots \text{(IV)}$$

Solved Example

Example 9: Using Ferrari's method solve the biquadratic equation $x^4+2x^3-7x^2-8x+12=0$.

Answer: The given biquadratic equation is

$$x^4+2x^3-7x^2-8x+12=0 \qquad \ldots (1)$$

Adding x^2 both side to (1) to make the perfect square, we get

$$x^4+2x^3+x^2=7x^2+8x-12+x^2$$
$$(x^2+x)^2=8x^2+8x-12$$
$$(x^2+x+\lambda)^2=8x^2+8x-12+\lambda^2+2\lambda(x^2+x)$$

$(x^2+x+\lambda)^2=(8+2\lambda)x^2+(8+2\lambda)x+\lambda^2-12$ (2)

Since, R.H.S is a perfect square*i.e.*, $B^2-4AC=0$, from which we get cubic in λ by which λ can be found.Although it is not easy to solve the cubic in λ, so we use the other method. If the R.H.S. is to be a perfect square, then coefficient of x^2 and constant term are also be perfect square. To solve this cubic by trial method we put the coefficient of x^2 perfect squares numbers like 1, 4, 9, 16, $\frac{1}{4},\frac{1}{9},\frac{1}{16}$ etc. and also see the value of λ obtained thus also makes the constant term a perfect square number.

Let $8+2\lambda=1$ i.e., $\lambda=-\frac{7}{2}$.

Since, this value makes the constant term also in perfect square i.e., $\lambda^2-12=\frac{1}{4}$.

Now, put the value of λ in equation (2).

$$f(x)\equiv\left(x^2+x-\frac{7}{2}\right)^2-\left(x+\frac{1}{2}\right)^2=0$$

i.e., $(x^2+2x-3)(x^2-4)=0$

$\Rightarrow x=\pm 2,-3,1.$

Example 10: Using Ferrari's method solve the biquadratic equation $x^4-8x^3-12x^2+60x+63=0$.

Answer: The given biquadratic equation is $x^4-8x^3-12x^2+60x+63=0$ … (1)

Adding $16x^2$ both side to (1) to make the perfect square, we get

$$x^4-8x^3+16x^2=28x^2-60x+63$$
$$(x^2-4x)^2=28x^2-60x+63$$
$$(x^2-4x+\lambda)^2=28x^2-60x+63+2\lambda(x^2-4x)+\lambda^2$$

$$(x^2-4x+\lambda)^2=(28+2\lambda)x^2-(60+8\lambda)x+\lambda^2-63$$
… (2)

Now we put $28+2\lambda=1,4,9$ i.e., a perfect square number and see that for which value of λ, constant term of the equation becomes a perfect square.

$$\therefore \lambda=-\frac{27}{4},-12,-\frac{19}{2}$$

Here it can be easily seen that for $\lambda=-12$, we have the constant term $\lambda^2-63=144-63=81$, which is perfect square number.

Now, put the value of λ in (2). Then the reducible equation is,

$$(x^2-4x-12)^2=4x^2+36x+81=(2x+9)^2$$

$$(x^2-4x-12)^2-(2x+9)^2=0$$
$$\left(x^2-4x-12-2x-9\right)\left(x^2-4x-12+2x+9\right)=0$$
$$\left(x^2-6x-21\right)\left(x^2-2x-3\right)=0$$

On solving we get, $x=3\pm\sqrt{30},3,-1$

Example 11: Using Ferrari's method solve the biquadratic equation $x^4-10x^3+44x^2-104x+96=0$.

Answer: The given biquadratic equation is
$$x^4-10x^3+44x^2-104x+96=0 \qquad \ldots (1)$$

Adding $25x^2$ both side to (1) to make the perfect square, we get

$$x^4-10x^3+25x^2=-44x^2+104x-96+25x^2$$
$$(x^2-5x)^2=-19x^2+104x-96$$
$$(x^2-5x+\lambda)^2=-19x^2+104x-96+2\lambda(x^2-5x)+\lambda^2$$

$$(x^2-5x+\lambda)^2=(-19+2\lambda)x^2+(104-10\lambda)x+\lambda^2-96$$
… (2)

Now we put $2\lambda-19=1,4,9$ i.e., a perfect square number and see that for which value of λ, constant term of the equation becomes a perfect square.

$$\therefore \lambda=10,\frac{23}{2},14$$

Here it can be easily seen that for $\lambda = 10$, we have the constant term $\lambda^2 - 96 = 100 - 96 = 4$, which is perfect square number.

Now, put the value of λ in (2). Then the reducible equation is,

$$(x^2 - 5x + 10)^2 = x^2 + 4x + 4 = (x+2)^2$$

$$(x^2 - 5x + 10)^2 - (x+2)^2 = 0$$
$$\left(x^2 - 5x + 10 - x - 2\right)\left(x^2 - 5x + 10 + x + 2\right) = 0$$
$$\left(x^2 - 6x + 8\right)\left(x^2 - 4x + 12\right) = 0$$

On solving we get, $x = 2 \pm i2\sqrt{2}, 4, 2$

Self Cheque Questions

1. If solution of the cubic equation $z^3 + 6z - 7 = 0$ is $z = u^{1/3} + u^{1/3}$ then if $u = 8$ and $v = ...$

2. In the solution of the biquadratic equation $x^4 + bx^3 + cx^2 + dx + e = 0$ by Descarte's method, we first remove the second term by diminishing its roots by *h*, then *h* = ……..

3. In the solution of the biquadratic equation $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$ by Descarte's method, we first remove the second term by diminishing its roots by *h*, then *h* = ……..

4. Roots of the equation $f(x) = 0$ if $f(x) = x^4 - 3x^2 - 6x - 2 = \left(x^2 + 2x + 2\right)\left(x^2 - 2x - 1\right)$ are ……

5. If $1 + i\sqrt{2}$ and $1 - \sqrt{2}$ are the roots of the biquadratic equation $x^4 - 2x^2 + 8x - 3 = 0$, then other two roots are ………

6. If $x^4 - 8x^2 - 24x + 7 \equiv (x^2 + px + q)(x^2 - px + q')$, then $q + q' - p^2 = -8, p(q' - q) = ...$ and $qq' = 7$

2.7 *SUMMARY*

After completion of this unit learners are able to solve the,

- Cubic equation by using the Cardan's method.

- Biquadratic equation by using the Euler's method.
- Biquadratic equation by using the Descarte's method.
- Biquadratic equation by using the Ferrari's method.

2.8 GLOSSARY

- Cardan's method
- Euler's solution
- Descarte's method
- Ferrari's method

2.9 REFERENCES

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- Press, W. H.; Teukolsky, S. A.; Vetterling, W. T.; Flannery, B. P. (2007), "Section 5.6 Quadratic and Cubic Equations", Numerical Recipes: The Art of Scientific Computing (3rd ed.), New York: Cambridge University Press, ISBN 978-0-521-88068-8
- Rechtschaffen, Edgar (July 2008), "Real roots of cubics: Explicit formula for quasi-solutions", Mathematical Gazette, Mathematical Association, **92**: 268–276, doi:10.1017/S0025557200183147, ISSN 0025-5572, S2CID 125870578

2.10 SUGGESTED READING

- 500 years of NOT teaching THE CUBIC FORMULA. What is it they think you can't handle? – YouTube video by Mathologer about the history of cubic equations and Cardano's solution, as well as Ferrari's solution to quartic equations

2.11 TERMINAL QUESTION

Objective Question

1. If the solution of the cubic equation $x^3 - 12x - 65 = 0$ is $x = u^{1/3} + u^{1/3}$, then which of the following will be the quadratic equation whose roots are *u* and *v*.

a) $t^2-64t+65=0$ **b)** $t^2-65t+64=0$

c) $t^2+65t-64=0$ **d)** $t^2-65t-64=0$

2. If we diminish the rootof the cubic equation $64x^3-144x^2+108x-27=0$ by h, then what will be the value of h to remove the second term of the equation.

a) $h=-\frac{3}{4}$ **b)** $h=\frac{4}{3}$

c) $h=\frac{3}{4}$ **d)** $h=-\frac{4}{3}$

3. Find the values of q and $q^{'}$ if

$x^4-8x^2-24x+7\equiv\left(x^2+px+q\right)\left(x^2-px+q^{'}\right)$ and $p=4$

a) $q=1,q^{'}=7$ **b)** $q=7,q^{'}=1$

c) $q=-1,q^{'}=-7$ **d)** $q=-7,q^{'}=-1$

Fill the correct option to make the following statement complete and correct.

1. If the solution of the cubic equation $z^3+3Hz+G=0$ is $z=u^{1/3}+u^{1/3}$ and if $u+v=-G$ then $uv=$ …

2. If the solution of the cubic equation $z^3-21z-344=0$ is $z=u^{1/3}+u^{1/3}$ then u and v are the roots of the quadratic equation $z^2-344z+....=0$

3. If the roots of cubic equation $z^3-6z-9=0$ are 3, $\frac{1}{2}(-3+\sqrt{3}i)$ and $\frac{1}{2}(-3-\sqrt{3}i)$. Then the roots of the equation $9x^3+6x^2-1=0$ are ……

4. Roots of the cubic $z^3+3Hz+G=0$ are all real if $G^2+4H^3\leq$ ……

Write T for true statement and F for False statement

1. A cubic equation with real coefficient has at least one real root.

2. All roots of cubic equation may be imaginary.

3. The cubic equation $z^3+3Hz+G=0$ has two equal roots if $G^2+4H^3\neq 0$.

4. The cubic equation $z^3+3Hz+G=0$ has two imaginary roots if $G^2+4H^3>0$.

5. To solve the cubic equation $x^3+px^2+qx+r=0$ by using Cardon's method, we first reduce it to the form $z^3+bz^2+c=0$

Solve the following questions

Solve the following cubic equation by cardon's method.

1. $x^3-21x-344=0$ **2.** $28x^3-9x^2+1=0$

3. $x^3-15x^2-357x+5491=0$ **4.** $x^3-12x^2-6x-10=0$

5. $x^3-18x-35=0$ **6.** $x^3+3x^2-27x+104=0$

7. $2x^3+3x^2+3x+1=0$ **8.** $64x^3-144x^2+108x-27=0$

Solve the following cubic equation by Descarte's method.

1. $x^4-5x^2-6x-5=0$ **2.** $x^4-8x^2-24x+7=0$

3. $x^4-12x+3=0$ **4.** $x^4+12x-5=0$

5. $x^4-6x^3+3x^2+22x-6=0$ **6.** $x^4-10x^2-20x-16=0$

7. $x^4-2x^2+8x-3=0$ **8.** $x^4+8x^3+9x^2-8x-10=0$

9. $x^4-12x-5=0$ **10.** $x^4-3x^2-6x-2=0$

Solve the following cubic equation by Ferrari's method.

1. $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$ **2.** $2x^4 + 6x^3 - 3x^2 + 2 = 0$

3. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ **4.** $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$

5. $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ **6.** $x^4 + 4x^3 + 12x^2 - 8x + 95 = 0$

7. $x^4 + 12x - 5 = 0$

2.12 *ANSWERS*

Answer of self cheque questions:

1. -1

2. $-b/4$ **3.** -2

4. $-1 \pm i, 1 \pm \sqrt{2}$ **5.** $1 - i\sqrt{2}$ *and* $-1 + \sqrt{2}$

6. -24

Answer of objective questions

1. (b) **2.** (c)

3. (b)

4. 0 **5.** -1

6. $-b/4$ **7.** -2

8. $-1 \pm i, 1 \pm \sqrt{2}$ **9.** $1 - i\sqrt{2}$ *and* $-1 + \sqrt{2}$

10. -24

Answer of fill in the blanks question

1. $-H^3$ **2.** 343

3. $1/3, -1/6(3 + \sqrt{3}i)$ *and* $-1/6(3 - \sqrt{3}i)$

4. 0

Answer of true and falls questions

1. T **2.** F **3.** F **4.** T **5.** F

Solution of equations by Cardon's method

1. $8, -4 \pm i\,3\sqrt{3}$ **2.** $-1/4, 1/7(2 \pm i\sqrt{3})$ **3.** $-19, 17, 17$

4. $4 + u^{1/3} + v^{1/3}, 4 + \omega u^{1/3} + \omega^2 v^{1/3}, 4 + \omega^2 u^{1/3} + \omega v^{1/3}$

5. $5, -5/2 \pm i\sqrt{3}/2$ **6.** $-8, 1/2(5 \pm i\,3\sqrt{3})$ **7.** $-1/2, -1/2 \pm i\sqrt{3}/2$

8. $2a\cos A, 2a\cos\left(\dfrac{2\pi}{3} \pm A\right)$ **9.** $-(a+b+c), -(a+b\omega+c\omega^2), -(a+b\omega^2+c\omega)$

Solution of equations by Descarte's method

1. $\dfrac{-1 \pm i\sqrt{3}}{2}, \dfrac{1 \pm \sqrt{21}}{2}$ **2.** $-2 \pm i\sqrt{3}, 2 \pm \sqrt{3}$

3. $\dfrac{-\sqrt{6} \pm \sqrt{-6-4\sqrt{6}}}{2}, \dfrac{\sqrt{6} \pm \sqrt{4\sqrt{6}-6}}{2}$

4. $-1 \pm \sqrt{2}, 1 \pm 2i$ **5.** $1 \pm \sqrt{7}, 2 \pm \sqrt{3}$

6. $4, -2, -1 \pm i$ **7.** $1 \pm i\sqrt{2}, -1 \pm \sqrt{2}$

8. $\pm 1, -4 \pm \sqrt{6}$ **9.** $-1 \pm 2i, 1 \pm \sqrt{2}$

10. $-1 \pm i, 1 \pm \sqrt{2}$

Solution of equations by Ferrari's method

1. $2, 4, 2 \pm i\,2\sqrt{2}$ **2.** $-2 \pm \sqrt{2}, 1/2(1 \pm i)$

3. 1, 2, 3, 4

4. $3, -1, 3 \pm \sqrt{30}$ **5.** $\frac{3 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{13}}{2}$

6. $-3 \pm i\sqrt{10}, 1 \pm 2i$ **7.** $-1 \pm \sqrt{2}, 1 \pm 2i$

BLOCK II:

ALGEBRA OF MATRICES

UNIT 3: ALGEBRA OF MATRICES

CONTENTS:

3.1 *INTRODUCTION*

In this unit we investigate the matrix and algebraic operation define on them, the matrix may be viewed in rectangular form, the linear system of equation and there solution may be efficiently investigate using the properties of matrix, consider the system of equation $x + 2y = 5$, $3x + 7y = 9$ here x, y are unknowns and there coefficient are taken from any field then the arrangement of these equations in rectangular form $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ is example of matrix

3.2 *OBJECTIVES*

After reading this unit you will be able to:

- Understand matrix and their types.
- Use of operations like addition of matrices, multiplication of matrices etc.
- Find transpose of a matrix.
- Find conjugate of a matrix.
- Know about symmetric and skew symmetric matrices.
- Know about the Hermition and skew Hermition matrices.
- Understand some special type of matrices.

3.3 *MATRIX*

Definition: A rectangular representation of a set of mn numbers into m-rows and n columns is known as matrix representation of order $m \times n$ and represented as follows:

$[a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$ or $\|A\|_{m\times n}$

Usually matrix is denoted by A, B, C… etc. A matrix **A** can be represented as:

$$A = \left[a_{ij}\right]_{m\times n}$$

i ⇒ i^{th}row (i^{th} horizontal line)

j ⇒ j^{th}column (j^{th} vertical line)

The element of matrix can be taken from the any field.

If the element of matrices are be taken from real field then the matrix is known as real matrix,

A matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}_{m\times n}$$

Generally there are two types of matrices

(i) Row Matrix:

Definition: A matrix contains only one row and any number of columns is known as row matrix.

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times n}$ is row matrix of order $1\times n$.

(ii) Column Matrix:

Definition: A matrix contain only one column and any number of rows is called column matrix

$A = \begin{bmatrix} 1 \\ 2 \\ n \end{bmatrix}_{n\times 1}$ is column matrix of order $n\times 1$.

3.4 *SUB MATRIX OF THE MATRIX*

Definition: A matrix obtained by omitting zero or more rows but not all and simultaneously omitting zero or more columns but not all is known as sub matrix of a original matrix.

Example: - if A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ Then

A_1= (1 2 3) is sub matrix of A (omitting 2^{nd}row)

A_2=$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$is sub matrix of A (omitting 3^{rd}column)

A_3= (4) is sub matrix of A (omitting 1^{st} row, 2^{nd} column and 3^{rd}column)

A_4 = $\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$ is not sub matrix of A

Note: Every matrix is sub matrix of itself.

3.5 *EQUALITY OF MATRIX*

If A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{p\times q}$ then two matrix A and B are equal if

$m = p$, $n = q$ and

$a_{ij} = b_{ij}\ \forall i, j$

Example 1 A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$ and B = $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}_{2\times 2}$ are not equal because A and B have different order

A = $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 6 & 5 \end{pmatrix}$ are not equal because $a_{21} \neq b_{21}$

A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ are equal matrix

3.6 *TYPES OF MATRICES*

(i) Null Matrix:

Definition: A matrix is said to be null matrix if it's all entries are zero.

$$A = \left\{[a_{ij}]_{\mathrm{m\times n}} \mid a_{ij} = 0 \;\forall\; i,j\right\}$$

(ii) Square Matrix:

Definition: A matrix is said to be square matrix if matrix have same number of rows and columns.

$A = [a_{ij}]_{\mathrm{m\times n}}$ is square matrix if $m = n$

For example: $A_1 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ is square matrix

$A_2 = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ is not a square matrix

(iii) Identity Matrix:

Definition: A matrix is said to be identity matrix if it's all elements in principal diagonal is 1 and remaining element are 0.

i.e. $$A = \left\{[a_{ij}]_{m\times n} \middle| \begin{matrix} a_{ij} = 1 & if\ i = j \\ a_{ij} = 0 & if\ i \neq j \end{matrix}\right\}$$

(iv) Upper Triangular matrix:

Definition: A matrix $A = [a_{ij}]_{\mathrm{m\times n}}$ is said be upper triangular matrix if,

$$A = \left\{[a_{ij}]_{\mathrm{m\times n}} \mid \begin{matrix} \mathrm{a_{ij}} = 0 & \text{if } \mathrm{i} > j \\ \mathrm{a_{ij}} = 0 \text{ or may not be zero} & \mathrm{i} \leq \mathrm{j} \end{matrix}\right.$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3}$

(v) Lower Triangular Matrix:

Definition: A matrix $A = [a_{ij}]_{\mathrm{m\times n}}$ is said to be lower triangular matrix if

$$A = \left\{(a_{ij}) \mid \begin{matrix} a_{ij} = 0 & \text{if } i < j \\ a_{ij} = 0 \text{ or may not be zero if } i \geq j \end{matrix}\right.$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, where $a_{12} = a_{13} = a_{23} = 0$ and $a_{11}, a_{22}, a_{33}, a_{21}, a_{31}, a_{32}$ are any number

i.e. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is lower triangular matrix

(vi) Strict Upper Triangular Matrix:

Definition: A matrix $A = [a_{ij}]_{\mathrm{m\times n}}$ is said to be strict upper triangular matrix if

$$A = \left\{(a_{ij}) \mid \begin{matrix} a_{ij} = 0 & if\ i \geq j \\ a_{ij} = 0 \text{ or may not be zero } if\ i < j \end{matrix}\right.$$

(vii) Strict Lower Triangular Matrix:

Definition: A matrix $A = [a_{ij}]_{\mathrm{m\times n}}$ is said to be strict lower triangular matrix if

$$\mathrm{A} = \left\{(\mathrm{a_{ij}}) \mid \begin{matrix} \mathrm{a_{ij}} = 0 & \text{if } \mathrm{i} \leq \mathrm{j} \\ \mathrm{a_{ij}} = 0 \text{ or may not be zero if } \mathrm{i} > \mathrm{j} \end{matrix}\right.$$

Example: $A = \begin{bmatrix} 0 & 5 & 6 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 7 & 0 & 0 \\ 7 & 5 & 0 \end{bmatrix}$

A is strict upper triangular and *B* is strict lower triangular matrix

3.7 *ADDITION OF MATRICES*

Let A and B be two same order matrices then there sum is defined to be the matrix of same order obtained by adding the corresponding element of A and B.

i.e. if $A = [a_{ij}]_{\mathrm{m\times n}} B = [b_{ij}]_{\mathrm{m\times n}}$

Then $A+B = [c_{ij}]_{\mathrm{m\times n}}$ where $c_{ij} = a_{ij} + b_{ij}$

$$A = \begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots \dots & a_{2n} \\ a_{m1} & a_{m2} \dots \dots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \dots & b_{1n} \\ \vdots & & \\ b_{m1} & b_{m2} \dots & b_{mn} \end{bmatrix}$$

$$\text{Then } A+B = \begin{bmatrix} a_{11} + b_{11} & a_{12+}b_{12} & \cdots \\ \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots \end{bmatrix}$$

3.8 *SUBTRACTION OF TWO MATRICES*

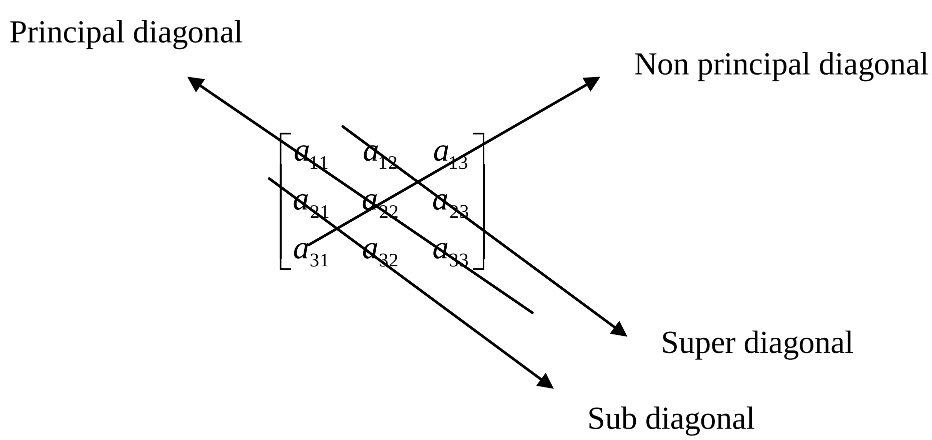
If A and B are any two same order matrix $m\times n$ then their subtraction is defined to the matrix of same order obtained by subtraction of corresponding of A and B.

i.e. if $A = [a_{ij}]_{\mathrm{m\times n}} B = [b_{ij}]_{\mathrm{m\times n}}$

Then A-$B = [c_{ij}]_{\mathrm{m\times n}}$ where $c_{ij} = a_{ij} - b_{ij}$

3.9 *PRINCIPAL DIAGONAL OF ANY MATRIX*

Definition: $A = [a_{ij}]_{\mathrm{m\times n}}$ be any matrix then the line along a_{ij} (such that $i = j$) is known as principal diagonal of square matrix.



3.10 *PROPERTIES OF MATRIX ADDITION*

(i) Matrix Addition Is Commutative: Let A and B are any two matrix of order $m\times n$

$A=\lfloor a_{ij}\rfloor_{m\times n}$ and $B=\lfloor b_{ij}\rfloor_{m\times n}$

Then $A+B = \lfloor a_{ij}\rfloor_{m\times n} + \lfloor b_{ij}\rfloor_{m\times n}$

$= \lfloor c_{ij}\rfloor_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$

$= \lfloor a_{ij}+b_{ij}\rfloor_{m\times n}$

$= \lfloor b_{ij}+a_{ij}\rfloor_{m\times n}$

$= \lfloor b_{ij}\rfloor_{m\times n} + \lfloor a_{ij}\rfloor_{m\times n}$

$= B + A$, hence

$\boxed{\mathrm{A} + B = B + A}$

(ii) Matrix Addition Is Associative: If A, B, C be three matrices of order $m\times n$

$A=\lfloor a_{ij}\rfloor_{m\times n}$, $B=\lfloor b_{ij}\rfloor_{m\times n}$, $C=\lfloor c_{ij}\rfloor_{m\times n}$

Then $(A + B) + C = \left(\lfloor a_{ij}\rfloor_{m\times n} + \lfloor b_{ij}\rfloor_{m\times n}\right) + \lfloor c_{ij}\rfloor_{m\times n}$

$= \lfloor a_{ij}+b_{ij}\rfloor_{m\times n} + \lfloor c_{ij}\rfloor_{m\times n}$

$= \lfloor (a_{ij}+b_{ij})+c_{ij}\rfloor_{m\times n}$ (by the definition of sum of two matrices)

$= \lfloor a_{ij}+(b_{ij}+c_{ij})\rfloor_{m\times n}$

$= \lfloor a_{ij}\rfloor_{m\times n} + \left(\lfloor b_{ij}+c_{ij}\rfloor_{m\times n}\right)$

$= \lfloor a_{ij}\rfloor_{m\times n} + \left(\lfloor b_{ij}\rfloor_{m\times n} + \lfloor c_{ij}\rfloor_{m\times n}\right)$

$= A + (B + C)$

$$\boxed{(A + B) + C = A + (B + C)}$$

(iii) Existence Of Additive Identity: $A = \lfloor a_{ij} \rfloor_{m\times n}$ is a matrix and $O = \lfloor b_{ij} \rfloor_{m\times n}$ such that $b_{ij} = 0\ \forall i, j$

Then $\quad$ A + O = $\lfloor a_{ij} \rfloor_{m\times n} + \lfloor b_{ij} \rfloor_{m\times n}$

$$= \lfloor a_{ij} + b_{ij} \rfloor_{m\times n} \quad = \quad \lfloor b_{ij} + a_{ij} \rfloor_{m\times n}$$

$$= \lfloor b_{ij} \rfloor_{m\times n} + \lfloor a_{ij} \rfloor_{m\times n} = O + \lfloor a_{ij} \rfloor_{m\times n} = O + A$$

$$\boxed{A + O = O + A}$$

O is additive identity of matrix A

(iv) Existence Of Additive Inverse: If $A = \lfloor a_{ij} \rfloor_{m\times n}$ and $B = \lfloor b_{ij} \rfloor_{m\times n}$ where, $b_{ij} = -a_{ij}$

Then $\quad A + B = \lfloor a_{ij} \rfloor_{m\times n} + \lfloor b_{ij} \rfloor_{m\times n}$

$$= \lfloor a_{ij} + b_{ij} \rfloor_{m\times n} = \lfloor a_{ij} + (-a_{ij}) \rfloor_{m\times n}$$

$$= \lfloor a_{ij} - a_{ij} \rfloor_{m\times n}$$

$$= [0]_{m\times n} = O$$

So $B = -A$ is additive inverse of A

3.11 *MULTIPLICATION OF A MATRIX WITH A SCALAR*

If k is a scalar and $A = \lfloor a_{ij} \rfloor_{m\times n}$ is a matrix then,

$kA = \lfloor b_{ij} \rfloor_{m\times n}$ where, $b_{ij} = ka_{ij}$

Example: $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ then $kA=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}_{2\times 3}$

3.12 *PROPERTIES OF MULTIPLICATION OF MATRIX WITH A SCALAR*

(i) If K_1 and K_2 are two scalar and A is any matrix of order $m\times n$ then,

$(K_1+K_2)A= \ K_1A+K_2A$

Proof Let $A=\lfloor a_{ij}\rfloor_{m\times n}$ then,

$(K_1+K_2)A=(K_1+K_2)\ \lfloor a_{ij}\rfloor_{m\times n}$

$$= \lfloor (K_1+K_2)a_{ij}\rfloor_{m\times n}$$

$$= \lfloor K_1(a_{ij})+K_2(a_{ij})\rfloor_{m\times n}$$

$$= \lfloor K_1a_{ij}\rfloor_{m\times n} + \lfloor K_2a_{ij}\rfloor_{m\times n}$$

$$= K_1\lfloor a_{ij}\rfloor_{m\times n} + K_2\lfloor a_{ij}\rfloor_{m\times n}$$

$$= \mathrm{K}_1A+\mathrm{K}_2A$$

(ii) If A and B are two matrices of same order then

$a(A+B)=a.A+a.B$ where a is any scalar

Proof Let A and B are two matrices of order $m\times n$

$A=\lfloor a_{ij}\rfloor_{m\times n}$, $B=\lfloor b_{ij}\rfloor_{m\times n}$ then

$a(A+B) \ = a\left(\lfloor a_{ij}\rfloor_{m\times n}+\lfloor b_{ij}\rfloor_{m\times n}\right)$

$$=a\lfloor a_{ij}+b_{ij}\rfloor_{m\times n} \text{ (by matrix addition)}$$

$$= \lfloor a(a_{ij}+b_{ij})\rfloor_{m\times n} \quad \text{(by distributive scalar}$$

multiplication)

$$= \lfloor aa_{ij} + ab_{ij} \rfloor_{m\times n}$$

$= \lfloor aa_{ij} \rfloor_{m\times n} + \lfloor ab_{ij} \rfloor_{m\times n}$ (by definition of distribution law)

$$= a\lfloor a_{ij} \rfloor_{m\times n} + a\lfloor b_{ij} \rfloor_{m\times n} = aA + aB$$

3.13 *MULTIPLICATION OF TWO MATRICES*

Two matrices can be multiplied only when the number of columns in first matrix (called pre factor) is equal to the number of rows in the second (called post factor) such matrix are said to be comfortable for multiplication.

$A = \lfloor a_{ij} \rfloor_{m\times n}$, $B = \lfloor b_{jk} \rfloor_{n\times p}$

Then $AB = [c_{ik}]_{m\times p}$

Where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots.$

$$c_{ik} = \sum_{i=1}^{n} a_{ij}b_{jk}$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

Then $AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}_{3\times 3}$$

Commutative Matrix With Respect To Multiplication: If A and B are any two matrices of same order then

If $AB = BA$ Then A and B are said to be commutative matrices.

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow AB = BA$

Anti-Commute Or Skew Commute Matrices: If A and B are any two matrices then $AB = -BA$ Then A and B are said to be anti-commutative.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB = -BA$

Important Note: Two matrix may or may not be commutative

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$AB \neq BA$

And if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow AB = BA$

Note: The product of two non-zero matrices may be zero (null) matrix.

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{2\times 2}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2}$ $A \neq 0, B \neq 0$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2}$

3.14 *PROPERTIES OF MULTIPLICATION OF TWO MATRICES*

Multiplication of matrices is distributive with respect to addition i.e. if *A*, *B* and *C* are any three matrices and confirmable for addition and multiplication then

$A(B + C) = AB + AC$

Proof: Let A, B, C are any three matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively

$A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$, $C = [c_{jk}]_{n\times p}$

Then $B + C = [b_{jk}]_{n\times p} + [c_{jk}]_{n\times p} = [b_{jk} + c_{jk}]_{n\times p}$

Then $(i.k)^{\text{th}}$ element of $A(B + C) = \sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk})$

$$= \sum_{j=1}^{n} (a_{ij}b_{jk} + a_{ij}c_{jk}) \qquad \text{(by distribution law)}$$

$$= \sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk}$$

$= (i.k)^{\text{th}}$ element of $AB + (i.k)^{\text{th}}$ element of AC

$= (i.k)^{\text{th}}$ element of $(AB + AC)$

$\Rightarrow (i.k)^{\text{th}}$ element of $A(B + C)$ and $(i.k)^{\text{th}}$ element of $(AB + AC)$ are same hence,

$$\boxed{A(B + C) = AB + AC}$$

Example: If *A* and *B* are any square matrix of order *n* Then show that

1. $(A + B)^2 = A^2 + AB + BA + B^2$

2. $(A + B)(A - B) = A^2 - AB + BA - B^2$

Solution: $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$

Then $A + B = C = [c_{ij}]_{n\times n}$

Therefore we have

1. $(A+B)^2 = (A + B)(A + B) = (A+B)A + (A+B)B$

$= A.A + B.A + A.B + B.B$ (by distribution law)

$= A^2 + BA + AB + B^2$

2. $(A + B)(A - B) = (A + B)A + (A + B)(-B)$

$= A.A + B.A + A(-B) + B(-B)$

$= A^2 + BA - AB - B^2$

Note:

- The sum of two upper triangular (lower triangular) matrix is also upper triangular (lower triangular)
- The product of two upper (lower triangular) matrix is also upper (lower triangular) matrix
- If A is upper (lower) triangular matrix and K is any positive integer than A^k is also upper (lower) triangular matrix.

3.15 *SOME SPECIAL TYPE OF MATRICES*

(i) Diagonal Matrix: A square matrix is said to be diagonal matrix if it is both upper and lower triangular matrix.

Or

A square matrix is said to be diagonal matrix if all principal diagonal are zero or may not be zero but remaining element are zero, i.e.

$$A=\left\{(a_{ij})\middle| \begin{matrix} a_{ij}=0 \text{ or may not be zero } if\ i=j \\ a_{ij}=0 \qquad\qquad\qquad if\ i\neq j \end{matrix}\right\}$$

Example: $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ is a diagonal matrix

(ii) Scalar Matrix: A square diagonal matrix is said to be scalar matrix if it's all principal diagonal elements are equaland remaining element are zero.

i.e. $A=\left\{(a_{ij})\middle| \begin{matrix} a_{ij}=K \text{ (constant) } if\ i=j \\ a_{ij=0} \qquad\qquad if\ i\neq j \end{matrix}\right\}$

Example: $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$ is a scalar matrix

Example: If A is any scalar matrix then total number of non-trivial entry of A is =….

Solution: $A=[a_{ij}]_{\text{n}\times\text{n}}$ be any scalar matrix then only one entry taken independently so total number of non-trivial entry of any scalar matrix is 1

$A=\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}_{3\times 3}$ (Here k is taken independently)

3.16 *TRACE OF MATRICES*

If $A=[a_{ij}]_{\text{n}\times\text{n}}$ is square matrix then trace of A is sum of all principal diagonal element

i.e. $$\text{Trace of } A=\sum_{\text{j}=1}^{\text{n}} a_{ii}$$

Example: $A=\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}_{3\times 3}$ Then trace A =?

Solution: Trace A = 2 + 6 + 1 = 9

3.17 *TRANSPOSE OF A MATRIX*

If $A = \left[a_{ij}\right]_{\mathrm{m\times n}}$ be any matrix then transpose of A is obtained by interchanging its rows and columns and is denoted by the symbol A^I or A^T

i.e. $A = \left[a_{ij}\right]_{\mathrm{m\times n}}$ Then $A^T = \left[b_{ij}\right]_{\mathrm{n\times m}}$ where $b_{ij} = a_{ji}$

$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ Then $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Example: If A and B any two matrices and confirmable to addition and multiplication then show that

(i) $(A + B)^T = A^T + B^T$
(ii) $(AB)^T = B^T A^T$
(iii) $(KA)^T = KA^T$
(iv) $(A^T)^T = A$

Solution:

(i) If A and B are any two matrix of order $m\times n$ then $A + B$ will be a matrix of order $m\times n$ and $(A + B)^{\mathrm{T}}$ will be a matrix of order $n\times m$

Let a_{ij} is $(i, j)^{\mathrm{th}}$ element of $(A + B)$ where $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$

Then $(j, i)^{\mathrm{th}}$ element of $(A + B)^T$

$=(i, j)^{\mathrm{th}}$ element of $(A + B) = a_{ij} + b_{ij}$

$=(i, j)^{\mathrm{th}}$ element of A + $(i, j)^{\mathrm{th}}$ element of B

$=(j, i)^{\mathrm{th}}$ element of A^T + $(j, i)^{\mathrm{th}}$ element of B^T

$\Rightarrow (j, i)^{\mathrm{th}}$ element of $(A^T + B^T)$

Thus the matrices $(A + B)^T$ and $A^T + B^T$ are of the same order and their $(j, i)^{\mathrm{th}}$ element are equal hence $(A + B)^T = A^T + B^T$

(ii) Let $A = \lfloor a_{ij} \rfloor_{m\times n}$ and K is any scalar then KA is also a matrix of order $m\times n$ consequently $(KA)^{\mathrm{T}}$will be a matrix of type $n\times m$

$=(i,j)^{\text{th}}$element of $KA = K[(i,j)^{\text{th}}$element of $A]$

$=K[(j,i)^{\text{th}}$element of $A^T] = (j,i)^{\text{th}}$ element of $K(A^T)$

$= (j,i)^{\text{th}}$element of $(KA)^T$

$\Rightarrow$ Matrix $(KA)^T$ and $KA^T$are of the same order and their $(j,i)^{\text{th}}$ elements are equal so $(KA)^T = KA^T$

(iii) Let $A = \lfloor a_{ij} \rfloor_{m\times n}$ and $B = \lfloor b_{jk} \rfloor_{n\times k}$ then A^T is a matrix of order $n\times m$ and B^T is a matrix of order $k\times n$

$A^T = \lfloor c_{ji} \rfloor_{n\times m}$ where $c_{ji} = a_{ij}$

$B^T = \lfloor d_{kj} \rfloor_{k\times n}$ where $d_{kj} = b_{jk}$

AB is a matrix of order $m\times k$

$(AB)^T$will be of order $k\times m$

Now $(k,i)^{\text{th}}$element of $(AB)^{\mathrm{T}}=(i,k)^{\text{th}}$ element of AB

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$= \sum_{j=1}^{n} c_{ji} d_{kj}$$

$$= \sum_{j=1}^{n} d_{kj} c_{ji} = (k,i)^{\text{th}} \text{ element of } A^T B^T$$

$$= (\mathrm{k},\mathrm{i})^{\text{th}}\text{element of } \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$$

$\Rightarrow$The matrices $(AB)^T$and $B^T A^T$are of the same order and their $(k,i)^{\text{th}}$ elements are same

Hence $(AB)^T = B^T A^T$

(iv) A = $\lfloor a_{ij} \rfloor_{m\times n}$ then $(A^T)^T$ will be of order $m \times k$ also $(i, j)^{\text{th}}$ element of $(A^T)^T$= $(j, i)^{\text{th}}$element of A^T = $(i, j)^{\text{th}}$element of A

Hence $(A^T)^T = A$

3.18 *CONJUGATE OF THE MATRIX*

Let $z = x + iy$ be any complex number then conjugate of Z is the mirror image of complex number Z about real axis.

Let A = $\lfloor a_{ij} \rfloor_{m\times n}$ is any matrix then conjugate of A is the matrix obtained by replacing its elements by the corresponding conjugate number it is denoted by $\bar{A}$

i.e. $\bar{A}$ = $\lfloor b_{ij} \rfloor_{n\times m}$ where $b_{ij} = \overline{a_{\iota\jmath}} \forall i, j$

Example: If $A = \begin{bmatrix} 2 & 5+6i \\ 7-3i & 3+4i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} \bar{2} & \overline{5+6\iota} \\ \overline{7-3\iota} & \overline{3+4\iota} \end{bmatrix}$

$$2 = 2 + 0i \Rightarrow \bar{2} = \overline{2+0\iota} = 2 - 0i = 2$$

$$\overline{5+6\iota} = 5 - 6i$$

$\overline{7-3\iota} = 7+3i$

$\overline{3+4\iota} = 3-4i$

$$\text{So, } \bar{A} = \begin{bmatrix} 2 & 5-6i \\ 7+3i & 3-4i \end{bmatrix}$$

Example: If A and B are any two matrices and confirmable to matrix addition and multiplication and K is any complex number then show that

(i) $\overline{(\bar{A})} = A$

(ii) $(\overline{A+B}) = \bar{A} + \bar{B}$

(iii) $(\overline{AB}) = \bar{A}\bar{B}$

(iv) $(\overline{KA}) = \bar{K}\bar{A}$

Solution:

(i) $A = \lfloor a_{ij} \rfloor_{m\times n}$ then $\bar{A} = \lfloor \overline{a_{ij}} \rfloor_{m\times n}$

A and $\overline{(\bar{A})}$ are of same order

Now $(i,j)^{\text{th}}$ element of $\overline{(\bar{A})}$ is a conjugate complex of the $(i,j)^{\text{th}}$element of $\bar{A}$

= the conjugate element of $\overline{a_{ij}}$

= $\overline{(\overline{a_{ij}})}$ = a_{ij}

= $(i,j)^{\text{th}}$ element if A

Hence $\overline{(\bar{A})}$ = A

(ii) Let A = $\lfloor a_{ij} \rfloor_{m\times n}$ and B = $\lfloor b_{ij} \rfloor_{m\times n}$ then $\bar{A}$ and $\bar{B}$ are the matrix of order $m\times n$

Now $(i,j)^{\text{th}}$element of $(\overline{A+B})$

= the conjugate element of $(i,j)^{\text{th}}$element of $(A + B)$

= the conjugate element of $(a_{ij} + b_{ij})$

=$(\overline{a_{ij} + b_{ij}})$ because $\overline{Z_1 + Z_2}$ = $\overline{Z_1}$ + $\overline{Z_2}$

= $(\overline{a_{ij}} + \overline{b_{ij}})$

= the conjugate of $(i,j)^{\text{th}}$ element of A + the conjugate of $(i,j)^{\text{th}}$ element of B

= $(i,j)^{\text{th}}$ element of $(\bar{A} + \bar{B})$

Hence $(\overline{A+B})$ = $\bar{A}$ + $\bar{B}$

(iii) Let $A = \lfloor a_{ij} \rfloor_{m\times n}$ and B = $\lfloor a_{jk} \rfloor_{n\times p}$ are two matrices

Then $\bar{A}$ and $\bar{B}$ are also matrices of order $m\times n$ and $n\times p$ respectively

⇒ $(\overline{AB})$ and $(\bar{A}\bar{B})$ are of same order

The conjugate of $(i,k)^{\text{th}}$ element of $(\overline{\text{AB}})$

$= \text{the conjugate of } \sum_{j=1}^{n} a_{ij}\, b_{jk}$

$$= \left(\overline{\sum_{j=1}^{n} a_{ij} b_{jk}}\right) = \sum_{j=1}^{n} \overline{a_{ij} b_{jk}} = \sum_{j=1}^{n} \overline{a_{ij}}\,\overline{b_{jk}}$$

= $(i,k)^{\text{th}}$element of $\bar{A}\bar{B}$

Hence $(\overline{AB})$ = $\bar{A}\,\bar{B}$

(iv) Let A = $\lfloor a_{ij} \rfloor_{m\times n}$ and let K is any complex number then $(\overline{KA})$ and $(\bar{K}\bar{B})$ will be a matrix of order $m\times n$

Now $(i,j)^{\text{th}}$element of $(\overline{KA})$

= The conjugate of $(i,j)^{\text{th}}$element of (KA) = $\left(\overline{Ka_{ij}}\right) = \overline{K}\,\overline{a_{ij}}$

= $\bar{K}$ (The conjugate of $(i,j)^{\text{th}}$element of A

Hence $(\overline{KA})$ = $(\bar{K}\bar{A})$

3.19 *TRANSPOSED CONJUGATE OF A MATRIX*

Transpose of the conjugate of a matrix A is called transposed conjugate of A i.e.

If A is any matrix the transposed conjugate matrix of A is obtained by interchanging rows and columns and taking conjugate of each element

It is denoted by A^{θ} or by A^{*}

A = $\lfloor a_{ij} \rfloor_{m\times n}$ then A^{θ} = $\lfloor b_{ji} \rfloor_{n\times m}$ where $b_{ji} = \overline{a_{ij}}$

Example: if A = $\begin{bmatrix} 2+3i & 4 & 5 \\ 0 & 6+7i & 9i \\ 2 & 5+9i & 7 \end{bmatrix}$ Then

$$A^{\theta} = \begin{bmatrix} 2-3i & 0 & 2 \\ 4 & 6-7i & 5-9i \\ 5 & -9i & 7 \end{bmatrix}$$

Example: If A and B are any two matrices confirmable to matrix addition and multiplication Then proof that

1. $(A^{\theta})^{\theta} = A$

2. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

3. $(AB)^{\theta} = B^{\theta}A^{\theta}$

4. $(KA)^{\theta} = \bar{K}A^{\theta}$

Proof:

1. $(A^{\theta})^{\theta} = \overline{((\overline{A^T})^T)} = (\overline{\overline{A^T}})^T$

$= \overline{(\bar{A})}$ $\quad \because (A^T)^T = A$

$= A$ $\quad \because \overline{(\bar{A})} = A$

2. $A+B)^{\theta} = \left(\overline{(A+B)^T}\right)$

$= \overline{(A^T + B^T)}$ $\quad \because (A+B)^{T} = A^T + B^T$

$= \overline{A^T} + \overline{B^T}$ $\quad \because (\overline{A+B}) = (\bar{A}+\bar{B})$

$= A^{\theta} + B^{\theta}$

$(A+B)^{\theta} = A^{\theta} + B^{\theta}$

3. $(AB)^{\theta} = \overline{(AB)}^{T}$

$= \left(\overline{B^T A^T}\right) \left[\because (AB)^T = B^T A^T\right]$

$= \left(\overline{B^T}\right)\left(\overline{A^T}\right)$

$= B^{\theta} A^{\theta}$

4. $(KA)^{\theta} = \left(\overline{KA}\right)^{\theta}$

$$= \left(\overline{K}(\overline{A})^T\right)$$

$$= \overline{K}A^\theta$$

$$(K\,A)^\theta = \overline{K}A^\theta$$

3.20 *SYMMETRIC MATRIX*

Definition: A square matrix A is said to be symmetric matrix if its $(i,j)^{\text{th}}$ element is the same as its $(j,i)^{\text{th}}$element

i.e. if $A = \left[a_{ij}\right]_{m\times m}$ then $a_{ij} = a_{ji}\ \forall\ i,j$

Example: $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix} = A$$

3.21 *SKEW SYMMETRIC MATRIX*

Definition: A skew matrix $A = \left[a_{ij}\right]_{m\times m}$ is said to be skew symmetric matrix if $(i,j)^{\text{th}}$element of A is the negative of $(j,i)^{\text{th}}$ element of A

i.e. if $A = \left[a_{ij}\right]$ then $a_{ij} = -a_{ji}\ \forall\ i,j$

Example: $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} = -A$$

$$\boxed{-A = A^T}$$

Example: If A and B are two square symmetric matrices of same order then what can be say about

1. $A + B$ **2.** AB

3. KA **4.** A^K where $K \geq 1$

Solution: A and B are two same order square symmetric matrices

So $A^T = A$ and $B^T = B$

1. $(A + B)^T = A^T + B^T = A + B$

$(A + B)$ is also symmetric matrix of same order

2. $(A\ B)^T = B^T A^T$

$= B\ A$

Case 1: If A and B are commutative then $A\ B = B\ A$

$\Rightarrow (AB)^T = B\ A = (A\ B)$

$\Rightarrow AB$ is symmetric

Case 2: If A and B are anti-commutative then

$AB = -BA$

$\Rightarrow (AB)^T = BA = -(AB)$

$\Rightarrow AB$ is skew symmetric

Case 3: If A and B are neither commutative nor anti-commutative then (AB) is neither symmetric nor skew symmetric

Case 4: IF one of A or B is null matrix then

$AB = 0 = BA$

$(AB)^T = BA = 0 = -0 = AB = -AB$

So (AB) is both symmetric and skew symmetric

3. $(K\ A)^T = K^T A^T$

$= K\ A^T \qquad = KA$

So KA is symmetric matrix

4. $(A^K)^T = (A^T)^K = (A)^K$ $\because A^T = A$

Where $K \geq 1$

Hence $(A)^K$ is symmetric matrix.

Example: If A and B are two symmetric matrices of order n, then show that $AB + BA$ is symmetric matrix

Solution: A and B are two symmetric matrices then $A^T = \mathrm{A}$ and $B^T = B$

Since both are of same order matrix, so multiplication and addition are confirmable

Now $(\mathrm{AB} + \mathrm{BA})^{\mathrm{T}} = (\mathrm{A\,B})^{\mathrm{T}} + (\mathrm{B\,A})^{\mathrm{T}}$

$= \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}} + \mathrm{A}^{\mathrm{T}}\mathrm{B}^{\mathrm{T}} = \mathrm{BA} + \mathrm{AB}$

$(\mathrm{AB} + \mathrm{BA})^{\mathrm{T}} = (\mathrm{AB} + \mathrm{BA})$ ($\therefore$ addition of matrices is commutative)

Hence $AB + BA$ is symmetric matrix

Example: If A and B are two skew symmetric matrices of same order then prove that (KA) and (A + B) are also skew symmetric matrices where K is any constant

Solution: A and B are two skew symmetric matrix

Then $A^T = -A$ and $B^T = -B$

Now $(K\,A)^T = K^T A^T = K\,A^T$ $= K(-A)$ $= -KA$ $= -(KA)$

Hence (KA) is skew symmetric matrix

Again $(A + B)^T = A^T + B^T$ $= -A + (-B)$ $= -(A + B)$

Hence ($A + B$) is also skew symmetric matrix

Example: If A and B are two skew symmetric matrix of same order then prove that (AB) may or may not be skew symmetric

Solution: If A and B are two skew symmetric matrix of same order $\Rightarrow A^T$ = - A and

B^T = - B Then AB is called matrix of same order

Now $(AB)^T = B^T A^T$ = (-B) (-A) = BA

Case 1: If A and B are commutative

Then $AB = -A$

So $(AB)^T = BA = AB$

Hence AB is symmetric matrix

Case 2: If A and B are anti-commutative

Then $AB = -BA$

So $(AB)^T = (BA) = -(AB)$

Hence (AB) is skew symmetric matrix

Case 3: If either $A = 0$ or $B = 0$

Then $AB = 0 = BA$

So $(AB)^T = (BA) = 0 = -(AB)$

In this case AB both symmetric and skew symmetric matrix.

Hence the product of two skew symmetric matrix need not be skew symmetric.

3.22 *HERMITIAN MATRICES*

A square matrix A = $\lfloor a_{ij} \rfloor_{m\times m}$ is said to be Hermitian matrix if its $(i, j)^{\text{th}}$ element is the conjugate of its $(j, i)^{\text{th}}$ element

i.e. $A = \lfloor a_{ij} \rfloor_{m\times m}$ then A is Hermitian of $A^{\theta} = A$

Example: $A = \begin{bmatrix} 2 & 2+3i & 3+4i \\ 2-3i & 3 & 4+5i \\ 3-4i & 4-5i & 4 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 2 & 2-3i & 3-4i \\ 2+3i & 3 & 4-5i \\ 3+4i & 4+5i & 4 \end{bmatrix}$

$$\overline{(A^T)} = \begin{bmatrix} 2 & 2+3i & 3+4i \\ 2-3i & 3 & 4+5i \\ 3-4i & 4-5i & 4 \end{bmatrix} = A$$

$A^\theta = A$ hence A is Hermitian matrix

Example: Prove that the principal diagonal elements of Hermitian matrices are real.

Solution: Let A be any square matrix, $A = \lfloor a_{ij} \rfloor_{m\times m}$

Let $a_{ii} = x_i + iy_i$ are principal diagonal element of A

A is Hermitian matrix so $A^\theta = \overline{(A^T)}$

a_{ii} is principal diagonal element of A

$\bar{a}_{ii}$ is principal element of A^θ

But $A^\theta = A$

$\bar{a}_{ii} = a_{ii}$

$x_i - iy_i = x_i + iy_i$

$2iy_i = 0$

$y_i = 0 \quad \forall i$

Hence the principal diagonal elements of Hermitian matrix are real.

Example: If A is any Hermitian matrix then what can be say about (KA) where K is any complex constant.

Solution: A is any Hermitian matrix

$\Rightarrow A^{\theta} = A$

Now $(KA)^{\theta} = \overline{(KA)^{T}} = \overline{KA^{T}} = \overline{K}A^{\theta} = \overline{K}A$

Case 1: K is any real number

Then $\overline{K} = K$

And $(KA)^{\theta} = KA$

So (KA) is Hermitian matrix

Case 2: If K is any complex number whose real and imaginary part both are non-zero

$K = (a + ib)$ say

$\overline{K} = a - ib$

$(KA)^{\theta} = \overline{K}A = (a - ib) \neq (KA)$

In this case (KA) is not Hermitian matrix.

Case 3: If K is purely imaginary then $(KA)^{\theta} = -KA$

Hence in this case KA is Skew Hermitian matrix.

3.23 *SKEW HERMITIAN MATRIX*

Definition: A square matrix A = $\lfloor a_{ij} \rfloor_{m\times m}$ is said to be skew hermitian matrix if its $(i, j)^{\text{th}}$ element is negative of conjugate if its $(j, i)^{\text{th}}$ element

i.e. if $A = \lfloor a_{ij} \rfloor_{m\times m}$

Then $a_{ij} = -\ \overline{a_{ji}}$

Example: $A = \begin{bmatrix} 0 & a+ib & c+id \\ -a+ib & 0 & e+if \\ -c+id & -e+if & 0 \end{bmatrix}_{3x3}$ where a, b, c, d, e, f all are real constant

Then $A^T = \begin{bmatrix} 0 & -a+ib & -c+id \\ a+ib & 0 & -e+if \\ c+id & e+if & 0 \end{bmatrix}$

$$\overline{A^T} = \begin{bmatrix} 0 & -a-ib & -c-id \\ a+ib & 0 & -e-if \\ c-id & e-if & 0 \end{bmatrix}$$

$$A^\theta = - \begin{bmatrix} 0 & a+ib & c+id \\ -a+ib & 0 & e+if \\ -c+id & -e+if & 0 \end{bmatrix} = -A$$

Hence A is skew Hermitian matrix.

Example: If A and B are skew Hermitian matrices of same order then prove that (A + B) is also skew Hermitian matrix

Solution: A and B any two square skew Hermitian matrix of same order so (A + B) is confirmable to addition and

$A^\theta = -A$, $B^\theta = -B$

Now $(A+B)^\theta = \overline{(A+B)^T} = \overline{A^T + B^T} = \overline{A^T} + \overline{B^T}$

$= A^\theta + B^\theta = (-A) + (-B) = -(A+B)$

Hence (A + B) is skew Hermitian matrix.

Example: If A is any skew Hermitian matrix and K is any complex constant then what can be says about KA?

Solution:A is any skew Hermitian matrix

So $A^\theta = -A$

Now $(KA)^\theta = \overline{(KA)^T} = \overline{KA^T} = \overline{K}\,\overline{A^T} = \overline{K}A^\theta = -\overline{K}A$

Case 1: If K is real constant

Then $\overline{K} = K$

And $(KA)^\theta = -\overline{K}A = -KA$

So KA is Skew Hermitian matrix.

Case 2: If K is purely imaginary

Then $\bar{K}$ = -K

And $(KA)^{\theta}$ = - $\bar{K}$ A = $-(-K)A$ = KA

So (KA) is Hermitian matrix.

Case 3: If K = $a + ib$ where a, b both are non-zero real number

Then $\bar{K}$ = (a - ib)

And $(KA)^{\theta} = -\bar{K}\ A$ = = $-(a - ib)A$ $\neq KA$ or $\neq$ -KA

So in this case (KA) is neither Hermitian nor skew Hermitian matrix

Case 4: If K = 0 or A is null matrix

Then $(\mathrm{KA})^{\theta} = -\bar{\mathrm{K}}\mathrm{A}$

$= 0 = \mathrm{KA} = -0 = -\mathrm{KA}$

So in this case (KA) is hermitian and skew hermitian both

3.24 *ORTHOGONAL MATRIX*

A square matrix `A´ is said to be orthogonal matrix if AA^T = I = A^TA

Note: If A is orthogonal matrix then the sum of square of each row's or column's element is equal to 1 and the product of corresponding different row's or column's is equal to zero.

Example: Show that the matrix $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal matrix

Solution: Let A = $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now $AA^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: If A and B are any two orthogonal matrix of same size then prove that (AB) is also orthogonal matrix

Solution: A and B are two orthogonal matrix

$\Rightarrow \quad AA^T = I = A^T A$

And $\quad BB^T = I = B^T B$

Since A and B are of same order so AB is confirmable.

Now $(AB)(AB)^T = AB(B^T A^T)$

$= A(\mathrm{BB^T})\mathrm{A^T}$

$= AI\,A^T$

$= AA^T = I$

$(AB)^T(AB) = (B^T A^T)\,(AB)$

$= B^T\,(A^T A)\,B$

$= B^T I\,B$

$= B^T B$

$= I$

We have $(AB)\,(AB)^T = I = (AB)^T(AB)$

Hence (AB) is orthogonal matrix.

3.25 *UNITARY MATRIX*

Definition: A square matrix A is said to be unitary matrix if $AA^\theta = I = A^\theta A$

Example: $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$$A^T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \qquad \overline{A^T} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$$

$$AA^\theta = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^\theta A = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^\theta = I = A^\theta A$$

$\Rightarrow A$ is unitary matrix

Example: If A is unitary matrix then for what value of K, (KA) is also unitary matrix.

Solution: A is unitary matrix so $AA^\theta = A^\theta A = I$

Let (KA) is also unitary matrix so $(KA)\,(KA)^\theta = (KA)^\theta(KA) = I$

$(KA)\,(KA)^\theta = (KA)\left(\bar{K}A^\theta\right) = K\,\bar{K}\,A\,A^\theta = I$

$K\,\bar{K}\,I = I$

$K\,\bar{K} = I$

So K is unit modulus

Hence (KA) is also unitary matrix if K is of unit modulus.

3.26 *IDEMPOTENT MATRIX*

Definition: A matrix A is said to be idempotent matrix if $A^2 = A$

Example: Consider a matrix $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 16 & -12 & -4 & +3 \\ 48 & -36 - 12 & & +9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$$

Hence A is idempotent matrix.

Example: If A and B are two square matrices show that $AB = A$ and $BA = B$ Then show that both A and B are idempotent matrices.

Solution: Since $AB = A$……….. (1)

Post multiply by A in both side in equation (1)

$$AB = A$$

$$(AB)\, A = A.A$$

$$A\,(BA) = A.A$$

$AB = A^2$ $\qquad\because\ BA = B$

$$\boxed{A = A^2} \qquad \because\ AB = A$$

Again BA = B

Post multiple by B in both side

$$(BA)B = B.B$$

$$B\,(AB) = A^2$$

$$BA = B^2$$

$\Rightarrow \boxed{B = B^2}$ Hence A and B both are idempotent

Example: If A and B are two idempotent matrices of same order then prove that if $(A + B)$ is idempotent then AB and BA both are null matrices.

Solution: Let A and B are two idempotent matrix

$\Rightarrow \quad A^2 = A$ and $\quad B^2 = B$

Let $(A + B)$ is also idempotent matrix

$\Rightarrow (A + B)^2 = (A + B)$

$(A + B)\,(A + B) = (A + B)$

$A.A + A.B + B.A + B.B = A + B$

$A^2 + AB + BA + B^2 = A + B$

$A + AB + BA + B = A + B \qquad \because A^2 = A, B^2 = B$

$AB + BA = O$……………………………… (1)

Pre multiply both sides by A in (1)

Then $A.(AB) + A(BA) = A.0 = O$

$A^2B + ABA = O \qquad \because A^2 = A$

$AB + ABA = O$…………………………….. (2)

Again post multiply both sides by A in (1)

Then $ABA + BA.A = 0.A = O$

$ABA + BA^2 = O \qquad \because A^2 = A$

$ABA + BA = O$……………………………….. (3)

From (2) and (3)

$AB = BA$ put in (1) then $\boxed{AB = BA = O}$

3.27 *INVOLUTORY MATRIX*

Definition: A square matrix A is said to be involutory matrix if $A^2 = I$

Example: Consider a matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then show that A is involutory.

Solution: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$A^2 = I$

Hence *A* is involutory matrix.

Example: If *A* and *B* are two involutory matrix of same order then prove that if (*A* + *B*) is involutory then *AB* + *BA* = -*I*

Solution: Let *A* and *B* are two involutory matrix ⇒ $A^2 = I = B^2$

Again let (*A* + *B*) is involutory matrix

⇒ $(A + B)^2 = I$

$(A + B)(A + B) = I$

$A.A + A.B + B.A + B.B = I$

$A^2 + AB + BA + B^2 = I$ $\quad \because A^2 = I = B^2$

$I + AB + BA + I = I$

$AB + BA = I - 2I$

$AB + BA = -I$

3.28 *NILPOTENT MATRIX*

Definition: A square matrix *A* is said to be nilpotent matrix if there exist a positive integer *n* such that A^n= O (O is null matrix). The smallest positive integer m such that $A^m = 0$ then *m* is called index of null matrix *A*.

Example: Show that the matrix $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}$ is nilpotent matrix with index 3.

Solution: A = $\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Then $A^2 = A.A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence A is nilpotent matrix

Since least positive integer is 3 such that $A^3 = 0$, so index A is 3.

Example: If A and B are two nilpotent matrix of same order then show that $A + B$ and AB may or may not be nilpotent

Solution:

(1) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

A and B both are nilpotent matrix.

$$A^2 = O = B^2$$

Now $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

$(A + B)^2 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(A + B)$ is nilpotent

$$\text{AB} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(AB) is nilpotent

(2) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}$

Clearly A and B are nilpotent because

$$A^2 = B^2 = \text{null matrix (O)}$$

But $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$ is not nilpotent matrix

Because no positive integer m exist such that $(A + B)^{\mathrm{m}} = \mathrm{O}$

(3) Let $A = \begin{bmatrix} 0 & 0 \\ 15 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 16 \\ 0 & 0 \end{bmatrix}$ A and B both are nilpotent.

But $(A.B) = \begin{bmatrix} 0 & 0 \\ 15 & 0 \end{bmatrix} \begin{bmatrix} 0 & 16 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 240 \end{bmatrix}$ is not nilpotent matrix.

Example: If A and B are any two nilpotent matrices of same order and commute to each other, then show that (A +B) is also nilpotent matrix.

Solution: A and B are any two nilpotent matrices and

Let index of $A = m_1$, index of $B = m_2$

$\Rightarrow$ $\mathrm{A}^{\mathrm{m_1}} = \mathrm{O} = \mathrm{B}^{\mathrm{m_1}}$

$\Rightarrow$ $\mathrm{A}^{\mathrm{m_1+k_1}} = \mathrm{O} = \mathrm{A}^{\mathrm{m_2+k_2}}$ Where $\mathrm{k_1}$ and $\mathrm{k_2}$ are positive integer

Now $(\mathrm{A} + \mathrm{B})^{\mathrm{m_1+m_2-1}}$

$$= \sum_{\mathrm{r}=0}^{\mathrm{m_1+m_2-1}} {}^{\mathrm{m_1+m_2-1}}{}_{\mathrm{r}}\mathrm{C}\, \mathrm{A}^{\mathrm{r}} \mathrm{B}^{(\mathrm{m_1+m_2-1})-\mathrm{r}} \ldots\ldots\ldots\ldots\ldots (1)$$

Case 1: If $\mathrm{r} > \mathrm{m_1}$ then $\mathrm{A}^{\mathrm{r}} = \mathrm{O}$

Then $(\mathrm{A} + \mathrm{B})^{\mathrm{m_1+m_2-1}} = \mathrm{O}$

Hence (A + B) is nilpotent matrix

Case 2: If $r < m_1$ Then

$$\begin{aligned} (m_1 + m_2 - 1) - r &> (m_1 + m_2 - 1) - m_1 && (\because r < m_1) \\ &> m_2 - 1 \\ &\geq m_2 && (m_1, m_2 \in I) \end{aligned}$$

The greater number of $m_2 - 1$ is either m_2 or $> m_2$

and $B^{m_2} = O$

so, $B^{(m_1+m_2-1)-r} = O$

Hence (A + B) is nilpotent matrix

The index of (A + B) is less than or equal to $m_1 + m_2 - 1$.

3.29 SUMMARY

In this unit we learned the concept of algebra of matrix,along with some important matrices that will further help us understand the matrix in all its forms, like orthogonal matrix, idempotent matrix , involutory matrix, nilpotent matrix etc.

3.30 GLOSSARY

1. **Non trivial entries:** The entries of matrix over any field which we can take independently

2. **Trivial entries:** The entries of matrix over any field which can not be taken independently

3. **Trace:** Sum of all principal diagonal entries

3.31 *SELF ASSESMENT QUESTIONS*

1.31.1 Multiple Choice Questions:

1. $A = \begin{bmatrix} p & 2 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 7 & 5 \end{bmatrix}$ then for what values of P, A and B are equal matrix

(a) 5 (b) 2

(c) 7 (d) None of these

2. If A is any matrix of order $m \times n$ and all entry of matrix A is equal, then total number of sub matrix of A is equal to

(a) n (b) $m.n$

(c) $m + n$ (d) m

3. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Then

(a) $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. The number of possible different element in a square matrix of order n in which $a_{pq} = a_{rs}$ where, $p + q = r + s$ is

(a) n (b) $2n$

(c) $2n - 1$ (d) 2

5. Total number non trivial entry in upper triangular matrix of order n is

(a) n^2 (b) $\frac{n(n+1)}{2}$

(c) $\frac{n(n-1)}{2}$ (d) n

6. A and B are two involuntary matrix, $(A + B)$ is also involuntary if

(a) $AB + BA = O$ (b) $AB + BA = I$

(c) $AB + BA = -I$ (d) $AB + BA = 2I$

7. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ then trace (AA^T) is

(a) 10 (b) 5

(c) 3 (d) 2

Answers:

1. (d) **2.** (b) **3.** (d) **4.** (c)

5. (c) **6.** (c) **7.** (a)

1.31.2 Fill In The Blanks:

Fill in the blanks ‘’………’’ so that the following statements are complete and correct

1. *A* square matrix is said to be idempotent if $A^2 =$ ………

2. Trace (*A* + *B*) = ………..

3. The diagonal element of Hermitian matrix is ……….

4. If *A* and *B* are two nilpotent matrix and confirmable to multiplication then (*AB*) is also nilpotent if *AB* = ……

5. The sum of square of each rows of orthogonal matrix is ………..

6. *A* square matrix is skew symmetric if $A^T =$ ………

7. If *A* and *B* are two symmetric matrices then $(K_2AB + K_2BA)$ is also symmetric if ……….

Answers:

1. *A* **2.** Trace *A* + Trace *B* **3.** Real **4.** *BA*

5. **1** **6.** $-A$ **7.** $K_1 = K_2$

3.32 REFERENCE

1. Linear Algebra, Vivek.sahai & Vikas Bist :Narosa publishing House

2. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

3. Schaum's out line (Linear Algebra)

3.33 SUGGESTED READINGS

1. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

2. Schaum's out line (Linear Algebra)

3.34 TERMINAL QUESTIONS

1.34.1 Short answer type questions:

1. Show that the matrix $\begin{bmatrix} 0 & 5 & -9 \\ -5 & 0 & 7 \\ 9 & -7 & 0 \end{bmatrix}$ is skew symmetric

2. Show that the matrix $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is orthogonal

3. If A and B are symmetric matrices of order n, Then show that $AB + BA$ is symmetric and $AB - BA$ is skew symmetric.

4. Express the following matrix as the sum of a symmetric and a skew symmetric matrix $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

5. If A is a square symmetric of order n, then show that trace of AA^T is equal to sum of square of each element of A

6. If A is an idempotent matrix, show that $B = I–A$ is also idempotent matrix and $AB = BA = O$ (Null Matrix)

7. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ satisfies the equation $A^3 - 14A^2 = O$

1.34.2 Long answer type questions:

1. If $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then show that $A_{\theta_1} A_{\theta_2} = A_{\theta_1 + \theta_2}$

2. Prove that the product of two matrices $\begin{bmatrix} \cos^2 \propto & \cos \propto \sin \propto \\ \cos \propto \sin \propto & \sin^2 \propto \end{bmatrix}$ and $\begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$ Is a zero matrix when α and β differ by an odd multiple of $\pi/2$

3. If $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ then find the values of A^3

4. If A and B are two nilpotent matrices then show that if (AB) is nilpotent then index of (AB) ≤ (index A, index B)

5. Show that every square matrix is uniquely expressively as the sum of two matrices one is hermitian and other is skew hermitian

6. If $A = \begin{bmatrix} 5 & 2-3i & 7+6i \\ 2+3i & 10 & i \\ 7-6i & -i & 15 \end{bmatrix}$ is hermitian matrix

7. $A = \begin{bmatrix} 0 & 5 & 6 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & h \end{bmatrix}$ then find the value of a, b, c, e and h

8. Show that principal diagonal element of skew symmetric matrix is zero

9. Show that every square matrix is uniquely expressible as the sum of symmetric and skew symmetric matrix.

10. Prove that the principal diagonal element of skew Hermitian matrix are purely imaginary or zero.

11. If *A* is skew hermitian matrix then what values of *n* show that A^n is also skew hermitian matrix

UNIT 4: DETERMINANTS

CONTENTS:

4.1 *INTRODUCTION*

In this unit we show that how to find the determinant of the matrix, we emphasize that an $n \times n$ array of scalars enclosed by straight lines called determinant of order n, the determinant function was first discovered during the investigation of system of linear (Homogeneous and Non Homogeneous) Equation.

We solved the determinant of matrix of order $1, 2, 3 \ldots$ and then we define a determinant of general $n \times n$ matrix.

4.2 *OBJECTIVE*

After reading this unit you will be able to:

- Understand minors and cofactors.
- Find determinant value of a square matrix.
- Understand properties of determinant and its uses.
- Find product of the two determinant and its uses.
- Know about singular and nonsingular matrices.
- Find solution system of non-homogeneous linear using Cramer's Rule.
- Find Adjoint of a square matrix.

4.3 *DETERMINANT*

Definition: Each *n*-square matrix is assigned a special scalar is called determinant of A, and it is denoted by $|A|$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} \ldots & a_{nn} \end{vmatrix}$$

4.3.1 DETERMINANTS OF ORDERS 1

If $A = [a_{11}]_{1\times 1}$ then $|A| = a_{11}$

4.3.2 DETERMINANTS OF ORDERS 2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| =$ (Product of principal diagonal element) – (Product of non-principal diagonal element)

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

Example: - if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then, |A|= (2.5) – (4.3)

= 10 – 12

= -2

4.3.3 DETERMINANTS OF ORDERS 3

Let $A = \lfloor a_{ij} \rfloor_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $|A| = a_{11}.a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

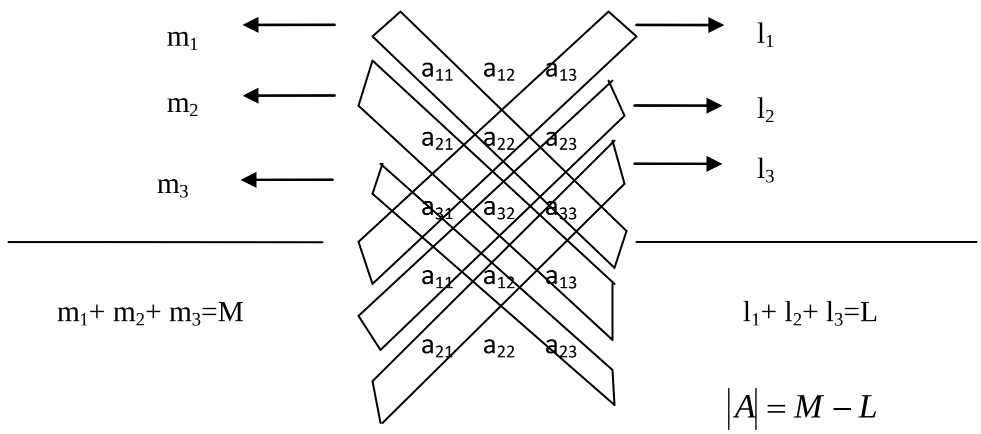
Or

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{a33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Or

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then arrange these number in rows and columns and first two rows again write in last



4.4 *MINORS AND COFACTORS*

Consider the determinant of 3×3 matrix (in general)

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then if we leave the column and the row passing through the elementa_{ij}, then the second order determinant is called minor of the element a_{ij} and it is denoted by M_{ij}

For example: The minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

The minor of the element $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$

Cofactors: The minor M_{ij} multiplied by $(-1)^{i+j}$ is called cofactor of the element a_{ij} and it is denoted by A_{ij}

$$A_{ij} = (-1)^{i+j} . M_{ij}$$

For example: - The cofactor of the element $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

The cofactor of the element $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

4.5 DEFINITION OF DETERMINANTS IN TERMS OF COFACTOR

Let A be any n-row's square matrix then the determinants of A is the sum of the product of the element of any column or any row with their corresponding cofactor

i.e. $|A| = \sum_{i=1\,or\,j=1}^{n} a_{ij} A_{ij}$ where, either *i* or *j* is fixed

Example:

1. If $\mathrm{i} = 1$ then

$$|\mathrm{A}| = \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{a_{ij}}\,\mathrm{A_{ij}} = \mathrm{a_{11}A_{11}} + \mathrm{a_{12}A_{12}} + \mathrm{a_{13}A_{13}}$$

2. Write the cofactors and minors of each element of the matrix $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: The matrix of the element $\mathrm{a_{11}} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 6 - 3 = 3 = \mathrm{M_{11}}$

The matrix of the element $\mathrm{a_{12}} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 + 6 = 9 = \mathrm{M_{12}}$

The matrix of the element $\mathrm{a_{13}} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5 = \mathrm{M_{13}}$

The matrix of the element $\mathrm{a_{21}} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3 + 1 = 4 = \mathrm{M_{21}}$

The matrix of the element $\mathrm{a_{22}} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1 = \mathrm{M_{22}}$

The matrix of the element $a_{23} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 = M_{23}$

The matrix of the element $a_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 = M_{31}$

The matrix of the element $a_{32} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4 = M_{32}$

The matrix of the element $a_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 = M_{33}$

The cofactor of the element $a_{11} = (-1)^{1+1} m_{11} = 3 = A_{11}$

The cofactor of the element $a_{12} = (-1)^{1+2} m_{12} = -9 = A_{12}$

The cofactor of the element $a_{13} = (-1)^{1+3} m_{13} = -5 = A_{13}$

The cofactor of the element $a_{21} = (-1)^{2+1} m_{21} = -4 = A_{21}$

The cofactor of the element $a_{22} = (-1)^{2+2} m_{22} = 1 = A_{22}$

The cofactor of the element $a_{23} = (-1)^{2+3} m_{23} = 3 = A_{23}$

The cofactor of the element $a_{31} = (-1)^{3+1} m_{31} = -5 = A_{31}$

The cofactor of the element $a_{32} = (-1)^{3+2} m_{32} = 4 = A_{32}$

The cofactor of the element $a_{33} = (-1)^{3+3} m_{33} = 1 = A_{33}$

4.6 PROPERTIES OF DETERMINANTS

Theorem 1: The value of determinant does not change when rows and columns are interchange

Proof: Let A be any square matrix of order *n*

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}_{n*n}$$

Then

$$|A| = \sum_{i=1}^{n} \sum_{j=1} a_{ij}\, A_{ij} \text{ where } A_{ij} \text{ is cofactor of } a_{ij}$$

Let us take a matrix of order 3 for example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Hence the determinants of any matrix A and its transpose matrix $A^T$are equal.

Theorem 2: If any two columns or rows of a determinant are interchanged then the values of determinant is negative multiple of determinant of original matrix.

Proof: - Consider a matrix A of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then } |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \dots\dots..(1)$$

Now interchanging any two rows or columns

$$R_1 \leftrightarrow R_2$$

Then new matrix $A_1 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A|$ interchange

$$= a_{21}(a_{12}a_{33} - a_{32}a_{13}) - a_{22}(a_{11}a_{33} - a_{31}a_{13}) + a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

$$= -[a_{32}a_{13}a_{21} - a_{21}a_{33}a_{12} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}]$$
$$- [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})]$$

So, $|A_1| = -|A|$

Note:

1. If any row or column in any matrix is multiplied by any scalar K then determinant of the matrix is K times of the determinants of the original matrix

For example: $\begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

2. If all the elements of matrix multiplied by constant K then determinant is equal to K^n time of the value of determinant of original matrix, where n is order of matrix.

i.e. $|KA| = K^n|A|$

3. If any two rows or columns are identical of any matrix then determinant is zero.

Theorem 3: If in a determinant each element in any row or column consists of the sum of two terms, then determinant can be written as sum of two determinants

Proof: Let $A = \begin{bmatrix} a_{11} + a & a_{12} & a_{13} \\ a_{21} + b & a_{22} & a_{23} \\ a_{31} + c & a_{32} & a_{33} \end{bmatrix}$

Expanding the determinant along the first column

$$|A| = (a_{11} + a) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{21} + b) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (a_{31} + c) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + c \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix}$$

Theorem 4: If the element of any row or column added by K time the corresponding element of any other row or column, then determinants of the matrix are same

Proof: -Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} + Ka_{12} & a_{12} & a_{13} \\ a_{21} + Ka_{22} & a_{22} & a_{23} \\ a_{31} + Ka_{32} & a_{32} & a_{33} \end{bmatrix}$

Then $B = \begin{vmatrix} a_{11} + Ka_{12} & a_{12} & a_{13} \\ a_{21} + Ka_{22} & a_{22} & a_{23} \\ a_{31} + Ka_{32} & a_{32} & a_{33} \end{vmatrix} =$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} Ka_{12} & a_{12} & a_{13} \\ Ka_{22} & a_{22} & a_{23} \\ Ka_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$= |A| + K \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$= |A| + KO$ [$\therefore$ If any two columns are identical then determinant will be zero]

$= |A|$

Example: If $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ then show that

$$|A| = (a-b)(b-c)(c-a)$$

Solution: $|A| = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Applying $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ then we get

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

Expanding the determinant along the first column

$$|A| = 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - a\begin{vmatrix} 0 & b^2-a^2 \\ 0 & c^2-a^2 \end{vmatrix} + a^2\begin{vmatrix} 0 & b-a \\ 0 & c-a \end{vmatrix}$$

$$= (b-a)(c^2-a^2)-(b^2-a^2)(c-a)-0+0$$

$$= (b-a)(c-a)(c+a)-(b-a)(b+a)(c-a)$$

$$= (b-a)(c-a)\{(c+a)-(b+a)\}$$

$= (b-a)(c-a)(c-b)$

$= (a-b)(b-c)(c-a)$

4.7 VANDERMODE MATRIX

A matrix is at form

$$A = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_n & \alpha_n^2 & \alpha_n^{n-1} \end{bmatrix}$$ is called vandermode matrix

And its determinant

$$|A| = \prod_{1<i<j\le n} (\alpha_j - \alpha_i)$$

Example 4: Find the determinant of

$$\Delta n = \begin{vmatrix} 0 & 0 & \cdots & a \\ 0 & 0 & \cdots & a0 \\ \cdots & \cdots & \iddots & \cdots \\ a & 0 & 0 & 0 \end{vmatrix}$$

Solution: $\Delta_n = (-1)^{n+1} a \, \Delta_{n-1}$

$$= (-1)^{n+1} a (-1)^{(n+1)-1} \Delta_{n-2}$$

$$= (-1)^{n+1} a (-1)^{(n+1)-1} a (-1)^{(n+1)-2} \Delta_{n-3}$$

$$\Delta_n = (-1)^{(n+1)+n+(n-1)+\,---4} a^{n-2} \Delta_2$$

Where $\Delta_2 = (-1)^3 a^2$

Then $\Delta_n = (-1)^{\left[\frac{(n+1))(n+2)}{2} - 1\right]}$

Example 5: Let A be a square matrix of order n, then show that

1. $|\bar{A}| = \overline{|A|}$ **2.** $|A^{\theta}| = \overline{|A|}$

Solution 1. let $A = [a_{ij}]_{n\times n}$ $A = [a_{ij}]_{n*n}$ then $\bar{A} = [\overline{a_{ij}}]_{n\times n}$

So $|\bar{A}| = |\overline{a_{ij}}| = \overline{|a_{ij}|} = \overline{|A|}$

2. A be a square matrix of order n, and $A^{\theta} = \overline{A^T}$

So $|A^{\theta}| = |\overline{A^T}| = |\bar{A}^T| = |\bar{A}| = \overline{|A|}$

$\because$ $|A^T| = |A|$ and $|\bar{A}| = \overline{|A|}$

Example 6: Show that the determinant of Hermitian matrix always a real number

Solution: Let A be a Hermitian matrix

Then $A^{\theta} = A$

$$|A^{\theta}| = |A|$$

$$\left|\overline{A^T}\right| = |A|$$

$$\overline{|A|} = |A|$$

Let $x + iy$ is the determinant of A

$$|A| = x + iy$$

$$\overline{|A|} = x - iy$$

But $\overline{|A|} = |A|$

$$x - iy = x + iy$$

$$2iy = 0$$

$$y = 0$$

$$|A| = x + i0 = x$$

Example 7: Show that the determinant of Skew symmetric matrix of odd order is zero.

Solution: Let A be a skew symmetric of odd order

$$A^T = -A$$

$$|A^T| = |-A| = |(-1)A| = (-1)^n|A| \qquad \because |KA| = K^n|A|$$

$$|A| = (-1)^n|A| \qquad \because |A^T| = |A|$$

Since n is odd so $(-1)^n = -1$

Now $\quad |A| = -|A|$

$$2|A| = 0$$

$$|A| = 0$$

4.8 PRODUCT OF THE TWO DETERMINANT OF THE SAME ORDER

Example 8: If A and B are two square matrices of same order then prove that

$$|AB| = |A||B|$$

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$A.B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

Now we know that

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\qquad |B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$

$$|A||B| = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{vmatrix}$$

$$\text{Hence } |AB| = |A||B|$$

Rule: Let A and B are only two matrices of same order

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\qquad$ and $\quad |B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$

then

$$|A||B| = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{vmatrix}$$

In general this is simply row by column multiplication

Example 9: If A be a square matrix of order n then show that

$$|A^K| = |A|^K$$

Solution: Let A and B are two square matrices of order n

Then we know that $|A.B| = |A||B|$

If we replace B with A then

$$|A.A| = |A||A|$$

$|A^2| = |A|^2$

In similar way $|A^K| = |A|^K$

Example 10: Show that the determinant of an idempotent matrix is either 0 or 1

Solution: Let A is an idempotent matrix, then

$A^2 = A$

$|A^2| = |A|$

$|A|^2 = |A|$

$$|A|(|A| - 1) = 0$$

$|A| = 0$ or $|A| - 1 = 0$

$|A| = 0$ or $|A| = 1$

Note: It is necessary condition the determinant of idempotent matrix is 0 or 1 but not sufficient.

For Example: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then $|A| = 0$ but $A^2 \neq A$

Example 11: Show that the determinant of orthogonal matrix is either 1 or -1

Solution: Let A is an orthogonal matrix, then

$A\,A^T = I$

$$|A\,A^T| = |I|$$

$$|A||A^T| = 1$$

$$|A||A| = 1 \qquad \because\ |A^T| = |A|$$

$$|A|^2 = 1 \qquad |I| = 1$$

$$|A| = 1 \text{ or } -1$$

Note: Determinant of a diagonal matrix, upper triangular matrix, lower triangular matrix is the product of principal diagonal elements.

Example 12: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ Then $|A| = ?$

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ is upper triangular matrix so its determinant values are the product of principal diagonal matrix

Hence $|A| = 1 \cdot 5 \cdot 9 = 45$

Example 13: Show that the value of determinant of skew Hermitian matrix of order *n*, is either 0 (zero) or purely imaginary if *n* is odd and real, if *n* is even.

Solution: Let A be a skew Hermitian matrix and

$$|A| = x + iy$$

$$A^\theta = -A \qquad \text{(By definition of skew Hermitian matrix)}$$

$$\left|A^\theta\right| = |-A|$$

$$\left|\overline{A^T}\right| = (-1)^n|A|$$

$$\overline{|A|} = (-1)^n|A|$$

Case 1: If *n* is even then,

$$\overline{|A|} = |A| \quad \Rightarrow \quad x - iy = x + iy \quad \Rightarrow \quad y = 0 \quad \text{so} \quad |A| = x$$

$\Rightarrow \ |A|$is real

Case 2: If *n* is odd then,

$|\overline{A}| = |A| \Rightarrow x - iy = -(x + iy) \Rightarrow 2x = 0 \Rightarrow x = 0$

$$\overline{|A|} = |A| \quad \Rightarrow \quad x - iy = -(x + iy) \quad \Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0$$

$|A| = iy$ If $y = 0$ then $|A| = 0$

If $y \neq 0$ then $|A|$ is purely imaginary.

4.9 NON SINGULAR MATRIX AND SINGULAR MATRIX

Non- Singular Matrix: A matrix '*A*' is said to be non-singular matrix if its determinant is non zero.

Singular Matrix: A matrix '*A*' is said to be singular matrix if its determinant is zero.

4.10 LINEAR EQUATION (HOMOGENEOUS AND NON-HOMOGENOUS EQUATION)

Linear homogenous equation: The equation is of the form $ax + by + cz = 0$ is called linear homogenous equation in x, y, z.

Linear non-homogenous equation: The equation is of the form $ax + by + cz = B$ where $B \neq 0$ is called non-homogenous equation in x, y, z.

4.11 SYSTEM OF NON-HOMOGENOUS LINEAR EQUATION (CRAMER'S RULE)

If we have n linear simultaneous equation in n variable$x_1, x_2, x_3 \cdots\cdots x_n$

i.e. $a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

$$\text{Let} \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

Suppose A_{ij} is the cofactor of element a_{ij} in Δ then multiplying this given equation by $A_{11}, A_{21}, A_{31} \cdots\cdots A_{n1}$ and adding

$$x_1(a_{11}A_{11} + a_{21}A_{21} + \cdots a_{n1}A_{n1}) + x_2(a_{12}A_{11} + a_{22}A_{21} + \cdots a_{n2}a_{n1}) + \cdots + x_n(a_{1n}A_{11} + a_{2n}A_{21} + \cdots A_{nn}a_{nn})$$

$$x_1\Delta + x_2(0) + x_3(0) + \cdots = b_1A_{11} + b_2A_{21} + \cdots A_nA_{n1}$$

$x_1\Delta = \Delta_1$ where, Δ_1 is the determinant obtained by replacing first column element of Δ by $b_1, b_2 \cdots b_n$ then $x_1 = \frac{\Delta_1}{\Delta}$

Again multiplying these equations by $A_{12}, A_{22}, \cdots A_{n2}$ and adding then we get

$$x_2\Delta = \Delta_2 \quad \Rightarrow \quad x_2 = \frac{\Delta_2}{\Delta}$$

Where Δ_2 is determinant obtained by replacing second column element of Δ by $b_1, b_2 \cdots b_n$

In similar way, we get

$$x_3\Delta = \Delta_3 \quad \Rightarrow \quad x_3 = \frac{\Delta_3}{\Delta}$$

$\cdots \quad \cdots$

$$x_n\Delta = \Delta_n \quad \Rightarrow \quad x_n = \frac{\Delta_n}{\Delta}$$

This method of solving n simulations linear non-homogenous equation provided $|A| \neq 0$ where A is the coefficient matrix. This method is known as Cramer's rule

Example 14: Solve the following system at equation by Cramer's rule

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

Solution: The coefficient matrix of given system at non-homogenous linear equation is

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$\Delta = |A| = 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2(1+1) + 3(-1-1) \quad = 4 - 6 \quad = -2 \quad \neq 0$$

Therefore the system of non-homogenous linear has unique solution

Now using Cramer's rule

$$\Delta_1 = \begin{bmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = -2, \qquad \Delta_2 = \begin{bmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix} = -4, \qquad \Delta_3 = \begin{bmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{bmatrix} = -6$$

Hence the solution is

$$x = \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1, \qquad y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2, \qquad z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3$$

4.12 ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n*n}$ be a square matrix of order *n* then the transpose of a matrix $B=[A_{ij}]_{n*n}$ where A_{ij} is the cofactor of the element a_{ij} called Adjoint of matrix A and it is denoted by Adj*A* or adj*A*.

If $$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

Then the cofactor matrix $C = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix}_{n\times n}$

Then adj$A$= transpose of the matrix C

adj$A$= $C = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix}_{n\times n}$

Example 17: Find the adjoint of the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: Let us find the cofactor A_{11} , A_{12} , A_{13} etc at the element of $|\mathrm{A}|$ we have

$A_{11} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$, $\quad$ $\mathrm{A}_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$, $\mathrm{A}_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$,

$$\mathrm{A}_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3 \,, \mathrm{A}_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \,, \mathrm{A}_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1 \,,$$

$$\mathrm{A}_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \,, \mathrm{A}_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \,, \mathrm{A}_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \,,$$

Therefore the matrix C formed at the cofactor of the element of $|\mathrm{A}|$ is

$$C = \begin{vmatrix} -5 & 1 & 3 \\ 3 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now adj$A$ is the transpose of the matrix C.

adj$A$= 3

Example 18: Prove that at $\mathrm{x} = 4$ the values of given determinant

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Solution: We have $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} \quad \text{by}$$

$R_2 \to -\frac{1}{2}R_2, R_3 \to -\frac{1}{6}R_3$

Solving the determinant along the first row then we get

$$(x-2)(30-24) - (2x-3)(10-6) + (3x-4)(4-3)$$

$$6(x-2) - 4(2x-3) + 1.(3x-4)$$

Put $x = 4$ then the value of determinant

= 6(4-2) - 4(8-3) + (3.4 - 4)

= 6.2 - 4(5) + 8

=12 - 20 + 8

=0

Example 19: If $\begin{vmatrix} t_1 & t_1^2 & 1+t_1^3 \\ t_2 & t_2^2 & 1+t_2^3 \\ t_3 & t_3^2 & 1+t_3^3 \end{vmatrix} = 0$ then prove that

$$t_1.t_2.t_3 = -1 \text{ where } t_1 \neq t_2 \neq t_3$$

Solution: We have $\begin{vmatrix} t_1 & t_1^2 & 1+t_1^3 \\ t_2 & t_2^2 & 1+t_2^3 \\ t_3 & t_3^2 & 1+t_3^3 \end{vmatrix} = \begin{vmatrix} t_1 & t_1^2 & 1 \\ t_2 & t_2^2 & 1 \\ t_3 & t_3^2 & 1 \end{vmatrix} + \begin{vmatrix} t_1 & t_1^2 & t_1^3 \\ t_2 & t_2^2 & t_2^3 \\ t_3 & t_3^2 & t_3^3 \end{vmatrix}$

(By theorem (properties of determinants))

$$=\begin{vmatrix} t_1 & t_1^2 & 1 \\ t_2 & t_2^2 & 1 \\ t_3 & t_3^2 & 1 \end{vmatrix} + t_1t_2t_3 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

(By taking t_1, t_2, t_3 common from first row, second row and third row of the second determinant)

$$=\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} + t_1t_2t_3 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_2 & t_3^2 \end{vmatrix}$$

(By $C_1 \leftrightarrow C_3$ then $C_3 \leftrightarrow C_2$ of the first determinant so determinant is unchanged)

$$= (1 + t_1.t_2.t_3) \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_2 & t_3^2 \end{vmatrix}$$

By vandermode matrix the value of above determinant is

$$= (1 + t_1.t_2.t_3)(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \quad \text{but} \quad t_1 \neq t_2 \neq t_3$$

So $(t_1 - t_2).(t_2 - t_3)(t_3 - t_1) \neq 0$

So if $\begin{vmatrix} t_1 & t_1^2 & 1 + t_1^3 \\ t_2 & t_2^2 & 1 + t_2^3 \\ t_3 & t_3^2 & 1 + t_3^3 \end{vmatrix} = 0$ then $(1 + t_1.t_2.t_3)$ must be zero.

Hence $t_1.t_2.t_3 = -1$

Example 20: Prove that if $x \neq y \neq z$ and $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = 0$ then $yz + zx + xy = 0$

Solution: We have $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = 0$

Multiplying by x, y, z in first, second and third column of the determinant from left side respectively then we get

$$\frac{1}{x.y.z}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix} = 0$$

Taking xyz common from 3^{rd} row at the above determinant

$\frac{x.y.z}{x.y.z}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = 0$ $R_1 \leftrightarrow R_3$ after that $R_2 \leftrightarrow R_3$

Then determinant is $(-1)^2$ time the original determinant.

$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = 0$ $\quad (C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$$\begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix} = 0$$

Expanding along first row

$$= \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ (y-x)(y^2+xy+x^2) & (z-x)(z^2+zx+x^2) \end{vmatrix} = 0$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix} = 0$$

Taking $(y-x)$ and $(z-x)$ is common from first and second column

$$= (y-x)(z-x)\{(y+x)(z^2+zx+x^2) - (y^2+xy+x^2)(z+x)\} = 0$$

$$=(y-x)(z-x)(z-y)(yz^2+xz^2-zy^2-xy^2)=0$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx) = 0$$

But $x-y \neq 0, y-z \neq 0, z-x \neq 0$ because x, y, z all are distinct, so $(xy+yz+zx) = 0$.

Example 21: Solve the following system of linear equation by Cramer's rule

$$2x - y + 3z = 8$$

$$-x + 2y + z = 4$$

$$3x + y - 4z = 0$$

Solution: We have $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2(-8-1) + 1(+4-3) + 3(-1-6)$

$$\Delta = -18 + 1 + (-21)\Delta = -38$$

Thus $\Delta \neq 0$ and therefore the system has a unique solution given by

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \qquad \text{i.e.}$$

$$\frac{x}{\begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix}} = -\frac{1}{38}$$

$$\text{Or given by } \frac{x}{-76} = \frac{y}{-76} = \frac{z}{-76} = \frac{1}{-38}$$

Hence $x = 2,\ y = 2,\ z = 2$

4.13 USEFUL METHOD FOR FINDING THE VALUE OF DETERMINANT OF ORDER 4 OR MORE

Let A be any non zero square matrix of order n $A = \lfloor a_{ij} \rfloor_{n \times n}$ with $n > 1$

Step 1: Choose an element in A such that $a_{ij} = 1$ or if nonexistent, $a_{ij} \neq 0$

Step 2: Using a_{ij} as a swivel, apply elementary row or column operations to put 0's in all the other positions in the column or row containing a_{ij}

i.e. if we apply row operation then to put 0 in all the other position in the column and similar for column operation

Step 3: Expand the determinant by the column or row (according to our selection of operation) containing a_{ij}

Example 22: Find the determinant of a matrix A of order 4×4

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$$

Solution: $|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$

Step 1: Choose an element a_{23} because $a_{23} = 1$

Step 2: Apply row operation and put 0's in all the other positions in third column

Apply $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 + R_2$

$$|A| = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Step 3: Now expanding the determinant by the third column

$$|A| = (-1)^{2+3} \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= -(4 - 18 + 5 - 30 - 3 + 4) \quad = 38$$

4.14 DETERMINANTS AND VOLUME:

Let A is any square matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Let $t_1 = (a_{11}, a_{12}, \cdots\ a_{1n})$

$$t_2 = (a_{21}, a_{22}, \cdots\ a_{2n})$$

$$\cdots \qquad\qquad \cdots$$

$$t_n = (a_{n1}, a_{n2}, \cdots\ a_{nn})$$

Then the determinant are related to the notions of area and volume

Let U be the parallelepiped determined by

$$U = \{a_1 t_1 + a_2 t_2 + \cdots a_n t_n : 0 \leq a_i \leq 1 \ \forall\, i = 1, 2, \ldots n\}$$

When $n = 2$ then U is parallelogram

Let V denote the volume of U then

V = Absolute volume of determinant of A

Example 23: Let $t_1 = (1,1,1) t_2 = (1,1,0) t_3 = (0,2,3)$

Then find the volume of the parallelepiped in three dimension space

Solution: $t_1 = (1,1,1) t_2 = (1,1,0) t_3 = (0,2,3)$

So the volume is the absolute volume of

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 1(3-0) - 1(3-0) + 1(2-0)$$

$$= 3 - 3 + 2 \quad = 2$$

Hence volume $V = |2| = 2$

Example 24: Find the value of $|A|$ where

$A = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$ Where w is the cube root of unity.

Solution: Cube root of unity in complex number system is solution of the equation$z^3 - 1 = 0$, then the values of z satisfied the above equation is called cube root of unity.

Now $z^3 - 1 = 0$, $z^3 = 1 \because \cos 0 + i \sin 0 = 1$

$$z^3 = \cos 0 + i \sin 0$$

$\because$ cos and sin are periodic function

So $\cos(0) = \cos(0 + 2k\pi)$ and $\sin(0) = \sin(0 + 2k\pi)$

So $z^3 = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi) = \cos 2k\pi + i \sin 2k\pi$

$$z = \left(\cos(2k\pi) + i(\sin 2k\pi)\right)^{1/3} = \cos\frac{2k\pi}{3} + i \sin\frac{2k\pi}{3}$$

$[\cos\theta + i \sin\theta = e^{i\theta}$, $(\cos\theta + i\sin\theta)^n = e^{in\theta}$ *or* $e^{in\theta} = \cos n\theta + i \sin n\theta]$

Put $k = 0$ then $z = 1$

$k = 1$ then $z = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3} = \omega$

$k = 2$ then $z = \cos\dfrac{(2\pi)^2}{3} + i\sin\dfrac{(2\pi)^2}{3} = \omega^2$

Hence cube root of unity are ω and ω^2.

And also $1 + \omega + \omega^2 = 0$

Now the given determinant applying $c_1 \to c_1 + c_2 + c_3$

Then we get

$$|A| = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \text{ or } |A| = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix}$$

$$|A| = 0$$

Example 25: Evaluate $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Solution: Let us denote the given determinant by $\det(A)$

$$\det(A) = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

Applying row transformation by using $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$

Then we get

$$\det(A) = \begin{vmatrix} -a^2 & ab & ac \\ ab - a^2 & -b^2 + ab & bc + ac \\ ac - a^2 & bc + ab & -c^2 + ac \end{vmatrix}$$

Taking a, b, c are common from first, second, third columns respectively

Then we get

$$\det(A) = abc \begin{vmatrix} -a & a & a \\ b - a & -b + a & b + a \\ c - a & c + a & -c + a \end{vmatrix}$$

Now applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

Then $$\det(A) = a.b.c \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Taking common a, b, c from first, second and third row respectively

Then $$\det(A) = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Expanding along first row

$$\det(A) = 4a^2b^2c^2$$

4.15 SUMMARY

In this unit we learned to find the value of determinant of any matrix, which will help us in solving the linear equation, it will also be helpful to understand the concept of eigen value and rank of matrix.

4.16 GLOSSARY

1. **Identical row or column:** Any two row or column are same

2. **Parallelepiped:** A solid body of which each face is a parallelogram,

3. **Absolute:** Free from imperfection.

4.17 *SELF ASSESMENT QUESTIONS*

2.17.1 Multiple choice question:

1. The values of det A, where $A = \begin{vmatrix} l & m & n & p \\ o & t & u & q \\ o & o & v & r \\ o & o & o & s \end{vmatrix}$

(a) l.m (b) l.t

(c) l.m.n.p (d) l.t.v.s

2. The value of t show that $\begin{vmatrix} t-4 & 3 \\ 2 & t-9 \end{vmatrix} = 0$

(a) 3, 10 (b) 5, 7

(c) 8, 9 (d) 1, 2

3. If $A = (a_{ij})_{6*6}$ such that $a_{ii} = 1$ and $a_{ij} = 2$ if $i + j = 7$ otherwise zero then det of A is

(a) 3 (b) 9

(c) -27 (d) 27

4. Determinant of Nilpotent matrix will be

(a) A prime number (b) Multiple of 2

(c) Always 1 (d) None

5. Determinant of Skew symmetric matrix of order 3 is

(a) 3 (b) 5

(c) 1 (d) 0

6. If A is any non singular square matrix f order 3, then determinant of $\mathrm{ad_j(A)}$ is

(a) $|\mathrm{A}|$ (b) $|\mathrm{A}|^2$

(c) $|\mathrm{A}|^3$ (d) None

7. If $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ then cofactor of element $\mathrm{a_{13}}$ is

(a) $|\mathrm{A}|$ (b) $|\mathrm{A}|^2$

(c) $|\mathrm{A}|^3$ (d) None

8. If A is any Square matrix of order n and determinant of $\mathrm{A^T}$ is

(a) 1 (b) 0

(c) 4 (d) None

9. If $\mathrm{A} = \begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$ then determinant of A is

(a) abc (b) $\mathrm{a(b-c)}$

(c) $\mathrm{abc(a-b)(b-c)}$ (d)$\mathrm{abc(a-b)(b-c)(c-a)}$

10. If $\begin{vmatrix} u & -6 & -1 \\ 2 & -3u & u-3 \\ -3 & 2u & u+2 \end{vmatrix} = 0$ then the value of U

(a) $1, 2, -3$ (b) $3, 7, 8$

(c) $5, 6, -1$ (d) $-1, -2, 3$

ANSWERS:

1. (d)	2. (a)	3. (c)	4. (d)	5. (d)
6. (b)	7. (b)	8. (a)	9. (d)	10. (a)

4.17.2 Fill in the blanks:

Fill in the blanks '…….' So that the following statements are complete and correct

1. A is square matrix of order n and $\left|A^{\theta}\right| = \cdots \ldots$

2. The value of determinant ………… When rows and columns are interchanged

3. $\begin{vmatrix} y+z & x & y \\ z=x & z & x \\ x+y & y & z \end{vmatrix} \neq 0$ then $x+y+z$ is………and x, z are…………

4. If A and B be two Square matrix of same order then $|A.B| = \cdots \ldots$

5. Determinant of hermition matrix is always…………

6. If $4x - 3y = 15$ and $2x + 5y = 1$ then $x = \cdots \ldots$ and $y \ldots \ldots$

7. A is idempotent matrix of order n and its determinant is ……..or…….

8. If A is non zero and $|A| = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$ is also……

ANSWERS:

1. $\overline{|A|}$	**2.** Does not change	**3.** Non zero, distinct	**4.** $|A|.|B|$
5. Real	**6.** 3, 1	**7.** 1, 0	**8.** Non zero

4.18 REFERENCES

1. Linear Algebra, Vivek.sahai & Vikas Bist :Narosa publishing House

2. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

3. Schaum's out line (Linear Algebra)

4.19 SUGGESTED READINGS

1. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

2. Schaum's out line (Linear Algebra)

4.20 *TERMINAL QUESTIONS*

4.20.1 Short answer type questions:

1. Show that if $A = \begin{vmatrix} u^3 & 3u^2 & 3u & 1 \\ u^2 & u^2+2u & 2u+1 & 1 \\ u & 2u+1 & u+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$ then determinant of A is $(u-1)^6$

2. Evaluate $\begin{vmatrix} 1 & yz & x(y+z) \\ 1 & za & y(z+a) \\ 1 & xy & z(x+y) \end{vmatrix}$

3. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ y+z & t+x & t+x & x+y \\ t & x & y & z \end{vmatrix} = 0$

4. Show that $|adjA| = |A|^{n-1}$, where n is a order of matrix A

5. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 7 & 8 & 0 & 0 \\ 9 & 8 & 7 & 5 \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 77 & 99 & 55 \\ 10 & 20 & 125 \\ 87 & 119 & 180 \end{vmatrix}$

7. If $D_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ and $D_2 = \begin{vmatrix} -x & a & -p \\ y & -b & q \\ z & -c & r \end{vmatrix}$ then show that $D_1 = D_2$

ANWERS:2. 0

4.20.2Long answer type questions:

1. Show that at least one real number x, show that $\det A$ is zero where

$$A = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}$$

2. Solve the following system of linear equation by Cramer's rule

$$2x - y + 3z = 8$$

$$-2x + 2y + z = 4$$

$$3x + y - 4z = 0$$

3. Solve the following system of linear equation by Cramer's rule

$$x + y + z = 9$$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

4. Solve the following system of linear equation by Cramer's rule

$$x + y + 4z = 6$$

$$3x + 2y - 2z = 9$$

$$5x + y + 2z = 13$$

5. Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$

6. Show that the determinant of any matrix A, whose first row is the sum of other row is zero.

7. Show that $\begin{vmatrix} 4 & 5 & 6 & a \\ 5 & 6 & 7 & b \\ 6 & 7 & 8 & c \\ a & b & c & 0 \end{vmatrix} = (a - 2b + c)^2$

8. Prove that $\begin{vmatrix} 1+ \ x & 1 & 1 & 1 \\ 1 & 1+\ y & z & t \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1+u \end{vmatrix} = xyzu\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u}\right)$

ANSWERS:

2. $x = 2, y = 2, z = 2$

3. $x = 1, y = 3, z = 5$

4. $x = 2, y = 2, z = {}^{1}/_{2}$

5. $\begin{bmatrix} 7 & -11 & -5 \\ 8 & -14 & -5 \\ -6 & -13 & -5 \end{bmatrix}$

UNIT 5: APPLICATION OF MATRICES

CONTENTS:

5.2 INTRODUCTION

System of linear Equations (Homogeneous and Non Homogeneous) plays a very important role in subject of mathematics. Many problems in mathematics reduce to finding the solution of linear Equation, all our system of linear Equation involve scalar may come from the number system

In this unit we will focus on the solution of system of linear equation, but the rank of matrix plays an important role to solve system of linear equation. So first of all we will discuss Rank of matrix.

5.1 OBJECTIVES

After reading this unit you will be able to

- Understand rank of a matrix.
- Use elementary transformation of matrices.
- Transform of matrix in Echelon form.
- Reduce a matrix in normal form.
- Understand linear dependence and linear independence of vectors.
- Solve homogenous linear equations.
- Solve non homogeneous linear equation.

5.3 RANK OF A MATRIX

If A is any arbitrary matrix of order $m\times n$ then the determinant of Square sub matrix of the matrix is known as a minor of the matrix A, if the Sub matrix of order k is taken then its determinant is known as k-rowed minor of a matrix of order $m\times n$, and the number of different option we have

$$m_{C_{m-k}}.n_{C_{n-k}}$$

And rank of A is non-negative integer r if there exist at least one non singular sub matrix of the given matrix of order r and all the $(r + 1)$ rowed minor are Zero "0".

i.e. the rank of matrix is the order of highest order non singular Sub square matrix of the given matrix, and Rank of A is denoted by $\rho(A)$

Note: (1) A is any matrix of order $m\times n$ then the Rank of A is less than or equal to minimum of m and n i.e. if $A=\left(a_{ij}\right)_{m\times n}$ then $\rho(A)\le \min\{m,n\}$

(2) Rank of every non singular matrix of order n is equal to n, because every matrix is itself a sub matrix of given matrix that is

If $A=\left(a_{ij}\right)_{m\times n}$ and $\det(A)\ne 0$ then $\rho(A)=n$

(3) If A is Singular matrix then Rank of A is less than order of matrix A

i.e., if $A = (a_{ij})_{n\times n}$ and $\det(A) = 0$ then $\rho(A) < n$

(4) If A is Null matrix then Rank of A is Zero.

Example 1: Find the Rank of matrix $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Highest possible order of Sub matrix is 3 and total number of sub matrix of order 3 is $3_{C_3}.3_{C_3} = 1$. So it is A itself

Now $|A| = 2(2-1) - 2(4-0) + 3(2-0)$

$$= 2 - 8 + 6$$

$$= 0$$

So Rank of A is not equal to 3

Now again total number of sub matrices of order 2 is $3_{C_2}.3_{C_2} = 9$

But we have a sub matrix of order 2 whose determinant is non zero $\Rightarrow \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$

So Rank of $A = 2$

Example 2: Find the Rank of matrix $A = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \\ 1 & \omega & \omega^2 \end{vmatrix}$ where ω is cube root of unity

Solution: $A = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \\ 1 & \omega & \omega^2 \end{vmatrix}$

We have $|A| = 1(\omega^2 - \omega^3) - \omega(\omega^3 - \omega^2) + \omega^2(\omega^2 - 1)$

$= \omega^2 - \omega^3 - \omega^4 + \omega^3 + \omega^4 - \omega^2$

$= 0$ $[\because \omega^3 = 1, \omega^4 = \omega]$

But there is at least one minor of order 2 of the matrix A namely $\begin{vmatrix} 1 & \omega \\ \omega & 1 \end{vmatrix}$ which is not equal to zero

Hence Rank of A is 2

5.4 ELEMENTARY TRANSFORMATION

The transformation over a matrix which does not affect rank of a matrix is called elementary transformation over the matrix. There are 3 elementary row transformation and corresponding 3 elementary column transformation.

If we denote i^{th} row by R_i and i^{th} column by C_i

Then elementary row transformations are

(1) $R_i \leftrightarrow R_j$ (The interchanging of any two rows)

(2) $R_i \to R_i + kR_j$ where k is non zero constant

(3) $R_i \to kR_i$ where k is non zero constant

And elementary columns transformations are

(1) $C_i \leftrightarrow C_j$

(2) $C_i \to C_i + kC_j$

(3) $C_i \to kC_i$ where k is any non zero constant

Elementary Matrix: A matrix obtained by performing a single elementary transformation over identity matrix is called elementary matrix.

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ interchange second and third row we get an Identity matrix

So A is an example of elementary matrix.

5.5 ECHELON FORM OF A MATRIX

If A is any matrix then by applying elementary row transformation over the matrix in such a way

(1) All zero row of the matrix are at the bottom

(2) Leading non- zero element in each row is 1

(3) Numbers of zero before first non-zero element in successive row is more

Some authors do not require (2) condition

Note: (1) The Rank of matrix is equal to the Rank of matrix in Echelon form.

(2) Total number of non-zero rows and total number of non-zero columns decide the Rank of matrix in Echelon form

$\rho(A) = \min\{$ non zero rows, non zero columns$\}$

Example 3: Find the Rank of matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 3 & 5 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 3 & 5 \end{bmatrix}$

Applying $R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 5R_1$

$$A \sim \begin{vmatrix} 1 & 2 & 3 \\ 0 & -7 & -10 \\ 0 & -7 & -10 \end{vmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & 10 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{by} \quad R_3 \to R_3 - R_2$$

Now E-transformation do not change the Rank of matrix and we see that 2×2 sub matrix $\begin{bmatrix} 0 & 2 \\ 0 & -7 \end{bmatrix}$ is non singular, therefore the rank of this Echelon matrix is 2, hence rank of A is 2 .

In other words total number of non zero rows in Echelon matrix is 2, and total number of non zero columns is 3.

$$\text{So Rank of A} = \text{minimum}\{2, 3\} = 2$$

Example 4: Find the Rank of matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

Solution: Given matrix

Performing the elementary operations

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Again performing the operation $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in Echelon form and number of Non Zero rows in this matrix is 3 and number of Non Zero columns in this matrix is 4.

So $\quad \text{Rank A} = \min\{4, 3\} = 3$

Hence $\quad \text{Rank A} = 3$

Row Rank: If we apply elementary row transformation only on A, where A is a matrix of order $n \times n$,then maximum number of Non zero row r which can be always obtain or number of linearly independent row is called row rank of the matrix, and number of dependent row $(\mathrm{n} - \mathrm{r})$ in the matrix A is called row Nullity, and sum of row rank and row nullity is equal to total number of rows in given matrix.

Column Rank: Number of independent columns in matrix is known as column rank of the matrix and total number of dependent column are known as column Nullity of matrix and sum of column rank and column nullity is equal to total number of column in the matrix.

Example 5: Find the possible values of row rank, column rank, row nullity and column nullity of the matrix

$$A = \begin{bmatrix} 1 & 5 & 7 & 6 & 8 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 7 & 5 & 9 \end{bmatrix}$$

Solution: The given matrix is already Echelon form, hence

Row rank is 3

So row nullity is equal to $3 - 3 = 0$

Since there are 3 linearly independent columns so column rank is 3 and column nullity is $5 - 3 = 2$

Example 6: Find the row rank, column nullity, column rank and row nullity of the matrix

$$A = \begin{bmatrix} 1 & 5 & 11 & 6 \\ 3 & 7 & 2 & 1 \\ 5 & 17 & 24 & 13 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 5 & 11 & 6 \\ 3 & 7 & 2 & 1 \\ 5 & 17 & 24 & 13 \end{bmatrix}$

Since $R_3 = R_2 + 2R_1$

So R_3 is linearly dependent row and other rows are linearly independent,

Hence row rank is $3 - 1 = 2$

And row nullity is $3 - 2 = 1$

Similarly column rank is 2

And column nullity is $4 - 2 = 2$

5.6. REDUCTION TO NORMAL FORM

Every matrix of order $m \times n$ of rank r can be reduced to normal form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ by finite elementary transformation, where I_r is the unit matrix of order r

Proof: Let $A = (a_{ij})_{m\times n}$ matrix with rank rthen two cases are possible

Case 1. If A is Null matrix then A is of the form $A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 \end{pmatrix}_{m\times n}$ so

Rank of A is 0

Then we have nothing to prove.

Case 2. If A is not Null matrix so at least one element in A is non-zero.

$$\Rightarrow \text{say} a_{ij} \neq 0$$

$$\Rightarrow \mathrm{i^{th} row is nonzero}$$

Interchanging the i^{th} row with first row and j^{th} column with first column.

Then we obtain another matrix B, whose leading element is non zero

let $a_{ij} = a \quad (a \neq 0)$

Then multiplying by $\frac{1}{a}$ in first row of matrix B.

$$\frac{1}{a} b_{ij} = [1, \frac{b_{12}}{a}, \frac{b_{13}}{a}, \ldots, \frac{b_{1n}}{a}]$$

Let C be any matrix whose first row is $\frac{1}{a} b_1$ so

$$C = \begin{bmatrix} 1 & C_{12} & C_{13} & \ldots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \ldots & C_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \because \frac{b_{ij}}{a} = C_{ij}$$

Now applying row and column transformation by suitable multipliers and subtracting by suitable rows and column such that which all element of first row and first column is zero except the leading element so new matrix D is of the form

$$D = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & & \ldots & \\ 0 & & & & \\ \vdots & \ldots & & T & \\ 0 & & & & \end{bmatrix}$$

Where T is a matrix of order $(m-1) \times (n-1)$

If T is Non-zero matrix then again apply the same process and if T is null matrix then we get a required result.

If rank of $T = r$, then continuing this process we shall finally obtain a matrix is of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Hence every matrix of order $m \times n$ of rank r can be reduced into Normal form

Example 7: If $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ find the rank of matrix and reduce to normal form

Solution: $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

Again $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now using Column transformation

$C_2 \to C_2 + C_1$ and $C_3 \to C_3 - 2C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now $C_2 \to \dfrac{C_2}{3}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Hence Rank of A is 2.

Theorem 1: The rank of a product of two matrices cannot exceed the rank of matrix.

Proof: let A and B be two matrix of order $m \times n$ and $n \times p$ respectively

Let r_1 and r_2 be the rank of A and B respectively and let r be the Rank of $(A.B)$

Then to prove $r \leq r_1 and r_2$

Since $A = (a_{ij})_{m \times n}$ and Rank (A) = r_1

So $A \sim \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}$

$B = (b_{ij})_{n \times p}$ and Rank (B) = r_2 so $B \sim \begin{bmatrix} I_{r_2} & 0 \\ 0 & 0 \end{bmatrix}$

Let $AB = (C_{ij})_{m \times p}$

$$AB = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{r_2} & 0 \\ 0 & 0 \end{bmatrix}$$

Since matrix $\begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix}$ has only r_1 non-zero rows and $p - r_2$ zero column

So AB cannot have more than r_1 non-zero row

So $\text{Rank}AB \leq r_1$

So $\text{Rank}(AB) \leq \text{Rank of the prefactor}A$

Again since $\text{Rank}(AB) = \text{Rank}(AB)^T = \text{Rank}(B^T A^T)$

Rank (AB) $\leq$ Rank of B^T = Rank of B i.e. Rank (AB) $\leq r_2$

$$\text{Rank}(AB) \leq \text{RankofB}^T = \text{RankofB}$$

Hence Rank (AB) $\leq r_1$ and Rank (AB) $\leq r_2$

5.7 VECTOR SPACE OF N-TUPLES

Any ordered *n*-tuple of number is called vector of *n*-tuples. For example $x_1, x_2, \dots x_n$ be any *n*-number then the ordered *n*-tuple $Y = (x_1, x_2, \dots x_n)$ is called a vector of *n* tuples.

(1) Equality of two Vectors: Let $S = (s_1, s_2, ..., s_n)$ and $T = (t_1, t_2, ..., t_m)$ are said to be equal if and only if $m = n$ and $s_i = t_i\ \forall\ i$

For example if $S = \{1, 2, 3\} T = \{2, 3, 4\}$ then S and T are not equal

$X = (1, 2, 4)$ and $Y = (1, 2, 4)$ then X and Y are equal vector.

(2) Addition and Subtraction of two vectors: If $S = \{s_1, s_2, \dots s_n\}$ and $T = \{t_1, t_2, \dots t_n\}$

Then $S \pm T = (s_1 \pm t_1, s_2 \pm t_2, ..., s_n \pm t_n)$

For example $S = (1, 2, 3)$ and $T = (2, 3, 4)$

Then $S + T = (1+2, 2+3, 3+4) = (3, 5, 7)$

(3) Multiplication of a Vector by any Scalar: If K be any scalar and $S = (s_1, s_2, ..., s_n)$ be

n-vector then $KS = (Ks_1, Ks_2, ..., Ks_n)$

(4) Linear dependence and linear independence of vectors:

(i) Linearly dependent set of vectors:

Definition: Let $X = \{x_1, x_2, \ldots x_n\}$be any set of vector and $c_1, c_2, \ldots c_n$are any scalar and if $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ thenall$c_i$are non zero then we can say X is linearly dependent set of vector.

For example: -

Let $X = \{(1,2,3), (1,2,4), (2,4,7)\}$

Then $c_1(1,2,3) + c_2(1,2,4) + c_3(2,4,7) = 0$

Where $c_1 = 1, c_2 = 1, c_3 = -1$

Since all $c_i$are non zero so X is the set of linearly dependent vectors.

(ii) Linearly independent set of vectors:

Definition: Let $X = \{x_1, x_2, \ldots x_n\}$ be the set of n vector is said to be linearly independent

If $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ where $c_i$are Scalar then $c_1 = c_2 = \cdots c_n = 0$

For example:

$$X = \{(1,0,0), (0,1,0), (0,0,1)\}$$

Let $c_1,\ c_2,\ c_3$ are Scalar and $c_1x_1 + c_2x_2 + c_3x_3 = 0$

$$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$$

$$c_1 = 0,\ c_2 = 0,\qquad c_3 = 0$$

Hence X is linearly independent set of vectors.

Example 8: Show that the set containing a zero vector is linearly dependent.

Solution: Let $S = \{x_1, x_2, ..., x_n\}$ be an *n*-vector whose term is zero.

Let $c_1, c_2, \ldots, c_n$ are any Scalar and $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$ for any value of c_i

So c_i is not necessarily zero

Hence a set containing zero vector is linearly dependent.

5.8 SYSTEM OF LINEAR EQUATION

There are two types of linear equations. One is Homogeneous linear equation and other is Non Homogeneous linear Equation, we shall first discuss the Homogeneous system of equation and its solution.

(1) Homogeneous linear equation: A linear equation in unknowns $x_1, x_2, \ldots x_n$ is an equation that can be put in the standard form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

A system of linear Homogeneous Equation is a list of linear equation with same unknown. Put into standard form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1)$$

The coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_2 & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$$

And the vector of unknowns is $X = [x_1, x_2, \ldots x_n]^T$ and $O = [0, 0, 0, \ldots 0]^T$

Where A, X, O are $m \times n, n \times 1, m \times 1$ matrices respectively. So the general form of system of Homogeneous equation is

$$A\,X = O$$

Since if $X = (0, 0, 0, \dots 0)$ then $x_1 = 0 = x_2 = x_3 = \dots = x_n$ are the solution of Equation (1) and it is called trivial solution.

If $x \neq 0$ and x_1, x_2 are two solutions of equation (1) then their linear combination

$c_1 x_1 + c_2 x_2$ is also solution of equation (1) where c_1 and c_1 are any arbitrary constant

We have $AX = 0$

Put $X = c_1 x_1 + c_2 x_2$ then

$$A(c_1 x_1 + c_2 x_2) = A\,c_1 x_1 + A\,c_2 x_2$$

$$= c_1 A\,x_1 + c_2 A\,x_2 \because \quad A\,x_1 = 0, \; A\,x_2 = 0$$

$$= c_1 . 0 + c_2 . 0$$

$$= 0$$

Hence $c_1 x_1 + c_2 x_2$ is also a solution of equation (1)

Some important conclusion about the behavior of the solution of System of Homogeneous linear Equation

$$\text{Let} \left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \qquad \dots \qquad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array}\right\} \dots\dots\dots\dots\dots (1)$$

Are the systems of Homogeneous Equation, where the coefficient matrix is:

$$A = \begin{bmatrix} a_{11} & a_2 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

And $X = [x_1, x_2, \ldots x_n]^T$

Since Rank of A is less than or equal to minimum of m and n

i.e. $\rho(A) \leq \min\{m, n\}$

Let rank of $A = r$ then two possibilities are there either $r = n$ or $r < n$

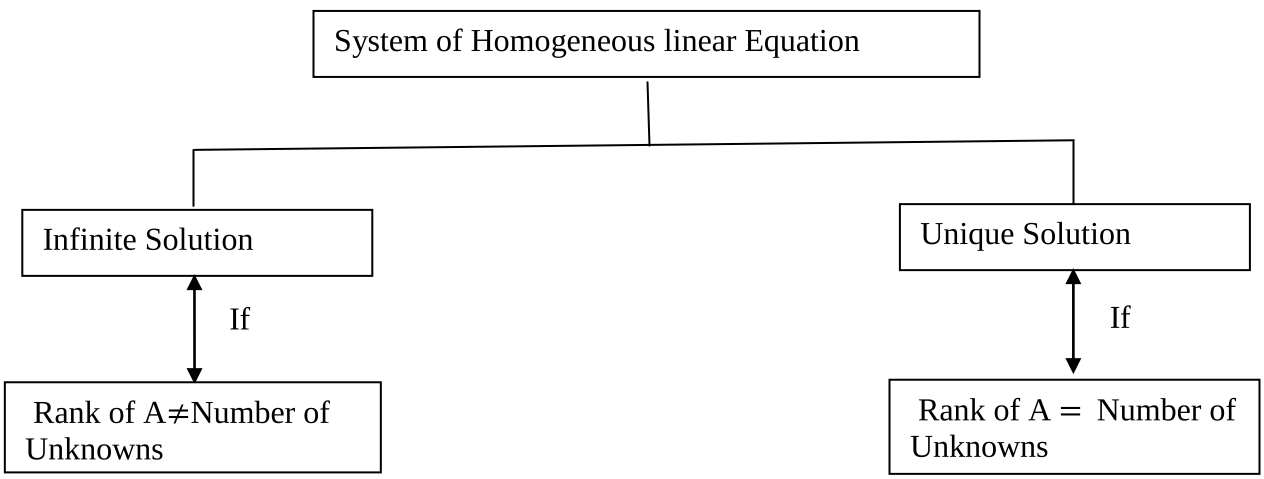
Case 1: If $r = n$ then number of linearly independent solution is $n - r = n - n = 0$

So in this case only Zero solution is a solution of equation (1)

i.e. $X = (0, 0, \ldots 0)^T$

Case 2: If $r < n$

Then $n - r$ linearly independent solution and the linear combination of any two solution is again a solution, so in this case Equation (1) has infinite many solutions.



Example 9: Solve the following System of Equation

$$x + 2y + z = 0$$

$$x + y + z = 0$$

$$x + y + 3z = 0$$

Solution: The given System of Equation can be written in the general form

$$A\,X = 0$$

Where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ AND $X = [X, Y, Z]^T$

Now we are interested to find out the Rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

Then $A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

Again $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Above is the Echelon form of the coefficient matrix A, we have rank of A is equal to minimum of number of non zero row and column.

So Rank of A is 3 and it is equal to Number of unknown

Hence, there is only one solution.

$X = (0, 0, 0)$ i.e. $x = 0,$ $y = 0,$ $z = 0,$ which is trivial solution.

Example 10: Solve completely the System of Equation

$$x + 2y + 3z = 0, \qquad x + y + z = 0, \qquad 2x + 3y + 4z = 0$$

Solution: The given System of Equation can be written as the general form $A\,X = 0$

Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, $X = [X, Y, Z]^T$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We shall reduce the matrix A into Echelon form by using Elementary row transformation

Now using $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

We have $AX = 0$ reduced into

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing $R_3 \to R_3 - R_2$, we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix being of rank 2, which is less than number of unknowns. Hence the given System of linear Equation has infinite solution

And solution is

$$x + 2y + 3z = 0, \qquad -y - 2z = 0$$

Thus $\qquad y = -2z$, put $z = c$ (constant) $\qquad$ then

$$y = -2c$$

$$x = -2y - 3z$$

$$= -2(-2c) - 3c$$

$$x = 4c - 3c = c$$

Hence $x = c$, $y = -2c$, $z = c$ where c is any arbitrary constant.

Example 11: Solve the System of linear Equation

$$x + y + z = 0$$

$$x + 2y + z = 0$$

$$2x + y + 2z = 0$$

$$2x + 3y + 2z = 0$$

Solution: The System of linear Equation can be written as $A\,X = 0$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix} \qquad X = [x, y, z]^T \qquad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We shall interested first to find the Rank of Coefficient matrix A, by using E-row transformation

$R_2 \to R_2 - R_1$

$R_3 \to R_2 - 2R_1$

$R_4 \to R_4 - 2R_1$

We have $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Again $R_3 \to R_3 + R_2$

$R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Above is the Echelon form of the Coefficient matrix A, we have Rank f A is 2 which is less than Number of unknown. Hence given System is of Equation has infinite many solutions.

Therefore,

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\Rightarrow x+y+z=0$ and $y=0$

$x + z = 0 \qquad \Rightarrow x = -z$

Put $z = c$ then $x = -c$

Hence solution is $x = -c, \quad z = c, \quad y = 0$ where c is any arbitary constant

Example 12: Find the solution of System of linear Equation

$$3t_1 + 4t_2 - t_3 - 6t_4 = 0$$

$$2t_1 + 3t_2 + 2t_3 - 3t_4 = 0$$

$$2t_1 + t_2 - 14t_3 - 9t_4 = 0$$

$$t_1 + 3t_2 + 13t_3 + 3t_4 = 0$$

Solution: The given System of Equation can be written as $A\,X = 0$ where

$$A = \begin{bmatrix}3 & 4 & -1 & -6\\2 & 3 & 2 & -3\\2 & 1 & -14 & -9\\1 & 3 & 13 & 3\end{bmatrix}, \qquad X = [t_1, t_2, t_3, t_4]^T \text{and} O = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Now first to find out the rank of matrix by using elementary row transformation.

Applying $R_1 \leftrightarrow R_4$ we get

$$A \sim \begin{bmatrix}1 & 3 & 13 & 3\\2 & 3 & 2 & -3\\2 & 1 & -14 & -9\\3 & 4 & -1 & -6\end{bmatrix}$$

Again by $\quad R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1 3$

Then $A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 2 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$

Again $R_2 \to -\frac{1}{3}R_1,\ R_3 \to -\frac{1}{5}R_3,\ R_4 \to -\frac{1}{5}R_4$

Then $A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix}$

Again $R_3 \leftarrow R_3 - R_2$

$R_4 \leftarrow R_4 - R_2$

Then $A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In the above Echelon form of the coefficient matrix, we have rank of A is 2 which is less than number of unknowns, hence the given system of equation has infinite solution. And the solution is:

$$\left.\begin{aligned} t_1 + 3t_2 + 13t_3 + 3t_4 = 0 \\ t_2 + 8t_3 + 3t_4 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1)$$

Solve the equation (1) by second equation,

$$3t_4 = -t_2 - 8t_3 \Rightarrow t_4 = -\frac{1}{3}t_2 - \frac{8}{3}t_3$$

Put $t_2 = k_1 \text{and} t_3 = k_2 \text{then}$

$$t_4 = -\frac{1}{3}k_1 - \frac{8}{3}k_2$$

From first Equation of (1) $t_1 = -3t_2 - 13t_3 - 3t_4$

$$= -3k_1 - 13k_2 - 3\left(-\frac{1}{3}k_1 - \frac{8}{3}k_2\right)$$

$$= -3k_1 - 13k_2 + k_1 + 8k_2$$

$$= -2k_1 - 5k_2$$

Hence solution is

$$t_1 = -2k_1 - 5k_2, \qquad t_2 = k_1, \qquad t_3 = k_2, \quad t_4 = -\frac{1}{3}k_1 - \frac{8}{3}k_2$$

Where k_1 and k_2 are any arbitrary constant.

Note: The System of Equation $AX = O$ where A is coefficient matrix, If A is square matrix of order 'n' then the solution of $AX = O$ depend on the determinant of A. If determinant of A is zero then $AX = O$ has infinite solutions and if determinant of A is non zero then $AX = O$ has trivial solution (Zero solution).

Example 13: Find the condition of two equations:

$$ax + by = 0$$

$$cx + dy = 0$$

has infinite solution or zero solution.

Solution: Given equations are $\quad ax + by = 0$

$$cx + dy = 0$$

The system of linear homogeneous equations can be written as $AX = O$

Where $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad X = [x, y]^T \text{and} O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since A is Square matrix of order 2, so its values of determinant decide the solution of system of given linear Equation

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

So if $ad - bc = 0$ then the solution of System of linear Equation is infinite, and if $ad - bc \neq 0$ then the solution of system of linear equation is unique (zero) solution,

(ii) System of linear Non Homogeneous Equation: A system of linear equation is a list of linear equations of the form:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots \qquad\qquad \ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

These equation are non homogeneous equations if all coefficients on the right hand side are not zero

i.e. $b_i \neq 0$ at least for one i.

(a) Linear Homogeneous equation in one unknown: Consider the equation $kx = c$ then

(1) If $k \neq 0$ then $x = \frac{c}{k}$ is a unique solution of $kx = c$

(2) If $k = 0$ and $c \neq 0$ then $kx = c$ has no solution.

(3) If $k = 0$ and $c = 0$ then $kx = c$ has infinite solution because $\forall x \quad 0 \,.\, x = 0$

(b) Linear Homogeneous equation in two unknown: Consider a System of two equations in two unknown x and y

$ax + by = c_1 L_1 \longrightarrow$

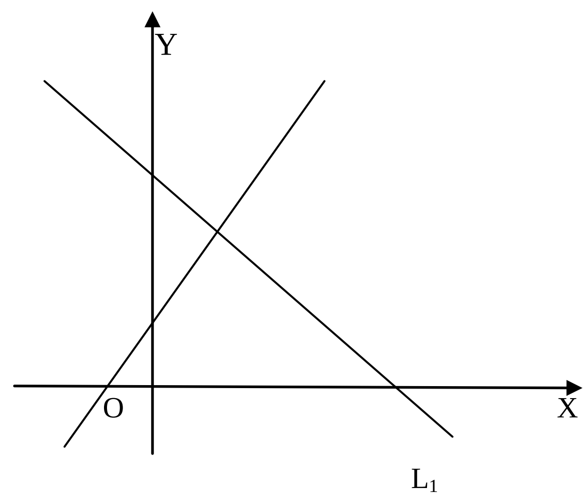
$cx + dy = c_2 L_2 \longrightarrow$

where a, b, c, d are Non Zero, then we are interested to discuss following three cases:

(1) $\frac{a}{c} \neq \frac{b}{d}$ (2) $\frac{a}{c} = \frac{b}{d} \neq \frac{c_1}{c_2}$ (3) $\frac{a}{c} = \frac{b}{d} = \frac{c_1}{c_2}$

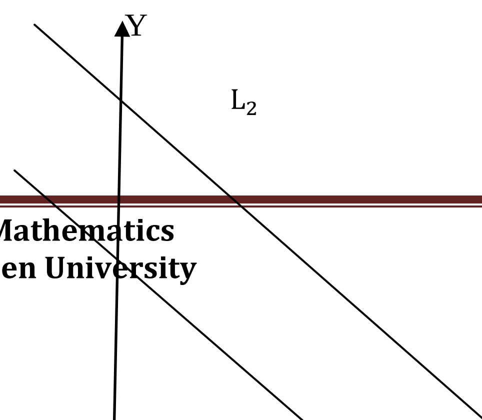
Case 1: If $\frac{a}{c} \neq \frac{b}{d}$ i.e. $a\,d - b\,c \neq 0$

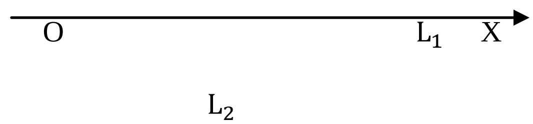
Hence two lines intersect in one pair so the system of linear Equation has unique solution



Unique Solution

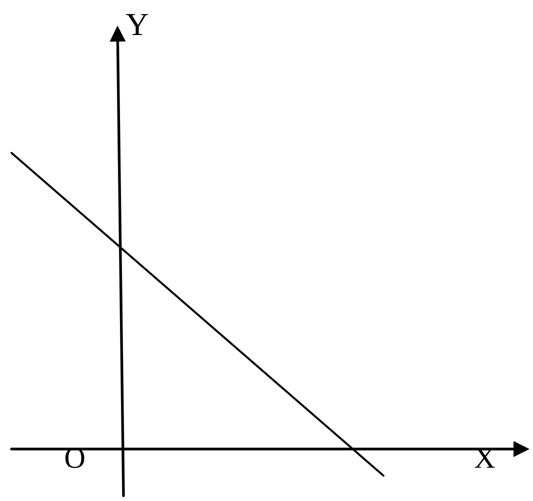
Case 2: If $\frac{a}{c} = \frac{b}{d} \neq \frac{c_1}{c_2}$ then two lines are parallel and not intersect in XY plane, so in this System of equation has no solution





No Solution

Case 3: If $\frac{a}{c} = \frac{b}{d} = \frac{c_1}{c_2}$ in this case two lines $ax + by = c_1$ and $cx + dy = c_2$ are coincide; hence we get a infinite solutions.



L_1 and L_2

Infinite Solution

(iii) Linear Homogeneous Equation in n-unknowns: Consider the system of Equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\dots \qquad \qquad \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Be a System of m Non Homogeneous equation into n-unknown $x_{1,}x_2, \ldots\ x_n$

The general form of the above equation is $A\,X = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \qquad X = [x_1, x_2, \ldots\ x_n]^T \qquad \text{and} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Where A is called Coefficient matrix and

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots\, b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots\, b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & \vdots\, b_m \end{bmatrix} \text{ is called the augmented matrix}$$

Theorem: The system of Non Homogeneous linear Equation $AX = B$ has solution if and only if the Rank of Coefficient matrix and augmented matrix are same.

Proof: Let
$$a_{11}x_1 + a_{12}x_2 + \ \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ \ldots + a_{2n}x_n = b_2$$
$$\ldots. \qquad\qquad \ldots.$$
$$a_{m1}x_1 + a_{m2}x_2 + \ \ldots + a_{mn}x_n = b_m$$

Be a System of Non Homogeneous Equation in n-unknowns $x_{1,}x_2, \ldots x_n$

Let $c_{1,}c_2, \ldots c_n$ are the column vectors of the coefficient matrix A. Then $A\,X = B$ is equivalent to

$$[c_1, c_2, \ldots, c_n]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = B$$

i.e. $c_1x_1 + c_2x_2 + \ \ldots c_nx_n = B$ ………………………… (1)

Let Rank of $A = r_1$ and rank of $[A \vdots B] = r_2$. Since Rank of $A = \ r_1$ so $r_1$column are linearly independent and $n - r_1$ column are linear combination of these $r_1$columns. So, we can suppose without loss of generality first r_1 columns are linearly independent.

The condition of Necessary: If the system of equation has solution then there must exist

n-scalar$c_1, c_2, \dots c_n$ such that

$c_1C_1 + c_2C_2 + \cdots + c_nC_n = B$………………. (2)

Since $C_{r+1}, C_{r+2}, \dots C_n$ is a linear combination of first r_1 column vectors from (2) B is also a linear combination of first r_1 columns.

So augmented matrix $[A : B]$ has$(n + 1 - r_1)$ column which are linearly dependent, but Rank of $[A : B] = r_2$ so the augmented matrix has r_2 linearly independent column, but B is linear combination of first r_1 column

Hence Rank $[A : B] = r_2$ must be equal to r_1 i. e. Rank $[A : B] =$ Rank$[A]$

The condition of Sufficient: Let matrix A and augmented matrix $[A : B]$ have same rank then maximum number of independent column of matrix $[A : B]$ is equal to its rank say r

Therefore the column B should be also expressed as the linear combination of the first r-column of matrix$[A : B]$ already from a linearly independent set.

Thus there exist r scalar $t_1, t_2, \dots t_r$ such that

$t_1C_1 + t_2C_2 + \cdots + t_rC_r = B$………………. (3)

Or $t_1C_1 + t_2C_2 + \cdots + t_rC_r + 0C_{r+1} + 0C_{r+2} + 0C_r = B$ …………………. (4)

From Equation (1) and (4)

$$x_1 = t_1$$

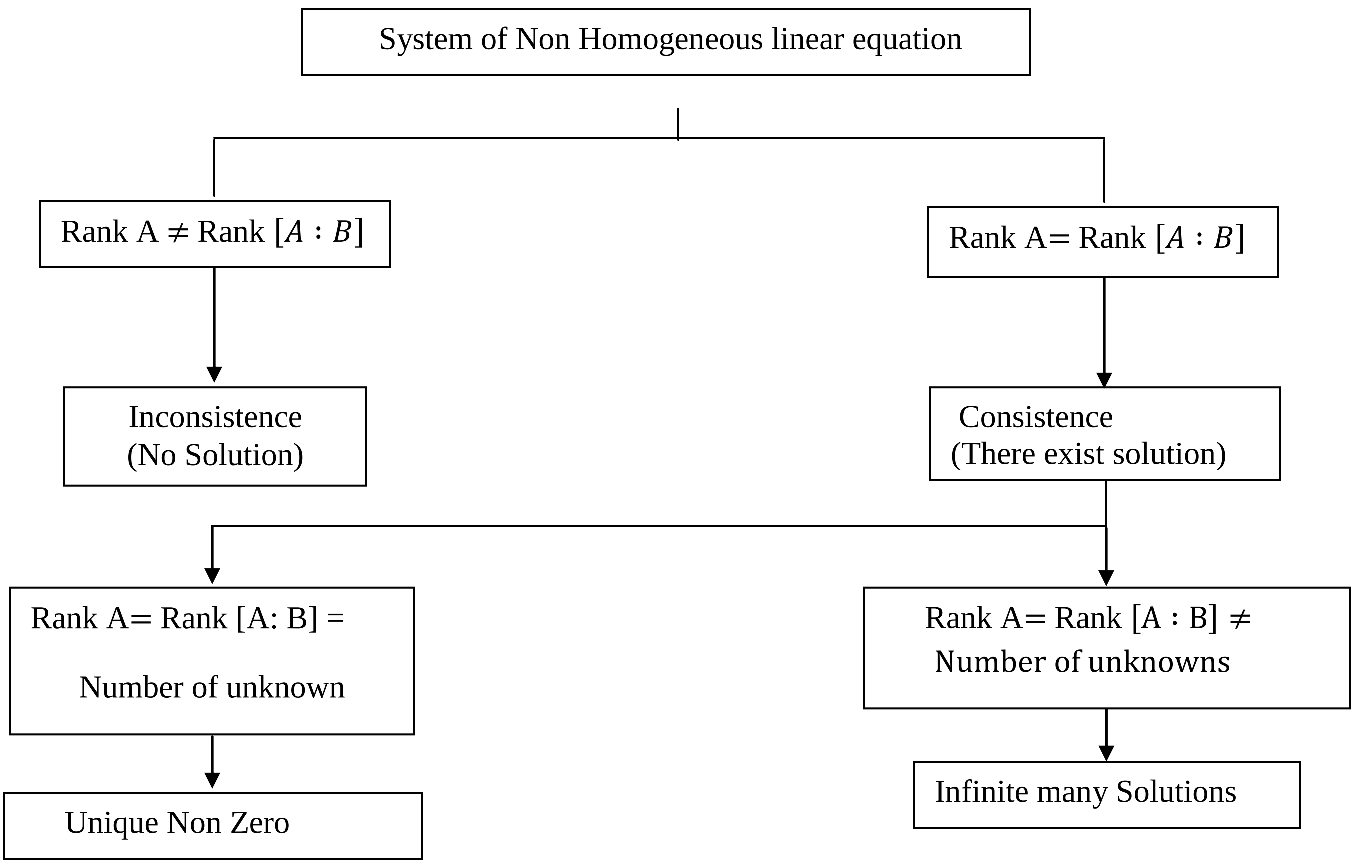
$$x_2 = t_2$$

….

$$x_r = t_r$$

$$x_{r+1} = x_{r+2} = \cdots = x_n = 0$$

Hence $AX = B$ has a solution.



Example 14: Solve the Equation $\quad x + y + z = 1$

$$2x + y + 3 = 2$$

$$3x + 2y + 4z = 4$$

Solution: This given System of Non Homogeneous linear Equation can be written as $AX = B$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$, $\quad X = [x, y, z]^T$ and $B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

The augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 3 & \vdots & 2 \\ 3 & 2 & 4 & \vdots & 4 \end{bmatrix}$

Now reduce the augmented matrix into Echelon form by applying E – row transformation using

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

We have $\quad [A : B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 1 & \vdots & 0 \\ 0 & -1 & 1 & \vdots & 1 \end{bmatrix}$

Again using $R_3 \to R_3 - 3R_2$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

Above is the Echelon form of the matrix $[A : B]$ and rank of $[A : B]$ is 3 but Rank A is 2

Since $Rank\ A \neq Rank\,[A : B]$, therefore the given Equations are inconsistence i.e. the given system of Equation has no solution.

Example 15: Solve the System of linear equation $\quad x + y + z = 2$

$$x + 2y + 3z = 2$$

$$2x + y + z = 1$$

Solution: The given System of equation can be written as single matrix equation $\mathrm{AX} = \mathrm{B}$

Where $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$, $\quad \mathrm{X} = [\mathrm{x}, \mathrm{y}, \mathrm{z}]^{\mathrm{T}} \mathrm{and} \mathrm{B} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

The augmented matrix $[\mathrm{A} : \mathrm{B}] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 1 & 2 & 3 & \vdots & 2 \\ 2 & 1 & 1 & \vdots & 1 \end{bmatrix}$

Now reduce the augmented matrix into Echelon form by applying E-row transformation only

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

We have $[\mathrm{A} : \mathrm{B}] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -3 \end{bmatrix}$

Again applying $R_3 \to R_3 + R_2$

$$[\mathrm{A} : \mathrm{B}] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -3 \end{bmatrix}$$

Above augmented matrix $[\mathrm{A} : \mathrm{B}]$ is the Echelon form,

so Rank of $[\mathrm{A} : \mathrm{B}] = \text{total number of Non Zero rows in Echelon form} = 3$

by the same elementary E-row transformation we get

$$\mathrm{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of
$\mathrm{A} = \text{total number of Non Zero row in Ehelon form of the matrix A}$

so, rank of $A = 3$

Here the number of unknowns is also 3 which is x, y, z

Since $Rank\ A = Rank\ [A:B]$ =Number of unknowns

Therefore the given Equations are consistence and have unique solution.

We see that the given System of Equation is equivalent to the matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

So the System of Equation which is Equivalent to matrix equation

$$x + y + z = 2$$

$$y + 2z = 0$$

$$z = -3$$

$\because y + 2z = 0, \quad y = -2z = 6, \quad x = 2 - y - z, \quad x = -1$

Hence $x = -1$, $y = 6$, $z = -3$ are the solution of given system of equation.

Example 16: Show that the Equation $2x + 3y + z = 1$

$$x + y + z = 2$$

$$3x + 4y + 2z = 3$$

are consistent and find the solution.

Solution: The given System of Equation can be written as $AX = B$

Where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, $\quad X = [x, y, z]^T$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The augmented matrix $[A : B] = \begin{bmatrix} 2 & 3 & 1 & \vdots & 1 \\ 1 & 1 & 1 & \vdots & 2 \\ 3 & 4 & 2 & \vdots & 3 \end{bmatrix}$

Now reduce the augmented matrix into Echelon form by applying E-row transformation only applying $R_2 \leftrightarrow R_1$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 2 & 3 & 1 & \vdots & 1 \\ 3 & 4 & 2 & \vdots & 3 \end{bmatrix}$$

Again applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

Then $$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 1 & -1 & \vdots & -3 \end{bmatrix}$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \quad \text{by} \quad R_3 \to R_3 - 2R_2$$

Above augmented matrix $[A : B]$ is in the Echelon form.

So $\text{Rank}[A : B] =$ total numbers of Non Zero rows in Echelon form $= 2$

By same E – row transformation in A then we get $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So $\text{Rank}A =$ total numbers of non zero rows in Echelon form A $= 2$

Since $\text{Rank}A = \text{Rank}[A : B]$, therefore the given equation are consistent.

Since number of unknowns in given System of Equation is 3 which is greater than Rank A, therefore the given System of Equation will have infinite many solutions.

The matrix equation of given System of Equation is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

This matrix is equivalent to $x + y + z = 2$

$$y - z = -3$$

$$\because y = -3 + z$$

$$x = 2 - y - z = 2 - (-3 + z) - z$$

$$= 2 + 3 - 2z = 5 - 2z$$

Taking $z = c(\text{constant})$ then we get

$$y = c - 3$$

$$x = 5 - 2c$$

Hence the solution of the given System of Equation is:

$x = 5 - 2c, \quad y = c - 3 \quad z = c.$

Example 17: If A be a $m \times n$ matrix of rank n with real entries then show that if System $AX = B$ has solution then it is unique.

Solution: Let $AX = B$ has solution then it is possible only if Rank of A is equal to rank of augmented matrix $[A : B]$

We know that if $A = (a_{ij})_{m \times n}$ then

Rank of $A \leq \min\{m, n\}$ and given Rank of A is equal to n so m must be greater than n

Since $\text{Rank} A = \text{Rank of} [A : B] = n = \text{Number of unknowns}$

So solution is unique.

Example 18: Investigate for what condition of λ and μ the given System of equation

$$x + y + 2z = 2$$

$$x + 2y + z = 3$$

$2x + 2y + \lambda z = \mu$

(1) No solution (2) An infinite solution (3) Unique solution

Solution: The matrix form of given System of linear Equation is $AX = B$

Where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & \lambda \end{bmatrix}$, $X = [x, y, z]^T$ and $B = \begin{bmatrix} 2 \\ 3 \\ \mu \end{bmatrix}$

The augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 2 & \vdots & 2 \\ 1 & 2 & 1 & \vdots & 3 \\ 2 & 2 & \lambda & \vdots & \mu \end{bmatrix}$

We shall first to reduce the matrix $[A : B]$ into Echelon form by applying elementary transformation only

applying $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 2 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & 1 \\ 0 & 0 & \lambda - 4 & \vdots & \mu - 4 \end{bmatrix}$$

Above is the Echelon form of the augmented matrix $[A : B]$

Case 1: If $\lambda \neq 4$ then Rank of [A:B] = Number of unknowns = 3= Rank A

So in this case the given System of Equation has unique solution.

Case 2: If $\lambda = 4$ but $\mu \neq 4$ then

$$\text{Rank of } A \neq \text{Rank of } [A : B]$$

So in this case System of Equation is Inconsistent hence no solution

Case 3: If $\lambda = \mu = 4$ then

$$\text{Rank of } A = \text{Rank of } [A : B] \text{ but not equal to number of unknowns}$$

Hence in this case System of equation have infinite solution.

3.9 *SUMMARY*

In this unit we learned to how to find rank of any matrix, after that we learned to solve the linear equation with the help of rank of matrix, in further classes we will be able to understand linearly dependent and independent sets and solutions very well with the help of rank of matrix

3.10 *GLOSSARY*

1. **Inconsistent solution:**If a system has no solution

2. **Augmented matrix:** A matrix obtained by appending the columns of given marix

3. **Unknown:** An unknown is variable in an equation which has to solved for

3.11 *SELF ASSESMENT QUESTIONS*

3.11.1Multiple choice questions:

1. The Rank of Null matrix of order *n* is

(a) 1 (b) 0

(c) $n-1$ (d) n

2. Which of the following matrix is elementary?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 5 & 1 \end{bmatrix}$ (d) none of these

3. The rank of matrix $A = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 3 & 6 & 11 & 9 \\ 6 & 1 & 26 & 30 \end{bmatrix}$ is

(a) 1 (b) 2

(c) 0 (d) 3

4. If A is an Square matrix of order 5 and Rank 3 then Rank of $adj(A)$ is

(a) 0 (b) 3

(c) 4 (d) 5

5. If the Nullity of the matrix $\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$ is 1 then the value of $k$is

(a) 0 (b) 1

(c) -1 (d) 2

6. Let $A = \left(a_{ij}\right)$ be a $n * n$ matrix such that $a_{ij} = 1 \quad \forall\, i, j$ then Nullity of A is

(a) 1 (b) 2

(c) $n - 1$ (d) n

7. If $x + y = a,\ \ x + by = 3$ then the System f Equation has unique solution if

(a) $a \in 1\, r and b = 1$(b) For all a but $b \neq 1$

(c) For all a and b (d) Always unique solution

8. Let A be a $m * n$ matrix of rank n with real entries, choose the correct statement

(a) $AX = B$hassolutionforanyB (b) $AX = 0$ does not have a solution

(c) If $AX = B$ has a solution then it is unique (d) None of these

9. Let A and B are two matrix such that $\mathrm{BA} + \mathrm{B}^2 = \mathrm{I} - \mathrm{BA}^2$whereIisthen $*$ nidentity matrix then

(a) $\mathrm{BX} = 0$ haszerosolution (b) $\mathrm{AX} = 0$ has always zero solution

(c) Rank of $B = 1$ (d) Rank of $A = n$

10. Which of the following matrix has same row space at $\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix}$

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

ANSWERS:

1. (b) 2. (c) 3. (b) 4. (a) 5. (c)

6. (c) 7. (b) 8. (c) 9.(a) 10. (a)

3.11.2 Fill in the Blanks:

1. No Skew Symmetric matrix of Rank …………..

2. Rank A $+$ Rank B $\geq$ …………

3. A is invertible idempotent matrix then Rank of A ………..

4. If A is any matrix of order $m * n$ and $m > n$ *then* $AX = 0$ has ………. solution

5. If A is any invertible matrix of order n then $AX = B$ has ………… solution

ANSWERS:

1. 1 **2.** Rank $(A + B)$ **3.**Order of matrix **4.**No **5.**Unique

3.12 *REFERENCE*

1. Linear Algebra, Vivek.sahai & Vikas Bist :Narosa publishing House

2. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

3. Schaum's out line (Linear Algebra)

3.13 *SUGGESTED READINGS*

1. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

2. Schaum's out line (Linear Algebra)

3.14 *TERMINAL QUESTIONS*

3.14.1 Short answer type questions:

1. Find the values of λ , the Equation $x + y + z = 1, \quad x + 2y + 4z = \lambda, \quad x + 4y + 10z = \lambda^2$, has unique solution

2. Show that the Equations $x + y + z = 1, \quad x + 2y + 3z = 5, \quad 2x + 4y + 4z = 7$ are not consistent

ANSWERS:1. No values of λ such that the system of equation has unique solution

3.14.1 Long answer type questions:

1. Find the Rank of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}$

2. Find the Rank of matrix $A = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

3. Examine if the System of Equations
$x + y + 4z = 6, \qquad x + 2y = 7, \qquad 2x + 3y + 4z = 13$ is consistent?

Find the solution if it is consistent

ANSWERS: **3.** 2 **4.** 3

UNIT 6: EIGEN VALUES AND EIGEN VECTORS

CONTENTS:

6.1 *INTRODUCTION*

If A is Square matrix then we are interested to find out a non zero vector X such that $AX = \boldsymbol{\lambda}X$ then $\boldsymbol{\lambda}$ is called eigenvalues of A corresponding eigenvector X

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Where X is non zero vector and $AX = \lambda X$

i.e.$AX - \lambda X = 0$

$\Rightarrow (A - \lambda I)X = 0$

Is the form of Homogenous System of equation and has non zero solution if,

Rank $(A - \lambda I) < n$

It is possible only when $|A - \lambda I| = 0$

$|A - \lambda I| = 0$ is called characteristic equation of matrix A in variable λ, there roots or zeros or solutions of characteristic equation are called Eigen values or Latent roots.

6.2 *OBJECTIVES*

After reading this unit you will be able to:

- Find eigenvalues of a square matrix
- Find eigenvector of a square matrix
- Understand algebraic and geometric multiplicity of eigen root (values)
- Understand various properties of eigenvalues and eigen vectors
- Know about Cayley Hamilton theorem and verify it.

6.3 *SPECTRUM OF ANY MATRIX*

A is any matrix of order n, then spectrum of A is the set of all eigen values of A.

Example 1: Determine the eigen value of matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Solution: Let λ is eigen value of matrix A, then the characteristic equation of A is $|A - \lambda I| = 0 |A - \lambda I| = \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix}$

$$= (-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & -\lambda \\ 2 & -1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda + 2) + 2(-1 + 2\lambda) \qquad = -\lambda^3 + 6\lambda - 4$$

The root of $|A - \lambda I| = 0$

$-\lambda^3 + 6\lambda - 4 = 0$ or $\lambda^3 - 6\lambda + 4 = 0$

$$(\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0 \lambda = 2, \quad -1 + \sqrt{3}, \quad -1 - \sqrt{3}$$

Hence the Eigen values of the matrix A are

$$2, \quad -1 + \sqrt{3}, \quad -1 - \sqrt{3}$$

Theorem 1: Let X is a eigen vector of A, then X can't corresponding to more than one eigen values of A.

Proof: Let X be an eigen vector of A, corresponding two eigen values are λ_1 and λ_2

Then $AX = \lambda_1 X$ ………………….. (1)

$AX = \lambda_2 X$ ………………….. (2)

From (1) and (2)

$$\lambda_1 X = \lambda_2 X \qquad (\lambda_1 - \lambda_2)X = 0$$

But X is non zero vector

So $(\lambda_1 - \lambda_2) = 0$ $\Rightarrow \lambda_1 = \lambda_2$

Hence cannot corresponding to more than one eigen value of A

Theorem 2: If X is eigen vector of a matrix A corresponding to the eigen valueλ, then KX is also an eigen vector of A corresponding to the same eigen values λ, where K is any non zero scalar

Proof: Let X is eigen vector of a matrix A corresponding eigen values λ

Then $AX = \lambda X$; $X \neq 0$

Since K is any non zero scalar

So, $KAX = K\lambda X$ $A(KX) = \lambda(KX)$

Hence $(KX)$is also eigen vector of A corresponding same eigen value λ.

Note: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

Then $(A - \lambda I) = \begin{bmatrix} a_{11-\lambda} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22-\lambda} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn-\lambda} \end{bmatrix}$ + polynomial over $\lambda \le (n-2)$

Then $(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)\begin{bmatrix} a_{33} - \lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n3} & a_{n4} & a_{nn} - \lambda \end{bmatrix}$ + polynomial over $\lambda \le (n-2)$

$= (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-1} (a_{11} + a_{22} + a_{33} + \dots + a_{nn}) +$ polynomial of degree $\le (n-2) \dots$ (1)

if $\lambda_1, \lambda_2 \dots . \lambda_n$ are eigen values of square matrix A, then

$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$ …… (2)

Coefficient of $\lambda^{n-1} = (-1)^n (-\lambda_1 - \lambda_2 \dots - \lambda_n) = (-1)^{n+1} (\lambda_1 + \lambda_2 \dots + \lambda_n)$

But coefficient of λ^{n-1} in equation (1) is $(-1)^{n-1}(a_{11} + a_{22} + \cdots a_{nn})$

So $a_{11} + a_{22} + \dots a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$

$$\boxed{\text{Trace } A = \text{Sum of Eigen values}}$$

Put $\lambda = 0$ in equation (2) then we get

$$|A - 0I| = (-1)^n (\lambda_1)(\lambda_2) \dots (\lambda_n)(-1)^n$$

$$|A| = \lambda_1 . \lambda_2 . \lambda_3 \dots \lambda_n$$

So, $\boxed{\text{Determinant of A} = \text{Product of Eigen values of A}}$

6.4 *ALGEBRAIC AND GEOMETRIC MULTIPLICITY OF EIGEN ROOT OR CHARACTERISTIC ROOT*

Let A be any square matrix of order n, and λ is root of order K of the characteristic polynomial $|AX - \lambda I| = 0$. Then K is called algebraic multiply of eigen value λ, total number of linearly independent eigen vector corresponding eigen value λ is called geometric multiplicity of λ i.e. total number of linearly independent solution $|AX - \lambda I|X = 0$

$$n - \text{Rank}(A - \lambda I) = \text{Geometric multiplicity of } \lambda$$

Theorem 3: The eigen vector corresponding distinct eigen value of a matrix is linearly independent.

Proof: Let A be a Square matrix of order n, and $X_1, X_2, \ldots X_n$ are the eigen vector of matrix A corresponding to distinct eigen value $\lambda_1, \lambda_2, \ldots \lambda_n$.

Then $\quad AX_1 = \lambda_1 X_1$

$$AX_2 = \lambda_2 X_2$$

$$\ldots \ldots$$

$$AX_n = \lambda_n X_n$$

And we have to prove the set of vector

$S = \{X_1, X_2, X_3, \ldots., X_n\}$ is linearly independent.

If the set $S = \{X_1, X_2, X_3, \ldots., X_n\}$ are linearly dependent then at least one X_i is a linear combination of remaining vector of S.

Then we choose the set S_1 is subset of S

$S_1 = \{X_1, X_2, X_3, \ldots., X_k\}$ is linearly independent.

But the set $S_2 = \{X_1, X_2, X_3, \ldots., X_k, X_{k+1}\}$ are linearly dependent.

Then we can choose some Scalar

$$a_1, a_2, \quad \ldots a_{k,} a_{k+1}$$

Show that $\quad a_1x_2 + a_2x_2 + \cdots + a_{k+1}x_{k+1} = 0 \quad$ ……… (1)

Pre multiply both sides by A, then we get

$$A(a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1}) = A.0$$

$$a_1Ax_1 + a_2Ax_2 + \cdots + a_kAx_k + a_{k+1}Ax_{k+1} = 0$$

Since $AX_1 = \lambda_i X_i \quad ; \quad i = 1, 2, 3, \ldots n$

Then $a_1\lambda_1x_1 + a_2\lambda_2x_2 + \cdots + a_k\lambda_kx_k + a_{k+1}\lambda_{k+1}x_{k+1} = 0$ ………. (2)

Multiply equation (1) by λ_{k+1}

Then $\quad a_1\lambda_{k+1}x_1 + a_2\lambda_{k+1}x_2 + \cdots + a_k\lambda_{k+1}x_k + a_{k+1}\lambda_{k+1}x_{k+1} = 0$ ………. (3)

Subtracting equation (3) from (2)

$$a_1(\lambda_1 - \lambda_{k+1})x_1 + a_2(\lambda_2 - \lambda_{k+1})x_2 + \ldots + a_k(\lambda_k - \lambda_{k+1})x_k = 0$$

Since the set $S_1 = \{X_1, X_2, X_3, \ldots, X_k\}$ is linearly independent.

So all the coefficient of x_i are zero

$$a_i(\lambda_i - \lambda_{k+1}) = 0 \qquad \forall\, i = 1, 2, 3, \ldots k$$

Put the values of a_i in equation (1) then we get

$$\lambda_{k+1} = \lambda_{k+1} \qquad \text{but} \qquad \lambda_{k+1} \neq 0$$

$$\lambda_{k+1} = 0$$

but $\lambda_i \neq \lambda_{k+1} \forall\, i = 1,2,3,\ldots,k$

therefore $a_i = 0 \,\forall\, i = 1,2,3,\ldots,k$

Now put $a_i = 0 \,\forall\, i = 1,2,3,\ldots,k$ in equation (1) we get

$a_{k+1}x_{k+1} = 0$

$\Rightarrow a_{k+1} = 0$, since $x_{k+1} \neq 0$

This is not possible, since S_2 is linearly dependent. So, our assumption is wrong hence the set of vector $\mathrm{S} = \{\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \ldots \mathrm{X}_\mathrm{n}\}$ is linearly independent,

Theorem 4: λ is eigen value of any square matrix A if and only if there exist a non zero vector X such that $\mathrm{AX} = \lambda\mathrm{X}$

Proof: Suppose λ is Eigen value of matrix A, and then we can say λ is root of the characteristic equation $|\mathrm{A} - \lambda\mathrm{I}| = 0$

Since $|\mathrm{A} - \lambda\mathrm{I}| = 0$ so Rank $(\mathrm{A} - \lambda\mathrm{I}) < order\ of\ matrixA$

Therefore the linear Homogeneous equation $|\mathrm{A} - \lambda\mathrm{I}|\mathrm{X} = 0$

$$\text{So } \mathrm{AX} = \lambda\mathrm{X}$$

Conversely, suppose there exist a non-zero vector X satisfied $\mathrm{AX} - \lambda\mathrm{X}$ where λ is a scalar

So $\quad \mathrm{AX} - \lambda\mathrm{IX} = 0 \Rightarrow (\mathrm{A} - \lambda\mathrm{I})\mathrm{X} = 0$

X is non zero if Rank of $(\mathrm{A} - \lambda\mathrm{I})$ is less than order of matrix

$$\Longrightarrow |\mathrm{A} - \lambda\mathrm{I}| = 0$$

Hence λ is eigen value of matrix A.

Example 2: Determine the characteristic roots and corresponding characteristic vector of the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Solution: The characteristic equation of the matrix A is $|\mathrm{A} - \lambda\mathrm{I}| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 2-\lambda & 2 \\ 3 & 3 & 3-\lambda \end{vmatrix} = 0 \quad$ or

$$(1-\lambda)\{(3-\lambda)(2-\lambda) - 6\} - 1\{(3-\lambda).2 - 6\} + 1\{6 - 3(2-\lambda)\} = 0$$

$$(1-\lambda)\{\lambda^2-5\lambda\}+2\lambda+3\lambda=0$$

$$-\lambda^3+\lambda^2+5\lambda^2=0$$

$\lambda^2(\lambda-6)=0$

Hence the characteristic roots of A are 0, 0, 6

The eigen vector $X=[x_1,x_2,x_3]^T$ of A corresponding to the eigen value 0 are given by the non zero solution of the equation

$$(A-0I)X=0$$

$$\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\begin{bmatrix}1&1&1\\2&2&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ by $R_3\to R_3-3R_1$

Or $\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ by $R_2\to R_2-2R_1$

The rank of coefficient matrix is 1, therefore these linear equation have $3-1=2$ linearly independent solution. The equation can be written as:

$X_1+X_2+X_3=0$

Put $X_2=a$ and $X_3=b$ then $X_1=-(a+b)$

Where a and b are any scalar, therefore $X_1=[-(a+b),a,b]^T$ is an eigen vector of A corresponding to the eigen values 0

Let $a=1,\ b=1$ then $X_1=[-2,1,1]^T$

$a=0,\ b=1$ then $X_2=[-1,0,1]^T$ are two linearly independent eigen vector of A corresponding eigen value 0

If P_1,P_2 are scalar not both are equal to zero then $P_1X_1+P_2X_2$ gives all the eigen vectors of A corresponding to eigen values 2

The eigen vector of A corresponding to the eigen value 6 are given by the non zero solution of the equation

$$(A-6I)X=0$$

$$\begin{bmatrix}-5 & 1 & 1\\ 2 & -4 & 2\\ 3 & 3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Or $$\begin{bmatrix}-5 & 1 & 1\\ 2 & -4 & 2\\ 3 & 3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Or $$\begin{bmatrix}-5 & 1 & 1\\ 0 & -18 & 12\\ 0 & 18 & 12\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$ by $R_2 \to 5R_2 + 2R_1$, $R_3 \to 5R_3 + 3R_1$

Or $$\begin{bmatrix}-5 & 1 & 1\\ 0 & -18 & 12\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$ by $R_2 \to 2R_1 - R_2$

The Rank of coefficient matrix of these equations is 2. Therefore these equations have $3-2=1$ linearly independent solution. These equations can be written as

$$-5x_1 + x_1 + x_2 = 0$$

$$-18x_2 + 12x_3 = 0$$

From last equation we get $x_2 = \frac{2}{3}x_3$

Put $x_3 = 1$ then $x_2 = {}^2/_3$ and first equation $x_1 = {}^1/_3$

So $x_3 = \left[{}^1/_3, {}^2/_3, 1\right]^T$ is eigen vector of A corresponding eigen value 6.

If P is any non zero scalar then Px_3 is also eigen vector of A corresponding to the eigen value 6.

Theorem 5: The eigen values of a Hermitian matrix are real.

Proof: Let A is a Hermitian matrix and λ is eigen value of A corresponding eigen vector X. Then $A^\theta = A$ ………… (1)

$AX = \lambda X$ ………. (2)

Pre multiplying both sides of equation (2) by X^θ, then

$$X^\theta AX = X^\theta \lambda X$$

$X^\theta AX = X^\theta \lambda X$ ………… (3)

Taking conjugate and transpose of both side of equation (3), then we get

$$\left(X^\theta AX\right)^\theta = \left(\lambda X^\theta X\right)^\theta$$

$$X^\theta A^\theta X = \bar{\lambda} X^\theta X \because \left(X^\theta\right)^\theta = X$$

$X^\theta AX = \bar{\lambda} X^\theta X$ ……………… (4) $\quad \because \; A^\theta = A$

From (3) and (4) we have

$$\lambda\, X^\theta X = \bar{\lambda} X^\theta X$$

$\left(\lambda - \bar{\lambda}\right) X^\theta X = 0$, but X is non zero vector so $\quad X^\theta X \neq 0$

$$\Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$$

Let $\quad \lambda = x + iy$

$\bar{\lambda} = x - iy, \lambda = \bar{\lambda}$

$$\Rightarrow x + iy = x - iy$$

$$\Rightarrow iy = 0$$

$$\Rightarrow y = 0$$

$$\lambda = x + i0$$

$\lambda = x$ $\qquad$ Hence λ is real.

Theorem 6: The eigen values of a skew Hermitian matrix are either zero or pure imaginary

Proof: Let A is skew Hermitian matrix then $A^{\theta} = -A$

Now $(iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -i(-A) = (iA)$

⇒ (iA) is Hermitian matrix when A is Skew Hermitian and we know that the eigen values of Hermitian matrix is real,

So the eigen values of (iA) is real.

Let λ is eigen value of A then (iλ) is eigen value of (iA)

But $i\lambda$ is real so

$$i\lambda = \bar{i}\bar{\lambda}$$

$i\lambda = -i\lambda$ if λ is real

$$2i\lambda = 0\lambda = 0$$

$\lambda = x + iy$ if λ is complex content and $y \neq 0$

$$i\lambda = ix - y$$

$\overline{(i\lambda)} = -y - ix$

$(i\lambda) = \bar{i}\bar{\lambda}$ Because (iλ) is real

$$ix - y = -y - ix$$

$$2ix = 0x = 0$$

$$\lambda = 0 + iy$$

$\lambda = iy \Longrightarrow$ is pure imaginary provided $\lambda \neq 0$

Hence the eigen values of skew Hermitian matrix are either zero or purely imaginary.

Theorem 7: Eigen values of a unitary matrix are of unit modules.

Proof: Let A is a unitary matrix and λ is Eigen value of A Corresponding Eigen vector X, then

$A^{\theta}A = AA^{\theta} = I$ …………….. (1)

$AX = \lambda X$ ……………………. (2)

Taking conjugate and transpose both sides of equation (2) then

$$(AX)^{\theta} = (\lambda X)^{\theta}$$

$X^{\theta}A^{\theta} = \bar{\lambda}X^{\theta}$ …………………(3)

Multiplying equation (2) and (3)

$$\left(X^{\theta}A^{\theta}\right)(AX) = \left(\bar{\lambda}X^{\theta}\right)(\lambda X)$$

$$\left(X^{\theta}A^{\theta}\right)(AX) = \lambda\bar{\lambda}X^{\theta}X$$

$$X^{\theta}\left(A^{\theta}A\right)X = \lambda\bar{\lambda}X^{\theta}X$$

$$X^{\theta}IX = |\lambda|^2X^{\theta}X \because A^{\theta}A = I, \qquad \lambda\bar{\lambda} = |\lambda|^2$$

$$X^{\theta}X = |\lambda|^2X^{\theta}X$$

$$(1 - |\lambda|^2)X^{\theta}X = 0$$

But $X^{\theta}X \neq 0$ because X is non zero vector

So, $(1 - |\lambda|^2) = 0$

$$|\lambda|^2 = 1$$

Hence λ is at unit modules.

Example 3: If A is Square matrix of order 4 and spectrum of A is $\{1, 2, 3\}$ and trace of A, determinant of A are 9 and 18 respectively then find the all eigen values of A.

Solution: Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are eigen values A, then

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \text{trace of matrix A} = 9$$

$$\lambda_1.\lambda_2.\lambda_3.\lambda_4 = \text{determinant of A} = 18$$

Given spectrum of A is $\{1, 2, 3\}$ so three eigen values of A are 1, 2, 3

So $1 + 2 + 3 + \lambda_4 = 9$

$$\lambda_4 = 3$$

$$1.2.3.3 = 18$$

So eigen values of A are $1, 2, 3, 3$.

Example 4: Show that matrix A and its transpose matrix A^T have same eigen values.

Solution: Let λ is Eigen values of A so λ is the root of characteristic Equation

$|A - \lambda I| = 0$ ………………… (1)

Now $(A - \lambda I)^T = (A^T - \lambda I)$

And we know that determinant of any matrix and its transpose are equal, therefore

$$|(A - \lambda I)^T| = |A - \lambda I|$$

$$|A^T - \lambda I| = |A - \lambda I|$$

$\because$ $|A - \lambda I| = 0$ So $|A^T - \lambda I| = 0$

$\Rightarrow$ λ is root of $|A^T - \lambda I| = 0$

Hence λ is also Eigen value of A^T.

Example 5: Show that the characteristic roots of A^θ are conjugates of the characteristic roots of matrix A.

Solution: Let λ is Eigen value of matrix A

$\Rightarrow$ λ is the root of the equation $|A - \lambda I| = 0$

Now $(A - \lambda I)^\theta = A^\theta - (\lambda I)^\theta$

$= \left(A^{\theta} - \bar{\lambda}I\right)$ ………………. (1)

And $\left|(A - \lambda I)^{\theta}\right| = \left|\overline{A - \lambda I}\right| \because \left|A^{\theta}\right| = |\bar{A}|$

From equation (1)

$$\left|A^{\theta} - \bar{\lambda}I\right| = \left|\overline{A - \lambda I}\right|$$

If $|A - \lambda I| = 0 \quad \Rightarrow \left|\overline{A - \lambda I}\right| = 0 \quad \Leftrightarrow \left|A^{\theta} - \bar{\lambda}I\right| = 0$

So $\bar{\lambda}$ is root at $\left|A^{\theta} - \bar{\lambda}I\right| = 0$ hence $\bar{\lambda}$ is eigen value of A^{θ}.

Note:

1. 0 is the eigen value of matrix A if matrix has determinant zero.

2. Eigen value of upper /lower triangular matrix are just principal diagonal element.

3. Eigen values of strict upper or strict lower triangular matrix are zero.

4. Eigen values of Nilpotent matrix are always zero.

5. Eigen values of idempotent matrix are either 0 or 1.

6. Eigen values of involutory matrix are either 1 or -1.

Example 6: If λ is the eigen value of a non singular matrix A, then prove that ${|A|}/{\lambda}$ is a eigen value of adj A.

Solution: Let A be any $n \times n$ non singular matrix and λ is eigen value of A

So λ must be non zero and there exist a non zero vector X such that

$AX = \lambda X$ ……………………. (1)

Pre multiplying both sides in equation (1) by $(\text{adj}A)$

So, $(\text{adj}A)(AX) = (\text{adj}A)(\lambda X)$

$$[(\text{adj}A)A]X = \lambda(\text{adj}A)X$$

$|A|I_n X = \lambda(adjA) \because \quad \mathrm{A(adj\ A) = (adj\ A).A = |A|I_n}$

$$\frac{|A|}{\lambda} X = (\mathrm{adj}\ A)X$$

Hence $\frac{|A|}{\lambda}$ is eigen value of matrix (adj A).

Example 7: If A is square matrix of order 3 with eigen values are 1, 2, 3 then what are the possible eigen values of $\mathrm{adj}(A^{-1})$.

Solution: A is Square matrix of order 3 with eigen values 1, 2, 3

So determinant of A is the product of its eigen value

$$|A| = 1.2.3 = 6$$

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{6}$$

And $1, 2, 3$ are eigen values of A then $1, \frac{1}{2}, \frac{1}{3}$ are eigen values of A^{-1}

Say $\quad \lambda_1 = 1, \qquad \lambda_2 = \frac{1}{2}, \qquad \lambda_3 = \frac{1}{3}$

Finally, eigen values of $\mathrm{adj}(A^{-1})$ are

$\frac{|A^{-1}|}{\lambda_1}, \quad \frac{|A^{-1}|}{\lambda_2}, \quad \frac{|A^{-1}|}{\lambda_3}$ which is $\frac{\left(\frac{1}{6}\right)}{1}, \quad \frac{\left(\frac{1}{6}\right)}{\left(\frac{1}{2}\right)}, \quad \frac{\left(\frac{1}{6}\right)}{\left(\frac{1}{3}\right)}$

$\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$ are required eigen values of $\mathrm{adj}(A^{-1})$.

6.5 *THE CAYLEY-HAMILTON THEOREM*

Every square matrix satisfies its characteristic equation i.e. if for a square matrix A of order *n*.

$|A - \lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + \ \ldots + a_n]$ then the matrix equation is

$X^n + a_1X^{n-1} + \ \ldots + a_nI = 0$ is satisfied by A

$$A^n + a_1A^{n-1} + \ \ldots + a_nI = 0$$

Proof: Let A is a square matrix of order n and λ is eigen value of A then element of $(A - \lambda I)$ are at most of the first degree in λ, and the element of $\text{adj}(A - \lambda I)$ are of degree $n - 1$ or less in terms of λ.

So $\text{adj}(A - \lambda I)$ can be written as

$$\text{adj}(A - \lambda I) = C_0\lambda^{n-1} + C_1\lambda^{n-2} + \ ... + C_{n-2}\lambda + C_{n-1}$$

Where $C_0,\ C_1,\ ...\ C_{n-1}$ are square matrices of order n

Now $(A - \lambda I)\ \text{adj}(A - \lambda I) = \ |A - \lambda I|I \because A\ \text{adj}A = |A|I$

Where I is the identity matrix of order n

$$(A - \lambda I)(C_0\lambda^{n-1} + C_1\lambda^{n-2} + \ ... + C_{n-2}\lambda + C_{n-1})$$
$$= (-1)^n[\lambda^n + a_1\lambda^{n-1} + \ ... + a_n]I$$

Comparing coefficients of like powers of λ on both sides, then we get

$$-IC_0 = (-1)^n I$$

$$AC_0 - IC_1 = (-1)^n I$$

$$AC_1 - IC_2 = (-1)^n a_2 I$$

$$...\ \ ....$$

$AC_{n-1} = (-1)^n a_n I$ n

Pre multiplying these equations by $A^n, A^{n-1}, ... I$ respectively and adding them

$$0 = (-1)^n[A^n + a_1A^{n-1} + \ ... + a_nI]$$

Thus $A^n + a_1A^{n-1} + a_2A^{n-2} + \ ... + a_n + A^1 + a_nI = 0$

Hence every Square matrix satisfies its characteristic equation.

Example 8: find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and verify that it is satisfied by A.

Solution: We have $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ the characteristic matrix of A

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

The characteristic polynomial of A

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

$$= (1-\lambda)[(6-\lambda)(3-\lambda) - 0] - 2[0-0] + 4[0-0]$$

$$= (1-\lambda)(6-\lambda)(3-\lambda)$$

∵The characteristic equation of A is $|A - \lambda I| = 0$

i.e.$(1-\lambda)(6-\lambda)(3-\lambda) = 0$ ………………………… (1)

The roots of this equation are 1, 3, 6

Hence the Eigen values of A are 1, 3, 6

From equation (1) the characteristic equation of the matrix A is

$$(\lambda-6)(\lambda-3)(\lambda-1) = 0$$

Or

$$(\lambda^2 - 3\lambda - 6\lambda + 18)(\lambda - 1) = 0$$

Or

$$(\lambda^2 - 9\lambda + 18)(\lambda - 1) = 0$$

Or

$\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

We are now to verify that

$$A^3 - 10A^2 + 27A - 18I = 0$$

We have

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = A * A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 38 \\ 0 & 9 & 45 \\ 0 & 0 & 36 \end{bmatrix} \qquad A^3$$

$$= A^2 * A = \begin{bmatrix} 1 & 8 & 38 \\ 0 & 9 & 45 \\ 0 & 0 & 36 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 26 & 272 \\ 0 & 27 & 315 \\ 0 & 0 & 216 \end{bmatrix}$$

Now we can verify that $A^3 - 10A^2 + 27A - 18\,I$

$$\begin{bmatrix} 1 & 26 & 272 \\ 0 & 27 & 315 \\ 0 & 0 & 216 \end{bmatrix} - \begin{bmatrix} -10 & -80 & -380 \\ 0 & -90 & -450 \\ 0 & 0 & -360 \end{bmatrix} + \begin{bmatrix} 27 & 54 & 108 \\ 0 & 81 & 135 \\ 0 & 0 & 162 \end{bmatrix}$$
$$- \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 9: Obtain the characteristic equation and find the eigen value of the matrix

$$A = \begin{bmatrix} 1 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

The characteristic matrix of $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} = 0$$

$$(8-\lambda)\begin{vmatrix} -3-\lambda & -2 \\ -4 & 1-\lambda \end{vmatrix} + 8\begin{vmatrix} 4 & -2 \\ 3 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 4 & -3-\lambda \\ 3 & -4 \end{vmatrix} = 0$$

Or

$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$ ………… ……. (1)

Or

$$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$

The root of this equation is 1, 2, 3

Hence $(1-\lambda)(\lambda^2 - 5\lambda + 6)$ is characteristic equation and 1, 2, 3 are eigen values of A

Example 10: Show that 0 is a eigen value of matrix A if and only if matrix is singular.

Solution: We know that product of eigen values is determinant of matrix

Let $\lambda_1, \lambda_2, \ldots\ \lambda_n$are eigen values of matrix A.

Then

$$\lambda_1.\lambda_2.\lambda_3\ \ldots\ \lambda_n = |A|$$

Since 0 eigen value is zero so $|A| = 0 \quad \Rightarrow$ A is singular

Conversely, let A is singular $\Rightarrow\ |A| = 0$

The characteristic equation of A is

$$(-1)^n[\lambda^n + \lambda^{n-1}a_n + \lambda^{n-2}a_{n-1} + \ \ldots + \lambda a_1 + a_0] = 0$$

Since $a_0 = |A|$

So $\lambda^n + \lambda^{n-1}a_n + \lambda a_n = 0$ ……………… (1)

0 is the root of equation (1), hence $\lambda = 0$ is eigen value of A.

Example 11: Show that matrix A and $C^{-1}AC$ have the same eigen values.

Solution: Let $B = C^{-1}AC$ then the characteristic matrix of B is

$$(B - \lambda I) = (C^{-1}AC - \lambda I)$$

Since C is invertible so $C^{-1}C = I = CC^{-1}$

So $(B - \lambda I) = (C^{-1}AC - C^{-1}\lambda C) \qquad = C^{-1}(A - \lambda I)C$

Taking determinant both sides

$$|B - \lambda I| = |C^{-1}(A - \lambda I)C| \because \quad |A.B| = |A||B|$$

$$= |C^{-1}||A - \lambda I||C||A^{-1}| = |A|^{-1}$$

$$= \frac{1}{|C|}|A - \lambda I||C| \because \quad |C| \neq 0$$

So

$$|B - \lambda I| = |A - \lambda I|$$

Thus the matrix A and B have same characteristic equation. Hence A and $B = C^{-1}AC$ have same eigen value.

Example 12: Obtain the characteristic equation of matrix $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$ and verify that it is satisfied by A and hence find its inverse.

Solution: We have $|B - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix}$

$$= (1-\lambda)\big((6-\lambda)(4-\lambda) - 0\big) - 2(0-0) + 3(0-0)$$

$$= (1-\lambda)(6-\lambda)(4-\lambda)$$

∵ The characteristic equation of A is

$(1-\lambda)(6-\lambda)(4-\lambda)=0$ ……………….. (1)

Or

$$(6-\lambda-6\lambda+\lambda^2)(4-\lambda)=0$$

$$(6-7\lambda+\lambda^2)(4-\lambda)=0$$

$$24-6\lambda-28\lambda+7\lambda^2+4\lambda^2-\lambda^3=0$$

$$24-34\lambda+11\lambda^2-\lambda^3=0$$

Or

$\lambda^3-11\lambda^2+34\lambda-24=0$ ……………………... (2)

By Cayley – Hamilton theorem, every square matrix satisfies its characteristic equation.

So

$A^3-11A^2+34A-24\,I=0$ ………………………. (3)

Verification of equation (3) we have

$$A=\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{vmatrix}$$

$$A^2=\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{vmatrix}\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{vmatrix}=\begin{vmatrix}1 & 10 & 31\\ 0 & 16 & 50\\ 0 & 0 & 36\end{vmatrix}$$

$$A^3=\begin{vmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{vmatrix}\begin{vmatrix}1 & 10 & 31\\ 0 & 16 & 50\\ 0 & 0 & 36\end{vmatrix}=\begin{vmatrix}1 & 42 & 239\\ 0 & 64 & 380\\ 0 & 0 & 216\end{vmatrix}$$

$$A^3 - 11A^2 + 34A - 24\,I$$
$$= \begin{vmatrix} 1 & 42 & 239 \\ 0 & 64 & 380 \\ 0 & 0 & 216 \end{vmatrix} - \begin{vmatrix} 11 & 110 & 341 \\ 0 & 176 & 550 \\ 0 & 0 & 396 \end{vmatrix} + \begin{vmatrix} 34 & 68 & 102 \\ 0 & 136 & 170 \\ 0 & 0 & 204 \end{vmatrix} - \begin{vmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{vmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

From equation (3)

$$A^3 - 11A^2 + 34A - 24\,I = 0$$

Since all eigen values are non zero so determinant of $A \neq 0$ hence A^{-1} exist

Pre multiplying both sides by A^{-1} in equation (3)

Then

$$A^2 - 11\,A + 34\,I - 24\,A^{-1} = 0 \qquad A^2 - 11\,A + 34\,I = 24\,A^{-1}$$

$$A^{-1} = \frac{1}{24}[A^2 - 11\,A + 34\,I]$$

$$A^{-1} = \frac{1}{24}\left\{\begin{bmatrix} 1 & 10 & 31 \\ 0 & 16 & 50 \\ 0 & 0 & 36 \end{bmatrix} - \begin{bmatrix} 11 & 22 & 33 \\ 0 & 44 & 55 \\ 0 & 0 & 66 \end{bmatrix} + \begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}\right\}$$

$$= \frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

6.6 *CHARACTERISTIC POLYNOMIALS OF DEGREE 2 AND 3*

There are simple formulas for the characteristic polynomials of matrices of order 2 and 3.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then simple characteristic equation of A

$\Longrightarrow \ \lambda^2 - (\text{trace A})\lambda + \det(A) = 0$

If B is of order 3 then $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

The characteristic equation of B is

$$\Longrightarrow \ \lambda^3 - (\text{trace B})\lambda^2 + \text{trace (adj B)}\lambda - \det B = 0$$

If A_{11}, A_{22},A_{33} are the cofactor of element b_{11}, b_{22},b_{33} then characteristic equation of B is

$$\Longrightarrow \ \lambda^3 - (\text{trace B})\lambda^2 + (A_{11} + A_{22} + \ A_{33})\lambda - \det B = 0$$

Example 13: If A is any matrix of order 3 and the Eigen values of A are $1, 2, 3$ then find the characteristic equation of A.

Solution: The characteristic equation of A is

$$\lambda^3 - (\text{trace A})\lambda^2 + \text{trace (adj A)}\lambda - \det A = 0$$

where, λ is eigen value of A

we know that if λ is eigen value of A then

$\frac{|A|}{\lambda}$ is eigen value of (adjA)

So, trace of $A = \text{sum of Eigen values} = 1 + 2 + 3 = 6$

$$\text{det of } A = \text{product of Eigen values} = 1.2.3 = 6$$

$$\text{Eigen values of adj } A = \frac{6}{1}, \frac{6}{2}, \frac{6}{3} = 6, 2, 3$$

So trace of adj $A = 6 + 2 + 3 = 11$

Hence characteristic equation of A is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Example 14: If A is a square matrix of order n and $A^2 = I$ then find the determinant and trace of matrix.

Solution: Let $A = (a_{ij})_{n\times n}$ and $A^2 = I$

And we know that if $A^2 = I$ then possible Eigen value of A are 1 or -1

Let algebraic multiplicity of Eigen values of -1 is k_1 and 1 is k_2

So $k_1 + k_2$ is equal to order of matrix and

$$\frac{1+1+1+\cdots+1}{k_2 \text{ time}} + \frac{(-1)+(-1)+\cdots+(-1)}{k_1 \text{ time}} = \text{trace of A}$$

$$k_2 - k_1 \text{ is trace of matrix A}$$

Determinant of A is product of Eigen values so

$$|A| = \frac{1\,.\,1\,.\,1\,...\,1}{k_2 \text{ time}} + \frac{(-1)\,.\,(-1)\ ...\ (-1)}{k_1 \text{ time}} = (-1)^{k_1}$$

Example 15: Find the trace of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{20}$

Solution: Let $A = B^{20}$ where $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Eigen values of B are 2, 2, and 3

So, eigen values of B^{20} are 2^{20}, 2^{20}, 3^{20}

And we know that trace (B^{20}) is equal to sum of its eigen values

So trace A $= 2^{20} + 2^{20} + 3^{20}$

$$= 2\,.\,2^{20} + 3^{20}$$

Example 16: Let B be a real $n\times 1$vector such that $B^TB = I$ and if $A = I - 2BB^T$ then show that A is involuntary matrix with trace $(n-2)$.

Solution: Let $B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix}_{n*1}$ then $B^T = [x_1, x_2, \dots\ x_n]$

Since $B^T\,B = I$

Now $B^T\,B = [x_1, x_2, \dots\ x_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1^2 + x_2^2 + \ \dots + x_n^2] = [1]_{1*1}$

So $x_1^2 + x_2^2 + \ \dots + x_n^2 = 1$

Since $A = I - 2B\,B^T$

$$A = I - 2\begin{bmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & x_n^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix} - 2\begin{bmatrix} x_1^2 & 0 & 0 & \dots \\ 0 & x_2^2 & 0 & \dots \\ \vdots & & & \\ 0 & 0 & \dots & x_n^2 \end{bmatrix}$$

Trace of $A = (1 - 2x_1^2) + (1 - 2x_2^2) + \ \dots + (1 - 2x_n^2)$

$$= n - 2(x_1^2 + \ \dots + x_n^2) = n - 2$$

Now $A^2 = (I - 2B\,B^T)(I - 2B\,B^T)$

$$A^2 = I$$

6.7 *SUMMARY*

In this unit ,we learned the concept of eigenvalue and eigenvector, additionally learned about algebraic and geometric multiplicity of eigenvalue , which helps us more to find the eigenvalue of any matrix ,the concept of eigenvalue is very important to understand linear algebra very well. To cheak your progress by solving all question given below,

6.8 *GLOSSARY*

Spectrum of matrix: Set of all eigenvalues

Geometric multiplicity of eigenvalue: The dimension of the eigen space of eigenvalue

6.9 *SELF ASSESMENT QUESTIONS*

6.9.1 Multiple choice Questions:

1. If 0 is eigen value of matrix A then $\det A$ is.

(a) 1 (b) 2

(c) 3 (d) 0

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ then if $\lambda_1, \lambda_2, \lambda_3$are eigen values of A then $\lambda_1 + \lambda_2 + \lambda_3$ is equal to

(a) 1 (b) 14

(c) 15 (d) 0

3. Eigen values of idempotent matrix are

(a) 1or -1 (b) 0 or -1

(c) 0 or 1 (d) 0 only

4. If P is Algebraic multiplicity and Q is geometric multiplicity then the relation between P and Q is

(a) $P \leq Q$ (b) $P = Q$

(c) $P < Q$ (d) $P \geq Q$

5. Eigen values of matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$is

(a) 1, 2, 3 (b) 3, 0

(c) 1, 3 (d) 4, 0

6. Trace of any idempotent matrix A of order n is

(a) Sum of its Eigen values (b) n

(c) Rank A (d) Product of Eigen values

7. If λ is eigen value of A then eigen value of 3A is

(a) 3λ (b) 2λ

(c) λ (d) 0

8. If $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$ is characteristic equation of matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ then

$P + Q + R$ is equal to

(a) 2 (b) 3

(c) 4 (d) 5

9. If A and B are two square matrices of the same order then if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A B then eigen values of BA is

(a) $\lambda_1, \lambda_2, \ldots \lambda_n$ (b) $\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \ldots \frac{\lambda_n}{2}$

(c) $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$ (d) $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots \sqrt{\lambda_n}$

10. If A is any orthogonal matrix then eigen value of A is

(a) $1, -1$ (b) $i, -i$

(c) $2, 2i$ (d) (a)and (b)both

ANSWERS:

1. (d) 2. (c) 3. (c) 4. (d) 5. (b)

6. (c)	7. (a)	8. (b)	9. (a)	10. (d)

6.9.2 Fill in the blanks:

1. Eigen values of upper triangular matrix is …………

2. Trace of null matrix is …………

3. Eigen value of skew Hermitian matrix are ……….. or ………..

4. If $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ then $|A^2 + 5\,I|$ is ………..

5. If A is idempotent matrix with rank 3 then trace of A is ………..

6. If $A = \left(a_{ij}\right)_{n*n}$: $a_{ij} = I \;\forall\; i, j$ if λ is non zero eigen value then $\lambda =$ …………..

7. If A is involuntary matrix of order n and k is algebraic multiplicity of -1 then trace of A…….

8. If S is the set of all matrices and S is defined A $S = \left\{ A \mid A = \begin{bmatrix} 1 & a & b \\ t & 0 & c \\ c & d & 1 \end{bmatrix} s.t. A^2 = I \right\}$ then cardinality of S is ………..

9. If A be a $n \times n$ matrix which is both Hermitian and Unitary than trace of $A^2$is ………

10. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then characteristic polynomial is …………..

ANSWERS:

1. Principal diagonal element	2. 0	3. Zero or purely imaginary	4. 54	5. 3

6. n 7. $n-2k$ 8. Null set 9. n 10. $(x-1)^3$

6.9.3 True and False questions:

Write T for true and F for false statement

1. If $1, -2, 3$ are the eigen values of matrix A of order 3 then

$A^{-1} = \frac{1}{6}(5\,\mathrm{I} + 2\mathrm{A} - \mathrm{A}^2)$T/F

2. If A is an idempotent matrix of order n then $\mathrm{Rank}\,(\mathrm{A}) + \mathrm{Rank}(\mathrm{A}-\mathrm{I}) = \mathrm{n}$ T/F

3. Eigen values of identity matrix is 0 T/F

4. If A is invertible idempotent matrix then eigen values of A are both 0 and 1 T/F

5. The possible eigen values of nilpotent matrices are zero only T/F

ANSWERS:

1. T 2. T 3. F 4. F 5. T

6.10 *REFERENCE*

1. Linear Algebra, Vivek.sahai & Vikas Bist :Narosa publishing House

2. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

3. Schaum’s out line (Linear Algebra)

6.11 *SUGGESTED READINGS*

1. Matrices .A.R.Vasishtha &A.K.Vasishtha :Krishna Parakashan Media

2. Schaum's out line (Linear Algebra)

6.12 *TERMINAL QUESTIONS*

6.12.1 Short answer type questions:

1. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then verify $A^2 - 5A + 7\,I = 0$

2. Find the all eigen value of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

3. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ hence or otherwise evaluate A^{-1}

4. Show that the characteristic roots of a triangular matrix are just the principal diagonal element

5. Determine the eigen vector of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

ANSWERS: **2.** 3, 6, 2 **3.** $A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$

6.12.2 Long answer type questions:

1. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of the n-square matrix A and K is a Scalar then prove that eigen values of $A - kI$ are λ_{1-K}, $\lambda_{2-K}, \ldots \lambda_{n-K}$

2. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}

3. Verify the Cayley-Hamilton theorem for a matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$

4. Show that if all eigen values of idempotent matrix are zero then matrix is invertible

5. Show that the matrix $A = \begin{bmatrix} 1+x^2 & 1 \\ 2 & 2x \end{bmatrix}$ has at least one eigen value is 0 for some real number x

BLOCK III: TRIGNOMETRICAL AND HYPERBOLIC FUNCTIONS

UNIT 7: EXPONENTIAL AND TRIGONOMETRICAL VARIABLES

Contents:

7.1 *OBJECTIVE*

After reading this unit you will be able to:

- Understand complex number and their geometric representation
- Do addition, substraction, multiplication and division of complex numbers
- Convert a complex number into modulus amplitude form (polar form)
- Find out square root of a complex number.

7.2 INTRODUCTION

We know that the square of a real number is always non-negative e.g. $(5)^2 = 25$ and $(-5)^2 = 25$. Therefore, square root of 25 is ± 5. What about the square root of a negative number? It is clear that a negative number cannot have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for positive square root of – 1 that is $i = \sqrt{-1}$.

7.3 COMPLEX NUMBER

A number of the form $a+ib$, is called a complex number where a, b are real numbers and $i = \sqrt{-1}$. If $z = a+ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as $\text{Re}(z) = a$, $\text{Im}(z) = b$.

If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and 3i is a purely imaginary number because its real part is zero.

7.4 REPRESENTATION OF COMPLEX NUMBER IN ARGAND PLANE

A complex number $z = x+iy$ written as an ordered pair (x, y) can be represented by a point P whose Cartesian coordinates are (x, y) referred to axes $X^{'}$OX and $Y^{'}$OY, usually called the real and the imaginary axes. The plane of $X^{'}$OX and $Y^{'}$OY is called the Argand diagram or the complex plane.

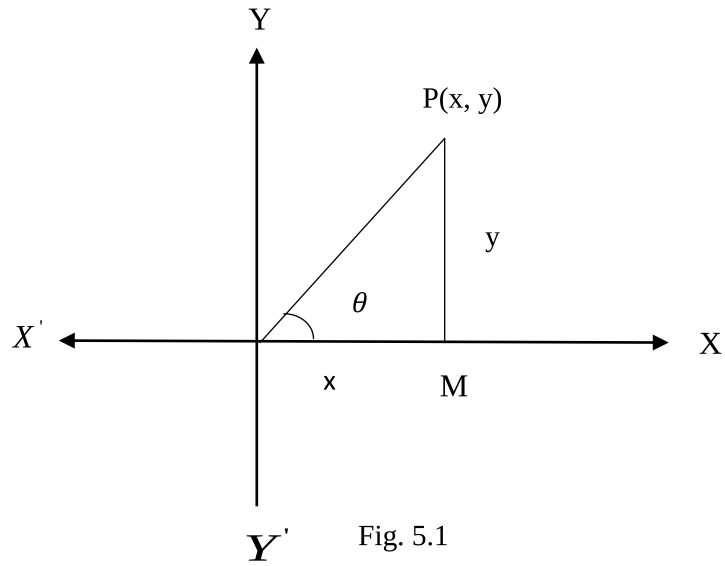


Fig. 5.1

7.5 ADDITION OF COMPLEX NUMBERS

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then

$$z_1 + z_2 = (a+c) + i(b+d)$$

Procedure: In addition of complex numbers we add real parts with real parts and imaginary parts.

7.6 ADDITION OF COMPLEX NUMBERS BY GEOMETRY

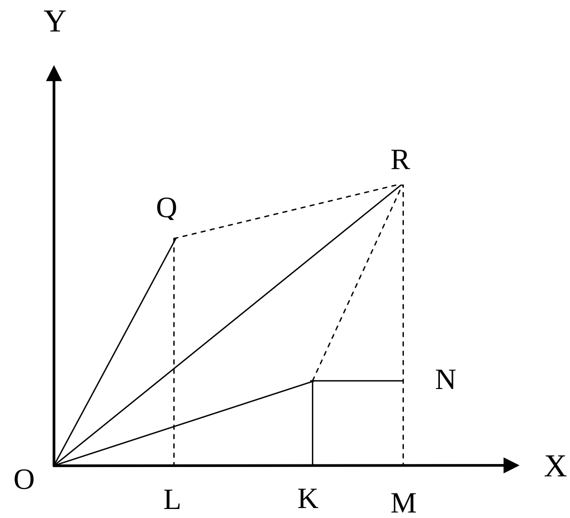


Fig. 5.2

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers represented by the points P and Q on the Argand diagram.

Suppose the parallelogram $OPRQ$. Draw the PK, RM, QL, perpendiculars on OX. Also draw $PN \perp RM$.

Since $\Delta OLQ \cong \Delta PNR$ so, $OQ = PR, KM = OL, NR = LQ$

$$OM = OK + KM = OK + OL = a + c$$
$$RM = MN + NR = KP + LQ = b + d$$

Hence the coordinates of R are $(a+c, b+d)$ and it represented the complex numbers.

$(a+c) + i(b+d) = (a+id) + (c+id) = z_1 + z_2$

Thus the sum of two complex numbers is represented by the extremity of the diagonal of the parallelogram formed by $OP(z_1)$ and $OQ(z_2)$ as adjacent sides.

$|z_1 + z_2| = OR$ and $amp(z_1 + z_2) = \angle ROM$.

Properties:

1. As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.

2. Addition of complex numbers is commutative, i.e., $z_1 + z_2 = z_2 + z_1$

3. Addition of complex numbers is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
4. For any complex number $z = a + ib$, there exist a complex number $(0+i0)$ *i.e.*, 0 such that $z + 0 = 0 + z$, known as identity element for addition.
5. For any complex number $z = a + ib$, there always exists a number $-z = -a - ib$, such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z .

7.7 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a + ib$ and $z_2 = c + id$, be two complex numbers. Then

$$\begin{aligned} z_1 \times z_2 &= (a+ib)\times(c+id) \\ &= ac + iad + ibc + i^2bd \\ &= ac + i(ad+bc) + (-1)bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

7.8 MULTIPLICATION OF COMPLEX NUMBERS (POLAR FORM)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$a = r_1\cos\theta_1, \qquad b = r_1\sin\theta_1$$

$$c = r_2\cos\theta_2, \qquad d = r_2\sin\theta_2$$

$$z_1 = a + ib = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = c + id = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 \times z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

7.9 MULTIPLICATION OF COMPLEX NUMBERS (GEOMETRICAL REPRESENTATION)

Let P and Q represent the complex numbers

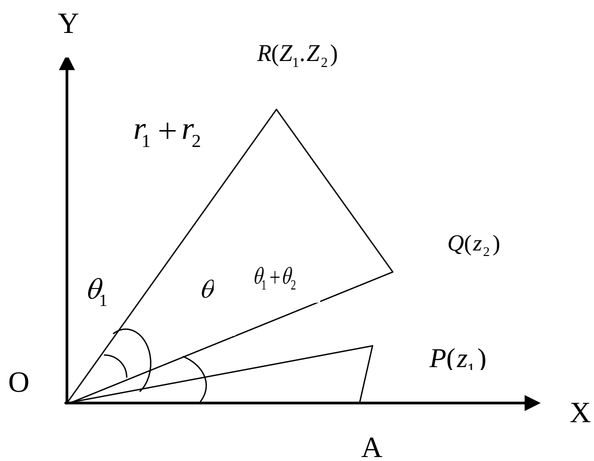


Fig. 5.3

$z_1 = a+ib = r_1(\cos\theta_1 + i\sin\theta_1)$

$$z_2 = c+id = r_2(\cos\theta_2 + i\sin\theta_2)$$

Cut off $OA = 1$ along x-axis. Construct ΔORQ on OQ similar to ΔOAP

So that $\quad \dfrac{OR}{OP} = \dfrac{OQ}{OA} \quad \Rightarrow \quad \dfrac{OR}{OP} = \dfrac{OQ}{1}$

$\Rightarrow \quad OR = OP.OQ = r_1.r_2$

$\Rightarrow \quad \angle XOR = \angle AOQ + \angle QOR = \theta_1 + \theta_2$

Hence the product of two complex numbers $z_1.z_2$ is represented by the point R, such that

(a) $|z_1.z_2| = |z_1|.|z_2|$

(b) $Arg(z_1, z_2) = Arg(z_1) + Arg(z_2)$

Properties:

1. As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.

2. Multiplication of complex numbers is commutative, *i.e.* $z_1 \times z_2 = z_2 \times z_1$

3. Multiplication of complex numbers is associative, *i.e.* $(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$

4. For any complex number $z = a + ib$, there exists a complex number $(1 + 0i)$ *i.e.*1 such that $z \times 1 = 1 \times z = z$ is known as identity element for multiplication.

5. For any non zero complex number $z = a + ib$, there exists a complex number $\frac{1}{z}$ such that $z \times \frac{1}{z} = \frac{1}{z} \times z = 1$ i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.

6. For any three complex numbers z_1, z_2 and z_3

$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$

and $(z_1 + z_2) \times z_3 = z_1 \times z_3 + z_2 \times z_3$

i.e., for complex numbers multiplication is distributive over addition.

7.10 EQUALITY OF COMPLEX NUMBERS

Two complex numbers $a + ib$ and $c + id$ are said to be equal if

$a + ib = c + id$

$\Rightarrow a - c = i(d - b)$

$\Rightarrow (a-c)^2 = -(d-b)^2$ or $(a-c)^2 + (d-b)^2 = 0$

Here sum of two positive numbers is zero. This is only possible if each number is zero.

That is

$(a-c)^2 = 0 \quad \Rightarrow \quad a = c$

and $(d-b)^2 = 0 \quad \Rightarrow \quad b = d$.

7.11 SUBTRACTION

$z_1 - z_2 = [(a+ib) - (c+id)] = (a-c) + i(b-d)$

That is, in subtraction of complex numbers we subtract real part from real part and imaginary part from imaginary part.

For example, suppose two complex numbers $z_1 = \frac{3}{4} - \frac{7}{3}i$ and $z_2 = -\frac{5}{3} + \frac{11}{5}i$ then

$$\text{Subtract } z_2 - z_1 = \left(\frac{-5}{3} + \frac{11}{5}i\right) - \left(\frac{3}{4} - \frac{7}{3}i\right)$$

$$= \left(\frac{-5}{3} - \frac{3}{4}\right) + \left(\frac{11}{5} + \frac{7}{3}\right)i$$

$$= \left(\frac{-20-9}{12}\right) + \left(\frac{33+35}{15}\right)i = \left(\frac{-29}{12}\right) + \left(\frac{68}{15}\right)i$$

7.12 SUBTRACTION OF A COMPLEX NUMBER BY GEOMETRY

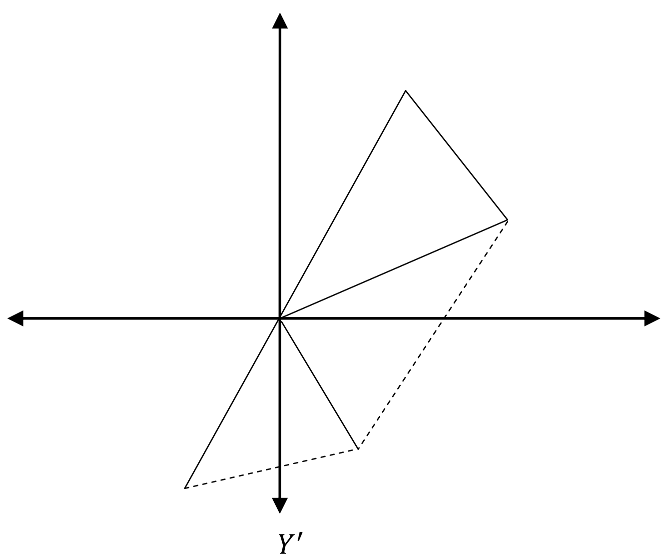


Fig. 5.4

Let P and Q represent the complex numbers

$$z_1 = a + ib \qquad \text{and} \qquad z_2 = c + id$$

Then $z_1 - z_2 = z_1 + (-z_2)$, $z_1 - z_2$ means the addition of z_1 and $-z_2$. $-z_2$ is represented by OQ' formed by producing OQ to OQ' such that $OQ = OQ'$.

Complete the parallelogram $OPRQ'$, then the sum of z_1 and $-z_2$ represented by OR.

7.13 CONJUGATE OF A COMPLEX NUMBER

Let $z = a + ib$, be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, that is, $\bar{z} = a - ib$.

In the other hand, two complex numbers which differ only in the sign of imaginary parts are called conjugate of each other.

Suppose a pair of complex numbers $z = a + ib$ and $\bar{z} = a - ib$ are said to be conjugate of each other.

Sum $= (a+ib)+(a-ib) = 2a$

(Real)

Product $= (a+ib)\times(a-ib) = a^2 + b^2$

(Real)

7.14 POWER OF i

Some time we need various power of i.

We know that $i = \sqrt{-1}$.

On squaring both sides, we get

$$i^2 = -1$$

Multiplying by i both sides, we get

$$i^3 = -i$$

Again $\quad i^4 = (i^3).(i) = (-i).(i) = -(i^2) = -(-1) = 1$

$$i^5 = (i^4).(i) = (1).(i) = i$$

$$i^6 = (i^5).(i) = (i).(i) = -i^2 = -1$$

$$i^7 = (i^6).(i) = (-1).(i) = -i$$

$$i^8 = (i^7).(i) = (-i).(i) = -(i^2) = -(-1) = 1.$$

7.15 i (IOTA) AS AN OPERATOR

Multiplication of a complex number by i

Let $z = a + ib = r(\cos\theta + i\sin\theta)$

$$i = 0 + i.1 = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$i.z = r(\cos\theta + i\sin\theta).\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right].$$

Hence a complex number multiplied by i give the rotation to the complex number by $\frac{\pi}{2}$ is anticlockwise direction without change in magnitude.

7.16 DIVISION OF A COMPLEX NUMBER

To divide a complex number $z_1 = a + ib$ by $z_2 = c + id$, we write it as

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

To simplify further, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{ac - iad + ibc - i^2bd}{(c)^2 - (id)^2}$$

$$= \frac{ac - i(ad - bc) + bd}{c^2 - i^2d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i$$

For example, suppose two complex numbers are $1+i$ by $3+4i$, then

$$\text{Division } = \frac{1+i}{3+4i} = \frac{1+i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{3 - 4i + 3i - 4i^2}{9 - 16i^2}$$

$$= \frac{3 - i + 4}{9 + 16} = \frac{7}{25} - \frac{1}{25}i\,.$$

7.17 DIVISION OF A COMPLEX NUMBER BY ALGEBRA

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}[(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

The modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument of the quotient is the difference of their arguments.

7.18 DIVISION OF A COMPLEX NUMBER BY GEOMETRY

Let P and Q represent the complex numbers

$$z_1 = a + ib$$
$$= r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = c + id$$
$$= r_2(\cos\theta_2 + i\sin\theta_2)$$

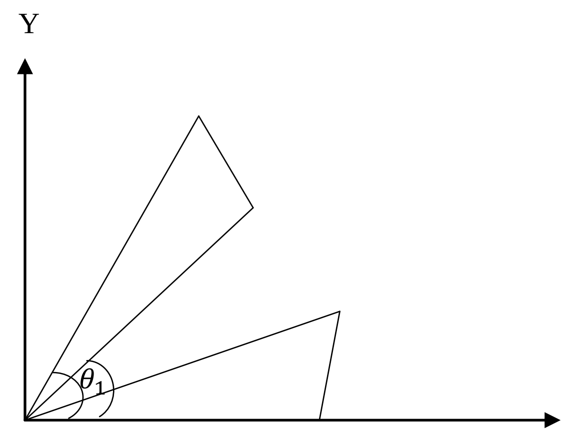


Fig. 7.5

Cut off $OA = 1$ along x-axis, construct ΔOAR on OA similar to ΔOQP

So that $\dfrac{OR}{OA} = \dfrac{OP}{OQ} \quad \Rightarrow \quad \dfrac{OR}{1} = \dfrac{OP}{OQ}$

$\Rightarrow \qquad OR = \dfrac{OP}{OQ}. = \dfrac{r_1.}{r_2}$

$\Rightarrow \qquad \angle AOR = \angle QOP = \angle AOP - \angle AOQ = \theta_1 - \theta_2$

$\therefore$ R represents the number $\dfrac{r_1}{r_2}[(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

Hence the complex number $\dfrac{z_1}{z_2}$ is represented by the point R.

Example 1. Subtract the two complex numbers

$$z_1 = \frac{1}{2} + \frac{3}{5}i \qquad \text{and} \qquad z_2 = \frac{3}{5} - \frac{5}{7}i$$

Solution: Given that $z_1 = \dfrac{1}{2} + \dfrac{3}{5}i$ and $z_2 = \dfrac{3}{5} - \dfrac{5}{7}i$

We have $z_1 - z_2 = \left(\dfrac{1}{2} + \dfrac{3}{5}i\right) - \left(\dfrac{3}{5} - \dfrac{5}{7}i\right) = \left(\dfrac{1}{2} - \dfrac{3}{5}\right) + \left(\dfrac{3}{5} + \dfrac{5}{7}\right)i$

$$= \left(\frac{5-6}{10}\right) + \left(\frac{21+25}{35}\right)i = \frac{-1}{10} + \frac{46}{35}i$$

Example 2. Add the following complex numbers

$$z_1 = 3 + \frac{2}{5}i, \quad z_2 = 6 - \frac{4}{7}i, \quad z_3 = \frac{5}{2} - \frac{7}{3}i, \; z_4 = \frac{-10}{3} - 4i.$$

Solution: We have

$$z_1 + z_2 + z_3 + z_4 = \left(3 + \frac{2}{5}i\right) + \left(6 - \frac{4}{7}i\right) + \left(\frac{5}{2} - \frac{7}{3}i\right) + \left(\frac{-10}{3} - 4i\right)$$

$$= \left(3 + 6 + \frac{5}{2} - \frac{10}{3}\right) + \left(\frac{2}{5} - \frac{4}{7} - \frac{7}{3} - 4\right)i$$

$$= \frac{49}{6} - \frac{683}{105}i$$

Example 3. Simplify the following complex numbers

(a) i^{49} (b) i^{103}

Solution: (a) We divide 49 by 4 and we have

$49 = 4 \times 12 + 1$

$\Rightarrow \qquad i^{49} = i^{4\times 12+1}$

$= (i^4)^{12}.(i) = i$

$i^{49} = i$

(b) We divide by 103 by 4 and we have

$103 = 4 \times 25 + 3$

$\Rightarrow \quad i^{103} = i^{4\times25+3}$

$= (i^4)^{25}.(i^3) = (i)^{25}.(-i) = -i$

$i^{103} = -i.$

Example 4. Find the multiply between the two complex numbers

$z_1 = 3 + 9i$ and $z_2 = 5 - 7i$

Solution: Given that $z_1 = 3 + 9i, \quad z_2 = 5 - 7i$, we have

$$z_1 \times z_2 = (3+9i) \times (5-7i) = 15 - 21i + 45i - 63i^2$$

$$= 15 - 21i + 45i - 63(-1) = 15 - 21i + 45i + 63$$

$$= 78 + 24i$$

Example 5. Find the conjugate of the complex number $9 + 2i$.

Solution: Let $z = 9 + 2i$

To obtain the conjugate number of $z = 9 + 2i$, then change the sign of imaginary parts.

Conjugate of $z = \bar{z} = 9 - 2i.$

Example 6. Divide $z_1 = 2 + i$ by $z_2 = 3 + 5i$.

Solution: Given that $z_1 = 2 + i,\ z_2 = 3 + 5i.$

$$\frac{z_1}{z_2} = \frac{2+i}{3+5i}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{2+i}{3+5i} \times \frac{3-5i}{3-5i} = \frac{6-10i+3i-5i^2}{9-25i^2}$$

$$= \frac{6-10i+3i+5}{9+25} = \frac{11-7i}{34} = \frac{11}{34} - \frac{7}{34}i$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{11}{34} - \frac{7}{34}i$$

Example 7. Express $\dfrac{(5+i).(3-i)}{(3+4i).(2-3i)}$ in the form of $a + ib$.

Solution: We have

$$\frac{(5+i).(3-i)}{(3+4i).(2-3i)} = \frac{15-5i+3i-i^2}{6-9i+8i-12i^2} = \frac{15-5i+3i+1}{6-9i+8i+12}$$

$$= \frac{16-2i}{18-i} = \frac{16-2i}{18-i} \times \frac{18+i}{18+i}$$

$$= \frac{288+16i-36i-2i^2}{18^2-i^2} = \frac{288+16i-36i+2}{324+1}$$

$$= \frac{290 - 20i}{325} = \frac{290}{325} - \frac{20}{325} i = \frac{58}{65} - \frac{4}{65} i$$

Example 8. If $a = \cos\alpha + i\sin\alpha$, prove that $1 + a + a^2 = (1 + 2\cos\alpha)(\cos\alpha + i\sin\alpha)$.

Solution: Here we have $a = \cos\alpha + i\sin\alpha$

$$1 + a + a^2 = 1 + (\cos\alpha + i\sin\alpha) + (\cos\alpha + i\sin\alpha)^2$$

$$= 1 + \cos\alpha + i\sin\alpha + \cos^2\alpha + 2i\sin\alpha\cos\alpha - \sin^2\alpha$$

$$= (\cos\alpha + i\sin\alpha) + \cos^2\alpha + 2i\sin\alpha\cos\alpha + (1 - \sin^2\alpha)$$

$$= (\cos\alpha + i\sin\alpha) + \cos^2\alpha + 2i\sin\alpha\cos\alpha + \cos^2\alpha$$

$$= (\cos\alpha + i\sin\alpha) + 2\cos^2\alpha + 2i\sin\alpha\cos\alpha$$

$$= (\cos\alpha + i\sin\alpha) + 2\cos\alpha(\cos\alpha + i\sin\alpha)$$

$$= (1 + 2\cos\alpha)(\cos\alpha + i\sin\alpha)$$

Example 9. Solve for θ such that the expression $\dfrac{4 + 5i\sin\theta}{2 - i\sin\theta}$ is imaginary.

Solution: We have

$$\frac{4 + 5i\sin\theta}{2 - i\sin\theta} = \frac{4 + 5i\sin\theta}{2 - i\sin\theta} \times \frac{2 + i\sin\theta}{2 + i\sin\theta}$$

$$= \frac{8 + 4i\sin\theta + 10i\sin\theta + 5i^2\sin^2\theta}{2^2 - i^2\sin^2\theta}$$

$$= \frac{8 - 5\sin^2\theta + 14i\sin\theta}{4 + \sin^2\theta}$$

If $8 - 5\sin^2\theta = 0$, then $\dfrac{8 - 5\sin^2\theta + 14i\sin\theta}{4 + \sin^2\theta}$ is purely imaginary.

$$\sin^2\theta = \frac{8}{5} \quad \Rightarrow \quad \sin\theta = \sqrt{\frac{8}{5}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\sqrt{\frac{8}{5}}\right)$$

Example 10. If $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, prove that

$$\frac{a + ib}{1 + c} = \frac{1 + iz}{1 - iz}.$$

Solution: Given that $b + ic = (1 + a)z$,

$$\Rightarrow \quad z = \frac{b + ic}{1 + a}$$

$$\Rightarrow \qquad \frac{1+iz}{1-iz}=\frac{1+i\dfrac{b+ic}{1+a}}{1-i\dfrac{b+ic}{1+a}}=\frac{1+a-c+ib}{1+a+c-ib}$$

$$=\frac{1+a-c+ib}{1+a+c-ib}\times\frac{1+a+c+ib}{1+a+c+ib}$$

$$=\frac{(1+a+ib)^2-c^2}{(1+a+c)^2+b^2}$$

$$=\frac{1+a^2-b^2+2a+2ib+2iab-c^2}{1+a^2+c^2+2a+2c+2ac+b^2}$$

$$=\frac{1+a^2-b^2-c^2+2a+2ib+2iab}{1+(a^2+b^2+c^2)+2a+2c+2ac}$$

Substituting $a^2+b^2+c^2=1$ in the above expression, we get

$$\frac{1+iz}{1-iz}=\frac{1+a^2-(1-a^2)+2a+2ib+2iab}{1+1+2a+2c+2ac}$$

$$=\frac{2(a^2+a+ib+iab)}{2(1+a+c+ac)}=\frac{2(1+a)(a+ib)}{2(1+a)(1+c)}=\frac{a+ib}{1+c}$$

$$\therefore \qquad \frac{1+iz}{1-iz}=\frac{a+ib}{1+c}.$$

Example 11. If $z=\cos\alpha+i\sin\alpha$, prove that

(a) $\dfrac{2}{1+z}=1-i\tan\dfrac{\alpha}{2}$ **(b)** $\dfrac{1+z}{1-z}=i\cot\dfrac{\alpha}{2}$

Solution: Given that $z=\cos\alpha+i\sin\alpha$, we have

(a) $$\frac{2}{1+z}=\frac{2}{1+(\cos\alpha+i\sin\alpha)}=\frac{2}{(1+\cos\alpha)+i\sin\alpha}$$

$$=\frac{2}{(1+\cos\alpha)+i\sin\alpha}\times\frac{(1+\cos\alpha)-i\sin\alpha}{(1+\cos\alpha)-i\sin\alpha}$$

$$=\frac{2[(1+\cos\alpha)-i\sin\alpha]}{(1+\cos\alpha)^2+\sin^2\alpha}=\frac{2[(1+\cos\alpha)-i\sin\alpha]}{1+\cos^2\alpha+2\cos\alpha+\sin^2\alpha}$$

$$=\frac{2[(1+\cos\alpha)-i\sin\alpha]}{1+1+2\cos\alpha}=\frac{2[(1+\cos\alpha)-i\sin\alpha]}{2+2\cos\alpha}$$

$$=\frac{(1+\cos\alpha)-i\sin\alpha}{1+\cos\alpha}=\frac{1+\cos\alpha}{1+\cos\alpha}-\frac{i\sin\alpha}{1+\cos\alpha}$$

$$=1-i\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+2\cos^2\frac{\alpha}{2}-1}=1-i\tan\frac{\alpha}{2}$$

$\therefore \quad \dfrac{2}{1+z}=1-i\tan\dfrac{\alpha}{2}$

(b) $\dfrac{1+z}{1-z}=\dfrac{(1+\cos\alpha)+i\sin\alpha}{(1+\cos\alpha)-i\sin\alpha}$

$$=\frac{2\cos^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}-2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$=\frac{\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2}\right)}$$

$$=\cot\frac{\alpha}{2}\frac{(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2})}{(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2})}\times\frac{(\sin\frac{\alpha}{2}+i\cos\frac{\alpha}{2})}{(\sin\frac{\alpha}{2}+i\cos\frac{\alpha}{2})}$$

$$=\cot\frac{\alpha}{2}\frac{(\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}+i\cos^2\frac{\alpha}{2}+i\sin^2\frac{\alpha}{2}-\sin\frac{\alpha}{2}\cos\frac{\alpha}{2})}{(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2})}$$

$$=\cot\frac{\alpha}{2}\left\{i\frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}\right\}=i\cot\frac{\alpha}{2}$$

$\therefore \quad \dfrac{1+z}{1-z}=i\cot\dfrac{\alpha}{2}$

Example 12. If $a=\cos\alpha+i\sin\alpha, \quad b=\cos\beta+i\sin\beta,$ prove that

$$\frac{a-b}{a+b}=i\tan\left(\frac{\alpha-\beta}{2}\right)$$

Solution: We have

$$\frac{a-b}{a+b}=\frac{(\cos\alpha+i\sin\alpha)-(\cos\beta+i\sin\beta)}{(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)}$$

$$= \frac{(\cos\alpha - \cos\beta) + i(\sin\alpha - \sin\beta)}{(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)}$$

$$= \frac{\left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + 2i\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]}{\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2i\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]}$$

$$= \frac{2i\sin\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]}{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]}$$

$$= i\tan\left(\frac{\alpha-\beta}{2}\right)$$

$$\therefore \quad \frac{a-b}{a+b} = i\tan\left(\frac{\alpha-\beta}{2}\right)$$

Example 13. If $a + ib = \dfrac{3}{2+\cos\theta + i\sin\theta}$, prove that

$$(a-1)(a-3) + b^2 = 0.$$

Solution: We have

$$a + ib = \frac{3}{2+\cos\theta + i\sin\theta}$$

…………(1)

So that
$$a - ib = \frac{3}{2+\cos\theta - i\sin\theta}$$

…………(2)

Multiplying (1) and (2), we get

$$a^2 + b^2 = \frac{9}{5+4\cos\theta}$$

…………..(3)

Adding (1) and (2), we get

$$2a = \frac{6[2+\cos\theta]}{5+4\cos\theta}$$

Now $(a-1)(a-3) + b^2 = a^2 - 4a + 3 + b^2$

$= a^2 + b^2 - 2\times 2a + 3$

$$= \frac{9}{5+4\cos\theta} - \frac{2\times 6(2+\cos\theta)}{5+4\cos\theta} + 3 = 0$$

$$(a-1)(a-3) + b^2 = 0$$

Exercise 1

1. Express $\frac{(6+i).(2-i)}{(4+3i).(1-2i)}$ in the form of $a+ib$.

Ans.: $\frac{6}{5}+\frac{1}{5}i$

2. Solve for θ such that the expression $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is imaginary.

Ans.: $\theta=\frac{\pi}{3}$

3. Find the complex conjugate of $\frac{2+3i}{1-i}$ **Ans.:** $-\frac{1}{2}-\frac{5}{2}i$

4. If $z=1+i$, find (a) z^2 (b) $\frac{1}{z}$ and plot them on the Argand diagram.

Ans.:(a) $2i$ **(b)** $\frac{1}{2}-\frac{i}{2}$

5. If $a+ib=\frac{1}{x+iy}$, prove that, $(a^2+b^2)(x^2+y^2)=1$.

6. Find the value of $x^2-6x+13$, when $x=3+2i$.

Ans.: 0

7. If $\alpha+i\beta=\frac{1}{a+ib}$, prove that $(\alpha^2+\beta^2)(a^2+b^2)=1$.

8. If $\frac{1}{\alpha+i\beta}+\frac{1}{a+ib}=1$, where α,β,a,b are real, express b in terms of α,β.

Ans.: $\frac{-\beta}{\alpha^2+\beta^2-2\alpha+1}$

9. Subtract $z_1=\frac{3}{4}-\frac{7}{3}i$ from $z_2=\frac{-5}{3}+\frac{11}{5}i$.

Ans.: $\frac{-29}{12}+\frac{68}{15}i$

10. Multiply (3+4i) by (7-3i) **Ans.:** $33+19i$

11. Divide (1+i) by (3+4i)

Ans.: $\frac{7}{25}-\frac{1}{25}i$

12. Express the following in the form $a+ib$, where a and b are real:

(a) $\frac{2-3i}{4-i}$ **Ans.:** $\frac{11}{17}-\frac{10}{17}i$ **(b)** $\frac{(3+4i)(2+i)}{1+i}$

Ans.: $\frac{13}{2}-\frac{9}{2}i$

(c) $\frac{(1+2i)^2}{(1+i)(2-i)}$ **Ans.:** $-\frac{7}{2}+\frac{1}{2}i$. (d) $\frac{1}{(1-2i)(2+3i)}$

Ans.: $\frac{8}{65}-\frac{1}{65}i$

13. If $(x+iy)^{\frac{1}{3}}=a+ib,$ prove that $4(a^2-b^2)=\frac{x}{a}+\frac{y}{b}$.

14. If $(x+iy)^3=u+iv,$ prove that $\frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$.

15. Find the values of a and $b,$ if $\frac{(1+i)a-2i}{3+i}+\frac{(2-3i)b+i}{3-i}=i.$ **Ans.:** $a=3,\ b=-1$

16. If $a+ib=\frac{(x+i)^2}{2x^2+1}$, prove that $a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$.

7.19 MODULUS OF A COMPLEX NUMBER

Let $z=a+ib$ be a complex number.

Putting $a=r\cos\theta$ and $b=r\sin\theta$ so that $r=\sqrt{a^2+b^2}$ then

$$\cos\theta=\frac{a}{\sqrt{a^2+b^2}},\ \sin\theta=\frac{b}{\sqrt{a^2+b^2}}$$

the positive value of the root being taken.

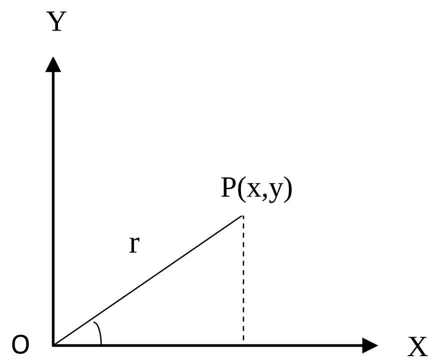


Fig. 5.6

Then r is said to be modulus or absolute value of the complex number $a+ib$ and is denoted by $|a+ib|$. Modulus of z is denoted by $|z|$ and if

$|z| = |a+ib|$

$\Rightarrow |z| = \sqrt{a^2+b^2}$

Let $z = a+ib$ be a complex number then its magnitude is defined by the real number $\sqrt{a^2+b^2}$.

In the set of complex numbers $z_1 > z_2$ or $z_1 < z_2$ are meaningless but

$$|z_1| > |z_2| \text{ or} |z_1| < |z_2|$$

are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

7.20 ARGUMENT OF A COMPLEX NUMBER

If $z = a+ib$ then angle θ given by $\tan\theta = \dfrac{b}{a}$ is said to be the argument or amplitude of the complex number z and is denoted by $\arg(z)$ or $amp(z)$. In case of $a = 0$ (where $b \neq 0$), $\arg(z) = +\pi/2$ or $-\pi/2$ depending upon $b > 0$ or $b < 0$ and the complex number is called purely imaginary. If $b = 0$ (where $a \neq 0$), then $\arg(z) = 0$ or π depending upon $a > 0$ or $a < 0$ and the complex number is called purely real. The argument of the complex number 0 is not defined. It means the value of θ lying in the range $-\pi < \theta \leq \pi$ is said to be the principal value of the argument. The principal value of θ is written either between 0 and π or 0 and $-\pi$.

Example 1. Express the following complex numbers in the modulus-amplitude form:

(i) $1-i$ **(ii)** $-\sqrt{3}+i$ **(iii)** $\dfrac{(1+i)(2-i)}{(3+i)}$

Solution:(i) Let $1-i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r\cos\theta = 1 \quad$ and $\quad r\sin\theta = -1$

Squaring and adding, we get

$$r = \sqrt{1+1} = \sqrt{2}$$

Again dividing, we get

$$\tan\theta = -1 = \tan\frac{7\pi}{4} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{4}$$

$$1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

(ii) Let $-\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r\cos\theta = -\sqrt{3}$ and $\quad r\sin\theta = 1$

Squaring and adding, we get

$$r = \sqrt{3+1} = \sqrt{4} = 2$$

Again dividing, we get

$$\tan\theta = \frac{-1}{\sqrt{3}} = \tan\frac{5\pi}{6} \qquad \Rightarrow \qquad \theta = \frac{5\pi}{6}$$

$$-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

(iii) We have $\dfrac{(1+i)(2-i)}{(3+i)} = \dfrac{2-i+2i-i^2}{3+i}$.

$$= \frac{3+i}{3+i}$$

$$= 1 = i(\cos 0 + i\sin 0)$$

Example 2. Find the modulus and principal argument of the complex number:

$$1+\cos\alpha + i\sin\alpha \qquad \left(0 < \alpha < \frac{\pi}{2}\right)$$

Solution: Let $1+\cos\alpha + i\sin\alpha = r(\cos\theta + i\sin\theta)$

Equating real and imaginary part, we get

$$1+\cos\alpha = r\cos\theta$$

..........................(1)

$$\sin\alpha = r\sin\theta$$
……………………..(2)

Squaring and adding (1) and (2), we get

$$r^2(\cos^2\theta + \sin^2\theta) = (1+\cos\alpha)^2 + (\sin\alpha)^2$$

$$r^2 = 1 + \cos^2\alpha + 2\cos\alpha + \sin^2\alpha$$

$$r^2 = 2(1+\cos\alpha) = 2\left(1 + 2\cos^2\frac{\alpha}{2} - 1\right) = 4\cos^2\frac{\alpha}{2}$$

$$r = 2\cos\frac{\alpha}{2}$$

From (1) we get, $\cos\theta = \dfrac{1+\cos\alpha}{r} = \dfrac{1+2\cos^2\frac{\alpha}{2} - 1}{2\cos\frac{\alpha}{2}} = \cos\dfrac{\alpha}{2}$

……………..……..(3)

From (2) we get, $\sin\theta = \dfrac{\sin\alpha}{r} = \dfrac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}} = \sin\dfrac{\alpha}{2}$

……………….(4)

Argument

$$= \tan^{-1}\left(\frac{\sin\alpha}{1 = \cos\alpha}\right) = \tan^{-1}\left(\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+2\cos^2\frac{\alpha}{2} - 1}\right) = \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$

General value of argument $= 2\pi k + \dfrac{\alpha}{2}$

$\theta = \dfrac{\alpha}{2}$ is satisfied both equations, (1) and (2).

Example 3. Find the modulus and principal argument of the complex

$$\frac{1+2i}{1-(1-i)^2}$$

Solution: We have

$$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-1+1+2i}$$

$$= \frac{1+2i}{1+2i} = 1 = 1 + 0i$$

$\Rightarrow$ $$\left|\frac{1+2i}{1-(1-i)^2}\right| = |1+0i| = \sqrt{1^2} = 1$$

Principal argument of $\dfrac{1+2i}{1-(1-i)^2} =$ Principal argument of $1+0i$

$$= \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0^0.$$

Hence modulus $= 1$ and principal argument $= 0^0$.

Exercise 2

1. Find the modulus and principal argument of the following complex numbers:

(a) $-\sqrt{3}-i$ **Ans.:** $2, \dfrac{5\pi}{6}$ **(b)** $\dfrac{(1+i)^2}{1-i}$

Ans.: $\sqrt{2}, \dfrac{3\pi}{4}$

(c) $\sqrt{\dfrac{1+i}{1-i}}$ **Ans.:** $1, \dfrac{\pi}{4}$ **(d)** $\tan\alpha - i$ **Ans.:** $\sec\alpha, -\left(\dfrac{\pi}{2}-\alpha\right)$

(e) $1-\cos\alpha + i\sin\alpha$ **Ans.:** $2\sin\dfrac{\alpha}{2}, \dfrac{\pi-\alpha}{2}$

(f) $(4+2i)(-3+i\sqrt{2})$ **Ans.:** $2\sqrt{55}, \tan^{-1}\left(\dfrac{3-2\sqrt{2}}{6+\sqrt{2}}\right)$

2. Find the modulus of the following complex numbers:

(a) $\overline{(7-i^2)}+(6-i)-(4-3i)$

Ans.: $4\sqrt{5}$

(b) $\overline{(5-6i)}+(5+6i)+(8-i)$

Ans.: $\sqrt{185}$

(c) $\overline{(9-i)}+(8-i^3)-(7i^2+5)$

Ans.: $\sqrt{365}$

(d) $(5+6i^{11})+(8i^3+i^5)+(i^2-i^4)$

Ans.: $\sqrt{178}$

3. If $\arg.(z+2i) = \dfrac{\pi}{4}$ and $\arg.(z-2i) = \dfrac{3\pi}{4}$, find z.

Ans.: $z = 2$

4. Express $\frac{1+7i}{(2-i)^2}$ in the modulus-argument form. **Ans.:** $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

5. Express $-5-12i$ in the modulus-argument form.

Ans.: $13(\cos\alpha+i\sin\alpha)$, where $\cos\alpha=-\frac{5}{13}$ and $\cos\alpha=-\frac{12}{13}$, α lying between $\frac{\pi}{2}$ and π.

6. Show that $\arg z+\arg\bar{z}=2n\pi$, where n is any integer.

7. Show that the equation of a circle in the Argand plane can be put in the form

$$z\bar{z}+b\bar{z}+\bar{b}z+c=0,$$

Where c is a real and b a complex constant.

7.21 SOME PROPERTIES OF MODULUS AND ARGUMENTS OF COMPLEX NUMBERS

Theorem 1. Modulus and Arguments of the conjugate of a Complex Numbers.

If z is any non zero complex numbers, then

$$|\bar{z}|=|z| \qquad \text{and} \qquad \arg\bar{z}=-\arg z$$

Proof: Let $|z|=r$ and $\arg z=\theta$.

Then from modulus argument of a complex number, we have

$$z=r(\cos\theta+i\sin\theta)$$

$$\therefore \qquad \bar{z}=r(\cos\theta-i\sin\theta)=r[\cos(-\theta)+i\sin(-\theta)]$$

Which is modulus argument from for $\bar{z}$.

Hence, $|\bar{z}|=r=|z|$ and $\arg\bar{z}=-\theta=-\arg z$.

Theorem 2. Modulus and Arguments of the Product of two Complex Numbers.

If z_1 and z_2 are any two non-zero complex numbers, then

$$|z_1.z_1|=|z_1|.|z_2| \qquad \text{and}$$

$$\arg(z_1.z_1)=\arg(z_1)+\arg(z_2)$$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $|z_1| = r_1$, $|z_2| = r_2$, $\arg z_1 = \theta_1$ $\arg z_2 = \theta_2$

We have $z_1 z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$

$$= r_1 r_2[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)]$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

From this represented of $z_1 z_2$ in the modulus-argument, we get

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$

Theorem 3. Modulus and Argument of the Quotient of two Complex Numbers.

If z_1, z_2 be any two complex numbers, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

So that $|z_1| = r_1$ $|z_2| = r_2$, $\arg z_1 = \theta_1$ $\arg z_2 = \theta_2$,

we get

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}.\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}.\frac{[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}{1}$$

$$= \frac{r_1}{r_2}.[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

From this represented of $\dfrac{z_1}{z_2}$ in the standard polar form, we observe that

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \qquad \Rightarrow \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and $\qquad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 \qquad \Rightarrow$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Theorem 4. Triangle Inequality

The modulus of the sum of two complex numbers can never exceed the sum of their moduli, that is $|z_1 + z_2| \le |z_1| + |z_2|$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

So that $|z_1| = r_1$ and $|z_2| = r_2$, we get

Now $\quad z_1 + z_2 = (r_1\cos\theta_1 + ir_1\sin\theta_1) + (r_2\cos\theta_2 + ir_2\sin\theta_2)$

$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i.(r_1\sin\theta_1 + r_2\sin\theta_2)$$

$\therefore \qquad |z_1 + z_2| = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2}$

$$= \sqrt{r_1^{\ 2} + r_2^{\ 2} + 2r_1r_2\cos(\theta_1 - \theta_1)}$$

$[\therefore \cos(\theta_1 - \theta_1) \le 1]$

$$\le \sqrt{r_1^2 + r_2^2 + 2r_1r_2} = \sqrt{(r_1 + r_2)^2}$$

$$\le r_1 + r_2 = |z_1| + |z_2|$$

Hence, $|z_1 + z_2| \le |z_1| + |z_2|$.

Theorem 5. The modulus of the difference of two complex numbers can never be less than the difference of their moduli, that is

$$|z_1 - z_1| \ge |z_1| - |z_2|$$

Proof: Suppose z_1, z_2 be any two complex numbers, we get

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

So that $|z_1| = r_1$ and $|z_2| = r_2$, we get

Now $\quad z_1 - z_2 = (r_1\cos\theta_1 + ir_1\sin\theta_1) - (r_2\cos\theta_2 + ir_2\sin\theta_2)$

$$= (r_1\cos\theta_1 - r_2\cos\theta_2) + i.(r_1\sin\theta_1 - r_2\sin\theta_2)$$

$\therefore \qquad |z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$

$$= \sqrt{r_1^{\ 2} + r_2^{\ 2} - 2r_1r_2\cos(\theta_1 - \theta_1)}$$

$[\therefore \cos(\theta_1 - \theta_1) \le 1]$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1 r_2} = \sqrt{(r_1 - r_2)^2}$$

$$\geq r_1 - r_2 = |z_1| - |z_2|$$

Hence, $|z_1 - z_2| \geq |z_1| - |z_2|$.

7.22 SQUARE ROOT OF A COMPLEX NUMBER

Let $a + ib$ be a Complex Number and its Square Root is $x + iy$. that is

$\sqrt{a + ib} = x + iy,$ where $x, y \in R.$

……………(1)

Squaring both sides of (1), we get

$$a + ib = (x + iy)^2$$

$$\Rightarrow \quad a + ib = x^2 - y^2 + 2ixy$$

……………(2)

Equating real and imaginary parts of (2), we get

$$a = x^2 - y^2$$

……………(3)

and $b = 2xy$

……………(4)

Also, we know that

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2$$

From using (3) and (4), we get

$$(x^2 + y^2)^2 = a^2 + b^2$$

$$\Rightarrow \quad x^2 + y^2 = \sqrt{a^2 + b^2}$$

……………(5)

Adding (3) and (5), we get

$$2x^2 = a + \sqrt{a^2 + b^2}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

Subtracting (3) from (5), we get

$$2y^2 = \sqrt{a^2 + b^2} - a$$

$$\Rightarrow y = \pm\sqrt{\frac{\sqrt{a^2 + b^2} - a^2}{2}}$$

Positive and negative values can be checked by satisfying equation (4).

Example 1. Find the square root of the complex number $5 + 12i$.

Solution: Let $\sqrt{5 + 12i} = x + iy$ (square)

$\Rightarrow \quad 5+12i=(x+iy)^2$

$\Rightarrow \quad 5+12i=(x^2-y^2)+2ixy$

Equating real and imaginary parts, we get

$x^2-y^2=5$

…………..(1)

and $\quad 2xy=12$

Also, we know that

$$x^2+y^2=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{(5)^2+(12)^2}$$

$$=\sqrt{25+144}=\sqrt{169}=13$$

$\Rightarrow \quad x^2+y^2=13$

…………..(2)

Adding (1) and (2), we get

$2x^2=5+13=18 \quad \Rightarrow \quad x^2=9 \quad \Rightarrow \quad x=\sqrt{9}=\pm 3$

Subtracting (1) from (2), we get

$2y^2=13-5=8 \quad \Rightarrow \quad y^2=4 \quad \Rightarrow$

$y=\sqrt{4}=\pm 2$

Since, xy is positive so x and y are of same sign. Hence, $x=\pm 3,\ y=\pm 2$

$\therefore \quad \sqrt{5+12i}=\pm 3\pm 2i \quad \Rightarrow \quad (3+2i) \quad$ or $\quad -(3+2i).$

Example 2. Find the radius and center of the circle

$$\left|\frac{z-i}{z+i}\right|=5$$

Solution: We have $\left|\frac{z-i}{z+i}\right|=5$

$\Rightarrow \quad |z-i|=5|z+i|$

$\Rightarrow \quad |z-i|^2=25|z+i|^2 \qquad \left[\because |z|^2=z.\bar{z}\right]$

$\Rightarrow \quad (z-i)\overline{(z-i)}=25(z+i)\overline{(z+i)}$

$\Rightarrow \quad (z-i)(\bar{z}+i)=25(z+i)(\bar{z}-i)$

$\Rightarrow \quad z\bar{z}+zi-i\bar{z}+1=25(z\bar{z}-zi+i\bar{z}+1)$

$\Rightarrow \quad 24z\bar{z}-26zi+26i\bar{z}+24=0$

$\Rightarrow \quad 24z\bar{z}-i26(z-\bar{z})+24=0$

$[z.\bar{z}=(x+iy)(x+iy)=(x^2+y^2),$

$z-\bar{z}=(x+iy)-(x-iy)=2iy]$

$\Rightarrow \quad 24(x^2+y^2)+52y+24=0$

$\Rightarrow \qquad x^2+y^2+\dfrac{13}{6}y+1=0$

Compare $x^2+y^2+2gx+2fy+c=0$

Here $\quad g=0,\ f=\dfrac{13}{12}$ and $c=1.$

This is the required equation of the circle.

Centre of the circle $=(-g,-f)=\left(0,-\dfrac{13}{12}\right)$

and radius $=\sqrt{g^2+f^2-c}=\sqrt{0^2+\left(-\dfrac{13}{12}\right)^2-1}=\sqrt{\dfrac{25}{144}}=\dfrac{5}{12}$

Example 3. Find the locus of the points z satisfying the condition

$$\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}.$$

Solution: We have $\dfrac{z-1}{z+1}=\dfrac{a+ib-1}{a+ib+1}=\dfrac{(a-1)+ib}{(a+1)+ib}$

$$=\frac{(a-1)+ib}{(a+1)+ib}\times\frac{(a+1)-ib}{(a+1)-ib}$$

$$=\frac{a^2-1+b^2+ib[(a+1)-(a-1)]}{(a+1)^2+b^2}$$

$$=\frac{(a^2+b^2-1)+2ib}{(a+1)^2+b^2}$$

$\therefore \qquad \arg\left(\dfrac{z-1}{z+1}\right)=\tan^{-1}\left(\dfrac{2b}{a^2+b^2-1}\right)$

Hence $\qquad \dfrac{\pi}{3}=\tan^{-1}\left(\dfrac{2b}{a^2+b^2-1}\right)$

$$\tan\frac{\pi}{3}=\frac{2b}{a^2+b^2-1}. \qquad \Rightarrow$$

$$\sqrt{3}=\frac{2b}{a^2+b^2-1}.$$

$\Rightarrow \qquad \dfrac{2}{\sqrt{3}}b=a^2+b^2-1 \qquad \Rightarrow$

$$a^2+b^2-\left(\frac{2}{\sqrt{3}}\right)b-1=0$$

This is the required locus which is a circle.

Example 4. Find the radius and center of the circle

$$z.\bar{z}-(2+3i)z-(2-3i)\bar{z}+9=0$$

Solution: Given that

$$z.\bar{z}-(2+3i)z-(2-3i)\bar{z}+9=0$$

$$\Rightarrow z\bar{z}-2(z+\bar{z})-3i(z-\bar{z})+9=0 \qquad \text{..............................(1)}$$

Here, $z.\bar{z}=(a+ib)(a+ib)=(a^2+b^2),\ z-\bar{z}=(a+ib)-(a-ib)=2ib$

And $z+\bar{z}=(a+ib)+(a-ib)=2a$

From (1), becomes

$$\Rightarrow \quad a^2+b^2-2(2a)+6b+9=0$$

$$\Rightarrow \quad a^2+b^2-4a+6b+9=0$$

This is the Cartesian equation of the circle.

Compare with the standard equation of the circle

$$a^2+b^2+2ga+2fb+c=0$$

Here $g=-2,\ f=3$ and $c=9.$

Coordinate of the Centre of the circle $=(-g,-f)=(2,-3).$

and radius $=\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+(3)^2-9}=\sqrt{4+9-9}=2$.

Example 5. If z_1 and z_2 are any two complex numbers, prove that

$$|z_1+z_1|^2+|z_1-z_1|^2=2\left(|z_1|^2+|z_2|^2\right)$$

Solution: Let $z_1=a+ib$ and $z_2=c+id$

$$|z_1+z_2|^2=|(a+ib)+(c+id)|^2$$

$$|z_1+z_2|^2=|(a+c)+i(b+d)|^2$$

$$|z_1+z_2|^2=(a+c)^2+(b+d)^2$$

Similarly $|z_1-z_2|^2=(a-c)^2+(b-d)^2$

And $|z_1|^2=a^2+b^2$ $\qquad |z_2|^2=c^2+d^2$

L.H.S. $=|z_1+z_1|^2+|z_1-z_1|^2=(a+c)^2+(b+d)^2+(a-c)^2+(b-d)^2$

$$=a^2+c^2+2ac+b^2+d^2+2bd+a^2+c^2-2ac+b^2+d^2-2bd$$

$$=2[a^2+c^2+b^2+d^2]=2[(a^2+b^2)+(c^2+d^2)]$$

$$=2[|z_1|^2+|z_2|^2]=\textbf{R.H.S.}$$

Example 6. If z_1 and z_2 are any two complex numbers, such that

$|z_1+z_2|=|z_1-z_2|$, prove that

$$\arg z_1 - \arg z_2 = \frac{\pi}{2}$$

Solution: Let $z_1 = a + ib$ and $z_2 = c + id$

Given that $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow$ $|(a+ib)+(c+id)| = |(a+ib)-(c+id)|$

$\Rightarrow$ $|(a+c)+i(b+d)| = |(a-c)+i(b-d)|$

$\Rightarrow$ $(a+c)^2 + (b+d)^2 = (a-c)^2 + (b-d)^2$

$\Rightarrow$

$$a^2 + c^2 + 2ac + b^2 + d^2 + 2bd = a^2 + c^2 - 2ac + b^2 + d^2 - 2bd$$

$\Rightarrow$ $2ac + 2bd = -2ac - 2bd$

$\Rightarrow$ $4ac + 4bd = 0$

$\Rightarrow$ $ac + bd = 0$

Now $\arg z_1 - \arg z_2 = \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(\frac{d}{c}\right)$

$$= \tan^{-1}\left[\frac{\frac{b}{a} - \frac{d}{c}}{1 + \left(\frac{b}{a}\right)\left(\frac{d}{c}\right)}\right] = \tan^{-1}\left[\frac{bc - ad}{ac + bd}\right]$$

$$= \tan^{-1}\left[\frac{bc - ad}{0}\right] = \tan^{-1}\infty = \frac{\pi}{2}$$

$$\arg z_1 - \arg z_2 = \frac{\pi}{2}$$

Proved

Example 7. Find the complex number z if $\arg(z+1) = \frac{\pi}{6}$ and $\arg(z-1) = \frac{2\pi}{3}$.

Solution: Let $z = a + ib$

$\therefore$ $z + 1 = (a+1) + ib$

……………….(1)

By the given condition

$$\arg(z+1) = \tan^{-1}\left(\frac{b}{a+1}\right) = \frac{\pi}{6}$$

$$\left(\frac{b}{a+1}\right) = \tan 30^0 = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}b = a + 1$$

Now $(z-1) = (a-1) + ib$

………………..(2)

$$\tan^{-1}\left(\frac{b}{a-1}\right) = \frac{2\pi}{3} \quad \Rightarrow \quad \left(\frac{b}{a-1}\right) = \tan 120^0$$

$\Rightarrow$ $\left(\frac{b}{a-1}\right) = -\cot 30^0 = -\sqrt{3}$

$\Rightarrow$ $-b = -\sqrt{3a} + 3$

$\Rightarrow$ $-\sqrt{3}b = 3a - 3$

…………………………(3)

Adding (2) and (3), we get

$$0 = 4a - 2 \quad \Rightarrow \quad 4a = 2 \quad \Rightarrow$$

$$a = \frac{1}{2}$$

Putting $a = \frac{1}{2}$ in (2), we get

$$\sqrt{3}b = \frac{1}{2} + 1 \quad \Rightarrow \quad \sqrt{3}b = \frac{3}{2} \quad \Rightarrow$$

$$b = \frac{\sqrt{3}}{2}$$

Putting the values of a and b in (1), we get

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

7.23 POLAR FORM

As we have discussed above

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$\Rightarrow$ $x + iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ (Exponential form)

On solving these equations, we get the value of θ and r which satisfied both the equations, we get

$$\theta = \tan^{-1}\frac{y}{x} \quad \text{and} \quad r = \sqrt{x^2 + y^2}.$$

Types of Complex numbers:

(a) Cartesian Form: $x + iy$ (b) Polar Form: $r(\cos\theta + i\sin\theta)$ (c) Exponential Form: $re^{i\theta}$

Example 1. Express in polar form $\left(1-\sqrt{2}\right)+i$.

Solution: Let $(1-\sqrt{2})+i = r(\cos\theta + i\sin\theta)$

$\therefore \quad 1-\sqrt{2} = r\cos\theta$

……..…………(1)

$1 = r\sin\theta$

………………..(2)

Squaring and adding (1) and (2), we get

$$r^2(\cos^2\theta + \sin^2\theta) = (1-\sqrt{2})^2 + 1^2$$

$$\Rightarrow \quad r^2 = 1 - 2\sqrt{2} + 2 + 1$$

$$\Rightarrow \quad r = \sqrt{4-2\sqrt{2}}$$

Putting the values of r in (1) and (2), we get

$$\cos\theta = \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{4-2\sqrt{2}}}$$

Hence, the polar form is $\sqrt{4-2\sqrt{2}}\left(\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} + i\frac{1}{\sqrt{4-2\sqrt{2}}}\right)$

7.24 SUMMARY

In this unit we have calculated the complex problem of mathematics through the trigonometry and also have been studied exponential and trigonometrical variables. In the other fields that use trigonometry or trigonometric functions and variables include music theory,geodesy, audio synthesis, architecture, electronics,biology,medical imaging (CT scans and ultrasound), chemistry,number theory (and hence cryptology),seismology,meteorology,oceanography,image compression,phonetics,economics,electrical engineering, mechanical engineering, civil engineering,computer graphics,cartography,crystallographyand game development.

7.25 *GLOSSORY*

Exponential: Very rapid increase.

Adjacent side: For a given acute angle in a right triangle, the adjacent side to that angle is the side that, along with the hypotenuse, forms that acute angle.

Identity: An equation that is true for any possible value of the variable.

Amplitude: Half the difference between the maximum and the minimum values of a periodic function.

Conterminal angle: The description of two angles drawn in standard position that share their terminal side.

Cycle: Any part of a graph of a periodic function that is one period long.

7.26 SELF ASSESSMENT QUESTIONS

7.26.1 Multiple Choice Questions:

1. The imaginary part of the complex number $\dfrac{1+7i}{(2-i)^2}$ is

(a) $2i$ (b) i

(c) $\dfrac{2}{3}i$ (d) $\dfrac{1}{2}i$

2. The modulus of the complex number $1-i$ is

(a) $\sqrt{2}$ (b) $\sqrt{\dfrac{1}{2}}$

(c) 2 (d) $\dfrac{1}{2}$

3. The argument of the complex number $-3i$ is

(a) $\dfrac{\pi}{2}$ (b) $-\dfrac{\pi}{2}$

(c) $-\pi$ (d) π

4. If$\sin(x+iy)=p+iq$, where p and $q$are real, then

(a) $q=\sin x\cos y$ (b)
$q=\cos x\sin y$

(c) $q = \sin x \cosh y$ (d) $q = \cos x \sinh y$

5. The exponential valueof $\cos x$ is

(a) $\dfrac{e^{i\theta} + e^{-i\theta}}{2}$ (b) $\dfrac{e^{\theta} + e^{-\theta}}{2}$

(c) $\dfrac{e^{i\theta} - e^{-i\theta}}{2}$ (d) None of these

6. The complex conjugate of $\cos(x - iy)$ is

(a) $\sin(x - iy)$ (b) $\cos(x + iy)$

(c) $\cos(x \pm iy)$ (d) None of these

7. Polar form of the complex number $(-1 + i\sqrt{3})$ is

(a) $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ (b) $\sqrt{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

(c) $2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ (d) $\sqrt{2}\left(\cos\dfrac{2\pi}{3} - i\sin\dfrac{2\pi}{3}\right)$

8. Modulus of $\dfrac{1+i}{1-i}$ is

(a) -1 (b) 0

(c) 1 (d) 2

ANSWERS:

1. (b)	**2.** (a)	**3.** (b)	**4.** (d)
5. (a)	**6.** (b)	**7.** (c)	**8.** (c)

7.26.2 Fill in the blanks:

1. Principal argument of $\frac{1+i}{1-i}$ is……

2. If z_1 and z_2 are any two complex numbers then $\arg\left(\frac{z_1}{z_2}\right)$ is……..

3. Radius of the circle $\left|\frac{z-i}{z+i}\right| = 5$ is……….

4. Centre of the circle $\left|\frac{z-i}{z+i}\right| = 5$ is……….

5. $z+\bar{z}=0$ if and only if………..

6. The real part of $\frac{(1+i)^2}{(3-i)}$ is……..

7. The locus represented by $|z-i| = |z+i|$ if……….

8. If $e^{i\theta} = \cos\theta + i\sin\theta$, then the value of $\sin\theta$ is…………

ANSWERS:

1. $\frac{\pi}{2}$	**2.** $\arg(z_1)-\arg(z_2)$	**3.** 5/12	**4.** (0, -13/12)
5. $\text{Re}(z)=0$	**6.** None of these	**7.** a straight line through the origin	**8.** $\frac{e^{i\theta}-e^{-i\theta}}{2i}$

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7.28 SUGGESTED READINGS

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2. Trigonometry and Algebra – Bhupendra S., Pundir S.K., Pragati Prakashan, ISBN: 978-93-5006-429-0.

3. Elementary Algebra and trigonometry – Vasishtha & Vasishtha, Krishna Publication Ltd. Meerut.

7.29 TERMINAL QUESTIONS

7.29.1 Short answer type questions:

1. Find the modulus and principal argument of the following complex numbers

(a) $-\sqrt{3}-i$ **(b)** $\sqrt{\dfrac{1+i}{1-i}}$ **(c)** $\dfrac{(1+i)^2}{1-i}$

2. Prove that $\left|\dfrac{z-1}{\bar{z}-1}\right|=1$

3. If z_1 and z_2 are any two complex numbers, then prove that

$$|z_1|-|z_2| \le |z_1-z_2| \le |z_1|+|z_2|$$

4. If $z=(a\cos\theta)+i(a\sin\theta)$, prove that $\left(\dfrac{z}{\bar{z}}+\dfrac{\bar{z}}{z}\right)=2\cos 2\theta$.

5. Express the following complex numbers into polar form:

(a) $\dfrac{1+i}{1-i}$ (b) $\dfrac{2+3i}{3-7i}$

6. Find the smallest positive integer n for which $\left(\dfrac{1+i}{1-i}\right)^n = I$.

7. Find the square root of the following complex number

(a) $1+i$ **(b)** $\frac{2+3i}{5-4i}+\frac{2-3i}{5+4i}$ **(c)** $-2+2\sqrt{3}i$

(d) $a^2-1+i(2a)$ **(e)** $-4-3i.$ **(f)** $3+4i\sqrt{7}$

ANSWERS:

1. (a): $2,-\frac{5\pi}{6}$, **(b)** $1, \frac{\pi}{4}$ **(c)** $\sqrt{2},\frac{3\pi}{4}$

5.(a) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (b) $r=\sqrt{754},\ \theta=\tan^{-1}\left(-\frac{23}{15}\right)$

6. $n=4.$

7.(a) $\left\{\pm\sqrt{\frac{\sqrt{2}+1}{2}}\pm\sqrt{\frac{\sqrt{2}-1}{2}}i\right\}$ **(b)** $\pm\frac{2}{\sqrt{41}}i$ **(c)** $\pm(1+\sqrt{3}i)$ **(d)** $\pm(a+i)$

(e) $\pm\left(\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}}i\right)$ (f) $\pm(\sqrt{7}+2i)$

7.29.2 Long answer type questions:

1. If z_1, z_2 and z_3 are the vertices of an equilateral triangle, prove that

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1.$$

2. If $\left|\frac{z-1}{z+1}\right|=2,$ prove that the locus of z on the Argand plane is a circle, whose center has affix $\left(-\frac{5}{3},0\right)$ and whose radius is $\frac{4}{3}$.

3. If $amp\left(\frac{1+i}{1-i}\right)=\frac{\pi}{4}$, prove that locus of z on the Argand plane is a circle, whose center has affix $(-1,0)$ and whose radius is $\sqrt{2}$.

4. Find the locus of the complex number z, if $\arg\left(\frac{z-3i}{z-3}\right)=\frac{\pi}{3}$.

5. Find the radius and center of the circle

(a) $\left|\frac{z+i}{z-i}\right|=4$ **(b)** $\left|\frac{z+2i}{4z-3i}\right|=1$

(c) $\left|\dfrac{z+2i}{z-3i}\right| = 2$

6. Find the radius and center of the circle
$z.\bar{z} + (2+i)z + (2-i)\bar{z} + 4 = 0$

7. Find the locus of the points z satisfying the condition
$\arg\left(\dfrac{z-2}{z+3}\right) = \dfrac{\pi}{6}$.

ANSWERS:

4. $a^2 + b^2 - (3-\sqrt{3})a - (3-\sqrt{3})b - 3\sqrt{3} = 0.$

5. **(a)** radius $= \dfrac{8}{15}$, center $= \left(0, \dfrac{34}{30}\right)$ **(b)** radius $= \dfrac{\sqrt{193}}{3}$, center $= \left(0, -\dfrac{28}{30}\right)$

(c) radius $= 5\sqrt{2}$, center $= (0,8)$

6. radius $= 2$, center $= (-2,2)$

7. $x^2 + y^2 + x - 2\sqrt{3}y - 6 = 0$

UNIT 8: HYPERBOLIC FUNCTION

CONTENTS

8.1 *OBJECTIVE*

After reading this unit you will be able to:

- Use De-Moivre's theorem
- Understand circular function and hyperbolic functions
- Understand relations between hyperbolic and circular function
- Expand sinh$x$ and cosh$x$
- Know periodicity of exponential and hyperbolic functions.

8.2 *INTRODUCTION*

In the previous unit we studied about complex number and their operations like addition, multiplication, subtraction and division. Also we know about the trigonometric ratios for real angle θ.

Now the question arises what will be the values of these trigonometric ratios when we use complex angles. To clear this concept hyperbolic functions developed by **Vincenzo Riccati and Johann Heinrich Lambert** in 1760s. Hyperbolic functions are analogs of ordinary trigonometric or circular function.

8.3 *EULER'S EXPONENTIAL VALUES (FUNCTION)*

When x is real, then we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ (1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots(3)$$

On putting $x = i\theta$ in the equation (1), we get

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots\ldots\ldots\ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{(i\theta)^6}{6!} + \ldots\ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots\right)$$

From (2) and (3), we get

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \ldots\ldots\ldots\ (4)$$

Similarly, $e^{-i\theta} = \cos\theta - i\sin\theta$ ....(5)

From (4) and (5), we get

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Where θ may be real or complex. These formulae are known as **Euler Exponential values**.

$$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} \qquad \text{and} \qquad \cot\theta = \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}$$

8.4 *DE-MOIVRE'S THEOREM (BY EXPONENTIAL FUNCTION)*

Theorem. Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: We know that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

If n is a fraction, then $\cos\theta + i\sin\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

8.5 *DE MOIVRE'S THEOREM (BY INDUCTION)*

Statement: For any rational number n the values or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof.Case I: Let n be a non-negative integer. By actual multiplication, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$+ i(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

Similarly, we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$$

$$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

Continuing in this way, we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)........(\cos\theta_n + i\sin\theta_n)$$

$$= \cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)$$

Substituting $\theta_1 = \theta_2 = ...\theta_n = \theta,$ we get

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\,.$$

Case II: Let n be a non-negative integer, say $n = -m$ where m is a positive integer. Then we have

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{(\cos m\theta + i\sin m\theta)}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \times \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta - i^2 \sin^2 m\theta)} = \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos n\theta + i\sin n\theta$$

Hence, the theorem is true for negative integers also.

Case III: Let n be a proper fraction $\frac{p}{q}$ where p and q are integers. Then we can select q to be positive integer, p may be positive or negative integer.

Now, $$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q.\frac{\theta}{q} + i\sin q.\frac{\theta}{q}$$

$$= \cos\theta + i\sin\theta$$

Taking the q root of both sides, we get

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$$

Raising both sides to the power p, we get

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \cos p.\frac{\theta}{q} + i\sin p.\frac{\theta}{q}$$

Therefore, one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Where n is the proper fraction, thus the theorem is true for all rational values of n.

Example 1. Express $\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form $a + ib$.

Solution: Given that $\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$

$$\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left(\cos\theta + \frac{1}{i}\sin\theta\right)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[(\cos(-\theta) + i\sin(-\theta)]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{\left[(\cos\theta + i\sin\theta)^{-1}\right]^4} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos\theta + i\sin\theta)^{12} = \cos 12\theta + i\sin 12\theta$$

Example 2. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then prove that

$$x^p.y^q + \frac{1}{x^p.y^q} = 2\cos(p\theta + q\phi).$$

Solution: We have

$$x + \frac{1}{x} = 2\cos\theta \quad \Rightarrow \quad x^2 - 2x\cos\theta + 1 = 0$$

$$\Rightarrow \quad x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}$$

Putting i^2 for -1 and considering the positive sign, we get

$x = \cos\theta + i\sin\theta$ and similarly

$y = \cos\phi + i\sin\phi$

Now, $x^p = (\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$

And $y^q = (\cos\phi + i\sin\phi)^q = \cos q\phi + i\sin q\phi$

Therefore $x^p.y^q = (\cos p\theta + i\sin p\theta).(\cos q\phi + i\sin q\phi)$

$$= \cos(p\theta + q\phi) + i\sin(p\theta + q\phi)$$

……….(1)

And also $$\frac{1}{x^p.y^q} = [\cos(p\theta + q\phi) + i\sin(p\theta + q\phi)]^{-1}$$

$$= \cos(p\theta + q\phi) - i\sin(p\theta + q\phi)$$

………..(2)

Adding equation (1) and (2), we get

$$x^p.y^q + \frac{1}{x^p.y^q} = = \cos(p\theta + q\phi) + i\sin(p\theta + q\phi)$$

$$+ \cos(p\theta + q\phi) - i\sin(p\theta + q\phi)$$

$$x^p.y^q + \frac{1}{x^p.y^q} = 2\cos(p\theta + q\phi).$$

Example 3. Prove that the general values of θ which satisfied the equation

$$(\cos\theta + i\sin\theta).(\cos 2\theta + i\sin 2\theta)...............(\cos n\theta + i\sin n\theta) = 1$$

Is $\frac{4m\pi}{n(n+1)}$, where m is any integer.

Solution: We have

$$(\cos\theta + i\sin\theta).(\cos 2\theta + i\sin 2\theta)...............(\cos n\theta + i\sin n\theta) = 1$$

$$(\cos\theta + i\sin\theta).(\cos\theta + i\sin\theta)^2...............(\cos\theta + i\sin\theta)^n = 1$$

$$(\cos\theta + i\sin\theta)^{1+2+........+n} = 1$$

$$(\cos\theta + i\sin\theta)^{\frac{n(n+1)}{2}} = (\cos 2m\pi + i\sin 2m\pi)$$

$$\cos\left(\frac{n(n+1)}{2}\right)\theta + i\sin\left(\frac{n(n+1)}{2}\right)\theta = \cos 2m\pi + i\sin 2m\pi$$

$$\left(\frac{n(n+1)}{2}\right)\theta = 2m\pi \quad \Rightarrow \quad \theta = \frac{4m\pi}{n(n+1)}$$

Example 4. Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^2\frac{\theta}{2}\cos\frac{n\theta}{2}$

where n is an integer.

Solution: L.H.S. $= (1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= \left[1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n + \left[1 + 2\cos^2\frac{\theta}{2} - 1 - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n$$

$$= \left[2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n + \left[2\cos^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n$$

$$= \left(2\cos\frac{\theta}{2}\right)^n\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]^n + \left(2\cos\frac{\theta}{2}\right)^n\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right] + 2^n\cos^n\frac{\theta}{2}\left[\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left[2\cos\frac{n\theta}{2}\right]$$

$$= 2^{n+1} \cos^n \frac{\theta}{2}\cos\frac{n\theta}{2} = \text{R.H.S.}$$

Example 5. By using De Moivre's theorem, solve

$$x^4 - x^3 + x^2 - x + 1 = 0$$

Solution: Given that $x^4 - x^3 + x^2 - x + 1 = 0$

We know that

$$x^5 + 1 = (x+1).(x^4 - x^3 + x^2 - x + 1)$$

$$(x^4 - x^3 + x^2 - x + 1) = \frac{x^5+1}{x+1} = 0$$

$$\Rightarrow \quad \frac{x^5+1}{x+1} = 0 \quad \Rightarrow \quad x^5 + 1 = 0$$

$$\Rightarrow \quad x^5 = -1 = \cos\pi + i\sin\pi = \cos(2r\pi + \pi) + i\sin(2r\pi + \pi)$$

$$\Rightarrow \quad x = \left(\cos(2r\pi + \pi) + i\sin(2r\pi + \pi)\right)^{\frac{1}{5}}$$

$$\Rightarrow \quad x = \cos\frac{(2r+1)}{5}\pi + i\sin\frac{(2r+1)}{5}\pi$$

Giving the values of $r = 0, 1, 2, 3, 4.$

$$x = \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right), \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right), (\cos\pi + i\sin\pi),$$

$$\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right),$$

$$\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$$

$$= \left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right), \left(\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}\right) \text{ and } -1$$

But $x = -1$ does not satisfy the given equation, therefore the required roots are

$$\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right), \left(\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}\right)$$

Example 6. If ω is a cube root of unity, prove that

$$(1-\omega)^6 = -27$$

Solution: Let $x^3 = 1$

$$\Rightarrow \qquad x = (1)^{\frac{1}{3}} = (\cos 0 + i\sin 0)^{\frac{1}{3}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}}$$
$$= \cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)$$

Putting $n = 0, 1, 2$, the cube roots of unity are

$$x_0 = 1$$
$$x_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \omega$$
$$x_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \omega^2$$

Now, $\quad 1 + \omega + \omega^2 = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$$= 1 + \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right) + \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$$
$$= 1 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$
$$= 1 - 2\cos\frac{\pi}{3} = 1 - 2\left(\frac{1}{2}\right) = 0$$

$$1 + \omega + \omega^2 = 0 \qquad \Rightarrow \qquad 1 + \omega^2 = -\omega$$
…………(1)

Now, $\quad (1 - w)^6 = [(1 - w)^2]^3 = [1 - 2w + w^2]^3 = [-w - 2w]^3$

$$= (-3\omega)^3 = -27\omega^3 = -27$$

Hence, $\quad (1 - \omega)^6 = -27$

Exercise 1

1. Evaluate $\left(\dfrac{1 + \cos\alpha + i\sin\alpha}{1 + \cos\alpha + i\sin\alpha}\right)^n$ **Ans.:** $\cos n\alpha + i\sin n\alpha$

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove that

(a) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{2}$

(b) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(c) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$

(d) $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$

3. If n be a positive integer, prove that $(1+i)^n+(1-i)^n = 2^{\frac{n+2}{2}}\cos\dfrac{n\pi}{4}$

4. If $x+\dfrac{1}{x}=2\cos\alpha,\ y+\dfrac{1}{y}=2\cos\beta,\ z+\dfrac{1}{z}=2\cos\gamma,$ then show that

$$xyz+\frac{1}{xyz}=2\cos(\alpha+\beta+\gamma)$$

5. If α,β are roots of the equation $x^2-2x+4=0,$ prove that

$$\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}$$

6. Simplify **(a)** $\dfrac{(\cos 2\theta - i\sin 2\theta)^7.(\cos 3\theta+i\sin 3\theta)^{-5}}{(\cos 4\theta+i\sin 4\theta)^{12}.(\cos 5\theta+i\sin 5\theta)^6}$ **Ans.:** $\cos 107\theta - i\sin 107\theta$

(b) $[(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)]^n+[(\cos\theta+\cos\phi)-i(\sin\theta+\sin\phi)]^n$

Ans.: $2^{n+1}\cos\dfrac{1}{2}(\theta-\phi)\cos\dfrac{n}{2}(\theta-\phi)$

(c) $\dfrac{(\cos 3\theta+i\sin 3\theta)^5.(\cos\theta-i\sin\theta)^3}{(\cos 5\theta+i\sin 5\theta)^7.(\cos 2\theta-i\sin 2\theta)^5}$ **Ans.:** $\cos 13\theta - i\sin 13\theta$

(d) $\dfrac{(\cos\theta+i\sin\theta)^8.(\cos 3\theta-i\sin 3\theta)^2}{(\cos 2\theta+i\sin 2\theta)^5.(\cos 4\theta-i\sin 4\theta)^7}$ **Ans.:** $\cos 20\theta + i\sin 20\theta$

7. Solve by De-Moivre's theorem

(a) $x^9-x^5+x^4-1=0$ **Ans.:** -1, 1, $\pm i, \cos\dfrac{\pi}{5}\pm i\sin\dfrac{\pi}{5},\ \cos\dfrac{3\pi}{5}\pm i\sin\dfrac{3\pi}{5}$

(b) $x^7-1=0$ **Ans.:** $\cos\dfrac{2n\pi}{7}\pm i\sin\dfrac{2n\pi}{7}$ where $n=0,1,2,3,4,5,6$

(c) $x^7+x^4+x^3+1=0$ **Ans.:** $x=\dfrac{1\pm i}{\sqrt{2}},\dfrac{-1\pm i}{\sqrt{2}},\dfrac{1\pm i\sqrt{3}}{2},-1$.

8.6 *CIRCULAR FUNCTIONS*

Circular Functions are already discussed in the form of Exponential function that is Euler's Exponential values in the term of Circular Functions:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

If $\theta = z$, then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

8.7 *HYPERBOLIC FUNCTIONS*

Any group of functions of an angle expressed as a relationship between the distances of a point on a hyperbola to the origin and coordinate axes. The group of functions includes the hyperbolic function, whether x be real or complex is said to be Hyperbolic function such as

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} \qquad \sec hx = \frac{2}{e^{x} + e^{-x}}$$

$$\cos echx = \frac{2}{e^{x} - e^{-x}}$$

$$\cosh x + \sinh x = \frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2} = e^{x}$$

$$\cosh x - \sinh x = \frac{e^{x} + e^{-x}}{2} - \frac{e^{x} - e^{-x}}{2} = e^{-x}$$

$$(\cosh x + \sinh x)^{n} = \cosh nx + \sinh nx$$

8.8 *RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTIONS*

Hyperbolic functions can be expressed in the term of Circular Functions as follows:

We know that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \text{ Put } x = ix$$

$$\sin ix = \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

$$\sin ix = \frac{i[e^{-x} - e^{x}]}{2i^2} = \frac{i[e^{-x} - e^{x}]}{-2}$$

$$\sin ix = \frac{i[e^{x} - e^{-x}]}{2} = i\sinh x \qquad \Rightarrow$$

$$\sinh ix = i\sin x$$

Similarly, $\cos ix = \cosh x \qquad \Rightarrow$

$$\cosh ix = \cos x$$

$$\tan ix = i\tanh x \qquad \Rightarrow$$

$$\tanh ix = i\tan x$$

8.9 *SOME FORMULAE OF HYPERBOLIC FUNCTIONS*

1. $\cosh^2 x - \sinh^2 x = 1$ **2.** $\sec h^2 x = 1 - \tanh^2 x$

3. $\operatorname{cos}ech^2 x = \coth^2 x - 1$ **4.**

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ **6.**

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

7. $\sinh 2x = 2\sinh x \cosh x$ **8.** $\cosh 2x = 2\cosh^2 x - 1$

9. $\cosh 2x = \cosh^2 x + \sinh^2 x$ **10.** $\cosh 2x = 1 + 2\sinh^2 x$

11. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$ **12.**

$$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$$

13. $\sinh x - \sinh y = 2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$ **14.**

$\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$

15. $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

Proof of some important above formulae:

Example 1. Prove that $\cosh^2 x - \sinh^2 x = 1$

Proof: L.H.S. $= \cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = \frac{4}{4} = 1 =$$

L.H.S.

Example 2. Prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$

Proof: R.H.S. $= \cosh^2 x + \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 + \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2(e^{2x} + e^{-2x})}{4}$$

$$= \frac{(e^{2x} + e^{-2x})}{2} = \cosh 2x = \text{L.H.S.}$$

Example 3. Prove that $\sinh 2x = 2\sinh x \cosh x$

Proof: R.H.S. $= 2\sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right)$

$$= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x = \text{L.H.S.}$$

Example 4. Prove that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Proof: R.H.S. $= \sinh x \cosh y + \cosh x \sinh y$

$$= .\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{e^{x+y} + e^{x-y} - e^{-(x-y)} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{-(x-y)} - e^{-(x+y)}}{4}$$

$$= \frac{2e^{(x+y)} - 2e^{-(x+y)}}{4} = \frac{e^{(x+y)} - e^{-(x+y)}}{2}$$
$$= \sin(x+y) = \text{L.H.S.}$$

8.10 *EXPANSIONS FOR* $\sinh x$ *AND* $\cosh x$

We have

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\right)\right]$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots.$$

And, again $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\right)\right]$$
$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^4}{6!}. + \ldots..$$

8.11 *PERIODICITY OF* e^z

Suppose $z = a + ib$, then

Now, $e^z = e^{x+iy} = e^x.e^{iy}$

$$= e^x(\cos y + i\sin y)$$
$$= e^x[\cos(2n\pi + y) + i\sin(2n\pi + y)]$$
$$= e^x.e^{i(2n\pi+y)}$$
$$= e^{(x+iy)+i.2n\pi} = e^{z+2.n\pi i}$$

Which shows that e^z is periodic function of period $2n\pi i$.

$$e^{2.n\pi i} = \cos 2n\pi + i\sin 2n\pi = 1 + i.0 = 1,$$
$$e^{-2.n\pi i} = \cos 2n\pi - i\sin 2n\pi = 1 - i.0 = 1$$
$$\Rightarrow e^{\pm 2.n\pi i} = 1$$

8.12 *PERIODS OF HYPERBOLIC FUNCTIONS*

We know that the Euler's theorem

$e^{2.n\pi i} = \cos 2n\pi + i\sin 2n\pi = 1 + i.0 = 1,$ Where n is being any integer.

and $e^{-2.n\pi i} = \cos 2n\pi - i\sin 2n\pi = 1 - i.0 = 1$

$$\therefore \quad \sinh(x + 2n\pi i) = \frac{e^{(x+2n\pi i)} - e^{-(x+2n\pi i)}}{2}$$

$$= \frac{e^x e^{2n\pi i} - e^{-x} e^{-2n\pi i}}{2} = \frac{e^x.1 - e^{-x}.1}{2}$$

$$= \frac{e^x - e^{-x}}{2} = \sinh x$$

Similarly $\cosh(x + 2n\pi i) = \dfrac{e^x - e^{-x}}{2} = \cosh x$

Again $\tanh(x + n\pi i) = \dfrac{\sinh(x + n\pi i)}{\cosh(x + n\pi i)}$

$$= \frac{\dfrac{e^{(x+n\pi i)} - e^{-(x+n\pi i)}}{2}}{\dfrac{e^{(x+n\pi i)} + e^{-(x+n\pi i)}}{2}} = \frac{e^{(x+n\pi i)} - e^{-(x+n\pi i)}}{e^{(x+n\pi i)} + e^{-(x+n\pi i)}}$$

$$= \frac{\sinh x}{\cosh x} = \tanh x$$

Hence $\sinh x$ and $\cosh x$ are periodic functions of periods $2\pi i$ and $\tanh x$ is a periodic functions of period πi.

Example 1. Separate the following into real and imaginary parts of the Circular functions.

(a) $\sin(x + iy)$ (b) $\cos(x + iy)$

(c) $\tan(x + iy)$

(d) $\cot(x + iy)$ (e) $\sec(x + iy)$

(f) $\cos ec(x + iy)$

Proof: (a) We have

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + \cos x(i\sinh y)$$
$$= \sin x \cosh y + i\cos x \sinh y$$

(b) $\cos(x + iy) = \cos x \cos iy - \sin x \sin iy$

$$= \cos x \cosh y - \sin x(i\sinh y)$$

$$= \cos x \cosh y - i \sin x \sinh y$$

(c) $$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$= \left(\frac{\sin 2x}{\cos 2x + \cosh 2y}\right) + i.\left(\frac{\sinh 2y}{\cos 2x + \cosh 2y}\right)$$

(d) $$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)} = \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\sin 2x - \sin 2iy}{\cos 2iy - \cos 2x} = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$$

$$= \left(\frac{\sin 2x}{\cosh 2y - \cos 2x}\right) - i.\left(\frac{\sinh 2y}{\cosh 2y - \cos 2x}\right)$$

(e) $$\sec(x+iy) = \frac{1}{\cos(x+iy)} = \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2[\cos x\cos(iy) + \sin x\sin(iy)]}{\cos 2x + \cos(2iy)}$$

$$= \frac{2[\cos x\cosh y + i\sin x\sinh y]}{\cos 2x + \cosh 2y}$$

$$= \left(\frac{2\cos x\cosh y}{\cos 2x + \cosh 2y}\right) + i.\left(\frac{2\sin x\sinh y}{\cos 2x + \cosh 2y}\right)$$

(f) $$\cos ec(x+iy) = \frac{1}{\sin(x+iy)} = \frac{2\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$

$$= \frac{2[\sin x\cos(iy) - \cos x\sin(iy)]}{\cos(2iy) - \cos 2x}$$

$$= \frac{2[\sin x\cos(iy) - \cos x\sin(iy)]}{\cosh 2y - \cos 2x}$$

$$= \left(\frac{2\sin x\cosh y}{\cosh 2y - \cos 2x}\right) - i\left(\frac{2\cos x\sinh y}{\cosh 2y - \cos 2x}\right)$$

Example 2. Separate the following into real and imaginary parts of the Hyperbolic functions.

(a) $\sinh(x+iy)$ **(b)** $\cosh(x+iy)$ **(c)** $\tanh(x+iy)$

Proof:(a) We have

$$\sinh(x+iy) = \sinh x\cosh(iy) + \cosh x\sinh(iy)$$

$$= \sinh x \cos y + i \sin y \cosh x$$

(b) $\cosh(x+iy) = \cosh x \cosh(iy) - \sinh x \sinh(iy)$

$$= \cosh x \cos y - i \sinh x \sinh y$$

(c) $\tanh(x+iy) = \dfrac{\sinh(x+iy)}{\cosh(x+iy)} = \dfrac{-i \sin i(x+iy)}{\cos i(x+iy)}$

$$= \frac{-i\sin(ix-y)}{\cos(ix-y)} = \frac{-2i\sin(ix-y)\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)}$$

$$= -i\frac{\sin 2ix - \sin 2y}{\cos 2ix + \cos 2y} = -i\left[\frac{\sinh 2x - \sin 2y}{\cosh 2x + \cos 2y}\right] = \frac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$$

$$= \left(\frac{\sinh 2x}{\cosh 2x + \cos 2y}\right) + i.\left(\frac{\sin 2y}{\cosh 2x + \cos 2y}\right)$$

Example 3. Prove that

$$(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$$

Proof: We have

L.H.S. $= (\cosh x - \sinh x)^n$

$$= \left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right]^n$$

$$= \left[\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right]^n$$

$$= \left[\frac{2e^{-x}}{2}\right]^n = \left(e^{-x}\right)^n = e^{-nx}$$

……………..(1)

R.H.S. $= \cosh nx - \sinh nx$

$$= \left[\frac{e^{nx} + e^{-nx}}{2} - \frac{e^{nx} - e^{-nx}}{2}\right]$$

$$= \left[\frac{2e^{-nx}}{2}\right] = e^{-nx}$$

……………..(2)

From (1) and (2), we have

L.H.S. = R.H.S.

Example 4. If $x = 2\sin\theta\cosh\phi,$ show that $y = \cos\theta \sinh\phi$

$$\operatorname{cosec}(\theta - i\phi) + \operatorname{cosec}(\theta + i\phi) = \frac{4x}{x^2 + y^2}$$

Proof: We know that

$$\operatorname{cosec}(\theta+i\phi)=\frac{1}{\sin(\theta+i\phi)}=\frac{1}{\sin\theta\cos i\phi+\cos\theta\sin i\phi}$$

$$=\frac{1}{\sin\theta\cosh\varphi+i\cos\theta\sinh\phi}$$

$$=\frac{1}{\frac{x}{2}+i\frac{y}{2}}=\frac{2}{x+iy}$$

…………..(1)

Similarly $\operatorname{cosec}(\theta-i\phi)=\dfrac{2}{x-iy}$

……………(2)

Adding (1) and (2), we get

$$\operatorname{cosec}(\theta-i\phi)+\operatorname{cosec}(\theta+i\phi)=\frac{2}{x-iy}+\frac{2}{x+iy}=\frac{4xy}{x^2+y^2}$$

Example 5. If $\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha,$ show that

$\cos 2\theta\cosh 2\phi=3.$

Proof: Given that

$$\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha,$$

$$\Rightarrow \quad \sin\theta\cos(i\phi)+\cos\theta\sin(i\phi)=\tan\alpha+i\sec\alpha,$$

$$\Rightarrow \quad \sin\theta\cosh\phi+i\cos\theta\sinh\phi=\tan\alpha+i\sec\alpha$$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi=\tan\alpha$$

……….(1)

$$\cos\theta\sinh\phi=\sec\alpha$$

………(2)

Now, we know that

$$\sec^2\alpha-\tan^2\alpha=1$$

…………(3)

From (1), (2) and (3), we get

$$\cos^2\theta\sinh^2\phi-\sin^2\theta\cosh^2\phi=1$$

$$\Rightarrow \quad \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\sinh 2\phi-1}{2}\right)-\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{\sinh 2\phi+1}{2}\right)=1$$

$$\Rightarrow \quad 2\cos 2\theta\cosh 2\phi-2=4$$

$$\Rightarrow \quad 2\cos 2\theta\cosh 2\phi=6$$

$$\cos 2\theta\cosh 2\phi=3$$

Example 6. If $\tan(A+iB) = x+iy$, prove that

(a) $\tan 2A = \dfrac{2x}{1-x^2-y^2}$ (b) $\tanh 2B = \dfrac{2x}{1+x^2+y^2}$

Proof: We have $\tan(A+iB) = x+iy$,

So that $\tan(A-iB) = x-iy$

We have $\tan 2A = \tan[(A+iB)+(A-iB)]$

$$\tan 2A = \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2+y^2)} = \frac{2x}{1-x^2-y^2}$$

Again $\tan 2iB = \tan[(A+iB)-(A-iB)]$

$$\tan 2iB = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+x^2+y^2}$$

$$i\tanh 2B = \left(\frac{2y}{1+x^2+y^2}\right)i$$

$$\tanh 2B = \left(\frac{2y}{1+x^2+y^2}\right)$$

Example 7. If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$, show that

(a) $\theta = \dfrac{n\pi}{2}+\dfrac{\pi}{4}$ (b) $\phi = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$

Solution: Given that $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$,

So that $\tan(\theta-i\phi) = \cos\alpha - i\sin\alpha$,

Hence $\tan 2\theta = \tan[(\theta+i\phi)+(\theta-i\phi)]$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1-(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)}$$

$$= \frac{2\cos\alpha}{0} = \infty$$

$$\tan 2\theta = \tan\frac{\pi}{2}$$

$$2\theta = n\pi + \frac{\pi}{2} \qquad \Rightarrow \qquad \theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

(b) Again $\tan 2i\phi = \tan[(\theta+i\phi)-(\theta-i\phi)]$

$$= \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\cos\alpha+i\sin\alpha-\cos\alpha+i\sin\alpha}{1+(\cos\alpha+i\sin\alpha)(\cos\alpha-i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1+(\cos^2\alpha+\sin^2\alpha)}$$

$$= \frac{2i\sin\alpha}{2} = i\sin\alpha$$

$$\tan 2i\phi = i\sin\alpha \qquad \Rightarrow \qquad i\tanh 2\phi = i\sin\alpha \qquad \Rightarrow$$

$$\tanh 2\phi = \sin\alpha$$

$\tanh 2\phi = \sin\alpha$ that is

$$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\sin\alpha}{1}$$ Cross-multiplying, we get

$$e^{2\phi}-e^{-2\phi} = \sin\alpha(e^{2\phi}+e^{-2\phi})$$

$$\Rightarrow \qquad (1-\sin\alpha)e^{2\phi} = (1+\sin\alpha)e^{-2\phi}$$

$$\Rightarrow \qquad e^{4\phi} = \frac{(1+\sin\alpha)}{(1-\sin\alpha)} = \frac{1-\cos\left(\frac{\pi}{2}+\alpha\right)}{1+\cos\left(\frac{\pi}{2}+\alpha\right)}$$

$$= \frac{2\sin^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)} = \tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$\Rightarrow \qquad e^{2\phi} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \qquad \Rightarrow \qquad 2\phi = \log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$\Rightarrow \qquad \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

Example 8. If $\cos(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$ show that

$$\phi = \frac{1}{2}\log\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}$$

Proof: Given that $\qquad \cos(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$

$\Rightarrow \qquad \cos\theta\cos(i\phi) - \sin\theta\sin(i\phi) = \rho(\cos\alpha + i\sin\alpha)$

$\Rightarrow \qquad \cos\theta\cosh\phi - i\sin\theta\sinh\phi = \rho(\cos\alpha + i\sin\alpha)$

Equating real and imaginary part on both sides, we get

$$\cos\theta\cosh\phi = \rho\cos\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

And $\qquad \sin\theta\sinh\phi = -\rho\sin\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$

From (1) and (2), we get

$$\frac{\cos\theta\cosh\phi}{\sin\theta\sinh\phi} = \frac{\rho\cos\alpha}{-\rho\sin\alpha}$$

$\Rightarrow \qquad \frac{\cos\theta\cosh\phi}{\sin\theta\sinh\phi} = \frac{\cos\alpha}{-\sin\alpha}$

$\Rightarrow \qquad \frac{\cosh\phi}{\sinh\phi} = \frac{\cos\alpha\sin\theta}{-\sin\alpha\cos\theta}$

$\Rightarrow \qquad \frac{\frac{e^{\phi}+e^{-\phi}}{2}}{\frac{e^{\phi}-e^{-\phi}}{2}} = \frac{\cos\alpha\sin\theta}{-\sin\alpha\cos\theta}$

$\Rightarrow \qquad \frac{e^{\phi}+e^{-\phi}}{e^{\phi}-e^{-\phi}} = \frac{\cos\alpha\sin\theta}{-\sin\alpha\cos\theta}$

By using componendo and dividend rule, we get

$$\frac{(e^{\phi}+e^{-\phi})+(e^{\phi}-e^{-\phi})}{(e^{\phi}+e^{-\phi})-(e^{\phi}-e^{-\phi})} = \frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta}$$

$\Rightarrow \qquad \frac{2e^{\phi}}{2e^{-\phi}} = \frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta}$

$\Rightarrow \qquad e^{2\phi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} \qquad \Rightarrow$

$$2\phi = \log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

Hence $$\phi = \frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

Example 9. If $\sin(\alpha + i\beta) = x + iy$, prove that

(a) $\dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1$ (b) $\dfrac{x^2}{\sin^2\alpha} - \dfrac{y^2}{\cos^2\alpha} = 1$

Proof: Given that $\sin(\alpha + i\beta) = x + iy$

$\Rightarrow$ $$x + iy = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$$

Equating real and imaginary parts, we get

$$x = \sin\alpha\cosh\beta, \qquad y = \cos\alpha\sinh\beta \quad \ldots\ldots\ldots\ldots\ldots(1)$$

(a) From (1) we get

$$\sin\alpha = \frac{x}{\cosh\beta}$$

$$\cos\alpha = \frac{y}{\sinh\beta}$$

Squaring and adding, $$\sin^2\alpha + \cos^2\alpha = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$$

$\Rightarrow$ $$1 = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$$

(b) Again, from (1) we get

$$\cosh\beta = \frac{x}{\sin\alpha}$$

$$\sinh\beta = \frac{y}{\cos\alpha}$$

Squaring and substracting, $$\cosh^2\beta - \sinh^2\beta = \frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha}$$

$$1 = \frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha}$$

Example 10. If $\tan(x + iy) = \sin(u + iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tanh v}$

Proof: Given that $\tan(x + iy) = \sin(u + iv)$

$\Rightarrow$ $$\frac{\sin(x+iy)}{\cos(x+iy)} = \sin(u + iv)$$

$\Rightarrow \qquad \dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} = \sin u\cosh v + i\cos u\sinh v$

$\Rightarrow \qquad \dfrac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \sin u\cosh v + i\cos u\sinh v$

$\Rightarrow \qquad \dfrac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} = \sin u\cosh v + i\cos u\sinh v$

$\Rightarrow \dfrac{\sin 2x}{\cos 2x + \cosh 2y} + i\dfrac{\sinh 2y}{\cos 2x + \cosh 2y} = \sin u\cosh v + i\cos u\sinh v$

Equating real and imaginary parts, we get

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \sin u\cosh v$$

……………………(1)

$$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \cos u\sinh v$$

…………………..(2)

Dividing (1) by (2), we get

$$\frac{\sin 2x}{\sinh 2y} = \frac{\sin u\cosh v}{\cos u\sinh v} \qquad \Rightarrow$$

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

Example 11. If $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$, prove that

$$r^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta), \qquad \text{and} \qquad \tan\alpha = \tanh\phi\cot\theta$$

Solution: Given that

$$\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$$

$\Rightarrow \qquad \sin\theta\cos(i\phi) + \cos\theta\sin(i\phi) = r(\cos\alpha + i\sin\alpha)$

$\Rightarrow \qquad \sin\theta\cosh\phi + i\cos\theta\sinh\phi = r\cos\alpha + ir\sin\alpha$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi = r\cos\alpha$$

……………….(1)

$$\cos\theta\sinh\phi = r\sin\alpha$$

……………….(2)

On squaring and adding equation (1) and (2), we get

$$r^2(\cos^2\alpha + \sin^2\alpha) = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$

$\Rightarrow \qquad r^2 = \left(\dfrac{1-\cos 2\theta}{2}\right)\left(\dfrac{1+\cosh 2\phi}{2}\right) + \left(\dfrac{1+\cos 2\theta}{2}\right)\left(\dfrac{\cosh 2\phi - 1}{2}\right)$

$$=\frac{1}{4}[1+\cosh 2\phi-\cos 2\theta-\cos 2\theta\cosh 2\phi+\cosh 2\phi-1+\cos 2\theta\cosh 2\phi-\cos 2\theta]$$

$$=\frac{1}{4}[2\cosh 2\phi-2\cos 2\theta]=\frac{1}{2}[\cosh 2\phi-\cos 2\theta]$$

Again dividing equation (2) by equation (1), we get

$$\tan\alpha=\frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi}=\tanh\phi\cot\theta$$

12. If $\sin(\theta+i\phi)\sin(\alpha+i\beta)=1$, prove that

$$\tanh^2\phi\cosh^2\beta=\cos^2\alpha$$

Solution: Given that

$$\sin(\theta+i\phi)\sin(\alpha+i\beta)=1,$$

$$\Rightarrow \quad \sin(\theta+i\phi)=\frac{1}{\sin(\alpha+i\beta)}=\frac{1}{\sin(\alpha+i\beta)}\times\frac{\sin(\alpha-i\beta)}{\sin(\alpha-i\beta)}$$

$$\Rightarrow \quad \sin\theta\cosh\phi+i\cos\theta\sinh\phi=\frac{\sin\alpha\cosh\beta-i\cos\alpha\sinh\beta}{\sin^2\alpha+\sinh^2\beta}$$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi=\frac{\sin\alpha\cosh\beta}{\sin^2\alpha+\sinh^2\beta}$$

…………(1)

$$\cos\theta\sinh\phi=-\frac{\cos\alpha\sinh\beta}{\sin^2\alpha+\sinh^2\beta}$$

…………(2)

Eliminating θ from (1) and (2), we get

$$1=\left(\frac{\sin\alpha\cosh\beta}{\cosh\phi(\sin^2\alpha+\sinh^2\beta)}\right)^2+\left(\frac{-\cos\alpha\sinh\beta}{\sinh\phi(\sin^2\alpha+\sinh^2\beta)}\right)^2$$

$\Rightarrow$

$$(\sin^2\alpha+\sinh^2\beta)^2=\sin^2\alpha\cosh^2\beta\,\mathrm{sec}h^2\phi+\cos^2\alpha\sinh^2\beta\,\mathrm{cos}ech^2\phi$$

$\Rightarrow$

$$(\sin^2\alpha+\sinh^2\beta)^2=\sin^2\alpha\cosh^2\beta(1-\tanh^2\phi)+\cos^2\alpha\sinh^2\beta(\coth^2\phi-1)$$

$\Rightarrow$

$$\sin^4\alpha+\sinh^4\beta+2\sin^2\alpha\sinh^2\beta=\sin^2\alpha\cosh^2\beta-\sin^2\alpha\cosh^2\beta\tanh^2\phi$$

$$+\cos^2\alpha\sinh^2\beta\coth^2\phi - \cos^2\alpha\sinh^2\beta$$

$\Rightarrow$

$$(\sin^4\alpha + \sin^2\alpha\sinh^2\beta - \sin^2\alpha\cosh^2\beta) + (\sinh^4\beta + \sin^2\alpha\sinh^2\beta + \cos^2\alpha\sinh^2\beta)$$

$$= \cos^2\alpha\sinh^2\beta\coth^2\phi - \sin^2\alpha\cosh^2\beta\tanh^2\phi$$

$\Rightarrow$

$$\{(\sin^4\alpha - \sin^2\alpha(\cosh^2\beta - \sinh^2\beta)\} + \{(\sinh^4\beta + \sinh^2\beta(\sin^2\alpha + \cos^2\alpha)\}$$

$$= \cos^2\alpha\sinh^2\beta\coth^2\phi - \sin^2\alpha\cosh^2\beta\tanh^2\phi$$

$\Rightarrow \quad \sin^2\alpha(\sin^2\alpha - 1) + \sinh^2\beta(\sinh^2\beta + 1)$

$= \cos^2\alpha\sinh^2\beta\coth^2\phi - \sin^2\alpha\cosh^2\beta\tanh^2\phi$

$\Rightarrow \quad -\sin^2\alpha\cos^2\alpha + \sinh^2\beta\cosh^2\beta$

$= \cos^2\alpha\sinh^2\beta\coth^2\phi - \sin^2\alpha\cosh^2\beta\tanh^2\phi$

$\Rightarrow$

$$\cosh^2\beta(\sinh^2\beta + \sin^2\alpha\tanh^2\phi) - \cos^2\alpha(\sin^2\alpha + \sinh^2\beta.\coth^2\phi) = 0$$

$\Rightarrow$

$$\cosh^2\beta(\sinh^2\beta + \sin^2\alpha\tanh^2\phi) - \cos^2\alpha\coth^2\phi(\sin^2\alpha\tanh^2\phi + \sin^2\beta) = 0$$

$\Rightarrow \quad (\cosh^2\beta - \cos^2\alpha\coth^2\phi)(\sinh^2\beta + \sin^2\alpha\tanh^2\phi) = 0$

If $(\cosh^2\beta - \cos^2\alpha\coth^2\phi) = 0,$ then

$(\sinh^2\beta + \sin^2\alpha\tanh^2\phi) \neq 0$

Now, $\cosh^2\beta = \cos^2\alpha\coth^2\phi$

$\Rightarrow \quad \tanh^2\phi\cosh^2\beta = \cos^2\alpha$

8.13 SUMMARY

A group of functions of an angle expressed as a relationship between the distances of a point on a hyperbola to the origin and to the coordinate axes. The group includes sinh (hyperbolic function), cosh (hyperbolic cosine), tenh(hyperbolic tangent), sech (hyperb olic secant), cosech (hyperbolic cosecant), and coth (hyperbolic cotangent). The Hyperbolic functions also satisfy identities analogous to those of the ordinary trigonometric functions and have important physical applications. And a function of an angle expressed as a relationship between the distances from a point on a hyperbola to the origin and to the coordinate axes, as hyperbolic sine or hyperbolic cosine: often expressed as combinations of exponential functions. For example, the hyperbolic cosine function may be used to describe the shape of the curve formed by a high-voltage line suspended between two towers.

8.14 GLOSSARY

Hypotenuse: The side opposite the right angle in any right triangle. The hypotenuse is the longest side of any right triangle.
Sine: If A is an acute angle of a right triangle, then the sine of angle A is the ratio of the length of the side opposite angle A over the length of the hypotenuse.
Hyperbola: Origin and to the coordinate axes, as hyperbolic sine or hyperbolic cosine.

8.15 SELF ASSESSMENT QUESTIONS

8.15.1 Multiple Choice Questions:

1. The value of $e^{\sin(i\theta)}$ is

(a) $\cos(\sinh\theta) - i\sin(\sinh\theta)$ (b) $\cos(\sinh\theta) + i\sin(\sinh\theta)$

(c) $\cos(\sinh\theta) + \sin(\sinh\theta)$ (d) $\sin(\sinh\theta) - i\cos(\sinh\theta)$

2. Value of $\sinh x$ is

(a) $\dfrac{e^x - e^{-x}}{2}$ (b) $\dfrac{e^{-x} - e^x}{2}$

(c) $\dfrac{e^x + e^{-x}}{2}$ (d) $\dfrac{e^x + e^{-x}}{2i}$

3. The value of $e^{\theta + i\pi}$ is equal to

(a) $e^{-\theta}$ (b) $-e^{-\theta}$

(c) e^{θ} (d) $-e^{-\theta}$

4. If θ is real then

(a) $\sin(i\theta) = i\sinh\theta$ (b) $\cos(i\theta) = i\cosh\theta$

(c) $\tan(i\theta) = \tanh\theta$ (d) $\cot(i\theta) = i\coth\theta$

5. The period of e^{z} is

(a) π (b) 2π

(c) πi (d) $2\pi i$

6. The real part of $\exp\left(e^{i\theta}\right)$ is

(a) $e^{\cos\theta}\cos(\sin\theta)$ (b) $e^{\cos\theta}\cos(\cos\theta)$

(c) $e^{\cos\theta}\sin(\cos\theta)$ (d) $e^{\cos\theta}\sin(\sin\theta)$

7. $\cosh(\theta + i\phi)$ is equal to

(a) $\cosh\theta\cos\phi - i\sinh\theta\sin\phi$ (b) $\cosh\theta\cos\phi + i\sinh\theta\sin\phi$

(c) $\cosh\theta\cos\phi + \sinh\theta\sin\phi$ (d) None of these

8. Euler's formula is

(a) $e^{i\theta} = \sin\theta + i\cos\theta$ (b) $e^{\theta} = \cos\theta + i\sin\theta$

(c) $e^{i\theta} = \cos\theta + i\sin\theta$ (d) $e^{\theta} = \sin\theta + i\cos\theta$

ANSWERS:

1. b	**2.** a	**3.** b	**4.** a
5. d	**6.** a	**7.** c	**8.** c

8.15.2Fill in the blanks:

1. The series $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$ is absolutely convergent for…….

2. If $Ae^{2i\theta} + Be^{-2i\theta} = 5cos2\theta - 7isin2\theta,$ then……..

3. If $\cos(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$, then the value of $\rho\sin\alpha$ is………

4. $\cosh^2 z - \sinh^2 z$ is equal to………

5. $\cos(\alpha + i\beta)$ is equal to

6. The periods of $\sinh z$, $\cosh z$ and $\tanh z$ are

7. The value of $\tan\theta$ is

ANSWERS:

1. for all values of z **2.** A = -1, B = 6 **3.** $-\sin\theta\sinh\phi$ **4.** 1

5. $\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$ **6.** $2\pi i, 2\pi i, \pi i$ **7.** $\frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} - e^{-i\theta})}$

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8.17 SUGGESTED READINGS

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8.18 TERMINAL QUESTIONS

8.18.1 Short answer type questions

1. Separate the equation $\sec(x+iy)$ into real and imaginary parts.

2. If $\tan(\theta+i\phi)=\tan\alpha+i\sec\alpha$, prove that

(a) $e^{2\phi}=\pm\cot\dfrac{\alpha}{2}$ **(b)** $2\theta=n\pi+\dfrac{\pi}{2}+\alpha$

3. If $\sin(\theta+i\phi)=\cos\alpha+i\sin\alpha$, then prove that, $\cos^2\theta=\sinh^2\phi=\pm\sin\alpha$

4. If $\tan(\theta+i\phi)=x+iy$, prove that

(a) $x^2+y^2+2x\cot 2\theta=1$ **(b)** $x^2+y^2-2y\coth 2\phi=-1$

5. If $A+iB=C\tan(x+iy)$, then prove that, $\tan 2x=\dfrac{2CA}{C^2-A^2-B^2}$

6. If $u=\log_e\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$, prove that, $\tanh\dfrac{u}{2}=\tan\dfrac{\theta}{2}$

7. If $\tanh x=\dfrac{1}{2}$, find the value of $2x$.

8. If $\operatorname{cosec}\left(\dfrac{\pi}{4}+ix\right)=u+iv$, prove that, $(u^2+v^2)^2=2(u^2-v^2)$

ANSWERS:

1. $\dfrac{2\cos x\cosh 2y+2i\sin x\sinh y}{\cos 2x+\cosh 2y}$ **7.** $\log_e^3$

8.18.2 Long answer type questions

1. If $\tan\left(\frac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2+2x=1$.

2. Show that $\tan\left(\frac{u+iv}{2}\right)=\frac{\sin u+i\sinh v}{\cos u+\cosh v}$

3. If $\cos(\theta+i\phi)=\cos\alpha+i\sin\alpha$, prove that

(a) $\cos 2\theta+\cosh 2\phi=2$ **(b)** $\sin^4\theta=\sin^2\alpha$ **(c)** $\sinh^4\phi=\sin^2\alpha$.

4. If $\frac{u-1}{u+1}=\sin(x+iy)$, find u.

5. Solve the equation $17\cosh x+18\sinh x=1$, for real values of x.

6. If $\tan(\alpha+i\beta)=i$, where α and β are real, then prove that α is indeterminate and β is infinite.

7. If $\tan(\theta+i\phi)=\sin(x+iy)$, prove that, $\coth y\sinh 2\phi=\cot x\sin 2\theta$

8. If $\cos^{-1}(x+iy)=A+iB$, prove that

(a) $x^2\sec h^2B+y^2\cos ech^2B=1$ **(b)** $x^2\sec^2 A-y^2\cos ec^2 A=1$.

ANSWERS:

4. $\tan^{-1}\frac{2\cos x\sinh y}{\cos^2 x-\sinh^2 y}$ **5.** $-\log 5$

UNIT 9: INVERSE HYPERBOLIC AND TRIGONOMETRIC FUNCTION AND LOGARITHM OF COMPLEX NUMBER

CONTENTS

9.1 *OBJECTIVE*

After reading this unit you will be able to:

- Know about inverse functions, inverse circular functions, and inverse hyperbolic functions.
- Relation between inverse hyperbolic and circular functions
- Find logarithm of complex quantity and some important results of logarithm.

9.2 *INTRODUCTION*

In the last unit we studied about the circular function and hyperbolic functions of complex numbers and we separate their real and imaginary parts. Inverse of these functions of complex numbers is of equally importance as it provides us help to solve trigonometric equations. We already know about the inverse trigonometric functions on real numbers that this is the extension of it.

9.3 *INVERSE FUNCTIONS*

If $\sin\theta = z \quad \Rightarrow \quad \theta = \sin^{-1} z,$

So here θ is called the Inverse functions of z .that is

$\sin\theta = \frac{1}{2},$ then $\theta = \sin^{-1}\frac{1}{2},$ therefore θ is called the Inverse functions of $\frac{1}{2}$.

Similarly, we can define inverse function $\cos\theta,\ \tan\theta,\ \cot\theta,\ \sec\theta,$ $\cos ec\theta$ etc.

9.4 *INVERSE CIRCULAR FUNCTIONS OF COMPLEX QUANTITIES*

Suppose $\sin(x+iy) = u+iv,$ then $(x+iy)$ is said to be the sine inverse of $(u+iv)$ and is denoted by $\sin^{-1}(u+iv)$. Thus

$$\sin(x+iy) = u+iv \quad \Rightarrow \quad (x+iy) = \sin^{-1}(u+iv)$$

Also, if $\sin(x+iy) = u+iv,$ then

$u+iv = \sin[n\pi + (-1)^n(x+iy)]$, Where n is any integer.

Therefore the general value of inverse sine of $(u+iv)$ is

$$[n\pi + (-1)^n(x+iy)],$$

Therefore the inverse sine of $(u+iv)$ is many-valued function. Its principal values of $[n\pi + (-1)^n(x+iy)]$ is that for which the real part lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The principal value is denoted by $\sin^{-1}(u+iv)$, then

$$\sin^{-1}(u+iv) = n\pi + (-1)^n\sin^{-1}(u+iv)$$

Similarly if $\cos(x+iy)=u+iv,$ then the general value of **cosine inverse** of $u+iv$ is

$$\cos^{-1}(u+iv)=2n\pi\pm\cos^{-1}(u+iv)$$

and the principal value is that for which its real part lies between 0 and π.
Again, if $\tan(x+iy)=u+iv,$ then the general value of **tangent inverse** of $u+iv$ is

$$\tan^{-1}(u+iv)=n\pi+\cos^{-1}(u+iv)$$

and the principal value is that for which its real part lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

9.5 *INVERSE HYPERBOLIC FUNCTIONS*

Let x and y be any two complex numbers. If $y=\sinh x,$ then x is said to be the inverse sine hyperbolic of y and is

$$x=\sinh^{-1} y$$

The other inverse hyperbolic functions $\cosh^{-1} y, \tanh^{-1} y, \coth^{-1} y,$ $\sec h^{-1} y,$ and $\cos ech^{-1} y$ are defined similarly.
The inverse hyperbolic functions can also be expressed as the logarithm functions as

(a) To prove that $\sinh^{-1} x=\log\left(x+\sqrt{x^2+1}\right)$

Proof: Let $\sinh^{-1} x=y \quad\Rightarrow\quad x=\sinh y$

$$\Rightarrow \quad x=\frac{e^y-e^{-y}}{2} \quad\Rightarrow\quad e^y-e^{-y}=2x$$

$$\Rightarrow \quad e^{2y}-2xe^y-1=0$$ (This is quadratic in e^y)

$$\Rightarrow \quad e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}=x\pm\sqrt{x^2+1}$$ (Taking logarithm and positive sign only)

$$\Rightarrow \quad y=\log\left(x\pm\sqrt{x^2+1}\right) \quad\Rightarrow$$

$$\sinh^{-1} x=\log\left(x+\sqrt{x^2+1}\right)$$

(b) To prove that $\cosh^{-1} x=\log\left(x+\sqrt{x^2-1}\right)$

Proof: Let $\cosh^{-1} x=y \quad\Rightarrow\quad x=\cosh y$

$\Rightarrow \quad x = \dfrac{e^y + e^{-y}}{2} \quad \Rightarrow \quad e^y + e^{-y} = 2x$

$\Rightarrow \quad e^{2y} - 2xe^y + 1 = 0$ (This is quadratic in e^y)

$\Rightarrow \quad e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$ (Taking logarithm and positive sign only)

$\Rightarrow \quad y = \log\left(x \pm \sqrt{x^2 - 1}\right) \quad \Rightarrow$

$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$

(c) To prove that $\tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$

Proof: Let $\tanh^{-1} x = y \quad \Rightarrow \quad x = \tanh y$

$\Rightarrow \quad x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

Applying componendo and dividendo, we get

$\Rightarrow \quad \dfrac{1+x}{1-x} = \dfrac{e^y}{e^{-y}} = e^{2y} \quad \Rightarrow \quad 2y = \log\left(\dfrac{1+x}{1-x}\right)$

(Taking logarithm)

$\Rightarrow \quad y = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right) \quad \Rightarrow$

$\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$

(d) Similarly, we can prove that $\coth^{-1} x = \dfrac{1}{2}\log\dfrac{x+1}{x-1}$

(e) To prove that $\operatorname{sec}h^{-1} x = \log\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$

Proof: Let $\operatorname{sec}h^{-1} x = y \quad \Rightarrow \quad x = \operatorname{sec}hy$

$\Rightarrow \quad x = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad x = \dfrac{2e^y}{e^{2y} + 1}$

$\Rightarrow \quad xe^{2y} - 2e^y + x = 0$ (This is quadratic in e^y)

$\Rightarrow \quad e^y = \dfrac{2 \pm \sqrt{4 - 4x^2}}{2x} = \dfrac{1 \pm \sqrt{1-x^2}}{x}$

$\Rightarrow \qquad e^y = \dfrac{1 \pm \sqrt{1-x^2}}{x}$ (Taking logarithm and positive sign only)

$\Rightarrow \qquad y = \log\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) \qquad \Rightarrow$

$$\sec h^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

(f) Similarly, we can prove that $\cos ech^{-1} x = \log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$

9.6 *RELATION BETWEEN INVERSE HYPERBOLIC AND CIRCULAR FUNCTIONS*

(a) If $x = \sinh y,$ then

$ix = i \sinh y = \sin(iy) \qquad \Rightarrow \qquad iy = \sin^{-1}(ix)$

$\Rightarrow \qquad y = \dfrac{1}{i}\sin^{-1}(ix) = -i\sin^{-1}(ix) \qquad \Rightarrow$

$$y = \sinh^{-1} x = -i\sin^{-1}(ix).$$

(b) If $x = \cosh y,$ then

$x = \cos(iy) \qquad \Rightarrow \qquad iy = \cos^{-1} x$

$\Rightarrow \qquad y = \dfrac{1}{i}\cos^{-1} x = -i\cos^{-1} x. \qquad \Rightarrow$

$$y = \cosh^{-1} x = -i\cos^{-1} x$$

(c) If $x = \tanh y,$ then

$ix = i\tanh y = \tan(iy) \qquad \Rightarrow \qquad iy = \tan^{-1}(ix)$

$\Rightarrow \qquad y = \dfrac{1}{i}\tan^{-1}(ix) = -i\tan^{-1}(ix) \qquad \Rightarrow$

$$y = \tanh^{-1} x = -i\tan^{-1}(ix).$$

Example 1. Separate $\sin^{-1}(\alpha + i\beta)$ into real and imaginary parts.

Proof: Let $\sin^{-1}(\alpha + i\beta) = x + iy \qquad \Rightarrow$

$$\alpha + i\beta = \sin(x + iy)$$

$\Rightarrow \qquad \alpha + i\beta = \sin x \cos iy + \cos x \sin iy$

$$= \sin x \cosh y + i\cos x \sinh y$$

Equating real and imaginary parts, we get

$$\alpha = \sin x \cosh y \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

and $$\beta = \cos x \sinh y \quad \ldots\ldots\ldots\ldots\ldots..(2)$$

We know that $\cosh^2 y - \sinh^2 y = 1$

From (1) and (2), we get

$$\left(\frac{\alpha}{\sin x}\right)^2 - \left(\frac{\beta}{\cos x}\right)^2 = 1$$

$$\Rightarrow \quad \alpha^2 \cos^2 x - \beta^2 \sin^2 x = \sin^2 x \cos^2 x$$

$$\Rightarrow \quad \alpha^2 (1-\sin^2 x) - \beta^2 \sin^2 x = \sin^2 x (1-\sin^2 x)$$

$$\Rightarrow \quad \sin^4 x - (\alpha^2 + \beta^2 + 1)\sin^2 x + \alpha^2 = 0$$

(This is quadratic equation in $\sin^2 x$)

$$\sin^2 x = \frac{(\alpha^2+\beta^2+1) \pm \sqrt{(\alpha^2+\beta^2+1)^2 - 4\alpha^2}}{2}$$

$$\Rightarrow \quad \sin x = \sqrt{\frac{(\alpha^2+\beta^2+1) \pm \sqrt{(\alpha^2+\beta^2+1)^2 - 4\alpha^2}}{2}}$$

$$\Rightarrow \quad x = \sin^{-1}\sqrt{\frac{(\alpha^2+\beta^2+1) \pm \sqrt{(\alpha^2+\beta^2+1)^2 - 4\alpha^2}}{2}}$$

We know that $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \quad \left(\frac{\alpha}{\cosh y}\right)^2 + \left(\frac{\beta}{\sinh y}\right)^2 = 1$$

(From (1) and (2))

$$\Rightarrow \quad \alpha^2 \sinh^2 y + \beta^2 \cosh^2 y = \sinh^2 y \cosh^2 y$$

$$\Rightarrow \quad \alpha^2 \sinh^2 y + \beta^2 (1+\sinh^2 y) = \sinh^2 y (1+\sinh^2 y)$$

$$\Rightarrow \quad \sinh^4 y - (\alpha^2+\beta^2-1)\sinh^2 y - \beta^2 = 0$$

(This is quadratic equation in $\sinh^2 y$)

$$\sinh^2 y = \frac{(\alpha^2+\beta^2-1) \pm \sqrt{(\alpha^2+\beta^2-1)^2 + 4\beta^2}}{2}$$

$$\Rightarrow \quad \sinh y = \sqrt{\frac{(\alpha^2+\beta^2-1) \pm \sqrt{(\alpha^2+\beta^2-1)^2 + 4\beta^2}}{2}}$$

$$\Rightarrow \quad y = \sinh^{-1}\sqrt{\frac{(\alpha^2+\beta^2-1) \pm \sqrt{(\alpha^2+\beta^2-1)^2 + 4\beta^2}}{2}}$$

Real part $=\sin^{-1}\sqrt{\dfrac{(\alpha^2+\beta^2+1)\pm\sqrt{(\alpha^2+\beta^2+1)^2-4\alpha^2}}{2}}$

Imaginary part $=\sinh^{-1}\sqrt{\dfrac{(\alpha^2+\beta^2-1)\pm\sqrt{(\alpha^2+\beta^2-1)^2+4\beta^2}}{2}}$

Example 2. Separate $\tan^{-1}(\alpha+i\beta)$ into real and imaginary parts.

Proof: Let $\tan^{-1}(\alpha+i\beta)=x+iy$

……………(1)

$\Rightarrow \quad (\alpha+i\beta)=\tan(x+iy)$

So that $\quad (\alpha-i\beta)=\tan(x-iy)$

Now, $\quad \tan 2x=\tan[(x+iy)+(x-iy)]=\dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$=\frac{\alpha+i\beta+\alpha-i\beta}{1-(\alpha+i\beta)(\alpha-i\beta)}=\frac{2\alpha}{1-\alpha^2-\beta^2}$$

$\Rightarrow \quad \tan 2x=\dfrac{2\alpha}{1-\alpha^2-\beta^2} \quad \Rightarrow$

$$2x=\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$$

$\Rightarrow \quad x=\dfrac{1}{2}\tan^{-1}\left(\dfrac{2\alpha}{1-\alpha^2-\beta^2}\right)$

……………..(2)

And $\quad \tan 2iy=\tan[(x+iy)-(x-iy)]$

$$=\frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$=\frac{\alpha+i\beta-\alpha+i\beta}{1+(\alpha+i\beta)(\alpha-i\beta)}=\frac{2i\beta}{1+\alpha^2+\beta^2}$$

$\Rightarrow \quad i\tanh 2y=\dfrac{2i\beta}{1+\alpha^2+\beta^2} \quad \Rightarrow$

$$\tanh 2y=\frac{2\beta}{1+\alpha^2+\beta^2}$$

$\Rightarrow \quad 2y=\tanh^{-1}\left(\dfrac{2\beta}{1+\alpha^2+\beta^2}\right) \quad \Rightarrow$

$$y=\frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right)\text{……….(3)}$$

$$\tan^{-1}(\alpha+i\beta)=\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)+\frac{i}{2}\tanh^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right)$$

Example 3. Show that $\tan^{-1}\left(i\frac{x-a}{x+a}\right)=-\frac{i}{2}\log\frac{a}{x}$

Solution: Let $\tan^{-1}\left(i\frac{x-a}{x+a}\right)=y \Rightarrow$

$$\left(i\frac{x-a}{x+a}\right)=\tan y$$

$$\Rightarrow \left(\frac{x-a}{x+a}\right)=\frac{1}{i}\tan y=-i\tan y \Rightarrow$$

$$\left(\frac{x-a}{x+a}\right)=\frac{-i\sin y}{\cos y}$$

By using componendo and dividendo, we get

$$\Rightarrow \frac{(x+a)-(x-a)}{(x+a)+(x-a)}=\frac{\cos y+i\sin y}{\cos y-i\sin y}$$

$$\Rightarrow \frac{2a}{2x}=\frac{e^{iy}}{e^{-iy}} \Rightarrow \frac{a}{x}=e^{2iy} \quad \text{(Taking logarithm)}$$

$$\Rightarrow 2iy=\log_e\left(\frac{a}{x}\right) \Rightarrow$$

$$y=\frac{1}{2i}\log_e\left(\frac{a}{x}\right)=-\frac{i}{2}\log_e\left(\frac{a}{x}\right)$$

$$\tan^{-1}\left(i\frac{x-a}{x+a}\right)=-\frac{i}{2}\log\frac{a}{x}$$

Example 4. Show that

(a) $\cosh^{-1}\sqrt{1+x^2}=\sinh^{-1}x$ **(b)** $\cosh^{-1}\sqrt{1+x^2}=\tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution: (a) Let $\cosh^{-1}\sqrt{1+x^2}=y$ ……………(1)

$$\Rightarrow \sqrt{1+x^2}=\cosh y$$

On squaring on both sides, we get

$$1+x^2=\cosh^2 y \Rightarrow$$

$$x^2=\cosh^2 y-1=\sinh^2 y$$

$\Rightarrow \qquad x = \sinh y \qquad \Rightarrow$

$y = \sinh^{-1} x$

$$\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1} x$$

[By equation (1)]

(b) Dividing (3) by (2), we get

$$\frac{\sinh y}{\cosh y} = \frac{x}{\sqrt{1+x^2}} \qquad \Rightarrow$$

$$\tanh y = \frac{x}{\sqrt{1+x^2}}$$

$\Rightarrow \qquad y = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \qquad \Rightarrow$

$$\cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Example 5. Prove that

$$\tan^{-1}(\cot\theta \tanh\phi) = \frac{1}{2i}\log\frac{\sin(\theta+i\phi)}{\sin(\theta-i\phi)}$$

Proof: We have

$$\textbf{R.H.S.} = \frac{1}{2i}\log\frac{\sin(\theta+i\phi)}{\sin(\theta-i\phi)}$$

$$= \frac{1}{2i}\log\frac{\sin\theta\cos(i\phi)+\cos\theta\sin(i\phi)}{\sin\theta\cos(i\phi)-\cos\theta\sin(i\phi)}$$

$$= \frac{1}{2i}\log\frac{\sin\theta\cosh\phi + i\cos\theta\sinh\phi}{\sin\theta\cosh\phi - i\cos\theta\sinh\phi}$$

(Divided by $\sin\theta\cosh\phi$)

$$= \frac{1}{2i}\log\frac{1+i\cot\theta\tanh\phi}{1-i\cot\theta\tanh\phi}$$

$$= \frac{1}{i}\left(\frac{1}{2}\log\frac{1+i\cot\theta\tanh\phi}{1-i\cot\theta\tanh\phi}\right)$$

$$= \frac{1}{i}\tanh^{-1}(i\cot\theta\tanh\phi)$$

$$\left[\because \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}\right]$$

$$= -i\tanh^{-1}(i\cot\theta\tanh\phi)$$

…………….(1)

Now, Let $\tanh^{-1}(i\cot\theta\tanh\phi) = x$
...……………..(2)

$$i\cot\theta\tanh\phi = \tanh x = \frac{1}{i}\tan(ix)$$

$[\because \tan(i\theta) = i\tanh\theta]$

$\Rightarrow$ $i^2\cot\theta\tanh\phi = \tan(ix)$ $\Rightarrow$
$-\cot\theta\tanh\phi = \tan(ix)$

$\Rightarrow$ $ix = \tan^{-1}(-\cot\theta\tanh\phi)$ $\Rightarrow$
$ix = -\tan^{-1}(\cot\theta\tanh\phi)$ (From 2)

$\tanh(i\cot\theta\tanh\phi) = -\tan^{-1}(\cot\theta\tanh\phi)$
…………(3)

From equation (1) and (3), we get

Hence, $\tan^{-1}(\cot\theta\tanh\phi) = \dfrac{1}{2i}\log\dfrac{\sin(\theta+i\phi)}{\sin(\theta-i\phi)}$

Example 6. Prove that

$$\tan^{-1}(\cos\theta + i\sin\theta) = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Solution: Let $\tan^{-1}(\cos\theta + i\sin\theta) = x + iy$
……………..(1)

$\Rightarrow$ $(\cos\theta + i\sin\theta) = \tan(x+iy)$

So that $(\cos\theta - i\sin\theta) = \tan(x-iy)$

Now $\tan 2x = \tan[(x+iy)+(x-iy)] = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{1-(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{1-1} = \frac{2\cos\theta}{0} = \infty$$

$\Rightarrow$ $\tan 2x = \infty$ $\Rightarrow$ $2x = n\pi + \dfrac{\pi}{2}$ $\Rightarrow$ $x = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

$$\tan 2iy = \tan[(x+iy)-(x-iy)]$$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{1+(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)}$$

$$= \frac{2i \sin \theta}{1 + (\cos^2 \theta + \sin^2 \theta)} = \frac{2i \sin \theta}{1+1} = i \sin \theta$$

$\Rightarrow$ $\quad i \tanh 2y = i \sin \theta \quad \Rightarrow$

$\tan 2hy = \sin \theta$

$\Rightarrow$ $\quad \dfrac{e^{2y} - e^{-2y}}{e^{2y} + e^{-2y}} = \sin \theta \quad \Rightarrow$

$\dfrac{e^{2y} + e^{-2y}}{e^{2y} - e^{-2y}} = \dfrac{1}{\sin \theta}$

By using componendo and dividendo rule, we get

$$\frac{e^{2y}}{e^{-2y}} = \frac{1+\sin \theta}{1-\sin \theta} \quad \Rightarrow \quad e^{4y} = \frac{1+\sin \theta}{1-\sin \theta}$$

$\Rightarrow$ $\quad e^{2y} = \left(\dfrac{1+\sin \theta}{1-\sin \theta}\right)^{\frac{1}{2}} \quad \Rightarrow$

$2y = \dfrac{1}{2} \log\left(\dfrac{1+\sin \theta}{1-\sin \theta}\right)$

$$= \frac{1}{2} \log\left(\frac{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} + 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} - 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}\right)$$

$$= \frac{1}{2} \log \frac{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)^2}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2} = 2 \times \frac{1}{2} \log \frac{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)}$$

$\Rightarrow$ $\quad 2y = \log \dfrac{\left(1 + \tan \frac{\theta}{2}\right)}{\left(1 - \tan \frac{\theta}{2}\right)} = \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

$\Rightarrow$ $\quad y = \dfrac{1}{2} \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

Hence from (1) $\tan^{-1}(\cos \theta + i \sin \theta) = x + iy$

$$\tan^{-1}(\cos \theta + i \sin \theta) = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2} \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Example 7. Prove that $\quad \sinh^{1} x = \tanh^{-1} \dfrac{x}{\sqrt{1+x^2}}$

Solution: Let $\sinh^{1} x = y,$ then $x = \sinh\ y$

Now, **R.H.S.** $= \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}} = \tanh^{-1}\dfrac{\sinh y}{\sqrt{1+\sinh^2 y}}$

$$= \tanh^{-1}\left(\frac{\sinh y}{\cosh y}\right) = \tanh^{-1}(\tanh y) = y = \sinh^{-1} x =$$

L.H.S.

Example 8. If $x > y$, then prove that

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4} + \frac{i}{2}\log\frac{x+y}{x-y}$$

Proof: Let $A + iB = \tan^{-1}\left(\dfrac{x+iy}{x-iy}\right)$

…………….(1)

$$= \tan^{-1}\left(\frac{x+iy}{x-iy}\times\frac{x+iy}{x+iy}\right)$$

$$= \tan^{-1}\frac{(x+iy)^2}{x^2+y^2} = \tan^{-1}\left(\frac{x^2+(iy)^2+2ixy}{x^2+y^2}\right)$$

$\Rightarrow$ $A + iB = \tan^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2} + i\dfrac{2xy}{x^2+y^2}\right)$

So that $A - iB = \tan^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2} - i\dfrac{2xy}{x^2+y^2}\right)$

Now $\tan 2A = \tan[(A+iB)+(A-iB)]$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{\left(\dfrac{x^2-y^2}{x^2+y^2} + i\dfrac{2xy}{x^2+y^2}\right) + \left(\dfrac{x^2-y^2}{x^2+y^2} - i\dfrac{2xy}{x^2+y^2}\right)}{1-\left(\dfrac{x^2-y^2}{x^2+y^2} + i\dfrac{2xy}{x^2+y^2}\right)\left(\dfrac{x^2-y^2}{x^2+y^2} - i\dfrac{2xy}{x^2+y^2}\right)}$$

$$= \frac{2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{1 - \left(\frac{(x^2 - y^2)^2}{(x^2 + y^2)^2} + \frac{4x^2y^2}{(x^2 + y^2)^2}\right)}$$

$$= \frac{2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{1 - \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2}} = \frac{2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)}{1 - 1} = \infty$$

$$\tan 2A = \tan \frac{\pi}{2} \qquad \Rightarrow \qquad A = \frac{\pi}{4}$$

Again $\tan 2iB = \tan[(A + iB) - (A - iB)]$

$$= \frac{\tan(A + iB) - \tan(A - iB)}{1 + \tan(A + iB)\tan(A - iB)}$$

$$= \frac{\left(\frac{x^2 - y^2}{x^2 + y^2} + i\frac{2xy}{x^2 + y^2}\right) - \left(\frac{x^2 - y^2}{x^2 + y^2} - i\frac{2xy}{x^2 + y^2}\right)}{1 + \left(\frac{x^2 - y^2}{x^2 + y^2} + i\frac{2xy}{x^2 + y^2}\right)\left(\frac{x^2 - y^2}{x^2 + y^2} - i\frac{2xy}{x^2 + y^2}\right)}$$

$$= \frac{\left(i\frac{4xy}{x^2 + y^2}\right)}{1 + \left(\frac{(x^2 - y^2)^2}{(x^2 + y^2)^2} + \frac{4x^2y^2}{(x^2 + y^2)^2}\right)}$$

$$= \frac{\left(i\frac{4xy}{x^2 + y^2}\right)}{1 + \left(\frac{(x^2 - y^2)^2}{(x^2 + y^2)^2}\right)} = \frac{\left(i\frac{4xy}{x^2 + y^2}\right)}{1 + 1}$$

$$\Rightarrow \qquad \tan 2iB = \frac{\left(i\frac{4xy}{x^2 + y^2}\right)}{1 + 1} = i\frac{2xy}{x^2 + y^2}$$

$$\Rightarrow \qquad i\tanh 2B = i\frac{2xy}{x^2 + y^2} \qquad \Rightarrow$$

$$\tanh 2B = \frac{2xy}{x^2 + y^2}$$

$$\Rightarrow \qquad \frac{e^{2B} - e^{-2B}}{e^{2B} + e^{-2B}} = \frac{2xy}{x^2 + y^2}$$

By using componendo and dividendo rule, we get

$$\frac{(e^{2B}+e^{-2B})+(e^{2B}-e^{-2B})}{(e^{2B}+e^{-2B})-(e^{2B}-e^{-2B})}=\frac{x^2+y^2+2xy}{x^2+y^2-2xy}$$

$$\Rightarrow \qquad \frac{2e^{2B}}{2e^{-2B}}=\frac{(x+y)^2}{(x-y)^2} \qquad \Rightarrow \qquad e^{4B}=\frac{(x+y)^2}{(x-y)^2}$$

$$\Rightarrow \qquad e^{2B}=\frac{(x+y)}{(x-y)} \qquad \Rightarrow \qquad 2B=\log\frac{(x+y)}{(x-y)}$$

$$\Rightarrow \qquad B=\frac{1}{2}\log\frac{(x+y)}{(x-y)}$$

Hence from equation (1), we get

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right)=A+iB=\frac{\pi}{4}+\frac{i}{2}\log\frac{x+y}{x-y}$$

Example 9. Prove that $\sinh^{-1}(\cot x)=\log(\cot\, x+\operatorname{cosec} x)$

Proof: Let $\sinh^{-1}(\cot x)=y,$ then $\cot x=\sinh\, y$ ………………..(1)

$$\therefore \qquad \cosh y=\sqrt{1+\sinh^2 y}=\sqrt{1+\cot^2 x}=\operatorname{cosec} x$$ ………………..(2)

Adding equation (1) and (2), we get

$$\sinh\, y+\cosh y=\cot x+\operatorname{cosec} x$$

$$\Rightarrow \qquad \frac{e^y-e^{-y}}{2}+\frac{e^y+e^{-y}}{2}=\cot x+\operatorname{cosec} x$$

$$\Rightarrow \qquad e^y=\cot x+\operatorname{cosec} x$$

(Taking logarithm)

$$\Rightarrow \qquad y=\log(\cot\, x+\operatorname{cosec} x)$$

(From 1)

$$\sinh^{-1}(\cot x)=\log(\cot\, x+\operatorname{cosec} x)$$

Example 10. If $\sinh^{-1}(\theta+i\phi)=\alpha+i\beta,$ then prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2-x(1+\theta^2+\phi^2)+\theta^2=0.$$

Solution: We have $\sinh^{-1}(\theta+i\phi)=\alpha+i\beta$

So that $(\theta+i\phi)=\sin(\alpha+i\beta)$

$$\Rightarrow \qquad (\theta+i\phi)=\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta$$

Equating real and imaginary parts, we have

$$\theta = \sin\alpha\cosh\beta \qquad \text{……………(1)}$$

$$\phi = \cos\alpha\sinh\beta \qquad \text{……………(2)}$$

Now, $1+\theta^2+\phi^2 = 1+\sin^2\alpha\cosh^2\beta+\cos^2\alpha\sinh^2\beta$

$$= 1+\sin^2\alpha\cosh^2\beta+(1-\sin^2\alpha)(\cosh^2\beta-1)$$

$$= 1+\sin^2\alpha\cosh^2\beta+\cosh^2\beta-1-\sin^2\alpha\cosh^2\beta+\sin^2\alpha$$

$$= \sin^2\alpha+\cosh^2\beta$$

Thus $\sin^2\alpha+\cosh^2\beta = 1+\theta^2+\phi^2$ ………………(3)

And from (1), we get

$$\theta^2 = \sin^2\alpha\cosh^2\beta \qquad \text{………………(4)}$$

Therefore, $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2 - x(1+\sin^2\alpha+\cosh^2\beta)+\theta^2 = 0$$

From (3) and (4), we get

$$x^2 - x(1+\theta^2+\phi^2)+\theta^2 = 0.$$

9.7 *LOGARITHM OF COMPLEX QUANTITY*

If $u+iv = e^{x+iy}$, then $(x+iy)$ is said to be logarithm of $(u+iv)$ to the base e such that

$$x+iy = \log(u+iv) \qquad \text{………………..(1)}$$

Now, $(u+iv) = e^{x+iy}\times 1$

$\Rightarrow$ $(u+iv) = e^{x+iy}\times e^{2n\pi i}$ Where n is an integer

$\Rightarrow$ $\log(u+iv) = \log e^{x+(y+2n\pi)i}$ $\Rightarrow$ $\log(u+iv) = x+(y+2n\pi)i$, $[\because \log e = 1]$

$\Rightarrow$ $\log(u+iv) = (x+iy)+2n\pi i$ (By using from equation1)

$\Rightarrow$ $\log(u+iv) = \log(u+iv)+2n\pi i$

Here logarithm of a complex quantity is a many valued function. When n = 0 then, $\log(u+iv)$ is called principle value. General value can be find by adding $2n\pi i$ in principle value and written as $Log_e(u+iv)$.

$Log_e(u+iv) = 2n\pi i + \log_e(u+iv)$

Note: We know that the following result will also be true:

$$\log_e mn = \log_e m + \log_e n$$

$$\log_e \frac{m}{n} = \log_e m - \log_e n$$

$$\log_e m^n = n \log_e m$$

9.8 *BY LOGARITHM TO SEPARATE THE REAL AND IMAGINARY PARTS*

Let $x = r\cos\theta$ and $y = r\sin\theta$

Squaring and adding, we get

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2}$$

And also on dividing, we get

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ $\log_e(x+iy) = \log_e r(\cos\theta + i\sin\theta)$

$$= \log_e r + \log_e(\cos\theta + i\sin\theta)$$

$$= \log_e r + \log_e(\cos\theta + i\sin\theta)$$

$$= \log_e r + \log_e e^{i\theta} = \log_e r + i\theta$$

$$= \log_e \sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}$$

$$\log_e(x+iy) = \log_e \sqrt{x^2 + y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

and $Log_e(x+iy) = \log\sqrt{x^2 + y^2} + i\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi i$

9.9 *SOME IMPORTANT RESULT OF THE LOGARITHM*

(a) Logarithm of a real negative number: Let x be a positive real number so that $-x$ negative real number, we have

$$-x = x\times(-1) = x\times(\cos\pi + i\sin\pi)$$

$$\Rightarrow \quad -x = x\times e^{\pi i}$$

$$\Rightarrow \quad \log(-x) = \log x + \log e^{\pi i} = \log x + \log(\cos\pi + i\sin\pi)$$

$$= \log x + \log\{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\}$$

$$= \log x + \log\{\cos(2n+1)\pi + i\sin(2n+1)\pi\}$$

$$= \log x + \log e^{(2n+1)\pi i} = \log x + (2n+1)\pi i$$

$$\Rightarrow \quad \log(-x) = \log x + (2n+1)\pi i$$

(b) Logarithm of a real positive number: Let x be a positive real number, we have

$$x = x\times 1 = x\times(\cos 0 + i\sin\pi 0)$$

$$\Rightarrow \quad \log x = \log x + \log(\cos 0 + i\sin 0)$$

$$= \log x + \log\{\cos(2n\pi+0) + i\sin(2n\pi+0)\}$$

$$= \log x + \log(\cos 2n\pi + i\sin 2n\pi)$$

$$= \log x + \log e^{2n\pi i} = \log x + 2n\pi i$$

$$\Rightarrow \quad \log x = \log x + 2n\pi i$$

(c) Logarithm of a purely imaginary number: Let x be a positive real number so that ix be a purely imaginary number, we have

$$xi = x\times i = x\times\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \log(xi) = \log x + \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \log x + \log\left\{\cos\left(2n\pi+\frac{\pi}{2}\right) + i\sin\left(2n\pi+\frac{\pi}{2}\right)\right\}$$

$$= \log x + \log\left\{\cos\left(2n+\frac{1}{2}\right)\pi + i\sin\left(2n+\frac{1}{2}\right)\pi\right\}$$

$$= \log x + \log e^{\left(2n+\frac{1}{2}\right)\pi} = \log x + i\left(2n+\frac{1}{2}\right)\pi$$

$$\Rightarrow \quad \log(xi) = \log x + i\left(2n+\frac{1}{2}\right)\pi$$

Example 1. Express in the term of $(a+ib)$

(a) $\log_e(-5)$ (b) $\log_e i$

(c) $\log_e 5$

Solution: (a) $\log_e(-5) = \log_e\{5\times(-1)\} = \log_e\{5\times(\cos\pi + i\sin\pi)\}$

$$= \log_e\{5\times(\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\}$$

$$= \log_e 5 + \log_e\{(\cos(2n+1)\pi + i\sin(2n+1)\pi\}$$

$$= \log_e 5 + i(2n+1)\pi$$

(b) We have $\log_e i = \log_e\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\}$

$$= \log_e\left\{\cos\left(2n\pi+\frac{\pi}{2}\right) + i\sin\left(2n\pi+\frac{\pi}{2}\right)\right\}$$

$$= \log_e\left\{\cos\left(2n+\frac{1}{2}\right)\pi + i\sin\left(2n+\frac{1}{2}\right)\pi\right\}$$

$$= \log_e e^{\left(2n+\frac{1}{2}\right)\pi i} = \frac{1}{2}(4n+1)\pi i$$

(c) We have $\log_e(5) = \log_e(5\times 1) = \log 5 + \log(\cos 0 + i\sin 0)$

$$= \log 5 + \log\{\cos(2n\pi+0) + i\sin(2n\pi+0)\}$$

$$= \log 5 + \log(\cos 2n\pi + i\sin 2n\pi)$$

$$= \log 5 + \log e^{2n\pi i} = \log 5 + 2n\pi i$$

$\Rightarrow$ $\log 5 = \log 5 + 2n\pi i$

Example 2. Prove that $\log\frac{x+iy}{x-iy} = 2i\tan^{-1}\frac{y}{x}$

Solution: We know that

$$\log_e(x+iy) = \log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

Taking $\text{L.H.S.} = \log\frac{x+iy}{x-iy}$

$$= \log(x+iy) - \log(x-iy)$$

$$= \left[\log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\right] - \left[\log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(-\frac{y}{x}\right)\right]$$

$$= \log_e\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) - \log_e\sqrt{x^2+y^2} - i\left(-\tan^{-1}\frac{y}{x}\right)$$

$$= i\tan^{-1}\left(\frac{y}{x}\right) + i\tan^{-1}\left(\frac{y}{x}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right) =$$

R.H.S.

Example 3. Prove that $\log(1+i) = \frac{1}{2}\log 2 + i\left(2n + \frac{1}{4}\right)\pi$

Solution: Let $1 + i = r(\cos\theta + i\sin\theta)$, then

$$r\cos\theta = 1, \qquad 1 = r\sin\theta$$

Now, squaring and adding, we get

$$r^2(\cos^2\theta + \sin^2\theta) = 1 + 1 \quad \Rightarrow \quad r = \sqrt{2}$$

Again, dividing, we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{1}{1} \quad \Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\therefore \qquad 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left[\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(2n + \frac{1}{4}\right)\pi + i\sin\left(2n + \frac{1}{4}\right)\pi\right]$$

$$\log(1+i) = \log\left\{\sqrt{2}\left[\cos\left(2n + \frac{1}{4}\right)\pi + i\sin\left(2n + \frac{1}{4}\right)\pi\right]\right\}$$

$$= \log\left\{\sqrt{2}e^{\left(2n+\frac{1}{4}\right)\pi i}\right\} = \log\sqrt{2} + \log e^{\left(2n+\frac{1}{4}\right)\pi i}$$

$$= \frac{1}{2}\log 2 + i\left(2n + \frac{1}{4}\right)\pi$$

$$\Rightarrow \qquad \log(1+i) = \frac{1}{2}\log 2 + i\left(2n + \frac{1}{4}\right)\pi$$

Example 4. Prove that $\log\tan\left(\frac{\pi}{4} + i\frac{\theta}{2}\right) = i\tan^{-1}(\sinh\theta)$

Solution: $\quad$ L.H.S. $= \log\tan\left(\frac{\pi}{4} + i\frac{\theta}{2}\right) = \log\dfrac{\tan\frac{\pi}{4} + \tan\left(i\frac{\theta}{2}\right)}{1 - \tan\frac{\pi}{4}.\tan\left(i\frac{\theta}{2}\right)}$

$$= \log \frac{1 + i\tanh\frac{\theta}{2}}{1 - i\tanh\frac{\theta}{2}}$$

$$= \log\left(1 + i\tanh\frac{\theta}{2}\right) - \log\left(1 - i\tanh\frac{\theta}{2}\right)$$

$$= \left\{\frac{1}{2}\log\left(1 + \tanh^2\frac{\theta}{2}\right) + i\tan^{-1}\left(\tanh\frac{\theta}{2}\right)\right\}$$

$$- \left\{\frac{1}{2}\log\left(1 + \tanh^2\frac{\theta}{2}\right) - i\tan^{-1}\left(\tanh\frac{\theta}{2}\right)\right\}$$

$$= 2i\tan^{-1}\left(\tanh\frac{\theta}{2}\right) = i\left[2\tan^{-1}\left(\tanh\frac{\theta}{2}\right)\right]$$

$$= i\left[\tan^{-1}\left(\frac{2\tanh\frac{\theta}{2}}{1 - \tanh^2\frac{\theta}{2}}\right)\right]$$

$$= i\tan^{-1}\left(\frac{2\sinh\frac{\theta}{2}\cosh\frac{\theta}{2}}{\cosh^2\frac{\theta}{2} - \sinh^2\frac{\theta}{2}}\right)$$

$$= i\tan^{-1}(\sinh\theta) = \text{R.H.S.}$$

Hence, $$\log\tan\left(\frac{\pi}{4} + i\frac{\theta}{2}\right) = i\tan^{-1}(\sinh\theta)$$

Example 5. Prove that $\log\left(1 + e^{i\theta}\right) = \log\left(2\cos\frac{\theta}{2}\right) + \frac{1}{2}i\theta$

Solution: L.H.S. $= \log\left(1 + e^{i\theta}\right) = \log(1 + \cos\theta + i\sin\theta)$

$$= \log\left\{2\cos^2\frac{\theta}{2} + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right\}$$

$$= \log\left\{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right\}$$

$$= \log\left\{2\cos\frac{\theta}{2}\left(e^{i\frac{\theta}{2}}\right)\right\} = \log\left(2\cos\frac{\theta}{2}\right) + \log\left(e^{i\frac{\theta}{2}}\right)$$

$$= \log\left(2\cos\frac{\theta}{2}\right) + i\frac{\theta}{2} = \text{R.H.S.}$$

Hence, $\log\left(1+e^{i\theta}\right)=\log\left(2\cos\dfrac{\theta}{2}\right)+\dfrac{1}{2}i\theta$

Example 6. Prove that $\log(1+i)=\dfrac{1}{2}\log 2+\left(2n\pi+\dfrac{\pi}{4}\right)i$

Solution: We know that

$$\log_e(x+iy)=\log_e\sqrt{x^2+y^2}+i\left[2n\pi+\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Here $x=1,\quad y=1$

$\therefore$
$$\log_e(1+i)=\log_e\sqrt{1^2+1^2}+i\left[2n\pi+\tan^{-1}\left(\frac{1}{1}\right)\right]$$
$$=\log_e\sqrt{2}+i(2n\pi+\tan^{-1}1)$$
$$=\frac{1}{2}\log_e 2+i\left(2n\pi+\frac{\pi}{4}\right)$$

Hence, $\log(1+i)=\dfrac{1}{2}\log 2+\left(2n\pi+\dfrac{\pi}{4}\right)i$

Example 7. Separate $\log\sin(x+iy)$ into real and imaginary parts.

Solution: We have

$$\sin(x+iy)=\sin x\cos(iy)+\cos x\sin(iy)$$
$$=\sin x\cosh y+i\cos x\sinh y$$

$\Rightarrow \sin x\cosh y+i\cos x\sinh y=r(\cos\theta+i\sin\theta)$

Equating real and imaginary parts, we get

$$\sin x\cosh y=r\cos\theta \qquad \text{………………….(1)}$$

$$\cos x\sinh y=r\sin\theta \qquad \text{………………(2)}$$

Adding and squaring (1) and (2), we get

$$r^2(\sin^2\theta+\cos^2\theta)=\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y$$

$$r^2=\frac{1}{4}[(1-\cos 2x)(\cosh 2y+1)+(1+\cos 2x)(\cosh 2y-1)]$$

$$=\frac{1}{4}[2(\cosh 2y-\cos 2x)]$$

$$r^2=\frac{1}{2}[(\cosh 2y-\cos 2x)] \qquad \text{…………………(3)}$$

And again divided (2) by (1), we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{\cos x \sinh y}{\sin x \cosh y}$$

$$\tan\theta = \cot x \tanh y$$

………………..(4)

Now, $\sin(x+iy) = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log\sin(x+iy) = \log re^{i\theta}$$

$$= \log r + \log e^{i\theta}.e^{2n\pi i} = \log r + (2n\pi + \theta)i$$

$\Rightarrow$

$$\log\sin(x+iy) = \log\sqrt{\frac{1}{2}(\cosh 2y - \cos 2x)} + i\left[2n\pi + \tan^{-1}(\cot x \tanh y)\right]$$

Example 8. Prove that $\tan\left(i\log\frac{x-iy}{x+iy}\right) = \frac{2xy}{x^2-y^2}$

Solution: Let $x = r\cos\theta, \quad y = r\sin\theta$

Now $\frac{x-iy}{x+iy} = \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$

And dividing, we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \quad \Rightarrow \quad \tan\theta = \frac{y}{x}$$

Taking L.H.S., $= \tan\left(i\log\frac{x-iy}{x+iy}\right) = \tan\left(i\log e^{-2i\theta}\right)$

$$= \tan\left(i(-2i\theta)\right) = \tan\left(-i^2(2\theta)\right)$$

$$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\frac{y}{x}}{1-\frac{y^2}{x^2}} = \frac{2xy}{x^2-y^2}$$

Hence, $\tan\left(i\log\frac{x-iy}{x+iy}\right) = \frac{2xy}{x^2-y^2}$

Example 9. If $\log_e \sin(x+iy) = \alpha + i\beta,$ then prove that

$$\alpha = \frac{1}{2}\log_e\left(\frac{\cosh 2y - \cos 2x}{2}\right)$$

Solution: From example 7,

$$\log\sin(x+iy) = \log\sqrt{\frac{1}{2}(\cosh 2y - \cos 2x)} + i\left[2n\pi + \tan^{-1}(\cot x \tanh y)\right]$$

Hence, $\alpha = \log\sqrt{\frac{1}{2}(\cosh 2y - \cos 2x)} = \frac{1}{2}\log\left(\frac{\cosh 2y - \cos 2x}{2}\right)$

Example 10. If $\log_e \sin(\theta + i\phi) = \alpha + i\beta,$ then prove that

$$\phi = \frac{1}{2}\log_e\left[\frac{\cos(\theta-\beta)}{\cos(\theta+\beta)}\right]$$

Solution: Given that

$$\log_e \sin(\theta + i\phi) = \alpha + i\beta$$

$$\Rightarrow \quad \log_e\left[\sin\theta\cos(i\phi) + \cos\theta\sin(i\phi)\right] = \alpha + i\beta$$

$$\Rightarrow \quad \sin\theta\cosh\phi + i\cos\theta\sinh\phi = e^{(\alpha+i\beta)}$$

$$\Rightarrow \quad \sin\theta\cosh\phi + i\cos\theta\sinh\phi = e^{\alpha}e^{i\beta}$$

$$\Rightarrow \quad \sin\theta\cosh\phi + i\cos\theta\sinh\phi = e^{\alpha}(\cos\beta + i\sin\beta)$$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi = e^{\alpha}\cos\beta$$

……………..(1)

$$\cos\theta\sinh\phi = e^{\alpha}\sin\beta$$

……………..(2)

Divided (1) by (2), we get

$$\frac{\sin\theta\cosh\phi}{\cos\theta\sinh\phi} = \frac{e^{\alpha}\cos\beta}{e^{\alpha}\sin\beta}$$

$$\Rightarrow \quad \coth\phi = \frac{\cos\theta\cos\beta}{\sin\theta\sin\beta} \quad \Rightarrow \quad \frac{e^{\phi}+e^{-\phi}}{e^{\phi}-e^{-\phi}} = \frac{\cos\theta\cos\beta}{\sin\theta\sin\beta}$$

Using componendo and dividendo rule, we get

$$\frac{e^{\phi}+e^{-\phi}+e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}-e^{\phi}+e^{-\phi}} = \frac{\cos\theta\cos\beta+\sin\theta\sin\beta}{\cos\theta\cos\beta-\sin\theta\sin\beta}$$

$$\Rightarrow \quad \frac{2e^{\phi}}{2e^{-\phi}} = \frac{\cos(\theta-\beta)}{\cos(\theta+\beta)} \quad \Rightarrow \quad e^{2\phi} = \frac{\cos(\theta-\beta)}{\cos(\theta+\beta)}$$

Hence, $\phi = \frac{1}{2}\log_e\left[\frac{\cos(\theta-\beta)}{\cos(\theta+\beta)}\right]$

Example 11. If $i^{x+iy} = x + iy,$ then show that $x^2 + y^2 = e^{-(4n+1)y\pi}$.

Solution: We have $x + iy = e^{(x+iy)\log i} = e^{(x+iy)\left[2n\pi+\frac{\pi}{2}\right]i}$

$$= e^{(x+iy)\left[2n+\frac{1}{2}\right]\pi i}$$

$= e^{(ix-y)\left[2n+\frac{1}{2}\right]\pi}$

$$= e^{-\left[2n+\frac{1}{2}\right]\pi y} e^{ix\left[2n+\frac{1}{2}\right]}$$

9.10 *SUMMARY*

In this unit, we have discussed the **inverse hyperbolic functions** are the inverse functions of the hyperbolic functions. For a given value of a hyperbolic function, the corresponding inverse hyperbolic function provides the corresponding hyperbolic angle.**Trigonometric functions** are also known as **Circular Functions** can be simply defined as the functions of an angle of a triangle. It means that the relationship between the angles and sides of a triangle are given by these trig functions. The basic trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. Also, read trigonometric identities here.Since any nonzero complex number has infinitely many **complex logarithms**, the complex logarithm cannot be defined to be a single-valued function on the complex numbers, but only as a multi-valued function. Settings for a formal treatment of this are, among others, the associated Riemann surface, branches, or partial inverses of the complex exponential function.

9.11 *GLOSSARY*

1. **Trigonometric functions:** A function of an angle expressed as the ratio of two of the sides of a right triangle that contains that angle; the sine, cosine, tangent, cotangent, secant, cosecant.
2. **Logarithm:** Inverse function to exponential.
3. ***Inverse hyperbolic functions*:**Inverse *functions* of the hyperbolic may be solved in terms of e^x, the square root and the *logarithm.*

9.12 REFERENCES:-

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9.13 SELF ASSESSMENT QUSETIONS

9.13.1 Multiple choice questions:

1. Value of $\sinh^{-1} x$ is

(a) $\log\{x+\sqrt{x^2+1}\}$ (b) $\log\{x+\sqrt{x^2-1}\}$

(c) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ (d) None of these

2. If $\tan^{-1}\left(\frac{1}{2n^2}\right)=\tan^{-1}(2n+1)-\tan^{-1}y$, then the value of y is

(a) $n-1$ (b) $n+1$

(c) $2n-1$ (d) $2n$

3. If $\tan^{-1}\left(\frac{1}{1+n(n+1)}\right)=\tan^{-1}(n+1)-\tan^{-1}y$, then the value of y is

(a) n (b) 1

(c) $n+1$ (d) $2n-1$

4. The real part of $\sin^{-1}(\cos\theta+i\sin\theta)$ is

(a) $\cos^{-1}\sqrt{\sin\theta}$ (b) $\log\{\sqrt{\sin\theta}+\sqrt{1+\sin\theta}$

(c) $\sin^{-1}\sqrt{\sin\theta}$ (d) $\log\{\sqrt{\cos\theta}+\sqrt{1+\cos\theta}$

5. If $x>0$, then $\log(-x)$ is equal to

(a) $-\log(x)$ (b) $-\log(x)+i\pi$

(c) $\log(x)-i\pi$ (d) $\log(x)+i\pi$

6. The value of $\log(\cos\theta+i\sin\theta)$ is

(a) θ (b) $i\theta$

(c) $-i\theta$ (d) $-\theta$

7. $\log(\pm i)$ is equal to

(a) $\pm i$ (b) $\pm\frac{1}{2}\pi i$

(c) $\pm\pi i$ (d) $\pm 2\pi i$

8. The general value of $\log\sqrt{i}$ is

(a) $\frac{1}{4}(8n+1)\pi i$ (b) $\frac{1}{8}(8n+1)\pi i$

(b) (c) $\frac{1}{4}(2n+1)\pi i$ (d) $\frac{1}{8}(2n+1)\pi i$

ANSWERS:

1. a	**2.** c	**3.** a	**4.** a
5. d	**6.** b	**7.** b	**8.** a

9.13.2 Fill in the blanks:

1. Value of $\log(1+e^{i\theta})$ is

2. The relation between $\log_e z$ and $\log_a z$ is

3. If $z = re^{i\theta}$, then $\log z$ is equal to

4. The value of $\sec^{-1} x + \operatorname{cosec}^{-1} x$ is

5. The value of $\tan^{-1}\frac{2x}{1-x^2}$ is

6. The value of $\log(-1)$ is

7. The value of $\log(1+i)$ is

8. If $\log(x-iy) = A+iB$ where A and B are real, then

ANSWERS:

1. $\log\left(2\cos\frac{\theta}{2}\right)+\frac{i\theta}{2}$

2. $\log_a z=\frac{\log_e z}{\log_e a}$

3. $\log r+i\theta$

4. $\frac{\pi}{2}$

5. $2\tan^{-1}x$

6. $-\frac{\pi}{2}i$

7. $\frac{1}{2}\log 2+i\left(2n\pi+\frac{\pi}{4}\right)$

8. $A=\frac{1}{2}\log(x^2+y^2)$

9.14 SUGGESTED READINGS

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9.15 TERMINAL QUESTIONS

9.15.1 Short answer type questions

1. Prove that $\sin^{-1}(\operatorname{cosec}\theta)=\frac{\pi}{2}+i\log\cot\frac{\theta}{2}$

2. Separate into real and imaginary parts $\sin^{-1}(\cos\theta+i\sin\theta)$

3. Prove that $\tanh^{-1}(\sin\theta)=\cosh^{-1}(\sec\theta)$

4. If $\tan(\theta+i\phi)=\sin(x+iy)$, then prove that $\coth y\sinh 2\phi=\cot x\sin 2\theta.$

5. Prove that $\tan^{-1}\frac{3-2i}{3+2i}=\frac{\pi}{4}+\frac{i}{2}\log 5$

6. Prove that $\tan^{-1}(\sinh x) = \frac{1}{i}\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)$

7. Prove that, **(a)** $\tanh^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

(b) $\coth^{-1}(2\operatorname{cosec}^2 x - 1) = \log(\sec x)$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

8. If $\cos^{-1}(u+iv) = \alpha + i\beta,$ then prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2 - (1+u^2+v^2)x + u^2 = 0.$$

9. If $\cosh x = \sec\theta,$ then prove that $x = \log(\sec\theta \pm \tan\theta)$

10. Prove that $\sinh^{-1}(\cot x) = \log(\cot x + \operatorname{cosec} x)$

11. Prove that $\tan^{-1}(\cos\theta + i\sin\theta) = n\pi + \frac{\pi}{4} - \frac{i}{2}\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

12. Prove that $\sin^{-1}(\operatorname{cosec}\theta) = \{2n+(-1)^n\}\frac{\pi}{2} + i(-1)^n\log\cot\frac{\theta}{2}$.

13. If $\sin^{-1}(\cos\theta + i\sin\theta) = x+iy$, show that

(a) $x = \cos^{-1}\sqrt{\sin\theta}$ **(b)** $y = \log\left(\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right)$

14. If $\cosh^{-1}(x+iy) + \cosh^{-1}(x-iy) = \cosh^{-1} a,$

prove that $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$

15. Prove that $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\operatorname{cosec}\theta)$

16. Solve the general value of $\log_e i.$

17. Express $\log_e(-3)$ in the term of $a+ib.$

18. If $\log_e\sin(\theta + i\phi) = \alpha + i\beta,$ then prove that $\cos 2\theta = \cosh 2\phi - 2e^{2\alpha}$

ANSWERS: **2.** $\cos^{-1}\sqrt{\sin\theta} + i\log[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}]$

11. Hint: $\tan^{-1}(\cos\theta + i\sin\theta) = n\pi + \tan^{-1}(\cos\theta + i\sin\theta)$

12. Hint: $\sin^{-1}(\operatorname{cosec}\theta) = n\pi + (-1)^n\sin^{-1}(\operatorname{cosec}\theta)$ **16.** $\frac{(4n+1)}{2}\pi i$ **17.** $\log_e 3 + (2n+1)\pi i$

9.15.2 Long answer type questions

1. If $\log_e \log_e(x+iy) = \alpha + i\beta,$ then show that $y = x\tan\left(\tan\beta \log_e \sqrt{x^2+y^2}\right)$

2. Prove that $\log_e i = \frac{4m+1}{4n+1}$, where m and n are any integers and $i = \sqrt{-1}.$

3. Prove that

(a) $\log_e(-1) = i\pi$ **(b)** $\log_e i = \frac{i\pi}{2}$ **(c)** $i\log_e \frac{x-i}{x+i} = \pi - 2\tan^{-1} x$

4. Prove that $\log_e\left(\frac{1}{1-e^{i\theta}}\right) = \log_e\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) + i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

5. Show that $\log_e \sin(x+iy) = \frac{1}{2}\log_e \frac{\cosh 2y - \cos 2x}{2} + i\tan^{-1}(\cot x \tanh y)$

6. Prove that $\log_e \frac{(a-b)+i(a+b)}{(a+b)+i(a-b)} = i\left(2n\pi + \tan^{-1}\frac{2ab}{a^2-b^2}\right)$

7. Prove that $\log_e\left(\frac{\sin(x+iy)}{\sin(x-iy)}\right) = 2i\tan^{-1}\left(\cot x \tanh y\right)$

8. Prove that $\log_e(\cos\theta + i\sin\theta) = i\theta,$ if $-\pi < \theta \le \pi.$

9. Prove that $\log_e(1+i\tan\theta) = \log_e \sec\theta + i\theta.$

10. Prove that $\log_e(1+\cos 2\theta + i\sin 2\theta) = \log_e(2\cos\theta) + i\theta,$ if $-\pi < \theta \le \pi.$

11. Prove that $\log_e\left(\log_e \sin(x+iy)\right) = \frac{1}{2}\log_e(p^2+q^2) + i\tan^{-1}\left(\frac{q}{p}\right)$

Where, $p = \frac{1}{2}\log_e\left(\frac{\cosh 2y - \cos 2x}{2}\right)$ and $q = \tan^{-1}(\cot x \tanh y)$.

BLOCK IV: SUMMATION OF SERIES AND INFINITE PRODUCT

UNIT 10: SUMMATION OF SERIES

CONTENTS:

10.1 OBJECTIVES

After reading this unit you will be able to:

- Expand trigonometric functions
- Find summation of sine and cosine series by using
  - (i) Geometric progression
  - (ii) Binomial series
  - (iii) Exponential series
  - (iv) Logarithmic series
- Find sum of series by the difference method.

10.2 INTRODUCTION

In this chapter, we shall study with the expansion and summing up finite or infinite trigonometric series. There are two different methods for summation, known as the $C+iS$ method and difference method. First we shall discussexpansion of Trigonometric series and again summing up trigonometric series with $C+iS$ method and difference method.

10.3 EXPANSION OF TRIGONOMETRIC FUNCTIONS

(a) Expansion of $\sin n\theta$ and $\cos n\theta$ in the powers of $\sin\theta$ and $\cos\theta$

Trigonometric Functions can be easily expansion of $\sin n\theta$ and $\cos n\theta$ in the powers of $\sin\theta$ and $\cos\theta$ with the help of De-Moivre's theorem, such that

Mathematically, $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

By Using the Binomial theorem, we get

$$\cos n\theta + i\sin n\theta = {}^nC_0(\cos\theta)^n + {}^nC_1(\cos\theta)^{n-1}(i\sin\theta) + {}^nC_2(\cos\theta)^{n-2}(i\sin\theta)^2$$

$$+ {}^nC_3(\cos\theta)^{n-3}(i\sin\theta)^3 + \dots\dots\dots\dots\dots + {}^nC_n(\cos\theta)^{n-n}(i\sin\theta)^n$$

$$= \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta + {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta$$

$$+ {}^nC_4\cos^{n-4}\sin^4\theta + {}^nC_5\cos^{n-5}\sin^5\theta + \dots\dots\dots\dots + (i\sin\theta)^n$$

$$= \left(\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - {}^nC_6\cos^{n-6}\theta\sin^6\theta + \dots\right)$$

$$+ i\left({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \dots\dots\right)$$

Equating real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - {}^nC_6\cos^{n-6}\theta\sin^6\theta + \dots\dots\dots\dots \quad (1)$$

$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta - \ldots\ldots\ldots\ldots\ldots\ldots$ (2)

Now every $\sin^2\theta$ replace by $(1-\cos^2\theta)$ in (1) and every $\cos^2\theta$ replace by $(1-\sin^2\theta)$ in (2), we get the expansions of $\cos n\theta$ in the powers of $\cos\theta$ and $\sin n\theta$ in the powers of $\sin\theta$.

Dividing (2) by (1), we get

$$\frac{\sin n\theta}{\sin n\theta} = \frac{{}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta - \ldots\ldots\ldots\ldots\ldots\ldots}{\cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - {}^nC_6 \cos^{n-6}\theta \sin^6\theta + \ldots\ldots\ldots\ldots}$$

Again dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \ldots\ldots\ldots\ldots\ldots\ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \ldots\ldots\ldots\ldots}$$

Example 1. Expand $\cos 6\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution: We have $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

By Using the Binomial theorem, we get

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta + {}^6C_1 \cos^5\theta(i\sin\theta) + {}^6C_2 \cos^4\theta(i\sin\theta)^2 + {}^6C_3 \cos^3\theta(i\sin\theta)^3$$

$$+{}^6C_4 \cos^2\theta(i\sin\theta)^4 + {}^6C_5 \cos\theta(i\sin\theta)^5 + {}^6C_6 (i\sin\theta)^6$$

$$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$$
$$+15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$$

Equating real and imaginary parts, we get

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

Example 2. Expand $\tan 5\theta$ in the powers of $\tan\theta$.

Solution: We know that

$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \ldots\ldots\ldots\ldots\ldots\ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \ldots\ldots\ldots\ldots}$$

$$\tan 5\theta = \frac{{}^5C_1 \tan\theta - {}^5C_3 \tan^3\theta + {}^5C_5 \tan^5\theta}{1 - {}^5C_2 \tan^2\theta + {}^5C_4 \tan^4\theta}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

Example 3. Prove that $\dfrac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta.$

Solution: We know that

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta - \ldots$$

$$\sin 6\theta = {}^6C_1 \cos^{6-1}\theta \sin\theta - {}^6C_3 \cos^{6-3}\theta \sin^3\theta + {}^6C_5 \cos^{6-5}\theta \sin^5\theta$$

[Now put $n = 6$]

$$= 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$$

Now, **L.H.S.** $= \dfrac{\sin 6\theta}{\cos\theta} = \dfrac{6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta}{\cos\theta}$

$$= \frac{6\cos^5\theta \sin\theta}{\cos\theta} - \frac{20\cos^3\theta \sin^3\theta}{\cos\theta} + \frac{6\cos\theta \sin^5\theta}{\cos\theta}$$

$$= 6\cos^4\theta \sin\theta - 20\cos^2\theta \sin^3\theta + 6\sin^5\theta$$

$$= 6(\cos^2\theta)^2 \sin\theta - 20(1-\sin^2\theta)\sin^3\theta + 6\sin^5\theta$$

$$= 6(1-\sin^2\theta)^2 \sin\theta - 20(1-\sin^2\theta)\sin^3\theta + 6\sin^5\theta$$

$$= 6(1+\sin^4\theta - 2\sin^2\theta)\sin\theta - 20(1-\sin^2\theta)\sin^3\theta + 6\sin^5\theta$$

$$= 6\sin\theta + 6\sin^5\theta - 12\sin^3\theta - 20\sin^3\theta + 20\sin^5\theta + 6\sin^5\theta$$

$$= 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta.$$

Exercise 1

1. Expand $\sin 4\theta$ in terms of $\sin\theta$. **Ans.:** $4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta$

2. Expand $\tan 9\theta$ in powers of $\tan\theta$. **Ans**

$$= \frac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1 - 36\tan^2\theta + 126\tan^4\theta - 84\tan^6\theta + 9\tan^8\theta}$$

3. Prove that**:**

(a) $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

(b) $\sin 7\theta = 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta$

(c) $\dfrac{\sin 7\theta}{\cos\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$

(d)

$$\sin 8\theta = \cos^8\theta - 28\cos^6\theta \sin^2\theta + 70\cos^4\theta \sin^4\theta - 28\cos^2\theta \sin^6\theta + \sin^8\theta$$

(e) $1 + \cos 10\theta = 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2$

4. Prove that $\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$.

5. Prove that $\dfrac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta.$

(b) Expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of sines and cosines of multiples of θ

Mathematically, we can explain

Let $\quad x = \cos\theta + i\sin\theta,$ then $\quad \dfrac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta \quad \text{and}$$

$$x - \frac{1}{x} = 2i\sin\theta$$

Again $\quad x^n = \cos n\theta + i\sin n\theta,$ and

$\dfrac{1}{x^n} = \cos n\theta - i\sin n\theta,$ then

$$x^n + \frac{1}{x^n} = 2\cos n\theta \quad \text{and}$$

$$x^n - \frac{1}{x^n} = 2i\sin n\theta$$

To expand $\cos^n\theta:$ start from $\quad (2\cos\theta)^n = \left(x + \dfrac{1}{x}\right)^n$

Expand right hand side and substitute the value of $\left(x + \dfrac{1}{x}\right), \left(x^2 + \dfrac{1}{x^2}\right)$ etc.

To expand $\sin^n\theta:$ start from $\quad (2i\sin\theta)^n = \left(x - \dfrac{1}{x}\right)^n$

Expand right hand side and substitute the value of $\left(x - \dfrac{1}{x}\right), \left(x^2 - \dfrac{1}{x^2}\right)$ etc.

Example 1. Express $\sin^5\theta$ in the terms of sines of multiples of θ

Solution: We Know $\quad (2i\sin\theta)^5 = \left(x - \dfrac{1}{x}\right)^5$

$\Rightarrow$

$$32i\sin^5\theta = x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$\Rightarrow$ $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta \Rightarrow$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Example 2. Prove that

$$-2^{12}\cos^6\theta\sin^7\theta = \sin 13\theta - \sin 11\theta - 6\sin 9\theta + 6\sin 7\theta + 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta$$

Solution: We have

$$x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

$$\frac{1}{x^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \qquad \text{and} \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2\cos\theta)^6(2i\sin\theta)^7 = \left(x + \frac{1}{x}\right)^6\left(x - \frac{1}{x}\right)^7 = \left(x^2 - \frac{1}{x^2}\right)^6\left(x - \frac{1}{x}\right)$$

$$= \left[x^{12} + 6x^{10}\left(-\frac{1}{x^2}\right) + 15x^8\left(-\frac{1}{x^2}\right)^2 + 20x^6\left(-\frac{1}{x^2}\right)^3 + 15x^4\left(-\frac{1}{x^2}\right)^4 + 6x^2\left(-\frac{1}{x^2}\right)^5 + \left(-\frac{1}{x^2}\right)^6\right]\left(x - \frac{1}{x}\right)$$

$$= \left[x^{12} - 6x^8 + 15x^4 - 20 + \frac{15}{x^4} - \frac{6}{x^8} + \frac{1}{x^{12}}\right]\left(x - \frac{1}{x}\right)$$

$$= x^{13} - 6x^9 + 15x^5 - 20x + \frac{15}{x^3} - \frac{6}{x^7} + \frac{1}{x^{11}} - x^{11} + 6x^7 - 15x^3 + \frac{20}{x} - \frac{15}{x^5} + \frac{6}{x^9} - \frac{1}{x^{13}}$$

$$= \left(x^{13} - \frac{1}{x^{13}}\right) - \left(x^{11} - \frac{1}{x^{11}}\right) - 6\left(x^9 - \frac{1}{x^9}\right) + 6\left(x^7 - \frac{1}{x^7}\right) + 15\left(x^5 - \frac{1}{x^5}\right) - 15\left(x^3 - \frac{1}{x^3}\right) - 20\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 13\theta - 2i\sin 11\theta + 6(2i\sin 9\theta) + 6(2i\sin 7\theta) + 15(2i\sin 5\theta) - 15(2i\sin 3\theta) - 20(2i\sin\theta)$$

$$\Rightarrow \quad -2^{12}\cos^6\theta\sin^7\theta = \sin 13\theta - \sin 11\theta + 6\sin 9\theta + 6\sin 7\theta + 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta$$

Exercise 2

1. Express $\sin^7\theta$ in the terms of sines of multiples of θ.

Ans.:

$$-\frac{1}{64}(\sin 7\theta - 7\sin 5\theta + 12\sin 3\theta - 35\sin\theta)$$

2. Express $\cos^8\theta$ as a sum of cosines of multiples of θ.

Ans.:

$$\frac{1}{128}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)$$

3. Expand $\cos^5\theta\ \sin^7\theta$ in the series of sines and of multiples of θ.

Ans.:

$$-\frac{1}{2^{11}}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$$

4. Prove that**:**

(a) $2^7\cos^3\theta\sin^5\theta = \sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta$

(b) $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$

(c) $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

(d) $\sin^5\theta = 2^{-4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

10.4 SUMMATION OF SINES AND COSINES SERIES

Here, we shall discuss important methods for summing up trigonometric series which may be finite or infinite. There are two important methods for summation. These are:

(a) $C+iS$ method (b) The difference method.

$C+iS$ **Method:** Sum of cosines series is denoted by C and Sum of sines series is denoted by S. Now summing the trigonometric method of the series is connected with some standard series and its effect will be complex quantities.

Suppose we have to find C, the sum of cosines series. Then write a similar series of sines, S. Multiply the sines series by i and add to series of cosines, we will get the $C+iS$ method is

Sum of series $= C+iS$

Consider cosine series

$$C = a\cos\alpha + a^2\cos(\alpha+\beta) + a^3\cos(\alpha+2\beta) + ...$$

and sines series

$$S = a\sin\alpha + a^2\sin(\alpha+\beta) + a^3\sin(\alpha+2\beta) + ...$$

These series may be finite or infinite. Then sum up the cosine and sines series may be complex

$$C+iS = a(\cos\alpha + i\sin\alpha) + a^2\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\} + a^3\{\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\} + ...$$

The sum of series is calculated by using any one of the following series**:**

(a) Series in Geometric Progression **(b)** Binomial series or which can be reduced to it

(c) Exponential series or the allied series **(d)** Logarithmic series

10.5 SUMMATION ON GEOMETRIC PROGRESSION OR ARITHMETIC O-GEOMETRIC SERIES

We now that a geometric progression is of the form

$$a,\ ar,\ ar^2,\ ar^3, ...,\ ar^n, ...\infty$$

Whose common ratio is r, then n^{th} term is ar^{n-1}

Sum of geometric progression in n^{th} term is

$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$ **(if *r*< 1)** or $= a.\left(\frac{r^n - 1}{r-1}\right)$ **(if *r*> 1)**

Sum of geometric progression in infinite series is

$$a + ar + ar^2 + ...\infty = \frac{a}{1-r}$$ **,**provided $|r| < 1$

An arithmetic o-geometric series is of the form

$$a,\ (a+d)r,\ (a+2d)r^2,\ (a+3d)r^3, ...$$

Example 1. Sum the series n terms and to infinity

$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + ...$**,** where a is less then unity.

Solution: Let

$$C = 1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + ... + a^{n-1}\cos(n-1)\theta$$

and

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + ... + a^{n-1}\sin(n-1)\theta$$

$\therefore$

$$C + iS = 1 + a(\cos\theta + i\sin\theta) + a^2(\cos 2\theta + i\sin 2\theta) + ... + a^{n-1}\{\cos(n-1)\theta + i(n-1)\theta\}$$

$$= 1 + ae^{i\theta} + a^2e^{2i\theta} + ... + a^{n-1}e^{i(n-1)\theta}$$

This is a geometric series, whose first term is 1 and common ratio is $ae^{i\theta}$, we get

$$C + iS = \frac{a(1-r^n)}{1-r} = \frac{1.(1-a^ne^{in\theta})}{1-ae^{i\theta}}$$

$$= \frac{(1-a^ne^{in\theta})}{(1-ae^{i\theta})} \times \frac{(1-ae^{-i\theta})}{(1-ae^{-i\theta})}$$

$$= \frac{1-ae^{-i\theta} - a^ne^{in\theta} + a^{n+1}e^{i(n-1)\theta}}{1-a(e^{i\theta}+e^{-i\theta}) + a^2e^{i\theta}.e^{-i\theta}}$$

$$= \frac{1-a(\cos\theta + i\sin\theta) - a^n(\cos n\theta + i\sin n\theta) + a^{n+1}\{\cos(n-1)\theta + i\sin(n-1)\theta}{1-2a\cos\theta + a^2}$$

Equating real and imaginary part, we get

$$C_n = \frac{1-a\cos\theta - a^n\cos n\theta + a^{n+1}\cos(n-1)\theta}{1-2a\cos\theta + a^2}$$

$$S_n = \frac{a\sin\theta - a^n\sin n\theta + a^{n+1}\sin(n-1)\theta}{1-2a\cos\theta + a^2}$$

When $n \to \infty$ then both a^n and $a^{n+1} \to 0$, we get

$$C_\infty = \frac{1-a\cos\theta}{1-2a\cos\theta + a^2}$$ and

$$S_\infty = \frac{a\sin\theta}{1-2a\cos\theta + a^2}$$

Example 2. Sum the series

$$1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + ...ad.\inf .$$

Solution: Let $$C = 1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + ...ad.\inf .$$

And $$S = \frac{\sin\theta}{\cos\theta} + \frac{\sin 2\theta}{\cos^2\theta} + \frac{\sin 3\theta}{\cos^3\theta} + ...ad.\inf .$$

$\therefore$

$$C+iS=1+\frac{(\cos\theta+i\sin\theta)}{\cos\theta}+\frac{(\cos 2\theta+i\sin 2\theta)}{\cos^2\theta}+\frac{(\cos 3\theta+i\sin\theta)}{\cos^3\theta}+........ad.\inf .$$

This is a geometric series, whose first term is 1 and common ratio is $\frac{(\cos\theta+i\sin\theta)}{\cos\theta}$, we get

$$C+iS=\frac{a}{1-r}=\frac{1}{1-\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)}$$

$$=\frac{\cos\theta}{\cos\theta-\cos\theta-i\sin\theta}$$

$$=\frac{i\cos\theta}{-i^2\sin\theta}=i\frac{\cos\theta}{\sin\theta}=i\cot\theta$$

$$\Rightarrow C+iS=0+i\cot\theta$$

Equating real parts, we get $C=0.$

Example 3. Sum the series

$$\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{2^2}\sin 3\theta+..................ad.\inf .$$

Solution: Let $S=\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{2^2}\sin 3\theta+..................ad.\inf .$

and

$$C=\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2^2}\cos 3\theta+..................ad.\inf .$$

$\therefore$

$$C+iS=(\cos\theta+i\sin\theta)+\frac{1}{2}(\cos 2\theta+i\sin 2\theta)+\frac{1}{2^2}(\cos 3\theta+i\sin 3\theta)+.......ad.\inf .$$

This is a geometric infinite series, whose first term is $(\cos\theta+i\sin\theta)$ and common ratio is $\frac{1}{2}(\cos\theta+i\sin\theta)$, we get

$$C+iS=\frac{a}{1-r}=\frac{\cos\theta+i\sin\theta}{1-\frac{1}{2}\left(\cos\theta+i\sin\theta\right)}$$

$$=\frac{2(\cos\theta+i\sin\theta)}{2-\left(\cos\theta+i\sin\theta\right)}=\frac{2(\cos\theta+i\sin\theta)}{(2-\cos\theta)-i\sin\theta}$$

$$=\frac{2(\cos\theta+i\sin\theta)}{(2-\cos\theta)-i\sin\theta}\times\frac{(2-\cos\theta)+i\sin\theta}{(2-\cos\theta)+i\sin\theta}$$

$$= \frac{2[(2\cos\theta - \cos^2\theta - \sin^2\theta) + i(2\sin\theta - \sin\theta\cos\theta + \sin\theta\cos\theta]}{4 - 4\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{2[(2\cos\theta - 1) + 2i\sin\theta]}{4 - 4\cos\theta + 1} \quad \Rightarrow$$

$$C + iS = \frac{(4\cos\theta - 2) + i(4\sin\theta)}{5 - 4\cos\theta}$$

Equating imaginary parts, we get

$$S = \frac{4\sin\theta}{5 - 4\cos\theta}$$

Example 4. Sum the series

$$1 + \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \dots\dots\dots\dots ad.\inf .$$

where a is less then unity.

Solution: Let

$$C = 1 + \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \dots\dots\dots\dots ad.\inf .$$

and

$$S = \cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cos^3\theta\sin 3\theta + \dots\dots\dots\dots ad.\inf .$$

$\therefore$

$$C + iS = 1 + \cos\theta(\cos\theta + i\sin\theta) + \cos^2\theta(\cos 2\theta + i\sin 2\theta) + \cos^3\theta(\cos 3\theta + i\sin 3\theta) + \dots ad.\inf$$

$$= 1 + \cos\theta.e^{i\theta} + \cos^2\theta.e^{2i\theta} + \cos^3\theta.e^{3i\theta} + \dots ad.\inf$$

This is a geometric infinite series, whose first term is 1 and common ratio is $\cos\theta.e^{i\theta}$, we get

$$C + iS = \frac{a}{1-r} = \frac{1}{1 - \cos\theta.e^{i\theta}}$$

$$= \frac{1}{1 - \cos\theta(\cos\theta + i\sin\theta)} = \frac{1}{(1 - \cos^2\theta) - i\sin\theta\cos\theta}$$

$$= \frac{1}{\sin^2\theta - i\sin\theta\cos\theta} = \frac{1}{\sin\theta(\sin\theta - i\cos\theta)}$$

$$= \frac{1}{\sin\theta(\sin\theta - i\cos\theta)} \times \frac{(\sin\theta + i\cos\theta)}{(\sin\theta + i\cos\theta)}$$

$$= \frac{(\sin\theta + i\cos\theta)}{\sin\theta(\sin^2\theta - i^2\cos^2\theta)} = \frac{(\sin\theta + i\cos\theta)}{\sin\theta(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{(\sin\theta + i\cos\theta)}{\sin\theta} = 1 + i\cot\theta \quad \Rightarrow$$

$$C + iS = 1 + i\cot\theta$$

Equating real parts, we get $C = 1.$

Example 5. Obtain the sum of the series

$$3\sin\theta + 5\sin 2\theta + 7\sin 3\theta + ...to\ n - terms$$

Solution: Let $S = 3\sin\theta + 5\sin 2\theta + 7\sin 3\theta + ...to\ n - terms$

And $C = 3\cos\theta + 5\cos 2\theta + 7\cos 3\theta + ...to\ n - terms$

$\therefore$

$$C + iS = 3(\cos\theta + i\sin\theta) + 5(\cos 2\theta + i\sin 2\theta) + 7(\cos 3\theta + i\sin 3\theta) + ...$$

$$n - terms \quad = 3e^{i\theta} + 5e^{2i\theta} + 7e^{3i\theta} + ... + (2n+1)e^{ni\theta} \quad ....(1)$$

This is an arithmetic o-geometric series whose common ratio is $e^{i\theta}$ and multiplying both sides by $e^{i\theta}$, we obtain

$$(C + iS)e^{i\theta} = 3e^{2i\theta} + 5e^{3i\theta} + 7e^{4i\theta} + ... + (2n+1)e^{(n+1)i\theta} \quad ....(2)$$

Subtracting (1) by (2), we obtain

$$(C + iS)(1 - e^{i\theta}) = 3e^{i\theta} + 2e^{2i\theta} + 2e^{3i\theta} + ... + 2e^{ni\theta} - (2n+1)e^{(n+1)i\theta}$$

$$= e^{i\theta} + 2\{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + ...\ n - terms\} - (2n+1)e^{(n+1)i\theta}$$

$$= e^{i\theta} + \frac{2e^{i\theta}(1 - e^{ni\theta})}{1 - e^{i\theta}} - (2n+1)e^{(n+1)i\theta}$$

$$= e^{i\theta} - (2n+1)e^{(n+1)i\theta} - \frac{2(1 - e^{ni\theta})}{1 - e^{-i\theta}}$$

$$\Rightarrow \quad C + iS = \frac{e^{i\theta} - (2n+1)e^{(n+1)i\theta}}{1 - e^{i\theta}} - \frac{2(1 - e^{ni\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})}$$

$$= \frac{\{e^{i\theta} - (2n+1)e^{(n+1)i\theta}\}(1 - e^{-i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} - \frac{2(1 - e^{ni\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})}$$

$$= \frac{e^{i\theta} - 1 - (2n+1)e^{(n+1)i\theta} + (2n+1)e^{ni\theta} - 2(1 - e^{ni\theta})}{1 - (e^{i\theta} + e^{-i\theta}) + 1}$$

$$\Rightarrow \quad C + iS = \frac{e^{i\theta} - (2n+1)e^{(n+1)i\theta} + (2n+3)e^{ni\theta} - 3}{2(1 - \cos\theta)}$$

$$\Rightarrow$$

$$C + iS = \frac{(\cos\theta + i\sin\theta) - (2n+1)\{\cos(n+1)\theta + i\sin(n+1)\theta\} + (2n+3)\{\cos n\theta + i\sin n\theta\} - 3}{2(1 - \cos\theta)}$$

Equating imaginary parts on both sides, we obtain

$$S = \frac{\sin\theta - (2n+1)\sin(n+1)\theta + (2n+3)\sin n\theta}{2(1 - \cos\theta)}$$

Example 6. Obtain the sum of the series $\cos\theta + 2\cos 2\theta + 3\cos 3\theta + ...to\ n-terms$

Solution: Let $C = \cos\theta + 2\cos 2\theta + 3\cos 3\theta + ...to\ n-terms$

and $S = \sin\theta + 2\sin 2\theta + 3\sin 3\theta + ...to\ n-terms$

$\therefore$

$C + iS = (\cos\theta + i\sin\theta) + 2(\cos 2\theta + i\sin 2\theta) + 3(\cos 3\theta + i\sin 3\theta) + ...to$
$n-terms$ $= e^{i\theta} + 2e^{2i\theta} + 3e^{3i\theta} + ... + ne^{ni\theta}$ ……..(1)

This is an arithmetic o-geometric series whose common ratio is $e^{i\theta}$ and multiplying both sides by $e^{i\theta}$, we obtain

$$(C + iS)e^{i\theta} = e^{2i\theta} + 2e^{3i\theta} + 3e^{4i\theta} + ... + ne^{(n+1)i\theta}$$ ……..(2)

Subtracting (1) by (2), we obtain

$$(C + iS)(1 - e^{i\theta}) = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + ... + e^{ni\theta} - ne^{(n+1)i\theta}$$

$$= \{e^{i\theta} + e^{2i\theta} + e^{3i\theta} + ...to\ n-terms\} - ne^{(n+1)i\theta}$$

$$= \frac{e^{i\theta}(1 - e^{ni\theta})}{1 - e^{i\theta}} - ne^{(n+1)i\theta} = -ne^{(n+1)i\theta} - \frac{(1 - e^{ni\theta})}{1 - e^{-i\theta}}$$

$$\Rightarrow \qquad C + iS = \frac{-ne^{(n+1)i\theta}}{1 - e^{i\theta}} - \frac{(1 - e^{ni\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})}$$

$$= \frac{\{-ne^{(n+1)i\theta}\}(1 - e^{-i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} - \frac{(1 - e^{ni\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})}$$

$$= \frac{-ne^{(n+1)i\theta} + ne^{ni\theta} - (1 - e^{ni\theta})}{1 - (e^{i\theta} + e^{-i\theta}) + 1} = \frac{(n+1)e^{ni\theta} - ne^{(n+1)i\theta} - 1}{2(1 - \cos\theta)}$$

$$\Rightarrow C + iS = \frac{(n+1)\{\cos n\theta + i\sin n\theta\} - n\{\cos(n+1)\theta + i\sin(n+1)\theta\} - 1}{2(1 - \cos\theta)}$$

Equating real parts on both sides, we obtain

$$C = \frac{(n+1)\cos n\theta - n\cos(n+1)\theta - 1}{2(1 - \cos\theta)}$$

Example 7. Obtain the sum of the series

$$1 - 2\cos\theta + 3\cos 2\theta - 4\cos 3\theta + ...to\ n-terms$$

Solution: Let $C = 1 - 2\cos\theta + 3\cos 2\theta - 4\cos 3\theta + ...to$
$n-terms$

And $S = -2\sin\theta + 3\sin 2\theta - 4\sin 3\theta + ...to\ n-terms$

$\therefore$

$C+iS = 1-2(\cos\theta + i\sin\theta) + 3(\cos 2\theta + i\sin 3\theta) - 4(\cos 3\theta + i\sin 3\theta)........to$
$n-terms$

$$= 1 - 2e^{i\theta} + 3e^{2i\theta} - 4e^{3i\theta} + ... + (-1)^{n-1} ne^{(n-1)i\theta}$$
……..(1)

This is an arithmetic o-geometric series whose common ratio is $-e^{i\theta}$, we obtain

$$(C+iS)(-e^{i\theta}) = -e^{i\theta} + 2e^{2i\theta} - 3e^{3i\theta} + ... + (-1)^{n-1}(n-1)e^{(n-1)i\theta} + (-1)^n ne^{ni\theta}$$
…..(2)

Subtracting (1) by (2), we obtain

$$(C+iS)(1+e^{i\theta}) = \{1 - e^{i\theta} + e^{2i\theta} - e^{3i\theta} + ... + (-1)^{n-1} e^{(n-1)\theta}\} - (-1)^n ne^{ni\theta}$$

$$= \frac{1.\{1-(-e^{i\theta})^n\}}{1-(-e^{i\theta})} - (-1)^n ne^{ni\theta}$$

$$= \frac{1-(-1)^n e^{ni\theta}}{1+e^{i\theta}} - (-1)^n ne^{ni\theta}$$

$$\Rightarrow \qquad C+iS = \frac{1-(-1)^n e^{ni\theta}}{(1+e^{i\theta})^2} - \frac{(-1)^n ne^{ni\theta}}{(1+e^{i\theta})}$$

$$= \frac{1+(-1)^{n-1} e^{ni\theta}}{(1+e^{i\theta})^2} + \frac{(-1)^{n-1} ne^{ni\theta}}{(1+e^{i\theta})} \qquad [\text{Putting } (-1)^n = (-1).(-1)^{n-1}]$$

$$= \frac{[1+(-1)^{n-1} e^{ni\theta}] + (1+e^{i\theta})[(-1)^{n-1} ne^{ni\theta}]}{(1+e^{i\theta})^2}$$

$$= \frac{[1+(-1)^{n-1}[(n+1)e^{ni\theta} + ne^{(n+1)i\theta}]}{e^{i\theta}\left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\right)^2}$$

$$= \frac{[e^{-i\theta} + (-1)^{n-1}[(n+1)e^{(n-1)i\theta} + ne^{ni\theta}]}{\left(2\cos\frac{\theta}{2}\right)^2}$$

$$= \frac{(\cos\theta - i\sin\theta) + (-1)^{n-1}[(n+1)\{\cos(n-1)\theta + i\sin(n-1)\theta\} + n\{\cos n\theta + i\sin n\theta\}]}{4\cos^2\frac{\theta}{2}}$$

$$= \frac{(\cos\theta - i\sin\theta) + (-1)^{n-1}[(n+1)\{\cos(n-1)\theta + i\sin(n-1)\theta\} + n\{\cos n\theta + i\sin n\theta\}]}{2(1+\cos\theta)}$$

Equating real parts on both sides, we get

$$C = \frac{\cos\theta + (-1)^{n-1}\{(n+1)\cos(n-1)\theta + n\cos n\theta\}}{2(1+\cos\theta)}$$

Exercise 3

1. Sum of series

(a) $\cos\theta\sin\theta + \cos 2\theta\sin^2\theta + \cos 3\theta\sin^3\theta + ...ad.\inf$. **Ans.:** $\frac{\sin\theta(\cos\theta - i\sin\theta)}{1-\sin 2\theta + \sin^2\theta}$

(b) $\sin\theta + \sin(\theta+\phi) + \sin(\theta+2\phi) + ...to\ n-term$. **Ans.:** $\sin\left\{\theta + (n-1)\frac{\phi}{2}\right\}\sin\frac{n\phi}{2}\cos ec\frac{\phi}{2}$

2. Find the sum of the series

$$\cos n\alpha + \cos\alpha\cos(n-1)\alpha + \cos^2\alpha\cos(n-2)\alpha + ... + \cos^n\alpha$$

Ans.: $\frac{\sin(n+1)\alpha}{\sin\alpha}$

3. Sum of series $\sin^2 1 + \sin^2 2 + \sin^2 3 + ... + \sin^2 40$.**Ans.:** $20 - \frac{\cos 41\sin 40}{\sin 1}$

4. Show that $\cos^2\alpha + \cos^2\left(\alpha + \frac{\pi}{n}\right) + \cos^2\left(\alpha + \frac{2\pi}{n}\right) + ... + to$ $n-terms = \frac{\pi}{2}$

5. Sum of series $\sin^4\alpha + \sin^4\left(\alpha + \frac{2\pi}{n}\right) + \sin^4\left(\alpha + \frac{4\pi}{n}\right) + ... +$ $n-terms$ **Ans.:** $\frac{3n}{8}$

6. If $1+\frac{\cos\theta}{\cos\theta}+\frac{\cos 2\theta}{\cos^2\theta}+\frac{\cos 3\theta}{\cos^3\theta}+.......+n-terms$, then prove that $C=\frac{\sec^n\theta\sin n\theta}{\tan\theta}$

7. Sum the series $1+c\cosh\theta+c^2\cosh 2\theta+c^3\cosh 3\theta+.............+n-terms$

Ans.: $\frac{1-c\cosh\theta-c^n\cosh n\theta+c^{n+1}\cosh(n-1)\theta}{1-2c\cosh\theta+c^2}$

8. Sum of series $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta+...to\ n-terms$. **Ans.:** $\frac{n}{2}+\frac{\cos(n+1)\theta\sin n\theta}{2\cos\theta}$.

9. Find the sum of series

$$1+\frac{2}{2}\cos\theta+\frac{3}{2^2}\cos 2\theta+\frac{4}{2^3}\cos 3\theta+.............ad.\inf$$ **Ans.:** $\frac{4(4-4\cos\theta+\cos 2\theta)}{(5-4\cos\theta)^2}$

10. Obtain the sum of the series

$$\sin\theta+2\sin 2\theta+3\sin 3\theta+........to\ n-terms$$ **Ans.:** $\frac{(n+1)\sin n\theta-n\sin(n+1)\theta}{2(1-\cos\theta)}$

10.6 SUMMATION DEPENDING UPON BINOMIAL SERIES

Using the some binomial series we can solve the sum of series as given below:

(a) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.......\infty$

(b) $(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+.......\infty$

(c) $(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+.........\infty$

(d) $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2.4}x^2+\frac{1.3}{2.4.6}x^3-.........\infty$

(e) $(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+........\infty$

(f) $(1+x)^{\frac{1}{3}} = 1+\frac{1}{3}x-\frac{1.2}{3.6}x^2+\frac{1.2.5}{3.6.9}x^3-........\infty$

(g) $(1-x)^{-\frac{1}{3}} = 1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+........\infty$

1. Sum the series $1+\frac{1}{2}\cos 2\theta+\frac{1.3}{2.4}\cos 4\theta+\frac{1.3.5}{2.4.6}\cos 6\theta+........\infty$

Solution: Let $C = 1+\frac{1}{2}\cos 2\theta+\frac{1.3}{2.4}\cos 4\theta+\frac{1.3.5}{2.4.6}\cos 6\theta+........\infty$

And $S = \frac{1}{2}\sin 2\theta+\frac{1.3}{2.4}\sin 4\theta+\frac{1.3.5}{2.4.6}\sin 6\theta+........\infty$

$\therefore$

$$C+iS = 1+\frac{1}{2}(\cos 2\theta+i\sin 2\theta)+\frac{1.3}{2.4}(\cos 4\theta+i\sin 4\theta)+\frac{1.3.5}{2.4.6}(\cos 6\theta++i\sin 6\theta)+...\infty$$

$$= 1+\frac{1}{2}e^{2i\theta}+\frac{1.3}{2.4}e^{4i\theta}+\frac{1.3.5}{2.4.6}e^{6i\theta}+...........\infty$$

$$= (1-e^{2i\theta})^{-\frac{1}{2}} = (1-\cos 2\theta-i\sin 2\theta)^{-\frac{1}{2}}$$ [By using binomial theorem]

$$= (2\sin^2\theta-i.2\sin\theta\cos\theta)^{-\frac{1}{2}} = (2\sin\theta)^{-\frac{1}{2}}(\sin\theta-i.\cos\theta)^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{-\frac{1}{2}}$$

$$= (2\sin\theta)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

Equating real parts on both sides, we get $C = (2\sin\theta)^{-\frac{1}{2}}\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

2. Sum the series $1+\frac{1}{2}\cos 2\theta-\frac{1}{2.4}\cos 4\theta+\frac{1.3}{2.4.6}\cos 6\theta-........\infty$,

where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Solution: Let $C = 1+\frac{1}{2}\cos 2\theta-\frac{1}{2.4}\cos 4\theta+\frac{1.3}{2.4.6}\cos 6\theta-........\infty$

And $$S=\frac{1}{2}\sin 2\theta-\frac{1}{2.4}\sin 4\theta+\frac{1.3}{2.4.6}\sin 6\theta-........\infty$$

$\therefore$

$$C+iS=1+\frac{1}{2}(\cos 2\theta+i\sin 2\theta)-\frac{1}{2.4}(\cos 4\theta+i\sin 4\theta)+\frac{1.3}{2.4.6}(\cos 6\theta++i\sin 6\theta)-...\infty$$

$$=1+\frac{1}{2}e^{2i\theta}-\frac{1}{2.4}e^{4i\theta}+\frac{1.3}{2.4.6}e^{6i\theta}-...........\infty$$

$$=(1+e^{2i\theta})^{\frac{1}{2}}=(1+\cos 2\theta+i\sin 2\theta)^{\frac{1}{2}}$$ [By using binomial theorem]

$$=(2\cos^2\theta+2i\sin\theta\cos\theta)^{\frac{1}{2}}=(2\cos\theta)^{\frac{1}{2}}(\cos\theta+i\sin\theta)^{\frac{1}{2}}$$

$$=(2\cos\theta)^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$

Equating real parts on both sides, we get

$$C=(2\cos\theta)^{\frac{1}{2}}\cos\frac{\theta}{2}=\sqrt{2\cos\theta.\cos^2\frac{\theta}{2}}=\sqrt{\cos\theta(1+\cos\theta)}$$

Exercise 4

1. Sum the series $\sin\theta+n\sin(\theta+\phi)+\frac{n(n-1)}{2!}\sin(\theta+2\phi)+........to(n+1)terms$

Ans.: $2^n\cos^n\frac{\phi}{2}\sin\left(\theta+\frac{n\phi}{2}\right)$

2. Sum the series $1+\frac{1}{3}y\cos\alpha+\frac{1.4}{3.6}y^2\cos 2\alpha+\frac{1.4.7}{3.6.9}y^3\cos 3\alpha+........\infty, if\ \ y<1$.

Ans.: $(1-2y\cos\alpha+y^2)^{-\frac{1}{6}}\cos\left[\frac{1}{3}\tan^{-1}\left\{\frac{y\sin\alpha}{1-y\cos\alpha}\right\}\right]$

3. Sum the series $n\sin\alpha+\frac{n(n-1)}{1.2}\sin 2\alpha+\frac{n(n-1)(n-2)}{1.2.3}\sin 3\alpha+........to\ n-terms$

Ans.: $\left(2\cos\frac{\alpha}{2}\right)^n\sin\frac{n\alpha}{2}$

4. Sum the series

$\sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1.3}{2.4}\sin 5\alpha + \frac{1.3.5}{2.4.6}\sin 7\alpha + ...\infty$ **Ans.:** $(2\sin\alpha)^{-\frac{1}{2}}\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

5. Sum the series $1 - \frac{1}{2}\cos\alpha + \frac{1.3}{2.4}\cos 2\alpha - \frac{1.3.5}{2.4.6}\cos 3\alpha + ...\infty$ **Ans.:** $(2\cos\alpha)^{-\frac{1}{2}}\cos\frac{\alpha}{4}$

6. Sum the series $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + ...\infty$

Ans.: $\left(2\sin\frac{\alpha}{2}\right)^{-n}\sin\frac{n(\pi-\alpha)}{2}$

10.7 SUMMATION DEPENDING UPON EXPONENTIAL SERIES

If x is any complex number, then we using some exponential to solve the summation of series as given below:

(a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...ad.\inf$. **(b)** $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...ad.\inf$.

(c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....ad.\inf$. **(d)** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....ad.\inf$.

(e) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...ad.\inf$. **(f)** $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...ad.\inf$.

1. Find the sum of series $1 + \frac{c^2\cos 2\alpha}{2!} + \frac{c^4\cos 4\alpha}{4!} + ...ad.\inf$.

Solution: Let $C = 1 + \frac{c^2\cos 2\alpha}{2!} + \frac{c^4\cos 4\alpha}{4!} + ...ad.\inf$.

and $S = \frac{c^2 \sin 2\alpha}{2!} + \frac{c^4 \sin 4\alpha}{4!} + \text{..........}ad.\inf$.

$\therefore$

$$C + iS = 1 + \frac{c^2(\cos 2\alpha + i\sin 2\alpha)}{2!} + \frac{c^4(\cos 4\alpha + i\sin 4\alpha)}{4!} + \text{..........}ad.\inf \ .$$

$$= 1 + \frac{c^2 e^{2i\alpha}}{2!} + \frac{c^4 e^{4i\alpha}}{4!} + \text{..........}ad.\inf \ .$$

$$= \cosh(ce^{i\alpha}) = \cosh\{c(\cos\alpha + i\sin\alpha)\}$$

$$= \cos\{ic(\cos\alpha + i\sin\alpha)\} = \cos(ic\cos\alpha - c\sin\alpha)$$

$\Rightarrow$ $C + iS = \cosh(c\cos\alpha)\cos(c\sin\alpha) + i\sinh(c\cos\alpha)\sin(c\sin\alpha)$

Equating real parts on both sides, we get

Hence, $C = \cosh(c\cos\alpha)\cos(c\sin\alpha)$

2. Sum the series

$$\cos\alpha + \sin\alpha\cos 2\alpha + \frac{\sin^2\alpha}{1.2}\cos 3\alpha + \text{..........}\infty$$

Solution: Let

$$C = \cos\alpha + \sin\alpha\cos 2\alpha + \frac{\sin^2\alpha}{1.2}\cos 3\alpha + \text{.........}\infty$$

and $S = \sin\alpha + \sin\alpha\sin 2\alpha + \frac{\sin^2\alpha}{1.2}\sin 3\alpha + \text{..........}\infty$

$\therefore$

$$C + iS = (\cos\alpha + i\sin\alpha) + \sin\alpha(\cos 2\alpha + i\sin 2\alpha) + \frac{\sin^2\alpha}{2!}(\cos 3\alpha + i\sin 3\alpha) + \text{....}\infty$$

$$= e^{i\alpha} + \sin\alpha e^{2i\alpha} + \frac{\sin^2\alpha}{2!}e^{3i\alpha} + \text{....}\infty$$

$$= e^{i\alpha}\left[1 + \frac{\sin\alpha}{1!}e^{i\alpha} + \frac{\sin^2\alpha}{2!}e^{2i\alpha} + \text{........}\infty\right]$$

$$= e^{i\alpha}e^{\sin\alpha . e^{i\alpha}} = e^{i\alpha}e^{\sin\alpha.(\cos\alpha + i\sin\alpha)} = e^{\sin\alpha.\cos\alpha}e^{i(\alpha + \sin^2\alpha)}$$

$\Rightarrow$ $C + iS = e^{\sin\alpha.\cos\alpha}.[\cos(\alpha + \sin^2\alpha) + i\sin(\alpha + \sin^2\alpha)]$

Equating real parts on both sides, we get

Hence, $C = e^{\sin\alpha.\cos\alpha}.\cos(\alpha + \sin^2\alpha)$

Exercise 5

1. Sum of the series $\sin\alpha + \frac{\sin 2\alpha}{2!} + \frac{\sin 3\alpha}{3!} + \ldots\ldots\ldots\infty$ **Ans.:** $e^{\cos\alpha}.\sin(\sin\alpha)$

2. Sum the series $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{2!}\cos 3\beta. + \ldots\infty$

Ans.: $e^{-\cos\alpha.\cos\beta}.\cos(\cos\alpha\sin\beta)$

3. Sum the series $\sin\alpha - c\sin(\alpha+\beta) + \frac{c^2}{2!}\sin(\alpha+2\beta) + \ldots\infty$ **Ans.:** $-e^{(\alpha - c\cos\beta)}.\sin(c\sin\beta)$

4. Sum of the series $\sin\alpha - \frac{\sin 2\alpha}{2!} + \frac{\sin 3\alpha}{3!} - \ldots\ldots\ldots\infty$ **Ans.:** $e^{-\cos\alpha}.\sin(\sin\alpha)$

5. Sum the series $\cos\alpha + c\cos(\alpha+\beta) + \frac{c^2}{2!}\cos(\alpha+2\beta) + \ldots.\infty$ **Ans.:** $e^{c\cos\beta}.\cos(\alpha + c\sin\beta)$

10.8 SUMMATION DEPENDING UPON LOGARITHMIC SERIES

We know that the Logarithmic series

(a) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty$

(b) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\infty$

(c) $\log(1+x) + \log(1-x) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\infty\right)$

(d) $\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty\right)$

(e) $\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty$

(f) $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

(g) $\log\left\{\frac{(x+iy)}{(x-iy)}\right\} = 2i\tan^{-1}\left(\frac{y}{x}\right)$

1. Find the sum of the series

$$c\sin\alpha - \frac{c^2}{2}\sin 2\alpha + \frac{c^3}{3}\sin 3\alpha - ....\infty$$

Solution: Let

$$S = c\sin\alpha - \frac{c^2}{2}\sin 2\alpha + \frac{c^3}{3}\sin 3\alpha - ....\infty$$

and $\quad C = c\cos\alpha - \frac{c^2}{2}\cos 2\alpha + \frac{c^3}{3}\cos 3\alpha - ....\infty$

$\therefore$

$$C + iS = c(\cos\alpha + i\sin\alpha) - \frac{c^2}{2}(\cos 2\alpha + i\sin 2\alpha) + \frac{c^3}{3}(\cos 3\alpha + i\sin 3\alpha) - ....\infty$$

$$= ce^{i\alpha} - \frac{c^2}{2}e^{2i\alpha} + \frac{c^3}{3}e^{3i\alpha} - ....\infty$$

$$= ce^{i\alpha} - \frac{1}{2}(ce^{i\alpha})^2 + \frac{1}{3}(ce^{i\alpha})^3 - ....\infty$$

$$= \log(1 + ce^{i\alpha}) = \log(1 + c\cos\alpha + ic\sin\alpha)$$

$$= \frac{1}{2}\log\{(1 + c\cos\alpha)^2 + c^2\sin^2\alpha\} + i\tan^{-1}\left(\frac{c\sin\alpha}{1 + c\cos\alpha}\right)$$

$$\left[\therefore \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Equating imaginary parts on the both sides, we get

Hence, $\quad S = \tan^{-1}\left(\frac{c\sin\alpha}{1 + c\cos\alpha}\right)$

2. Find the sum of the series

$$c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + ...\infty$$

Solution: Let

$$S = c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + ...\infty$$

and $\quad C = c\cos\alpha + \frac{c^3}{3}\cos 3\alpha + \frac{c^5}{5}\cos 5\alpha + ...\infty$

$\therefore$

$$C + iS = c(\cos\alpha + i\sin\alpha) + \frac{c^3}{3}(\cos 3\alpha + i\sin 3\alpha) + \frac{c^5}{5}(\cos 5\alpha + i\sin 5\alpha) + ...\infty$$

$$= ce^{i\alpha} + \frac{c^3}{3}e^{3i\alpha} + \frac{c^5}{5}e^{5i\alpha} + ...\infty$$

$$\left[\therefore \frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right]$$

$$= \frac{1}{2}\log\left\{\frac{1+ce^{i\alpha}}{1-ce^{i\alpha}}\right\} = \frac{1}{2}\log(1+ce^{i\alpha}) - \frac{1}{2}\log(1-ce^{i\alpha})$$

$$= \frac{1}{2}\log\{(1+c\cos\alpha) + ic\sin\alpha\} - \frac{1}{2}\log\{(1-c\cos\alpha) - ic\sin\alpha\}$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\{(1+c\cos\alpha)^2 + c^2\sin^2\alpha\} + i\tan^{-1}\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right)\right]$$

$$-\frac{1}{2}\left[\frac{1}{2}\log\{(1-c\cos\alpha)^2 + c^2\sin^2\alpha\} - i\tan^{-1}\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\left\{(1+2c\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha)\right\} + i\tan^{-1}\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right)\right]$$

$$-\frac{1}{2}\left[\frac{1}{2}\log\left\{(1-2c\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha)\right\} - i\tan^{-1}\left(\frac{c\sin\alpha}{1-c\cos\alpha}\right)\right]$$

$$= \frac{1}{4}\left[\log\left\{(1+2c\cos\alpha + c^2)\right\} - \log\left\{(1-2c\cos\alpha + c^2)\right\}\right]$$

$$+ i\frac{1}{2}\left[\tan^{-1}\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right) + \tan^{-1}\left(\frac{c\sin\alpha}{1-c\cos\alpha}\right)\right]$$

Equating imaginary parts on both sides, we get

$$S = \frac{1}{2}\left[\tan^{-1}\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right) + \tan^{-1}\left(\frac{c\sin\alpha}{1-c\cos\alpha}\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right)+\left(\frac{c\sin\alpha}{1-c\cos\alpha}\right)}{1-\left(\frac{c\sin\alpha}{1+c\cos\alpha}\right)\times\left(\frac{c\sin\alpha}{1-c\cos\alpha}\right)}\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{c\sin\alpha\{(1-c\cos\alpha)+(1+c\cos\alpha)\}}{(1-c^2\cos^2\alpha)-c^2\sin^2\alpha}\right] = \frac{1}{2}\tan^{-1}\left[\frac{2c\sin\alpha}{1-c^2}\right]$$

Exercise 6

1. Find the sum of the series

$$a\cos\alpha - \frac{a^2}{2}\cos 2\alpha + \frac{a^3}{3}\cos 3\alpha - \text{..........}\infty$$ **Ans.:**

$\frac{1}{2}\log(1+2a\cos\alpha+a^2)$

2. Prove that, if $\theta < \frac{\pi}{4}$, $\log\sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \text{......}$

3. Sum of the following series

(a) $a\sin\alpha - \frac{a^3}{3}\sin 3\alpha + \frac{a^5}{5}\sin 5\alpha - \text{..........}\infty$ **Ans.:**

$\frac{1}{4}\log\left\{\frac{1+2a\sin\alpha+a^2}{1-2a\sin\alpha+a^2}\right\}$

(b) $\cos\alpha - \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha - \text{....}\infty$ **Ans.:**

$\log 2 + \log\cos\frac{\alpha}{2}$

(c) $\cos 2\alpha + \frac{1}{3}\cos 4\alpha + \frac{1}{5}\sin 6\alpha + \text{....}\infty$ **Ans.:**

$\frac{1}{2}\left\{\cos\alpha\log\cot\frac{\alpha}{2} - \frac{\pi}{2}\sin\alpha\right\}$

(d) $\cos^2\theta - \frac{1}{3}\cos^3\theta\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta - \text{....}\infty$ **Ans.:**

$\frac{1}{2}\tan^{-1}(2\cot^2\theta)$

(e) $\cos\frac{\pi}{3} + \frac{1}{3}\cos\frac{2\pi}{3} + \frac{1}{5}\cos\frac{3\pi}{3} + \frac{1}{7}\cos\frac{4\pi}{3}\text{....}\infty$ **Ans.:**

$\frac{1}{8}\left\{2\sqrt{3}\log(2+\sqrt{3}) - \pi\right\}$

4. If the series $u = \cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - ....\infty$, then prove that $u = \frac{\pi}{4}$.

5. If the series $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - ....\infty = v$, show that $\tanh 2v = \sin\theta$

10.9 THE DIFFERENCE METHOD

In this method to sum of series, every term is split into the difference of two terms, such that one term of each expression should appear as one of the expressions of the next term with sign changed. In this way we added equipped together all the terms of the series, all expression cancel in pairs diagonal wise expect two, one each from the first and last term. This method is to split the n^{th} term of the series as the difference of two terms, and then the splitting mode can be found by putting $n = 1$ in the solution and reduce to first term of the series.

Suppose the sum of series

$$S_n = T_1 + T_2 + T_3 + ...+ T_n$$ (Sum of the first n terms of the series)

The series is expressed as

$$T_1 = f(2) - f(1)$$

$$T_2 = f(3) - f(2)$$

$$T_3 = f(4) - f(3)$$

... ...

... ...

$$T_{n-1} = f(n) - f(n-1)$$

and $T_n = f(n+1) - f(n)$

On addition, we get

$$T_1 + T_2 + T_3 + ... + T_n = f(n+1) - f(1)$$

If the given series is convergent and sum to infinity reduce as follows

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} [f(n+1) - (1)]$$

1. Find the sum of the series to n terms

$$\tan^{-1}\frac{4}{1+3.4}+\tan^{-1}\frac{6}{1+8.9}+\tan^{-1}\frac{8}{1+15.16}..........$$

Solution: We have the n terms

Let $\quad T_n=\tan^{-1}\dfrac{2(n+1)}{1+n(n+2)(n+1)^2}$

$$=\tan^{-1}\left\{\frac{(n+1)(n+2)-n(n+1)}{1+n(n+2)(n+1)^2}\right\}=\tan^{-1}\left[\frac{x-y}{1+xy}\right]$$

where, $x=(n+2)(n+1)$ and $y=n(n+1)$

hence, $T_n=\tan^{-1}x-\tan^{-1}y$

$\therefore \quad T_n=\tan^{-1}[(n+1)(n+2)]-\tan^{-1}[n(n+1)]$

Now putting $n=1,2,3,............,n$, we have

$$T_1=\tan^{-1}3.2-\tan^{-1}2.1,$$

$$T_2=\tan^{-1}4.3-\tan^{-1}3.2,$$

$$T_3=\tan^{-1}4.5-\tan^{-1}4.3,$$

$$\ldots \quad \ldots \quad \ldots$$

$$T_n=\tan^{-1}[(n+1)(n+2)]-\tan^{-1}[n(n+1)]$$

Adding all above term, we get

$$S_n=\tan^{-1}[(n+2)(n+1)]-\tan^{-1}2.1=\tan^{-1}(n^2+3n+2)-\tan^{-1}2$$

$$=\tan^{-1}\frac{n^2+3n}{1+2(n^2+3n+2)}=\tan^{-1}\frac{n^2+3n}{2n^2+6n+5}$$

2. Sum of the series

$$\tan^{-1}\frac{1}{3+3.1+1^2}+\tan^{-1}\frac{1}{3+3.2+2^2}+\tan^{-1}\frac{1}{3+3.3+3^2}+.....+\tan^{-1}\frac{1}{3+3.n+n^2}$$

Solution: Let $\quad T_n=\tan^{-1}\dfrac{1}{3+3.n+n^2}=\tan^{-1}\dfrac{1}{1+(n^2+3n+2)}$

$$=\tan^{-1}\frac{1}{1+(n+2)(n+1)}=\tan^{-1}\frac{(n+2)-(n+1)}{1+(n+2)(n+1)}$$

$\Rightarrow \quad T_n=\tan^{-1}(n+2)-\tan^{-1}(n+1)$

Putting $n=1,2,3,............,n$, we have

$$T_1=\tan^{-1}3-\tan^{-1}2,$$

$$T_2=\tan^{-1}4-\tan^{-1}3,$$

$$T_3=\tan^{-1}5-\tan^{-1}4,$$

$$\ldots \quad \ldots \quad \ldots$$

$$T_n = \tan^{-1}(n+2) - \tan^{-1}(n+1)$$

Adding all above term, we get

$$S_n = \tan^{-1}(n+2) - \tan^{-1} 2$$

3. Sum of the series

$$\tan^{-1}\frac{12}{31} + \tan^{-1}\frac{12}{139} + \ldots\ldots\ldots\ldots\ldots + \tan^{-1}\frac{12}{36n^2-5}$$

Solution: Let $T_n = \tan^{-1}\dfrac{12}{36n^2-5} = \tan^{-1}\dfrac{12}{4+(36n^2-9)}$

$$= \tan^{-1}\frac{3}{1+(9n^2-\frac{9}{4})} = \tan^{-1}\frac{3}{1+\left(3n+\frac{3}{2}\right)\left(3n-\frac{3}{2}\right)}$$

$$= \tan^{-1}\frac{\left(3n+\frac{3}{2}\right)-\left(3n-\frac{3}{2}\right)}{1+\left(3n+\frac{3}{2}\right)\left(3n-\frac{3}{2}\right)}$$

$\Rightarrow$ $$T_n = \tan^{-1}\left(3n+\frac{3}{2}\right) - \tan^{-1}\left(3n-\frac{3}{2}\right)$$

Putting $n = 1,2,3,\ldots\ldots\ldots\ldots, n$, we have

$$T_1 = \tan^{-1}\frac{9}{2} - \tan^{-1}\frac{3}{2},$$

$$T_2 = \tan^{-1}\frac{15}{2} - \tan^{-1}\frac{9}{2},$$

$$T_3 = \tan^{-1}\left(\frac{21}{2}\right) - \tan^{-1}\left(\frac{15}{2}\right)$$

$$\ldots \quad \ldots \quad \ldots$$

$$T_n = \tan^{-1}\left(3n+\frac{3}{2}\right) - \tan^{-1}\left(3n-\frac{3}{2}\right)$$

Adding all above term, we get

$$S_n = \tan^{-1}\left(3n+\frac{3}{2}\right) - \tan^{-1}\left(\frac{3}{2}\right)$$

$$= \tan^{-1}\left[\frac{\left(3n+\frac{3}{2}\right)-\left(\frac{3}{2}\right)}{1+\left(3n+\frac{3}{2}\right)\times\left(\frac{3}{2}\right)}\right]$$

$$= \tan^{-1}\frac{3n}{1+\frac{9}{2}n+\frac{9}{4}} = \tan^{-1}\frac{12n}{4+18n+9}$$

$$S = \tan^{-1}\frac{12n}{18n+13}$$

10.10 SUMMARY

This unit defined that summations of infinite sequences are called series. They involve the concept of limit, and are not considered in this article. The summation of an explicit sequence is denoted as a succession of additions. For example, summation of [1, 2, 4, 2] is denoted 1 + 2 + 4 + 2, and results in 9, that is, 1 + 2 + 4 + 2 = 9. The common pattern in an arithmetic sequence is that the same number is added or subtracted to each number to produce the next number. This is called the common difference. In other words, summation is the addition of a sequence of any kind of numbers, called addends or summands; the result is their sum or total.

10.11 GLOSSARY

1. ***Series:*** *Get* when you add up all the terms of a *sequence.*
2. ***Sine:*** *L*ength of the opposite side angle over the length of the hypotenuse.
3. **Cosine**: Length of the side adjacent to angle over the length of the hypotenuse.

10.12 SELF ASSESSMTS QUESTIONS

10.12.1 Multiple choice questions

1. $C+iS$ method of finding the sum of these series which involve:

(a) sine and tangents of multiple of angles

(b) cosine and cotangent of multiple of angles

(c) sine and cosine of multiple of angles

(d) tangent and cotangent of multiple of angles

2. If $C+iS$ method of finding the sum, the resulting series is $a+ar+ar^2+.....$to n terms then we use the formula:

(a) $S_n=\dfrac{a(r^n-1)}{1-r}$ (b) $S_n=\dfrac{a(1-r^n)}{r-1}$

(c) $S_n=\dfrac{a(1-r^n)}{1-r}$ (d) $S_n=\dfrac{a}{1-r}$

3. If $\tan^{-1}\dfrac{1}{1+n(n+1)}=\tan^{-1}(n+1)-\tan^{-1}y$ then the value of y is

(a) n (b) $n+1$

(c) n^2+n+1 (d) $n-1$

4. If $\tan^{-1}\dfrac{1}{2n^2}=\tan^{-1}(2n+1)-\tan^{-1}y$ then the value of y is

(a) $2n$ (b) $n+1$

(c) $n-1$ (d) $2n-1$

5. Sum of series $\cos\alpha+\dfrac{1}{2}\cos 2\alpha+\dfrac{1}{2^2}\cos 3\alpha+.........\infty$ is

(a) $\dfrac{4\cos\alpha-2}{5+4\cos\alpha}$ (b) $\dfrac{4\cos\alpha+2}{5-4\cos\alpha}$

(c) $\dfrac{4\cos\alpha-2}{5-4\cos\alpha}$ (d) $\dfrac{2\cos\alpha-4}{5-4\cos\alpha}$

6. Sum of series $\dfrac{\sin\theta}{1!}+\dfrac{\sin 2\theta}{2!}+\dfrac{\sin 3\theta}{3!}+.........\infty$ is

(a) $e^{\cos\theta}\sin(\sin\theta)$ (b) $e^{\cos\theta}\sin(\cos\theta)$

(c) $e^{\cos\theta}\cos(\cos\theta)$ (d) $e^{\sin\theta}\sin(\sin\theta)$

7. Sum of series $\dfrac{\sin\theta}{1!}-\dfrac{\sin 2\theta}{2!}+\dfrac{\sin 3\theta}{3!}-\dfrac{\sin 4\theta}{4!}+\cos\theta.........\infty$ is

(a) $e\sin(\sin\theta)$ (b) $e^{-\cos\theta}\sin(\sin\theta)$

(c) $e^{\sin\theta}\cos(\sin\theta)$ (d) $e^{-\sin\theta}\cos(\cos\theta)$

8. If $|c|<1,$ than the sum of the infinite series $ce^{i\theta}+\frac{1}{2}c^2e^{i2\theta}+\frac{1}{3}c^3e^{i3\theta}+..............\infty$ is

(a) $\log(1+ce^{i\theta})$ (b) $\log(1-ce^{i\theta})$

(c) $-\log(1-ce^{i\theta})$ (d) $-\log(1+ce^{i\theta})$

10.12.2Fill in the blanks

1. If $|z|<1,$ than the sum of the infinite series $z+\frac{1}{3}z^3+\frac{1}{5}z^5+..............\infty$ is……

2. If $C+iS=\tan^{-1}(c\cos\theta+ic\sin\theta),$ then C is…………

3. Sum of series $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{21}+......$ to n terms is………

4. Sum of series taking the value of c not greater than unity numerically, $c\cos\theta-\frac{c^2}{2}\cos 2\theta+\frac{c^3}{3}\cos 3\theta-...........ad.\inf$. is………………

5. Sum of series $1+\frac{\cos\theta}{\cos\theta}+\frac{\cos 2\theta}{2!\cos^2\theta}+\frac{\cos 3\theta}{3!\cos^3\theta}+........ad.\inf$. is…….

6. Sum of series $\sin^4\theta+\sin^4\left(\theta+\frac{2\pi}{n}\right)+\sin^4\left(\theta+\frac{4\pi}{n}\right)+.........+n$ terms is………

7. Sum to infinite terms of the series $1+\frac{\cos\alpha}{\cos\alpha}+\frac{\cos 2\alpha}{\cos^2\alpha}+\frac{\cos 3\alpha}{\cos^2\alpha}+........\infty$ is

8. Sumof the series deduce to infinite $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{4}{33}+..........\tan^{-1}\frac{2^{n-1}}{1+2^{n-1}}$

ANSWERS:

1. $\frac{1}{2}\log\frac{1+z}{1-z}$ **2.** $\tan\theta$ **3.** $\tan^{-1}\frac{n}{n+2}$ **4.** $\frac{1}{2}\log(1+c^2+2c\cos\theta)$

5. $e\cos(\tan\theta)$ **6.** $\frac{3n}{8}$ **7.** 0 **8.** $\frac{\pi}{4}$

10.13 REFERENCES

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10.14 SUGGESTED READING

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10.15 TERMINAL QUESTIONS

10.15.1 Short answer type question

Sum of the following series:

1. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}....to\ n-terms$

2. $\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{2}{16}+...........\infty$

3. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{4}{33}+....+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$

4. $tan^{-1}\frac{1}{1+1+1^2}+tan^{-1}\frac{1}{1+2+2^2}+tan^{-1}\frac{1}{1+3+3^2}+\cdots to\ n-terms$

ANSWERS:1. $\tan^{-1}\frac{n}{n+2}$ **2.** $\tan^{-1}3$ **3.** $\tan^{-1}\frac{2^n-1}{1+2^n}$ **4.** $\tan^{-1}\frac{n}{n+2}$

10.15.2 Long answer type question

Sum of the following series:

1. $\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{4}{19}+\tan^{-1}\frac{4}{39}+......+\tan^{-1}\frac{4}{4n^2+3}$

2. $\tan\frac{\theta}{2}\sec\theta+\tan\frac{\theta}{2^2}\sec\frac{\theta}{2}+\tan\frac{\theta}{2^3}\sec\frac{\theta}{2^2}+....to\ n-terms$

3. $\frac{1}{2\cos\theta}+\frac{1}{2^2\cos\theta\cos2\theta}+\frac{1}{2^3\cos\theta\cos2\theta\cos2^2\theta}+...to\ n-terms$

4. $\cos ec\theta\cos ec2\theta+\cos ec2\theta\cos ec3\theta+....to\ n-terms$

ANSWERS:1. $\tan^{-1}\left(\frac{4n}{2n+5}\right)$ **2.** $\tan\theta$ **3.** $\sin\theta[\cot\theta-\cot2^2\theta]$ **4.** $\frac{1}{\sin\theta}[\cot\theta-\cot(n+1)\theta]$

UNIT 11: INFINITE PRODUCT AND GREGORY'S SERIES

CONTENTS:

11.1 *OBJECTIVE*

After reading this unit you will be able to:

- Expand $\sin\theta$ and $\cos\theta$ in form of infinite product.
- Expand $\sinh\theta$ and $\cosh\theta$ in form of infinite product.
- Understand some special series like
 (i) Greagory's series
 (ii) Euler's series

(iii) Machin's series

(iv) Rutherford's series

- Find value of π using these special series.

11.2 *INTRODUCTION*

This unit deals with expressions of $\sin\theta, \cos\theta, \sinh\theta$ and $\cosh\theta$ in the form of infinite product. It helps us to find value of π. Gregory's series, Euler's series, Machin's series and Rutherford's series are also discussed in this unit. Among then Gregory's series has a great importance,it is the infinite Taylor series expression of inverse tangent function.

11.3 *INFINITE PRODUCT*

11.3.1 Expansion of $\sin\theta$ in the form of Infinite Product:

To express $\sin\theta$ as an infinite product, we have

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\sin\frac{x}{2}\sin\frac{\pi + x}{2}$$

……….(1)

Putting $x = \theta$ in (1), we get

$$\sin\theta = 2\sin\frac{\theta}{2}\sin\frac{\pi+\theta}{2}$$

……….(2)

Putting $x = \frac{\theta}{2}$ in (1), we get

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}$$

……….(3)

Putting $x = \frac{\pi+\theta}{2}$ in (1), we get

$$\sin\frac{\pi+\theta}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

……….(4)

Putting the values of $\sin\frac{\theta}{2}$ and $\sin\frac{\pi+\theta}{2}$ in (2), we get

$$\sin\theta = 2^3 \sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

Continuing this process successively, we get

$$\sin\theta = 2^7 \sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}.............\sin\frac{7\pi+\theta}{2^3}$$

$$= 2^{(2^3-1)} \sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}.............\sin\frac{(2^3-1)\pi+\theta}{2^3}$$

....

....

$$= 2^{(p-1)} \sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}\sin\frac{3\pi+\theta}{p}.............\sin\frac{(p-1)\pi+\theta}{p}$$

……(5)

Where $p = 2^n$

The last factor on the right hand side of the (5) is

$$= \sin\left\{\frac{(p-1)\pi+\theta}{p}\right\} = \sin\left(\pi - \frac{\pi-\theta}{p}\right) = \sin\left(\frac{\pi-\theta}{p}\right)$$

The second last factor on the right hand side of the (5) is

$$= \sin\left\{\frac{(p-2)\pi+\theta}{p}\right\} = \sin\left(\pi - \frac{2\pi-\theta}{p}\right) = \sin\left(\frac{2\pi-\theta}{p}\right)$$

and so on.

Now we combine together the second last and the last factors, the third and so on. The uncombined factor is $\left[\frac{p}{2}+1\right]^{th}$ factor and its value is $\sin\left(\frac{\frac{p}{2}\pi+\theta}{p}\right)$.

Thus on combining the above pairs of factors (5) become

$$\sin\theta = 2^{(p-1)}\sin\frac{\theta}{p}\left\{\sin\frac{\theta+\pi}{p}\sin\frac{\pi-\theta}{p}\right\}\times\left\{\sin\frac{2\pi+\theta}{p}\sin\frac{2\pi-\theta}{p}\right\}......\sin\left\{\frac{\frac{p}{2}\pi+\theta}{p}\right\}$$

….(6)

But $\sin\left(\frac{\frac{\pi p}{2}+\theta}{p}\right)=\sin\left(\frac{\pi}{2}+\frac{\theta}{p}\right)=\cos\frac{\theta}{p}$

Since, by using $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B,$ then (6) reduces to

$$\sin\theta = 2^{p-1}\sin\frac{\theta}{p}\left\{\sin^2\frac{\pi}{p}-\sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p}-\sin^2\frac{\theta}{p}\right\}.................$$

$$................................\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-\sin^2\frac{\theta}{p}\right\}\cos\frac{\theta}{p}\ ……..(7)$$

Dividing both sides of (7) $\sin\frac{p}{2}$ and taking $\theta\to 0$, we get

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1, \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\sin\frac{\theta}{p}}=\lim_{\theta\to 0}\frac{p\left(\frac{\sin\theta}{\theta}\right)}{\left(\frac{\sin\frac{\theta}{p}}{\frac{\theta}{p}}\right)}=p \qquad \text{and}$$

$$\lim_{\theta\to 0}\sin^2\frac{\theta}{p}=0.$$

Now we have

$$p=2^{(p-1)}\sin^2\frac{\pi}{p}.\sin^2\frac{2\pi}{p}.\sin^2\frac{3\pi}{p}...........\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}$$

……….(8)

Dividing equation (7) by (8), we get

$$\sin\theta = p\sin\frac{\theta}{p}\cos\frac{\theta}{p}\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right\}\times\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{2\pi}{p}}\right\}\times\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{3\pi}{p}}\right\}....\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\left(\frac{p}{2}-1\right)\frac{\pi}{p}}\right\}$$

Now $p\to\infty$, then $$\lim_{p\to\infty}\left(p\sin\frac{\theta}{p}\right)=\lim_{p\to\infty}\left\{\frac{\sin\frac{\theta}{p}}{\frac{\theta}{p}}\right\}\theta=\theta$$

$$\lim_{\theta\to 0}\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}=\lim_{\theta\to 0}\left\{\frac{\left(\frac{\sin\theta}{\theta}\right)}{\frac{\theta^2}{p^2}}\right\}\left\{\frac{\frac{\pi^2}{p^2}}{\sin^2\frac{\pi}{p}}\right\}\left\{\frac{\theta^2}{\pi^2}\right\}=\frac{\theta^2}{\pi^2}\text{ etc.}$$

and so on and also $\lim\limits_{\theta\to 0}\cos\left(\frac{\theta}{p}\right)=1$, we have

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)..............\infty$$

$$\sin\theta=\theta\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right).$$

11.3.2 Expansion of $\cos\theta$ in the form of Infinite Product:

To express $\cos\theta$ as an infinite product, we have

Now, by using the result (5) of 9.3, we get

$$\sin\theta=2^{(p-1)}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}.............\sin\frac{(p-1)\pi+\theta}{p}$$

………(1)

Putting $\theta=\left(\frac{\pi}{2}+\theta\right)$, we get

$$\cos\theta=2^{(p-1)}\sin\left(\frac{\pi+2\theta}{2p}\right)\sin\frac{3\pi+2\theta}{2p}.............\sin\frac{(2p-1)\pi+2\theta}{2p}$$

………(2)

Now the last factor on the right hand sides of (2), we get

$$\sin\left\{\frac{(2p-1)\pi+2\theta}{2p}\right\}=\sin\left\{\frac{2p\pi-\pi+2\theta}{2p}\right\}$$

$$=\sin\left\{\pi-\frac{\pi-2\theta}{2p}\right\}=\sin\left\{\frac{\pi-2\theta}{2p}\right\}$$

Again the second factor from the end of (2), we get

$$\sin\left\{\frac{(2p-3)\pi+2\theta}{2p}\right\}=\sin\left\{\frac{2p\pi-3\pi+2\theta}{2p}\right\}$$

$$=\sin\left\{\pi-\frac{3\pi-2\theta}{2p}\right\}=\sin\left\{\frac{3\pi-2\theta}{2p}\right\}$$

Therefore, on combining first and last factor from the beginning and the end, and so on, we get

$$\cos\theta=2^{(p-1)}\left\{\sin\frac{\pi+2\theta}{2p}\sin\frac{\pi-2\theta}{2p}\right\}.\left\{\sin\frac{3\pi+2\theta}{2p}\sin\frac{3\pi-2\theta}{2p}\right\}.....$$

$$=2^{(p-1)}\left\{\sin^2\frac{\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\}.\left\{\sin^2\frac{3\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\}.....$$

……….(3)

Now putting $\theta=0$ we get

$$1=2^{(p-1)}\sin^2\frac{\pi}{2p}.\sin^2\frac{3\pi}{2p}\sin^2\frac{5\pi}{2p}........$$

..……..(4)

Dividing (3) by (4), we get

$$\cos\theta=\left\{1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right\}.\left\{1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{3\pi}{2p}}\right\}.........$$

………..(5)

Making $p\to\infty$, we get

$$\lim_{p\to\infty}\left\{\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right\}=\left(\frac{2\theta}{\pi}\right)^2=\frac{4\theta^2}{\pi^2}\quad,\quad\lim_{p\to\infty}\left\{\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{3\pi}{2p}}\right\}=\left(\frac{2\theta}{3\pi}\right)^2=\frac{4\theta^2}{9\pi^2}$$

and so on

From (5), we get

$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right).\left(1-\frac{4\theta^2}{3^2\pi^2}\right)..............\infty \qquad \Rightarrow$$

$$\cos\theta = \prod_{r=1}^{\infty}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$$

11.4 *EXPANSION OF* $\sinh\theta$ *AND* $\cosh\theta$ *IN THE FORM OF INFINITE PRODUCT*

(a) **To express** $\sinh\theta$ **as an infinite product:**

We know that

$$\sin\theta = \theta\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$$

Putting $\theta = i\theta$, we get

$$\sin i\theta = i\theta\prod_{r=1}^{\infty}\left(1+\frac{\theta^2}{r^2\pi^2}\right) \qquad \Rightarrow$$

$$\sinh\theta = \theta\prod_{r=1}^{\infty}\left(1+\frac{\theta^2}{r^2\pi^2}\right)$$

(b) **To express** $\cosh\theta$ **as an infinite product:**

We know that

$$\cos\theta = \prod_{r=1}^{\infty}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$$

Putting $\theta = i\theta$, we get

$$\cos i\theta = \prod_{r=1}^{\infty}\left(1-\frac{4i^2\theta^2}{(2r-1)^2\pi^2}\right) \qquad \Rightarrow$$

$$\cosh\theta = \prod_{r=1}^{\infty}\left(1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$$

11.5 *SOME STANDARD RESULTS OF INFINITE PRODUCT*

(a) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..........\infty=\frac{\pi^2}{6}$ **(b)**

$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+..........\infty=\frac{\pi^4}{90}$

(c) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+..........\infty=\frac{\pi^2}{8}$ **(d)**

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+..........\infty=\frac{\pi^4}{96}$

Example 1. Show that **(i)** $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..........\infty=\frac{\pi^2}{6}$ **.**

(ii) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+..........\infty=\frac{\pi^4}{90}$

Solution (i): We know that

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)..............\infty$$

$$\Rightarrow \quad \frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)..............\infty$$

……….(1)

And also, $\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-.........\infty$

$$\Rightarrow \quad \frac{\sin\theta}{\theta}=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-.........\infty$$

……….(2)

From (1) and (2), we get

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)..............\infty=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-............\infty$$

...............(3)

Taking logarithms of both sides, we get

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+.............\infty=\log\left[1-\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}+............\infty\right)\right]$$

$$\left(\frac{\theta^2}{\pi^2}+\frac{\theta^4}{2\pi^4}+.....\right)+\left(\frac{\theta^2}{2^2\pi^2}+\frac{\theta^4}{2.2^4\pi^4}+.....\right)+.....\infty$$

$$=\left[\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}+.....\infty\right)+\frac{1}{2}\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}+.....\infty\right)^2+....\infty\right]$$

............(4)

Equating the coefficients of θ^2 on both sides in equation (4), we get

$$\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+..................\infty=\frac{1}{3!}$$

$$\Rightarrow \quad \frac{1}{\pi^2}\left(\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+.......\infty\right)+......\infty=\frac{1}{3!}$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.........\infty+.....\infty=\frac{\pi^2}{6}$$

Solution (ii): Equating the coefficients of θ^4 on both sides in equation (4), we get

$$-\frac{1}{2\pi^2}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.......\infty\right]=-\left[-\frac{1}{120}+\frac{1}{72}\right]$$

$$\Rightarrow \quad \left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.........\infty\right]=2\pi^4\left[-\frac{1}{120}+\frac{1}{72}\right]$$

$$\Rightarrow \quad \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.........\infty=\frac{2\pi^4}{180}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.........\infty=\frac{\pi^4}{90}$$

Example 2. Show that

(a) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)+............\infty=\frac{2}{\pi}$

(b) $\frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\frac{6}{5}.\frac{6}{7}.\frac{8}{7}.\frac{8}{9}................\infty=\frac{\pi}{2}$

Solution:(a) We know that

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right).\left(1-\frac{\theta^2}{3^2\pi^2}\right).\left(1-\frac{\theta^2}{4^2\pi^2}\right)..............\infty$$

Substituting $\theta=\frac{\pi}{2}$ on both sides, we get

$$\sin\frac{\pi}{2}=\frac{\pi}{2}\left(1-\frac{\pi^2}{2^2\pi^2}\right)\left(1-\frac{\pi^2}{2^2 2^2\pi^2}\right)\left(1-\frac{\pi^2}{3^2 2^2\pi^2}\right).\left(1-\frac{\pi^2}{4^2 2^2\pi^2}\right)..............\infty$$

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{8^2}\right)..........................\infty$$

$$\frac{2}{\pi}=\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{4^2}\right).\left(1-\frac{1}{6^2}\right).\left(1-\frac{1}{8^2}\right)...............\infty$$

……….(1)

(b) From above result (1) can expressed in the form, we get

$$\frac{2}{\pi}=\left(\frac{2^2-1}{2^2}\right)\left(\frac{4^2-1}{4^2}\right)\left(\frac{6^2-1}{6^2}\right).\left(\frac{8^2-1}{8^2}\right)................\infty$$

$$\frac{2}{\pi}=\frac{3}{2^2}.\frac{15}{4^2}.\frac{35}{6^2}.\frac{63}{8^2}.\frac{99}{10^2}.............\infty$$

$$\frac{2}{\pi}=\frac{1.3}{2.2}.\frac{3.5}{4.4}.\frac{5.7}{6.6}.\frac{7.9}{8.8}.\frac{9.11}{10.10}.............\infty$$

$$\frac{\pi}{2}=\frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\frac{6}{5}.\frac{6}{7}.\frac{8}{7}.\frac{8}{9}............\infty$$

Example 3. Show that

$$\sqrt{2}=\frac{4}{3}.\frac{36}{35}.\frac{100}{99}.\frac{196}{195}.\frac{324}{323}.............\infty$$

Solution: We know that

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right).\left(1-\frac{4\theta^2}{3^2\pi^2}\right).\left(1-\frac{4\theta^2}{5^2\pi^2}\right).\left(1-\frac{4\theta^2}{7^2\pi^2}\right)...............\infty$$

Substituting $\theta=\frac{\pi}{4}$ on both sides, we get

$$\cos\frac{\pi}{4}=\left(1-\frac{4\pi^2}{1^2 4^2\pi^2}\right).\left(1-\frac{4\pi^2}{3^2 4^2\pi^2}\right).\left(1-\frac{4\pi^2}{5^2 4^2\pi^2}\right).\left(1-\frac{4\pi^2}{7^2 4^2\pi^2}\right).\left(1-\frac{4\pi^2}{9^2 4^2\pi^2}\right)........\infty$$

$$\frac{1}{\sqrt{2}}=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3^2 4}\right).\left(1-\frac{1}{5^2 4}\right)..\left(1-\frac{1}{7^2 4}\right).\left(1-\frac{1}{9^2 4}\right)...............\infty$$

$$=\left(1-\frac{1}{4.1}\right).\left(1-\frac{1}{4.9}\right).\left(1-\frac{1}{4.25}\right).\left(1-\frac{1}{4.49}\right).\left(1-\frac{1}{4.81}\right)...............\infty$$

$$=\left(\frac{4-1}{4}\right).\left(\frac{36-1}{36}\right).\left(\frac{100-1}{100}\right).\left(\frac{196-1}{196}\right).\left(\frac{324-1}{324}\right)...............\infty$$

$\Rightarrow \qquad \frac{1}{\sqrt{2}} = \frac{3}{4}.\frac{35}{36}.\frac{99}{100}.\frac{195}{196}.\frac{323}{324}....\infty$

$\Rightarrow \qquad \sqrt{2} = \frac{4}{3}.\frac{36}{35}.\frac{100}{99}.\frac{196}{195}.\frac{324}{323}....\infty$

Example 4. When n is very large, then prove that

$$\sqrt{\frac{(2n+1)\pi}{2}} = \frac{2.4.6..........2n}{1.3.5....(2n-1)} \text{ approximately.}$$

Solution: We know that, if n is very large

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right).\left(1-\frac{\theta^2}{3^2\pi^2}\right)......................\left(1-\frac{\theta^2}{n^2\pi^2}\right)$$

Substituting $\theta = \frac{\pi}{2}$ on both sides, we get

$$\sin\frac{\pi}{2} = \frac{\pi}{2}\left(1-\frac{\pi^2}{2^2\pi^2}\right).\left(1-\frac{\pi^2}{2^2 2^2\pi^2}\right).\left(1-\frac{\pi^2}{2^2 3^2\pi^2}\right)......................\left(1-\frac{\pi^2}{2^2 n^2\pi^2}\right).$$

$\Rightarrow \qquad 1 = \frac{\pi}{2}\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{4^2}\right).\left(1-\frac{1}{6^2}\right)......................\left(1-\frac{1}{(2n)^2}\right).$

$\Rightarrow$

$$1 = \frac{\pi}{2}\left(1-\frac{1}{2}\right).\left(1+\frac{1}{2}\right).\left(1-\frac{1}{4}\right).\left(1+\frac{1}{4}\right).\left(1-\frac{1}{6}\right).\left(1+\frac{1}{6}\right)......................\left(1-\frac{1}{2n}\right).\left(1+\frac{1}{2n}\right).$$

$\Rightarrow \qquad 1 = \frac{\pi}{2}.\frac{1}{2}.\frac{3}{2}.\frac{3}{4}.\frac{5}{4}.\frac{5}{6}.\frac{7}{6}......................\frac{2n-1}{2n}.\frac{2n+1}{2n}$

$\Rightarrow \qquad 1 = \frac{\pi}{2}.\frac{1^2.3^2.5^2..............(2n-1)^2(2n+1)}{2^2.4^2.6^2................(2n)^2}$

$\Rightarrow \qquad \frac{2}{(2n+1)\pi} = \frac{1^2.3^2.5^2..............(2n-1)^2}{2^2.4^2.6^2................(2n)^2}$

$\Rightarrow \qquad \frac{2}{(2n+1)\pi} = \left(\frac{1.3.5..............(2n-1)}{2.4.6................(2n)}\right)^2$

$\Rightarrow \qquad \frac{(2n+1)\pi}{2} = \left(\frac{2.4.6................(2n)}{1.3.5..............(2n-1)}\right)^2$

$\Rightarrow \qquad \sqrt{\frac{(2n+1)\pi}{2}} = \frac{2.4.6................(2n)}{1.3.5..............(2n-1)}$

Example 5. Show that

$$\cos\left(\frac{\pi}{2}\sin\theta\right)=\frac{\pi}{4}\cos^2\theta\left(1+\frac{\cos^2\theta}{2.4}\right).\left(1+\frac{\cos^2\theta}{4.6}\right)\ldots\ldots\ldots$$

Solution: We know that the cosine series as an infinite product

$$\cos x=\left(1-\frac{4x^2}{\pi^2}\right).\left(1-\frac{4x^2}{3^2\pi^2}\right).\left(1-\frac{4x^2}{5^2\pi^2}\right)\ldots\ldots\ldots\ldots\ldots$$

Substituting $x=\frac{\pi}{2}\sin\theta$ on both sides, we get

$$\cos\left(\frac{\pi}{2}\sin\theta\right)=\left(1-\frac{4\pi^2\sin^2\theta}{2^21^2\pi^2}\right).\left(1-\frac{4\pi^2\sin^2\theta}{2^23^2\pi^2}\right).\left(1-\frac{4\pi^2\sin^2\theta}{2^25^2\pi^2}\right)\ldots\ldots\ldots\ldots\ldots$$

$$=(1-\sin^2\theta).\left(1-\frac{\sin^2\theta}{3^2}\right).\left(1-\frac{\sin^2\theta}{5^2}\right)\ldots\ldots\ldots\ldots\ldots$$

$$=\cos^2\theta.\left(\frac{3^2-(1-\cos^2\theta)}{3^2}\right).\left(\frac{5^2-(1-\cos^2\theta)}{5^2}\right)\ldots\ldots\ldots\ldots\ldots$$

$$=\cos^2\theta.\left(\frac{8+\cos^2\theta}{3^2}\right).\left(\frac{24+\cos^2\theta}{5^2}\right)\ldots\ldots\ldots\ldots\ldots$$

$$\left(\frac{2.4.4.6\ldots\ldots\ldots\ldots}{3.3.5.5\ldots\ldots\ldots\ldots}\right)\cos^2\theta.\left(1+\frac{\cos^2\theta}{2.4}\right)\left(1+\frac{\cos^2\theta}{4.6}\right)$$

$\ldots\ldots\ldots(1)$

Also, we know that the sine series as an infinite product

$$\sin x=x\left(1-\frac{x^2}{\pi^2}\right).\left(1-\frac{x^2}{2^2\pi^2}\right).\left(1-\frac{x^2}{3^2\pi^2}\right)\ldots\ldots\ldots\ldots$$

Substituting $x=\frac{\pi}{2}$ on both sides, we get

$$\Rightarrow\quad 1=\frac{\pi}{2}\left(1-\frac{\pi^2}{2^2\pi^2}\right).\left(1-\frac{\pi^2}{2^22^2\pi^2}\right).\left(1-\frac{\pi^2}{2^23^2\pi^2}\right)\ldots\ldots\ldots\ldots\Rightarrow$$

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{4^2}\right).\left(1-\frac{1}{6^2}\right)\ldots\ldots\ldots\ldots\quad\Rightarrow$$

$$1=\frac{\pi}{2}\left(1-\frac{1}{4}\right).\left(1-\frac{1}{16}\right).\left(1-\frac{1}{36}\right)\ldots\ldots\ldots\ldots$$

$\Rightarrow \quad 1=\frac{\pi}{2}\left(\frac{3}{4}\right).\left(\frac{15}{16}\right).\left(\frac{35}{36}\right)..........\qquad \Rightarrow$

$$\frac{\pi}{4}=\frac{2}{3}.\frac{16}{15}.\frac{36}{35}..........$$

$$\Rightarrow \quad \frac{\pi}{4}=\frac{2.16.36..........}{3.15.35..........}=\frac{2.4.4.6.6..........}{3.3.5.5.7..........}$$

……….(2)

From (1) and (2), we get

$$\cos\left(\frac{\pi}{2}\sin\theta\right)=\frac{\pi}{4}\cos^2\theta\left(1+\frac{\cos^2\theta}{2.4}\right).\left(1+\frac{\cos^2\theta}{4.6}\right)..........$$

Example 6. Show that

$$\sin x+\cos x=\left(1+\frac{4x}{\pi}\right).\left(1-\frac{4x}{3\pi}\right).\left(1+\frac{4x}{5\pi}\right).\left(1-\frac{4x}{7\pi}\right)..........$$

and hence deduce that $\quad 1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.........=\frac{\pi^2}{32}$

Solution: We know that

$$\sin x+\cos x=\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)=\sqrt{2}\sin\frac{\pi}{4}\left(1+\frac{4x}{\pi}\right)$$

$$=\sqrt{2}.\frac{\pi}{4}\left(1+\frac{4x}{\pi}\right)\prod_{r=1}^{\infty}\left[1-\frac{\left\{\frac{\pi}{4}\left(1+\frac{4x}{\pi}\right)\right\}^2}{r^2\pi^2}\right]$$

..........(1)

Putting $x=0$ in (1), we get

$$1=\sqrt{2}\,\frac{\pi}{4}\prod_{r=1}^{\infty}\left[1-\frac{1}{(4r)^2}\right]$$

..........(2)

Dividing (1) by (2), we get

$$\sin x+\cos x=\left(1+\frac{4x}{\pi}\right)\prod_{r=1}^{\infty}\left[\frac{(4r)^2-\left(1+\frac{4x}{\pi}\right)^2}{(4r)^2-1}\right]$$

$$=\left(1+\frac{4x}{\pi}\right)\prod_{r=1}^{\infty}\left[\frac{\left(4r-1-\frac{4x}{\pi}\right)\left(4r+1+\frac{4x}{\pi}\right)}{(4r-1)(4r+1)}\right]$$

$$=\left(1+\frac{4x}{\pi}\right)\prod_{r=1}^{\infty}\left[\frac{\left(4r-1-\frac{4x}{\pi}\right)}{(4r-1)}\times\frac{\left(4r+1+\frac{4x}{\pi}\right)}{(4r+1)}\right]$$

$$=\left(1+\frac{4x}{\pi}\right)\prod_{r=1}^{\infty}\left(1-\frac{4x}{(4r-1)\pi}\right).\left(1+\frac{4x}{(4r+1)\pi}\right).$$

$$=\left(1+\frac{4x}{\pi}\right).\left(1-\frac{4x}{3\pi}\right).\left(1+\frac{4x}{5\pi}\right)\left(1-\frac{4x}{7\pi}\right)..........$$

……..(3)

Deduction: Taking logarithms on both sides of (3), we get

$$\log\left(1+\frac{4x}{\pi}\right)+\log.\left(1-\frac{4x}{3\pi}\right)+\log\left(1+\frac{4x}{5\pi}\right)+\log\left(1-\frac{4x}{7\pi}\right).........=\log(\sin x+\cos x)$$

$$=\frac{1}{2}\log(1+\sin 2x)=\frac{1}{2}\log\left[1+\left(2x-\frac{8x^3}{3!}+.......\right)\right]$$

$$=\frac{1}{2}\left[\left(2x-\frac{8x^3}{3!}+....\right)-\frac{1}{2}\left(2x-\frac{8x^3}{3!}+....\right)^2+\frac{1}{3}\left(2x-\frac{8x^3}{3!}+....\right)^3+....\right]$$

Here, equating the coefficients of x^3 and logarithm expansion on both sides, we get

$$\frac{1}{3}.\frac{4^3}{\pi^3}\left[1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.........\right]=\frac{1}{2}\left(-\frac{8}{3!}+\frac{8}{3}\right)=\frac{1}{2}\left(-\frac{8}{6}+\frac{8}{3}\right)=\frac{2}{3}$$

$$\Rightarrow\quad \frac{1}{3}.\frac{4^3}{\pi^3}\left[1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.........\right]=\frac{2}{3}\quad\Rightarrow$$

$$\left[1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.........\right]=2.\frac{\pi^3}{64}$$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}.........=\frac{\pi^2}{32}$$

Example 7. Show that

$$\sin \pi\theta + \cos \pi\theta = (1+4\theta)\prod_{r=1}^{\infty}\left[1-\frac{4\theta}{4r-1}\right]\cdot\left[1+\frac{4\theta}{4r+1}\right]$$

Solution: We have

$$\sin \pi\theta + \cos \pi\theta = \sqrt{2}\sin\left(\frac{\pi}{4}+\pi\theta\right) = \sqrt{2}\sin\left\{\frac{\pi}{4}(1+4\theta)\right\}$$

$$= \sqrt{2}\left\{\frac{\pi}{4}(1+4\theta)\right\}\prod_{r=1}^{\infty}\left[1-\frac{\left\{\frac{\pi(1+4\theta)}{4}\right\}^2}{r^2\pi^2}\right]$$

………(1)

Let $\theta = 0,$ then we have

$$1 = \frac{\pi}{4}\sqrt{2}\prod_{r=1}^{\infty}\left[1-\frac{1}{4^2\pi^2}\right]$$

………(2)

Dividing (1) by (2), we have

$$\sin \pi\theta + \cos \pi\theta = (1+4\theta)\prod_{r=1}^{\infty}\frac{(4r)^2-(1+4\theta)^2}{(4r)^2-1}$$

$$= (1+4\theta)\prod_{r=1}^{\infty}\left(\frac{4r-1-4\theta}{4r-1}\right)\cdot\left(\frac{4r+1+4\theta}{4r+1}\right)$$

$$= (1+4\theta)\prod_{r=1}^{\infty}\left(1-\frac{4\theta}{4r-1}\right)\cdot\left(1+\frac{4\theta}{4r+1}\right)$$

Exercise 1

1. Show that

(a) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}.........\infty = \frac{\pi^2}{8}$ **(b)**

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}.........\infty = \frac{\pi^4}{96}$

(c) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.........\infty = \frac{\pi^2}{12}$ **(d)**

$\frac{1}{1.2}+\frac{1}{2.4}+\frac{1}{3.6}+\frac{1}{4.8}+....... = \frac{\pi^2}{12}$

2. Show that

(a) $\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+....\infty=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right)$ **(b)**

$\frac{3}{2^4}+\frac{8}{3^4}+\frac{15}{4^4}+....\infty=\frac{\pi^2}{6}\left(1-\frac{\pi^2}{15}\right)$

(c) $1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}.........\infty=\frac{\pi}{2\sqrt{2}}$ **(d)**

$\frac{2}{1^4}+\frac{5}{2^4}+\frac{10}{3^4}+\frac{17}{4^4}.........=\frac{\pi^2}{6}+\frac{\pi^4}{90}$

3. Prove that

$$1+\sin x=\frac{1}{8}(\pi+2x)^2\left\{1-\frac{(\pi+2x)^2}{4^2\pi^2}\right\}^2\left\{1-\frac{(\pi+2x)^2}{8^2\pi^2}\right\}^2 ....$$

4. Prove that

(a) $\frac{\pi}{3}=\frac{36}{35}\cdot\frac{144}{143}\cdot\frac{324}{323}\cdot\frac{576}{575}........$ **(b)**

$\frac{\sqrt{3}}{2}=\frac{8}{9}\cdot\frac{80}{81}\cdot\frac{224}{225}\cdot\frac{440}{441}........$

5. If $2, 3, 5$ are all prime numbers, show that

$$\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\frac{5^2}{5^2+1}.......=\frac{\pi^2}{15}$$

6. Prove that

$$\cot x=\frac{1}{x}-\frac{2x}{\pi^2-x^2}-\frac{2x}{2^2\pi^2-x^2}.............$$

7. Prove that

$$\cos x+\cosh x=2\prod_{r=1}^{\infty}\left\{1+\frac{4x^2}{(2r-1)^2\pi^4}\right\}$$

8. Prove that $\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-..........=\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)$

11.6 *GREGORY'S SERIES*

Statement: If θ lies within the closed interval $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, i.e., if $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, then show that

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\frac{1}{7}\tan^7\theta+..........\infty.$$

Proof: We have

$$1+i\tan\theta = 1+i\frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta}(\cos\theta + i\sin\theta)$$

$$= \sec\theta . e^{i\theta}$$

Now, taking logarithm of both sides, we have

$$\log(1+i\tan\theta) = \log\sec\theta + i\theta$$

…………..(1)

Now, since θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, so $\tan\theta$ lies between –1 and 1, i.e., $\tan\theta$ is numerically not greater than 1.

We have from (1)

$$\log\sec\theta + i\theta = \log(1+i\tan\theta)$$

$$= i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta - \frac{1}{4}i^4\tan^4\theta + \frac{1}{5}i^5\tan^5\theta ....$$

$$= i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta + \frac{1}{5}i\tan^5\theta ....$$

$$= \left(\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + ...\right) + i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta .....\right)$$

Equating imaginary part on both sides, we get

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + ...\infty$$

.................................(2)

This is known as Gregory's series.

Now we put $\tan\theta = x$ so that $\tan^{-1}x = \theta,$ we obtain another form of Gregory's series, as

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...\infty \qquad \text{where } |x| \le 1.$$

Also, equating real parts on both sides of (2), we have

$$\log\sec\theta = \left(\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - ....\right)$$

11.7 *GENERAL THEOREM ON GREGORY'S SERIES*

Statement: If θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$ i.e., if $n\pi - \frac{\pi}{4} \le \theta \le n\pi + \frac{\pi}{4}$, then show that

$$n\pi - \theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + ...\infty .$$

Proof: We put $\theta - n\pi = \alpha$, then $\theta = n\pi + \alpha.$

Hence the given condition reduces to $-\frac{\pi}{4} \le \alpha \le \frac{\pi}{4}.$

Hence, $\quad 1 + i\tan\theta = 1 + i\tan(n\pi + \alpha)$

$$= 1 + i\tan\alpha = 1 + i\frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$$

$$= \sec\alpha . e^{i\alpha}$$

Now, taking logarithm of both sides, we have

$$\log(1 + i\tan\theta) = \log\sec\alpha + i\alpha$$

…………..(1)

$$\log\sec\alpha + i\alpha = \log(1 + i\tan\theta)$$

The expansion is valid because θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$ and implies that $\tan\theta$ is not numerically greater than 1.

$$\log\sec\alpha + i\alpha = i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta - \frac{1}{4}i^4\tan^4\theta + ...\infty$$

$$= i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta + ...\infty$$

Equating imaginary parts on both sides, we get

$$\alpha = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - ....\infty$$

$$\therefore \quad \theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - ....\infty$$

11.8 *VALUE OF* π

Using of Gregory's series is to obtain the value of π to the various decimal places. We have from the Gregory's series

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{..........}\infty$$

On putting $x = 1$, we get

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{..........}\infty$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{..........}\infty$$

This series does not converge rapidly and so a large number of terms will have to be taken in order to obtain π correct to any degree of accuracy.

11.9 *EULER'S SERIES*

To prove $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

We have $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \tan^{-1} 1 = \frac{\pi}{4}$

Expanding by Gregory's series, we get

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

$$\left[\therefore \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \text{.....}\infty\right]$$

$$= \left\{\frac{1}{2} - \frac{1}{3}.\left(\frac{1}{2^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}\right) - \text{......}\right\} + \left\{\frac{1}{3} - \frac{1}{3}.\left(\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{3^5}\right) - \text{......}\right\}$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}.\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \text{......}$$

From the above series, The value of π easily can be calculated and more rapidly convergent than the preceding one.

11.10 *MACHIN'S SERIES*

To prove $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

We have $4\tan^{-1}\frac{1}{5} = 2.2\tan^{-1}\frac{1}{5} = 2\tan^{-1}\frac{2.\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}$

$$\left[\therefore 2\tan^{-1}x = 2\tan^{-1}\frac{2.x}{1-x^2}\right]$$

$$= 2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{2.\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2} = \tan^{-1}\frac{120}{119}$$

Now $4\tan^{-1}\frac{1}{5} - \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}1$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}1$$

$$= \tan^{-1}\frac{\frac{120}{119}-1}{1+\frac{120}{119}} = \tan^{-1}\frac{1}{239}$$

Therefore, $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= 4\left\{\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5^3}\right) + \frac{1}{5}\left(\frac{1}{5^5}\right) - ......\infty\right\} - \left\{\frac{1}{293} - \frac{1}{3}\left(\frac{1}{293^3}\right) + \frac{1}{5}\left(\frac{1}{293^5}\right) - ......\infty\right\}$$

.

11.11 *RUTHERFORD'S SERIES*

To prove $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

We have

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = 4\tan^{-1}\frac{1}{5} - \left[\tan^{-1}\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right]$$

$$= 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4} \quad \text{[by Machin's series]}$$

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 4\left\{\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5^3}\right) + \frac{1}{5}\left(\frac{1}{5^5}\right) - \ldots\right\} - \left\{\frac{1}{70} - \frac{1}{3}\left(\frac{1}{(70)^3}\right) + \frac{1}{5}\left(\frac{1}{(70)^5}\right) - \ldots\right\}$$

$$+ \left\{\frac{1}{99} - \frac{1}{3}\left(\frac{1}{(99)^3}\right) + \frac{1}{5}\left(\frac{1}{(99)^5}\right) - \ldots\right\}$$

Example 1. Show that

$$\frac{\pi}{4} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \ldots\ldots\ldots$$

Solution: We have

$$\textbf{R.H.S.} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \ldots\ldots\ldots$$

$$= 2\left\{\frac{1}{3} - \frac{1}{3}.\frac{1}{3^3} + \frac{1}{5}.\frac{1}{3^5} - \ldots\ldots.\right\} + \left\{\frac{1}{7} - \frac{1}{3}.\frac{1}{7^3} + \frac{1}{5}.\frac{1}{7^5} - \ldots\ldots.\right\}$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$\left[\therefore \tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots..\infty\right]$$

$$= \tan^{-1}\frac{2.\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} + \tan^{-1}\frac{1}{7}$$

$$\left[\therefore 2\tan^{-1}x = 2\tan^{-1}\frac{2.x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$\left[\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-x.y}\right]$$

$$= \tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}} = \tan^{-1}\frac{25}{25} = \tan^{-1}1 = \frac{\pi}{4} = \textbf{L.H.S.}$$

Example 2. Show that

$$\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \text{..........}ad.\inf\ .$$

Solution: We have

$$\textbf{R.H.S.} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \text{..........}ad.\inf\ .$$

$$= \frac{1}{2}\left\{\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \text{..........}\infty\right\}$$

$$= \frac{1}{2}\left\{\frac{3-1}{1.3} + \frac{7-5}{5.7} + \frac{11-9}{9.11} + \text{..........}\infty\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{3}{1.3} - \frac{1}{1.3}\right) + \left(\frac{7}{5.7} - \frac{5}{5.7}\right) + \left(\frac{11}{9.11} - \frac{9}{9.11}\right) + \text{..........}\infty\right\}$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{..........}\infty\right)$$

$$= \frac{1}{2}\tan^{-1}1 = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8} = \textbf{R.H.S.}$$

Example 3. Prove that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \text{..........}\right)$$

Solution: We have

$$\textbf{R.H.S.} = 2\sqrt{3}\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \text{..........}\right)$$

$$= 2\sqrt{3}.\sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{3\left(\sqrt{3}\right)^3} + \frac{1}{5\left(\sqrt{3}\right)^5} - \frac{1}{7\left(\sqrt{3}\right)^7} + \text{..........}\right)$$

$$= 6.\left(\frac{1}{\sqrt{3}} - \frac{1}{3\left(\sqrt{3}\right)^3} + \frac{1}{5\left(\sqrt{3}\right)^5} - \frac{1}{7\left(\sqrt{3}\right)^7} + \text{..........}\right)$$

$$= 6.\tan^{-1}\frac{1}{\sqrt{3}} = 6\times\frac{\pi}{6} = \pi = \textbf{L.H.S.}$$

Example 4. Prove that

$$1-2\left\{\frac{1}{3.5}+\frac{1}{7.9}+\frac{1}{11.13}+..........\infty\right\}=\frac{\pi}{4}$$

Solution: We have

$$\textbf{L.H.S.} = 1-2\left\{\frac{1}{3.5}+\frac{1}{7.9}+\frac{1}{11.13}+..........\infty\right\}$$

$$=1-\left\{\frac{2}{3.5}+\frac{2}{7.9}+\frac{2}{11.13}+..........\infty\right\}$$

$$=1-\left\{\frac{5-3}{3.5}+\frac{9-7}{7.9}+\frac{13-11}{11.13}+..........\infty\right\}$$

$$=1-\left\{\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+..........\infty\right\}$$

$$=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-..........\infty$$

$$=\tan^{-1}1=\frac{\pi}{4}=\textbf{R.H.S.}$$

Example 5. If $x>0,$ prove that

$$\tan^{-1}x=\frac{\pi}{4}+\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-............$$

Solution: If $x>0,$ then $\frac{x-1}{x+1}$ lies between -1 and 1. We have

$$\frac{x-1}{x+1}<1 \qquad \Leftrightarrow \qquad \frac{(x-1)^2}{(x+1)^2}<1$$

$$\Leftrightarrow \quad (x-1)^2<(x+1)^2 \qquad \Leftrightarrow \qquad x^2-2x+1<x^2+2x+1$$

$$\Leftrightarrow \quad 4x>0 \qquad \Leftrightarrow \qquad x>0$$

$\therefore$ If $x>0,$ then $\left|\frac{x-1}{x+1}\right|<1.$

$$\textbf{R.H.S.}=\frac{\pi}{4}+\left\{\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-...........\right\}$$

$$=\frac{\pi}{4}+\tan^{-1}\left(\frac{x-1}{x+1}\right)$$ [By Gregory's series]

$$= \tan^{-1} 1 + \tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}\left\{\frac{1+\left(\frac{x-1}{x+1}\right)}{1-\left(\frac{x-1}{x+1}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{x+1+x-1}{x+1-x+1}\right\} = \tan^{-1}\frac{2x}{2} = \tan^{-1} x = \textbf{L.H.S.}$$

Example 6. When θ lies between 0 and $\frac{\pi}{2}$, prove that

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - ....\infty$$

Solution: We have $\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan^2\frac{\theta}{2}\right)$

………(1)

Given that if θ lies between 0 and $\frac{\pi}{2}$, then $\frac{\theta}{2}$ lies between 0 and $\frac{\pi}{4}$ so that $\tan^2\frac{\theta}{2} < 1$. Therefore $\tan^{-1}\left(\tan^2\frac{\theta}{2}\right)$ can be expanded by Gregory's series,

Now from (1), we have

$$\tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\tan^2\frac{\theta}{2}\right)$$

$$= \tan^2\frac{\theta}{2} - \frac{1}{3}\left(\tan^2\frac{\theta}{2}\right)^3 + \frac{1}{5}\left(\tan^2\frac{\theta}{2}\right)^5 - ....$$

$$= \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - ....$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1-\cos\theta}{1+\cos\theta}\right) = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - ....$$

Example 7. If $x < \sqrt{2} - 1$, prove that

$$2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - ....\infty\right) = \left(\frac{2x}{1-x^2}\right) - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - ....\infty$$

Solution: Given that if $x < \sqrt{2} - 1$, then $x < 1$, we have

$$x < \sqrt{2} - 1 \quad \Leftrightarrow \quad x + 1 < \sqrt{2}$$

$\Leftrightarrow \quad (x+1)^2 < 2 \qquad \Leftrightarrow \quad x^2 + 2x + 1 < 2$

$\Leftrightarrow \quad 2x < 1 - x^2 \qquad \Leftrightarrow \quad \dfrac{2x}{1-x^2} < 1.$

$\therefore$ If $x < \sqrt{2} - 1$, then $\left|\dfrac{2x}{1-x^2}\right| < 1$, we have

$$2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - ....\infty\right) = 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$$

……………(1)

Since $\tan^{-1}\dfrac{2x}{1-x^2}$ can be expanded by the Gregory's series, we have

$$\tan^{-1}\frac{2x}{1-x^2} = \left(\frac{2x}{1-x^2}\right) - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - ....$$

……………(2)

From (1) and (2), we get

$$2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - ....\infty\right) = \left(\frac{2x}{1-x^2}\right) - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - ....\infty$$

Example 8. Sum of series

(a) $1 - \dfrac{1}{3.4^2} + \dfrac{1}{5.4^4} - ......\infty$ (b) $\dfrac{1}{2^3} - \dfrac{1}{3.2^7} + \dfrac{1}{5.2^{11}} ....\infty$

Solution: (a) Given that

$$1 - \frac{1}{3.4^2} + \frac{1}{5.4^4} - ......\infty = 4\left[\frac{1}{4} - \frac{1}{3.4^3} + \frac{1}{5.4^5} - ......\infty\right]$$

$$= 4\tan^{-1}\frac{1}{4}$$

Since by the Gregory's series because $\dfrac{1}{4} < 1.$

(b) Given that

$$\frac{1}{2^3} - \frac{1}{3.2^7} + \frac{1}{5.2^{11}} - ......\infty = \frac{1}{2}\left[\frac{1}{2^2} - \frac{1}{3.2^6} + \frac{1}{5.2^{10}} - ......\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{(2^2)} - \frac{1}{3.(2^2)^3} + \frac{1}{5.(2^2)^5} - ......\infty\right]$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{(2^2)} = \frac{1}{2}\tan^{-1}\frac{1}{4}$$

Example 9. Prove that $1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+......\infty=\frac{\pi}{2\sqrt{2}}$

Solution: We know

$$\tan^{-1}x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-.....\infty$$

provided $|x|\le 1.$

Now substituting $x=\sqrt{i}$, we get

$$\tan^{-1}\sqrt{i}=\sqrt{i}-\frac{1}{3}\left(\sqrt{i}\right)^3+\frac{1}{5}\left(\sqrt{i}\right)^5-\frac{1}{7}\left(\sqrt{i}\right)^7.....\infty$$

$$=\sqrt{i}\left\{1-\frac{1}{3}i+\frac{1}{5}i^2-\frac{1}{7}i^3....\right\}$$

$$=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{1-\frac{1}{3}i+\frac{1}{5}i^2-\frac{1}{7}i^3....\right\}$$

$$=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left\{1-\frac{1}{3}i+\frac{1}{5}i^2-\frac{1}{7}i^3....\right\}$$

$$=\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\left\{1-\frac{1}{3}i+\frac{1}{5}i^2-\frac{1}{7}i^3....\right\}$$

Equating real parts on the both sides, we get

$$=\frac{1}{\sqrt{2}}\left\{1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}....\right\}=\text{real parts of }\tan^{-1}\sqrt{i}$$

…………..(1)

Again $\tan^{-1}\sqrt{i}=\tan^{-1}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=x+iy$ (say)

…………..(2)

So that $\tan^{-1}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=x-iy$

…………..(3)

Adding (2) and (3), we get

$$2x=\tan^{-1}\frac{2\cos\frac{\pi}{4}}{1-\left(\cos^2\frac{\pi}{4}-i\sin^2\frac{\pi}{4}\right)}=\tan^{-1}\infty=\frac{\pi}{2}$$

$\Rightarrow$ $x=\frac{\pi}{4}=$ real parts of $\tan^{-1}\sqrt{i}$

…………..(4)

Hence from (1) and (4), we get $$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+......\infty=\frac{\pi}{2\sqrt{2}}$$

11.12 *SUMMARY*

This unit is about going in-depth into the theory and application of infinite series and Gregory series. Gregory's series is an infinite Taylor series expansion of the inverse tangent function. It was discovered in 1668 by James Gregory. It was rediscovered a few years later by Gottfried Leibniz, who re obtained the Leibniz formula for π as the special case $x = 1$ of the Gregory series. Perhaps the most widely used technique in the physicist's toolbox is the use of infinite series (i.e. sums consisting formally of an infinite number of terms) to represent functions, to bring them to forms facilitating further analysis, or even as a prelude to numerical evaluation. The acquisition of skill in creating and manipulating series expansions is therefore an absolutely essential part of the training of one who seeks competence in the mathematical methods of physics. An important part of this skill set is the ability to recognize the functions represented by commonly encountered expansions, and it is also of importance to understand issues related to the convergence of infinite series.

11.13 *GLOSSARY*

Infinite series: S*um* of an *infinite* number.
***Infinite product*:** Limit of the partial *products* as n increases without bound.
Gregory's series: Expansion of the inverse tangent function.
Euler' series: Transform converges to a sum.
***Rutherford*:** Calculation of *Dictionary* of National Biography.

11.14 *SELF ASSESSMESNTS QUESTIONS*

11.14.1 Multiple choice questions

1. The expansion of $\cos\theta$ as an infinite product is given by

(a) $\cos\theta = \theta\prod_{r=1}^{\theta}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$ (b) $\cos\theta = \theta\prod_{r=1}^{\theta}\left(1+\frac{\theta^2}{r^2\pi^2}\right)$

(c) $\cos\theta = \prod_{r=1}^{\theta}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$ (d) $\cos\theta = \prod_{r=1}^{\theta}\left(1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$

2. If $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$, then the sum of the product of the squares of the reciprocals of every pair of positive integers is equal to

(a) $\frac{\pi^4}{120}$ (b) $\frac{\pi^2}{120}$

(c) $\frac{\pi^4}{384}$ (d) $\frac{\pi^2}{384}$

3. Walli's expansion for π is given by

(a) $\sqrt{\frac{\pi}{2}(2n+1)} = \frac{2.4.6...........(2n-2)2n}{1.3.5..........(2n-1)}$ (b) $\sqrt{\frac{\pi}{2}(2n-1)} = \frac{2.4.6...........(2n+2)2n}{1.3.5..........(2n+1)}$

(c) $\sqrt{\frac{\pi}{2}(n+1)} = \frac{2.4.6...........(n+1)n}{1.3.5..........(2n+1)}$ (d) $\sqrt{\frac{\pi}{2}(n-1)} = \frac{2.4.6...........(n-1)n}{1.3.5..........(n+1)}$

4. The expansion of $\sin\theta$ as an infinite product is

(a) $\sin\theta = \theta\prod_{r=1}^{\theta}\left(1-\frac{4\theta^2}{r^2\pi^2}\right)$ (b) $\sin\theta = \theta\prod_{r=1}^{\theta}\left(1+\frac{\theta^2}{r^2\pi^2}\right)$

(c) $\sin\theta = \theta\prod_{r=1}^{\theta}\left(1+\frac{4\theta^2}{r^2\pi^2}\right)$ (d) $\sin\theta = \theta\prod_{r=1}^{\theta}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$

5. The expansion $\log_e(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}$ holds if

(a) z is any complex number (b) $z \neq 0$ $|z| \leq 1$ and $z \neq -1$

(c) $|z| \leq 1$ and $z \neq 0$ (d) $z \neq 0$

6. Gregory's series $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - ....$ holds if θ satisfies

(a) $-\pi \leq \theta < \pi$ (b) $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$

(c) $-\frac{\pi}{4} \leq \theta < \frac{\pi}{4}$ (d) θ is any real

7. The sum of series $\frac{1}{2^3}-\frac{1}{3.2^7}+\frac{1}{5.2^{11}}-..........ad.\inf$. is

(a) $\frac{1}{4}\tan^{-1}\frac{1}{2}$ (b) $\tan^{-1}\frac{1}{2^3}$

(c) $\frac{1}{2}\tan^{-1}\frac{1}{4}$ (d) $\frac{1}{8}\tan^{-1}1$

8. The value of $\frac{\pi}{4}$. is

(a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+.....$ (b) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....$

(c) $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-.....$ (d) None of these

ANSWERS:

1. c	**2.** a	**3.** a	**4.**d
5. b	**6.** c	**7.** c	**8.** b

11.14.2 Fill in the blanks

1. The expansion of $\cosh\theta$ as an infinite product is ……………..

2. The value of $\frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - ......$ is……………..

3. If $-1 \le x \le 1$, then is equal to…………………………….

4. Rutherford's series evaluate value of is……………………..

5. The sum of series is…………………

6. If $x > 0$, then $tan^{-1}x$ is equal to ……………………….

7. The infinite series is…………………..

8. Euler's series is……………………………

ANSWERS:

1.

2. $\log\sec\theta$

3. $x - \frac{x^3}{3} + \frac{x^5}{4} - .....$

4. $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$

5. $\frac{\pi^2}{8}$

6. $\frac{\pi}{4} + \left(\frac{x-1}{x+1}\right) - \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5$

7. $\frac{\pi^2}{12}$

8. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} = \frac{\pi}{4}$

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11.17 *TERMINAL QUESTIONS*

11.17.1 Short answer type questions

1. Prove that, $\frac{\pi}{4}=\left[\frac{2}{3}+\frac{1}{7}\right]-\frac{1}{3}\left[\frac{2}{3^3}+\frac{1}{7^3}\right]+\frac{1}{5}\left[\frac{2}{3^5}+\frac{1}{7^5}\right]-......$

2. Prove that $1-\frac{1}{3^2}+\frac{1}{5}\left(\frac{1}{3^2}\right)-\frac{1}{7}\left(\frac{1}{3^3}\right)+......\infty=\frac{\pi\sqrt{3}}{6}$

3. Prove that if $x,\ y,\ z$ are cube roots of unity

$$\frac{\tan^{-1}x}{x}+\frac{\tan^{-1}y}{y}+\frac{\tan^{-1}z}{z}=3\left[1-\frac{1}{7}+\frac{1}{13}-\frac{1}{19}+\frac{1}{25}-........\right]$$

4. Prove that

$$\frac{\pi}{4}=\frac{17}{21}-\frac{713}{81\times 343}+............+\frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}9^{1-n}+7^{1-2n}\right\}+..........$$

5. Prove that

$$\frac{\pi}{4}=\left[\frac{1}{2}+\frac{1}{5}+\frac{1}{8}\right]-\frac{1}{3}\left[\frac{1}{2^3}+\frac{1}{5^3}+\frac{1}{8^3}\right]+\frac{1}{5}\left[\frac{1}{2^5}+\frac{1}{5^5}+\frac{1}{8^5}\right]-....$$

11.17.2Long answer type questions

1. If $\theta<\frac{\pi}{4}$, prove that

$$\log\sec\theta=\frac{1}{2}\tan^2\theta-\frac{1}{4}\tan^4\theta+\frac{1}{6}\tan^6\theta-..........$$

2. Expand $\tan^{-1}\theta\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ as a power series in $\tan\theta$.

3. If θ and $\tan^{-1}(\sec\theta)$ both lie between 0 and $\frac{\pi}{2}$, then

$$\tan^{-1}(\sec\theta)=\frac{\pi}{4}+\tan^2\frac{\theta}{2}-\frac{1}{3}\tan^6\frac{\theta}{2}+........$$

4. Find the sum of the series

$$\frac{7}{1.3.5}+\frac{19}{5.7.9}+\frac{31}{9.11.13}+..........\infty$$

5. If x lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, then prove that

$$\tan x-\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x-.......=\tanh x+\frac{1}{3}\tanh^3 x+\frac{1}{5}\tanh^5 x+.......$$

ANSWERS:

2. $n\pi+\frac{\pi}{4}+\tan\theta-\frac{1}{3}\tan^3\theta+...$ **4.** $1-\frac{\pi}{8}$

BLOCK V: VECTOR ANALYSIS

UNIT 12: VECTORS MULTIPLE PRODUCTS AND DIFFERENTIATION OF VECTORS

CONTENTS:

12.1 *INTRODUCTION*

There are two forms of vector multiplication. The cross product of two vectors and the dot product of two vectors are the two ways to multiply a vector since a vector contains both magnitude and direction. Given that the resulting value is a scalar quantity, the dot product of two vectors is also known as the scalar product. As the result is a vector that is perpendicular to these two vectors, the cross product is also known as the vector product.

In this unit learners will be learn more about the two-vector multiplication, including its working principle, attributes, applications, and examples.

12.2 *OBJECTIVES*

After reading this unit learners will be able to

- Memorized about the vector triple product and scalar triple product and also their geometrical representation.
- Analyze about the reciprocal system of vectors.
- Analyze the application of differentiation of vectors.
- Memorized the useful theorems and their application of vector triple product and scalar triple product.

12.3 *TRIPLE PRODUCT*

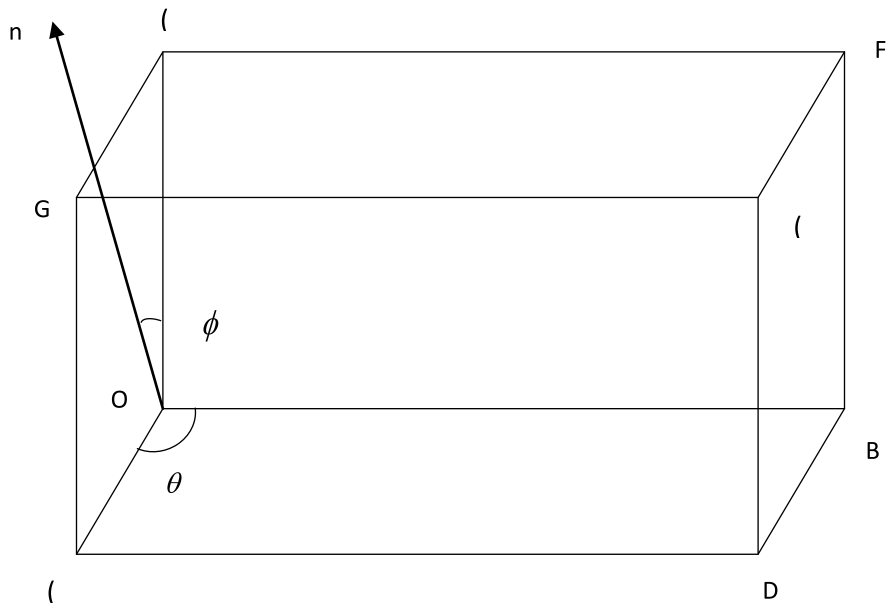
Triple Product: As we know that the vector product $\vec{a}\times\vec{b}$ and scalar product $\vec{a}.\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ are always a vector quantity and scalar quantity respectively. Therefore, if we multiply to these quantities with another vector quantity $\vec{c}$ by both vectorially and scalarly i.e., $(\vec{a}\times\vec{b})\times\vec{c}$ called **vector triple product** similarly $(\vec{a}\times\vec{b}).\vec{c}$ is called **scalar triple product**.

Remark: (***a***) Vector triple product is again a vector quantity.
(***b***) Scalar triple product again is a scalar quantity.
(***c***) $(\vec{a}.\vec{b})\times\vec{c}$ and $(\vec{a}.\vec{b}).\vec{c}$ are meaningless because scalar quantity $(\vec{a}.\vec{b})$ never be product vectorially and scalarly with any vector quantity. Similarly, the product $\vec{a}\times\vec{b}.\vec{c}$ is meaningless so it is meaningful only if it is written in some sense $(\vec{a}\times\vec{b}).\vec{c}$.

Scalar Triple Product: The scalar triple product is the scalar product of two vectors in which one of the vectors is itself vector product of two vectors. Thus if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then, $(\vec{a}\times\vec{b}).\vec{c}$ is called **scalar triple product**.

Some books named scalar triple product as **mixed product** because in this product both ‘cross’ and ‘dot’ signs involved.

12.3.1 *GEOMETRICAL INTERPRETATION OF SCALAR TRIPLE PRODUCT*



To explain geometrically to scalar triple product, we consider a parallelopiped whose edges and length are *OA*, *OB*, *OC* in the direction of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Let the volume of parallelopiped is *V* which is necessarily positive.

Let $\vec{a} \times \vec{b} = \vec{n}$, then from definition of vector product it is clear that $\vec{n}$ is perpendicular to the face *OADB* and $\left|\vec{n}\right|$ is measure as the area of parallelogram OADB. Since, by definition, vectors, $\vec{a}$, $\vec{b}$ and $\vec{n}$ form right-handed triad.

Let ϕ is the angle between the vectors $\vec{OC}$ and $\vec{n}$. Then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form right-handed or a left-handed triad according as ϕ to be acute and obtuse.

Now, $\left(\vec{a} \times \vec{b}\right).\vec{c} = \left|\left(\vec{a} \times \vec{b}\right)\right|.\left|\vec{c}\right| \cos\phi = \left|\vec{n}\right|\left|\vec{c}\right| \cos\phi$

$= $ (area of the parallelogram *OADB*).(OC $\cos\phi$) $\left[\because |c| = OC\right]$

So, according to the value of ϕ is acute or obtuse, $OC\cos\phi$ will depend positive or negative. Its absolute value gives the length of the perpendicular from *C* to the plane *OADB*.

So, *V* = (Area of the Parallelogram *OADB*).(Length of perpendicular from C to the parallelogram *OADB*).

Therefore, if ϕ is acute, $\left(\vec{a}\times\vec{b}\right).\vec{c} = +V$ i.e., if $\vec{a}$, $\vec{b}$ and $\vec{n}$ form right-handed triad.

And, if ϕ is obtuse, $\left(\vec{a}\times\vec{b}\right).\vec{c} = -V$ i.e., if $\vec{a}$, $\vec{b}$ and $\vec{n}$ form left-handed triad.

Since, we know that the vectors $\vec{a}$, $\vec{b}$ $\vec{c}$ are right-handed triad so, vectors $\vec{b}$, $\vec{c}$, $\vec{a}$ and $\vec{c}$, $\vec{a}$, $\vec{b}$ are also right-handed triad. Hence each product $\left(\vec{b}\times\vec{c}\right).\vec{a}$ and $\left(\vec{c}\times\vec{a}\right).\vec{b}$ will have the same value +*V* or -*V* according as $\vec{a}$, $\vec{b}$ $\vec{c}$ are left-handed triad.

Thus, $\left(\vec{a}\times\vec{b}\right).\vec{c} = \left(\vec{b}\times\vec{c}\right).\vec{a} = \left(\vec{c}\times\vec{a}\right).\vec{b}$

$\therefore$ $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ and $\vec{a}\times\vec{b} = -\vec{b}.\vec{a}$

$$\left(\vec{a}\times\vec{b}\right).\vec{c} = \vec{c}.\left(\vec{a}\times\vec{b}\right) = \left(\vec{b}\times\vec{c}\right).\vec{a} = \vec{a}.\left(\vec{b}\times\vec{c}\right) = \left(\vec{c}\times\vec{a}\right).\vec{b} = \vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$= -\left(\vec{b}\times\vec{a}\right).\vec{c} = -\vec{c}.\left(\vec{b}\times\vec{a}\right) = -\left(\vec{c}\times\vec{b}\right).\vec{a}$$

$$= -\vec{a}.\left(\vec{c}\times\vec{b}\right) = -\left(\vec{a}\times\vec{c}\right).\vec{b} = -\vec{b}.\left(\vec{a}\times\vec{c}\right)$$

From this we conclude that *value of scalar triple product depends on the cyclic order of the factors and is independent of the position of the dot and cross. These may be interchanged at pleasure. However, an analytic permutation of the three factors changes the value of the product in sign but not in magnitude*. The notation used to write scalar triple product is $\left(\vec{a}\times\vec{b}\right).\vec{c} = \left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{a},\vec{b},\vec{c}\right]$. This notation takes into consideration only the cyclic order of three vectors and disregards the unimportant position of dot and cross.

i.e., $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{b}\,\vec{c}\,\vec{a}\right] = \left[\vec{c}\,\vec{a}\,\vec{b}\right] = -\left[\vec{c}\,\vec{b}\,\vec{a}\right]$ etc.

Note 1: If $\hat{i}, \hat{j}, \hat{k}$ constitutes an orthogonal right-handed triad of unit vectors, then $\left[\hat{i}\,\hat{j}\,\hat{k}\right] = \left(\hat{i} \times \hat{j}\right).\hat{k} = \hat{k}.\hat{k} - = 1$

2: As nature of $\vec{a}$, $\vec{b}$, $\vec{c}$ are right-handed or left-handed, scalar triple product $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ will be decided to be positive or negative.

Distributive law for vector product:

To prove that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$, where $\vec{a}$, $\vec{b}$ $\vec{c}$ are any three vectors.

Let $\vec{r} \equiv \vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$ … (1)

Now scalar product both side by the vector $\vec{d}$, we get

$\vec{d}.\vec{r} = \vec{d}.\left[\vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}\right]$ … (2)

$\vec{d}.\vec{r} = \vec{d}.\left[\vec{a} \times (\vec{b} + \vec{c})\right] - \vec{d}.(\vec{a} \times \vec{b}) - \vec{d}.(\vec{a} \times \vec{c})$ [Scalar product follows the distributive law]

As we know that position of cross and dot can be interchanged without affecting its value.

$\vec{d}.\vec{r} = (\vec{d} \times \vec{a}).(\vec{b} + \vec{c}) - (\vec{d} \times \vec{a}).\vec{b} - (\vec{d} \times \vec{a}).\vec{c}$

$\vec{d}.\vec{r} = (\vec{d} \times \vec{a}).\vec{b} + (\vec{d} \times \vec{a}).\vec{c} - (\vec{d} \times \vec{a}).\vec{b} - (\vec{d} \times \vec{a}).\vec{c}$ [scalar product is distributive]

$= 0$

$\Rightarrow \vec{d} = 0$ or $\vec{r} = 0$ or $\vec{d}$ is perpendicular to $\vec{r}$. But we had taken $\vec{d}$ as arbitrary. So, we can choose it non zero and not perpendicular to $\vec{r}$.

So, $\vec{r} = 0$ i.e., $\vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$

$\Rightarrow \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$, hence proved.

Properties of scalar triple Product:

(1) If two vectors of a scalar product are equal then its value will be zero.

Proof: Let three vectors are $\vec{a}, \vec{a}, \vec{b}$ in which two vectors are equal. So, there scalar product is $\left[\vec{a}\,\vec{a}\,\vec{b}\right] = \left[\vec{a}, \vec{a}, \vec{b}\right] = \vec{a}.(\vec{a}\times\vec{b})$

Since we know that $(\vec{a}\times\vec{b})$ is the perpendicular vector to the plane of $\vec{a}$ and $\vec{b}$. It means dot product of vector $(\vec{a}\times\vec{b})$ with vector $\vec{a}$ and $\vec{b}$ will be zero.

$\Rightarrow \vec{a}.(\vec{a}\times\vec{b}) = 0$ i.e., $\left[\vec{a}\,\vec{a}\,\vec{b}\right] = 0$

(2) If two vectors are parallel then value of scalar triple product will be zero.

Proof: Let three vectors are $\vec{a}, \vec{b}, \vec{c}$ in which vector $\vec{b}$ is parallel to $\vec{a}$ i.e., $\vec{b} = k\,\vec{a}$. So there scalar product is,

$$\left[\vec{a}, \mathrm{k}\,\vec{a}, \vec{b}\right] = \vec{a}.(\mathrm{k}\,\vec{a}\times\vec{b})$$

$$= k\left[\vec{a}.(\vec{a}\times\vec{b})\right]$$

$$= k\left[\vec{a}, \vec{a}, \vec{b}\right] = k.0 = 0$$

(3) The necessary and sufficient condition for three non-parallel and non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be coplanar is that $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$=0

Proof: *Necessary Condition*: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ are three coplanar vectors. As we know that vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{a}$ and $\vec{b}$. Since vector $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar so, vector $\vec{c}$ will also perpendicular to the vector $(\vec{a}\times\vec{b})$.

As we know that if two vectors $\vec{\alpha}$, $\vec{\beta}$ are perpendicular then $\vec{\alpha}.\vec{\beta} = 0$

So, $(\vec{a}\times\vec{b}).\vec{c} = 0$

$\Rightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

Sufficient Condition: Let $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$ i.e., $\left(\vec{a}\times\vec{b}\right).\vec{c}=0$ it means vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{c}$. Since $(\vec{a}\times\vec{b})$ is also perpendicular to both the vector $\vec{a}$ and $\vec{b}$. Hence vector $(\vec{a}\times\vec{b})$ is perpendicular to the vector $\vec{a}$, $\vec{b}$, $\vec{c}$. It means vectors are on the same plane i.e., these are coplanar.

(4) As distributive law holds for both vector and scalar product, it holds for the scalar triple product.

Thus $\left[\vec{a},\vec{b}+\vec{d},\vec{c}+\vec{r}\right]=\left[\vec{a},\vec{b},\vec{c}\right]+\left[\vec{a},\vec{b},\vec{r}\right]+\left[\vec{a},\vec{d},\vec{c}\right]+\left[\vec{a},\vec{d},\vec{r}\right]$, the order of cycle of the factor being maintained in each term.

To express the scalar triple product in terms of rectangular component of the vector:

Let $\vec{a}=a_1i+a_2j+a_3k,\quad \vec{b}=b_1i+b_2j+b_3k,\quad \vec{c}=c_1i+c_2j+c_3k$

$$\because \vec{b}\times\vec{c}=\left(b_1i+b_2j+b_3k\right)\times\left(c_1i+c_2j+c_3k\right)=\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=(\mathrm{b}_2c_3-\mathrm{b}_3c_2)i-(\mathrm{b}_1c_3-\mathrm{b}_3c_1)j+(\mathrm{b}_1c_2-\mathrm{b}_2c_1)k$$

$$\vec{a}.(\vec{b}\times\vec{c})=\left(a_1i+a_2j+a_3k\right).(\mathrm{b}_2c_3-\mathrm{b}_3c_2)i-(\mathrm{b}_1c_3-\mathrm{b}_3c_1)j+(\mathrm{b}_1c_2-\mathrm{b}_2c_1)k$$

$$=a_1(\mathrm{b}_2c_3-\mathrm{b}_3c_2)-a_2(\mathrm{b}_1c_3-\mathrm{b}_3c_1)+a_3(\mathrm{b}_1c_2-\mathrm{b}_2c_1)$$

$$\left[\because i.i=j.j=k.k=1, i.j=j.k=k.i=0\right]$$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\vec{c}.\left(\vec{a}\times\vec{b}\right)=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It means position of dot and cross in scalar triple product is independent.

Expression of the scalar triple product in terms of three non-coplanar vectors *l, m, n*:

Let, $\vec{a}=a_1l+a_2m+a_3\mathrm{n},\quad \vec{b}=b_1l+b_2m+b_3\mathrm{n},\quad \vec{c}=c_1l+c_2m+c_3n$

So,

$$\vec{b}\times\vec{c}=\left(b_1 l+b_2 m+b_3 n\right)\times\left(c_1 l+c_2 m+c_3 n\right)$$

$$=b_1c_1 l\times l+b_1c_2 l\times m+b_1c_3 l\times n+b_2c_1 m\times l+b_2c_2 m\times m+b_2c_3 m\times n+b_3c_1 n\times l+b_3c_2 n\times m+b_3c_3 n\times n$$

$$\left[\because l\times l=0, l\times m=-m\times l\, etc.\right]$$

$$=b_1c_2 l\times m+b_1c_3 l\times n-b_2c_1 l\times m+b_2c_3 m\times n-b_3c_1 l\times n-b_3c_2 m\times n$$

$$=\left(b_1c_2-b_2c_1\right)l\times m+(b_2c_3-b_3c_2)m\times n+(b_1c_3-b_3c_1)l\times n$$

$$=(b_2c_3-b_3c_2)m\times n-(b_1c_3-b_3c_1)n\times l+\left(b_1c_2-b_2c_1\right)l\times m$$

$$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=\left(a_1 l+a_2 m+a_3\text{ n}\right).\left[(b_2c_3-b_3c_2)m\times n-(b_1c_3-b_3c_1)n\times l+\left(b_1c_2-b_2c_1\right)l\times m\right]$$

$$=a_1(b_2c_3-b_3c_2)[l\,m\,n]-a_2(b_1c_3-b_3c_1)[l\,m\,n]+a_3\left(b_1c_2-b_2c_1\right)[l\,m\,n]$$

[$\because [l\,m\,n]=[m\,n\,l]=[n\,l\,m]$ and scalar triple product in which two vectors are same is equal to zero i.e., $\because [l\,l\,n]=0$]

Hence, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[l\,m\,n]$

Solved Example

Example 1: If vectors $\vec{a}=2i-j+k,\ \ \vec{b}=i+2j-3k,\ \ \vec{c}=3i+pj+5k$ are coplanar then find the value of constant.

Answer: As we know that three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar then

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$$

Now, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix}=2(10+3p)+1(5+9)+1(p-6)$ = $7p$ + 28

Since, $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

$\Rightarrow 7p+28=0$

$\Rightarrow p=-4$

Example 2: If the vectors $\vec{a}=2i-4j+5k,\ \ \vec{b}=i-j+k,\ \ \vec{c}=3i-5j+2k$ are representing the edges of parallelopiped, then find its volume.

Answer: Since we know that the volume of parallelopiped is equal to the absolute value of scalar triple product of its edges i.e., $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}2 & -4 & 5\\ 1 & -1 & 1\\ 3 & -5 & 2\end{vmatrix}=2(-2+5)+4(2-3)+5(-5+3)$$

$$=6-4-10=-8$$

Since volume of surface never be a negative quantity, so required volume of parallelopiped is 8.

Example 3: Prove that the points $4i+5j+k,\ -(\mathrm{j}+\mathrm{k}),\ 3\mathrm{i}+9\,\mathrm{j}+4\,\mathrm{k}$ and $4(-i+j+k)$ are coplanar.

Answer: Let *A*, *B*, *C*, *D* are the four points whose position vectors position vectors are given from the origin *O*.

It means, $\vec{OA}=4i+5j+k,\ \vec{OB}=-(\mathrm{j}+\mathrm{k})$, $\vec{OC}=3i+9j+4k$ and $\vec{OD}=4(-i+j+k)$.

If we have to show that four points *A*, *B*, *C*, *D* are coplanar then we have only to prove that the vectors $\vec{AB},\vec{AC},\vec{AD}$ are coplanar.

Now, $\vec{AB}=\vec{OB}-\vec{OA}=-(j+k)-(4i+5j+k)=-4i-6j-2k=a$ (say)

$\vec{AC}=\vec{OC}-\vec{OA}=(3\mathrm{i}+9j+4k)-(4i+5j+k)=-i+4j+3k=b$ (say)

$\vec{AD}=\vec{OD}-\vec{OA}=(-4\mathrm{i}+4j+4k)-(4i+5j+k)=-8i-j+3k=c$ (say)

$$\left[\vec{AB},\vec{AC},\vec{AD}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}=-4(12+3)+6(-3+24)-2(1+32)$$

$$=-60+126-66=0$$

So, we can say that given four vectors are coplanar.

Example 4: Prove that the points $-a+4b-3c,\ 3a+2b-5c,\ -3a+8b-5c$ and $-3a+2b+c$ are coplanar.

Answer: Let *A*, *B*, *C*, *D* are the four points whose position vectors position vectors are given from the origin *O*.

It means, $\vec{OA}=-a+4b-3c,\ \vec{OB}=3a+2b-5c$, $\vec{OC}=-3a+8b-5c$ and $\vec{OD}=-3a+2b+c$.

If we have to show that four points *A*, *B*, *C*, *D* are coplanar then we have only to prove that the vectors $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar.

Now, $\vec{AB} = \vec{OB} - \vec{OA} = 3a + 2b - 5c - (-\mathrm{a} + 4\mathrm{b} - 3\mathrm{c}) = 4a - 2b - 2c$

$\vec{AC} = \vec{OC} - \vec{OA} = (-3\mathrm{a} + 8\mathrm{b} - 5\mathrm{c}) - (-\mathrm{a} + 4\mathrm{b} - 3\mathrm{c}) = -2a + 4b - 2c$

$\vec{AD} = \vec{OD} - \vec{OA} = (-3a + 2b + c) - (-\mathrm{a} + 4\mathrm{b} - 3\mathrm{c}) = -2a - 2b + 4c$

$$\left[\vec{AB}, \vec{AC}, \vec{AD}\right] = \begin{vmatrix} -4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} [a\ b\ c] = \{-4(16-4) + 2(-8-4) - 2(4+8)\}[a\ b\ c]$$

$$= \{-48 - 24 - 24\}[a\ b\ c] = 0$$

So, we can say that given four vectors are coplanar.

Example 5: Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$

Answer: $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = \left(\vec{a}+\vec{b}\right).\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]$

$$= \left(\vec{a}+\vec{b}\right).\left[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\right]$$

$$= \vec{a}.\left(\vec{b}\times\vec{c}\right) + \vec{a}.\left(\vec{b}\times\vec{a}\right) + \vec{a}.\left(\vec{c}\times\vec{c}\right) + \vec{a}.\left(\vec{c}\times\vec{a}\right)$$

$$+ \vec{b}.\left(\vec{b}\times\vec{c}\right) + \vec{b}.\left(\vec{b}\times\vec{a}\right) + \vec{b}.\left(\vec{c}\times\vec{c}\right) + \vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{a}\right] + \left[\vec{a}, \vec{c}, \vec{c}\right] + \left[\vec{a}, \vec{c}, \vec{a}\right]$$

$$+ \left[\vec{b}, \vec{b}, \vec{c}\right] + \left[\vec{b}, \vec{b}, \vec{a}\right] + \left[\vec{b}, \vec{c}, \vec{c}\right] + \left[\vec{b}, \vec{c}, \vec{a}\right]$$

(If two vectors of a scalar product are equal then its value will be zero.)

$$=\left[\vec{a},\vec{b},\vec{c}\right]+0+0+0+0+0+0+\left[\vec{b},\vec{c},\vec{a}\right]$$

$$=\left[\vec{a},\vec{b},\vec{c}\right]+\left[\vec{a},\vec{b},\vec{c}\right]$$

$$\left[\because\left[\vec{a},\vec{b},\vec{c}\right]=\left[\vec{b},\vec{a},\vec{c}\right]=\left[\vec{c},\vec{a},\vec{b}\right]\right]$$

$$= 2\left[\vec{a},\vec{b},\vec{c}\right]$$

Example 6: Prove that $[l\,m\,n][a\,b\,c]=\begin{vmatrix} l.a & l.b & l.c \\ m.a & m.b & m.c \\ n.a & n.b & n.c \end{vmatrix}$

Answer: Let $l=l_1i+l_2j+l_3k,\ m=m_1i+m_2j+m_3k,\ n=n_1i+n_2j+n_3k$

and $a=a_1i+a_2j+a_3k,\ b=b_1i+b_2j+b_3k,\ c=c_1i+c_2j+c_3k$

Now taking LHS = $[l\,m\,n][a\,b\,c]=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=\begin{vmatrix} l_1a_1+l_2a_2+l_3a_3 & l_1b_1+l_2b_2+l_3b_3 & l_1c_1+l_2c_2+l_3c_3 \\ m_1a_1+m_2a_2+m_3a_3 & m_1b_1+m_2b_2+m_3b_3 & m_1c_1+m_2c_2+m_3c_3 \\ n_1a_1+n_2a_2+n_3a_3 & n_1b_1+n_2b_2+n_3b_3 & n_1c_1+n_2c_2+n_3c_3 \end{vmatrix}$$

As we know that, $l.a=l_1a_1+l_2a_2+l_3a_3$

Similarly, we can write

$$L.H.S.=[l\,m\,n][a\,b\,c]=\begin{vmatrix} l.a & l.b & l.c \\ m.a & m.b & m.c \\ n.a & n.b & n.c \end{vmatrix}$$

VECTOR TRIPLE PRODUCT: Vector triple product is the vector product of two vectors in which one is itself the vector product of two vectors. Thus if $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors then the product of the form $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ and $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ etc. are called "Vector Triple Products".

Theorem 1: To prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Proof: Let $\vec{r} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$ and $\vec{b} \times \vec{c} = \vec{d}$

As we know that if $\vec{b} \times \vec{c} = \vec{d}$, it means $\vec{d}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. Also, $\vec{r} = \vec{a} \times \vec{d}$ i.e., $\vec{r}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{d}$. Now the vector $\vec{r}$ is perpendicular to the vector $\vec{d}$, whereas the vector $\vec{d}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. It means $\vec{r}$ must lie in the plane containing $\vec{b}$ and $\vec{c}$.

$\Rightarrow \vec{r} = l\,\vec{b} + m\,\vec{c}$, where, l and m are scalars. …… (1)

Since $\vec{r}$ is perpendicular to the vector $\vec{a}$, then $\vec{r} . \vec{a} = \vec{0}$

$$\Rightarrow \left(l\,\vec{b} + m\,\vec{c}\right).\vec{a} = l\left(\vec{b} . \vec{a}\right) + m\left(\vec{c} . \vec{a}\right)$$

Let, $\dfrac{l}{\left(\vec{c} . \vec{a}\right)} = \dfrac{-m}{\left(\vec{b} . \vec{a}\right)} = \lambda$ (say)

Putting the value of scalars l, m in (1)

$$\vec{r} = \lambda\left(\vec{c} . \vec{a}\right)\vec{b} - \lambda\left(\vec{b} . \vec{a}\right)\vec{c} = \lambda\left[\left(\vec{c} . \vec{a}\right)\vec{b} - \left(\vec{b} . \vec{a}\right)\vec{c}\right]$$

……(2)

Now we have to find the value of λ.

Consider unit vectors $\hat{j}$ and $\hat{k}$, the first parallel to $\vec{b}$ and second perpendicular to it in the plane containing $\vec{b}$ and $\vec{c}$. Then we may write $\vec{b} = b_2\,\hat{j}$ and $\vec{c} = c_2\,\hat{j} + c_3\,\hat{k}$, then remaining vector may be written as $\vec{a} = a_1\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$

Now, $\vec{b} \times \vec{c} = b_2\,\hat{j} \times \left(c_2\,\hat{j} + c_3\,\hat{k}\right) = b_2c_2\left(\hat{j} \times \hat{j}\right) + b_2c_3\left(\hat{j} \times \hat{k}\right) = b_2c_3\,\hat{i}$

$$\vec{r} = \vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(a_1\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}\right) \times b_2c_3\,\hat{i}$$

$$= a_1b_2c_3\,\hat{i} \times \hat{i} + a_2b_2c_3\,\hat{j} \times \hat{i} + a_3b_2c_3\,\hat{k} \times \hat{i} = a_3b_2c_3\,\hat{j} - a_2b_2c_3\,\hat{k} \qquad \text{…….(3)}$$

$$\left[\because \hat{i} \times \hat{i} = 0, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{i} = \hat{j}\right]$$

Also,

$$\vec{r}=\lambda\left(\vec{c}.\vec{a}\right)\vec{b}-\lambda\left(\vec{b}.\vec{a}\right)\vec{c}=\lambda\left[\left(c_2\hat{j}+c_3\hat{k}\right).\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)b_2\vec{\hat{j}}-b_2\vec{\hat{j}}.\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\left(c_2\hat{j}+c_3\hat{k}\right)\right]$$

$$\vec{r}=\lambda\left[c_2a_2b_2\hat{j}+c_3a_3b_2\hat{j}-b_2a_2c_2\hat{j}-b_2a_2c_3\hat{k}\right]$$

$$=\lambda\left[c_3a_3b_2\hat{j}-b_2a_2c_3\hat{k}\right]$$

………. (4)

Now from equation (3) and (4), we conclude that $\lambda=1$

Hence, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Corollary:

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=-\left[\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right]=-\left[\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}\right]=\left(\vec{c}.\vec{a}\right)\vec{b}-\left(\vec{c}.\vec{b}\right)\vec{a}$$

Solved Example

Example 7: Prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$.

Answer: We know that, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

……..(1)

Similarly, $\vec{b}\times\left(\vec{c}\times\vec{a}\right)=\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}$

…… (2)

and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}$

…… (3)

Now, adding equation (1), (2) and (3)

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}+\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}+\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}$$

As we know that vector scalar product is commutative i.e., $\vec{a}.\vec{b}=\vec{0}$

$$\Rightarrow\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$$

Example 8: Prove that the vectors $\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{b}\times\left(\vec{c}\times\vec{a}\right), \vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar.

Answer: Let $\vec{r_1}=\vec{a}\times\left(\vec{b}\times\vec{c}\right),\ \vec{r_2}=\vec{b}\times\left(\vec{c}\times\vec{a}\right),\ \vec{r_3}=\vec{c}\times\left(\vec{a}\times\vec{b}\right)$.

To prove that these vectors are coplanar, first we have to prove that $\vec{r_1}+\vec{r_2}+\vec{r_3}=0$. In the previous example we have already prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$ i.e., $\vec{r_1}+\vec{r_2}+\vec{r_3}=0$.

It means any one of these vectors can be expressed in terms of other two vectors. Hence these vectors are coplanar.

Example 9: If the vectors $\vec{a}=i-2j+k,\quad \vec{b}=2i+j+k,\quad \vec{c}=i+2j-k$ then find $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$.

Answer: We have $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

$$\left(\vec{a}.\vec{c}\right)\vec{b}=\left[(i-2j+k).(i+2j-k)\right](2i+j+k)=(1-4-1)(2i+j+k)$$

$$\left(\vec{a}.\vec{c}\right)\vec{b}=-8i-4j-4k$$

Similarly,

$$\left(\vec{a}.\vec{b}\right)\vec{c}=\left[(i-2j+k).(2i+j+k)\right](i+2j-k)=(2-2+1)(i+2j-k)$$

$$\left(\vec{a}.\vec{b}\right)\vec{c}=i+2j-k$$

Hence, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}=(-8i-4j-4k)-(i+2j-k)$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=-9i-6j-3k$$

Example 10: Show that $\left[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}\right]=\left[a\ b\ c\right]^2$ and also express the result in terms of determinants.

Answer: We know that, $\left[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}\right]=\left(\vec{a}\times\vec{b}\right).\left[\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)\right]$

Let $\vec{d} = \vec{b} \times \vec{c}$, then $\vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left(\vec{d} . \vec{a}\right)\vec{c} - \left(\vec{d} . \vec{c}\right)\vec{a}$

$$\vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left[\left(\vec{b} \times \vec{c}\right) . \vec{a}\right]\vec{c} - \left[\left(\vec{b} \times \vec{c}\right) . \vec{c}\right]\vec{a} = \left[\vec{b}\,\vec{c}\,\vec{a}\right]\vec{c} - \left[\vec{b}\,\vec{c}\,\vec{c}\right]\vec{a} = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

$$\left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \vec{d} \times \left(\vec{c} \times \vec{a}\right) = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

$$\left(\vec{a} \times \vec{b}\right) . \left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \left(\vec{a} \times \vec{b}\right) . \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{c}$$

Since we know that scalar triple product is the scalar.

$$\Rightarrow \left(\vec{a} \times \vec{b}\right) . \left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\left\{\left(\vec{a} \times \vec{b}\right) . \vec{c}\right\} = \left[\vec{a}\,\vec{b}\,\vec{c}\right]\left[\vec{a}\,\vec{b}\,\vec{c}\right]$$

$$\Rightarrow \left(\vec{a} \times \vec{b}\right) . \left[\left(\vec{b} \times \vec{c}\right) \times \left(\vec{c} \times \vec{a}\right)\right] = \left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$$

Let $\vec{a} = a_1 i + a_2 j + a_3\,\mathrm{k}$, $\vec{b} = b_1 i + b_2 j + b_3\,\mathrm{k}$, $\vec{c} = c_1 i + c_2 j + c_3 k$, then,

$$[a\,b\,c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

again $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2 a_3 - b_2 a_3\right) i + \left(b_1 a_3 - a_1 b_3\right) j + \left(a_1 b_2 - a_2 b_1\right) k$

similarly, $\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(b_2 c_3 - b_3 c_2\right) i + \left(c_1 b_3 - b_1 c_3\right) j + \left(b_1 c_2 - c_1 b_2\right) k$

and $\vec{c} \times \vec{a} = \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \left(c_2 a_3 - a_2 c_3\right) i + \left(a_1 c_3 - a_3 c_1\right) j + \left(c_1 a_2 - a_1 c_2\right) k$

Now, $\left[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\right] = \begin{vmatrix} a_2 b_3 - b_2 a_3 & b_1 a_3 - a_1 b_3 & a_1 b_2 - a_2 b_1 \\ b_2 c_3 - b_3 c_2 & c_1 b_3 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\ c_2 a_3 - c_3 a_2 & a_1 c_3 - a_3 c_1 & c_1 a_2 - c_2 a_1 \end{vmatrix}$

$$\left[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\right] = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

where the capital letters A_1, A_2, A_3 etc., denote the cofactor corresponding small letters a_1, a_2, a_3 etc. in the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Hence, $\left[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}\right] = [a\ b\ c]^2 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Scalar product of four vectors: If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four vectors then the product $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)$ or $\left(\vec{a}\times\vec{d}\right).\left(\vec{b}\times\vec{c}\right)$ is called scalar product of four vectors.

Theorem 2: Prove that $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$

Proof: Let $\vec{a}\times\vec{b} = \vec{r}$. Then $\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \vec{r}.\left(\vec{c}\times\vec{d}\right)$

As we know that position of dot and cross may be interchanged without altering the value of the product. Therefore,

$$\vec{r}.\left(\vec{c}\times\vec{d}\right) = \left(\vec{r}\times\vec{c}\right).\vec{d} = \left[\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right].\vec{d} = \left[\left(\vec{c}.\vec{a}\right)\vec{b} - \left(\vec{c}.\vec{b}\right)\vec{a}\right].\vec{d}$$

$$= \left(\vec{c}.\vec{a}\right)\left(\vec{b}.\vec{d}\right) - \left(\vec{c}.\vec{b}\right)\left(\vec{a}.\vec{d}\right)$$

$$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

This relation also known as Lagrange's Identity.

Vector Product of four Vectors: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be four vectors then the vector products of the vectors $\left(\vec{a}\times\vec{b}\right), \left(\vec{c}\times\vec{d}\right)$ i.e., $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ is known as vector product of four products.

Theorem 3: To prove that

(i) $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c} - \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}$

(ii) $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$

Proof: As we know that $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ is the vector quantity which is either written in terms of $\vec{c}$ and $\vec{d}$ or in terms of $\vec{a}$ and $\vec{b}$. To express the vector $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of, let us put $\vec{a}\times\vec{b}=\vec{l}$. Then

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\vec{l}\times\left(\vec{c}\times\vec{d}\right)=\left(\vec{l}.\vec{d}\right)\vec{c}-\left(\vec{l}.\vec{c}\right)\vec{d}$$

$$=\left[\left(\vec{a}\times\vec{b}\right).\vec{d}\right]\vec{c}-\left[\left(\vec{a}\times\vec{b}\right).\vec{c}\right]\vec{d}=\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d} \qquad \text{……..}$$

(1)

Similarly, we also express the vector $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)$ in terms of $\vec{a}$ and $\vec{b}$ which is,

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$$

……….. (2)

equating the equation (1) and (2)

$$\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}=\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}-\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}$$

$$\left[\vec{b}\ \vec{c}\ \vec{d}\right]\vec{a}-\left[\vec{a}\ \vec{c}\ \vec{d}\right]\vec{b}+\left[\vec{a}\ \vec{b}\ \vec{d}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}=0$$

Which is the required linear relation connecting the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

12.4 *RECIPROCAL SYSTEM OF VECTORS*

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors such that $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\neq 0$, then the three vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ defined as

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

are called reciprocal systems of vectors to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Example 11: To show that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$

Answer: $\vec{a}.\vec{a}' = \vec{a}.\dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\vec{a}.\left(\vec{b}\times\vec{c}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

Similarly, $\vec{b}.\vec{b}' = \vec{b}.\dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\vec{b}.\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{b}\ \vec{c}\ \vec{a}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

and $\vec{c}.\vec{c}' = \vec{c}.\dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\vec{c}.\left(\vec{a}\times\vec{b}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{c}\ \vec{a}\ \vec{b}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 1$

Example 12: The scalar product of any other pair of vectors, one from each system is zero, i.e., $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$

Answer: $\vec{a}.\vec{b}' = \vec{a}.\dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\vec{a}.\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{\left[\vec{a}\ \vec{c}\ \vec{a}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 0$

Similarly, we can prove the other results i.e., $\vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$.

Example 13: The product of the scalar triple product of three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and the scalar triple product of their reciprocal $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ is equal to 1 i.e., $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = 1$.

Answer: First we have to define scalar triple product $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = \vec{a}'.\left(\vec{b}'\times\vec{c}'\right)$

$$\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = \vec{a}'.\left(\vec{b}'\times\vec{c}'\right) = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}.\left[\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\times\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\right] = \frac{\left(\vec{b}\times\vec{c}\right).\left[\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}$$

As we know that

$$\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)=\left[\vec{c}\ \vec{a}\ \vec{b}\right]\vec{a}-\left[\vec{c}\ \vec{a}\ \vec{a}\right]\vec{b}=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}-0=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}$$

So,

$$\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]=\frac{\left(\vec{b}\times\vec{c}\right).\left[\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left(\vec{b}\times\vec{c}\right).\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\left(\vec{b}\times\vec{c}\right).\vec{a}\right\}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}$$

$$=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^3}=\frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

So, $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\dfrac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left[\vec{a}\ \vec{b}\ \vec{c}\right]=1$

Remarks 1: Since scalar triple product $\left[\vec{a}\ \vec{b}\ \vec{c}\right]\neq 0$, so we conclude that $\left[\vec{a}'\ \vec{b}'\ \vec{c}'\right]\neq 0$ i.e., $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are also non coplanar.

2: Since the vectors $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ similarly, we can say that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are also reciprocal to the vector $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$, this is known as symmetry property.

3: The unit vector along three-dimensional co-ordinate axes i.e., the orthonormal vector triads $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a self-reciprocal system.

Solved examples

Example 14: Find the set of reciprocal vectors of the vectors

$\vec{a}=2i+3j-k,\quad \vec{b}=i-j-2k,\quad \vec{c}=-i+2j+2k$.

Solution: Let $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are the system of reciprocal vectors of the given vectors, then

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\text{ and }\vec{c}'=\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2(-2+4) - 3(2-2) - 1(2-1) = 3 \neq 0$$

Now, $\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = (-2+4)i - (2-2)j + (2-1)k = 2i + k$

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{2}{3}i + \frac{1}{3}k$$

$$\vec{c} \times \vec{a} = \begin{vmatrix} i & j & k \\ -1 & 2 & -2 \\ 2 & 3 & -1 \end{vmatrix} = -8i + 3j - 7k$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{-8i + 3j - 7k}{3}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = (-6-1)i - (-4+1)j + (-2-3)k = -7i + 3j - 5k$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \frac{-7i + 3j - 5k}{3}$$

12.5 *DIFFERENTIATION OF VECTORS*

Vector Function: Let D be the subset of real numbers. If we associate unique vector $f(t)$ to each element t of D, then this rule defines a **vector function** of the scalar variable t. Here $f(t)$ is the vector quantity and thus f is a vector function.

As we know that every vector function can be easily expressed as a linear combination of three fixed non-coplanar vector. Thus, we can write

$$f(t) = f_1(t)i + f_2(t)j + f_3(t)k$$

where i, j, k denote the orthonormal right-handed triad.

Scalar field and vector field

If in the region *R* in the space corresponding to each point *P*(*x*, *y*, *z*) there is unique scalar *f*(*P*), then *f* is called a scalar point function and we say scalar field *f* has been defined in *R*.

e.g., $f(\text{x}, \text{y}, \text{z}) = x^3 - y^2 - 2xyz$ defines a scalar field.

If corresponding to each point *P*(*x*, *y*, *z*) in the region *R* there is unique vector *f*(*P*), then *f* is called a vector point function and we say vector field *f* has been defined in *R*.

e.g., $f(\text{x}, \text{y}, \text{z}) = x^3 i - y^2 j - 2xyz\, k$ defines a scalar field.

Limit and Continuity of a Vector Function:

Definition 1: A real number *l* is said to be limit of a vector function $f(t)$, when *t* tends to t_0, if for any small positive number $\in$, there exist a number δ such that

$$|f(t) - l| < \in \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

If vector function $f(t)$ tends to a limit *l* as *t* tends to t_0, we write it mathematically by,

$$\lim_{t \to t_0} f(t) = l$$

Definition 2: A vector function $f(t)$ is said to be continuous at a point t_0 of *t* if following are satisfied,

(i) $f(t)$ is defined at t_0

(ii) Corresponding to any small positive number $\in$, there exist a number δ such that

$$|f(t) - f(t_0)| < \in \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

A vector function $f(t)$ is said to be continuous if it is continuous for every value of *t* for which it has been defined.

Some important theorems:

Theorem 4: The Necessary and Sufficient condition for a vector function $f(t)$ to be continuous at $t = t_0$ is that $\lim_{t \to t_0} f(t) = f(t_0)$.

Theorem 5: The vector function $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$ is continuous if and only if $f_1(t), f_2(t), f_3(t)$ are continuous.

Theorem 6: Let $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$ and $l = l_1 i + l_2 j + l_3 k$, then the necessary and sufficient condition that $\lim_{t \to t_0} f(t) = l$ are

$$\lim_{t \to t_0} f_1(t) = l_1, \quad \lim_{t \to t_0} f_2(t) = l_2, \quad \lim_{t \to t_0} f_3(t) = l_3$$

Theorem 7: If $f(t)$, g(t) are scalar function of the scalar variable t and let $\psi(t)$ is a scalar function of scalar variable t, then

(i) $\lim_{t \to t_0}\left[f(t) \pm g(t)\right] = \lim_{t \to t_0} f(t) \pm \lim_{t \to t_0} g(t)$

(ii) $\lim_{t \to t_0}\left[f(t).g(t)\right] = \left[\lim_{t \to t_0} f(t)\right].\left[\lim_{t \to t_0} g(t)\right]$

(iii) $\lim_{t \to t_0}\left[f(t) \times g(t)\right] = \left[\lim_{t \to t_0} f(t)\right] \times \left[\lim_{t \to t_0} g(t)\right]$

(iv) $\lim_{t \to t_0}\left[\psi(t) g(t)\right] = \left[\lim_{t \to t_0} \psi(t)\right]\left[\lim_{t \to t_0} g(t)\right]$

(v) $\lim_{t \to t_0}\left|f(t)\right| = \left|\lim_{t \to t_0} f(t)\right|$

Note: Here we use application of these theorems without proof.

Derivative of a vector function with respect to a scalar:

Definition: Let $r = f(t)$ be a vector function of the scalar variable *t*, we define

$$r + \delta r = f(t + \delta t)$$

$$\delta r = f(t + \delta t) - f(t)$$

Consider the vector, $\dfrac{\delta r}{\delta t} = \dfrac{f(t+\delta t) - f(t)}{\delta t}$

If $\lim_{\delta t \to 0} \dfrac{\delta r}{\delta t} = \lim_{\delta t \to 0} \dfrac{f(t+\delta t) - f(t)}{\delta t}$ exist, then the value of this limit is called the derivative of the vector function r with respect to *t*, denoted by $\dfrac{dr}{dt}$ i.e.,

$$\frac{dr}{dt} = \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \lim_{\delta t \to 0} \frac{(\mathrm{r} + \delta r) - \mathrm{r}}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

If $\dfrac{dr}{dt}$ exists, then r is said to be differentiable. Since r is vector quantity so its will also a vector quantity.

If we again differentiate $\dfrac{dr}{dt}$, we get $\dfrac{d^2 r}{dt^2}$, the second derivative of r w.r.t. *t*, and so on differentiating successively *n* times we get,

$$\frac{dr}{dt}, \frac{d^2 r}{dt^2}, \frac{d^3 r}{dt^3}, \frac{d^4 r}{dt^4}, \ldots, \frac{d^n r}{dt^n}.$$

Differentiation Formulae:

Theorem 8: Let $\vec{a}, \vec{b}$ and $\vec{c}$ are differentiable vector function of a scalar *t* and $\psi(t)$ is a scalar function of same variable *t*, then

(i) $\frac{d}{dt}\left[a \pm b\right] = \frac{da}{dt} \pm \frac{db}{dt}$

(ii) $\frac{d}{dt}\left[a.b\right] = a.\frac{db}{dt} + \frac{da}{dt}.b$

(iii) $\frac{d}{dt}\left[a \times b\right] = a \times \frac{db}{dt} + \frac{da}{dt} \times b$

(iv) $\frac{d}{dt}\left[a\,b\,c\right] = \left[\frac{da}{dt}\;b\;c\right] + \left[a\;\frac{db}{dt}\;c\right] + \left[a\;b\;\frac{dc}{dt}\right]$

(v) $\frac{d}{dt}\left[\psi a\right] = \psi\frac{da}{dt} + \frac{d\psi}{dt}a$

(vi) $\frac{d}{dt}\left\{a \times \left(b \times c\right)\right\} = \frac{da}{dt} \times (b \times c) + a \times \left(\frac{db}{dt} \times c\right) + a \times \left(b \times \frac{dc}{dt}\right)$

Proof (*i*):

$$\frac{d}{dt}\left[a+b\right] = \lim_{\delta t \to 0}\frac{\left\{\left(a+\delta a\right)+\left(b+\delta b\right)\right\}-\left(a+b\right)}{\delta t} = \lim_{\delta t \to 0}\frac{\delta a+\delta b}{\delta t} = \lim_{\delta t \to 0}\frac{\delta a}{\delta t} + \lim_{\delta t \to 0}\frac{\delta b}{\delta t} = \frac{da}{dt} + \frac{db}{dt}$$

Similarly, we can prove that, $\frac{d}{dt}\left[a-b\right] = \frac{da}{dt} - \frac{db}{dt}$

Proof (*ii*):

$$\frac{d}{dt}\left[a.b\right] = \lim_{\delta t \to 0}\frac{\left\{\left(a+\delta a\right).\left(b+\delta b\right)\right\}-a.b}{\delta t} = \lim_{\delta t \to 0}\frac{a.b+a.\delta b+\delta a.b+\delta a.\delta b-a.b}{\delta t}$$

$$= \lim_{\delta t \to 0}\frac{a.\delta b+\delta a.b+\delta a.\delta b}{\delta t} = \lim_{\delta t \to 0}\left\{a.\frac{\delta b}{\delta t}+\frac{\delta a}{\delta t}.b+\frac{\delta a}{\delta t}.\delta b\right\}$$

$$= \lim_{\delta t \to 0}\left\{a.\frac{\delta b}{\delta t}+\frac{\delta a}{\delta t}.b+\frac{\delta a}{\delta t}.\delta b\right\} = \lim_{\delta t \to 0}a.\frac{\delta b}{\delta t} + \lim_{\delta t \to 0}\frac{\delta a}{\delta t}.b + \lim_{\delta t \to 0}\frac{\delta a}{\delta t}.\delta b$$

$$= a.\frac{db}{dt} + \frac{da}{dt}.b + \frac{da}{dt}.0$$

Since $\delta b \to 0$ as $t \to 0$

$$\frac{d}{dt}\left[a.b\right] = a.\frac{db}{dt} + \frac{da}{dt}.b$$

Note: As we know that vector dot product is commutative, then

$$\frac{d}{dt}\left[a.b\right] = a.\frac{db}{dt} + \frac{da}{dt}.b = \frac{d}{dt}\left[b.a\right]$$

Proof (*iii*):

$$\frac{d}{dt}\left[a \times b\right] = \lim_{\delta t \to 0}\frac{\left(a+\delta a\right)\times\left(b+\delta b\right)-a \times b}{\delta t} = \lim_{\delta t \to 0}\frac{a \times b+a \times \delta b+\delta a \times b+\delta a \times \delta b-a \times b}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{a \times \delta b + \delta a \times b + \delta a \times \delta b}{\delta t}$$

$$= \lim_{\delta t \to 0} \left\{ a \times \frac{\delta b}{\delta t} + \frac{\delta a}{\delta t} \times b + \frac{\delta a}{\delta t} \times \delta b \right\} = \lim_{\delta t \to 0} a \times \frac{\delta b}{\delta t} + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \times b + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \times \delta b$$

$$= a \times \frac{db}{dt} + \frac{da}{dt} \times b + \frac{da}{dt} \times 0$$

Since $\delta b \to 0$ as $t \to 0$

$$\frac{d}{dt}[a.b] = a \times \frac{db}{dt} + \frac{da}{dt} \times b$$

Note: As we know that vector cross product is not commutative, so we must have to maintain the order of the factor *a* and *b*.

Proof (*iv*): $\frac{d}{dt}[a\,b\,c] = \frac{d}{dt}\{a.(b \times c)\} = a.\frac{d}{dt}(b \times c) + \frac{da}{dt}.(b \times c)$

$$= a.\left(b \times \frac{dc}{dt} + \frac{db}{dt} \times c \right) + \frac{da}{dt}.(b \times c) = a.\left(b \times \frac{dc}{dt} \right) + a.\left(\frac{db}{dt} \times c \right) + \frac{da}{dt}.(b \times c)$$

$$= \left[a \; b \; \frac{dc}{dt} \right] + \left[a \; \frac{db}{dt} \; c \right] + \left[\frac{da}{dt} \; b \; c \right] = \left[\frac{da}{dt} \; b \; c \right] + \left[a \; \frac{db}{dt} \; c \right] + \left[a \; b \; \frac{dc}{dt} \right]$$

Proof (*v*):

$$\frac{d}{dt}[\psi a] = \lim_{\delta t \to 0} \frac{(\psi + \delta\psi)(a + \delta a) - \psi a}{\delta t} = \lim_{\delta t \to 0} \frac{\psi a + \psi \delta a + a \delta\psi + \delta\psi \delta a - \psi a}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\psi \delta a + a \delta\psi + \delta\psi \delta a}{\delta t} = \lim_{\delta t \to 0} \psi \frac{\delta a}{\delta t} + \lim_{\delta t \to 0} a \frac{\delta\psi}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\psi}{\delta t} \delta a$$

$$= \psi \frac{da}{dt} + \frac{d\psi}{dt} a + \frac{d\psi}{dt} 0 = \psi \frac{da}{dt} + \frac{d\psi}{dt} a$$

Proof (*vi*): $\frac{d}{dt}\{a \times (b \times c)\} = a \times \frac{d}{dt}(b \times c) + \frac{da}{dt} \times (b \times c)$

$$= a \times \left(\frac{db}{dt} \times c + b \times \frac{dc}{dt} \right) + \frac{da}{dt} \times (b \times c) = a \times \left(\frac{db}{dt} \times c \right) + a \times \left(b \times \frac{dc}{dt} \right) + \frac{da}{dt} \times (b \times c)$$

$$= \frac{da}{dt} \times (b \times c) + a \times \left(\frac{db}{dt} \times c \right) + a \times \left(b \times \frac{dc}{dt} \right)$$

Remarks 1: Derivative of a constant vector is always zero.

2: If $\vec{r}$ is a vector quantity and *s* is a scalar quantity, then we write

$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$, which is nothing but the multiplication of the vector $\frac{d\vec{r}}{ds}$ and scalar $\frac{ds}{dt}$.

3: If $\vec{r} = xi + yj + zk$, where the component *x*, *y*, *z* are scalar function of scalar variable *t* and *i*, *j*, *k* are unit vectors along the three co-ordinate axes then,

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

It means when we differentiate a vector, we should differentiate its components.

Sometime, we also use the notation,

$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$ and so on.

Some important theorems

Theorem 9: The necessary and sufficient condition for the vector function $a(t)$ to be constant is that $\frac{da}{dt} = 0$.

Proof: The condition is necessary: Let the vector function $a(t)$ is the constant vector corresponding to the scalar variable *t*, it means $a(t+\delta t) = a(t)$

Now, $\frac{da}{dt} = \lim_{\delta t \to 0} \frac{a(t+\delta t) - a(t)}{\delta t} = \lim_{\delta t \to 0} \frac{a(t) - a(t)}{\delta t} = 0$

The condition is sufficient: Let $\frac{da}{dt} = 0$, then, we have to prove that $a(t)$ is the constant vector. Let $a(t) = a_1(t)i + a_2(t)\,\mathrm{j} + a_3(t)\,\mathrm{k}$, then,

$\frac{da}{dt} = \frac{da_1}{dt}i + \frac{da_2}{dt}\mathrm{j} + \frac{da_3}{dt}\mathrm{k}$

Since, $\frac{da}{dt} = 0 \Rightarrow \frac{da_1}{dt}i + \frac{da_2}{dt}\mathrm{j} + \frac{da_3}{dt}\mathrm{k} = 0$

Now, equating both side coefficient of *i*, *j*, *k*, we get

$\frac{da_1}{dt} = 0$, $\frac{da_2}{dt} = 0$, $\frac{da_3}{dt} = 0$, Here a_1, a_2, a_3 are constant vector because they are independent from *t*. therefore $a(t)$ is the constant vector function.

Theorem 10: If $\vec{a}$ is differentiable vector function of the scalar variable t and if $\left|\vec{a}\right| = a$, then

(i) $$\frac{d\left(\vec{a}\right)^2}{dt} = 2\vec{a}\frac{d\vec{a}}{dt}$$

(ii) $$\vec{a}.\frac{d\vec{a}}{dt} = a\frac{da}{dt}$$

Proof: (*i*) As we know that $\vec{a}^2 = \vec{a}.\vec{a} = \left|\vec{a}\right|\left|\vec{a}\right|\cos 0 = aa = a^2$

Then, $\dfrac{d\vec{a}^2}{dt} = \dfrac{da^2}{dt} = 2a\dfrac{da}{dt}$

(ii) $$\frac{d\vec{a}^2}{dt} = \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 2\vec{a}.\frac{d\vec{a}}{dt}$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 2a\frac{da}{dt}$$

$$\vec{a}.\frac{d\vec{a}}{dt} = a\frac{da}{dt}$$

Theorem 11: If $\vec{a}$ has constant length (fixed magnitude), then $\vec{a}$ and $\dfrac{d\vec{a}}{dt}$ are perpendicular provided $\left|\dfrac{d\vec{a}}{dt}\right| \neq 0$.

Proof: We have given that $\left|\vec{a}\right| = a =$ constant. Then $\vec{a}.\vec{a} = a^2$ = constant.

Since, $\dfrac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0$$

Since, scalar dot product of two vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ is zero. It means vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular i.e., $\left|\frac{d\vec{a}}{dt}\right| \neq 0$

Theorem 12: The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt} = 0$.

Proof: Let $\vec{a}(t)$ is the vector function of the scalar variable *t* and it have constant magnitude i.e., $|\vec{a}| = a$ = constant.

$\Rightarrow \vec{a}.\vec{a} = a^2$ = constant

$$\Rightarrow \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$$

$$\Rightarrow 2\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0$$

Which is the necessary condition

Condition is sufficient: Let us assume that $\vec{a}.\frac{d\vec{a}}{dt} = 0$, then we have to prove that $\vec{a}(t)$ is a constant vector.

Since, $\vec{a}.\frac{d\vec{a}}{dt} = 0 \Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$

$$\Rightarrow \frac{d\left(\vec{a}.\vec{a}\right)}{dt} = 0 \quad \Rightarrow \vec{a}.\vec{a} = a^2 = \text{constant}$$

Theorem 13: If $\vec{a}(t)$ is the vector function of the scalar variable *t* and it is differentiable, then $\frac{d}{dt}\left(a \times \frac{da}{dt}\right) = a \times \frac{d^2a}{dt^2}$

Proof: $\frac{d}{dt}\left(a \times \frac{da}{dt}\right) = \frac{da}{dt} \times \frac{da}{dt} + a \times \frac{d^2a}{dt^2}$

As we know that $a \times a = 0$

$$\Rightarrow \frac{d}{dt}\left(a\times\frac{da}{dt}\right)=0+a\times\frac{d^2a}{dt^2}=a\times\frac{d^2a}{dt^2}$$

Theorem 14: The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant direction $\left(a\times\frac{da}{dt}\right)=0$.

Proof: Let $\vec{a}(t)$ be a vector function corresponding to the scalar variable *t*.

Let **A** be the unit vector in the direction of $\vec{a}(t)$ then, $\vec{a}(t)=\left|\vec{a}(t)\right|A$, If we consider *a* be the magnitude of the vector function $\vec{a}(t)$ i.e., $\vec{a}(t)=aA$

$$\frac{d\vec{a}}{dt}=a\frac{dA}{dt}+\frac{da}{dt}A$$

Hence, $\vec{a}\times\frac{d\vec{a}}{dt}=(aA)\times\left(a\frac{dA}{dt}+\frac{da}{dt}A\right)=a^2A\times\frac{dA}{dt}+a\frac{da}{dt}A\times A$

$$=a^2A\times\frac{dA}{dt}+0=a^2A\times\frac{dA}{dt}\qquad \text{[As we know that } a\times a=0\text{]}$$

……. (1)

The condition is necessary: Suppose $\vec{a}$ has constant direction, then **A** is constant vector because it has constant magnitude as well as constant direction. Therefore $\frac{dA}{dt}=0$

Hence, from (1) we get $\vec{a}\times\frac{d\vec{a}}{dt}=a^2A\times 0=0$

Thus, the condition is necessary.

The condition is sufficient: Let we consider, $\vec{a}\times\frac{d\vec{a}}{dt}=0$

Then from (1) we get, $a^2A\times\frac{dA}{dt}=0$ or $A\times\frac{dA}{dt}=0$ …. (2)

$\therefore$ A is of constant length, then $A.\frac{dA}{dt}=0$ …. (3)

From (2) and (3), we get $\frac{dA}{dt}=0$

Hence A is a constant vector it means direction of A is constant.

Note: (*i*) If $\vec{r}$ represent position vector of a particle at a time t with respect to the origin O, then $\delta\vec{r}$ represents small displacement at a particle in time δt. If $\vec{v}$ represents the velocity of the particle at P, then

$$\vec{v} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

(ii)If $\vec{a}$ represents the acceleration of the particle at time t, then

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

SELF CHECK QUESTIONS

Example 15: Find $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$, where

$\vec{r} = (t+1)i + (t^2+t+1)j + (t^3+t^2+t+1)k$

Solution: We have given that $\vec{r} = (t+1)i + (t^2+t+1)j + (t^3+t^2+t+1)k$

So, $\frac{d\vec{r}}{dt} = \frac{d}{dt}(t+1)i + \frac{d}{dt}(t^2+t+1)j + \frac{d}{dt}(t^3+t^2+t+1)k$

$$\frac{d\vec{r}}{dt} = \left(\frac{d}{dt}t + \frac{d}{dt}1\right)i + \left(\frac{d}{dt}t^2 + \frac{d}{dt}t + \frac{d}{dt}1\right)j + \left(\frac{d}{dt}t^3 + \frac{d}{dt}t^2 + \frac{d}{dt}t + \frac{d}{dt}1\right)k$$

$$\frac{d\vec{r}}{dt} = (1+0)i + (2t+1+0)j + (3t^2+2t+1+0)k = i + (2t+1)j + (3t^2+2t+1)k$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}i + \left(2\frac{d}{dt}t + \frac{d}{dt}1\right)j + \left(3\frac{d}{dt}t^2 + 2\frac{d}{dt}t + \frac{d}{dt}1\right)k$$

$$\frac{d^2\vec{r}}{dt^2} = 0 + (2+0)j + (6t+2+0)k = 2j + (6t+2)k$$

Example 16: If $\vec{r} = \sin t\, i + \cos t\, j + t\, k$, then find the following

(i) $\frac{d\vec{r}}{dt}$ **(ii)** $\frac{d^2\vec{r}}{dt^2}$

(iii) $\left|\frac{d\vec{r}}{dt}\right|$ **(iv)** $\left|\frac{d^2\vec{r}}{dt^2}\right|$

Solution: (i) $\frac{d\vec{r}}{dt} = i\frac{d}{dt}\sin t + j\frac{d}{dt}\cos t + k\frac{d}{dt}t = \cos t\, i - \sin t\, j + k$

(ii) $\dfrac{d^2\vec{r}}{dt^2} = \dfrac{d}{dt}(\cos t\, i - \sin t\, j + k) = i\dfrac{d}{dt}\cos t - j\dfrac{d}{dt}\sin t + k\dfrac{d}{dt}1$

$$\frac{d^2\vec{r}}{dt^2} = -\sin t\, i - \cos t\, j + 0k = -\sin t\, i - \cos t\, j$$

(iii) $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{(\cos t)^2 + (\sin t)^2 + (1)^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

(iv) $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

Example 17: If $\vec{r} = (\cos nt)i + (\sin nt)\, j$, show that $\vec{r}\times\dfrac{d\vec{r}}{dt} = nk$, where n is a constant.

Solution: We have given, $\vec{r} = (\cos nt)i + (\sin nt)\, j$

So, $\dfrac{d\vec{r}}{dt} = i\dfrac{d}{dt}\cos nt + j\dfrac{d}{dt}\sin nt = -n\sin nt\, i + n\cos nt\, j$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} i & j & k \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix} = 0i - 0j + \left(n\cos^2 nt + n\sin^2 nt\right)k = nk$$

Example 18: If $\vec{r} = (\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}$, where $\vec{a}$, $\vec{b}$ are constant vector and ω is a constant. Then show the following:

(i) $\dfrac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ and **(ii)**

$$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\vec{a}\times\vec{b}$$

Solution (*i*): We have given, if $\vec{a}$, $\vec{b}$ are constant vector and ω is a constant it means $\dfrac{d\vec{a}}{dt} = 0 = \dfrac{d\vec{b}}{dt}$

Since, $\vec{r} = (\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}$

Then, $\dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left((\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}\right) = \dfrac{d(\cos\omega t)}{dt}\vec{a} + \dfrac{d(\sin\omega t)}{dt}\vec{b}$

$$\frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right) = -\omega^2\cos\omega t\,\vec{a} - \omega^2\sin t\,\omega t\,\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\left(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b}\right) = -\omega^2\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$$

(ii) $\vec{r}\times\dfrac{d\vec{r}}{dt} = \left((\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}\right)\times\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right)$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) - \omega\sin^2\omega t\left(\vec{b}\times\vec{a}\right) = \omega\cos^2\omega t\left(\vec{a}\times\vec{b}\right) + \omega\sin^2\omega t\left(\vec{a}\times\vec{b}\right)$$

$$\vec{r}\times\frac{d\vec{r}}{dt} = \omega\left(\cos^2\omega t + \sin^2\omega t\right)\left(\vec{a}\times\vec{b}\right) = \omega\left(\vec{a}\times\vec{b}\right)$$

Example 19: If $\vec{a} = (\cos\theta)i + (\sin\theta)j + \theta k$, $\vec{b} = (\cos\theta)i - (\sin\theta)j - 3k$ and $\vec{c} = 2i + 3j - 3k$, find $\dfrac{d}{d\theta}\{a\times(b\times c)\} = (\cos\theta)i - (\sin\theta)j - 3k$ at $\theta = \dfrac{\pi}{2}$.

Answer:

$$b\times c = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix} = (3\sin\theta + 9)i + (3\cos\theta - 6)j + (3\cos\theta + 2\sin\theta)k$$

Now, $a\times(b\times c) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & 3\cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$

$= \left(3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta\right)i + \left(3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta\right)j +$
$(-6\sin\theta - 9\cos\theta)k$

$$\frac{d\{a\times(b\times c)\}}{d\theta}=\left(-6\cos\theta\sin\theta+2\cos^2\theta-2\sin^2\theta-3\cos\theta+3\theta\sin\theta+6\right)i$$
$$+\left(3\sin\theta+3\theta\cos\theta+9-3\cos^2\theta+3\sin^2\theta-4\sin\theta\cos\theta\right)j+\left(-6\cos\theta+9\sin\theta\right)k$$

Now, putting $\theta=\frac{\pi}{2}$, we get

$$=\left(4+\frac{3}{2}\pi\right)i+15j+9k$$

Example 20: Let a particle moves along the curve $x=t^3+1, y=t^2, z=2t+5$, where t represents the time. Then find the velocity and acceleration of the particle at t = 1, in the direction $i+j+3k$.

Answer: Since, we have given particle moves along the curve $x=t^3+1, y=t^2, z=2t+5$, then

$$\vec{r}=\left(t^3+1\right)i+t^2j+\left(2t+5\right)k$$

So, the velocity of the particle is

$$\vec{v}=\frac{d\vec{r}}{dt}=\left\{\frac{d}{dt}\left(t^3+1\right)\right\}i+\left\{\frac{d}{dt}t^2\right\}j+\left\{\frac{d}{dt}\left(2t+5\right)\right\}k$$

$$\vec{v}=3t^2i+2tj+2k$$

The velocity of particle at t = 1 is $\left(\vec{v}\right)_{t=1}=3i+2j+2k$

Similarly, the acceleration of the particle is

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\left\{\frac{d}{dt}3t^2\right\}i+\left\{\frac{d}{dt}2t\right\}j+\left\{\frac{d}{dt}2\right\}k=6ti+2j$$

The acceleration of particle at t = 1 is $\left(\vec{a}\right)_{t=1}=6i+2j$

Since, the unit vector in the direction of $i+j+3k$ is

$$=\frac{i+j+3k}{\sqrt{1^2+1^2+3^2}}=\frac{i+j+3k}{\sqrt{11}}$$

So, the component of velocity in the direction of given vector

$$=\vec{v}.\vec{b}=\frac{(3i+2j+2k).(i+j+3k)}{\sqrt{11}}=\frac{11}{\sqrt{11}}=\sqrt{11}$$

Similarly, the component of acceleration in the direction of given vector

$$=\vec{a}.\vec{b}=\frac{(6i+2j).(i+j+3k)}{\sqrt{11}}=\frac{8}{\sqrt{11}}$$

Example 21: If $\vec{r}$ is a vector function corresponding to a scalar variable *t*, $\vec{a}$ is a constant vector and *m* is a constant, then differentiate the following with respect to *t*:

(i) $\vec{r}.\vec{a}$ **(ii)** $\vec{r}\times\vec{a}$ **(iii)** $\vec{r}\times\frac{d\vec{r}}{dt}$

(iv) $\vec{r}.\frac{d\vec{r}}{dt}$ **(v)** $\vec{r}^2+\frac{1}{\vec{r}^2}$ **(vi)** $m\left(\frac{d\vec{r}}{dt}\right)^2$

(vii) $\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ **(viii)** $\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$

Answer (i): Let $\vec{R}=\vec{r}.\vec{a}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}.\vec{a}\right)=\frac{d\vec{r}}{dt}.\vec{a}+\vec{r}.\frac{d\vec{a}}{dt}=\frac{d\vec{r}}{dt}.\vec{a}$ $\qquad \because\left[\frac{d\vec{a}}{dt}=0\right]$

(ii) Let, $\vec{R}=\vec{r}\times\vec{a}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}\times\vec{a}\right)=\frac{d\vec{r}}{dt}\times\vec{a}+\vec{r}\times\frac{d\vec{a}}{dt}=\frac{d\vec{r}}{dt}\times\vec{a}$

(iii) Let, $\vec{R}=\vec{r}\times\frac{d\vec{r}}{dt}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}=\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$\because\left[\vec{a}\times\vec{a}=0\right]$

(iv) Let, $\vec{R}=\vec{r}.\frac{d\vec{r}}{dt}$

then, $\frac{d\vec{R}}{dt}=\frac{d}{dt}\left(\vec{r}.\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}+\vec{r}.\frac{d^2\vec{r}}{dt^2}=\left(\frac{d\vec{r}}{dt}\right)^2+\vec{r}.\frac{d^2\vec{r}}{dt^2}$

$\because\left[\vec{a}.\vec{a}=\vec{a}^2\right]$

(v) Let, $\vec{R}=\vec{r}^2+\dfrac{1}{\vec{r}^2}$

then, $\dfrac{d\vec{R}}{dt}=\dfrac{d}{dt}\left(\vec{r}^2+\dfrac{1}{\vec{r}^2}\right)=\dfrac{d}{dt}\vec{r}^2+\dfrac{d}{dt}\left(\dfrac{1}{\vec{r}^2}\right)=2\vec{r}\dfrac{d\vec{r}}{dt}-\dfrac{2}{\vec{r}^3}\dfrac{d\vec{r}}{dt}$

(vi) Let, $\vec{R}=m\left(\dfrac{d\vec{r}}{dt}\right)^2$

Then, $\dfrac{d\vec{R}}{dt}=\dfrac{d}{dt}\left\{m\left(\dfrac{d\vec{r}}{dt}\right)^2\right\}=m\left(2\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2}\right)=2m\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2}$

$\left[\dfrac{d\vec{r}^2}{dt}=2\vec{r}.\dfrac{d\vec{r}}{dt}\right]$

(vii) Let, $\vec{R}=\dfrac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$

Then,

$$\frac{d\vec{R}}{dt}=\frac{d}{dt}\left\{\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}\right\}=\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)}\frac{d}{dt}\left(\vec{r}+\vec{a}\right)+\left\{\frac{d}{dt}\left(\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)}\right)\right\}\left(\vec{r}+\vec{a}\right)$$

$$=\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)}\left(\frac{d}{dt}\vec{r}+\frac{d}{dt}\vec{a}\right)-\left\{\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)^2}\frac{d}{dt}\left(\vec{r}^2+\vec{a}^2\right)\right\}\left(\vec{r}+\vec{a}\right)$$

$$=\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)}\frac{d\vec{r}}{dt}-\frac{2\vec{r}.\frac{d\vec{r}}{dt}}{\left(\vec{r}^2+\vec{a}^2\right)^2}\left(\vec{r}+\vec{a}\right)$$

$$\left[\because \frac{d\vec{a}}{dt}=0,\frac{d}{dt}\vec{r}^2=2\vec{r}.\frac{d\vec{r}}{dt},\frac{d}{dt}\vec{a}^2=0\right]$$

(viii) Let, $\vec{R} = \dfrac{\vec{r}\times\vec{a}}{\vec{r}\,.\,\vec{a}}$

Then, $$\frac{d\vec{R}}{dt} = \frac{d}{dt}\left\{\frac{\vec{r}\times\vec{a}}{\vec{r}\,.\,\vec{a}}\right\} = \frac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\frac{d}{dt}\left(\vec{r}\times\vec{a}\right) + \left\{\frac{d}{dt}\left(\frac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\right)\right\}\left(\vec{r}\times\vec{a}\right)$$

$$= \frac{1}{\left(\vec{r}\,.\,\vec{a}\right)}\left(\frac{d}{dt}\vec{r}\times\vec{a} + \vec{r}\times\frac{d}{dt}\vec{a}\right) - \left\{\frac{1}{\left(\vec{r}\,.\,\vec{a}\right)^2}\frac{d}{dt}\left(\vec{r}\,.\,\vec{a}\right)\right\}\left(\vec{r}\times\vec{a}\right)$$

$$= \frac{\dfrac{d\vec{r}}{dt}\times\vec{a}}{\left(\vec{r}\,.\,\vec{a}\right)} - \left\{\frac{\dfrac{d\vec{r}}{dt}.\vec{a}}{\left(\vec{r}\,.\,\vec{a}\right)^2}\right\}\left(\vec{r}\times\vec{a}\right)$$

12.6 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of vector triple product and scalar triple product.
- The application of differentiation of vectors.

12.7 *GLOSSARY*

- **Vector triple product:** $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ is the vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.
- **Scalar triple product:** $\left(\vec{a}\times\vec{b}\right).\vec{c}$ is the vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

12.8 *REFERENCES*

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- Erwin. Kreyszig, "Advanced engineering mathematics, 10th eddition", 2009.
- A. R. Vasishtha, "Vector Calculus", 20th edition, Krishna publication, 2020.

12.9 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

12.10 *TERMINAL QUESTION*

Objective Question

1. The value of $i.(j\times k)+j.(k\times i)+k.(i\times j)$

a) 0 **b)** 1

c) 2 **d)** 3

2. The volume of parallelopiped whose edges are given by

$\vec{OA}=2i-3j, \vec{OB}=i+j-k, \vec{OC}=3i-k$ is

a) 1 **b)** 4

c) 2/7 **d)** None

3. If [*a*, *b*, *c*] is the scalar triple product of three vectors *a*, *b*, *c* then [*a*, *b*, *c*] is equal to

a) [*b*, *a*, *c*] **b)** [*c*, *b*, *a*]

c) [*b*, *c*, *a*] **d)** [*a*, *c*, *b*]

4. If *i*, *j*, *k* are the orthogonal right handed triad of unit vector and $\vec{a}$ is a vector then

$i\times\left(\vec{a}\times i\right)+j\times\left(\vec{a}\times j\right)+k\times\left(\vec{a}\times k\right)$ is equal to

a) $\vec{a}$ **b)** $2\vec{a}$

c) $3\vec{a}$ **d)** 0

5. If $\vec{r} = \vec{a} e^{\omega t} + \vec{b} e^{-\omega t}$, $\vec{a}, \vec{b}$ are constant vectors then $\dfrac{d^2 \vec{r}}{dt^2} - \omega^2 \vec{r}$ is equal to

a) 1 **b)** 0

c) 2 **d)** none of these

6. If a particle moves along the curve $\vec{r} = e^{-t} \cos t\, i + e^{-t} \sin t\, j + e^{-t} k$, then find the magnitude of velocity at $t = 0$ is

a) $2\sqrt{3}$ **b)** $\dfrac{\sqrt{3}}{2}$

c) $\sqrt{3}$ **d)** none of these

Fill in the blanks

1. If $\{i, j, k\}$ be a set of orthonormal unit vectors, then $[i, j, k]$ = ……….

2. If $\vec{A}, \vec{B}, \vec{C}$ be three non-coplanar vectors, then $\dfrac{\vec{A}.\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}.\vec{B}} + \dfrac{\vec{B}.\vec{A}\times\vec{C}}{\vec{C}.\vec{A}\times\vec{B}} =$ …………

3. For any three vectors $\vec{a}, \vec{b}, \vec{c}, \vec{a}\times\left(\vec{b}\times\vec{c}\right) + \vec{b}\times\left(\vec{c}\times\vec{a}\right) + \vec{c}\times\left(\vec{a}\times\vec{b}\right) =$ …………

4. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ are reciprocal system of vectors, then find $\vec{a}.\vec{a'} + \vec{b}.\vec{b'} + \vec{c}.\vec{c'} = \ldots$

5. If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\left(\vec{a}\times\vec{b}\right).\vec{c} =$ ………………

6. If $\vec{r} = 3i - 6t^2 j + 4tk$, then $\dfrac{d\vec{r}}{dt} =$ ……….; $\dfrac{d^2\vec{r}}{dt^2} =$ …………

7. If $\vec{u} = t^2 i - tj + (2t+1)k$, $\vec{v} = (2t-3)i + j - tk$, then $\dfrac{d}{dt}\left(\vec{u}.\vec{v}\right) =$ ……….

8. If $\vec{r} = (\cos\omega t)i + (\sin\omega t)j$, then $\vec{r}\times\dfrac{d^2\vec{r}}{dt^2} =$ …………

9. The necessary and sufficient condition for the vector $\vec{a}(t)$ to be constant direction is ...

True or False

1. If $x.\vec{a} = x.\vec{b} = x.\vec{c} = 0$, for some non-zero vectors x, then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

2. If $\{i, j, k\}$ be orthonormal set of unit vectors, then $i \times (j \times k) \neq 0$

3. The orthonormal unit vector triads, $\{i, j, k\}$ form a reciprocal system.

4. A vector is said to be constant only if its direction changes and magnitude is fixed.

5. The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant direction is $\vec{a}.\dfrac{d\vec{a}}{dt} = 0$

Short answer type question:

1. Prove that the identity $a \times [a \times (a \times b)] = (a\,.\,a)(b \times a)$

2. If a, b, c and a', b', c' are reciprocal system of vectors, prove that

(i) $a \times a' + b \times b' + c \times c' = 0$

(ii) $a' \times b' + b' \times c' + c' \times a' = \dfrac{a+b+c}{[a\,b\,c]}$

(iii) $a.a' + b.b' + c.c' = 3$

3. Show that $i.j \times k = 1$.

4. Show that $[\lambda a + \mu b, c, d] = \lambda[a, c, d] + \mu[b, c, d]$.

5. Prove that $[i - j, j - k, k - i] = 0$

Long answer type question:

1. Prove that the four points $6a - 4b + 10c, -5a + 3b + 10c, 4a - 6b - 10c$ and $2b + 10c$ are coplanar.

2. Prove that $a \times (b \times a) = (a \times b) \times a$

3. Prove that any three vectors A, B, C ,

$$(A\times B).((B\times C)\times(C\times A))=(A.B\times C)^2.$$

4. If $r=t^3i+(2t^3-\frac{1}{5t^2})j$, show that $r\times\frac{dr}{dt}=k$

5. If $r=e^{nt}a+e^{-nt}b)$ where a, b are constant vector, show that

$$\frac{d^2r}{dt^2}-n^2r=0$$

12.11 *ANSWERS*

Answer of objective type questions

1. (d) **2.** (b) **3.** (c) **4.** (b) **5.** (b) **6.** (c)

Answer of fill in the blanks

1. 1 **2.** 0 **3.** 0 **4.** 3 **5.** 0

6. $-12tj+4k;-12j$ **7.** $6t^2-10t-2$ **8. 0**

9. $\vec{a}\times\frac{d\vec{a}}{dt}=0$

Answer of true and false question.

1. *T* **2.** *F* **3.** *T* **4.** *F* **5.** *T*

UNIT 13: GRADIENT, DIVERGENCE AND CURL

CONTENTS:

13.1 *INTRODUCTION*

A gradient in calculus is the differential operator that is used to create a vector from a three-dimensional vector-valued function. The gradient is denoted by the symbol ∇ (nabla). For instance, if "f" is a function, then "∇f " is used to represent a function's gradient. Let's go into depth about the definition of a function's gradient, directional derivative, characteristics, and solved instances in this unit. Divergence and curl are the two essential operations performed on the vector field in mathematics. Both are important in calculus because they aid in the development of the higher-dimensional version of the calculus fundamental theorem. Divergence often explains the field's behaviour in relation to a point or away from it. The rotational extent of the field around a certain point is also measured using curl.

In rectilinear coordinates, the gradient of a scalar and the divergence and curl of a vector have a straightforward form. If the origin is moved or the coordinates are rotated while using rectilinear coordinates, the form is preserved. However, if you pick arbitrary coordinates, their shapes alter, and they appear fairly different even in polar coordinates in the plane, as well as in cylindrical and spherical coordinates in three dimensions.

13.2 *OBJECTIVES*

After reading this unit learners will be able to

- Implementation of application of vector triple product and scalar triple product in vector calculus
- Memorized about the basic differences and relations between the gradient, divergence and curl operators.
- Application of gradient, divergence and curl operators and there use in vector calculus.
- Memorized the useful theorems and their application of vector triple product and scalar triple product.

13.3 *PARTIAL DERIVATIVES OF VECTORS*

If a vector $\vec{r}$ is depending on two or more variable *i.e.*, $\vec{r} = f(x,y,z)$. Then partial derivative of $\vec{r}$ with respect to x is defined as

$$\frac{\partial \vec{r}}{\partial x} = \lim_{\delta x \to 0} \frac{f\left(x+\delta x, y, z\right) - f(x,y,z)}{\delta x}$$

if this limit exists. Thus $\frac{\partial \vec{r}}{\partial x}$ is just ordinary differentiation of $\vec{r}$ only with respect to the variable x and other variable y and z are regarded as constant. Similarly, we can find another partial derivative like $\frac{\partial \vec{r}}{\partial y}$ and $\frac{\partial \vec{r}}{\partial z}$.

Other higher order partial derivative can also define as,

$$\frac{\partial^2 \vec{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^2 \vec{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{r}}{\partial z}\right)$$

$$\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right), \frac{\partial^2 \vec{r}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial x}\right), \frac{\partial^2 \vec{r}}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{r}}{\partial y}\right)$$

If $\vec{r}$ has continuous partial derivatives of the second order, then,

$$\frac{\partial^2 \vec{r}}{\partial y \partial x} = \frac{\partial^2 \vec{r}}{\partial x \partial y}$$

If $\vec{r} = f(x, y, z)$, then total differential $d\vec{r}$ of $\vec{r}$ is given by,

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy + \frac{\partial \vec{r}}{\partial z} dz$$

The vector differential operator: The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Here, the symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its component along i, j, k

.

13.4 *GRADIENT OF SCALAR FIELD*

Let $f(x, y, z)$ be a differentiable function at each point (x, y, z) in the space (i.e., defines a differential scalar field). Then gradient of f, written as *grad* f or ∇f and defined as

$$\nabla f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Note: (*i*) Gradient of f is a vector quantity whose three successive components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

(*ii*) Gradient of a scalar field defines a vector field it means if f is a scalar point functionthen its gradient will be a vector point function.

Theorems involving gradient.

Theorem 1: Gradient of addition of two scalar point function: If f and g are two scalar point function then,

$$grad(f + g) = grad\ f + grad\ g \text{ or } \nabla(f + g) = \nabla f + \nabla g$$

Proof:

$$grad(f + g) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(f + g) = i\frac{\partial}{\partial x}(f + g) + j\frac{\partial}{\partial y}(f + g) + k\frac{\partial}{\partial z}(f + g)$$

$$= i\frac{\partial}{\partial x} f + i\frac{\partial}{\partial x} g + j\frac{\partial}{\partial y} f + j\frac{\partial}{\partial y} g + k\frac{\partial}{\partial z} f + k\frac{\partial}{\partial z} g$$

$$= \left(i\frac{\partial}{\partial x}f + j\frac{\partial}{\partial y}f + k\frac{\partial}{\partial z}f \right) + \left(i\frac{\partial}{\partial x}g + j\frac{\partial}{\partial y}g + k\frac{\partial}{\partial z}g \right)$$

$$= grad\ f + grad\ g$$

i.e., $\nabla(f+g) = \nabla f + \nabla g$

Similarly, we can prove that $grad(f-g) = grad\ f - grad\ g$ or $\nabla(f-g) = \nabla f - \nabla g$

Theorem 2: Gradient of a constant function: The necessary and sufficient condition for scalar point function to be constant is that $\nabla f_c = 0$

Proof: Let us suppose that f_c is a constant function it means its three successive components are zero i.e., $\frac{\partial f_c}{\partial x} = 0, \frac{\partial f_c}{\partial y} = 0, \frac{\partial f_c}{\partial z} = 0$.

$\because grad\ f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

Then gradient of constant function is, $grad\ f_c = i.0 + j.0 + k.0 = 0$ *i.e.*, $\nabla f_c = 0$

Conversely, let $\nabla f_c = 0$.

It means, $i\frac{\partial f_c}{\partial x} + j\frac{\partial f_c}{\partial y} + k\frac{\partial f_c}{\partial z} = 0i + 0j + 0k$

equating both side component of *i*, *j*, *k* we get, $\frac{\partial f_c}{\partial x} = 0, \frac{\partial f_c}{\partial y} = 0, \frac{\partial f_c}{\partial z} = 0$

$\Rightarrow f_c$ must be independent from *x*, *y*, *z*.

$\Rightarrow f_c$ must be constant function.

Hence the condition is sufficient.

Theorem 3: Gradient of the product of two scalar point function: If f and g are two scalar point function, then $grad(fg) = f\ grad\ g + g\ grad\ f$ or $\nabla(fg) = f\nabla g + g\nabla f$.

Proof: Wehave

$$\nabla(fg) = grad(fg) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right)(fg) = i\frac{\partial}{\partial x}(fg) + j\frac{\partial}{\partial y}(fg) + k\frac{\partial}{\partial z}(fg)$$

$$= i\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} \right) + j\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y} \right) + k\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z} \right)$$

$$= f\left(i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z} \right) + g\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right)$$

$\nabla(fg) = f\nabla g + g\nabla f$ or $grad(fg) = f\ grad\ g + g\ grad\ f$

In particular, if c is a constant function, then,
$\nabla(c\, f) = f\nabla c + c\nabla f = 0 + c\nabla f = c\nabla f$

Theorem 4: Gradient of the quotient of two scalar functions: If f and g are two scalar point function, then $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

Proof:

$$\nabla\left(\frac{f}{g}\right) = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(\frac{f}{g}\right) = i\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + j\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + k\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$\because \frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}, \frac{\partial}{\partial y}\left(\frac{f}{g}\right) = \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}, \frac{\partial}{\partial z}\left(\frac{f}{g}\right) = \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}$$

$$\Rightarrow \nabla\left(\frac{f}{g}\right) = i\left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right) + j\left(\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right) + k\left(\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)$$

$$\Rightarrow \nabla\left(\frac{f}{g}\right) = \frac{\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right)i + \left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right)j + \left(g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}\right)k}{g^2}$$

$$= \frac{g\frac{\partial f}{\partial x}i - f\frac{\partial g}{\partial x}i + g\frac{\partial f}{\partial y}j - f\frac{\partial g}{\partial y}j + g\frac{\partial f}{\partial z}k - f\frac{\partial g}{\partial z}k}{g^2}$$

$$= \frac{g\frac{\partial f}{\partial x}i + g\frac{\partial f}{\partial y} + g\frac{\partial f}{\partial z}k - \left(f\frac{\partial g}{\partial x}i + f\frac{\partial g}{\partial y}j + f\frac{\partial g}{\partial z}k\right)}{g^2}$$

$$= \frac{g\nabla f - f\nabla g}{g^2}$$

Solved Example

Example 1: Evaluate the value of $\frac{\partial^2}{\partial x\partial y}(A\times B)$ at (1, 0, -2) where,

$A = x^2yzi - 2xz^3j + xz^2k, B = 2zi + yj - x^2k$.

Answer: First we have to find the value of $A\times B$

So, $A\times B = \begin{vmatrix} i & j & k \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$=\left(2x^3z^3-xyz^2\right)i+\left(2xz^3+x^4yz\right)j+\left(x^2y^2z+4xz^4\right)k$$

Now,

$$\frac{\partial^2}{\partial x\partial y}(A\times B)=\frac{\partial^2}{\partial x\partial y}\left\{\left(2x^3z^3-xyz^2\right)i+\left(2xz^3+x^4yz\right)j+\left(x^2y^2z+4xz^4\right)k\right\}$$

$$\frac{\partial^2}{\partial x\partial y}(A\times B)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{\left(2x^3z^3-xyz^2\right)i+\left(2xz^3+x^4yz\right)j+\left(x^2y^2z+4xz^4\right)k\right\}\right]$$

$$=\frac{\partial}{\partial x}\left\{-xz^2i+x^4zj+2x^2yzk\right\}$$

$$=-z^2i+4x^3zj+4xyzk$$

Thus, value of $\frac{\partial^2}{\partial x\partial y}(A\times B)$ at the point (1, 0, -2) evaluated by putting

$x=1, y=0, z=-2$

$$\left[\frac{\partial^2}{\partial x\partial y}(A\times B)\right]_{(1,0,-2)}=-4i-8j$$

Example 2: Evaluate *grad f* at the point (1, -2, -1) where,

$f(x,y,z)=3x^2y-y^3z^2$.

Answer: We have given $f(x,y,z)=3x^2y-y^3z^2$

So, $grad\ f=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(3x^2y-y^3z^2\right)$

$$=i\frac{\partial}{\partial x}\left(3x^2y-y^3z^2\right)+j\frac{\partial}{\partial y}\left(3x^2y-y^3z^2\right)+k\frac{\partial}{\partial z}\left(3x^2y-y^3z^2\right)$$

$$=6xyi+\left(3x^2-3y^2z^2\right)j+2y^3z\,k$$

Thus, value of *grad f* at the point (1, -2, -1)

$$\left(grad\ f\right)_{(1,-2,-1)}=-12i-9j-16k$$

Example 3: If $r=\left|\vec{r}\right|$ where, $\vec{r}=xi+yj+zk$, then prove the following:

(i) $\nabla f(\text{r})=f^{'}(\text{r})\nabla r$ **(ii)** $\nabla r=\frac{1}{r}\vec{r}$ **(iii)** $\nabla f(\text{r})\times\vec{r}=0$

(iv) $\nabla\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}$ **(v)** $\nabla\log\left|\vec{r}\right|=\frac{\vec{r}}{r^2}$ **(vi)** $\nabla\vec{r}^n=nr^{n-2}\vec{r}$

Answer: Let $\vec{r} = xi + yj + zk$, then

$r = \left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}\ or\ r^2 = x^2 + y^2 + z^2$

(i) $\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) f(r) = i\frac{\partial}{\partial x} f(r) + j\frac{\partial}{\partial y} f(r) + k\frac{\partial}{\partial z} f(r)$

$= i\ f^{'}(r)\frac{\partial r}{\partial x} + j\ f^{'}(r)\frac{\partial r}{\partial y} + k\ f^{'}(r)\frac{\partial r}{\partial z}$

$= f^{'}(r)\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right) = f^{'}(r)\nabla r$

(ii) $\nabla \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\left(\sqrt{x^2 + y^2 + z^2} \right)$

$= i\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} + j\frac{\partial}{\partial y}\sqrt{x^2 + y^2 + z^2} + k\frac{\partial}{\partial z}\sqrt{x^2 + y^2 + z^2}$

$= i\frac{2x}{2\sqrt{x^2 + y^2 + z^2}} + j\frac{2y}{2\sqrt{x^2 + y^2 + z^2}} + k\frac{2z}{2\sqrt{x^2 + y^2 + z^2}}$

$= \frac{2xi + 2yj + 2zk}{2\sqrt{x^2 + y^2 + z^2}} = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}} = \frac{\vec{r}}{\left|\vec{r}\right|}$

(iii) As we know from **(i)** proof that $\nabla f(\mathrm{r}) = f^{'}(\mathrm{r})\nabla r$ and from (ii) proof that $\nabla r = \frac{1}{r}\vec{r}$

So, $\nabla f(\mathrm{r}) = f^{'}(\mathrm{r})\frac{1}{r}\vec{r}$

Now, $\nabla f(\mathrm{r}) \times \vec{r} = f^{'}(\mathrm{r})\frac{1}{r}\vec{r} \times \vec{r} = f^{'}(\mathrm{r})\frac{1}{r}\left(\vec{r} \times \vec{r} \right) = f^{'}(\mathrm{r})\frac{1}{r}.0 = 0$ [As we know $\vec{r} \times \vec{r} = 0$]

(iv) $\nabla\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$= i\frac{1}{r^2}\frac{\partial r}{\partial x} + j\frac{1}{r^2}\frac{\partial r}{\partial y} + k\frac{1}{r^2}\frac{\partial r}{\partial z}$

$= \frac{1}{r^2}\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right) = \frac{\nabla r}{r^2}$

Since we know that $\nabla r = \frac{1}{r}\vec{r}$

So, $\nabla\left(\frac{1}{r}\right)=\frac{\nabla r}{r^2}=\frac{1}{r^2}\left(\frac{1}{r}\vec{r}\right)=-\frac{\vec{r}}{r^3}$

(v) $$\nabla\log\left|\vec{r}\right|=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\log r=i\frac{\partial}{\partial x}\log r+j\frac{\partial}{\partial y}\log r+k\frac{\partial}{\partial z}\log r$$

$$=i\frac{1}{r}\frac{\partial r}{\partial x}+j\frac{1}{r}\frac{\partial r}{\partial y}+k\frac{1}{r}\frac{\partial r}{\partial z}=\frac{1}{r}\left(i\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+k\frac{\partial r}{\partial z}\right)$$

$$=i\frac{1}{r}\frac{\partial r}{\partial x}+j\frac{1}{r}\frac{\partial r}{\partial y}+k\frac{1}{r}\frac{\partial r}{\partial z}=\frac{1}{r}\left(i\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+k\frac{\partial r}{\partial z}\right)$$

$$=\frac{1}{r}\nabla r=\frac{\vec{r}}{r^2}$$

(vi) $$\nabla r^n=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)r^n=i\frac{\partial r^n}{\partial x}+j\frac{\partial r^n}{\partial y}+k\frac{\partial r^n}{\partial z}$$

$$=nr^{n-1}\frac{\partial r}{\partial x}i+nr^{n-1}\frac{\partial r}{\partial y}j+nr^{n-1}\frac{\partial r}{\partial z}k$$

$$=\left(\frac{\partial r}{\partial x}i+\frac{\partial r}{\partial y}j+\frac{\partial r}{\partial z}k\right)nr^{n-1}$$

$$=nr^{n-1}\nabla r=nr^{n-1}\frac{\vec{r}}{r}=nr^{n-2}\vec{r}$$

Example 4: If $f=\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k,$ then verify that $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$.

Answer: We have given, $f=\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k$

Then, $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2}{\partial y\partial x}\left\{\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k\right\}$

$$=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left\{\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k\right\}\right]$$

$$=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(2x^2y-x^4\right)i+\frac{\partial}{\partial x}\left(e^{xy}-y\sin x\right)j+\frac{\partial}{\partial x}x^2\cos y\,k\right]$$

$$=\frac{\partial}{\partial y}\left\{\left(4xy-4x^3\right)i+\left(ye^{xy}-y\cos x\right)j+2x\cos y\,k\right\}$$

$$=\frac{\partial}{\partial y}\left(4xy-4x^3\right)i+\frac{\partial}{\partial y}\left(ye^{xy}-y\cos x\right)j+\frac{\partial}{\partial y}2x\cos y\,k$$

$$=4xi+\left(e^{xy}+xye^{xy}-\cos x\right)j-2x\sin y\,k$$

Similarly, $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2}{\partial x\partial y}\left\{\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k\right\}$

$$=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left\{\left(2x^2y-x^4\right)i+\left(e^{xy}-y\sin x\right)j+x^2\cos y\,k\right\}\right]$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(2x^2y-x^4\right)i+\frac{\partial}{\partial y}\left(e^{xy}-y\sin x\right)j+\frac{\partial}{\partial y}x^2\cos y\,k\right]$$

$$=\frac{\partial}{\partial x}\left\{2x^2i+\left(xe^{xy}-\sin x\right)j-x^2\sin y\,k\right\}$$

$$=\frac{\partial}{\partial x}2x^2i+\frac{\partial}{\partial x}\left(xe^{xy}-\sin x\right)j-\frac{\partial}{\partial x}x^2\sin y\,k$$

$$=4xi+\left(e^{xy}+xye^{xy}-\cos x\right)j-2x\sin y\,k$$

Hence, we can easily see that $\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{\partial^2 f}{\partial x\partial y}$.

Equipotential surfaces or level surfaces: Let a scalar field $f(x,y,z)=c$ in the region *R*. The points which are satisfying the equation $f(x,y,z)=c$, constitutes family of surfaces in the three-dimensional space. The occurred surface of this family is called ***level surfaces***.

Any surface of this family is such a way that the value of the function *f* at any point of it is the same. Hence these surfaces are also called ***iso-f-surfaces***.

13.5 *DIVERGENCE OF A VECTOR POINT FUNCTION*

Let *V* be the differentiable vector point function. Then the divergence of *V* denoted as *divV* or $\nabla .V$ and defined as follows:

$$\nabla .V=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).V=i.\frac{\partial V}{\partial x}+j.\frac{\partial V}{\partial y}+k.\frac{\partial V}{\partial z}=\sum i.\frac{\partial V}{\partial x}$$

It should be noted that *divV* is always a scalar quantity.

Solenoidal vector: Adifferentiable vector point function *V* is said to be solenoidal if $divV=0$

Theorem 5: If $V=V_1i+V_2j+V_3k$ is differentiable vector point function, then

$$\nabla .V=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(V_1i+V_2j+V_3k\right)=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}=\sum\frac{\partial V_1}{\partial x}$$

Proof: We have given that, $V=V_1i+V_2j+V_3k$

Then,

$$\nabla . V=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(V_1 i+V_2 j+V_3 k\right)=i\frac{\partial}{\partial x}.\left(V_1 i\right)+j\frac{\partial}{\partial y}.\left(V_2 j\right)+k\frac{\partial}{\partial z}.\left(V_3 k\right)$$

$$=\frac{\partial V_1}{\partial x}(i.i)+\frac{\partial V_2}{\partial y}(j.j)+\frac{\partial V_3}{\partial z}(k.k)$$

$$=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}$$

Hence, $\nabla . V=\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}$

13.6 *CURL OF A VECTOR POINT FUNCTION*

Let F be any differentiable vector point function. Then the curl or sometime called rotation of F denoted as *curlF* or $\nabla \times F$ and defined as follows:

$$\nabla \times F=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times F=i\times\frac{\partial F}{\partial x}+j\times\frac{\partial F}{\partial y}+k\times\frac{\partial F}{\partial z}=\sum i\times\frac{\partial F}{\partial x}$$

It should be noted that *curlF* is always a vector quantity.

Irrotational vector: Adifferentiable vector point function F is said to be irrotational if $curl\, F=0$.

Theorem 6: If $F=f_1 i+f_2 j+f_3 k$ is differentiable vector point function, then

$$curl\, F=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)i+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)j+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)k$$

Proof: $\nabla \times F=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(f_1 i+f_2 j+f_3 k\right)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)i+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)j+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)k$$

OR

The *curlF* is also prove as,

$$\nabla \times F=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(f_1 i+f_2 j+f_3 k\right)$$

$$= i\frac{\partial}{\partial x}\times\left(f_1 i + f_2 j + f_3 k\right) + j\frac{\partial}{\partial y}\times\left(f_1 i + f_2 j + f_3 k\right) + k\frac{\partial}{\partial z}\times\left(f_1 i + f_2 j + f_3 k\right)$$

as we know that,

$i\times i = 0, j\times j = 0, k\times k = 0, i\times j = k, j\times i = -k, i\times k = j, k\times i = -j, j\times k = i, k\times j = -i$

so, $\nabla\times F = \left(\frac{\partial f_2}{\partial x}k - \frac{\partial f_3}{\partial x}j\right) + \left(-\frac{\partial f_1}{\partial y}k + \frac{\partial f_3}{\partial y}i\right) + \left(\frac{\partial f_1}{\partial z}j - \frac{\partial f_2}{\partial z}i\right)$

$$\nabla\times F = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)k$$

13.7 *LAPLACIAN OPERATOR*

The Laplacian Operators ∇^2 : The Laplacian operator mathematically denoted as ∇^2 and defined as, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If f is a scalar point function, then, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$, here $\nabla^2 f$ is also a scalar quantity.

If f is a vector point function, then, $\nabla^2 \vec{f} = \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2}$, here $\nabla^2 \vec{f}$ is also a vector quantity.

Laplace's Equation: The Laplace equation is defined as $\nabla^2 f = 0$. A function which satisfied Laplace equation is called **Harmonic function**.

Solved Example

Example 5: Show that $div\,\vec{r} = 3$.

Answer: As we know that $\vec{r} = xi + yj + zk$.

So, $div\,\vec{r} = \nabla.\vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(x i + y j + z k\right)$

$$= i\frac{\partial}{\partial x}.\left(x i + y j + z k\right) + j\frac{\partial}{\partial y}.\left(x i + y j + z k\right) + k\frac{\partial}{\partial z}.\left(x i + y j + z k\right)$$

$$= \frac{\partial x}{\partial x}(i.i) + \frac{\partial y}{\partial y}(\mathrm{j.j}) + \frac{\partial z}{\partial z}(\mathrm{k.k})$$

$$= 1 + 1 + 1 = 3$$

Example 6: Show that $curl\,\vec{r} = 0$.

Answer: $curl\,\vec{r} = \nabla\times\vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(xi + yj + zk)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0i + 0j + 0k = 0$$

Hence, we can say that $curl\,\vec{r} = 0$

Example 7: If $f = x^2yi - 2xz\,j + 2yz\,k$, then find the following:

(i) $div\,\vec{f}$ **(ii)** $curl\,\vec{f}$ **(iii)** $curl\,curl\,\vec{f}$

Answer: (i) We have given that, $f = x^2yi - 2xz\,j + 2yk$

Then, $div\,\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(x^2yi - 2xz\,j + 2yz\,k\right)$

$$= i\frac{\partial}{\partial x}.\left(x^2yi - 2xz\,j + 2yz\,k\right) + j\frac{\partial}{\partial y}.\left(x^2yi - 2xz\,j + 2yz\,k\right) + k\frac{\partial}{\partial z}.\left(x^2yi - 2xz\,j + 2yz\,k\right)$$

$$= 2xy + 0 + 2y = 2y(x+1)$$

(ii) $curl\,\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times\left(x^2yi - 2xz\,j + 2yz\,k\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = i\left(\frac{\partial}{\partial y}2yz + \frac{\partial y}{\partial z}2xz\right) - j\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial z}x^2y\right) + k\left(-\frac{\partial}{\partial x}2xz - \frac{\partial}{\partial y}x^2y\right)$$

$$= i\left(2z + 2x\right) + k\left(-2z - x^2\right) = 2(x+z)i - (2z + x^2)k$$

(iii) $curl\,curl\,\vec{f} = \nabla\times\left(\nabla\times\vec{f}\right) = \nabla\times\left[2(x+z)i - (2z+x^2)k\right]$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(x+z) & 0 & -(2z+x^2) \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-2z - x^2)\right) - \left[\frac{\partial}{\partial x}(-2z - x^2) - \frac{\partial}{\partial z}(2x + 2z)\right]j + \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right]k$$

$$= 0i - (-2x - 2)\,j + (0-0)k = (2x+2)\,j$$

Example 8: If the vector, $V=(x+3y)i+(y-2z)\,j+(x+az)k$, is solenoidal then find the constant a.

Answer: As we know that a vector will be called solenoidal if $divV=0$.

Now, $divV=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left[(x+3y)i+(y-2z)\,j+(x+az)k\right]=0$

$\Rightarrow\frac{\partial}{\partial x}(x+3y)i.i+\frac{\partial}{\partial y}(y-2z)\,j.j+\frac{\partial}{\partial z}(x+az)k.k=0\ \left[\because i.i=j.j=k.k=0\right]$

$\Rightarrow 1+1+a=0$

$\Rightarrow a=-2$

Example 9: If the vector, $V=(\sin y+z)i+(x\cos y-z)\,j+(x-y)k$, then show that V is irrotational.

Answer: As we know that a vector will be called irrotational if $curlV=0$
.

Now, $curlV=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left[(\sin y+z)i+(x\cos y-z)\,j+(x-y)k\right]$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y+z & x\cos y-z & x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]i-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]j+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]k$$

$=(-1+1)i-(1-1)\,j+\left(\cos y-\cos y\right)k=0$

Hence, $curlV=0$, which shows the vector V is irrotational.

Example 10: Show that, $\nabla^2\left(\frac{x}{r^3}\right)=0$.

Answer: As we know that, $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

So, $\nabla^2\left(\frac{x}{r^3}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)$

First, we evaluate, $\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right]=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}$$

$$\left[\because r^2=x^2+y^2+z^2, then\frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}$$

Again, $\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\}$

$$\left[\because r^2=x^2+y^2+z^2, then\frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left\{-\frac{3xy}{r^5}\right\}=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}$$

Similarly, we evaluate, $\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$

Therefore,

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{9x}{r^5}+\frac{15x^3}{r^7}+\left(-\frac{3x}{r^5}+\frac{15xy^2}{r^7}\right)+\left(-\frac{3x}{r^5}+\frac{15xz^2}{r^7}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}--\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}\left(x^2+y^2+z^2\right)$$

$$==-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0$$

Hence, $\nabla^2\left(\frac{x}{r^3}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=0$

SOME IMPORTANT RESULTS ON VECTOR IDENTITIES:

1. Prove that $div\left(A+B\right)=div\,A+div\,B$ or $\nabla.\left(A+B\right)=\nabla.A+\nabla.B$

Proof: $div\left(A+B\right)=\nabla.\left(A+B\right)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(A+B\right)$

$$=i.\frac{\partial}{\partial x}\left(A+B\right)+j.\frac{\partial}{\partial y}\left(A+B\right)+k.\frac{\partial}{\partial z}\left(A+B\right)$$

$$= i.\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial x}\right)+ j.\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial y}\right)+k.\left(\frac{\partial A}{\partial z}+\frac{\partial B}{\partial z}\right)$$

$$= i.\frac{\partial A}{\partial x}+i.\frac{\partial B}{\partial x}+ j.\frac{\partial A}{\partial y}+ j.\frac{\partial B}{\partial y}+k.\frac{\partial A}{\partial z}+k.\frac{\partial B}{\partial z}$$

$$=\left(i.\frac{\partial A}{\partial x}+ j.\frac{\partial A}{\partial y}+k.\frac{\partial A}{\partial z}\right)+\left(i.\frac{\partial B}{\partial x}+ j.\frac{\partial B}{\partial y}+k.\frac{\partial B}{\partial z}\right)$$

$$=\nabla.\mathrm{A}+\nabla.B = div\, A+div\, B$$

2. Prove that $curl\left(A+B\right)=curl\, A+curl\, B$ or $\nabla\times\left(A+B\right)=\nabla\times A+\nabla\times B$

Proof: $curl\left(A+B\right)=\nabla\times\left(A+B\right)=\left(i\frac{\partial}{\partial x}+ j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(A+B\right)$

$$= i\times\frac{\partial}{\partial x}\left(A+B\right)+ j\times\frac{\partial}{\partial y}\left(A+B\right)+k\times\frac{\partial}{\partial z}\left(A+B\right)$$

$$= i\times\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial x}\right)+ j\times\left(\frac{\partial A}{\partial y}+\frac{\partial B}{\partial y}\right)+k\times\left(\frac{\partial A}{\partial z}+\frac{\partial B}{\partial z}\right)$$

$$= i\times\frac{\partial A}{\partial x}+i\times\frac{\partial B}{\partial x}+ j\times\frac{\partial A}{\partial y}+ j\times\frac{\partial B}{\partial y}+k\times\frac{\partial A}{\partial z}+k\times\frac{\partial B}{\partial z}$$

$$=\left(i\times\frac{\partial A}{\partial x}+ j\times\frac{\partial A}{\partial y}+k\times\frac{\partial A}{\partial z}\right)+\left(i\times\frac{\partial B}{\partial x}+ j\times\frac{\partial B}{\partial y}+k\times\frac{\partial B}{\partial z}\right)$$

$$=\nabla\times\mathrm{A}+\nabla\times B = curl\, A+curl\, B$$

3. If A and ϕ are differentiable vector and scalar function respectively, then

$$div\left(\phi A\right)=\left(grad\,\phi\right).A+\phi\, divA \;\; or \;\; \nabla.\left(\phi A\right)=\left(\nabla\phi\right).A+\phi\left(\nabla.A\right)$$

Proof: As we know, $div\left(\phi A\right)=\nabla.\left(\phi A\right)=\left(i\frac{\partial}{\partial x}+ j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(\phi A\right)$

$$= i.\frac{\partial}{\partial x}\left(\phi A\right)+ j.\frac{\partial}{\partial y}\left(\phi A\right)+k.\frac{\partial}{\partial z}\left(\phi A\right)=\sum\left\{i.\left(\frac{\partial}{\partial x}\left(\phi A\right)\right)\right\}$$

$$=\sum\left\{i.\left(\frac{\partial\phi}{\partial x}A+\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$=\sum\left\{i.\left(\frac{\partial\phi}{\partial x}A\right)\right\}+\sum\left\{i.\left(\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$=\sum\left\{\left(\frac{\partial\phi}{\partial x}i\right).\mathrm{A}\right\}+\sum\left\{\phi\left(i.\frac{\partial A}{\partial x}\right)\right\}\left[\because a.(mb)=(ma).b=m(a.b)\right]$$

$$=\left\{\sum\left(\frac{\partial\phi}{\partial x}i\right)\right\}.A+\phi\sum\left(i.\frac{\partial A}{\partial x}\right)=(\nabla\phi).A+\phi(\nabla.\mathrm{A})$$

4. If A and ϕ are differentiable vector and scalar function respectively, then

$$curl(\phi A)=(grad\,\phi)\times A+\phi\,curlA \;\; or \;\; \nabla\times(\phi A)=(\nabla\phi)\times A+\phi(\nabla\times A)$$

Proof: As we know, $curl(\phi A)=\nabla\times(\phi A)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(\phi A)$

$$=i\times\frac{\partial}{\partial x}(\phi A)+j\times\frac{\partial}{\partial y}(\phi A)+k\times\frac{\partial}{\partial z}(\phi A)=\sum\left\{i\times\left(\frac{\partial}{\partial x}(\phi A)\right)\right\}$$

$$=\sum\left\{i\times\left(\frac{\partial\phi}{\partial x}A+\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$=\sum\left\{i\times\left(\frac{\partial\phi}{\partial x}A\right)\right\}+\sum\left\{i\times\left(\phi\frac{\partial A}{\partial x}\right)\right\}$$

$$=\sum\left\{\left(\frac{\partial\phi}{\partial x}i\right)\times\mathrm{A}\right\}+\sum\left\{\phi\left(i\times\frac{\partial A}{\partial x}\right)\right\}$$

$[\because a\times(mb)=(ma)\times b=m(a\times b)]$

$$=\left\{\sum\left(\frac{\partial\phi}{\partial x}i\right)\right\}\times A+\phi\sum\left(i\times\frac{\partial A}{\partial x}\right)=(\nabla\phi)\times A+\phi(\nabla\times\mathrm{A})$$

5. Prove that

$$div(A\times B)=B.curl\,A-A.curl\,B \;\; or \;\; \nabla.(A\times B)=B.(\nabla\times A)-A.(\nabla\times B)$$

Proof: As we know that, $div(A\times B)=\nabla.(A\times B)=\sum\left\{i.\frac{\partial}{\partial x}(A\times B)\right\}$

$$=\sum\left\{i.\left(\frac{\partial A}{\partial x}\times B+A\times\frac{\partial B}{\partial x}\right)\right\}=\sum\left\{i.\left(\frac{\partial A}{\partial x}\times B\right)\right\}+\sum\left\{i.\left(A\times\frac{\partial B}{\partial x}\right)\right\}$$

$$=\sum\left\{\left(i\times\frac{\partial A}{\partial x}\right).\mathrm{B}\right\}-\sum\left\{i.\left(\frac{\partial B}{\partial x}\times A\right)\right\}$$

$[\because a.(b\times c)=(\mathrm{a}\times\mathrm{b}).\mathrm{c} \text{ and } \mathrm{a}.(\mathrm{b}\times\mathrm{c})=-\mathrm{a}.(\mathrm{c}\times\mathrm{b})]$

$$=\left\{\sum\left(i\times\frac{\partial A}{\partial x}\right)\right\}.B-\sum\left\{\left(i\times\frac{\partial B}{\partial x}\right).A\right\}$$

$$=(curl\,A).B-\left\{\sum\left(i\times\frac{\partial B}{\partial x}\right)\right\}.A=(curl\,A).B-A.curl\,B$$

6. Prove that $curl(A\times B)=(B.\nabla)A-B\,div\,A-(A.\nabla)B+A\,div\,B$

Proof: As we know that, $curl\left(A\times B\right)=\nabla\times\left(A\times B\right)=\sum\left\{i\times\dfrac{\partial}{\partial x}\left(A\times B\right)\right\}$

$$=\sum\left\{i\times\left(A\times\frac{\partial B}{\partial x}+\frac{\partial A}{\partial x}\times B\right)\right\}=\sum\left\{i\times\left(A\times\frac{\partial B}{\partial x}\right)\right\}+\sum\left\{i\times\left(\frac{\partial A}{\partial x}\times B\right)\right\}$$

$$=\sum\left\{\left(i.\frac{\partial B}{\partial x}\right)A\right\}-\sum\left\{\left(A.i\right)\frac{\partial B}{\partial x}\right\}+\sum\left\{\left(B.i\right)\frac{\partial A}{\partial x}\right\}-\sum\left\{\left(i.\frac{\partial A}{\partial x}\right)B\right\}$$

$$=\left\{\sum\left(i.\frac{\partial B}{\partial x}\right)\right\}A-\left\{A.\sum i\frac{\partial}{\partial x}\right\}B+\left\{B.\sum i\frac{\partial}{\partial x}\right\}A-\left\{\sum\left(i.\frac{\partial A}{\partial x}\right)\right\}B$$

$$=\left(div\,B\right)A-\left(A.\nabla\right)B+\left(B.\nabla\right)A-\left(div\,A\right)B$$

7. Prove that $grad\left(A.B\right)=(B.\nabla)\mathrm{A}+(\mathrm{A}.\nabla)+\mathrm{B}\times\mathrm{curl\,A}+\mathrm{A}\times\mathrm{curl\,B}$

Proof: We have,

$$grad\left(A.B\right)=\nabla\left(A.B\right)=\sum i\frac{\partial}{\partial x}\left(A.B\right)=\sum i\left(A.\frac{\partial B}{\partial x}+\frac{\partial A}{\partial x}.B\right)$$

$$=\sum\left\{\left(A.\frac{\partial B}{\partial x}\right)i\right\}+\sum\left\{\left(B.\frac{\partial A}{\partial x}\right)i\right\} \qquad \text{…. (1)}$$

Since we know that,

$a\times\left(b\times c\right)=\left(a.c\right)b-\left(a.b\right)c\Rightarrow\left(a.b\right)c=\left(a.c\right)b-a\times\left(b\times c\right)$

Thus, $\sum\left\{\left(A.\dfrac{\partial B}{\partial x}\right)i\right\}=\sum\left\{\left(A.i\right)\dfrac{\partial B}{\partial x}\right\}+\sum\left\{A\times\left(i\times\dfrac{\partial B}{\partial x}\right)\right\}$

$$=\left\{A.\sum i\frac{\partial}{\partial x}\right\}B+A\times\sum\left(i\times\frac{\partial B}{\partial x}\right)=\left(A.\nabla\right)B+A\times\left(\nabla\times B\right) \qquad \text{…. (2)}$$

Similarly, $\sum\left\{\left(\mathrm{B}.\dfrac{\partial A}{\partial x}\right)i\right\}=\left(B.\nabla\right)A+B\times\left(\nabla\times A\right)$ ….

(3)

Now, putting the value of equation (2) and equation (3) in equation (1)

We get, $grad\,(A.B)=\left(A.\nabla\right)B+A\times\left(\nabla\times B\right)+\left(B.\nabla\right)A+B\times\left(\nabla\times A\right)$

8. Prove that $div\,grad\phi=\nabla^2\phi$ $i.e.,\nabla.\left(\nabla\phi\right)=\nabla^2\phi$

Proof: $div\,grad\phi=\nabla.\nabla\phi=\left(i\dfrac{\partial}{\partial x}+j\dfrac{\partial}{\partial y}+k\dfrac{\partial}{\partial z}\right).\left(i\dfrac{\partial\phi}{\partial x}+j\dfrac{\partial\phi}{\partial y}+k\dfrac{\partial\phi}{\partial z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)i.i+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)j.j+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)k.k$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi$$

9. Prove that, curl of the gradient of ϕ is zero, i.e.,

$\nabla\times(\nabla\phi)=0 \;\; i.e., curl\ grad\ \phi=0$

Proof: $curl\ grad\ \phi=\nabla\times(\nabla\phi)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right)$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}=\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)i+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)j+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)k$$

$-0i+0j+0k=0$

10. Prove that $div Curl\ A=0 \;\; i.e, \nabla.(\nabla\times A)=0$

Proof: Let $A=A_1 i+A_2 j+A_3 k$, Then

$$div Curl\ A=\nabla.(\nabla\times A)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left\{\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(A_1 i+A_2 j+A_3 k)\right\}$$

First we find out, $\nabla\times A=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(A_1 i+A_2 j+A_3 k)$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3\end{vmatrix}=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k$$

Now,

$$div Curl\ A=\nabla.(\nabla\times A)=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left\{\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)i+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)j+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)k\right\}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)(i.i)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)(j.j)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)(k.k)$$

$$=\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}$$

$$=\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_1}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_2}{\partial x\partial z}\right)=0$$

SOLVED EXAMPLE

Example 11: Find $\nabla\phi$ and $|\nabla\phi|$, where $\phi=\left(x^2+y^2+z^2\right)e^{\left(x^2+y^2+z^2\right)^{1/2}}$.

Solution: Let $r=xi+yj+zk$, then $r^2=x^2+y^2+z^2$.

So, we can write $\phi=r^2e^{-r}$, then $\nabla\phi=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$

We consider, $\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x}=\left(2re^{-r}-r^2e^{-r}\right)\frac{\partial r}{\partial x}=\left(2re^{-r}-r^2e^{-r}\right)\frac{x}{r}$

Similarly, $\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y}=\left(2re^{-r}-r^2e^{-r}\right)\frac{y}{r}$ and

$\frac{\partial\phi}{\partial z}=\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z}=\left(2re^{-r}-r^2e^{-r}\right)\frac{r}{z}$

Now, $\nabla\phi=\left(2re^{-r}-r^2e^{-r}\right)\frac{x}{r}i+\left(2re^{-r}-r^2e^{-r}\right)\frac{y}{r}j+\left(2re^{-r}-r^2e^{-r}\right)\frac{z}{r}k$

$$\nabla\phi=\left(2re^{-r}-r^2e^{-r}\right)\frac{\vec{r}}{r}=(2-r)e^{-r}\vec{r}$$

And $|\nabla\phi|=\left|\left(2re^{-r}-r^2e^{-r}\right)\frac{\vec{r}}{r}\right|=(2-r)e^{-r}\left|\vec{r}\right|=(2-r)re^{-r}$

Example 12: Prove that $div\left(r^n\vec{r}\right)=(n+3)r^n$.

Solution: We know that, $div\left(\phi\vec{A}\right)=\phi\left(div\vec{A}\right)+\vec{A}.\text{grad}\,\phi$

So, $div\left(r^n\vec{r}\right)=r^n\left(div\,\vec{r}\right)+\vec{r}.\text{grad}\,r^n$

Since,

$$div\,\vec{r}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(xi+yj+zk)=\frac{\partial}{\partial x}x(i.i)+\frac{\partial}{\partial y}\text{y(j. j)}+\frac{\partial}{\partial z}\text{z(k.k)}=3$$

And

$$\text{grad}\,r^n=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)r^n=i\frac{\partial}{\partial x}r^n+j\frac{\partial}{\partial y}r^n+k\frac{\partial}{\partial z}r^n=nr^{n-1}\left(\frac{\partial r}{\partial x}i+\frac{\partial r}{\partial y}j+\frac{\partial r}{\partial z}k\right)$$

$$=nr^{n-1}\left(\frac{x}{r}i+\frac{y}{r}i+\frac{z}{r}i\right)=nr^{n-1}\frac{\vec{r}}{r}$$

So, $\text{grad}\,r^n=nr^{n-2}\vec{r}$

Now,

$$div\left(r^n \vec{r} \right) = r^n \left(div\, \vec{r} \right) + \vec{r}\,.\text{grad}\, r^n = 3r^n + nr^{n-2}\, \vec{r}\,.\,\vec{r} = 3r^n + nr^n = \left(3+n\right) r^n$$

Example 13: Prove that $div\left(\dfrac{\vec{r}}{r^3} \right) = 0$.

Proof: As we know that, $div\left(r^n \vec{r} \right) = \left(n+3\right) r^n$ .... (1)

Now putting n = -3 in equation (1)

So, $div\left(r^{-3} \vec{r} \right) = \left(3-3\right) r^{-3} = 0$

Example 14: Prove that $div\left(\hat{r} \right) = \dfrac{2}{r}$.

Proof: We have $div\left(\dfrac{\vec{r}}{r} \right) = div\left(r^{-1} \vec{r} \right)$

As we know that, $div\left(r^n \vec{r} \right) = \left(n+3\right) r^n$ .... (1)

Now putting n = -1 in equation (1)

So, $div\left(\hat{r} \right) = div\left(r^{-1} \vec{r} \right) = \left(-1+3\right) r^{(-1)} = \dfrac{2}{r}$

Example 15: Prove that vector $f(r)\vec{r}$ is irrotational.

Proof: As we know that any vector $\vec{A}$ is irrotational if $curl\, \vec{A} = 0$

So, if we have to show that the vector $f(r)\vec{r}$ is irrotational we have to show

$$curl\left[f(r)\vec{r} \right] = 0$$

Since, $curl\left(\phi \vec{A} \right) = \left(grad\, \phi \right) \times A + \phi\, curl\, A$

Now, $curl\left[f(r)\vec{r} \right] = \left[grad\ f(r) \right] \times \vec{r} + f(r)\, curl\, \vec{r}$

$$= \left[f^{'}(r) grad\, \vec{r} \right] \times \vec{r} + f(r).0 \qquad \left[\because curl\, \vec{r} = 0 \right]$$

$$= \left[f'(r)\frac{\vec{r}}{r} \right] \times \vec{r} = f'(r)\frac{1}{r}\left(\vec{r} \times \vec{r}\right) = 0$$

Example 16: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Proof: As we know that if ϕ is a scalar function, then $\nabla^2\phi = \nabla.\left(\nabla\phi\right)$

So, $\nabla^2 f(r) = \nabla.\left(\nabla f(r)\right) = div\{grad\ f(r)\} = div\left\{ f'(r)\,\mathrm{grad}\,\mathrm{r}\right\}$

$$= div\left\{\frac{1}{r} f'(r)\vec{r}\right\} = \frac{1}{r} f'(r) div\,\vec{r} + \vec{r}.grad\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \vec{r}.\left[\frac{d}{dr}\left\{\frac{1}{r} f'(r)\right\} grad\, r\right] = \frac{3}{r} f'(r) + \vec{r}.\left[\left\{\frac{-1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right]\left(\vec{r}.\vec{r}\right)$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r)$$

Example 17: If $\nabla^2 f(r) = 0$, show that $f(r) = \frac{c_1}{r} + c_2$, where $r^2 = x^2 + y^2 + z^2$ and c_1, c_2 are arbitrary constant.

Answer: From the previous example we know that,

$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$, where $r^2 = x^2 + y^2 + z^2$

Since, we have given that $\nabla^2 f(r) = 0$, then $f''(r) + \frac{2}{r} f'(r) = 0$

$$\Rightarrow \frac{f''(r)}{f'(r)} = -\frac{2}{r}$$

Integrating with respect to r we get, $\log f'(r) = -2\log r + \log c = \log\frac{c}{r^2}$

$$\Rightarrow f'(r) = \frac{c}{r^2}$$

Again, integrating we get, $f(r) = -\frac{c}{r} + c_2$, where c_2 is a constant

After replacing constant $-c$ by c_1, we get $f(r) = \frac{c_1}{r} + c_2$

SELF CHECK QUESTION

Fill in the blanks:

1. If $F=(x^2+y^2)i-2xy\,j$, then $F.d\vec{r}=\ldots\ldots$

2. If $P=e^{xy}i+(x-2y)\,j+(x\sin y)k,$ then $\dfrac{\partial P}{\partial x}=\ldots\ldots\ldots$

3. If $\vec{a}$ is a constant vector then $grad\,(\vec{a}.\vec{r})=\ldots\ldots\ldots$

4. If $\vec{a}$ is a constant vector then $\nabla.(\vec{a}\times\vec{r})=\ldots\ldots\ldots$

5. If $\vec{r}=xi+yj+zk$, then the value of $div\,\vec{r}=$

6. If $\vec{A}=x^2zi-2y^3z^2\,j+xy^2zk$, then $div\,\vec{A}$ at (1, -1, 1) = ………

7. If $\vec{r}=xi+yj+zk$, then the value of $curl\,\vec{r}=\ldots\ldots\ldots$

8. For any vector $\vec{A}$, $div\,curl\,\vec{A}=\ldots\ldots$

9. A vector $\vec{V}$ is said to be solenoidal if ……

10. A vector $\vec{F}$ is said to be irrotational if ……

11. If $\phi=x^2y+2xy+z^2$, then $curl\,grad\,\phi=\ldots\ldots$

13.8 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of gradient, divergence, curl operators and Laplacian operators.
- The relations between gradient, divergence and curl operators.

13.9 *GLOSSARY*

- **Gradient of a vector function *f*:** ∇f
- **Divergence of a vector function *f*:** $\nabla.f$
- **Curl of a vector function *f*:** $\nabla\times f$

13.10 *REFERENCES*

- Spiegel, R. Murray (1959), *Vector Analysis*, Schaum's Outline Series.

- N. Saran and S. N. Nigam, *Introduction to vector analysis*, Pothishala Pvt. Ltd. Allahabad.
- Erwin. Kreyszig, "Advanced engineering mathematics, 10th eddition", 2009.
- A. R. Vasishtha, "Vector Calculus", 20th edition, Krishna publication, 2020.

13.11 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

13.12 *TERMINAL QUESTION*

Objective type question:

1. What will be the value of constant ***a***, if the vector

$\vec{V} = (x+3y)i + (y-2z)j + (x+az)k$ is solenoidal?

a. 0 **b.** 1

c. -2 **d.** 2

2. What will be the value of directional derivative of

$\phi(x,y,z) = x^2yz + 4xz^2$, at the point (1, -2, -1) in the direction of the vector $2i - j - 2k$.

a. 37/3 **b.** 1

c. -2 **d.** 2

3. Choose the correct value of $\nabla^2 r^n$, where $\vec{r} = xi + yj + zk$ and $r = \left|\vec{r}\right|$

a. $n(n+1)r^n$ **b.** $n(n+1)r^{n-1}$

c. $n(n+1)r^{n-2}$ **d.** none of these

4. Choose the correct value of $\nabla^2\left(\frac{1}{r}\right)$

a. $-2/r^3$ **b.** $2/r^3$

c. 0 **d.** none of these

5. Choose the correct value of $curl\,(\vec{r}\times\vec{a})$, where $\vec{r} = xi + yj + zk$ and $\vec{a}$ is constant vector.

a. $-\vec{a}$ **b.** $-2\vec{a}$

b. $-3\vec{a}$ **d.** none of these

6. Choose the correct value of $div\,\hat{r}$ is

a. $\frac{2}{r}$ **b.** $\frac{1}{r}$

c. 0 **d.** none of these

Find True and False Statement.

1. The vector $\vec{r} = xi + yj + zk$ is solenoidal

2. $div\vec{V} = 0$, if $\vec{V}$ is a constant vector.

3. $\vec{F} = 2xyzi + y^2zj - 2yz^2k$, is irrotational vector.

4. $curl\ grad\ \phi = 0$, if ϕ is a differential scalar function.

5. $div\ grad\ \phi = \nabla^2\phi$, if ϕ is a differential scalar function.

6. $\nabla.(A\times B) = A.(\nabla\times B) - B.(\nabla\times A)$

7. Function which satisfies the Laplace's equation is called harmonic function.

Short answer type question:

1. If $f = x^2y + 2xy + z^2$, then verify that $curl\ grad\ f = 0$

2. Show that $curl\,(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$ further prove that,

$\nabla\psi\times\nabla\phi = -curl\,(\phi\nabla\psi)$

3. If $\vec{a}$ is a constant vector then show that $curl\,(\vec{a}.\vec{r})\,\vec{a} = 0$

4. For the constant vector $\vec{a}$ prove the followings:

(i) $\nabla\left(\vec{a}.\vec{u}\right) = \left(\vec{a}.\nabla\right)\vec{u} + \vec{a}\times curl\,\vec{u}$

(ii) $\nabla.\left(\vec{a}\times\vec{u}\right)=-\vec{a}.curl\,\vec{u}$

(iii) $\nabla\times\left(\vec{a}\times\vec{u}\right)=\vec{a}\,div\,\vec{u}-\left(\vec{a}.\nabla\right)\vec{u}$

5. If $\vec{a}$ is a constant unit vector then show that

$\vec{a}.\left\{\nabla\left(\vec{v}.\vec{a}\right)-\nabla\times\left(\vec{v}\times\vec{a}\right)\right\}=div\,\vec{v}$

6. If $\vec{a}$ is a constantvector then show that $curl\,\vec{a}\,\phi(r)=\frac{1}{r}\phi^{'}(r)\,\vec{r}\times\vec{a}$

Long answer type question:

1. If $\vec{a}$ is a constantvector then show that,

$$curl\,r\left[r^n(\vec{a}\times\vec{r})\right]=(n+2)r^n\,\vec{a}-nr^{n-2}\left(\vec{r}.\vec{a}\right)\vec{r}$$

2. If $\nabla^2 f(r)=0$ show that $f(r)=c_1\log r+c_2$, where $r^2=x^2+y^2+z^2$, where c_1,c_2 are arbitrary constant.

3. Show that $\frac{1}{2}\nabla\vec{a}^2=\left(\vec{a}.\nabla\right)\vec{a}+\vec{a}\times curl\,\vec{a}$

4. Show that $\nabla^2\left[\nabla.\left(\frac{\vec{r}}{r^2}\right)\right]=2r^{-4}$

5. If $\vec{a}$ is a constant vector then find the value of $div\left\{\vec{a}\times\left(\vec{r}\times\vec{a}\right)\right\}$

6. Find $grad\left(div\,\vec{u}\right)$, where $\vec{u}=\left(1/r\right)\vec{r}$

13.13 *ANSWERS*

Answer of self cheque questions:

1. $\left(x^2+y^2\right)dx-2xydy$ **2.** $ye^{xy}i+j+\sin y\,k$

3. $\vec{a}$

4. 0 **5.** 3

6. -3

7. 0 **8.** 0

9. $div\vec{V} = 0$

10. $curl\,\vec{F} = 0$ **11.** 0

Answer of objective questions:

1. c **2.** a

3. c

4. c **5.** b

6. a

Answer of true and false questions:

1. F **2.** T

3. F

4. T **5.** T

6. F

7. T

Answer of long answer type questions:

Answers: **5.** $2a^2$ **6.** $-\frac{2}{r^3}\vec{r}$

UNIT-14: GREEN'S, GAUSS'S AND STOKE'S THEOREMS

CONTENTS:

14.1 INTRODUCTION

Green's theorem is mainly used for the integration of the line combined with a curved plane. This theorem shows the relationship between a line integral and a surface integral. It is related to many theorems such as Gauss theorem, Stokes theorem. Green's theorem is used to integrate the derivatives in a particular plane. If a line integral is given, it is converted into a surface integral or the double integral or vice versa using this theorem. In this unit, we will going to learn what is Green's theorem, its statement, formula, applications and examples in detail.

This unit finally begins to deliver on why we introduced div grad and curl. Two theorems, both of them over two hundred years old, are explained: Gauss' Theorem enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa. Why would we want to do that? Computational efficiency and/or numerical accuracy! Stokes' Law enables an integral taken around a closed curve to be replaced by one taken over any surface bounded by that curve.

14.2 *OBJECTIVES*

After reading this unit learners will be able to

- Memorized about the introduction of vector functions, line integrals, surface integrals and volume integrals.
- Analyze about the Green's theorem and applications of Green's theorem.
- Analyze about the Gauss divergence theorem and applications of this theorem.
- Analyze about the Stoke's theorem and applications of Stokes's theorem.

14.3 INTRODUCTION OF VECTOR FUNCTIONS

We shall usually define integration as the reverse process of differentiation. Let two vector functions $f(t)$ and $\mathrm{F}(t)$ of the scalar function t such that

$$\frac{d}{dt}\mathrm{F}(t) = f(t)$$

Here, $\mathrm{F}(t)$ is called the indefinite integral of $f(t)$ with respect to t and symbolically we denote

$$\int f(t)dt = \mathrm{F}(t) \qquad \ldots\ldots. (1)$$

The function $f(t)$ which to be integrated is called integrand. If c is arbitrary constant vector independent from t, then

$$\frac{d}{dt}\{\mathrm{F}(t)+c\} = f(t)$$

Which will equivalent to, $\int f(t)dt = \mathrm{F}(t)+\mathrm{c}$ ……. (2)

From above equation (2) it is obvious that the integral $\mathrm{F}(t)$ of $f(t)$ is indefinite to the extent of an additive arbitrary constant c. Therefore $\mathrm{F}(t)$ is called the indefinite integral of $f(t)$.

If $\frac{d}{dt}\text{F}(t) = f(t)$ for all values of t in the interval $[a,b]$, then the definite integral between the limits $t = a$ and $t = b$ can be defined as,

$$\int_a^b f(t)dt = \int_a^b \left\{\frac{d}{dt}\text{F}(t)\right\}dt = \left[F(t)+c\right]_a^b = F(b) - F(a)$$

Some important rule of integration (without proof)

1. If $f(t) = f_1(t)i + f_2(t)\,\text{j} + f_3(t)\,\text{k}$, then

$$\int f(t)dt = i\int f_1(t)dt + j\int f_2(t)dt + k\int f_3(t)dt$$

2. We have $\frac{d}{dt}(r.s) = \frac{dr}{dt}.s + r.\frac{ds}{dt}$ therefore,

$$\int\left(\frac{dr}{dt}.s + r.\frac{ds}{dt}\right)dt = r.s + c$$

3. We have $\frac{d}{dt}\left(\frac{dr}{dt}\right)^2 = 2\frac{dr}{dt}.\frac{d^2r}{dt^2}$ therefore,

$$\int\left(2\frac{dr}{dt}.\frac{d^2r}{dt^2}\right)dt = \left(\frac{dr}{dt}\right)^2 + c$$

4. $\left(\frac{dr}{dt}\right)^2 = \frac{dr}{dt}.\frac{dr}{dt}$

5. We have $\frac{d}{dt}\left(r\times\frac{dr}{dt}\right) = \frac{dr}{dt}\times\frac{dr}{dt} + r\times\frac{d^2r}{dt^2} = r\times\frac{d^2r}{dt^2}$ therefore,

$$\int\left(r\times\frac{d^2r}{dt^2}\right)dt = r\times\frac{dr}{dt} + c$$

6. We have $\frac{d}{dt}(a\times r) = \frac{da}{dt}\times r + a\times\frac{dr}{dt} = a\times\frac{dr}{dt}$ therefore,

$$\int\left(a\times\frac{dr}{dt}\right)dt = a\times r + c$$

7. We have $\frac{d}{dt}\left(\hat{r}\right) = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}$ therefore,

$$\int\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)dt = \hat{r} + c$$

8. If c is constant scalar and r a vector function of the scalar t then,

$\int cr\,dt = c\int r\,dt$

9. If r and s are two vector function of the scalar t then,

$\int (r+s)\,dt = \int r\,dt + \int s\,dt$

Solved Example

Example 1: If $f(t) = \left(t - t^2\right)i + 2t^3 j - 3k$ then find,

(i) $\int f(t)dt$ **(*ii*)** $\int_1^2 f(t)dt$

Answer (i):

$$\int f(t)dt = \int\left\{\left(t-t^2\right)i + 2t^3 j - 3k\right\}dt = i\int\left(t-t^2\right)dt + j\int 2t^3 dt + k\int -3dt$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)i + \frac{t^4}{2}j - 3tk + c$$

(ii): $$\int_1^2 f(t)dt = \int_1^2\left\{\left(t-t^2\right)i + 2t^3 j - 3k\right\}dt = i\int_1^2\left(t-t^2\right)dt + 2j\int_1^2 t^3 dt - 3k\int_1^2 dt$$

$$= \left[\left(\frac{t^2}{2} - \frac{t^3}{3}\right)i + \frac{t^4}{2}j - 3tk\right]_1^2 + c$$

$$= \left[\left(\frac{t^2}{2} - \frac{t^3}{3}\right)i + \frac{t^4}{2}j - 3tk\right]_1^2 + c$$

$$= \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_1^2 i + \left(\frac{t^4}{2}\right)_1^2 j - \left(3t\right)_1^2 k = -\frac{5}{6}i + \frac{15}{2}j - 3k$$

Example 2: Evaluate the value of r which satisfying the equation $\frac{d^2r}{dt^2} = a$, where a is a constant vector. It is given that at a time $t = 0, r = 0$ and $\frac{dr}{dt} = u$.

Answer: Given differential equation on r is $\frac{d^2r}{dt^2} = a$.

Now, integrating both side the equation with respect to t we get,

$\frac{dr}{dt} = ta + b$, here b is arbitrary constant vector.

Since it is given that at a time $t = 0, r = 0$ and $\frac{dr}{dt} = u$.

Then, $u = 0a + b$ *i.e.*, $b = u$

$\Rightarrow \frac{dr}{dt} = ta + u$

Again, integrating both side with respect to t we get.

$r = \frac{1}{2}t^2 a + tu + c$

at time $t = 0, \quad r = 0$

$\Rightarrow 0 = 0 + 0 + c$ or $c = 0$

So, $r = \frac{1}{2}t^2 a + tu$

Example 3: If $r(t) = 5t^2 i + tj - t^3 k$ then show that

$$\int_1^2 \left(r \times \frac{d^2 r}{dt^2} \right) dt = -14i + 75j - 15k$$

Answer: As we know that, $\int \left(r \times \frac{d^2 r}{dt^2} \right) dt = r \times \frac{dr}{dt} + c$

So, $\int_1^2 \left(r \times \frac{d^2 r}{dt^2} \right) dt = \left[r \times \frac{dr}{dt} \right]_1^2$

First, we evaluate $r \times \frac{dr}{dt}$.

Since, $r(t) = 5t^2 i + tj - t^3 k$ then $\frac{dr}{dt} = \frac{d}{dt}\left(5t^2 i + tj - t^3 k\right) = 10ti + j - 3t^2 k$

So, $r\times\frac{dr}{dt}=\left(5t^2i+tj-t^3k\right)\times\left(10ti+j-3t^2k\right)=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$

$$=\left(-3t^3+t^3\right)i-\left(-15t^4+10t^4\right)j+\left(5t^2-10t^2\right)k=-2t^3i+5t^4j-5t^2k$$

Now,

$$\left[r\times\frac{dr}{dt}\right]_1^2=\left[-2t^3i+5t^4j-5t^2k\right]_1^2=\left(-2t^3i+5t^4j-5t^2k\right)_{at\,t=2}-\left(-2t^3i+5t^4j-5t^2k\right)_{at\,t=1}$$

$$=\left(-16i+80j-20k\right)-\left(-2i+5j-5k\right)=-14i+75j-15k$$

Example 4: Prove that $\int_2^3\left(r.\frac{dr}{dt}\right)dt=10$, where $r(t)=\begin{cases}2i-j+2k, \text{ } if \text{ } t=2\\ 4i-2j+3k, \text{ } if \text{ } t=3\end{cases}$

Answer: We know that, $\int\left(r.\frac{dr}{dt}\right)dt=\frac{r^2}{2}+c$

So, $\int_2^3\left(r.\frac{dr}{dt}\right)dt=\left[\frac{r^2}{2}\right]_2^3$

When $t=2,\ r(t)=2i-j+2k$ then

$r^2=r.r=\left(2i-j+2k\right).\left(2i-j+2k\right)=4+1+4=9$

Similarly, $t=3,\ r(t)=4i-2j+3k$ then

$r^2=r.r=\left(4i-2j+3k\right).\left(4i-2j+3k\right)=16+4+9=29$

Thus, $\int_2^3\left(r.\frac{dr}{dt}\right)dt=\frac{1}{2}\left(29-9\right)=10$

Example 5: At a time $t\geq 0$, the acceleration of a particle is given by,

$$a=\frac{dv}{dt}=12\cos 2t\,i-8\sin 2t\,j+16t\,k$$

If at a time $t=0$, the velocity (v) and displacement (r) are zero, then find v and r at any time.

Answer: We have given that acceleration at a time is $a=\frac{dv}{dt}=12\cos 2t\,i-8\sin 2t\;j+16t\,k$

Integrating both side with respect to t we get,

$$v=\int(12\cos 2ti-8\sin 2tj+16tk)dt=\frac{12\sin 2ti}{2}+\frac{8\cos 2tj}{2}+\frac{16t^2k}{2}+c=6\sin 2ti+4\cos 2tj+8t^2k+c$$

At a time $t=0, v=0$, then $0=0+4j+0+c \Rightarrow c=-4j$

So, $v=6\sin 2ti+4\cos 2tj+8t^2k-4j=6\sin 2ti+(4\cos 2t-4)\,j+8t^2k$

Since, $v=\frac{dr}{dt}=6\sin 2ti+(4\cos 2t-4)\,j+8t^2k$

Integrating both side with respect to t we get,

$$r=\int\left(6\sin 2ti+(4\cos 2t-4)\,j+8t^2k\right)dt+d=-\frac{6\cos 2ti}{2}+\left(\frac{4\sin 2\mathrm{t}}{2}-4t\right)j+\frac{8t^3k}{3}+d$$

$$=-3\cos 2ti+(2\sin 2\mathrm{t}-4t)\,j+\frac{8}{3}t^3k+d$$

At a time $t=0, r=0$, then $0=-3+0+0+d \Rightarrow d=3i$

$$r=-3\cos 2ti+(2\sin 2\mathrm{t}-4t)\,j+\frac{8}{3}t^3k+3i$$

$$r=(3-3\cos 2t)i+(2\sin 2\mathrm{t}-4t)\,j+\frac{8}{3}t^3k$$

14.4 LINE INTEGRAL

An integral which is to be evaluated along a curve is called a line integral.

Let $r(t)=x(t)i+y(t)\,j+z(t)k$, be a position vector of (x,y,z) *i.e.*, $r=xi+y\,j+zk$, defines a piecewise smooth curve joining two points *A* and *B*. Let at the time $t=t_1$ point be at *A* and at the time $t=t_2$ point is at *B*. Suppose $F(x,y,z)=F_1i+F_2\,j+F_3k$ is a vector point function defined and continuous along *C*. If *s* denotes the arc length of the curve *C*, then

$\dfrac{dr}{ds} = t$ is a unit vector along the tangent to the curve *C* at the point *r*. The component of the vector *F* along the tangent is $F.\dfrac{dr}{ds}$. The integral of $F.\dfrac{dr}{ds}$ along *C* from *A* to *B* written as

$$\int_A^B \left[F.\frac{dr}{ds} \right] ds = \int_A^B F.dr = \int_C F.dr$$

is an example of a line integral. It is called the tangent line integral of *F* along *C*.

Remarks:

1. If the equation of the curve *C* given in the parametric form *i.e.*, $x = x(t), y = y(t), z = z(t)$

thus, we may write $\int_C F.dr = \int_{t=t_1}^{t=t_2} \left[F_1 \dfrac{dx}{dt} + F_2 \dfrac{dy}{dt} + F_3 \dfrac{dz}{dt} \right] dt$.

2. If *r* is the position vector of a point in *C* and let *F* be the force acting on the particle. Then the work done (*W*) by *F* in this displacement is given by the line integral,

$W = \int_C F.dr$, here *r* be taken in the sense of displacement.

14.5 SURFACE INTEGRAL

Any integral which evaluated over a surface is called a surface integral.

Let *S* is a finite surface area. Suppose $f(x, y, z)$ is a single valued function defined over *S*. If we divide the area of *S* into *m* small areas like $\delta S_1, \delta S_2, \delta S_3, ..., \delta S_m$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).

We define $f(P_k) = f(x_k, y_k, z_k)$

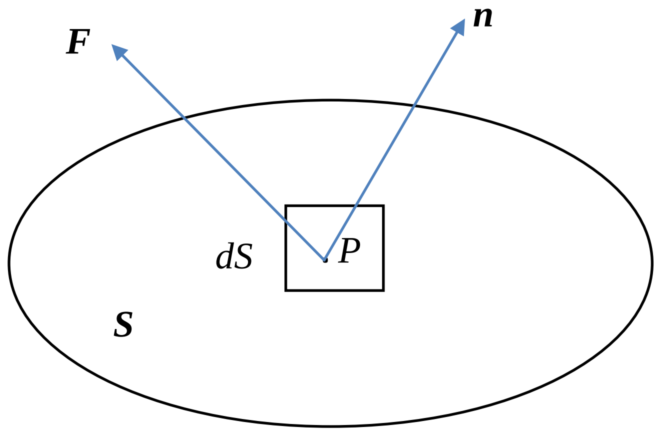
From the sum $\sum_{k=1}^{n} f(P_k) \delta S_k$

Taking limit of this sum as $n \to \infty$ in such a way that largest area δS_k approaches zero. If this limit is exist called surface integral of $f(x, y, z)$ over S and is denoted by

$$\iint_S f(x, y, z) dS$$

If the surface S is piecewise smooth and the function is continuous over S, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

Flux: Suppose a Piecewise smooth surface S and $F(x, y, z)$ is a vector function of defined position and continuous over ***S***. Let P be a point on the surface ***S*** and let n be the unit vector at P in the direction of outward drawn normal to the surface ***S*** at P. Then $F.n$ is the normal component of F at P. The integral of $F.n$ over ***S*** is,



$\iint_S F.n \ dS$, is called the flux of F over ***S*****.**

Let us associate with the differential of surface area ***dS*** a vector ***dS*** (called vector area) whose magnitude is ***dS*** and direction is that of ***n***. Then ***dS* = n** *dS*. Therefore, we can write,

$$\iint_S F.n \ dS = \iint_S F.dS$$

Let we consider at the point P the outward normal to the surface S makes the angle α, β, γ with the positive direction of x, y, z. If the direction cosines of the outward drawn normal are l, m, n, then

$$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

Also $n = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$

let $F(x, y, z) = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma = F_1 l + F_2 m + F_3 n$

Therefore, we can write

$$\iint_S F.n\ dS = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma\right) dS$$

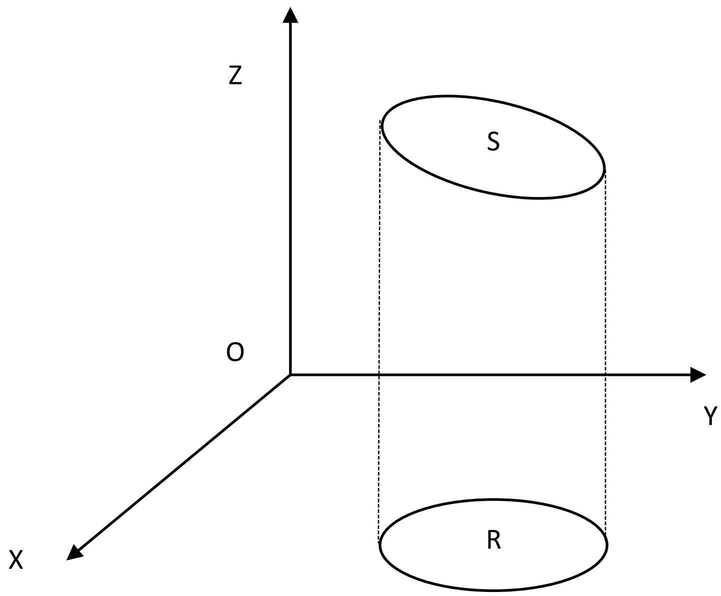
If we define $\iint_S F_1 \cos\alpha dS = \iint_S F_1 dydz$, $\iint_S F_2 \cos\beta dS = \iint_S F_2 dzdx$, $\iint_S F_3 \cos\gamma dS = \iint_S F_3 dxdy$

then,

$$\iint_S F.n\ dS = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy\right) dS$$

Note 1. Other surface integrals are $\iint_S f\ n\ dS, \iint_S F \times dS$

Note 2. Let we consider the surface S in such a way that any line perpendicular to the *xy*- plane meets S in no more than one point. Then the equation of surface S can be written in the form $z = h(x, y)$



Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle the undirected normal $\boldsymbol{n}$ at $P(x, y, z)$ to the surface $\boldsymbol{S}$ makes with z-axis, then it can be shown that

$$\cos\gamma \, dS = dx\,dy$$

where dS is the small element of area of surface S at the point P.

Therefore $dS = \dfrac{dx\,dy}{\cos\gamma} = \dfrac{dx\,dy}{|n.k|}$, where k is unit vector along z-axis.

Hence $\iint_S F.n\,dS = \iint_R F.\mathrm{n}\dfrac{dx\,dy}{|n.\mathrm{k}|}$

So, a double integral integrated over R can be used to evaluate the surface integral on S.

14.6 *VOLUME INTEGRAL*

The volume V that is enclosed by surface S. Let $f(x, y, z)$ be a single-valued positional function defined over V. Split the volume V into n volume components $\delta V_1, \delta V_2, ..., \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define

$$f(P_k) = (x_k, y_k, z_k)$$

From the sum $\sum_{k=1}^{n} f(P_k)\delta V_k$

Now taking the limit of this summation as $n \to \infty$ in a manner that the largest of the volumes δV_k approaches zero. This limit exists, is called the volume integral of $f(x, y, z)$ over *V* and is denoted by

$$\iiint_V f(x, y, z)\, dV$$

If the function $f(x, y, z)$ is continuous and surface is piecewise smooth over *V*, then the above limit exists *i.e.*, is independent of the choice of sub-division and points P_k. If V is the volume of small cuboids, then $dV = dx\, dy\, dz$, so, the will becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

If F(x, y, z) is a vector function, $\iiint_V F\, dV$ is the example of volume integral.

Solved Examples

Example 6: Find $\int_C F.dr$, where $F = x^2 i + y^3 j$ and curve *C* represents the parabola's arc $y = x^2$ in the *x*-*y* plane from (0,0) to (1,1).

Answer: Method 1. Since we have given curve *C* is parabola. First, we have to convert the equation of parabola in parametric form by putting $x = t$ and $y = t^2$.

So, $F = t^2 i + t^6 j$ and we know that $r(t) = xi + yj = ti + t^2 j$

Then $\frac{dr}{dt} = i + 2tj$

Now, $\left(F.\frac{dr}{dt}\right) dt = \left(t^2 i + t^6 j\right).\left(i + 2tj\right) dt = \left(t^2 + 2t^7\right) dt$

At the point (0,0), $t = x = 0$. At the point (1,1), $t = 1$

$$\therefore \int_C \left(F.\frac{dr}{dt}\right)dt = \int_0^1 \left(t^2 + 2t^7\right)dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Method 2. As we know that $r = xi + yj$

$\Rightarrow dr = dxi + dyj$

Thus $F.dr = \left(x^2 i + y^3 j\right).\left(dx\, i + dy\, j\right) = x^2 dx + y^3 dy$

$$\therefore \int_C F.dr = \int_C x^2 dx + y^3 dy$$

Now along the curve *C*, $y = x^2$. Therefore $dy = 2xdx$

$$\therefore \int_C F.dr = \int_0^1 \left[x^2 dx + x^6 (2x) dx\right] = \int_0^1 \left(2x^7 + x^2\right)dx$$

$$= \left[\frac{2x^8}{8} + \frac{x^3}{3}\right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Example 7: Find $\int_C F.dr$, where $F = \left(2x + yz\right)i + xz\, j + (xy + 2z)\mathrm{k}$ along the curve $x^2 + y^2 = 1, z = 1$ in the positive direction from (0,1,1) to (1,0,1).

Answer: Let the curve is denoted by *C* and the points *A* and *B* are the points (0,1,1) to (1,0,1) respectively.

As we know the position vector of a point is, $r = xi + yj + zk$

$\Rightarrow dr = dx\, i + dy\, j + dz\, k$

Thus

$$F.dr = \left(\left(2x + yz\right)i + xz\, j + (xy + 2z)\mathrm{k}\right).\left(dx\, i + dy\, j + dz\, k\right) = \left(2x + yz\right)dx + xzdy + (xy + 2z)dz$$

$$\therefore \int_C F.dr = \int_C \left(2x + yz\right)dx + xzdy + (xy + 2z)dz$$

Along the curve from *A* to *B*, *x* varies from 0 to 1, *y* varies from 1 to 0 and *z* remains constant i.e., $dz = 0$

$$\therefore \int_C F.dr = \int_0^1 (2x+y)\,dx + \int_1^0 xdy + 0$$

$$\left[\int_a^0 f(x)\,dx = -\int_0^a f(x)dx\right]$$

$$= \int_0^1 \left(2x + \sqrt{1-x^2}\right)dx - \int_0^1 \sqrt{1-y^2}\,dy$$

$$= \int_0^1 2x + \int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 \sqrt{1-y^2}\,dy$$ (Integration does not depend upon variable)

$$= \int_0^1 2xdx = \left[x^2\right]_0^1 = 1$$

Example 8: If C is the line segment of the line $y = 2x$ in the xy-plane from (-1, -2) to (1, 2), then find

Answer: Since we have given curve C is line. First, we convert the equation of line in parametric form by putting $x = t$ and $y = t$.

So, $r(t) = xi + yj = ti + 2tj$

Then $\dfrac{dr}{dt} = i + 2j$

As we know that, $\dfrac{dr}{dt} = \dfrac{dr}{ds}\dfrac{ds}{dt}$

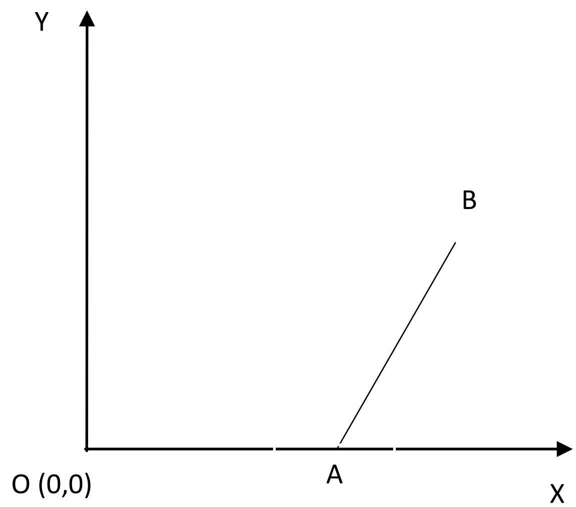
$\therefore \left|\dfrac{dr}{dt}\right| = \left|\dfrac{dr}{ds}\right|\dfrac{ds}{dt} = \dfrac{ds}{dt}$ (Because, $\dfrac{dr}{ds}$ is the unit vector)

$$\therefore \frac{ds}{dt} = \left|\frac{dr}{dt}\right| = \left|i + 2j\right| = \sqrt{5}$$

$$\int_C xy^3 ds = \int_C xy^3 \frac{ds}{dt}dt = \int_{-1}^1 t(2t)^3\sqrt{5}dt = 8\sqrt{5}\int_{-1}^1 t^4 dt = \frac{16}{\sqrt{5}}$$

Example 9: Find the value of $\int_C F.dr$, where C is the *xy*-plane curve formed by the straight lines from (0, 0) to (2, 0) and then to (3, 2), where $F = (2x + y)i + (3y - x)j$.

Answer: The figure of the path of curve C in the *xy*-plane is shown below. It consists straight lines *OA* and *AB*.



We have, $\int_C F.dr = \int_C [(2x + y)i + (3y - x)j].(dxi + dyj)$

$$= \int_C (2x + y)dx + (3y - x)dy$$

Along the straight-line *OA*, $y = 0, dy = 0$ and x varies from 0 to 2 and equation of the straight line *AB* is,

$$y - 0 = \frac{2-0}{3-2}(x - 2) \; i.e., \; y = 2x - 4$$

Along *AB*, $y = 2x - 4, dy = 2dx$ and x varies from 2 to 3.

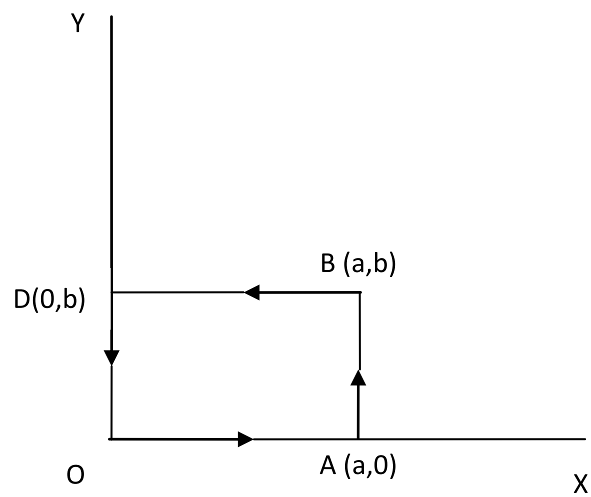
$$\int_C F.dr = \int_0^2 [(2x + 0)dx + 0] + \int_2^3 [(2x + 2x - 4)dx + 2(6x - 12 - x)dx]$$

$$= [x^2]_0^2 + \int_2^3 (14x - 28)dx = 4 + 14\int_2^3 (x - 2)dx$$

$$= 4 + 14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4 + 7 = 11$$

Example 10: If C is the rectangle bounded by $y = 0, x = a, y = b, \mathrm{x} = 0$ in the *xy*-plane then evaluate $\int_C F.dr$, where $F = \left(x^2 + y^2\right)i - 2xyj$.

Answer: The path of the integration C has been shown in figure which consists the straight lines *OA*, *AB*, *BD* and *DO*.



Now we have,

$$\int_C F.dr = \int_C \left[\left(x^2 + y^2\right)i - 2xyj\right].\left(dxi + dyj\right)$$

$$= \int_C \left(x^2 + y^2\right)dx + 2xydy$$

In the line *OA*, $y = 0, dy = 0$ and x varies from 0 to *a*.

In the line *AB*, $x = a, dx = 0$ and y varies from 0 to *b*.

In the line *BD*, $y = b, dy = 0$ and x varies from *a* to 0.

In the line *DO*, $x = 0, dx = 0$ and y varies from *b* to 0.

$$\int_C F.dr = \int_0^a x^2 dx - \int_0^b 2ay\,dy + \int_a^0 \left(x^2 + b^2\right)dx + \int_b^0 0\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2 x\right]_a^0 + 0 = -2ab^2$$

14.7 *GREEN'S THEOREM*

Let R be a closed bounded region in the x-y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. Then $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C \left(Mdx + Ndy\right)$, the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

Greens theorem in the plane (in vector notation):

We know that the position vector of a point is $r = xi + yj$ so that $dr = dxi + dyj$.

Let $F = Mi + Nj$. Then $Mdx + Ndy = \left(Mi + Nj\right).\left(dxi + dyj\right) = F.dr$

Since $curl\,F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z} i + \frac{\partial M}{\partial z} j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$

$$\left(\nabla \times F\right).k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Thus, the Green's theorem in plane can be written as

$$\iint_R \left(\nabla \times F\right).k\,dR = \oint_C F.dr \qquad \dots (1)$$

Where $dR = dxdy$ and k is the perpendicular unit vector to the xy-plane.

If s is the arc length of C and t denotes the unit tangent vector to C, then $dr = \frac{dr}{ds} ds = tds$. So, the equation (1) can also be rewritten as,

$$\iint_R (\nabla \times F).kdR = \oint_C F.\text{t}\, ds$$

Solved Examples

Example 11: If *C* is the closed curve of the region bounded by the straight line $y = x$ and the parabola $y = x^2$ then verify the Green's theorem in the plane $\oint_C (xy + y^2)\text{dx} + \text{x}^2\text{dy}$.

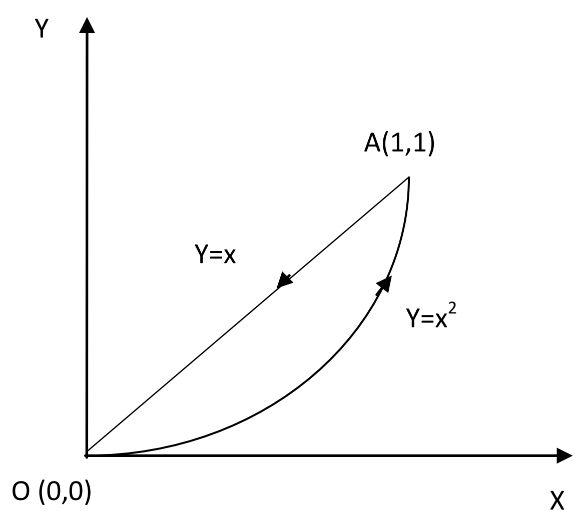
Answer: Since by the Green's function in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C (Mdx + Ndy) \quad \dots (1)$$

So, after comparing equation (1) with $\oint_C (xy + y^2)\text{dx} + \text{x}^2\text{dy}$ we get,

$$M = xy + y^2, \; N = \text{x}^2$$

The curves $y = x$ and $y = x^2$ intersect at the point (0,0) and (1,1) and positive direction in traversing *C* is as show in figure.



Thus, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}\left(xy + y^2\right) \right] dxdy$

$$= \iint_R (2x - x - 2y) dxdy = \iint_R (x - 2y) dxdy$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{x} (x - 2y) dydx = \int_0^1 \left[xy - y^2 \right]_{y=x^2}^{x} dx$$

$$\int_0^1 \left[xy - y^2 \right]_{y=x^2}^{x} dx = \int_0^1 \left[x^2 - x^2 - x^3 + x^4 \right] dx$$

$$= \int_0^1 \left(x^4 - x^3 \right) dx = \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Now we evaluate the line integral along C. Along the curve $y = x^2, dy = 2xdx$. Thus, along the curve $y = x^2$, the line integral equals

$$\int_0^1 \left[\left\{ (x)\left(x^2\right) + x^4 \right\} dx + x^2 (2x) dx \right] = \int_0^1 \left(3x^3 + x^4 \right) dx = \frac{19}{20}$$

Along $y = x, dy = dx$. Therefore, along the curve $y = x$, the line integral equals

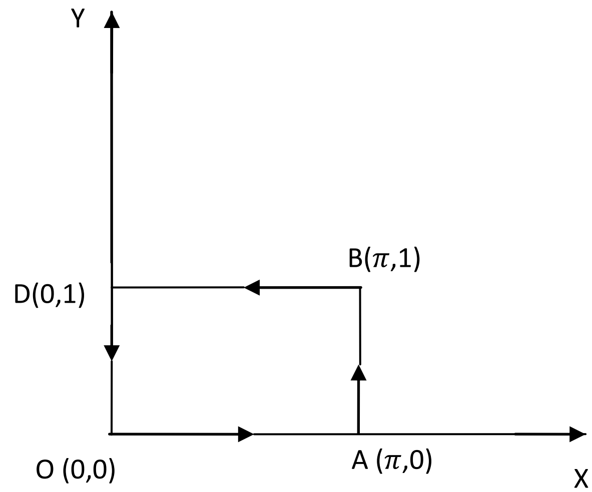
$$\int_1^0 \left[\left\{ (x)(x) + x^2 \right\} dx + x^2 dx \right] = \int_1^0 3x^4 dx = -1$$

Hence the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

Thus, the theorem is verified.

Example 12: Using the Green's theorem find the value of $\oint_C \left(x^2 - \cosh y \right) dx + \left(y + \sin x \right) dy$, where C is the rectangle having vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Answer: As we know by Green's theorem, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C \left(Mdx + Ndy \right)$ …. (1)



Comparing the equation (1) with the given equation $\oint_C \left(x^2 - \cosh y \right) dx + \left(y + \sin x \right) dy$ we get,

$$M = x^2 - \cosh y,\ N = y + \sin x$$

$$\therefore \frac{\partial M}{\partial y} = -\sinh y,\ \frac{\partial N}{\partial x} = \cos x$$

Thus, the given line integral is equal to

$$\iint_R \left(\cos x + \sinh y \right) dxdy = \int_{x=0}^{x=\pi} \int_{y=0}^{1} \left(\cos x + \sinh y \right) dydx$$

$$= \int_{x=0}^{\pi} \left[y\cos x + \cosh y \right]_{y=0}^{y=1} dx = \int_{x=0}^{\pi} \left[\cos x + \cosh 1 - 1 \right] dx$$

$$= \left[\sin x + x\cosh 1 - \mathrm{x} \right]_0^{\pi} = \cosh 1 - 1$$

Example 13: Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (xdy - ydx)$ also find the area generated by the ellipse $x = a\cos\theta, y = b\sin\theta$.

Answer: As we know for the plane region bounded by closed curve C, Green's theorem is

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (Mdx + Ndy)$$

putting $M = -y,\ N = x$ we get,

$$\oint_C (xdy - ydx) = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dxdy$$

$= 2\iint_R dxdy = 2A$, where $A = \frac{1}{2}\oint_C (xdy - ydx)$ is the area bounded by C.

The area of the ellipse $= \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\left(a\cos\theta\frac{dy}{d\theta} - b\sin\theta\frac{dx}{d\theta}\right)d\theta$

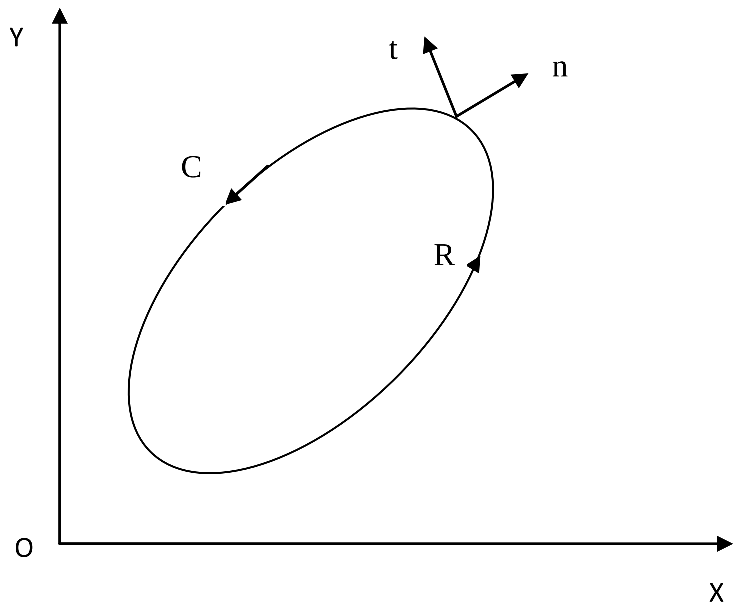
$$= \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\int_0^{2\pi}\left(ab\cos^2\theta + ab\sin^2\theta\right)d\theta = \frac{ab}{2}\int_0^{2\pi} d\theta = \pi ab.$$

Example 14: Prove that the Green's function may be written as $\iint_R divA\,dxdy = \oint_C A.n\,ds$, where $A = Ni - Mj$, n is the outward unit normal vector to C and s denotes the arc length of C.

Answer: Since we have, $A = Ni - Mj$

$$\therefore \quad divA = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$\therefore \quad \iint_R divA\,dxdy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (Mdx + Ndy)$, by Green's theorem



Since we can write, $Mdx + Ndy = (Mi + Nj).(dxi + dyj)$

$$= (Mi + Nj).\mathrm{dr} = \left\{(Mi + Nj).\frac{dr}{ds}\right\}ds$$

Let t is unit tangent vector to C, then $t = \frac{dr}{ds}$ and we also know that the unit vector k is perpendicular to xy-plane. Then, $t = k \times n$

So, $Mdx + Ndy = \left[(Mi + Nj).t\right]ds = \left[(Mi + Nj).(k \times n)\right]ds$

$$= \left[(Mi + Nj) \times k\right].nds = (Mi \times k + Nj \times k).nds = (Ni - Mj).nds = A.nds$$

Hence proved.

Note: Putting $A = \nabla\phi$ in previously discussed result, we get

$$\iint_V div(\nabla\phi) dx\, dy = \oint_C (\nabla\phi).nds$$

Or $\iint_V \nabla^2 \phi dx\, dy = \oint_C \frac{\partial \phi}{\partial n} ds$, since $\nabla \phi = \frac{\partial \phi}{\partial n} n$

14.8 *THE DIVERGENCE THEOREM OF GAUSS*

Assume that V is the volume enclosed by the closed, piecewise smooth surface S. Assume $F(x, y, z)$ is a continuous vector function of position with a continuous first partial derivative in V. Then

$$\iiint_V \nabla . F dV = \iint_S F . n\, ds$$

where n is the unit normal vector drawn outward from S.

Here, $F.n$ is normal component of vector F, consequently, the divergence theorem may be explained as follows:

The integral of the divergence of a vector F taken over the surface's enclosed volume is the same as the surface integral of the normal component of a vector F taken over a closed surface.

Divergence theorem in cartesian form:

Let $F = F_1 i + F_2 j + F_3 k$ is vector function.

Thus, $\nabla . F = divF = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the outward drawn unit normal n where α, β, γ are the angles which are taking with the positive direction of $\text{x, y, z} -$ axes.

$$n = \cos\alpha i + \cos\beta j + \cos\gamma k$$

$$\therefore F.n = (F_1 i + F_2 j + F_3 k).(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$$

$$= F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$$

Thus the divergence theorem can be rewritten as,

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dS$$

$$= \iint (F_1 dydz + F_2 dxdx + F_3 dxdy)$$

Remarks: *The divergence theorem is significant because it shows how a surface may be written as a volume integral and vice versa.*

Note: If a region V is surrounded by two closed surfaces S_1 and S_2, one of which falls within the other, the divergence theorem applies to that region.

Some important deduction from divergence theorem:

1. **Green's Theorem:** Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface, then

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi).nds$$

2. **Harmonic Function:** If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, then ϕ is called harmonic function. If ϕ and ψ are both harmonic functions, then $\nabla^2\phi = 0$, $\nabla^2\psi = 0$.

Since from Green's second identity, we get $\iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = 0$.

Note 1: $\iiint_V \nabla\phi\, dV = \iint_S \phi\, n\, dS$

2: $\iiint_V \nabla \times B\, dV = \iint_S n \times B\, dS$

Solved Examples

Example 15: For any closed surface *S*, prove that $\iint_S curlF.n\, ds = 0$

Solution: By divergence theorem, we have

$\iint_S curlF.n\, ds = \iiint_V (div\, curl\, F)\, dV$, where V is the volume enclosed by S

$= 0,$ since $div\, curl\, F = 0$.

Example 16: Evaluate $\iint_S r.n\,ds,$ where S is a closed surface.

Solution: $\iint_S r.n\,ds = \iiint_V \nabla.r\,dV = \iiint_V 3\,dV$, Since $\nabla.r = div\,r = 3$

$= 3V,$ where V is the volume enclosed by S .

Example 17: If $F = ax\,i + by\,j + cz\,k, a, b, c$ are constant then prove that

$\iint_S F.n\,ds = \frac{4}{3}\pi(a+b+c)$, where S is the surface of unit sphere.

Solution: By the divergence theorem we have,

$\iint_S F.n\,ds = \iiint_V (\nabla.F)\,dV$, where V is the volume enclosed by S.

$$= \iiint_V [\nabla.(ax\,i + by\,j + cz\,k)]dV = \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right]dV$$

$$= \iiint_V (a+b+c)dV = (a+b+c)V = (a+b+c)\frac{4}{3}\pi$$

Since the volume V is enclosed by a sphere of unit radius is equal to
$\frac{4}{3}\pi(1)^3$ *i.e.*, $\frac{4}{3}\pi.$

Example 18: If n is the unit outward drawn normal to any closed surface S, show that

$$\iiint_V div\,n\,dV = S\ .$$

Solution: Since we have the divergence theorem,

$$\iiint_V div\,n\,dV = \iint_S n.ndS = \iint_S dS = S$$

Example 19: Prove that, $\iiint_V \frac{dV}{r^2} = \iint_S \frac{r.n}{r^2}dS$.

Solution: $\iint_S \frac{r.n}{r^2} dS = \iint_S \frac{r}{r^2} .ndS = \iiint_V \nabla.\left(\frac{r}{r^2}\right) dV$, (by using divergence theorem)

So, $\nabla.\left(\frac{r}{r^2}\right) = \frac{1}{r^2}(\nabla.r) + r.\nabla\left(\frac{1}{r^2}\right)$

$$= \frac{3}{r^2} + r.\left(-\frac{2}{r^3}\nabla r\right) = \frac{3}{r^2} - \frac{2}{r^3}\left(r.\frac{r}{|r|}\right) = \frac{3}{r^2} - \frac{2}{r^4}r^2 = \frac{1}{r^2}$$

Example 20: Using divergence theorem, prove that the volume V of a region T bounded by a surface S is,

$$\iint_S x\, dydz = \iiint_V \frac{\partial}{\partial x}(x)\, dV = \iiint_V dV = V \qquad \ldots (1)$$

$$\iint_S y\, dzdx = \iiint_V \frac{\partial}{\partial y}(y)\, dV = \iiint_V dV = V \qquad \ldots (2)$$

$$\iint_S z\, dxdy = \iiint_V \frac{\partial}{\partial z}(z)\, dV = \iiint_V dV = V \qquad \ldots (3)$$

Adding equation (1), (2) and (3) we get the result

$$3V = \iint_S (x\, dydz + y\, dzdx + z\, dxdy)$$

$$V = \frac{1}{3}\iint_S (x\, dydz + y\, dzdx + z\, dxdy)$$

Example 21: If $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. Then verify the divergence theorem.

Solution: We have $div\, F = \nabla.F$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z$$

$\therefore$ Volume integral $= \iiint_V \nabla.F\, dV = \iiint_V 2(x + y + z)\, dV$

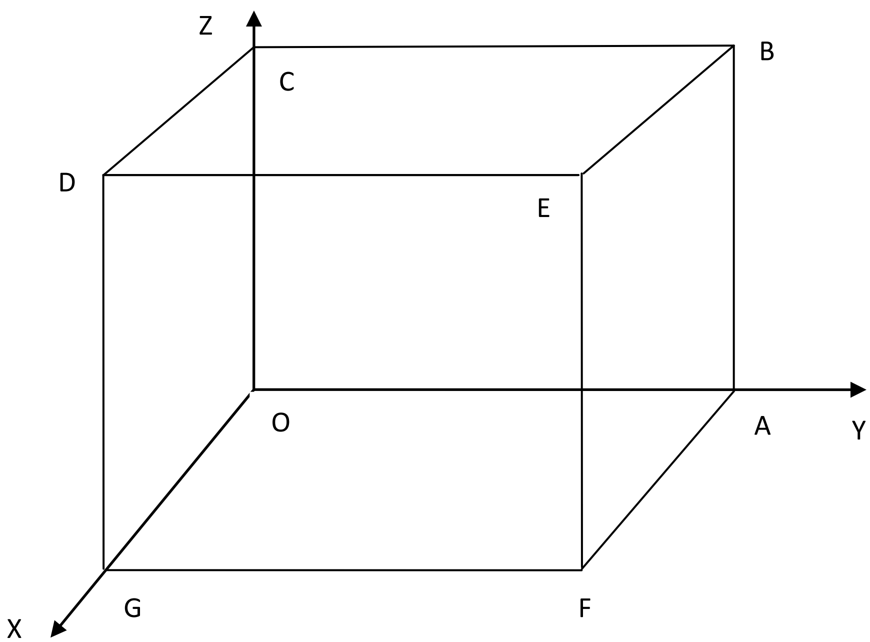
$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(x+y+z)dx\,dy\,dz = 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{x^2}{2}+yx+zx\right]_{x=0}^{a} dydz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2}+ax+az\right]dydz = 2\int_{z=0}^{c}\left[\frac{a^2}{2}y+a\frac{y^2}{2}+azy\right]_{y=0}^{b} dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right]dz = 2\left[\frac{a^2b}{2}z+\frac{ab^2}{2}z+ab\frac{z^2}{2}\right]_0^c$$

$$= [a^2bc+ab^2c+abc^2] = abc(a+b+c)$$

Surface integral: We shall now calculate $\iint_S F.n\,dS$ over the six faces of the rectangular parallelepiped.



Over the face DEG, $\hat{n} = i, x = a$.

Therefore, $\iint_{DEFG} F.n\,dS$

$$= \int_{z=0}^{c}\int_{y=0}^{b}\left[\left(a^2-yz\right)i+\left(y^2-za\right)j+\left(z^2-ay\right)k\right].idydz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}\left(a^2 - yz\right)dydz = \int_{z=0}^{c}\left[a^2 y - z\frac{y^2}{2}\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c}\left[a^2 b - \frac{zb^2}{2}\right]dz = \left[a^2 bz - \frac{z^2}{4}b^2\right]_0^c = a^2 bc - \frac{c^2 b^2}{4}$$

Over the face ABCO, $\hat{n} = -i, x = 0$. Therefore,

$$\iint_{ABCO} F.n\,dS = \iint [(0 - yz)i + ... + ...].(-i)dydz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yzdydz = \int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_{y=0}^{b} dz = \int_{z=0}^{c}\left[\frac{b^2}{2}z\right]dz = \frac{b^2 c^2}{4}$$

Over the face ABEF, $\hat{n} = j, y = b$. Therefore,

$$\iint_{ABEF} F.n\,dS = \int_{z=0}^{c}\int_{x=0}^{a}\left[\left(x^2 - bz\right)i + \left(b^2 - bz\right)j + \left(z^2 - bx\right)k\right].\,jdxdz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a}(b^2 - zx)dxdz = b^2 ca - \frac{a^2 c^2}{4}$$

Over the face OGDC, $\hat{n} = -j, y = 0$. Therefore,

$$\iint_{OGDC} F.n\,dS = \int_{z=0}^{c}\int_{x=0}^{a} zxdxdz = \frac{c^2 a^2}{4}$$

Over the face BCDE, $\hat{n} = k, z = c$. Therefore,

$$\iint_{BCDE} F.n\,dS = \int_{y=0}^{b}\int_{x=0}^{a}(c^2 - xy)dxdy = c^2 ab - \frac{a^2 b^2}{4}$$

Over the face AFGO, $\hat{n} = -k, z = 0$. Therefore,

$$\iint_{AFGO} F.n\,dS = \int_{y=0}^{b}\int_{x=0}^{a} xydxdy = \frac{a^2 b^2}{4}$$

Now adding six surface integrals, we get

$$\iint_S F.n\,dS = \left(a^2bc - \frac{c^2b^2}{4} + \frac{c^2b^2}{4}\right) + \left(b^2ca - \frac{a^2c^2}{4} + \frac{a^2c^2}{4}\right) + \left(c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}\right)$$

$$= abc(a+b+c)$$

Hence the theorem is verified.

Example 22: Evaluate $\iint_S x^2 dydz + y^2 dzdx + 2z(xy - x - y)dxdy$

Where S is the surface integral of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

Proof: By using divergence theorem, we convert the given surface integral into volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z(xy - x - y)\}\right] dV$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}[2x + 2y + 2xy - 2x - 2y]dx\,dy\,dz = 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xydx\,dy\,dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}y\right]_{x=0}^{1} dy\,dz = \int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz = \int_{z=0}^{1} 1/2\,dz = 1/2$$

14.9 STOKE'S THEOREM

Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $F(x, y, z)$ be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then

$$\oint_C F.dr = \iint_S (\nabla \times F).nds = \iint_S (curl\ F).ds$$

Where C is traversed in the position direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal n to S, has the surface on the left.

Note: $\oint_C F.dr = \oint_C \left(F.\frac{dr}{ds} \right) ds = \oint_C (F.t) ds,$ where t is unit tangent vector to C. Therefore $F.t$ is the component of $curl\, F$ in the direction of outward drawn normal vector n of S. Therefore in other words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector F taken around a simple closed curve C is equal to the surface integral of the normal component of the $curl$ of F taken over any surface S having C as its boundary.

Cartesian equivalent of stokes theorem:

Let $F = F_1 i + F_2 j + F_3 k$. Let outward drawn normal vector n of S make angles α, β, γ with positive directions of x, y, z axes. Then $n = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.

Also,

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i + \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial x} \right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k$$

$$(\nabla \times F).n = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos\gamma$$

Also, $F.dr = (F_1 i + F_2 j + F_3 k).(dx\, i + dy\, j + dz\, k) = F_1 dx + F_2 dy + F_3 dz$

So, Stoke's theorem can be rewritten as,

$$\oint_C F_1 dx + F_2 dy + F_3 dz = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \cos\gamma \right] dS$$

Note: *Green's theorem in plane is special case of Stoke's theorem*. If R is a region in the $xy-$plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_S (\nabla \times F).k\, dR = \oint_C F.dr$$

This is nothing but a special case of Stoke's theorem because here $k = n =$ outward drawn unit normal to the surface of region R .

Solved Example

Example 23: Prove that $\oint_C r\,.dr = 0$

Solution: By Stoke's theorem $\oint_C r\,.dr = \iint_S (curl\, r).n\, ds = 0$, since

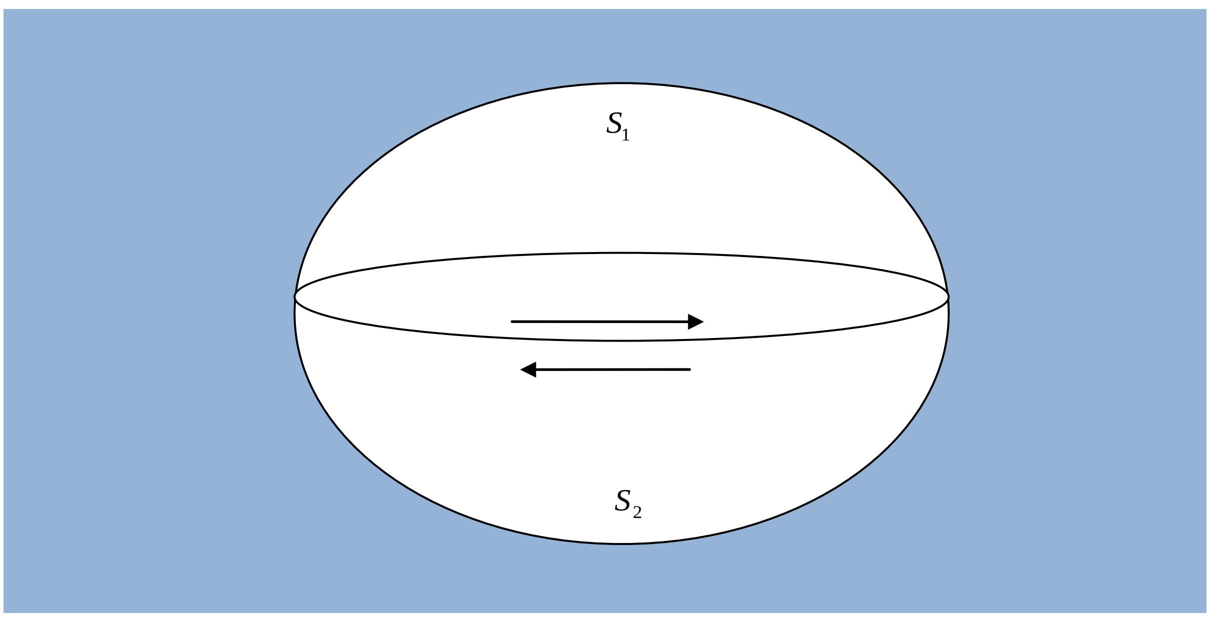
$curl\, r = 0$

Example 24: By Stoke's theorem prove that $div\, curl\, F = 0$

Solution: Let V is the volume enclosed by a closed surface. Then using the divergence theorem

$$\iiint_V \nabla.(curl\, F)\, dV = \iint_S (curl\, F).n\, dS$$

Divide the surface S in two section S_1 and S_2 by a closed curve C (as in figure). Then



$$\iint_S (curl\, F).n\, dS = \iint_{S_1} (curl\, F).n\, dS_1 + \iint_{S_2} (curl\, F).n\, dS_2 \qquad \ldots (1)$$

In the right hand side of equation (1) by Stoke's theorem

$$= \oint_C F\,.dr - \oint_C F\,.dr = 0$$

Here, the negative sign indicate because the positive directions about the boundaries of the two surfaces are opposite.

$$\iiint_V \nabla.(curl\ F)\, dV = 0$$

Now this equation is true for all volume element V. Therefore we have,

$$\nabla.(curl\ F)\, dV = 0 \quad or \quad div\, curl\ F = 0$$

Example 25: Using Stoke's theorem prove that $curl\ grad\ \phi = 0$.

Solution: Let S be the surface which is enclosed by a simple closed curve C. Then by Stoke's theorem, we have

$$\iint_S (curl\ grad\ \phi).n\, dS = \oint_C grad\phi.dr$$

Now

$$grad\phi.dr = \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right).(dx\, i + dy\ j + dz\ k) = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi$$

$\therefore \oint_C grad\phi.dr = \oint_C d\phi = [\phi]_A^A$, where A is any point on C

$$= 0$$

Therefore we have $\iint_S (curl\ grad\ \phi).n\, dS = 0$

Now this equation is true for all surface elements S.

Therefore we have, $(curl\ grad\ \phi) = 0$

Example 26: Verify the Stoke's theorem for $F = yi + zj + xk$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: The boundary C of S is a circle in the $xy-plane$ or radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1, z = 0$.

Let us assume $x = \cos t, y = \sin t, z = 0,\ 0 \le t < 2\pi$ are parametric equation of C. Then

$$\therefore \oint_C F.dr = \oint_C (yi + zj + xk).(dxi + dyj + dzk) = \oint_C (ydx + zdy + xdz) = \oint_C ydx,$$

Since on $C, z = 0$ and $dz = 0$.

$$= \int_0^{2\pi} \sin t \frac{dx}{dt} dt = \int_0^{2\pi} -\sin^2 t dt$$

$$= -\frac{1}{2}\int_0^{2\pi}(1 - \cos 2t)dt = -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = -\pi \qquad \ldots (1)$$

Now let us evaluate $\iint_S curl\, F.ndS$. We have

$$curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S curl\, F.ndS = \iint_{S_1} curl\, F.k\, dS \qquad \text{[Always remember]}$$

$$= \iint_{S_1} (-i - j - k).k\, dS = \iint_{S_1} (-1)\, dS = -\iint_{S_1} dS = -S_1$$

But $S_1 =$ area of a circle of radius $1 = \pi(1)^2 = \pi$

$$\therefore \iint_S curl\, F.ndS = -\pi \qquad \ldots (2)$$

Now from (1) and (2), the theorem is verified.

Example 27: Verify Stoke's theorem for $F = (2x - y)i - yz^2 j - y^2 zk$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: The boundary C of S is a circle in the $xy-$plane of radius unity and centre origin. Suppose $x = \cos t, y = \sin t, z = 0,\ 0 \le t < 2\pi$ is parametric equation of C. Then

$$\oint_C F.dr = \oint_C [(2x-y)i - yz^2 j - y^2 zk).(dxi + dyj + dzk) = \oint_C [(2x-y)dx - yz^2 dy - y^2 zdz]$$

$\oint_C (2x-y)dx$, since $z=0$ and $dz=0$

$$\int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}dt = \int_0^{2\pi} (2\cos t - \sin t)\sin t dt = \int_0^{2\pi} [\sin 2t - \frac{1}{2}(1-\cos 2t)]dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi} = -\left[\left(-\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}(\pi - 0) + \frac{1}{4}(0-0)\right] = \pi$$

… (1)

And

$$(\nabla \times F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)i - (0-0)j + (0+1)k = k$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is the closed surface.

Using the application of Gauss divergence theorem, we have

$$\iint_{S'} curl\, F.n\, dS = 0$$

$$\iint_S curl\, F.n\, dS = \iint_{S_1} curl\, F.n\, dS = 0 \qquad [\because S' \text{ consists of } S \text{ and } S_1]$$

$$\iint_S curl\, F.n\, dS - \iint_{S_1} curl\, F.k\, dS = 0 \qquad [\text{on } S_1, n = -k]$$

$$\iint_S curl\, F.n\, dS = \iint_{S_1} curl\, F.k\, dS$$

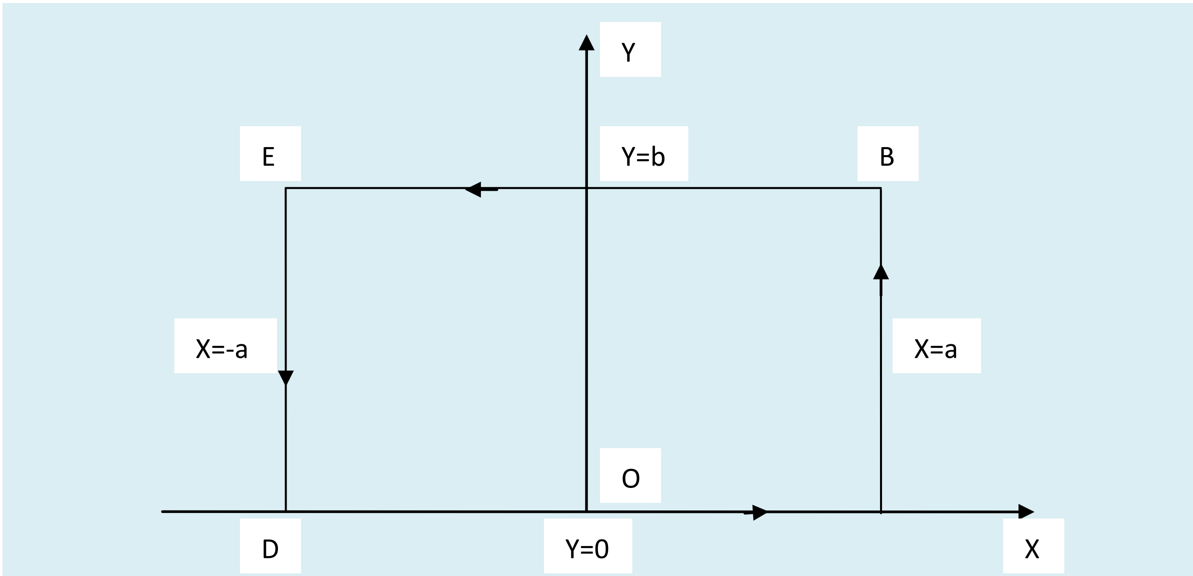
$$\therefore \iint_S curl\, F.n\, dS = \iint_{S_1} curl\, F.k\, dS = \iint_{S_1} k.k\, dS = \iint_{S_1} dS = S_1 = \pi$$

… (2)

Note that $S_1 =$ area of a circle of radius $1 = \pi(1)^2 = \pi$

Thus by equation (1) and equation (2) Stoke's theorem is verified.

Example 28: Verify Stokes theorem for $F = (x^2 + y^2)i - 2xyj$ taken round the rectangle bounded by $x = \pm a, y = 0, y = b$.



Solution: We have $curl\ F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)k = -4yk$

Also $n = k$

$$\therefore \iint_S (curl\ F).n\, dS = \int_{y=0}^{b} \int_{x=-a}^{a} (-4yk).k\, dxdy$$

$$= -4 \int_{y=0}^{b} \int_{x=-a}^{a} y\, dxdy = -4 \int_{y=0}^{b} [xy]_{x=-a}^{a} dy = -4 \int_{y=0}^{b} 2aydy = -4\left[ay^2\right]_0^b = -4ab^2$$

Also $\oint_C F.dr = \oint_C [(x^2 + y^2)i - 2xy\, j].(dxi + dy\, j) = \oint_C [(x^2 + y^2)dx - 2xydy]$

$$= \oint_{DA} [(x^2 + y^2)dx - 2xydy] + \int_{AB} + \int_{BE} + \int_{ED}$$

Along $DA, y = 0, dy = 0$. Along $AB, x = a, dx = 0$. Along $BE, y = b, dy = 0$. Along $ED, x = -a, dx = 0$.

$$\therefore \oint_C F.dr = \int_{x=-a}^{a} x^2 dx + \int_{y=0}^{b} -2ay dy + \int_{x=-a}^{a} (x^2+b^2)dx + \int_{y=b}^{0} 2ay dy .$$

$$= \int_{x=-a}^{a} x^2 dx - \int_{x=-a}^{a} (x^2+b^2)dx - 4a\int_0^b y dy$$

$$= -\int_{x=-a}^{a} x^2 dx - 4a\int_0^b y dy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2$$

Thus, $\oint_C F.dr = \iint_S (curl\ F).ndS$

Hence the theorem is verified.

SELF CHECK QUESTION

Fill in the blanks:

12. $\int_0^1 [ti + (t^2-2t)j]dt = 0$ **………………..**

13. If $F(t) = 3t^2 i + t\ j + 2k$ and $G(t) = 6t^2 i + (t-1)j + 3tk$, then $\int_0^1 \left(\frac{dF}{dt}.G + F.\frac{dG}{dt}\right)dt = 0$ **……….**

14. Any integral which is to be evaluated along a curve is called a ……………

15. Any integral which is to be evaluated over a surface is called a ……………

16. $\iint_S F.n\,dS$ is called the ………….. of F over S

17. For any Closed surface S, $\iint_S curlF.n\,dS =$ **……………..**

14.10 *SUMMARY*

After completion of this unit learners are able to memorize and analyze

- The application of line, surface and volume integrals.
- The applications of Green's theorem.
- The application of Gauss divergence theorem.
- The application of Stoke's theorem.
- Basic differences between line integral, surface integral, volume integral, Green's, Gauss and Stoke's theorem.

14.11 *GLOSSARY*

- Line integral
- Surface integral
- Volume integral
- Green's theorem
- Gauss divergence theorem
- Stoke's theorem

14.12 *REFERENCES*

- Spiegel, R. Murray (1959), *Vector Analysis*, Schaum's Outline Series.
- N. Saran and S. N. Nigam, *Introduction to vector analysis*, Pothishala Pvt. Ltd. Allahabad.
- Erwin. Kreyszig, "Advanced engineering mathematics, 10th eddition", 2009.
- A. R. Vasishtha, "Vector Calculus", 20th edition, Krishna publication, 2020.

14.13 *SUGGESTED READING*

- Shanti Narayan (2003), *A Textbook of Vector Calculus*, S. Chand Publishing.
- Shanti Narayan and P. K. Mittal (2010). *A textbook of matrices*, S. Chand Publishing.

14.14 *TERMINAL QUESTION*

Objective type question:

7. If C is the curve $x^2 + y^2 = 1, z = y^2$, then by Stoke's theorem $\oint_C (yzdx + zxdy + xydz)$ is

b. 0 **b.** 3
d. 5 **d.** None of these

8. If S denote the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$ then by application of Gauss divergence theorem the value of $\iint_S (xi + yj + zk).n\,dS$ is

b. a^3 **b.** $2a^3$

d. $3a^3$ **d.** 0

9. If $F(t) = ti + (t^2 - 2t)j + (3t^2 + 3t^3)k$, then the value of $\int_0^1 F(t)dt$ is

b. $\frac{1}{2}i + \frac{2}{3}j + \frac{7}{4}k$ **b.** $\frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$

d. $-\frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$ **d.** none of these

10. If $x^2 i + y^3 j$ and curve C is the arc of the parabola $y = x^2$ in the $xy-$plane from $(0,0)$ to $(1,1)$ then $\oint_C F.dr$ is

b. 7/12 **b.** 5/12

d. 11/12 **d.** none of these

11. The work done in moving a particle in a force field $F = 3x^2 i + (2xz - y)j + 3k$ along the line joining $(0,0,0)$ to $(2,1,3)$ is

c. 12 **b.** 16

d. 0 **d.** 20

12. For a closed surface S, the value of $\iint_S r\,.\,n\,dS$ is

b. V **b.** $2V$

d. $3V$ **d.** 0

Find True and False Statement.

8. $\int\left(2\frac{dr}{dt}.\frac{d^2r}{dt^2}\right)dt = \left(\frac{dr}{dt}\right)^2 + C$

9. If C is a simple closed curve, then $\oint_C F.dr$ is called the circulation of F about C.

10. If $F = axi + byj + czk, a, b, c$ are constant, then $\iint_S F\,.n\,dS = \frac{2}{3}\pi(a + b + c)$ where S is the surface of a unit sphere.

11. For any surface S, $\iint_S n\, dS = 0$
12. Green's theorem in plane is a special case of Stoke's theorem.
13. Green's theorem states that "the surface integral of the normal component of a vector F taken over a closed surface is equal to the integral of the divergence of F taken over the volume enclosed by the surface".

Short answer type question:

7. Evaluate $\int_C F.dr$ where F is $x^2y^2i + yj$ and C is $y^2 = 4x$ in the xy-plane from (0,0) to (4,4)

8. Evaluate $\int (xdy - ydx)$ around the circle $x^2 + y^2 = 1$.

9. Evaluate $\int_C F.dr$, where $F = yzi + zxj + xyk$ and C is the portion of the curve $r = a\cos ti + b\sin tj + ctk$ from $t = 0$ to $t = \pi/2$.

10. Evaluate $\int_C F.dr$, where $F = zi + xj + yk$ and C is the portion of arc of the curve $r = \cos ti + \sin tj + tk$ from $t = 0$ to $t = 2\pi$.

11. Verify Green's theorem in the plane for $\int_C [(2xy - x^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense.

12. Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$.

13. Evaluate by Green's theorem in the plane $\int_C (e^{-x}\sin y dx + e^{-x}\cos y dy]$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,\pi/2), (0,\pi/2)$ and $y = x^2$.

14. Verify divergence theorem for $F = (2x - z)i + x^2yj - xz^2k$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

15. Verify divergence theorem for $F = 4xzi - y^2yj + yzk$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

16. Prove that $\iint_S r \times n\, dS = 0$, for any closed surface S.

17. By using Gauss divergence theorem evaluate $\iint_S (xi + yj + z^2k).ndS$, where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$
18. Verify Stoke's theorem for $F = zi + xj + yk$ where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$.
19. Verify Stoke's theorem for $A = 2yi + 3xj - z^2k$ where S is upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.

Long answer type question.

7. Verify Stoke's theorem for the vector $B = zi + xj + yk$ taken over half of the sphere $x^2 + y^2 + z^2 = a^2$ lying about the $xy -$ plane.

8. Evaluate $\oint_C F.dr$ by stoke's theorem where $F = y^2i + x^2j - (x + z)k$ and C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0), (1,1,0)$.

9. Verify Stoke's theorem for $F = -y^3i + x^3j$, where S is the circular disc $x^2 + y^2 \le 1, z = 0$.

10. Evaluate $\oint_C (xydx + xy^2dy)$ by Stoke's theorem where C is the positively oriented square with vertices $(1,0), (-1,0), (0,1)$ *and* $(0,-1)$

11. Use Gauss divergence theorem to show that $\iint_S \{(x^3 - yz)i - 2x^2yj + 2k\}.ndS = \frac{1}{3}a^5$, where S denotes the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$

12. By Gauss divergence theorem, evaluate $\iint_S (xi + yj + z^2k).ndS$, where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

13. If $F = axi + byj + czk$, where a, b, c are constant, show that $\iint_S n.F\, dS = \frac{4\pi}{3}(a + b + c)$, S being surface of the sphere $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1$.

14. Verify Green's theorem in the plane to evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the surface enclosed by the x-axis and the semi circle $y = (1-x^2)^{1/2}$.

15. Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$, where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$. .

14.15 *ANSWERS*

Answer of self cheque questions:

1. $\frac{1}{2}i - \frac{2}{3}j$ **2.** 24 **3.** Line integral

5. Surface integral **5.** Flux **6.** 0

Answer of objective questions:

2. a **2.** c **3.** b

5. a **5.** b **6.** c

Answer of true and false questions:

2. T **2.** T **3.** F

5. F **5.** T **6.** F

Answer of short answer type questions:

1. 264 **2.** 2π **3.** 0

4. 3π **7.** $2(e^{-\pi} - 1)$ **11.** $7\pi/6$

Answer of long answer type questions:

2. 1/3 **3.** $3\pi/2$ **4.** 1/3

6. $7\pi/6$